PERFORMANCE ANALYSIS OF ELLIPTIC CURVE MULTIPLICATION ALGORITHMS FOR ELLIPTIC CURVE CRYPTOGRAPHY

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ABSTRACT

PERFORMANCE ANALYSIS OF ELLIPTIC CURVE MULTIPLICATION ALGORITHMS FOR ELLIPTIC CURVE CRYPTOGRAPHY

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Elliptic curve cryptography (ECC) has been introduced as a public-key cryptosystem, which offers smaller key sizes than the other known public-key systems at equivalent security level. The key size advantage of ECC provides faster computations, less memory consumption, less processing power and efficient bandwidth usage. These properties make ECC attractive especially for the next generation public-key cryptosystems. The implementation of ECC involves so many arithmetic operations; one of them is the elliptic curve point multiplication operation, which has a great influence on the performance of ECC protocols.

In this thesis work, we have studied on elliptic curve point multiplication methods which are proposed by many researchers. The software implementations of these methods are developed in C programming language on Pentium 4 at 3 GHz. We have used NIST-recommended elliptic curves over prime and binary fields, by using efficient finite field arithmetic. We have then applied our elliptic curve point multiplication implementations to Elliptic Curve Digital Signature

Algorithm (ECDSA), and compared different methods. The timing results are presented and comparisons with recent studies have been done.

Keywords: Elliptic Curve Cryptography, Elliptic Curve Point Multiplication, Prime Field, Binary Field, Software Implementation

ÖZ

ELİPTİK EĞRİ KRİPTOGRAFİSİNDE KULLANILAN ÇARPMA ALGORİTMALARININ BAŞARIM ÇÖZÜMLEMESİ

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Eliptik eğri kriptografisi (EEK), aynı güvenlik seviyesinde diğer asimetrik anahtar kripto sistemlerinden daha kısa anahtar boyları gerektiren bir asimetrik anahtar kripto sistemi olarak önerilmiştir. Anahtar boyunun kısalığı, hızlı hesaplamalar, daha az bellek alanı gereksinimi, daha az işlem gücü ve bant genişliği verimliliği sağlar. Yeni nesil asimetrik anahtar kripto sistemlerinde, EEK sahip olduğu bu özellikleriyle ilgi odağı olmuştur. EEK uygulaması birçok aritmetik işlemden oluşmaktadır. Eliptik eğri noktası çarpma işlemi bu işlemlerden biridir ve kripto protokollerindeki işlem hızında en etkin olanıdır.

Bu çalışmada, birçok araştırmacı tarafından önerilmiş eliptik eğri nokta çarpma metodları incelenmiştir. Bu metodların C programlama dilinde, 3 GHz'lik Pentium 4 işlemci üzerinde yazılım uygulaması yapılmıştır. Uygulamalarda NIST tarafından tavsiye edilien, asal ve ikili alanlarda tanımlanmış eliptik eğriler kullanılmıştır. Uygulaması yapılan eliptik eğri nokta çarpma metotları, Eliptik Eğri Sayısal İmza Algoritması'nda kullanılmış ve değişik metotlar karşılaştırılmıştır. Elde edilen zaman değerleri, son zamanlarda yapılan çalışmalarla karşılaştırılmıştır.

Anahtar sözcükler: Eliptik Eğri Kriptografisi, Eliptik Eğri Nokta Çarpma İşlemi, Asal Alan, İkili Alan, Yazılım Uygulama.

To my parents, Gülsen & Mahmut Acımış

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

AES	Advanced Encryption Standard
DES	Data Encryption Standard
DL	Discrete Logarithm
DLP	Discrete Logarithm Problem
DSA	Digital Signature Algorithm
EC	Elliptic Curve
ECC	Elliptic Curve Cryptography
ECDLP	Elliptic Curve Discrete Logarithm Problem
ECDSA	Elliptic Curve Digital Signature Algorithm
FIPS	Federal Information Processing Standards
IFP	Integer Factorization Problem
NAF	Non-Adjacent Form
NIST	National Institute of Standards and Technology
RSA	Rivest-Shamir-Adleman

CHAPTER 1

INTRODUCTION

For over a hundred years, mathematicians have studied elliptic curves. In 1985, elliptic curves were used independently by Neal Koblitz [Kob, 1987] and Victor Miller [Mil, 1986] to design public key cryptographic systems. Their proposal was using the group of points on an elliptic curve (EC) defined over a finite field to implement discrete log cryptosystems. Since then lots of research have been published on the security and efficient implementation of elliptic curve cryptography. In the late 1990, elliptic curve systems started receiving commercial acceptance. Standard organizations specified elliptic curve protocols and private companies included these protocols in their security products. When we compare the traditional public-key cryptographic algorithms, elliptic curve cryptography algorithms can achieve the same level of security with shorter key lengths. The shorter key lengths provide speed, less memory usage and less energy consumption for a cryptosystem. That is why elliptic curve cryptography has become a challenging interest for many researchers.

Since elliptic curve cryptography aspects are based on EC point multiplication operation, we focused on EC point multiplication methods. This thesis study has covered many elliptic curve multiplication methods, published until now [HanMenVan, 2004], [BrHankLopMen, 2001], [HankHernMen, 2000]. In this work, software implementations of Right-to-Left Binary, Left-to-Right Binary, Binary NAF (Non-Adjacent Form), Window NAF, Sliding Window, Montgomery, Fixed-base Window, Fixed-base NAF, Fixed-base Comb, and Fixed-base Comb (with two tables) methods have been done on Pentium 4 processor at 3 GHz and their performances have been measured. We have developed the EC point multiplication algorithms both for prime and binary fields. The main performance criterion is the speed of the algorithms, which is affected by the choice of the field, the methods used in performing the field operations, the coordinate system for EC point multiplication algorithms, and the algorithm parameters such as window width. We have optimized the speed performances of these EC point multiplication methods by suitable choice of the field arithmetic methods, point representation and window width of the algorithms. The comparisons of EC point multiplication methods, their superiority to each other and behavior according to the chosen field have been discussed.

The organisation of this thesis is as follows. Chapter 2 begins with the general overview of public-key cryptosystems. Then elliptic curve arithmetic is described, definitions and descriptions of the basic elliptic curve operations are stated [HankMenVan, 2004].

In Chapter 3, the description of the field arithmetic is given [HankHernMen, 2000], [BrHankLopMen, 2001]. The arithmetic operations and algorithms differ according to the binary field or prime field choice. EC point multiplication implementations are based on field operations. This chapter presents the performances of the implemented field operations. We continue by focusing on EC point multiplication operation. The algorithms are described theoretically in detail [HankMenVan, 2004]. We have implemented the EC point multiplication methods in C code according to these descriptions. The performance of the implementations have been measured and the results have been stated together with comparisons to the previous researches [BrHankLopMen, 2001], [HankHernMen, 2000], [YanShi, 2006].

In Chapter 4, the chosen EC point multiplication methods are applied to the signature generation and verification of Elliptic Curve Digital Signature Algorithm (ECDSA) and implementation timings are illustrated.

Chapter 5 presents the conclusions of this thesis work.

CHAPTER 2

PUBLIC-KEY CRYPTOGRAPHY AND ELLIPTIC CURVES

2.1 CRYPTOGRAPHY FUNDAMENTALS

The general definition of cryptography is the design and analysis of mathematical techniques that enable secure communications in the existence of adversaries. We can model the communication media for many different cases. For example, communication can be between two people via a cellular telephone network, or the communication between a web browser and a web site, or sending an email message to someone over the internet, or between a smart card and a computer. All these examples are vulnerable to eavesdrop. In today's world any communication scenario brings the requirement of security.

In a cryptographic communication system, we need to provide some fundamental aspects for secure communications given as follows:

1. Confidentiality
2. Data Integrity
3. Data origin authentication
4. Entity authentication
5. Non-repudiation

Symmetric-key Cryptography

In a symmetric-key cryptographic system, the entities first agree upon the keying material. Keying material should be secret and authentic. A symmetric-key encrytion scheme is used (i.e., DES, RC4, AES,..etc) for data achievement, data origin authentication, and message authentication codes. The major advantage of the symmetric-key cryptography is its high efficiency. However there exist significant drawbacks. One of them is the key distribution problem. Key distribution should be handled secretly and in authenticated form. The second drawback is the key management problem. In a network, cryptopraphic system can work only with different keying materials for each entity.

Public-key Cryptography

Public-key schemes require only the communication entities exchange keying material that is authentic but not secret. Each entity selects a single key pair (*e*,*d*) which consist of a public key *e* and related private key *d*. The private key is kept secret by the entity. The basic property of these keys is the computational infeasibility of having the private key from the knowledge of the public key. Public-key cryptography provides solutions to the problems of symmetric-key cryptography, specifically key distribution, key management, and the provision of non-repudiation.

Although we can eliminate the requirement for a secret channel for the distribution of keys by using public-key cryptography, implementing public-key infrastructure for distributing and managing public keys can be challenging in practice. Furthermore, public-key operations are usually slower than their symmetric-key counterparts. So, it is preferrable to use hybrid systems that benefit from the efficiency of symmetric-key algorithms and functionality of public-key algorithms.

2.2 USING PUBLIC-KEY CRYPTOSYTEMS

Public-key cryptosystems can provide all the services of information security. In this section we will describe how these services could be supplied by a public-key cryptosystem.

2.2.1 Encryption and decryption (confidentiality)

Assume we want to send a message to one of our friends. We will use the public key *e* of our friend to encrypt our message *M*. After encrypting the message, it is converted to ciphertext *C*. Now we can send our ciphered message *C* to our friend. The receiver will obtain the original message *M* by using his own secret private key. Private key is secret so that only the owner can have the enciphered message. Public-key encryption and decryption service the confidentiality required for a cryptosystem.

To have the system confidentiality, the users should know the public keys are authentic and belong to the specified users. At this point another concept is introduced: Electronic or digital certificates which are distributed by a trusted third party, should be used.

2.2.2 Digital signatures (authentication, integrity, non-repudiation)

Electronic form of handwritten signatures are called digital signatures. Since it is easy to have a fake electronic signature it must be produced in a systematic secure way. Public-key cryptographic systems are used for digital signature schemes. The use of public-key systems in signature algorithms is diferent than the use in encryption and decryption. Now let's suppose a signature requirement case where we want to sign a message *M*. First of all, we will compute the hash value $h(M)$ of the message by using a hash function. The signature *S* of message *M* is obtained by encrypting the hash value $h(M)$ with our own private key, *d*. We can then send the message *M* together with the signature *S*. After the message and the signature have been received, the receiver uses our public key *e* to transform signature to the hash value. The verification occurs if the recomputed hash value $h(M)$ and the decrypted value of *S* are equal. If the verification is completed with equality, the receiver accepts that our signature is valid. If the verification fails, the receiver understands that the signature is not ours and the message is not accepted.

From the signature scenario, we can conclude that the public-key signature scheme provides data origin authentication, data integrity and non-repudiation.

2.3 MATHEMATICAL PROBLEMS FOR PUBLIC-KEY CRYPTOSYSTEMS

The security level of a public-key cryptographic system is graded according to the hardness of the mathematical problem underlying the cryptographic algorithm. The commonly used public-key schemes provide their security by the mathematical problems which are given as follows.

1. Integer factorization problem (IFP).
2. Discrete logarithm problem (DLP).
3. Elliptic curve discrete logarithm problem (ECDLP).

All these problems are extremely difficult to solve by today's computing power provided that the big numbers used are big enough . The hardness of the integer factorization problem is essential for the security of RSA public-key encryption and signature schemes. The security of ElGamal public-key encryption and signature schemes and their variants such as Digital Signature Algorithm (DSA) depends on the hardness of DLP. Elliptic curve cryptography security depends on the intractibility of ECDLP. which is another form of classic DLP. Because of this analogy we are going to mention both problems.

2.3.1 Discrete logarithm problem (DLP)

Let $(G,*)$ be a multiplicative cyclic group of order p which is a prime. In the cyclic group, addition and multiplication are performed modulo p. The domain parameters are p, a given integer $g \in G$, the private key x randomly selected from the interval $[0, p-1]$, and the public key is $y = g^x \bmod p$. The discrete logarithm problem modulo *p* is to determine the integer *x* for a given pair of *g, y* and *p*. The integer *x* is called the discrete logarithm of *y* to the base *g*. The known algorithms to solve the discrete logarithm problem modulo *p* are not computationally efficient for large *p*.

2.3.2 Elliptic curve discrete logarithm problem (ECDLP)

Let *p* be a prime number and F_p be the field of integers modulo *p*. Say, the equation of a simple elliptic curve *E* over F_p is

$$y^2 = x^3 + ax + b \qquad (1.1)$$

where $a, b \in F_p$ satisfy $4a^3 + 27b^2 \neq 0$ (mod *p*). A pair $(x, y) \in F_p$ is a point on the curve if (x, y) satisfies equation (1.1). The elliptic curve discrete logarithm problem is as follows. Let P be a point on the elliptic curve. Multiplying P by *k* is simply addition of P to itself by *k* times. Suppose Q is a multiple of P, so that $Q = kP$ for some *k*. Then the "elliptic curve discrete logarithm problem" is finding *k* (private-key) where *P* and *Q* (public-key) are given.

2.4 WHY ELLIPTIC CURVE CRYPTOGRAPHY?

When selecting a family of public-key cryptosystem for an application we have to take some criteria in consideration. The basic ones are functionality, security and performance. The common public-key cryptosystems are RSA, Discrete Logarithm (DL) and Elliptic Curve Cryptography (ECC), which are used for encryption, signatures and key agreement schemes. These systems provide the expected functionality of public-key cryptography. Researchers have developed many techniques for designing and proving the security of these public-key systems. Since the performances of the public-key algorithms are directly affected by the hardness of the underlying mathematical problems, the choice of a system designer would depend on a comparison among them. The security and efficiency analysis of the common public-key cryptosystems are presented as follows.

2.4.1 Security

In order to break the cryptographic system, the underlying mathematical problem must be solved. It is known that the fastest algorithms known for solving integer factorization problem of RSA and DLP have subexponential expected running time. In order to solve ECDLP, the known fastest algorithms have exponential expected running time [HankMenVan, 2004]. So, solving ECDLP takes more time than integer factorization and DLP when the same key sizes are used. This advantage allows ECC to achieve the same level of security with smaller key sizes and higher computational efficiency. If we use 1024-bit modulus for RSA and DSA, the security level becomes comparable to ECC with 160-bit modulus [GuPaWaEbSh, 2004].

2.4.2 Efficiency

The underlying mathematical problem of a public-key cryptosystem determines the efficiency of the cryptosystem in a way. Because these problems dictate the sizes of domain parameters and keys, which in turn affect the performance of the arithmetic operations of the public-key crypto algorithms.

The parameter sizes, generally called the key sizes are listed in Table 2.1. The listing has been done according to equivalent security levels for RSA, DSA and ECC as symmetric-key encryption schemes, stated in the table. The comparison shows that elliptic curve cryptography algorithm uses smaller parameter sizes than RSA and DSA for the same security levels. That brings the advantage of faster computations, smaller keys and certificates. The smaller domain parameter sizes also provide bandwidth savings.

Table 2.1 RSA, DSA and ECC key sizes for equivalent security levels.

	Security level (bits)				
	80 (SKIPJACK)	112 (Triple-DES)	128 (AES-Small)	192 (AES-Medium)	256 (AES-Large)
RSA modulus DSA modulus	1024	2048	3072	8192	15360
ECC modulus ECDSA modulus	160	224	256	384	512

According to the key size, bandwidth requirements of ECC is said to provide greater efficiency than either integer factorization systems or discrete logarithm systems. This means that higher speeds, lower power consumptions and reduced code size are the advantages of ECC. Furthermore, the hardness of the underlying mathematical problem ECDLP appeals EC public-key cryptosystem for applications demanding high security. That is why elliptic curve cryptography has aroused much interest for many researchers.

2.5. ELLIPTIC CURVE ARITHMETIC

Cryptographic systems based on elliptic curves depend on point arithmetic on the selected elliptic curve. We can define elliptic curve arithmetic in terms of field operations. The chosen field will dominate the elliptic curve cryptographic mechanism and the field operations will directly affect the efficiency of the system. Efficient curve operations are crucial to the performance of ECC system. The curve arithmetic is built on not only field operations, but also on big numbers and modular arithmetic. Elliptic Curve Digital Signature Algorithm (ECDSA) is an ECC scheme, which needs a hash function and some modular operations.

Elliptic curve definitions and arithmetic rules will be described in the following subsections.

2.5.1 Elliptic curve definition

An elliptic curve *E* over a field *K* is defined by an equation

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{2.1}$$

Where $a_1, a_2, a_3, a_4, a_6 \in K$ and $\Delta \neq 0$, where Δ is the discriminant of *E* and is defined as follows:

$$\left.\begin{aligned}
&\Delta = -d_2^2d_8 - 8d_4^3 - 27d_6^2 + 9d_2d_4d_6 \\
&d_2 = a_1^2 + 4a_2 \\
&d_4 = 2a_4 + a_1a_3 \\
&d_6 = a_3^2 + 4a_6 \\
&d_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2
\end{aligned}\right\} \tag{2.2}$$

The points of an elliptic curve must satisfy the curve equation and must be in the same defined field. We can express this in the following way. Let *L* be any extension field of *K*, then the set of *L-rational points* on *E* is

$$E(L) = \{(x, y) \in L \times L : y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6 = 0\} \cup \{\infty\}$$

where ∞ is the point at infinity. The equation (2.1) is called Weierstrass equation. The condition of $\Delta \neq 0$ ensures that the elliptic curve is smooth. The smoothness provides that no points on the curve have two distinct tangent lines. We can give examples for elliptic curves over the field R of real numbers.

$$E_1 : y^2 = x^3 - x$$

$$E_2 : y^2 = x^3 + \frac{1}{4}x + \frac{5}{4}$$

We illustrated an elliptic curve over the field R of real numbers in Figure 2.1.

