ADAPTIVE CONTROL OF ACTIVE VEHICLE SUSPENSIONS

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ABSTRACT

ADAPTIVE CONTROL OF ACTIVE VEHICLE SUSPENSIONS

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In this thesis, an adaptive control scheme for the quarter car model with an active suspension has been implemented in MATLAB® environment with user interface tools and dynamic simulation facilities.

Stochastic linear optimal control theory has been employed in the optimization of the active suspension for varying vehicle speeds and different road disturbance characteristics, and the adaptive control scheme has been applied to maintain optimal performance with varying vehicle speeds and road surface qualities.

The identification of the road surface quality has been performed by the frequency analysis of the road signal reconstructed from the signals obtained from two accelerometers attached to the vehicle sprung and unsprung masses, and by

comparing this signal with the six previously obtained road types represented by empirical formulas in the frequency domain.

MATLAB® software has been used to solve the signal processing and control engineering problems, and also to obtain and present the results in a graphical user interface environment. Also, dynamic simulation of the results has been presented in the Simulink environment of MATLAB®.

The performances of the passive and adaptive active suspensions have been compared in time and frequency domains for different road disturbance characteristics and vehicle speeds, and the advantages of the adaptive active suspension implementation has been illustrated.

Keywords: Active Suspensions, Stochastic Optimal Control, Adaptive Control.

ÖZ

AKTİF ARAÇ SÜSPANSİYONLARININ ADAPTİF KONTROLU

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Bu çalışmada, aktif süspansiyonlu bir araç modelinin adaptif kontrolu kullanıcı arayüzlü ve dinamik simulasyon olanağına sahip MATLAB® ortamında gerçekleştirilmiştir.

Değişen araç hızlarında ve yol yüzey karakteristiklerinde aktif süspansiyonun optimize edilmesi için stokastik doğrusal optimal kontrol teorisi kullanılmıştır. Elde edilen optimum performansın değişen araç hızlarına ve yol yüzeyine karşı korunması için de adaptif kontrol uygulanmıştır.

Yol yüzey kalitesinin belirlenmesinde araç gövdesi ve dingil üzerine yerleştirilmiş iki ivmeölçerden alınan sinyallerden elde edilen yol sinyali kullanılmıştır. Bu sinyalin frekans analizi yapılmış ve daha önceden elde edilerek

emprik formüllerle gösterilen altı yol çeşidiyle frekans ortamında karşılaştırılması ile yol yüzey kalitesi belirlenmiştir.

Sinyal işleme ve kontrol mühendisliği problemlerinin çözümünde ve ayrıca sonuçların elde edilmesinde ve kullanıcı arayüzlü bir ortamda sunulmasında MATLAB® yazılımı kullanılmıştır. Ayrıca, sonuçların dinamik simulasyonu MATLAB® yazılımının Simulink ortamında gerçekleştirilmiştir.

Pasif ve adaptif aktif süspansiyonların değişik yol yüzey karakteristikleri ve taşıt hızı değerlerindeki karşılaştırılmaları frekans ve zaman ortamlarında yapılmış ve adaptif aktif süspansiyon uygulamasının avantajları gösterilmiştir.

Anahtar kelimeler: Aktif Süspansiyon, Stokastik Optimal Kontrol, Adaptif Kontrol.

To My Wife and Parents

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CHAPTER I

INTRODUCTION

Every vehicle experiences vibration while moving on a random road surface and this vibration is harmful for the passengers, load, and the vehicle durability. The most feasible way to reduce the vibration to a permissible level is to design and produce more advanced suspension systems.

In conventional (passive) suspension systems, the basic wheel movement is controlled by the suspension springs which are referred to as the energy storage devices, and the spring movement is controlled by the dampers which are referred to as the energy dissipating elements. Such control over the moving spring is produced by the damper resistance when hydraulic fluid is forced through a system of valves and orifices during suspension travel. The energy produced by the spring movement is dissipated by the damper in the form of heat energy and is passed into the surrounding air. A schematic representation of a passive suspension system is shown in Figure 1.