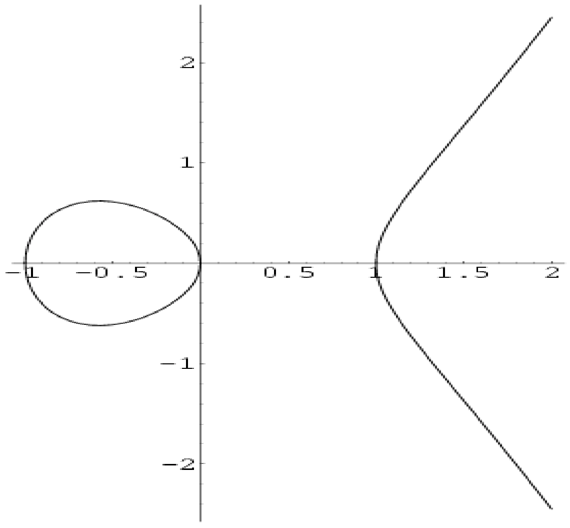

Figure 2.1 Elliptic curves over *R*.

We have given the definition of elliptic curves and illustrated an elliptic curve in the Figure 2.1. In cryptosystems we are going to use the elliptic curves which have simplified Weierstrass equations. We can divide these simplified forms into three and let us state the conditions and simplified forms of Weierstrass equation (2.1).

- If characteristic of field $K \neq 2,3$, then with the admissible change of variables

  $$(x,y) \to \left(\frac{x-3a_1^2-12a_2}{36}, \frac{y-3a_1x}{216} - \frac{a_1^3+4a_1a_2-12a_3}{24} \right)$$

  the elliptic curve equation is transformed to the following simplified equation.

  $$y^2 = x^3 + ax + b \qquad (2.3)$$

  where $a,b \in K$. The discriminant of this curve is $\Delta = -16(4a^3+27b^2)$.
- If characteristic of field $K = 2$ and $a_1 \neq 0$, then with the admissible change of variables

$$(x,y) \to \left(a_1^2 x + \frac{a_3}{a_1}, a_1^3 y + \frac{a_1^2 a_4 + a_3^2}{a_1^3} \right)$$

the elliptic curve equation transforms the curve equation to

$$y^2 + xy = x^3 + ax^2 + b \qquad (2.4)$$

where $a, b \in K$ and $b \neq 0$. This kind of curve is called *non-supersingular* and has $\Delta = b$. If $a_1 = 0$, the admissible change of variables

$$(x,y) \to (x + a_2, y)$$

transforms the curve equation to

$$y^2 + cy = x^3 + ax^2 + b \qquad (2.5)$$

where $a, b, c \in K$ and $c \neq 0$. Such curves are said to be supersingular and has discriminant $\Delta = c^4$.

- If characteristic of field K =3 and $a_1^2 \neq -a_2$, the admissible change of variables

$$(x,y) \to \left(x + \frac{a_4 - a_1 a_3}{a_1^2 + a_2}, y + a_1 x + a_1 \frac{a_4 - a_1 a_3}{a_1^2 + a_2} + a_3 \right)$$

transforms the curve equation to

$$y^2 = x^3 + ax^2 + b \qquad (2.6)$$

where $a, b \in K$ and $a, b \neq 0$. This kind of curve is called *non-supersingular* and has $\Delta = -a^3 b$. If $a_1^2 = -a_2$, the admissible change of variables

$$(x,y) \to (x, y + a_1 x + a_3)$$

transforms the curve equation to

$$y^2 = x^3 + ax + b \qquad (2.7)$$

where $a, b \in K$ and $a \neq 0$. Such curves are said to be supersingular and has discriminant $\Delta = -a^3$.

The above classifications have different arithmetic rules. We will use the arithmetic according to the curve which we will design our system on. Now let's mention the arithmetic of elliptic curves.

2.5.2 Group law for elliptic curves

Let *E* be an elliptic curve and *K* be the field which our curve is defined on. Adding two points in *E(K)* gives a third point in *E(K)*. The addition is performed according to a rule called *chord-tangent-rule*. The set of points in *E(K)* forms an abelian group with this addition property and the element ∞ which serves the identity element. Elliptic curve cryptosystems are constructed on abelian groups.

We can explain addition operation geometrically. Let $P = (x_1, x_2)$ and $Q = (x_1, x_2)$ be two points on an elliptic curve *E*. The sum of *P* and *Q* is defined as follows. First draw a line through *P* and *Q*. This line intersects the curve at a third point. The reflection of this point about the x-axis is the addition result point R. This is depicted in Figure 2.2.

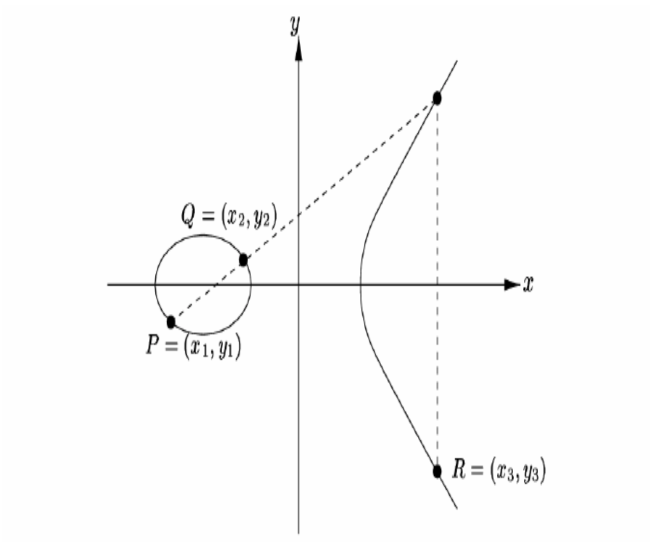

Figure 2.2 Geometric addition of elliptic curve points, *P+Q=R*.

Doubling of a point on an elliptic curve can be also explained geometrically. If we want to have the double of point *P* on elliptic curve *E*, we draw a tangent line to the elliptic curve *E* at point *P*. This line intersects the curve at a second point. Then the doubling result *R* is the reflection of this point about the x-axis. This geometrical explanation is depicted in Figure 2.3.

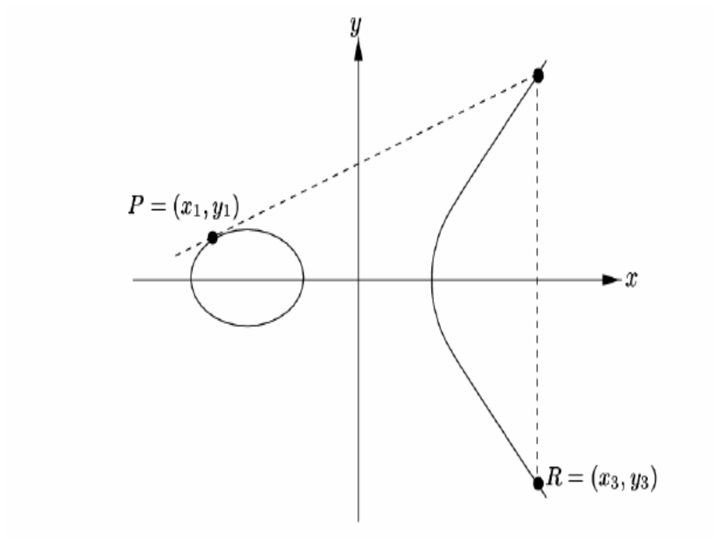

Figure 2.3 Geometric doubling of elliptic curve point, 2*P*=*R*

Algebraic formulas for elliptic curve arithmetic can be derived from the geometric description. We have stated the simplified Weierstrass equations in 2.5.1. The algebraic formulas for these cases in affine coordinates when the underlying field *K* characteristic is not 2 or 3, for non-supersingular elliptic curves *E* of the form (2.4) over $K = F_{2^m}$, and for supersingular curves *E* of the form (2.5) over K=F_{2^m} have differences. Now we will state the group law and the algebraic formulas for group operations.

Elliptic curves over prime fields *K, char (K)* ≠*2, 3, E*: $y^2 = x^3 + ax + b$

- Identity. $P + \infty = \infty + P = P$ for all $P \in E(K)$
- Negatives. If $P = (x, y) \in E(K)$, then $(x, y) + (x, -y) = \infty$.

  The point $-P = (x, -y)$ is the negative of point P and $-P \in E(K)$. Also $-\infty = \infty$.
- Point addition. Let *P* and *Q* be two different points on elliptic curve over field *K* and $P \neq \pm Q$. Then $P + Q = (x_3, y_3)$ where

$$x_3 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 - x_1 - x_2$$

$$y_3 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x_1 - x_3) - y_1$$

- Point doubling. Let $P = (x_1, y_1) \in E(K)$, where $P \neq -P$. Then $2P = (x_3, y_3)$ where

  $$x_3 = \left(\frac{3x_1^2 + a}{2y_1}\right)^2 - 2x_1$$

  $$y_3 = \left(\frac{3x_1^2 + a}{2y_1}\right)(x_1 - x_3) - y_1$$

Non-supersingular elliptic curves over F_{2^m}, *E*: $y^2 + xy = x^3 + ax^2 + b$

- Identity. $P + \infty = \infty + P = P$ for all $P \in E(F_{2^m})$.
- Negatives. If $P = (x, y) \in E(F_{2^m})$, then $(x, y) + (x, x + y) = \infty$. The point $-P = (x, x + y)$ is the negative of point P and $-P \in E(F_{2^m})$. Also $-\infty = \infty$.
- Point addition. Let *P* and *Q* be two different points on elliptic curve over field F_{2^m} and $P \neq \pm Q$. Then $P + Q = (x_3, y_3)$ where

  $$x_3 = \lambda^2 + \lambda + x_1 + x_2 + a$$

  $$y_3 = \lambda(x_1 + x_3) + x_3 + y_1$$

  where $\lambda = \left(\frac{y_1 + y_2}{x_1 + x_2}\right)$
- Point doubling. Let $P = (x, y) \in E(F_{2^m})$, where $P \neq -P$. Then $2P = (x_3, y_3)$ where

  $$x_3 = \lambda^2 + \lambda + a = x_1^2 + \frac{b}{x_1^2}$$

  $$y_3 = x_1^2 + \lambda x_3 + x_3$$

  where $\lambda = x_1 + \frac{y_1}{x_1}$

Supersingular elliptic curves over F_{2^m}, *E*: $y^2 + cy = x^3 + ax^2 + b$

- Identity. $P + \infty = \infty + P = P$ for all $P \in E(F_{2^m})$.
- Negatives. If $P = (x, y) \in E(F_{2^m})$, then $(x, y) + (x, y + c) = \infty$. The point $-P = (x, y + c)$ is the negative of point P and $-P \in E(F_{2^m})$. Also $-\infty = \infty$.
- Point addition. Let *P* and *Q* be two different points on elliptic curve over field F_{2^m} and $P \neq \pm Q$. Then $P + Q = (x_3, y_3)$ where

  $x_3 = \lambda^2 + x_1 + x_2 + a$

  $y_3 = \lambda(x_1 + x_3) + x_3 + y_1 + c$

  where $\lambda = \left(\dfrac{y_1 + y_2}{x_1 + x_2}\right)$
- Point doubling. Let $P = (x, y) \in E(F_{2^m})$, where $P \neq -P$. Then $2P = (x_3, y_3)$ where

  $$x_3 = \left(\frac{x_1^2 + a}{c}\right)^2$$

  $$y_3 = \left(\frac{x_1^2 + a}{c}\right)(x_1 + x_3) + y_1 + c$$

2.5.3 Point representation

In Section 2.5.2 we have given the algebraic formulas for elliptic curve addition and doubling operations. The presented formulas were for the elliptic curves defined over field *K* of characteristic neither 2 nor 3, and over binary fields. For these curves the formulas of point addition and point doubling includes a field inversion and many field multiplication operations. For some cases inversion in a field is more expensive than multiplication. Using projective coordinates for the point representation may be advantageous.

We will describe the projective coordinates which will be used to avoid the expense of field inversion. The benefit of using projective coordinates will be observed later in the implementation results sections. Let us define a field *K*, and positive integers *c* and *d*. The definition can be given for an equivalence relation ~ on the set $K^3 \setminus \{(0,0,0)\}$ of nonzero triples over *K* by

$(X_1,Y_1,Z_1) \sim (X_2,Y_2,Z_2)$ if $X_1 = \lambda^c X_2$, $Y_1 = \lambda^d Y_2$, $Z_1 = \lambda Z_2$ for some $\lambda \in K^*$.

We can represent the above expression by another notation as

$(X : Y : Z) = \{(\lambda^c X, \lambda^d Y, \lambda Z) : \lambda \in K^*\}$

where K^* represents the set of nonzero elements of the field *K*.

$(X : Y : Z)$ is called projective point and (X,Y,Z) is called a representative of $(X : Y : Z)$. The set of all projective points is denoted as *P*(*K*)

We will use these point representations for our elliptic curve systems. First Weierstrass equation (2.1) will be transformed. That is performed by replacing *x* by X / Z^c and *y* by Y / Z^d, and clearing denominators. After this formula transformation, we obtain a projective equation. If $(X,Y,Z) \in K^3 \setminus \{(0,0,0)\}$ satisfies the projective equation, we can say that the projective point $(X : Y : Z)$ lies on *E* [HankMenVan, 2004].

The known projective coordinates for elliptic curve $E : y^2 = x^3 + ax^2 + b$**,** are as follows [BrHankLopMen, 2001].

- **Standard projective coordinates:** The positive integers are *c*=1, *d*=1. The point at infinity ∞ is represented as (0:1:0) for this coordinate system. The negative of $(X : Y : Z)$ is $(X : -Y : Z)$.
- **Jacobian projective coordinates:** The positive integers are *c*=2, and *d*=3. The point at infinity is represented as (1:1:0) for this coordinate system. The negative of $(X : Y : Z)$ is $(X : -Y : Z)$.
- **Chudnovsky coordinates:** This representation is formed by representing Jacobian coordinates $(X : Y : Z)$ as $(X : Y : Z : Z^2 : Z^3)$.

The known projective coordinates for elliptic curve $E : y^2 + xy = x^3 + ax^2 + b$**,** are as follows [HankHernMen, 2000].

- **Standard projective coordinates:** The positive integers are c=1, d=1. The point at infinity ∞ is represented as (0:1:0) for this coordinate system. The negative of $(X:Y:Z)$ is $(X:X+Y:Z)$.
- **Jacobian projective coordinates:** The positive integers are c=2, and d=3. The point at infinity is represented as (1:1:0) for this coordinate system. The negative of $(X:Y:Z)$ is $(X:X+Y:Z)$.
- **Lopez-Dahab projective coordinates:** The positive integers are c=1, and d=2. The point at infinity is represented as (1:0:0) for this coordinate system. The negative of $(X:Y:Z)$ is $(X:X+Y:Z)$.

2.5.4 Point multiplication

The last topic for this section is multiplication operation over elliptic curves. The term point multiplication refers to computing $Q=kP$, where Q and P are points on an elliptic curve and k is an integer. This expression means that we add P to itself k times. This operation is also called scalar multiplication.

The point multiplication of elliptic curve points is explained as a straight sum. In fact, there are many efficient ways to compute the point multiplication of elliptic curves. Chapter 3 focuses on the underlying field arithmetic and elliptic curve point multiplication methods. In addition to the theoretical descriptions, the software implementation results are also presented in Chapter 3.

CHAPTER 3

FIELD ARITHMETIC AND ELLIPTIC CURVE POINT MULTIPLICATION

Elliptic curve point multiplication is a fundamental issue in elliptic curve cryptography. That is because it dominates the cryptographic schemes. The operation is kP, where k is an integer and P is a point on an elliptic curve E defined over a field. This operation is called point multiplication or scalar multiplication. In this chapter, the point multiplication techniques are presented and software implementation results are discussed. Before getting into EC point multiplication methods and software implementations, we describe the underlying finite field operations of these algorithms. In Section 3.1, finite field arithmetic which we have used for our EC software implementations is described shortly. The timings of field arithmetic operations are presented both for prime fields and binary fields in Subsection 3.1.3. Section 3.2 covers the theoretical description of different elliptic curve multiplication methods. The software implementation of each method has been done in C on a Pentium 4 processor at 3 GHz. Our main concern is the timing results of these implementations. We present the measured timing performances in Section 3.3.

3.1 FINITE FIELD ARITHMETIC

Since we will deal with the timings of EC point multiplication methods, it is important to have efficient field arithmetic implementations. We will discuss implementations in prime fields and binary fields. Efficient methods of addition, multiplication, inversion in both prime fields and binary fields are described in subsections 3.1.1 and 3.1.2 [BrHankLopMen, 2001], [HankHernMen, 2000].

3.1.1 Finite field arithmetic in prime fields

A prime field is formed of integers modulo *p*, {0,1,2 …*p*-1} and the operations of addition and multiplication performed modulo *p*. The order of the field is *p* and represented as F_p. The prime number *p* is the modulus of F_p. The reduction modulo *p* operation is dividing any integer by *p* and keeping the remainder of division for the result. Now let us describe how we perform prime field arithmetic in software. First we assume that the platform is 32-bit processor. The elements of F_p will be written in binary representation. The number $m = \lceil \log_2 p \rceil$ gives the number of bits that we are going to present our field elements. In our software we can store the prime field elements in $t = m/32$ number of words [Mur, 2003].

Addition and Subtraction

Addition in prime fields is the sum of two integers *mod p*. The prime numbers are added word by word and if the sum exceeds *p*-1, we subtract *p* from the sum. After each word addition, we carry a bit for the next word addition and add it to the next sum. Implementation of modular subtraction is similar to the implementation of modular addition. But the carry in addition is the borrow in subtraction. If we have a borrow from the word subtraction, we subtract it from the next word operation.

Multiplication and squaring

Field multiplication of prime field elements can be performed first multiplying two integers then reducing the product modulo *p*. Field squaring can be accomplished by first squaring the field element as an integer, then reducing the result modulo *p*.