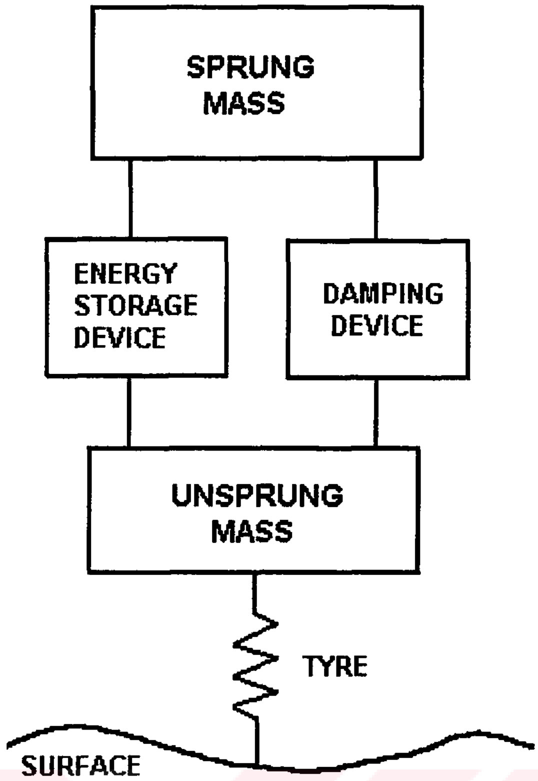

Figure 1. Passive Suspension System

During the motion of a vehicle, the sprung mass is propelled at widely varying speeds over diverse road surface types and through directional changes in complex combinations, including braking and acceleration. Thus suspension design involves consideration of a number of conflicting requirements in different operation sequences. To achieve good vibration isolation of the sprung mass over a wide range of frequencies, a soft suspension spring is needed, while to obtain good road holding capability near the natural frequency of the unsprung mass (wheel hop frequency), a stiff suspension spring is required. That is to say, good handling performance is achieved by incorporating stiff spring rates and small "bleed orifices" in the damping devices for high initial damping force buildup, at the cost of a bumpy and harsh ride. Comfortable ride performance is achieved by incorporating soft spring rates and large "bleed orifices" in the damping devices for low initial damping force, resulting in poor ride stability. To reduce the amplitude of vibration of the sprung mass near its natural frequency (body bounce frequency) a high damping ratio is needed, while to achieve good vibration isolation of the sprung mass in the high frequency region a low damping ratio is needed. On the other hand, to obtain satisfactory road holding capability, in the high frequency region, a high damping ratio is required. These conflicting requirements can never be fulfilled by passive suspensions since the characteristics of the spring and damper of the passive suspensions are fixed and cannot be modulated in response to the varying operating conditions of the vehicle [1]. Consequently, optimum ride comfort and performance for all road conditions, vehicle speeds, and cornering situations cannot adequately be provided by passive suspension systems.

In recent years, with the increasing importance given to customer satisfaction, the demands for comfort on rough roads and for security on smooth roads where the vehicle velocities are high have become so intense that the conventional passive suspension systems cannot meet these demands. With the intense development of cheaper microcomputing technologies, more advanced suspension systems, which can improve the overall performance of the vehicle, have become feasible.

In the light of these developments, the concept of "*active suspension*" to provide the vehicle with improved ride quality, handling, and performance under conflicting operating conditions, has emerged and various active suspension systems have been proposed or developed, [2, 3-13]. Active suspension systems measure the system states and produce a feedback control force between the vehicle body and the wheel according to these measurements. So, active suspensions consist of one or more active elements having their own power sources, in addition to the passive elements. Unlike the passive elements, active elements can apply forces which are proportional to the signals that are produced independently or measured at an arbitrary point of the vehicle [14]. The concept of an active suspension system is illustrated in Figure 2.

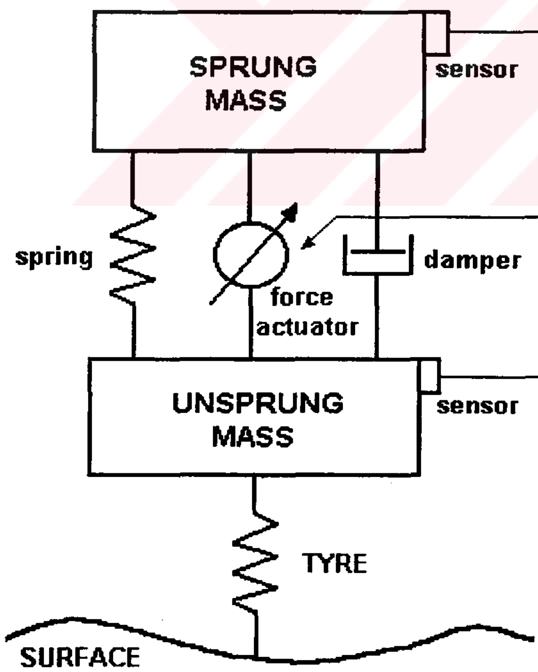

Figure 2. Active suspension system

As shown in Figure 2, the vibrations of the sprung and unsprung masses of the vehicle are continuously monitored by the sensors. Based on the signals obtained by the sensors and the prescribed control strategy, the force in the actuator is modulated to achieve improved ride, handling and performance. Furthermore, the results presented in reference [15] indicate that the addition of the active suspension increases the effectiveness of the newly developed control systems such as antilock brake system (ABS), automatic stability control (ASC), automatic stability control plus traction (ASC+T), and electronic damping control.

Keeping in mind that the active suspension gives the optimum performance for the vehicle with constant parameters and road input, it should be noted that the moving vehicle experiences changes in combinations of parameters and road inputs. In the classical linear optimal control procedure applied to the active suspensions, the model parameters are taken as constant and it is assumed that the road input, which is taken as white noise most of the time, is known completely, and consequently, an optimal state-feedback control gain matrix with constant elements is obtained. However, the dynamic behavior of the moving vehicle changes drastically with the following conditions.

- Varying vehicle parameters in response to accelerating (load shift from front to rear), braking (load shift from rear to front), cornering (load shift from one side to another), loading (variation of payload or number of passengers) or aging of the suspension elements in time,
- varying vehicle velocity, and

- varying road surface quality

Thus, the active suspension with fixed controller and parameters is insufficient in a global suspension optimization application. Also, it should be mentioned that for the active element to exert control force to control the unsprung mass motion, these forces should also be reacted against the sprung mass, and this reduces the ride comfort. This imposes a fundamental limit to what an active suspension can achieve in terms of satisfying the ride comfort and road holding simultaneously [1]. The conventional fixed parameter optimal control solutions are insensitive to parameter and road disturbance changes, thus an optimal controller for one vehicle configuration may be far from optimal if one or more of the vehicle parameters or road input changes.

Thus, for the active suspension to give the optimum performance in every vehicle configuration coping with the varying parameters, road spectra, and vehicle velocity it should adapt itself to the significant changes stated above. This problem is very suitable for the application of adaptive controllers, in which the controller parameters are changed, the controller is redesigned, and the system is adapted to variations mentioned above. An adaptive controller diagnoses the road disturbance variations, vehicle parameter variations, and structural variations, re-determines the controller parameters and/or configuration via on-line adaptive algorithms. So, adaptive control of active vehicle suspensions can be employed to compensate for the effects caused by the variations in vehicle parameters, vehicle velocity, and road surface spectra. With the rapidly developing microcomputer technology, the physical implementation of the adaptive active vehicle suspensions is getting more feasible day by day. Up to now, various adaptive control schemes for active suspensions have been suggested [16-26]. The basic adaptive control schemes are Gain Scheduling Adaptive Control, Self-Tuning Regulator, and Model Reference Adaptive Control which are explained in Chapter II in detail.