Modular Reduction

The modular reduction is part of field multiplication. In fact it is the expensive part of multiplication operation. Since we care about the speed of elliptic curve schemes we should have time efficient modular reduction. Barrett reduction is generally considered to be the fast reduction technique [HankMenVan, 2004].

3.1.2 Finite field arithmetic in binary fields

Binary fields are also called characteristic–two finite fields. The elements of F_{2^m} are often represented with binary polynomials whose coefficients are in the field $F_2 = \{0,1\}$ and degree is at most *m*-1. When programming, we can represent coefficients as bits of our words. For example, on 32-bit processor 0x20000005 represents the polynomial $x^{29} + x^2 + 1$ in $F_{2^{31}}$. A polynomial can be represented with $\lceil (m+1)/w \rceil$ words, where polynomial has the degree *m* and the word size is *w*.

Addition

The addition of two polynomials is bitwise xor operations. Let $t = \lceil (m+1)/w \rceil$ where *m* is the degree of a binary polynomial in a binary field and *w* is the word size. The addition of two polynomials whose degree is *m* needs *t* word operations.

Multiplication

When we multiply two polynomials of degree (m-1) we will obtain a polynomial of degree (2m-2). Since we are performing finite field operations, this product should be reduced with respect to an irreducible polynomial $f(x)$ of degree m. In order to have fast reduction operations, the irreducible polynomial with a few terms should be chosen. The modulation operation can be done during or after the polynomial multiplication.

The basic method for polynomial multiplications is shift-and-add method. Let a and b be two polynomials and c be their partial product. Shift-and-add method begins with setting c to 0 if $a_0 = 0$ or to b if $a_0 = 1$. Then method goes on with scanning bits of a. For each bit b is shifted to left by one, and if the bit is 1 we add this shifted b to c [YanShi, 2006].

Some other methods have been also developed for field multiplication. They are comb algorithms. With these methods faster polynomial multiplications can be performed. We told that we will store our binary polynomial in t number of words. The comb methods avoid us testing each bit one by one. Instead we test each bit 0 of all t words of a. Then the other bits are tested for each word. We can also test the bits from the most significant bit to the least significant bit. For left to right comb methods the shifting is done to partial product c. So the input values remain unchanged. Then we can employ sliding window technique to reduce shifting. We can scan the bits with a fixed size window and then multiply more than one bit with b at a time [LopDa, 2000]. Then c is shifted left by the fixed window size. Since our window size is fixed we can calculate the possible b products and store them in a table. So the shifting number is reduced by the cost of storage.

Another point for multiplication of polynomials is polynomial squaring. It can be performed faster than multiplication operation by inserting 0 bits between consecutive bits of the polynomial [HankMenVan, 2004].

Inversion

In Chapter 2, EC operations are described in detail. The definitions tell us that there exists field inversion in EC addition and doubling operations. The inverse of an element of $a \in F_{2^m}$ is the unique element of the same field that satisfies $a.a^{-1} = 1$ in F_{2^m}. The classical algorithm for computing multiplicative inverse is Extended Euclidean Algorithm (EEA) [HankHernMen, 2004]. Since EEA inversion is stated as the most time efficient inversion we have chosen this method for our field inversion implementation [YanShi, 2006].

3.1.3 Field arithmetic on Pentium processor

We have implemented both prime field and binary field arithmetic operations on a Pentium 4 processor at 3 GHz. The field arithmetic routines are written in C.

In Table 3.1 we have presented the timings of prime field operations which are addition, subtraction, modular reduction, multiplication, squaring and inversion. The operations are done in NIST prime fields. It should be noted that NIST primes provide fast reductions [BrHankLopMen, 2001]. The prime field arithmetic implementation is based on 16-bit word size. Same calculations are done in [BrHankLopMen, 2001] but with 32-bit word size and by using hand-coded assembly code. It is expected that using 32-bit word size and assembly programming should have better timings. However using Pentium 4 processor has better timings for addition, subtraction and modular reduction. So we have obtained noteworthy results for prime field operations. If we want to discuss about the timing characteristics of prime field operations we can conclude from the results that inversion and multiplication operation is the expensive field operations. A noticable point is squaring takes less execution time than multiplication.

Table 3.1 Execution time (μs) of field operations in $F_{192}, F_{224}, F_{256}, F_{384}, F_{521}$.

	F_{P192}	F_{P224}	F_{P256}	F_{P384}	F_{P521}
Addition	0.071	0.160	0.083	0.142	0.145
Subtraction	0.088	0.162	0.099	0.137	0.146
Reduction	0.216	0.2	0.2	0.27	0.216
Multiplication	1.5	3.1	3.1	7.8	10.9
Squaring	2.35	2.35	3.15	6.25	8.6
Inversion	150	160	160	310	620

In Table 3.2 binary field results are presented. We have also written binary field operation routines in C programming language. This time the word size is 32-bits. The field operaions are in $F_{2^{163}}, F_{2^{283}}, F_{2^{409}}$. The reduction polynomials are $x^{163} + x^7 + x^6 + x^3 + 1$, $x^{163} + x^{283} + x^{12} + x^7 + x^5 + 1$ and $x^{409} + x^{87} + 1$ respectively. Binary field software implementations are studied in [HankHernMen, 2000] and [Yan, Shi, 2006]. Hankerson et al. have made their implementations on Pentium II 400 MHz workstation in C code. Since we are using a better processor our field arithmetic timings are 30% percent faster. Yan and Shi have published their software implementation study of elliptic curve cryptography in April 2006. For their study they have used the same processor with us. Because of that our results are close to theirs.

The results show that addition, modular reduction and squaring operations are faster than multiplication and inversion operations. We have implemented two methods for binary field multiplication. They are Right-to-Left Comb Method and Left-to-Right Comb Method with 4-bit window. The Left-to-Right Comb method with window size 4 is approximately 40% faster. The precomputation process makes this method faster. So if we use this method for our field multiplication operations we need extra storage for precomputed values.

Table 3.2 Execution time (μs) of field operations in $F_{2^{163}}, F_{2^{283}}, F_{2^{409}}$

	$F_{2^{163}}$	F_{283}	F_{409}
Addition	0.016	0.031	0.047
Modular operation	0.075	0.08	0.16
Multiplication			
R-to-L Comb Method	4.7	9.4	15
L-to-R Comb Method with 4-bit window	1.6	4.7	9.4
Squaring	0.075	0.155	0.235
Inversion	148	387	939
Inversion with EEA Method	39	85	164

So far we have made field operation analysis. Now the next step will be the analysis of EC multiplication methods. We will cover these methods in the next section.

3.2 DESCRIPTION OF ELLIPTIC CURVE POINT MULTIPLICATION OPERATION ALGORITHMS

In this section different types of elliptic curve multiplication methods will be described. When we need to compute an elliptic curve multiplication we can have two cases. The first one is **unknown *P*** case that means we do not have any information about the elliptic curve point which is going to be multiplied by the scalar k. The multiplication methods which are suitable for unknown *P* case are Right-to-Left Binary Method, Left-to-Right Binary Method, Binary NAF Method, Window NAF Method, Sliding Window Method and Montgomery Method. These methods will be described in 3.2.1. The second case is **known *P*** case which means we have a priori information about the elliptic curve point which is going to be multiplied by scalar *k*. By using the a priori information EC point multiplication methods are developed through precomputation steps. The known elliptic curve point is used to form some data which will be used when EC point multiplication will be proceeded. So some memory will be required to store these precomputed data. The known *P* case methods are Fixed Base Windowing Method, Fixed Base NAF Windowing Method, Fixed Base Comb Method, and Fixed Base Comb with two tables Method. We will describe these methods in 3.2.2 step by step.

3.2.1 Point *P* is unknown

The first algorithms that we are going to introduce you, are the Right-to-Left Binary and Left-to-Right Binary methods. These methods perform EC point multiplication operation by processing the scalar *k* bit by bit from right to left or left to right. These two methods are the additive versions of repeated-square-and-multiply methods for exponentiation. The algorithm details are as follows [HankMenVan, 2004].

Algorithm 3.1 Right to left binary method for point multiplication

```
INPUT: $k = (k_{t-1},...,k_1,k_0)$ $P \in E(F_q)$
OUTPUT: $kP$
    1. $Q \leftarrow \infty$
    2. For $i$ from $0$ to $t$-1 do
      2.1 If $k_i = 1$ then $Q \leftarrow Q + P$
      2.2 $P \leftarrow 2P$
    3. Return ($Q$)
```

Algorithm 3.2 Left to right binary method for point multiplication

```
INPUT: $k = (k_{t-1},...,k_1,k_0)$ $P \in E(F_q)$
```

```
OUTPUT: $kP$
    1. $Q \leftarrow \infty$
    2. For $i$ from $t$-1 to $0$ do
        2.1 $Q \leftarrow 2Q$
        2.2 If $k_i = 1$ then $Q \leftarrow Q + P$
    3. Return $(Q)$
```

Since we will be interested in running time of the multiplication algorithms we can predict the execution time by looking at the algorithm steps. The scalar *k* has *t* bits and it is expected that *t*/2 number of ones may be in the binary representation of *k*. So the Alg.3.1 and Alg.3.2 may have the running time of $\frac{t}{2}A + tD$ where *A* represents elliptic curve point addition and *D* represents elliptic curve point doubling times.

The subtraction of points on an elliptic curve is just as efficient as addition. So a signed digit representation can be used for scalar *k*. A useful signed digit representation is the non-adjacent form (NAF). A non-adjacent form (NAF) of a positive integer *k* is an expression $k = \sum_{i=0}^{l-1} k_i 2^i$. The scalar *k* has unique NAF representation denoted as NAF(*k*) and in this representation, none of the consecutive bits are nonzero. NAF(*k*) has fewest nonzero digits than any signed representation of *k*. The length of the NAF(*k*) is almost one more digit more than the length of the binary representation of *k*. The attracted point of this representation is the density of nonzero digits. It is approximately 1/3 [MorOl, 1990]. Now, we will first describe how to represent a binary represented number in NAF, and then we will substitute this form to scalar *k* in elliptic curve multiplication methods.

Computing NAF

```
INPUT: a positive integer $k$
OUTPUT: $kP$
    1. 1. $Q \leftarrow \infty$
```

```
2. While $k \geq 1$ do
    2.1 If $k$ is odd then: $k_i \leftarrow 2-(k \bmod 4), k \leftarrow k-k_i$
    2.2 Else $k_i \leftarrow 0$
    2.3 $k \leftarrow k/2, i \leftarrow i+1$
3. Return $(k_{i-1}, k_{i-2}, ..., k_1, k_0)$
```

By using non-adjacent form Left-to-Right Method can be modified. The modified multiplication method is as follows.

Algorithm 3.3 Binary NAF method for point multiplication

```
INPUT: Positive integer $k$, $P \in E(F_q)$
OUTPUT: $kP$
1. Compute $NAF(k) = \sum_{i=0}^{l-1} k_i 2^i$
2. $Q \leftarrow \infty$
3. For i from $l$-1 to 0 do
    3.1 $Q \leftarrow 2Q$
    3.2 If $k_i = 1$ then $Q \leftarrow Q + P$
    3.3 If $k_i = -1$ then $Q \leftarrow Q - P$
4. Return $(Q)$
```

3.2.2 Window Methods

If we have some extra memory in our implementation platform, the running time of multiplication algorithms can be decreased by using a window method. In window methods *w* digits of *k* are processed at a time so the running times of algorithms decrease. The notation *w* means the window width. Before getting into EC point multiplication methods we want to describe a version of NAF representation called width-*w* NAF. In this representation we represent the scalar *k* by odd numbers as $k = \sum_{i=0}^{l-1} k_i 2^i$ where $|k_i| < 2^{w-1}$ and $k_{l-1} \neq 0$. The scalar *k* has a unique width-*w* NAF representation denoted as NAF_w(k). The average density of nonzero digits of width-*w* NAF representation is approximately 1/(*w*+1) [Sol, 2000]. The computation of width-*w* NAF representation is as follows.

Computing width-*w* NAF of a positive integer

```
INPUT: Window width $w$, a positive integer $k$.
 OUTPUT: $\text{NAF}_w(k)$
1. $i \leftarrow 0$
2. While $k \geq 1$ do
    2.1. If $k$ is odd then: $k_i \leftarrow k \bmod 2^w, k \leftarrow k - k_i$
    2.2. Else $k_i \leftarrow 0$
    2.3. $k \leftarrow k/2, i \leftarrow i+1$
3. Return $(k_{i-1}, k_{i-2}, ..., k_1, k_0)$
```

If *w*=2 the NAF_w(k) representation will be equal to NAF(k) representation. We have used NAF representation in Left-to-Right Binary Method for EC point multiplication. By using width-*w* NAF representation in this method we can generalize this EC point multiplication method. That is called Window NAF method. The steps of this multiplication method are described as follows.

Algorithm 3.4 Window NAF method for point multiplication

```
INPUT: Positive integer $k$, $P \in E(F_q)$
OUTPUT: $kP$
    1. Compute $NAF_w(k) = \sum_{i=0}^{l-1} k_i 2^i$
    2. Compute $P_i = iP$ for $i \in \{1,3,5,...,2^{w-1}-1\}$
    3. $Q \leftarrow \infty$
```

```
4. For $i$ from l-1 to 0 do
   4.1 $Q \leftarrow 2Q$
   4.2 If $k_i \neq 0$ then
           If $k_i > 0$ then $Q \leftarrow Q + P_{k_i}$
           Else $Q \leftarrow Q - P_{-k_i}$
5. Return($Q$)
```

The following method had been developed as an alternative to Window NAF method. It is another version of computed width-w NAF and using this representation in Left-to-Right Binary method. In this method we can use a sliding window on the NAF(k). The window moves left-to-right over the digits and is placed so that the value in the window is odd. Here we use NAF(k) and meet the odds as sliding our window from left-to-right. The steps of this algorithm are stated in detail in Alg.3.5 description [Sol, 2000].

Algorithm 3.5 Sliding window method for point multiplication

```
INPUT: Positive integer $k$, $P \in E(F_q)$
OUTPUT: $kP$
1. Compute $NAF(k) = \sum_{i=0}^{l-1} k_i 2^i$
2. Compute $P_i = iP$ for $i \in \{1,3,5,...,2(2^w - (-1)^w)/3 - 1\}$
3. $Q \leftarrow \infty$, $i \leftarrow l - 1$
4. While $i \geq 0$ do
   4.1. If $k_i = 0$ then $t \leftarrow 1, u \leftarrow 0$
        Else find the largest $t \leq w$ such that $u \leftarrow (k_i,...,k_{i-t+1})$ is odd.
   4.2. $Q \leftarrow 2^t Q$
   4.3. If $u > 0$ then $Q \leftarrow Q + P_u$; else if $u < 0$ then $Q \leftarrow Q - P_{-u}$
   4.4. $i \leftarrow i - t$
5. Return (Q)
```

Another algorithm will be described for non-supersingular elliptic curves over binary fields. This method is due to Lopez and Dahab and it is based on idea of Montgomery. Using the EC addition formulas we can have the *x* coordinate of an EC point addition operation result by using the *x*-coordinate of the two points and the *x*-coordinate of their subtraction result. Let *P* and *Q* be the two points that we want to add. Hence we only need to know the *x* coordinates of *P*, *Q* and *P*-*Q* to determine the *x*-coordinate of *P*+*Q*. Before getting into Montgomery multiplication method we want to describe addition and doubling computations. We have stated different point representations in Chapter 2. Here we will represent *x*-coordinate of *P* by *X*/*Z*. The coordinates of points *2P* and *P*+*Q* are as follows.

Let *D = 2.P*

$$X_D = X^4 + b.Z^4$$
$$Z_D = X^2.Z^2$$

and let *A = P+Q*

$$Z_A = (X_P.Z_Q + X_Q.Z_P)^2$$
$$X_A = x.Z_A + (X_P.Z_Q).(X_Q.Z_P)$$

We have stated the main operations of Montgomery multiplication method for elliptic curve points. Below EC point multiplication of Montgomery Method is described.

Algorithm 3.6 Montgomery point multiplication

```
INPUT: $k = (k_{t-1},...,k_1,k_0)_2$ with $k_{t-1} = 1$, $P = (x, y) \in E(F_{2^m})$
OUTPUT: $kP$
1.  $X_1 \leftarrow x, Z_1 \leftarrow 1, X_2 \leftarrow x^4 + b, Z_2 \leftarrow x^2$ (P and 2P is computed)
2.  For i from t-2 down to 0 do
    2.1. If $k_i = 1$ then
            $T \leftarrow Z_1, Z_1 \leftarrow (X_1Z_2 + X_2Z_1)^2, X_1 \leftarrow xZ_1 + X_1X_2TZ_2$
            $T \leftarrow X_2, X_2 \leftarrow X_2^4 + bZ_2^4, Z_2 \leftarrow T^2Z_2^2$
    2.2. else
        $T \leftarrow Z_2, Z_2 \leftarrow (X_1Z_2 + X_2Z_1)^2, X_2 \leftarrow xZ_2 + X_1X_2TZ_1$
        $T \leftarrow X_1, X_1 \leftarrow X_1^4 + bZ_1^4, Z_1 \leftarrow T^2Z_1^2$
3. Convert the results to affine coordinates $(x_3, y_3)$.
4. Return $(x_3, y_3)$.
```

3.2.3 Point *P* is known

We may have some cases that we know the point *P* a priori. With this information we can precompute some data based on *P* and store it. This can be done when there exists enough memory. The precomputed data will accelarate our point multiplication operation *kP*. In this section we will describe the EC point multiplication methods based on known point *P*.

The first method will be the Fixed-base Windowing Method. For this method we will precompute every multiple $2^i P$ and base-2^w representation of *k* will be used.

Algorithm 3.7 Fixed-base windowing method for point multiplication

```
INPUT: Window width $w, d = \lceil t/w \rceil$ , $k = (K_{d-1}, ..., K_1, K_0)_{2^w}$, $P \in E(F_q)$
OUTPUT: $kP$
1. Precomputation: Compute $P_i = 2^{wi} P, 0 \le i \le d-1$
2. $A \leftarrow \infty, B \leftarrow \infty$
3. For $j$ from $2^w - 1$ down to 1 do
    3.1. For each $i$ for which $K_i = j$ do: $B \leftarrow B + P_i$
    3.2. $A \leftarrow A + B$
4. Return ($A$).
```

The execution time of the Fixed-base windowing Method is expected as (2^w+d-3)A where A is EC point addition execution time. We can modify this method by using NAF representation instead of binary representation for k. The modified version of this method is described as follows.

Algorithm 3.8 Fixed-base NAF windowing method for point multiplication

```
INPUT: Window width w, positive integer $k$, $P \in E(F_q)$
OUTPUT: kP
1. Precomputation: Compute $P_i = 2^{wi} P, 0 \le i \le \lceil (t+1)/w \rceil$
2. Compute $NAF(k) = \sum_{i=0}^{l-1} k_i 2^i$
3. $d \leftarrow \lceil l/w \rceil$
5. By padding NAF (k) on the left with 0s if necessary, write
   $(k_{l-1},...,k_1,k_0) = K_{d-1} || ... || K_1 || K_0$ where each $K_i$ is a $\{0,\pm 1\}$ string of length
   d.
6. If $w$ is even then $I \leftarrow (2^{w+1}-2)/3$; else $I \leftarrow (2^{w+1}-1)/3$
7. $A \leftarrow \infty, B \leftarrow \infty$
8. For j from $I$ down to 1 do
   8.1. For each $i$ for which $K_i = j$ do: $B \leftarrow B + P_i$
   8.2. For each I for which $K_i = -j$ do $B \leftarrow B - P_i$
   8.3. $A \leftarrow A + B$
9. Return(A)
```

Another known point EC point multiplication style is fixed-base combing. In comb method we manipulate with scalar k again. We will form a matrix from the bits of k. The first thing is compute a parameter $d = \lceil t/w \rceil$ where t is the bit number of k. We will pad on the left of k with d. (w-t) number of zeros and divide the padded k into w bit strings. These strings will form the rows of our matrix and has the d number of bits. So we have a $w \times d$ matrix. For precomputation we will compute $[a_{w-1},...,a_1,a_0]P$ for all possible bit strings and store the results. This method is described in detail in Alg.3.9.

Algorithm 3.9 Fixed-base comb method for point multiplication

```
INPUT: Window width $w, d = \lceil t/w \rceil$ , $k = (k_{t-1},...,k_1,k_0)_{2^w}$ , $P \in E(F_q)$
OUTPUT: $kP$
1. Precomputation: Compute $[a_{w-1},...,a_1,a_0]P$ for all bit strings $(a_{w-1},...,a_1,a_0)$
   of length $w$.
2. By padding $k$ on the left with 0s if necessary, $k = K^{w-1} || ... || K^1 || K^0$ where
   each $K^j$ is a bit string of length $d$. Let $K_i^j$ denote the $i$th bit of $K^j$.
3. $Q \leftarrow \infty$
4. For $i$ from $d-1$ down to 0 do
   4.1. $Q \leftarrow 2Q$
   4.2. $Q \leftarrow Q + [K_i^{w-1},...,K_i^1,K_i^0]P$
5. Return($A$).
```

We can accelerate this method if we can tolerate some additional storage. We can use a second table of precomputations. So we can call fewer EC doubling operations. This method is described in Alg.3.10.

Algorithm 3.10 Fixed-base comb method (with two tables) for point multiplication

```
INPUT: Window width $w$, $d = \lceil t/w \rceil$ , $e = \lceil d/2 \rceil$ , $k = (k_{t-1},...,k_1,k_0)_{2^w}$ ,
$P \in E(F_q)$
OUTPUT: $kP$
1. Precomputation: Compute $[a_{w-1},...,a_1,a_0]P$ and $2^e[a_{w-1},...,a_1,a_0]P$ for all bit
   strings $(a_{w-1},...,a_1,a_0)$ of length $w$.
```

```
2. By padding $k$ on the left with 0s if necessary, $k = K^{w-1} \| \dots \| K^1 \| K^0$ where
   each $K^j$ is a bit string of length $d$. Let $K_i^j$ denote the $i$th bit of $K^j$.
3. $Q \leftarrow \infty$
4. For i from $e-1$ down to 0 do
        4.1 $Q \leftarrow 2Q$
        4.2 $Q \leftarrow Q + [K_i^{w-1},\dots,K_i^1,K_i^0]P + 2^e[K_{i+e}^{w-1},\dots,K_{i+e}^1,K_{i+e}^0]P$
5. Return ($Q$).
```

3.3 PERFORMANCE COMPARISON OF ELLIPTIC CURVE POINT MULTIPLICATION METHODS

We have stated the theoretical descriptions of elliptic curve multiplication methods. In this section we are going to examine these methods in real life. Elliptic curve point multiplication operation which is going to be handled in a cryptographic system should be implemented in a most appropriate and desired way. That is why the comparison work will be helpful in choosing the appropriate method to substitute in a real system. Our main aim is to find out the efficient and accelerated EC point multiplication method. We have chosen firstly to work on different fields with the idea of field operation speeds variety. Prime fields and binary fields became our computation area. In part 3.2.1 the EC point multiplication methods are examined over prime fields. The following part 3.2.2 will cover the EC point multiplication methods over binary fields. We will try to go further by including point representation schemes and window width effect for each field.

3.3.1 EC point multiplication over Prime Fields

Point multiplication over prime fields is done by field operations. First of all we choose our field for our system. In February 2000, FIPS 186-1 was revised by NIST to include the elliptic curve digital signature algorithm (ECDSA) as specified in ANSI X9.62 [ANSI, 1999] with further recommendations for the selection of underlying finite fields and elliptic curves; the revised standard is called FIPS 186-2 [NIST, 2000] [BrHankLopMen, 2001].

Table 3.3 NIST-recommended randomly chosen elliptic curves over prime fields F_{192}, F_{224}, F_{256}, F_{384}, F_{521}.

F_{192} : ***a*=-3,** $p_{192} = 2^{192} - 2^{64} - 1$ *b* = **0x64210519 E59C80E7 0FA7E9AB 72243049 FEB8DEEC C146B9B1** ***n*** **= 0xFFFFFFFF FFFFFFFF FFFFFFFF 99DEF836 146BC9B1 B4D22831**
F_{224} : ***a*=-3,** $p_{224} = 2^{224} - 2^{96} + 1$ ***b*** **= 0xB4050A85 0C04B3AB F5413256 5044B0B7 D7BFD8BA 270B3943** **2355FFB4** ***n*** **= 0xFFFFFFFF FFFFFFFF FFFFFFFF FFFF16A2 E0B8F03E 13DD2945** **5C5C2AD2**
F_{256} : ***a*=-3 ,** $p_{256} = 2^{256} - 2^{224} + 2^{192} + 2^{96} - 1$ ***b*** **= 0x5AC635D8 AA3A93E7 B3EBBD55 769886BC 651D060B CC53B0F6** **3BCE3C3E 27D2604B** ***n*** **= 0xFFFFFFFF 00000000 FFFFFFFF FFFFFFFF BCE6FAAD A7179E84** **F3B9CAC2 FC632551**
F_{384} : ***a*=-3,** $p_{384} = 2^{384} - 2^{128} - 2^{96} + 2^{32} - 1$ ***b*** **= 0XB3312FA7 E23EE7E4 988E056B E3F82D19 181D9C6E FE814112** **0314088F 5013875A C656398D 8A2ED19D 2A85C8ED D3EC2AEF** ***n*** **= 0xFFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF** **C7634D81 F4372DDF 581A0DB2 48B0A77A ECEC196A CCC52973**
F_{521} : ***a*=-3,** $p_{521} = 2^{521} - 1$ ***b*** **= 0x00000051 953EB961 8E1C9A1F 929A21A0 B68540EE A2DA725B 99B315F3** **B8B48991 8EF109E1 56193951 EC7E937B 1652C0BD 3BB1BF07 3573DF88** **3D2C34F1 EF451FD4 6B503F00** ***n*** **= 0x000001FF FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF** **FFFFFFFF FFFFFFFA 51868783 BF2F966B 7FCC0148 F709A5D0 3BB5C9B8** **899CA7AE BB6FB71E 91386409**

In Table 3.3 we have presented NIST-recommended randomly elliptic curves over prime fields. Our notation is as follows. For each of the prime fields, one randomly selected elliptic curve with the formula $y^2 + xy = x^3 + ax^2 + b$ was recommended and the coefficients of elliptic curve equation a and b will be denoted. The recommended elliptic curves have the coefficient a=-3. The reason of this selection is to have the Jacobian projective representation of elliptic curve. We know that Jacobian projective representation will make the field operation timings faster by reducing inverse operations. The number n is the prime number and order of base point of elliptic curve. For each field the prime modulo is also written.

The implementations are done according to these recommended prime fields. We have chosen $F_{192}, F_{224}, F_{256}, F_{384}, F_{521}$ NIST recommended fields for our prime field software implementations. This part of the section will cover the performance of the multiplication algorithms over these fields.

When focusing on elliptic curve multiplication, the first study became the basic methods. They are Right-to-Left (R-to-L) and Left-to-Right (L-to-R) methods. These methods are formed of repeated elliptic curve doubling and addition operations for each bit. Alg.3.1 and Alg.3.2 in Section 3.1.1 are the descriptions of these methods. In order to develop these methods for timing performance we can use NAF bit representation for our scalar *k*. This representation makes our nonzero bit density less than binary representation.

We have implemented these basic multiplication methods for affine and mixed (Jacobian-affine) coordinates. Table 3.4 shows the timing results of these methods for prime fields $F_{192}, F_{224}, F_{256}, F_{384}, F_{521}$. The k is chosen randomly in the field which the operations are done over. The execution times are given in milliseconds.

Table 3.4 Timing results (ms) of Right-to-Left, Left-to- Right and Binary NAF methods for elliptic curve point multiplication over prime fields.

Method	Coordinates	Prime fields		
		F_{192}	F_{224}	F_{384}
Right-to-Left	Affine	55	55.7	216
	Jacobian-affine	43	45.3	169
Left-to-Right	Affine	37.5	54.7	223
	Jacobian-affine	7.8	9.3	40.6
Binary NAF	Affine	34.3	51.6	206
	Jacobian-affine	6.8	9.69	35.7

The Right-to-Left and Left-to-Right methods do not have much superior to each other when affine coordinates are used. But if we use Jacobian coordinates for doubling stages and Jacobian-affine mixed coordinates for addition stages the results differ. Left-to-Right method timing results are better than the Right-to-Left method timing results when Jacobian-affine coordinates are used together. Since binary NAF method implementation requirements (e.g. no memory environment) are same with Right-to-Left and Left-to-Right methods, we can take this method in consideration for our performance analysis. The results, presented in Table 3.4 express that binary NAF method is better than Right-to-Left and Left-to-Right methods for both affine and mixed coordinates. That is why using binary NAF method for elliptic curve point multiplication is appropriate especially when we have memory constraints in our cryptographic system or we are going to do the point multiplication operation with unknown elliptic curve point.

Window Methods over Prime Fields

If we examine the detailed steps of window methods for elliptic curve multiplication operation we will recognize the precomputation step. Window NAF and Sliding Window methods are the first examples of elliptic curve multiplication methods which need precomputed data. The required memory for implementation of these kinds of methods is the one of the issues that we should investigate. If memory is a constraint for a cryptographic system the designer should choose the most suitable method. If memory is not a constraint in a cryptographic system we can prefer fixed base methods. Fixed base methods need more data storage with better timing results for EC point multiplication operation. In order to see the memory usage for these methods we have presented storage requirements. In Table 3.5, the storage requirements are stated for different window widths. The window width is shown by the letter *w*.

Table 3.5 Number of stored EC point data for window methods.

	w=3	w=4	w=5	w=6
WindowNAF	2	4	8	16
SlidingWindow	3	5	11	21
Fixed-base Window	63	47	38	31
Fixed-base WindowNAF	63	47	38	31
Fixed-base Comb[a]	14	30	62	126
Fixed-base Comb with 2 tables[a]	28	60	124	252

[a] The denoted number of points stored in precomputation phase are calculated for F_{192}.

Table 3.5 tells us which EC point multiplication method needs how much storage. The given values are the number of points that would be stored where a point occupies the memory of two times key length. That means if we work over F_{192}, we must provide 2*192 bits memory space for an EC point.

We will try to examine the timing performance of the methods listed in Table 3.5. It is obviously seen that these EC point multiplication methods can be chosen if we do not have any restrictions about memory. Our first approach when examining the timing performance is observing the window effect for each window method. Does window width positively or negatively affects the multiplication performance? Next part gives the investigated results about window width effect.

Window width effect over prime fields

In order to observe window width effect on elliptic curve multiplication methods over prime fields, we tested our algorithms for both affine and mixed coordinate system. We have seen that projective coordinates made time efficient the Left-to-Right, Right-to-Left and Binary NAF methods. That is why we also implemented the window methods in projective coordinates. Affine coordinated implementations are done for each multiplication operation. Jacobian and Cudnowsky projective coordinates are substituted to methods which are suitable for them. We have chosen F_{224} prime field for window width effect observation. In Table 3.6 the timing results are denoted with stating the coordinate system implemented. The suitable window width may differ for different EC point multiplication methods. We can have an opinion about the window width effect on window methods by examining Table 3.6, Figure 3.1 and Figure 3.2. We have listed the execution time values of all window methods both with affine and mixed coordinates in Table 3.6. Figure 3.1 represents the behavior of affine coordinated window methods with changing window width. For mixed coordinated window methods we have also plotted Figure 3.2 the behavior of the window methods with changing window width. According to the timing values we can choose the appropriate window width for EC point multiplication method which will dominate our cryptosystem.

Table 3.6 Timing results (ms) of window methods over F_{224} for different window widths.

		Window Width			
Methods	Coordinates	w=3	w=4	w=5	w=6
Window NAF	Affine Jacobian-affine	46.9 8.59	45.3 8.59	43.8 8.12	43.8 7.97
Sliding Window	Affine Jacobian-affine	45.4 7.8	43.8 9.4	42.2 7.8	42.2 7.8
Fixed base-window	Affine Cudnowsky-affine-Jacobian	10.9 3.1	11 3.1	12.5 3.1	15.6 4.7
Fixed-base NAF	Affine Cudnowsky-affine-Jacobian	9.4 2.3	10.9 2.65	10.9 2.81	12.5 4.7
Fixed-base Comb	Affine Jacobian-affine	22 4.6	17.8 3.91	14.5 3.1	12.5 3.2
Fixed-base Comb With 2 tables	Affine Jacobian-affine	15.6 3.6	12.5 3.13	10.9 2.5	9.4 2.19

Sliding Window and Window NAF methods do not have a great timing dependence on window width both affine and Jacobian-affine coordinates, but the choice of *w*=5 or *w*=6 would be the suitable choice for this field EC point multiplication. The window effect on Fixed-base Window and Fixed-base NAF method is also slight. It would be better to have smaller window widths for these methods. This also brings smaller data storage requirement. This scheme is not the same for comb methods. The enlarging window width makes these methods faster. As we stated before larger window widths for comb methods require the storage of precomputed data. In Figure 3.1 and Figure 3.2 the comb methods present similar behavior. We have computed the best timing result for Fixed Base Comb Method (with two tables) when window width is 6.

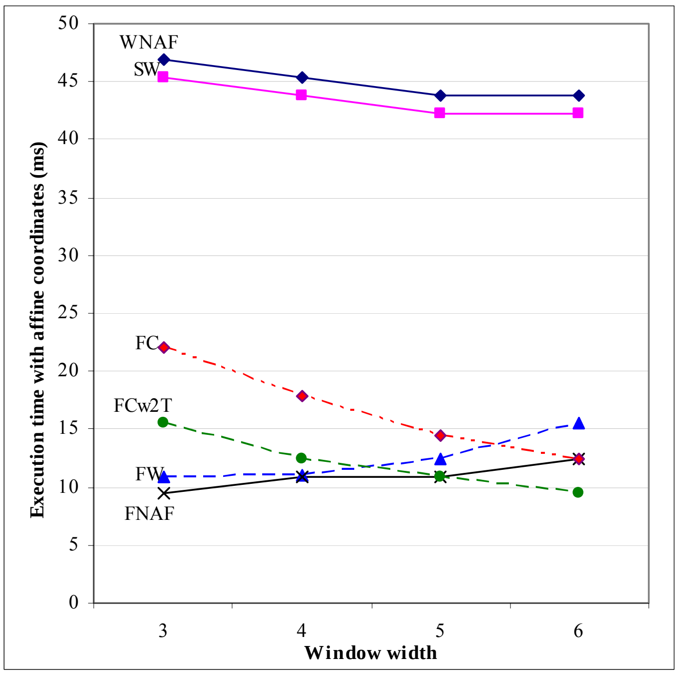

Figure 3.1 Timings of EC point multiplication window methods at affine coordinates versus window width.