In this study, an adaptive control of a quarter car vehicle model with active suspension oriented towards the compensation of the changes in road excitation characteristics and the vehicle velocity, rather than the compensation of the changes in vehicle model parameters, is implemented. Adaptive active control is applied to control the stationary response of a 2-DOF-vehicle model traversing with a constant velocity.

Perfect and complete measurements are assumed and the dynamics of the active element (actuator) is neglected for the sake of simplicity. It is assumed that power spectral density of the road irregularities is reconstructed from the accelerometer measurements at the vehicle body (sprung mass) and the axle (unsprung mass), and is modeled by a rational function. The road irregularity is presented as an output of a linear filter with Gaussian white noise input. Stochastic optimal linear control theory is used in the optimization process and a full-state feedback gain with constant elements is obtained. The quadratic performance index is chosen to be the ensemble average of a weighted sum of the mean square values of the sprung mass acceleration, suspension travel, road holding, and the control force.

The type of the road surface quality is identified by the comparison of the measured irregularity with the previously defined road types in frequency domain. After the identification of the road type is achieved, two different adaptation mechanisms can be applied.

In the first one named as "On-line adaptation", the weighing constants are varied according to the degree of ride comfort or road holding requirements. Following the determination of the weighing constants, the appropriate state-feedback control gain matrix is found by the real-time optimization process and optimal performance is achieved.

In the second one named as "Stored-data adaptation", the appropriate state feedback control gain matrix is directly picked up from a look-up table according to the degree of ride comfort or road holding requirements.

The performance of the adaptive active suspension is compared with that of the passive suspension and the performance improvement results have been obtained in MATLAB® environment with user interface. Furthermore, dynamic simulation capability in MATLAB® environment has been provided so that the results could be analyzed dynamically.

In Chapter II, general information on suspension types has been given, and literature review has been done. In Chapter III, dynamic model of the vehicle has been constructed and state equations have been derived. Road surface modeling and road type identification have been explained and the optimal control rule for the adopted vehicle model has been derived. In Chapter IV, the theory has been implemented on MATLAB® and in Chapter V, case studies for different road types have been handled.

CHAPTER II

REVIEW OF LITERATURE

The automotive suspension systems are classified into the following 4 main categories [27].

1. Passive Suspension Systems are composed of an energy storage device such as a spring, and an energy dissipating device such as a damper, parameters of which are constant.

2. Self-leveling Suspension Systems are a variation of the passive suspensions in which the primary lift component (such as an air spring) maintains the vehicle height disregarding the variations in payload. Air suspensions, which are self-leveling, are utilized on many heavy trucks and on a few luxury passenger cars. As the system pressure changes with load, the spring stiffness changes correspondingly maintaining the natural frequency of the suspension constant.

3. Semi-Active Suspension Systems consist of spring and damping elements, the parameters of which can be changed by an external control as shown in Figure 3 below.