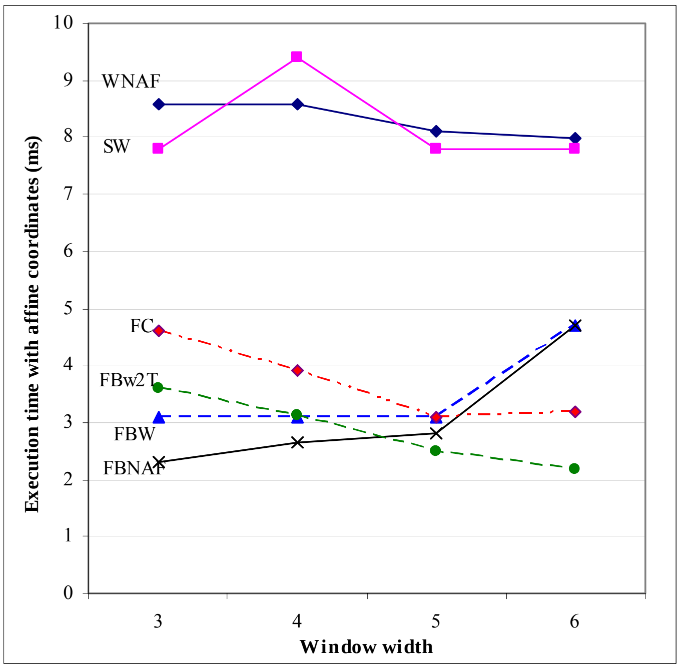

Figure 3.2 Timings of EC point multiplication window methods at mixed coordinates versus window width.

General Look to Execution Times of Elliptic Curve Multiplication Methods over Prime Fields

In this part of the chapter we will talk about the timing results of the elliptic curve multiplication methods over prime fields generally. In a cryptographic system design which we are going to use elliptic curve cryptography we should choose the most suitable multiplication operation for our system. That is because of the big role of elliptic curve multiplication operation for ECDSA (Elliptic Curve Digital Signature Algorithm) or elliptic curve key exchange algorithms. We have chosen the best timing results for different platforms. Table 3.7 represents the timing results of these chosen algorithms. Since using projective coordinates in EC point multiplications eliminates field inversion operation we prefer using mixed coordinates instead of using only affine coordinates.

After the evaluation of elliptic curve multiplication operation over prime fields, we will study binary field case. The next section will cover the same path of evaluation of this section.

Table 3.7 Timings (ms) of common multiplication methods for different prime fields.

Multiplication Method	Memory usage	F_{192}	F_{224}	F_{256}	F_{384}	F_{521}
Left-to-Right	NO	7.8	9.3	15.6	40.6	78
Window NAF	YES	6.09[b]	8.12[c]	11.7[c]	29.2[d]	57.35[d]
Fixed-base window	YES	1.5[a]	3.1[a]	3.6[c]	17.2[c]	17.2[b]
Fixed-base Comb with 2 Table[d]	YES	1.41	2.19	3.12	7.5	15.4

[a]The window width is 3, [b]The window width is 4, [c]The window width is 5, [d]The window width is 6.

3.2.2 EC point multiplication over Binary Fields

Point multiplication over binary fields is done by bit operations. We know that multiplication operation over elliptic curves is formed of several addition operations. In binary fields, addition of field elements is performed bitwise xor-ing the vector representations. The field F_{2^m} can be viewed as a vector space of dimension m over F_2. There exists a set of m elements $\{\alpha_0, \alpha_1, ..., \alpha_{m-1}\}$ in F_{2^m} such that each $\alpha \in F_{2^m}$ can be uniquely written in the form

$$\alpha = \sum_{i=0}^{m-1} a_i \alpha i, \ where \ a_i \in \{0,1\}$$

We can represent alpha as the binary vector $(a_0, a_1, ..., a_{m-1})$.

Firstly, we choose our binary field for our system. In February 2000, FIPS 186-1 was revised by NIST to include the elliptic curve digital signature algorithm (ECDSA) as specified in ANSI X9.62 [ANSI, 1999] with further recommendations for the selection of underlying finite fields and elliptic curves; the revised standard is called FIPS 186-2 [NIST, 2000]. There are 5 recommended binary fields. They are $F_{2^{163}}, F_{2^{233}}, F_{2^{283}}, F_{2^{409}}, F_{2^{571}}$. Our implementations are done over these recommended binary fields. In this part of the chapter, we are going to examine the EC point multiplication algorithms and their performances over binary fields.

We have chosen the $F_{2^{163}}, F_{2^{233}}, F_{2^{283}}, F_{2^{409}}, F_{2^{571}}$ NIST recommended fields for our calculations. Table 3.5 gives the parameters of these recommended fields. The implementations in this study are performed according to these parameters. The following notation is used. The elements of F_{2^m} are represented using a polynomial basis representation with reduction polynomial, *f*(*x*). An elliptic curve *E* over F_{2^m} is specified by the coefficients $a, b \in F_{2^m}$ of its equation $y^2 + xy = x^3 + ax^2 + b$. The number of points on *E* defined over F_{2^m} is nh, where *n* is prime, and *h* is the cofactor.

Table 3.8 NIST-recommended elliptic curves over $F_{2^{163}}, F_{2^{233}}, F_{283}, F_{2^{409}}, F_{571}$.

$F_{2^{163}}$ **: a=1,** $f(z)=z^{163}+z^{7}+z^{6}+z^{3}+1$
b = 0x00000002 0A601907 B8C953CA 1481EB10 512F7874 4A3205FD
n = 0x00000004 00000000 000000000 000292FE 77EC0C12 A4234C33
$F_{2^{233}}$ **: a=1,** $f(z)=z^{233}+z^{74}+1$
b = 0x00000066 647EDE6C 332C7F8C 0923BB58 213B333B 20E9CE42 81FE115F 7D8F90AD
n = 0x00000100 00000000 00000000 00000000 0013E974 E72F8A69 22031D26 03CFE0D7
F_{283} **: a=1 ,** $f(z)=z^{283}+z^{12}+z^{7}+z^{5}+1$
b = 0x027B680A C8B8596D A5A4AF8A 19A0303F CA97FD76 45309FA2 A581485A F6263E1 3B79A2F5
n = 0x03FFFFFF FFFFFFFF FFFFFFFF FFFFFFFF FFFFEF90 399660FC 938A9016 5B042A7C EFADB307
$F_{2^{409}}$ **: a=1 ,** $f(z)=z^{409}+z^{87}+1$
b = 0x0021A5C2 C8EE9FEB 5C4B9A75 3B7B476B 7FD6422E F1F3DD67 4761FA99 D6AC27C8 A9A197B2 72822F6C D57A55AA 4550AE31 7B13545F
n = 0x01000000 00000000 00000000 00000000 00000000 00000000 000001E2 5FA47C3C 9E052F83 8164CD37 D9A21173
F_{571} **: a=1 ,** $f(z)=z^{571}+z^{10}+z^{5}+z^{2}+1$
b = 0x02F40E7E 2221F295 DE297117 B7F3D62F 5C6A97FF CB8CEFF1 CD6BA8CE 4A9A18AD 84FFABBD 8EFA5933 2BE7AD67 56A66E29 4AFD185A 78FF12AA 520E4DE7 39BACA0C 7FFEFF7F 2955727A
n = 0x03FFFFFF FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF E661CE18 FF559873 08059B18 6823851E C7DD9CA1 161DE93D 5174D66E 8382E9BB 2FE84E47

As we did for prime fields we have studied elliptic curve multiplication implementations for binary fields. All the methods that we have described in section 3.1 are written in C programming language. We begin with the basic EC point multiplication methods which are Right-to-Left, Left-to-Right Binary and

Binary NAF, Montgomery. Since we can use these methods for platforms both with constraint and adequate memory, we call these methods basic methods. These EC point multiplication methods are especially for memoryless systems and for the case of unknown curve point. Both affine and projective coordinates are substituted for the methods. The Montgomery Method uses projective coordinates. The timings are affected by the coordinate choice. We stated our execution time results in Table 3.9 below.

Table 3.9 Timing results (ms) of Right to Left (RTL), Left to Right (LTR) and Binary NAF methods for elliptic curve point multiplication over binary fields.

Method	Coordinates	Binary fields		
		$F_{2^{163}}$	$F_{2^{283}}$	$F_{2^{409}}$
RTL	Affine	12.5	42	109
	Projective	7.8	26	68
LTR	Affine	12	40.6	112
	Projective	4.7	10.9	31
Binary NAF	Affine	11	37	96
	Projective	3.1	10.9	23
Montgomery	Projective	3.1	7.8	21.8

As expected, the performance decreases with the increasing field size. That refers to the key size in elliptic curve cryptography that will determine the cryptographic performance. In Table 3.6, it is obviously concluded that coordinate choice has a great effect on the timings. Using projective coordinates instead of affine coordinates gives at least 50% better results. When we use affine coordinates for these basic multiplication methods, the results do not present any superiority among different methods. Because of that, we can choose any of these methods for affine coordinated software. But if we can use projective coordinates, the methods will show better timing performance. Between the projective coordinated implementation results of these methods, Montgomery method should be the most appropriate choice.

As described in the previous sections, there is a concept of window width that provides faster EC point multiplication methods. Now, we will study this scheme and try to find a performance relation with the chosen window width in binary fields.

Window width effect over binary fields

In this part the described window methods for elliptic curve multiplications will be studied over binary fields. The window methods are Window NAF, Sliding window, Fixed Base Window , Fixed Base NAF, Fixed Base Comb, Fixed Base Comb (with two tables) methods. We have implemented each of these methods with different window widths. The results are presented in Table 3.7. The multiplications are done over $F_{2^{163}}$ field by using both affine and projective coordinates. From the results we can conclude that window width does not have a noticeable effect on multiplication durations when we use projective coordinates. If we use affine coordinates, *w*=4 for Fixed Base Window method, *w*=5 for Fixed Base NAF method, *w*=6 for Fixed Base Comb methods are the good choices for window widths to have fast elliptic curve multiplication operations. Widening window width affects comb methods in a good way that their execution times decrease.

Table 3.10 Timing results (ms) of window methods over $F_{2^{163}}$ for different window widths.

		Window Width			
Methods	Coordinates	w=3	w=4	w=5	w=6
Window NAF	Affine	9.4	9.4	7.8	9.3
	Projective	3.1	1.5	3.1	1.6
Sliding Window	Affine	9.4	9.4	7.8	9.4
	Projective	3.1	3.1	1.6	1.5
Fixed base-window	Affine	3.1	1.6	3.2	4.7
	Projective	1.6	1.6	3.1	1.6
Fixed-base NAF	Affine	2.35	2.3	2.5	3.44
	Projective	1.57	1.7	2.3	3.4
Fixed-base Comb	Affine	4.7	3.2	3.2	1.6
	Projective	1.6	1.5	1.6	1.6
Fixed-base Comb	Affine	3.1	3.1	1.5	1.6
(with 2 tables)	Projective	1.5	1.5	1.5	1.5

After investigating elliptic curve multiplication over binary fields we conclude that using projective coordinates make our multiplication performance better. So using projective coordinates instead of affine coordinates may be our first choice for our cryptographic system design. According to the memory constraints of the system it is better to use Montgomery method for memoryless systems. If we have enough memory for the precomputations we can use Fixed-base window or Fixed-base Comb methods. We have chosen the appropriate window widths for speed of the window methods

General Look to Execution Times of Elliptic Curve Multiplication Methods over Binary Fields

Binary field elliptic curve multiplications have been implemented for different binary fields, multiplication methods, and different multiplication method parameters (e.g. window width, coordinate system). A general look to our results will be figured and detailed in the following table. We have chosen the fast results for different possibility of conditions. If we want to design a cryptographic system based on elliptic curve cryptography, we can choose the appropriate EC point multiplication method by examining this table. Which method would be suitable for a design? To answer of this question we should check out the conditions that we have. For example if we do not have much memory to store some data, our choice would be a multiplication method without using any precomputation. The implementation complexity should be another choice and depends on the software designer's preference. There is also a case that we cannot precompute any data before multiplication operation proceeds. This is multiplication of unknown EC point. In order to draw a conclusion about the choice of multiplication method we should consider memory, software platform meaning the processor that EC algorithms will run over, and the case which we meet an unknown point.

Table 3.11 Timings (ms) of common EC point multiplication methods.

Method	Memory requirement	$F_{2^{163}}$	$F_{2^{283}}$	$F_{2^{409}}$
Left-to-Right	NO	12	10.9	31
Sliding Window	YES	1.6	7.4	18
Montgomery	NO	3.1	7.8	21.8
Fixed-base Window	YES	1.6	4.7	12.5
FBComb	NO	1.6	3.2	7.8

3.4 COMMENTS ON ELLIPTIC CURVE MULTIPLICATION IMPLEMENTATIONS

The efficiency of the underlying finite field operations supply the dominant performance of all elliptic curve operation schemes. If we ask the question of which field is the best choice, there can not be a single correct answer, since it depends on the constraints such as the processor type or memory requirements. In this study we have worked on Pentium 4 processor at 3 GHz and our code size is 200 KB for prime field implementations and 160 KB for binary field implementations .

Our main study is presented as the analysis of elliptic curve multiplication methods in Section 3.2 and in Section 3.3. We first approach the analysis over different fields. For both prime and binary fields, we have examined the multiplication method properties and their behaviours against the changing crucial parameters. The operations over binary fields are faster and easier to implement than the operations over prime fields of approximately the same size. Our timing results also show better performance of binary fields on PC.

The path of investigation of multiplication methods is routed through the known point and unknown point cases. Especially *known point* multiplications are significantly faster than the random *unknown point* multiplications. As we have stated many times, the aid of precomputation step supplies faster multiplication operations.

Window width effect was one of our study themes for timing analysis of the window multiplication methods. We have seen that for fixed-base methods, wider window widths affect the performance of multiplication operation positively.

We have seen that coordinate choice has a great effect on the timing performance of EC point multiplication operations. Using projective coordinates such as Jacobian or Cudnowsky for prime fields, and standart projective coordinates for binary fields make our multiplication performances faster. In Section 3.1, our field arithmetic software implementation study shows that the most expensive field operation is field inversion. Since projective coordinates elliminates the field inversion operation, their usage makes our implementations faster.

We have presented a general graph for the chosen multiplication algorithms in Figure 3.3. We have put together prime field and binary field EC point multiplication results. At implementation stages of this study, it is observed that using mixed coordinates instead of affine coordinates for appropriate stages of multiplication algorithms perform faster computations. So the results in Figure 3.3 belong to implementations which we have used mixed coordinate for EC point multiplication methods. For prime fields, affine coordinates are used with Jacobian and Cudnowsky coordinates. For the binary field methods, standart projective coordinates are used. We have chosen typical results in order to make good comparison.

The calculations are done in fields $F_{2^{163}}$, F_{192}, $F_{2^{283}}$, F_{256}, $F_{2^{409}}$, F_{384}. The NIST recommended curves stated in Table 3.3 and Table 3.8 are used for EC point multiplications and the scalar k is chosen randomly. From Figure 3.3 it is seen that the ranking of the methods does not change according to the field. The EC point multiplication implementations in binary fields are faster than the ones in prime fields of similar size. The results show that Fixed-base Comb Method (with two table) has the best timing performance for both binary and prime fields; however this timing performance is paid by using memory for data storage. Besides known point case, we may face unknown point EC point multiplications in EC cryptography. Then a multiplication method without precomputation should be performed. For such cases, Window NAF multiplication method seems to have faster peformances than Left-toRight Binary method as observed from Figure 3.3. In Figure 3.3, binary fields with 2^m elements are shown B-m whereas prime fields with m-bit primes are indicated by P-m. The represented elliptic curve multiplication methods are Left-toRight Binary Method, Window NAF Method, Fixed-base Window Method and Fixed-base Comb Method (with two tables). Correspondence between the column pattern and the method is given in the figure legend.

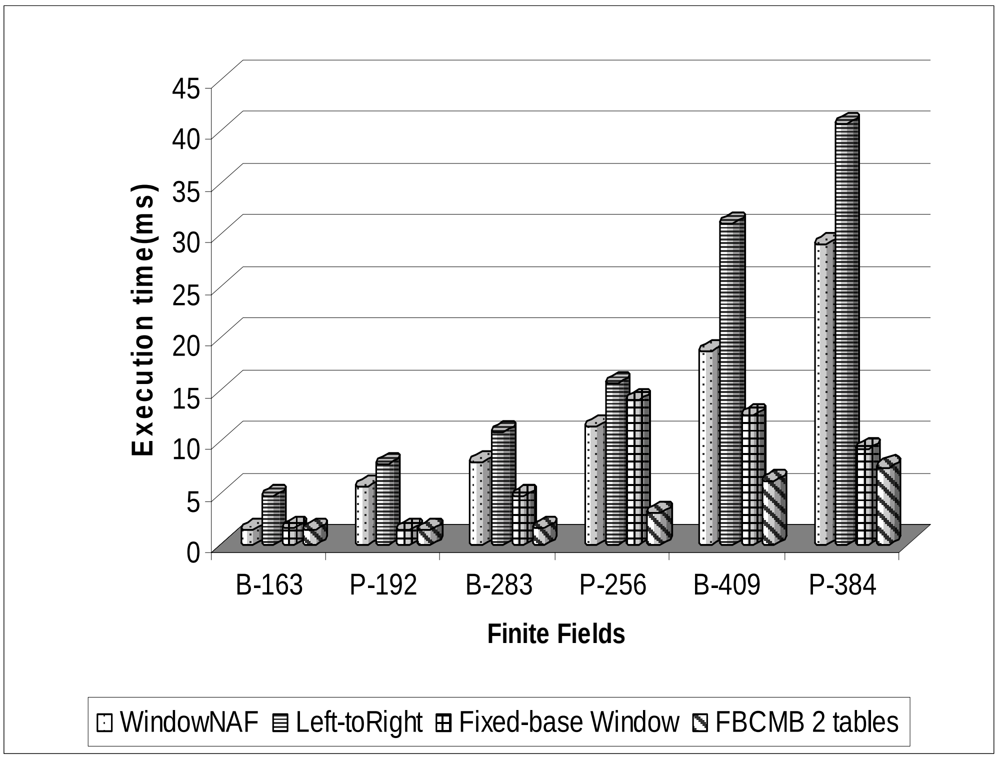

Figure 3.3. Execution times (ms) for Window NAF, Left-toRight, Fixed-base Window, Fixed-base Comb with two tables elliptic curve point multiplication methods over finite fields.

CHAPTER 4

ELLIPTIC CURVE DIGITAL SIGNATURE ALGORITHM (ECDSA)

Signature schemes in cryptography are the digital counterparts to handwritten signatures. We can use digital signatures to provide authentication, data integrity, and non-repudiation. In this chapter, we first describe the signature schemes and elliptic curve digital signature algorithm. We then substitute the elliptic curve multiplication methods that we have implemented into the digital signature algorithms. Our aim is to see the EC point multiplication affect on signature generation and verification. Basic definitions and algorithm descriptions are covered in Section 4.1. The results and the comments about ECDSA implementations are presented in Section 4.2.

4.1 SIGNATURE SCHEMES AND ECDSA DESCRIPTION

Digital signatures provide the validity of a document by ensuring the sender's identity and the unchanged data. Due to Goldwasser, Micali and Rivest there is a notion about the security of a signature scheme, which is defined as follows. A signature scheme is said to be secure if it is existentially unforgeable by a computationally bounded adversary who can mount an adaptive chosen message attack. This means that the adversary can obtain the signature of any message but can not produce a valid signature of any new message. In order to build this concept a signature scheme consists of four algorithms. These are domain parameter generation, key generation, signature generation and signature verification algorithms.

The Elliptic Curve Digital Signature Algorithm (ECDSA) is the analogue of the Digital Signature Algorithm (DSA). ECDSA is the standardized elliptic curve-based signature scheme in ANSI X9.62, FIPS 186-2, and IEEE 1363-2000, ISO/IEC 15946-2 standards. Before getting into implementation results we will describe the signature schemes. The first one is domain parameter generation. The domain parameters are field order (*q*), field representation, elliptic curve equation parameters (*a*,*b*), the base elliptic curve point (*G*), the prime order of the point and the cofactor (*h*=*N*/*n*). Alg.4.1 details each step of cryptographically secure domain parameter generation.

Algorithm 4.1 Domain parameter generation

INPUT: Field of order *q*, security level *L* satisfying $160 \le L \le \lfloor \log_2 q \rfloor$ and $2^L \ge 4\sqrt{q}$

OUTPUT: Domain parameters *D*=(*q*,*FR*, *a*,*b*,*G*,*n*,*h*)

1. Select coefficients *a* and *b* from F_q at random. Let *E* be the curve $y^2 = x^3 + ax + b$ in the case F_q is a prime field or $y^2 + xy = x^3 + ax^2 + b$ in the case F_q is a binary field.
2. Compute $N=\#E(F_q)$
3. Verify that *N* is divisible by a large prime n ($n > 2^{160}$ and $n > 4\sqrt{q}$). If it is not then return to step 1.
4. Verify that *n* does not divide $q^k - 1$ for each *k*, $1 \le k \le 20$. If not return to step 1.
5. Verify that $n \ne q$. If not, then return to step 1.
6. Select an arbitrary point $G' \in E(F_q)$ and set $G = (N/n)G'$. Repeat until $G \ne \infty$.
7. Return (*q*,*FR*,*a*,*b*,*G*,*n*,*h*)

After generation of domain parameters we can generate the keys for our cryptosystem. Generation of key has a procedure as follows.

Algorithm 4.2 Key pair generation

INPUT: Field order *q*, field representation FR for F_q, seed for random generations, curve equation coefficients *a* and *b*, base point *P*, the order of base point *n*, cofactor *h* ($h=\#E(F_q)/n$)

OUTPUT: Public key *Q*, private key *k*.

1. Select $k \in_R [1, n-1]$
2. Compute $Q=kP$
3. Return(Q,k)

The generated pair (Q,k) is the keys of a communicating entity. The private key is k and the public key is *Q*. The entity will use his private key to sign his message. For signature generation we use the hash value of the message, $H(m)$. The entity also publics the public key, *Q*. The receiver will use the public key to verify the sender. Next we will describe signature generation algorithm.

Algorithm 4.3 ECDSA Signature generation

INPUT: Domain parameters $D=(q,FR,a,b,G,n,h)$, private key *k*, message *m*

OUTPUT: Signature (r,s)

1. Select a random *k*, $1 \leq k \leq n-1$.
2. Compute $kG=(x_1,y_1)$ and convert x_1 to an integer $\overline{x_1}$.
3. Compute $r = x_1 \bmod n$. If $r=0$ then go back to step 1.
4. Compute $k^{-1} \bmod n$.
5. Compute $H(m)$ and convert this bit string into an integer *e*.
6. Compute $s = k^{-1}(e+dr) \bmod n$. If $s=0$ then go to step 1.
7. The signature for message *m* is (r,s).

Verification of a received signed message is done by using the same domain parameters and the public key of the sender. The ECDSA signature verification is done as follows.

Algorithm 4.4 ECDSA Signature verification

INPUT: Domain parameters *D*=(*q*,*FR*,*a*,*b*,*P*,*n*,*h*), public key *Q*, message *m*, signature (*r*,*s*).
OUTPUT: Acceptance or rejection of the signature.

```
1. Verify that $r$ and $s$ are integers in the interval $[1, n-1]$. If it fails reject the signature.
2. Compute $e=H(m)$.
3. Compute $w = s^{-1} \bmod n$
4. Compute $u_1 = ew \bmod n$ and $u_2 = rw \bmod n$.
5. Compute $X = u_1 P + u_2 Q$
6. If $X = \infty$ then reject the signature.
7. Convert the $x$-coordinate $x_1$ of $X$ to an integer $\overline{x_1}$, compute $v = \overline{x_1} \bmod n$
8. If $v = r$ then accept the signature
   Else reject the signature
```

We have implemented the signature generation and verification algorithms on Pentium 4 processor in binary fields. Our aim is to see the performance of our implemented elliptic curve multiplication methods. As we stated before, EC point multiplication influences elliptic curve cryptography schemes particularly. Next section will cover these results and conclusions

4.2 IMPLEMENTING ECDSA SIGNATURE GENERATION AND VERIFICATION ALGORITHMS

In this part of the chapter we present the results of our signature generation and verification implementation. The implementation is written in C on Pentium 4 processor.

Before getting into the timing results, let us describe the implementation steps. The implementation of signature verification is done according to Alg.4.3 [HankHernMen 2000]. If we try to interpret what is going on we can say that in Step 2, there is an elliptic curve point multiplication. The other steps are big number arithmetic and a hash function call. We can easily say that the dominant part of this algorithm is EC point multiplication. That is why it can be concluded that the timing of signature generation will depend on EC point multiplication timing. So here comes the importance of the efficiency of an EC point multiplication operation.

If we interpret signature verification algorithm in Alg.4.4 [HankHernMen, 2000], we observe two elliptic curve multiplication operations at Step 5. The other operations are big number operations and a hash function call. Since EC point multiplication operation takes more time than the other number operations, it takes most of the time of the verification algorithm. So it is again important to use a time efficient elliptic curve multiplication method for signature verification in an ECC system. Step 5 of the verification algorithm is $X = u_1P + u_2Q$. Here there are two elliptic curve points. The EC point P is our domain parameter and known a priori. We can use window methods or fixed-base methods for this multiplication. With known point, precomputation can be completed before verification is needed. The EC point Q is the public key of the sender, so it is not known before. The concept of unknown point multiplication will take place for this purpose. The appropriate multiplication should be chosen for this step among unknown point EC point multiplication methods.

In Table 4.1, the timings of our signature generation implementations are presented. The implementations are done over $F_{2^{163}}, F_{2^{283}}, F_{2^{409}}$ binary fields. We have used NIST recommended curves from Table 3.8. The results show that we have 15% better timing results than the results in [HankHernMen, 2000]. It is interesting that the best of timing results are found for the Fixed-base Comb Method (with two tables) [HankMenVan, 2004], which is not implemented in [HankHernMen, 2000].

Table 4.1. Timings (ms) of ECDSA signature generation algorithm with different elliptic curve multiplication methods.

Multiplication Method	Memory requirement	$F_{2^{163}}$	$F_{2^{283}}$	$F_{2^{409}}$
Montgomery	NO	3.33	9.9	24
Left-to-Right	NO	4.4	13.4	32
SlidingWindow[a]	YES	3.09	8.76	21.4
WindowNAF[a]	YES	3.01	8.84	21
Fixed-base Window[b]	YES	2.14	5.87	13.9
Fixed-base Comb[a]	YES	1.54	4.42	10.7
Fixed-base Comb (with 2 tables)[a]	YES	1.37	3.9	9.50

[a] The window width is w=6.
[b] The window width is w=3.

We illustrate the timings of our implementation for signature verification in Table 4.2. There are two EC point multiplications in ECDSA signature verification scheme, one of them is the known point type and the other one is the unknown point type. We can use two different multiplication methods for each case. Using a known point multiplication method would make our signature verification process faster. We have chosen the Montgomery and Left-to-Right Binary EC point multiplication methods which have fine timing performances, for the unknown point case. In [HankHernMen, 2000], similar measurements have been done with Montgomery method for the unknown point case and Fixed-base Comb method for the known point case. We have included Fixed-base Comb (with two tables) method for the known point case EC multiplication step with the expectation of the fastest result.

Table 4.2 Timings (ms) of ECDSA signature verification algorithm with different elliptic curve multiplication methods.

Multiplication Method	Memory requirement	$F_{2^{163}}$	$F_{2^{283}}$	$F_{2^{409}}$
Montgomery + Montgomery	NO NO	7.8	18	45
Left-to-Right + Left-to-Right	NO NO	9.3	25	64
SlidingWindow[a] + Montgomery	YES NO	6.3	18.9	42
Fixed-base Window[b] + Montgomery	YES NO	4.6	14	36
Fixed-base Comb[a] + Montgomery	YES NO	4.7	12.5	31.2
Fixed-base Comb with 2 tables[a] + Montgomery	YES NO	4.22	12.5	30.9

[a] The window width is w=6.
[b] The window width is w=3.

In Table 4.2, the measurements are presented by using both memory requiring and not memory requiring EC point multiplication methods for the known point multiplication step. The timing results show that using a memory requiring method makes signature verification process faster. That is why if memory is not constrained, a window method or comb method should be preferred for faster results. We have obtained 30% faster signature verification timings than the timings in [HankHernMen, 2000] when finite fields with large bit size are used. For 163-bit binary field we have measured 10% faster timing. When, the bit size gets larger, the efficiency of the methods is observed more precisely.

CHAPTER 5

CONCLUSION

In this thesis study the software implementation of EC point multiplication methods and their applications to the ECC protocols have been studied. Our first work is the implementation of finite field arithmetic algorithms, which underlie the elliptic curve operations. For prime fields, we have implemented efficient field arithmetic algorithms and obtained better timings than the timing results in [BrHankLopMen, 2001]. In binary fields, our implementations have had fine timing results which are better than the results in [HankHernMen, 2000] and similar to the results in [YanShi, 2006].

Our software implementations of the EC point multiplication methods have been chosen among the proposed ones in the literature. These methods are Right-to-Left Binary, Left-to-Right Binary, Binary NAF, Window NAF, Sliding Window, Montgomery, Fixed-base Window, Fixed-base NAF, Fixed-base Comb, and Fixed-base Comb (with two tables) methods. All these EC point multiplication methods have been proposed by the researchers according to the point knowledge (if it is known a priori or not) and memory usage facilities. For the known point case, precomputing some useful data makes the proposed methods more time efficient. We have measured and compared the timings of our implemented EC point multiplication methods. The measurements have been done for the stated finite fields, with pseudo random scalar multiplier k. We have seen that the point representation has a significant effect in timings and using mixed coordinates like affine coordinates with projective coordinates makes the algorithms faster. The windowing methods have been studied for the window widths between 3 and 6. Although the window width does not have a significant effect, the fixed-base window and comb methods have typical window widths for the fastest timing results.

After comparing the methods in binary fields and prime fields separately, we have made the comparisons of similar size binary and prime fields. It is observed and experienced that the EC multiplication operations are faster in binary fields than they are in prime fields.

The similar software implementation studies over prime and binary fields have been done on the same CPU platforms in [BrHankLopMen, 2001] and [HankHernMen, 2000]. We have presented approximately 25% better timing results for finite fields with large bit size by the aid of today's Pentium processor facilities.

The implemented EC point multiplication algorithms have been applied to the ECDSA algorithm. Utilizing different EC point multiplication methods, we have had better timings in signature generation and verification processes with fast methods. The influence of EC point multiplication operation on ECC protocols has been demonstrated.

As a final comment, the design of an ECC system requires many decisions. These decisions include the type of the underlying field, the algorithms of the field arithmetic, the algorithms of the elliptic curve arithmetic, the elliptic curve cryptography protocols, the computation platform, the memory constraints and the programming language. According to our study, we propose the following choices: The underlying field can be binary fields, since the execution timings of the field arithmetic and EC arithmetic are faster. Using affine coordinates together with projective coordinates gives faster EC point multiplication timings. The EC point multiplication method choice depends on the design platform memory constraints. Montgomery and Binary NAF methods are good choices for memoryless systems. If we have enough memory space for storage of pecomputations comb methods are the finest choices for EC point multiplication. We hope that our study may be a guide for the design decisions of an ECC system, since it contains many comparisons and underlies some critical points for the choices of EC point multiplication method, field, point representation, and window width.

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APPENDIX A

ROUTINES OF FIELD ARITHMETIC AND ELLIPTIC CURVE ARITMETIC OVER BINARY FIELDS

A.1 REPRESENTING BINARY FIELD ELEMENTS

```
/*** field2n.h ***/

#define WORDSIZE        (sizeof(int)*8)
#define NUMBITS         163
#define NUMWORD         (NUMBITS/WORDSIZE)
#define UPRSHIFT        (NUMBITS%WORDSIZE)
#define F2NMAXLONG      (NUMWORD+1)
#define MAXBITS         (F2NMAXLONG*WORDSIZE)
#define MAXSHIFT        (WORDSIZE-1)
#define MSB             (1L<<MAXSHIFT)
#define UPRBIT          (1L<<(UPRSHIFT-1))
#define UPRMASK         (~(-1L<<UPRSHIFT))
typedef short int INDEX;
typedef unsigned long ELEMENT;
typedef struct {
      ELEMENT e[F2NMAXLONG];
}  FIELD2N;
#define DBLBITS         2*NUMBITS
#define DBLWORD         (DBLBITS/WORDSIZE)
#define DBLSHIFT        (DBLBITS%WORDSIZE)
#define MAXDBL          (DBLWORD+1)
typedef struct {
      ELEMENT e[MAXDBL];
} DBLFIELD;
```

A.2 BINARY FIELD ARITHMETIC

```
void poly_add(FIELD2N *a,FIELD2N *b,FIELD2N *c)
{
  unsigned int i;
  for(i=0; i<F2NMAXLONG; i++)
    c->e[i] = a->e[i] ^ b->e[i];
}
```

```
void poly_mul_Left_to_Right_Comb_W4
      (FIELD2N *a, FIELD2N *b, DBLFIELD *c)
{
  DBLFIELD B[16];
  unsigned int i,j,w,W,index;
  int l,k;
  FIELD2N U;
  unsigned long mymask;
  unsigned int maskindex,uindex;
  if (EQU_ZERO(a))     {
    dblnull(c);
    return;
  }
  else if(EQU_ZERO(b)){
    dblnull(c);
    return;
  }
  if(EQU_NUMS(a,b))     {
    poly_square(a,c);
    return;
  }
  null(&U);
  w=4;W=32;
  dblnull(&B[0]);
  sngltodbl(b, &B[1]);
  for(i=2;i<16;i++)
  {
    if (i%2){
      for(j=0;i<DBLMAX;j++)
        B[i].e[j] =B[i-1].e[j]^B[1].e[j];
    }
    else{
      if (i==2) copy_dbl(&B[1],&B[2]);
      else copy_dbl(&B[i/2],&B[i]);
      mul_shift(&B[i]);
    }
  }
  dblnull(c);
  for(k=7;k>=0;k--)
  {
    for(j=0;j<F2NMAXLONG;j++)
    {
      uindex=0;
      for(index=0;index<4;index++)
      {
        maskindex = 4*k+index;
        mymask = (0x00000001)<<maskindex;
        uindex=uindex|
          (((a->e[NUMWORD-j]&mymask)>>maskindex)<<index);
      }
      for(l=DBLWORD-j;l>=0;l--)
        c->e[l] ^= B[uindex].e[l+j];
    }
    if(k!=0)
    for(index=0;index<4;index++)
      mul_shift(c);
  }
}
```

```
void poly_mul_Right_to_Left
      (FIELD2N *a,FIELD2N *b,DBLFIELD *c)
{
  ELEMENT mask;
  DBLFIELD B;
  unsigned int W = 32;
  unsigned int t,k,j;
  int l;
  if (NUMBITS%W) t = NUMBITS/W + 1;
  else t = NUMBITS/W;
  dblnull(c); // clear all bits in result
  sngltodbl(b, &B);
  mask = 1;
  for(k=0;k<W;k++)
  {
    for(j=0;j<t;j++)
      if (a->e[NUMWORD-j]&mask)
      {
        for(l=DBLWORD-j;l>=0;l--)
          c->e[l] ^= B.e[l+j];
      }
      if(k!=W-1)
        mul_shift(&B);
      mask<<=1;
  }
}

void poly_inversion_Extended_Eucledean_Alg
      (FIELD2N *a,FIELD2N *inverse)
{
  FIELD2N u,v,g1,g2,one,temp,g2shift;
  int j,i;
  copy(a,&u);
  copy(&poly_prime,&v);
  null(&g1);
  null(&g2);
  g1.e[NUMWORD]=1;
  null(&one);
  one.e[NUMWORD]=1;
  while(!EQU_NUMS(&u,&one))
  {
    j = degreeof(&u,NUMWORD)-degreeof(&v,NUMWORD);
    if (j<0){
      copy(&v,&temp);copy(&u,&v);
      copy(&temp,&u);copy(&g2,&temp);
      copy(&g1,&g2);copy(&temp,&g1);
      j=-j;
    }
    copy(&v,&temp);copy(&g2,&g2shift);
    for (i=0;i<j;i++)
      shift_left(&temp);
    i=0;
    for (i=0;i<F2NMAXLONG;i++)
      u.e[i]^= temp.e[i];
    for (i=0;i<j;i++)
      shift_left(&g2shift);
    for(i=0; i<F2NMAXLONG; i++)
      g1.e[i]^=g2shift.e[i];
  }
  copy(&g1,inverse);
  }

void poly_mod_163bitfield(DBLFIELD *num,FIELD2N  *mod)
{
  ELEMENT T;
  unsigned int i;
  INDEX deg_top,deg_bot;
  deg_top = degreeof( num, DBLWORD);
  deg_bot = degreeof( &poly_prime, NUMWORD);
  if (deg_top < deg_bot){
    dbltosngl(num, mod);
    return;
    }
  for(i=0;i<5;i++)
  {
    T=num->e[i];
    num->e[i+6] = num->e[i+6]^(T<<29);
    num->e[i+5] = num->e[i+5]^(T<<4)^(T<<3)^T^(T>>3);
    num->e[i+4] = num->e[i+4]^(T>>28)^(T>>29);
  }
  T=num->e[5]>>3;
  num->e[10] = num->e[10]^(T<<7)^(T<<6)^(T<<3)^T;
  num->e[9] = num->e[9]^(T>>25)^(T>>26);
  num->e[5] = num->e[5]&0x07;
  mod->e[5] = num->e[10];
  mod->e[4] = num->e[9];
  mod->e[3] = num->e[8];
  mod->e[2] = num->e[7];
  mod->e[1] = num->e[6];
  mod->e[0] = num->e[5];
}
```