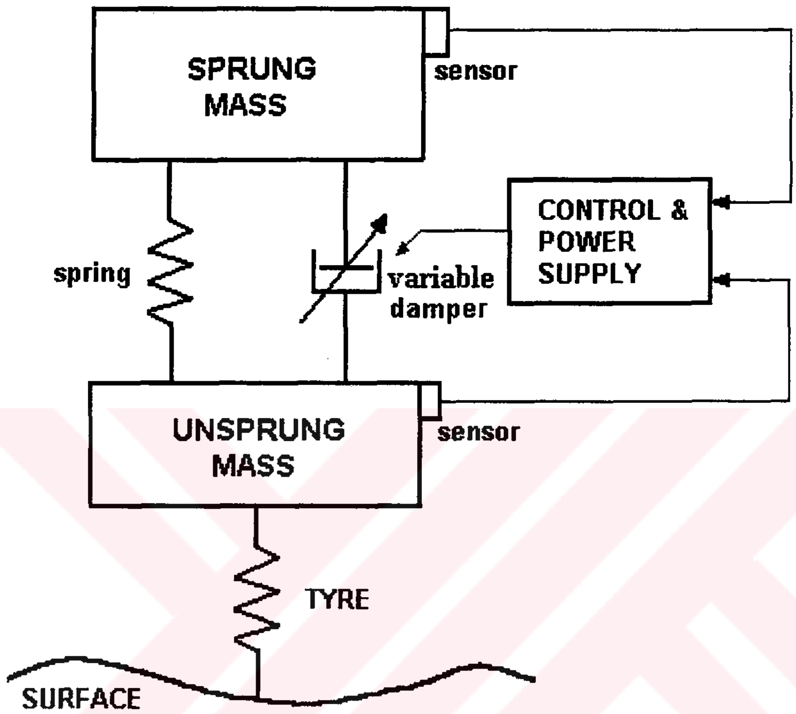

Figure 3. Semi-Active Suspension System

- Slow-active Suspension Systems are systems in which suspension damping and/or spring rate can be switched between several discrete levels according to the changes in driving conditions. Brake pressure, steering angle or suspension motions are typically used to change the control such that higher or lower levels of damping or stiffness are obtained.

- Low-bandwidth Suspension Systems are systems in which the spring rate and/or damping are varied continuously according to low-frequency sprung mass motions (1-3 Hz).

- High-bandwidth Suspension Systems are systems in which spring rate and/or damping are varied continuously according to both the low-frequency sprung mass motions (1-3 Hz) and the high frequency unsprung mass motions (10-15 Hz).

4. Full Active Suspension Systems consist of one or more actuators either added to the passive elements in parallel or established to the vehicle alone, and these actuators produce the desired forces in the suspension. The actuators are generally hydraulic cylinders, and external power is needed to operate the system. Full-active suspension systems can also be classified as low-bandwidth and high-bandwidth by similar definitions like those made in the semi-active suspension part.

The modes of performance that can be improved by active control are as follows:

- Ride Control: Considerable improvements in ride comfort can be achieved by reducing the vibration level caused by the road disturbance. Ride isolation properties superior to that of passive suspensions can be achieved.

- Height Control: The changes in load or aerodynamic forces change the vehicle height significantly, and active suspension can be adjusted to keep the vehicle height constant, which offers several advantages in performance. By the active suspensions, the vehicle can always operate at the design ride height, and this provides maximum strokes when going over the bumps, eliminates the road holding variations that occur because of the deviation from the design height. The height can also be reduced to prevent aerodynamic lift at high velocities or increased to provide a higher ground clearance and suspension stroke on rough roads.

- Roll Control: Roll motion causes load shifts from one side of the vehicle to the other during cornering, which in turn causes instability. Roll motion can be entirely eliminated by increasing damping or exerting anti-roll forces in the suspension during cornering. Roll control can be achieved by sensing the vehicle speed, steer angle, steer rate and/or lateral acceleration.

- Dive Control: Dive (forward pitch) motion is caused by the load shift from the rear of the vehicle to the front during braking, and is one of the important instabilities. Increasing damping or exerting anti-pitch forces during braking can reduce dive motion, and dive control can be activated by the brake light signal, brake pressure and/or longitudinal acceleration.

- Squat Control: Squat (rearward pitch) motion is caused by the load shift from the front of the vehicle to the rear during acceleration, and is one of the instabilities. Squat motion can be reduced by increasing damping or exerting anti-pitch forces during acceleration, and squat control can be activated by the throttle position, gear selection and/or longitudinal acceleration.

- Road Holding: Active suspensions have the potential to improve road holding by reducing the dynamic variations in wheel loads that are caused by the road irregularities.

Table 1. [27] presents the performance level that can be obtained with each level of advance in the suspension design.