A.3 ELLIPTIC CURVE ARITHMETIC OVER BINARY FIELDS

```
void ec_add (ECPOINT *p1,ECPOINT *p2,ECPOINT *p3,CURVE *curv)
{
    INDEX    i;
    FIELD2N  x1, y1, theta, onex, theta2;
    // check if p1 or p2 is point at infinity
    if (EQU_ZERO(&p1->x)&&EQU_ZERO(&p1->y))
   {
     copy_point( p2, p3);
     return;
    }
    if (EQU_ZERO(&p2->x)&&EQU_ZERO(&p2->y))
   {
     copy_point( p1, p3);
     return;
   }
    if (EQU_AFF_POINTS(p1,p2))
    {
      ec_double (p1,p3,curv);
      return;
    }
    memset(&x1,0,4*F2NMAXLONG);
    memset(&y1,0,4*F2NMAXLONG);
    poly_add(p1->x.e,p2->x.e,x1.e,F2NMAXLONG);
    poly_add(p1->y.e,p2->y.e,y1.e,F2NMAXLONG);
    poly_inversion_Extended_Eucledean_Alg (&x1,&onex);
    poly_mul( &onex, &y1, &theta);
    poly_square(&theta,&theta2);
    xor_3long
      (&theta.e[0],&p3->x.e[0],&theta2.e[0],&x1.e[0],F2NMAXLONG);
    poly_add(p1->x.e,p3->x.e,x1.e,F2NMAXLONG);
    poly_mul( &x1, &theta, &theta2);
    xor_3long(&theta2.e[0],&p3->x.e[0],&p1->y.e[0],
      &p3->y.e[0],F2NMAXLONG);
}
```

```
void ec_double (ECPOINT *p1,ECPOINT *p3,CURVE *curv)
{
    FIELD2N  x1, y1, theta, theta2, t1;
    INDEX   i;
    if (EQU_ZERO(&p1->x)){
      null(&p3->x);
      null(&p3->y);
      return;
    }
    poly_inversion_Extended_Eucledean_Alg
      (&p1->x,&x1);
    poly_mul( &x1, &p1->y, &y1);
    poly_add(y1.e,p1->x.e,theta.e,F2NMAXLONG);
    poly_square(&theta,&theta2);
    if(curv->form)
      xor_3long(&theta.e[0],&theta2.e[0],&curv-a2.e[0],
        &p3->x.e[0],F2NMAXLONG);
    else
      poly_add(theta.e,theta2.e,p3->x.e,F2NMAXLONG);
    theta.e[NUMWORD] ^= 1;
    poly_mul(&theta,&p3->x,&t1);
    poly_square(&p1->x,&x1);
    poly_add(x1.e,t1.e,p3->y.e,F2NMAXLONG);
}
```

```
void ec_subtraction
  (ECPOINT *p1, ECPOINT *p2, ECPOINT *p3,CURVE curv)
{
    ECPOINT   negp;
    copy ( &p2->x, &negp.x);
    null (&negp.y);
    poly_add(p2->x.e,p2->y.e,negp.y.e,F2NMAXLONG);
    ec_add (p1, &negp, p3, curv);
}
```

```
void ec_mul_LefttoRightBinaryMethod
      (FIELD2N k,ECPOINT *p,ECPOINT *r,CURVE *curv)
{
  ECPOINT     rtemp1,rtemp2,ptemp1,ptemp2;
  ELEMENT   notzero=1,j,i,checknum,bitcount=0;
  FIELD2N      knum;
  copy_point(p,&ptemp1);
  copy_point(p,&ptemp2);
  nullpoint(&rtemp1);
  nullpoint(&rtemp2);
  copy(&k,&knum);
  i=0;
  checknum = 0x80000000;
  bitcount=degreeof(&knum,NUMWORD)+1;
  j=bitcount%32;
  if (j==0) i=NUMWORD-(bitcount/32)+1;
  else      i=NUMWORD-(bitcount/32);
  checknum = (0x00000001<<(j-1));
  while (bitcount)
  {
    ec_double(&rtemp1,&rtemp2,curv);
    copy_point(&rtemp2,&rtemp1);
    if (knum.e[i] & checknum){
      ec_add(&rtemp1,&ptemp1,&rtemp2,curv);
      copy_point(&rtemp2,&rtemp1);
    }
    shift_left(&knum.e[0],F2NMAXLONG);
    bitcount--;
  }
  copy_point(&rtemp1,r);
}

void ec_mul_FixedBaseWindowMethod
      (FIELD2N k,ECPOINT *p1,ECPOINT *p2,CURVE mycurve)
{
  ECPOINT p1temp,p2temp,A,B,p3temp;
  INDEX i,j,num,mask,d,t;
  FIELD2N knum;
  ELEMENT checknum;
  int k_base_w[NUMBITS/2+1];
  nullpoint(&A);nullpoint(&B);
  nullpoint(&p1temp);nullpoint(&p2temp);
  nullpoint(&p3temp);
  copy_point(p1,&p1temp);
  copy_point(p2,&p2temp);
  copy(&k,&knum);
  checknum = 0x80000000;
  i=0;
  t=degreeof(&knum,NUMWORD)+1;
  if ((t%WINDOWWIDTH)!=0)     d = (t/WINDOWWIDTH)+1;
  else                              d = t/WINDOWWIDTH;
  // form scalar k as k = (K(d-1)....K1,K0)
  mask = 0x0000;
  for (i=0;i<WINDOWWIDTH;i++)
  {
    mask <<= 1;
    mask |= 0x0001;
  }
  j = 1;
  while(j<(d+1))
  {
    k_base_w[j] = knum.e[NUMWORD] & mask ;
    for(i=0;i<WINDOWWIDTH;i++)
      rot_right(&knum);
      j++;
  }
  k_base_w[0]=j;//length of the array
  num=(unsigned short)(pow(2,WINDOWWIDTH)-1);
  for (j=num;j>0;j--)
  {
    for (i=1;i<=k_base_w[0];i++)
    {
      if (j==k_base_w[i])
      {
        //POINTARR contains the precomputed data
        ec_add(&B,&POINTARR[i-1],&p2temp,&mycurve);
        copy_point(&p2temp,&B);
      }
    }
    ec_add(&A,&B,&p3temp,&mycurve);
    copy_point(&p3temp,&A);
  }
  copy_point(&A,p2);
}
```

void **ec_mul_FixedBaseCombMethodwith2tables**

```
      (FIELD2N k,ECPOINT* p1,ECPOINT *p2,CURVE mycurve)
{
  ECPOINT ptemp,ptemp2,Q;
  USHORT i,j,d,t,e,index,index2,ee;
   FIELD2N knum;
  char k_base_d[WINDOWWIDTH][NUMBITS+1];
  copy(&k,&knum);
  t=degreeof(&knum,NUMWORD)+1;
  t=NUMBITS;
  if (t>WINDOWWIDTH){
    if ((t%WINDOWWIDTH)!=0)  d =(t/WINDOWWIDTH)+1;
    else                           d = (t/WINDOWWIDTH);
  }
  else
    d = t;
  if((d%2)!=0)    e = d/2+1;
   else           e=d/2;
  if ((d%e)!=0)   ee = e-1;
   else           ee=e;
  for(i=0;i<WINDOWWIDTH;i++)
    for(j=0;j<d;j++)
    {
      k_base_d[i][j]=(char)(knum.e[NUMWORD] & 0x0001);
      rot_right(&knum);
    }
  nullpoint(&Q);
  for(i=e;i>0;i--)
  {
    ec_double(&Q,&ptemp,&mycurve);
    copy_point(&ptemp,&Q);
    index = 0;
    index2 = 0;
    for(j=0;j<WINDOWWIDTH;j++)
    {
      index=index+k_base_d[j][i-1]*
       (unsigned short)pow(2,j);
       if ((i-1+ee)>(e-1))
         index2=index2 + k_base_d[j][i-1+ee]*
           (unsigned short)pow(2,j);
       else
         index2 = 0;
    }
    ec_add(&Q,&POINTARR[index],&ptemp,&mycurve);
    ec_add(&ptemp,&POINTARR2[index2],&ptemp2,&mycurve);
    copy_point(&ptemp2,&Q);
  }
  copy_point(&Q,p2);
}
```

APPENDIX B

ROUTINES OF FIELD ARITHMETIC AND ELLIPTIC CURVE ARITHMETIC OVER PRIME FIELDS

B.1 REPRESENTING PRIME FIELD ELEMENTS

```
/*** fieldp.h ***/
typedef unsigned short clint;
typedef unsigned long clintd;
typedef clint CLINT[CLINTMAXSHORT];
typedef clint CLINTD[1 + (CLINTMAXDIGIT << 1)];
typedef clint CLINTQ[1 + (CLINTMAXDIGIT << 2)];
typedef clint *CLINTPTR;
#define WINDOWWIDTH 4
#define NUMBITS  192
typedef struct {
            CLINT x;
            CLINT y;
} ECPOINT;
typedef struct {
            unsigned int type;
            CLINT a2;
            CLINT a6;
} CURVE;
```

B.2 PRIME FIELD ARITHMETIC

```
int madd_l (CLINT aa_l, CLINT bb_l, CLINT c_l, CLINT m_l)
{
  CLINT a_l, b_l;
  clint tmp_l[CLINTMAXSHORT + 1];
  int i;
  if (EQZ_L (m_l)){
      return DIV_BY_ZERO;
  }
  copy(a_l, aa_l);copy(b_l, bb_l);
  if (GE_L (a_l, m_l) || GE_L (b_l, m_l))
    {
      if (a_l[0]!=b_l[0])
      add (a_l, b_l, tmp_l);
      else if ((a_l[0]%2)!=0)
        add (a_l, b_l, tmp_l);
           else
```

```
             AddBigNum(tmp_l,a_l,b_l);
       mod_l (tmp_l, m_l, c_l);
    }
  else
    {
      if (a_l[0]!=b_l[0])
        add (a_l, b_l, tmp_l);
      else if ((a_l[0]%2)!=0)
        add (a_l, b_l, tmp_l);
      else
        AddBigNum(tmp_l,a_l,b_l);
      if (GE_L(tmp_l,m_l))
        sub_l(tmp_l, m_l, tmp_l);
            //Underflow prevented
    copy (c_l, tmp_l);
    }
  return OK;
}

int msub_l(CLINT aa_l,CLINT bb_l,CLINT c_l,CLINT m_l)
{
  CLINT a_l, b_l, tmp_l;
  if (EQZ_L(m_l)){
      return DIV_BY_ZERO;
  }
  copy(a_l,aa_l);
  copyl(b_l,bb_l);
  if (GE_L(a_l,b_l)){
      sub(a_l,b_l,tmp_l);
      mod_l(tmp_l,m_l,c_l);
    }
  else {
      sub(b_l,a_l,tmp_l);
      mod_l(tmp_l,m_l,tmp_l);
      if(GTZ_L (tmp_l))
        sub (m_l, tmp_l, c_l);
      else
        SETZERO_L (c_l);
   }
  return OK;
}

int mmul_l (CLINT aa_l,CLINT bb_l,CLINT c_l,CLINT m_l)
{
  CLINT a_l, b_l;
  CLINTD tmp_l;
  if (EQZ_L (m_l))
    return DIV_BY_ZERO;
  copy (a_l, aa_l);
  copy (b_l, bb_l);
  mult (a_l, b_l, tmp_l);
  mod_l (tmp_l, m_l, c_l);
  return OK;
}
void mod_l (CLINT dv_l, CLINT ds_l, CLINT r_l)
{
  CLINTD junk_l;
  div_l (dv_l, ds_l, junk_l, r_l);
}
```

B.3 ELLIPTIC CURVE ARITHMETIC

```
void ec_add(ECPOINT *p1,ECPOINT *p2,ECPOINT *p3,CURVE mycurve)
{
  CLINT x1,y1,x2,y2,x3,y3,theta,theta2,inv,g,temp;
  ECPOINT temppoint;
  copy(x1,p1->x);
  copy (y1,p1->y);
  copy (x2,p2->x);
  copy (y2,p2->y);
  if (((x1[0]==0)&&(y1[0]==0))&&((x2[0]==0)&&(y2[0]==0))){
    null(&p3->x);
    null(&p3->y);
    return;
  }
  else if ((x1[0]==0)&&(y1[0]==0)){
    copy (p3->x,p2->x);
    copy (p3->y,p2->y);
    return;
  }
  else if ((x2[0]==0)&&(y2[0]==0)){
    copy (p3->x,p1->x);
    copy (p3->y,p1->y);
    return;
  }
  else if(equ_l(x1,x2)){
    if (equal(y1,y2)){
      ec_double (p1,&temppoint,mycurve);
      copy_point(&temppoint,p3);
      return;
    }
  else{
    null(&p3->x);
    null(&p3->y);
    return;
    }
  }
  msub_l(y2,y1,y3,myprime);
  msub_l(x2,x1,x3,myprime);
  inv_l(x3,myprime,g,inv);
  mmul_l(inv,y3,theta,myprime);
  mmul_l(theta,theta,theta2,myprime);
  msub_l(theta2,x1,x3,myprime);
  msub_l(x3,x2,temp,myprime);
  copy(x3,temp);
  msub_l(x1,x3,temp,myprime);
  mmul_l(theta,temp,y3,myprime);
  msub_l(y3,y1,temp,myprime);
  copy (y3,temp);
  copy (p3->x,x3);
  copy (p3->y,y3);
}

void ec_sub(ECPOINT *p1,ECPOINT *p2,ECPOINT *p3,CURVE mycurve)
{
  ECPOINT   negp,p3temp;
  CLINT zero,temp;
  null(&zero);
  copy(negp.x,p2->x);
  msub_l(zero,p2->y,temp,myprime);
  copy(negp.y,temp);
  ec_add(p1,&negp,&p3temp,mycurve);
  copy_point(&p3temp,p3);
}

void ec_double(ECPOINT *p1,ECPOINT *p3,CURVE mycurve)
{
  CLINT x1,y1,x3,y3,theta,theta2,inv,temp,temp2,temp3,temp4;
  CLINT two,three;
  u2clint_l(two,2);
  u2clint_l(three,3);
  cpy_l(x1,p1->x);
  cpy_l(y1,p1->y);
  if ((x1[0]==0)&&(y1[0]==0)){
    null(&p3->x); null(&p3->y);
    return;
  }
  msqr_l(x1,temp,myprime);
  mmul_l(three,temp,temp2,myprime);
  madd_l(temp2,mycurve.a2,temp3,myprime);
  mmul_l(two,y1,temp4,myprime);
  inv_l(temp4,myprime,temp2,inv);
  mmul_l(temp3,inv,theta,myprime);
  msqr_l(theta,theta2,myprime);
  mmul_l(two,x1,temp2,myprime);
  msub_l(theta2,temp2,x3,myprime);
  msub_l(x1,x3,temp,myprime);
  mmul_l(theta,temp,temp2,myprime);
  msub_l(temp2,y1,y3,myprime);
  copy(p3->x,x3);
  copy(p3->y,y3);
}

void ec_mul_SlidingWindowMethod
      (CLINT k,ECPOINT *p,ECPOINT *r,CURVE mycurve)
{
  char            blncd[NUMBITS+1];
  short int u,bit_count;
  USHORT    j,t,num1,num2,index;
  CLINT     number;
  ECPOINT   temp;
  u=0;
  t=1;
  copy(number,k);
  createNAF(number,blncd,2,&bit_count);
  createPOINT(r);
  bit_count--;
  while (bit_count >= 0)
  {
    if (blncd[bit_count]==0){
      t=1;u=0;
    }
    else {
      if (WINDOWWIDTH>bit_count)
        num2=bit_count+1;
      else
        num2=WINDOWWIDTH;
      for(t=num2;t>=0;t--)
        if (blncd[bit_count-t+1]&1)
          break;
      for (j=bit_count-t+1;j<=bit_count;j++)
        u =u+(short)pow(2,j-(bit_count-t+1))*blncd[j];
      }
      for (index = 0 ; index<t ; index++)
      {
        ec_double(r,&temp,mycurve);
        copy_point(&temp,r);
      }
      if (u>0)
      {
        num1 = u/2;
        ec_add(&temp,&POINTARR[num1],r,mycurve);
      }
      else if (u<0)
      {
        num1 = (-u)/2;
        ec_sub(&temp,&POINTARR[num1],r,mycurve);
      }
      else
        copy_point(&temp,r);
      bit_count = bit_count-t;
  }
}

void ec_mul_FixedBaseNAF
      (CLINT k,ECPOINT *p1,ECPOINT *p2,CURVE mycurve)
{
  char blncd[NUMBITS+1],k_base_w[NUMBITS+1];
  char negj;
  USHORT                    i,j,msbloc,andnum,max,t,precompcount,
bitcount,d,index,I;
  ECPOINT A,B,ptemp1,ptemp2;
  CLINT knum;
  copy(knum,k);
  msbloc = 0;
  andnum = 0x8000;
  for(i=0;i<(NUMBITS+1);i++)
  k_base_w[i] = 0;
  max = knum[0];
  createNAF(knum,blncd,2,&bitcount);
  if (bitcount%WINDOWWIDTH)
    d = bitcount/WINDOWWIDTH + 1;
  else
    d = bitcount/WINDOWWIDTH;
  if (knum[0]==0 )
    return;
  index = 0;
  j=0;
  while(index<(d+1))
  {
    for (i=0;i<WINDOWWIDTH;i++)
    {
       if (j<bitcount)
         k_base_w[index]    =    k_base_w[index]+(blncd[j++]    *
pow(2,i));
    }
  index++;
  }
  if (WINDOWWIDTH % 2)
    I = (pow(2,WINDOWWIDTH+1)-1)/3;
  else
    I = (pow(2,WINDOWWIDTH+1)-2)/3;
  createPOINT(&A);
  createPOINT(&B);
  for (j=I;j>0;j--)
  {
    negj = (~j+1)&0x00FF ;
    for (i=0;i<index;i++)
    {
      if (k_base_w[i] == j)
      {
        ec_add(&B,&POINTARR[i],&ptemp1,mycurve);
        copy_point(&ptemp1,&B);
      }
      else if (k_base_w[i] == negj)
      {
        ec_sub(&B,&POINTARR[i],&ptemp1,mycurve);
        copy_point(&ptemp1,&B);
      }
    }
    ec_add(&A,&B,&ptemp2,mycurve);
    copy_point(&ptemp2,&A);
  }
  copy_point(&A,p2);
}
```