Table 1. Performance comparison of suspension types

	Performance Mode					
Suspension Type	Ride	Height	Roll	Dive	Squat	Road holding
Passive	Performance is a compromise between all modes					
Self-leveling	High	High	NA	NA	NA	NA
Semi-active	Medium	NA	Low	Low	Low	Medium
Full-active	High	High	High	High	High	High

The concept of automatic suspension is first presented by a complete design in a paper by Federspiel-Labrosse in 1954, [28]. A similar study was performed by Osbon and Allen in 1965, [29]. Karnopp, Bender, Paul and Fenoglio at Massachusets Institute of Technology considered feedback automatic control systems for vehicles, [30, 31, 32, and 33]. Thompson [34] considered the optimal active suspension to control the bounce, roll, and pitch motions of the vehicles.

The implementation of broad bandwidth active suspension began in the early 1980's with a collaboration between Peter Wright at Lotus and David Williams at CIT, which resulted in a concept, termed "modal control" [35]. In this concept, sprung mass motions are decomposed into separate modes of motion with second order dynamics governing each sprung mass mode. Stiffness and damping can then be assigned independently to each mode. The objective is to make the vehicle soft in bounce and rigid in pitch and roll, structuring the MIMO system so that tunable parameters make physical sense. The prototype vehicle proved enormously successful and provoked much of the early industry interest in active suspensions.

Karnopp presented the notion of sky-hook damping [36]. He suggests that a force opposing and proportional to an inertially measured velocity will provide good vibration isolation. If a given second order system is more than critically damped, transient response will be dominated by the lowest frequency real pole. So, if the damping level is high enough, suspension springs re-center the sprung mass slowly to its static equilibrium height and the transient response can be approximated by a first order lag.

Hac studied the active suspension control of 2-DOF vehicle travelling on a stochastic roadway in [10]. By the use of stochastic optimal linear control theory, the suspension system is optimized with respect to ride comfort, road holding and suspension travel. The average behavior of the system is obtained by solving the Lyapunov equation. The effects of the values of the passive suspension elements on the active system performance are studied.

Later in 1997, Williams, and Haddad developed a model, which allows comparison of different active suspension control algorithms [4]. They have shown that by using the Lotus modal decomposition combined with the inner force loop, the inertial damping described by Karnopp is achieved and near optimal ride is performed, also resulting in beneficial handling effects.

It should be noted, however that the fully active suspensions need significant external power to operate, and also they are complex, unreliable, expensive and heavy. To reduce the complexity and cost while improving ride comfort and road holding, the concept of semi-active suspension has been proposed by Karnopp et al. [13]. In this concept, the conventional suspension spring is usually retained, while the damping force in the shock absorber can be adjusted according to the varying operating conditions. The force demand signal is generated from a control law derived for a fully active control but differs in that it only follows the active law when power dissipation is involved, and switches to zero, which in practice means to the minimum damper setting, when power input is demanded. The regulation of the damping force can be achieved by adjusting the orifice area in the damping device, thus varying the resistance to fluid flow. More recently, the variation of the resistance to fluid flow is achieved by special fluid types, the viscosity of which changes with the electrical voltage applied to them [1].

In comparison with a fully active system, a semi-active suspension system needs much less power to operate, and is less complex. Also, in [37], it is stated that when properly designed, the performance of a semi-active system may approach that of a fully active suspension under certain conditions.

Crolla and Hady also analyzed the semi active suspension arrangement which is controlled by a continuously variable damper and they have predicted the ride performance of a full vehicle model fitted with such a system [9]. The objective is to extend the quarter car model to cover the full vehicle case and to derive control laws which take account of the correlation between the roadway inputs at each of the four wheels. They also demonstrated the advantage of using wheelbase preview information, which says that rear input is a delayed version of that of the front wheel. A comparison of semi active and fully active systems has been done.

Also, in [3], [5], and [7] semi-active suspensions are utilized throughout the analysis.