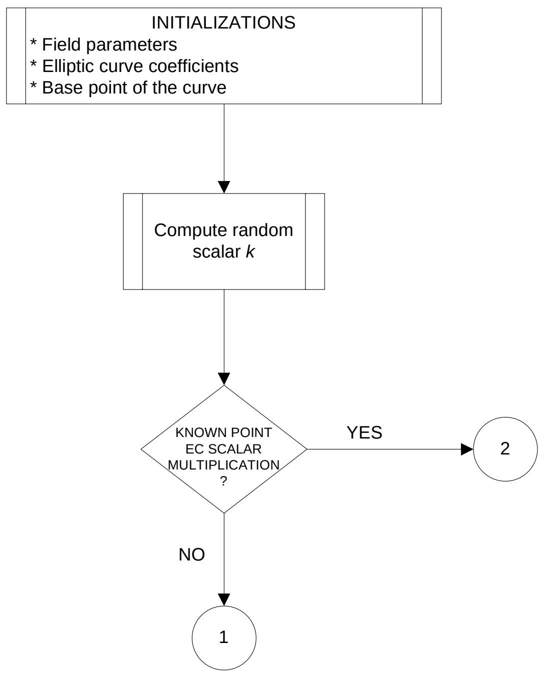
void **ec_mul_Fixedbase_NAF_Precomputation**

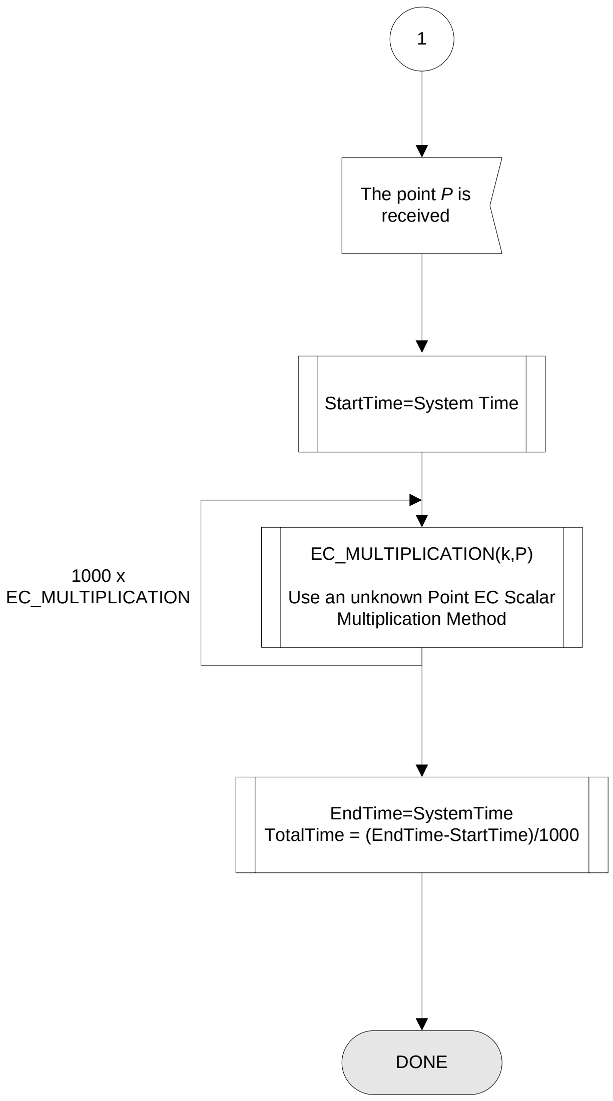
```
        (CLINT k,ECPOINT *p,CURVE mycurve)
  {
    CLINT knum;
    USHORT precompcount,t,i,j,msbloc,andnum,max;
    ECPOINT ptemp1,ptemp2;
    msbloc = 0;
    andnum = 0x8000;
    copy(knum,k);
    max = knum[0];
    for (msbloc=0;msbloc<16;msbloc++)
    {
      if(knum[max] & andnum )
        break;
      andnum = andnum >> 1;
    }
    t = knum[0] * 16 - msbloc ;
    if ((t+1)%WINDOWWIDTH)
      precompcount = (t+1)/WINDOWWIDTH + 1;
    else
      precompcount = (t+1)/WINDOWWIDTH;
    copy_point(p,&ptemp1);
    POINTARR[0] = ptemp1;
    for(i=1;i<precompcount;i++)
    {
      copy_point(&POINTARR[i-1],&ptemp1);
      for (j = 0 ; j<WINDOWWIDTH ; j++)
      {
        ec_double(&ptemp1,&ptemp2,mycurve);
        copy_point(&ptemp2,&ptemp1);
      }
      POINTARR[i] = ptemp1;
    }
  }

  void ec_mul_FixedBaseCombMethod_JacobianAffine
        (CLINT k,ECPOINT* p1,ECPOINT *p2,CURVE mycurve)
  {
    CLINT knum;
    ECPOINT ptemp;
    ECJPOINT Q,Qtemp;
    USHORT i,j,d,t,max,msbloc,andnum,index;
    char k_base_d[WINDOWWIDTH][NUMBITS+1];
    copy(knum,k);
    createPOINT(&ptemp);
    POINTARR[0] = ptemp;
    copy_point(p1,&ptemp);
    max = knum[0];
    andnum = 0x8000;
    for (msbloc=0;msbloc<16;msbloc++)
    {
      if(knum[max] & andnum )
        break;
      andnum = andnum >> 1;
    }
    t = knum[0] * 16 - msbloc ;
    if ((t%WINDOWWIDTH)!=0)    d =(t/WINDOWWIDTH)+1;
    else                             d = (t/WINDOWWIDTH);
    for(i=0;i<WINDOWWIDTH;i++)
      for(j=0;j<d;j++)
      {
        k_base_d[i][j] = knum[1] & 0x0001;
        shift_left(knum);
      }
    jacobian_infinity(&Q);
    for(i=d;i>0;i--)
    {
      point_double_jacobian(&Q,&Qtemp,mycurve);
      copy_point_jac(&Qtemp,&Q);
      index = 0;
      for(j=0;j<WINDOWWIDTH;j++)
        index = index + k_base_d[j][i-1]*(unsigned short)pow(2,j);
      ec_add_JacobianAffine(&Q,&POINTARR[index],&Qtemp,mycurve);
      copy_point_jacobian(&Qtemp,&Q);
    }
    jacobian_to_affine(&Q,p2);
  }
```

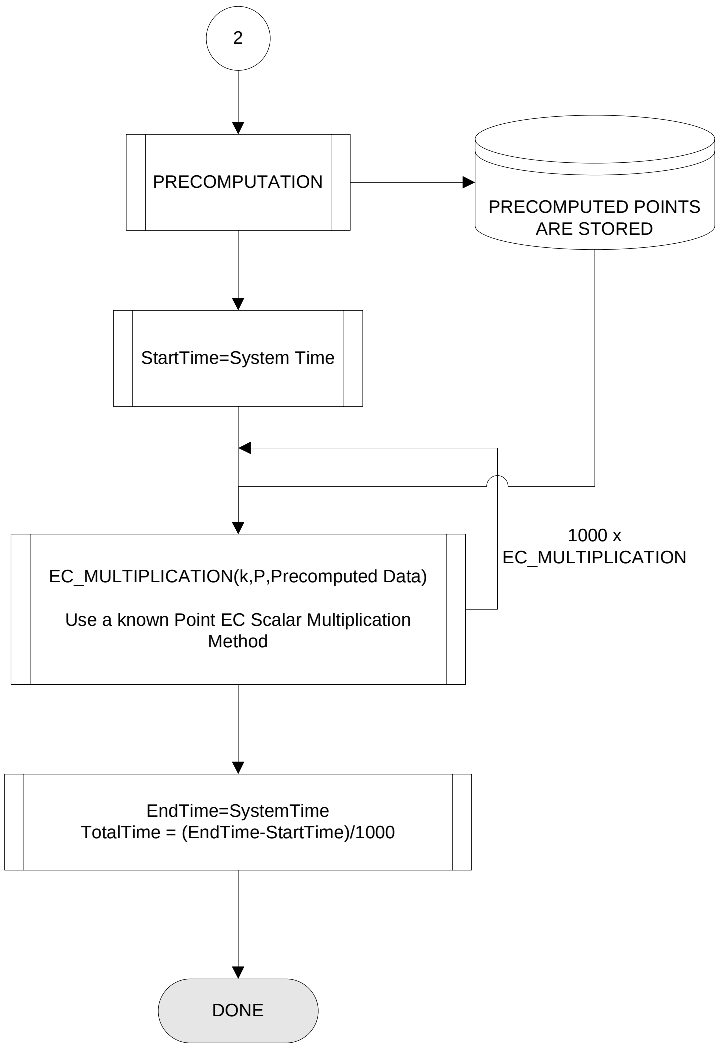
APPENDIX C

FLOWCHARTS

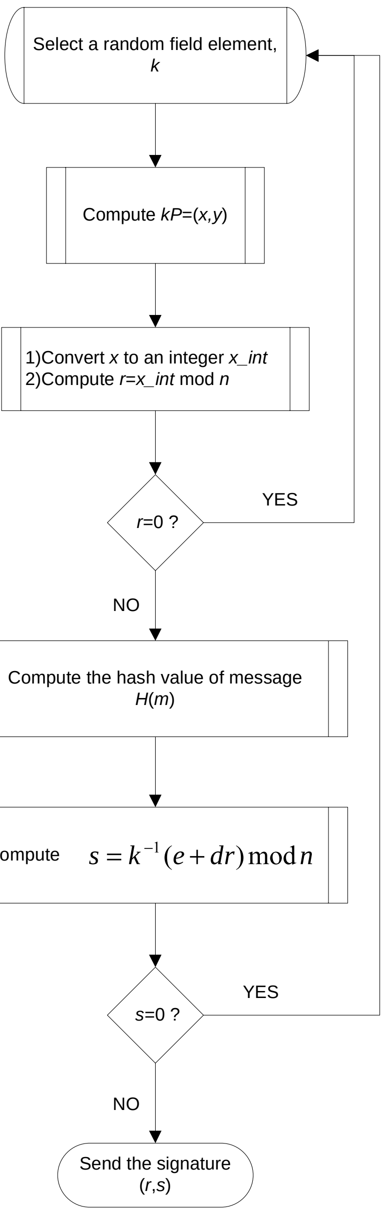
C. MEASURING THE TIMINGS OF ELLIPTIC CURVE POINT MULTIPLICATION METHODS


INITIALIZATIONS
* Field parameters
* Elliptic curve coefficients
* Base point of the curve
Compute random scalar *k*
KNOWN POINT EC SCALAR MULTIPLICATION ?
YES
2
NO
1

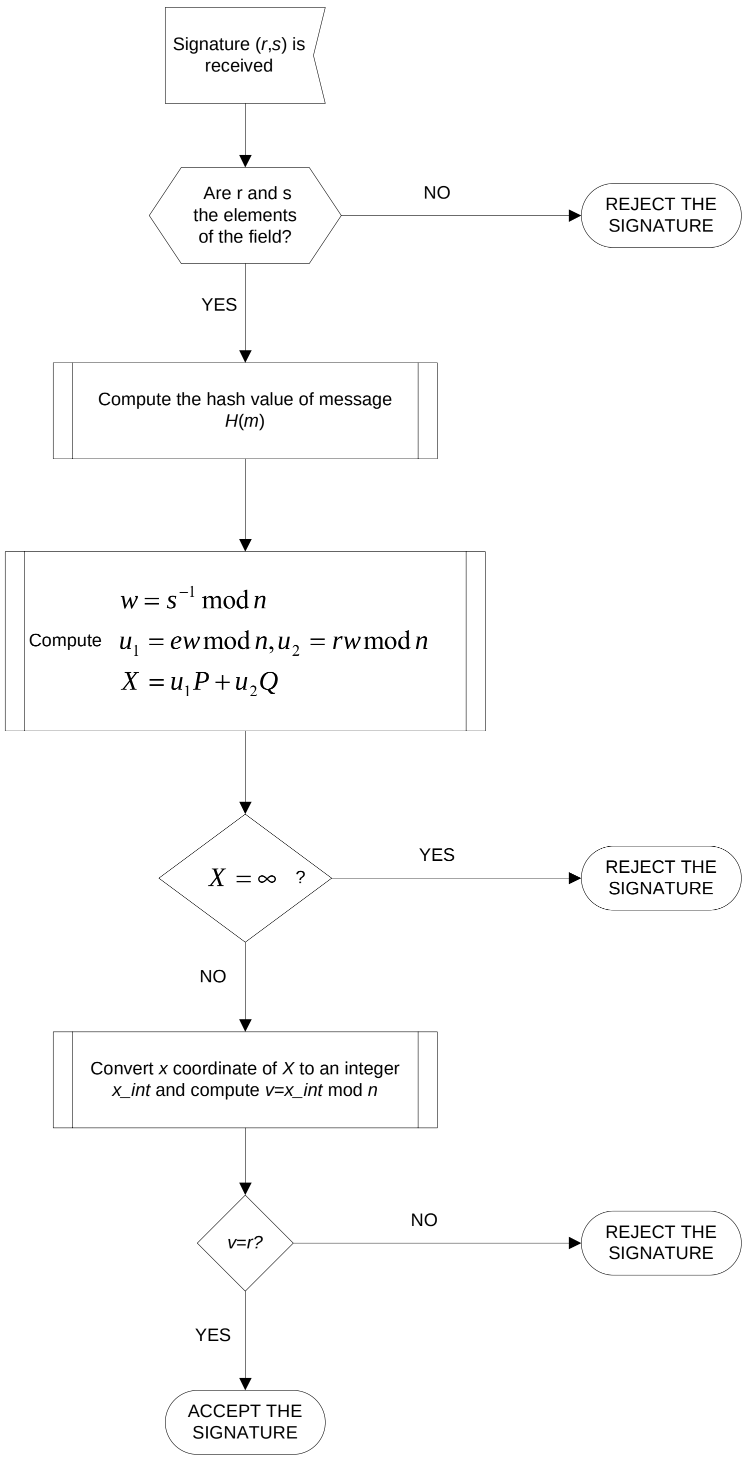

1
The point *P* is received
StartTime=System Time
1000 x EC_MULTIPLICATION
EC_MULTIPLICATION(k,P)
Use an unknown Point EC Scalar Multiplication Method
EndTime=SystemTime
TotalTime = (EndTime-StartTime)/1000
DONE


2
PRECOMPUTATION
PRECOMPUTED POINTS ARE STORED
StartTime=System Time
1000 x EC_MULTIPLICATION
EC_MULTIPLICATION(k,P,Precomputed Data)
Use a known Point EC Scalar Multiplication Method
EndTime=SystemTime
TotalTime = (EndTime-StartTime)/1000
DONE

C.2 ECDSA SIGNATURE GENERATION


Select a random field element, k
Compute $kP=(x,y)$
1)Convert x to an integer x_int
2)Compute $r=x\_int \bmod n$
$r=0$?
YES
NO
Compute the hash value of message $H(m)$
Compute $s = k^{-1}(e + dr) \bmod n$
$s=0$?
YES
NO
Send the signature (r,s)

C.3 ECDSA SIGNATURE VERIFICATION



Signature (*r*,*s*) is received
Are r and s the elements of the field?
NO
REJECT THE SIGNATURE
YES
Compute the hash value of message *H*(*m*)
Compute $w = s^{-1} \bmod n$
$u_1 = ew \bmod n, u_2 = rw \bmod n$
$X = u_1P + u_2Q$
$X = \infty$?
YES
REJECT THE SIGNATURE
NO
Convert *x* coordinate of *X* to an integer *x_int* and compute *v*=*x_int* mod *n*
v=*r*?
NO
REJECT THE SIGNATURE
YES
ACCEPT THE SIGNATURE