An interesting control scheme considered frequently in the literature for a vehicle suspension application is the preview control. In this scheme, the input from the road irregularities is assumed to be measured in front of the vehicle and this information is utilized by the controller to prepare the system for the oncoming input.

It was Bender [38] who first suggested the idea of applying preview control to vehicle suspensions traversing a road with a statistically described disturbance. He showed that the use of preview control in vehicle suspensions leads to enormous improvements in ride comfort and road holding properties.

Despite its great potential, the idea did not get so much interest at that time because of practical problems in implementing the scheme. With the development of microcomputer technology and with the improvements in the design of actuators and ultrasonic sensors, the sonar suspensions became implementable and preview control began to raise interest of the automotive industry.

Tomizuka [39] analyzed the preview control of vehicle suspension as considered by Bender from the viewpoint of discrete optimal control. The formulation and solution of a class of discrete optimal finite preview problems are given in a general form so that they apply to all problems in which preview is considered.

Huisman et al. [40] presented a continuous time control strategy for an active suspension with preview, based on optimal control theory. The active suspension is applied to a 2-DOF model of the rear side of the tractor of a tractor-semitrailer and the preview information is obtained from the measurements of the front wheel data (wheelbase preview information). A step function is used as the road input and considerable improvement is obtained in ride comfort and road holding with respect to that of a passive suspension. Also, the active suspension with preview is found to be superior to the active suspension without preview.

In the study of Hac [8], a linear optimal preview control problem of a vehicle suspension is formulated as a continuous time problem and solved in deterministic and stochastic environments taking the measurement errors into account as opposed to earlier studies. A 2-DOF vehicle model with active suspension is used in the simulation and the effects of preview information on ride comfort, road holding, suspension travel and power requirements are examined in time and frequency domains. The results show that road holding properties are improved best with preview control.

Narayanan and Senthil applied optimal preview control technique to 2-DOF active suspension system with variable velocity [6]. The nonstationary response of the accelerating vehicle is controlled by the use of stochastic optimal control theory and it is concluded that there is a critical length of preview beyond which there is no further significant improvement in performance quantities.

Here, it is worth to mention the work on the non-stationary response of vehicles. The response of a vehicle becomes non-stationary in the following situations

1. when the vehicle traverses a homogeneous rough road with variable velocity

2. when the vehicle traverses a nonhomogeneous road profile at constant or variable velocity

3. when the vehicle traversing a smooth road suddenly encounters a homogeneous rough road at constant velocity in which case the initial transient response is nonstationary which becomes stationary after a lapse of time.

The control of non-stationary response of vehicle models with active suspensions was first considered by Narayanan and Raju, [41, 2].

In vehicle suspension models, usually linear suspension elements are utilized for the sake of easiness. In practice, however, the suspension elements exhibit a nonlinear character. The problem of active control of the non-stationary response of vehicle models with nonlinear passive suspension elements moving with either constant or variable velocity was first considered in 1992 by Narayanan and Raju, [42]. They analyzed active control of nonstationary response of a single degree of freedom vehicle model with nonlinear passive suspension elements. They have demonstrated that by the application of the equivalent linearization technique of handling nonlinear random vibration problems, active control concepts can be extended directly to control the nonstationary response of vehicles with nonlinear passive elements such as quadratic damping and hysteretic stiffness. The performance of the active suspensions is much better than the performance of the corresponding passive suspensions.

Kim and Yoon investigated the effects of optimal-controlled and neuro-controlled suspensions using frequency shaped performance index with simultaneous control of sprung and unsprung masses for the improvement of the ride quality. The frequency shaped performance index allows one to emphasize the specific variables related to the vibrations of the specific bands of frequencies, especially frequency band of ride comfort. Artificial neural networks is used to construct the control law (to solve the complicated equations that give the control law) and to train the neuro controller to learn the relation of road input and the optimal control force. The performances of the active and neuro-controlled suspensions are compared with that of the passive suspension.

As it has been stated in Chapter I., most of the studies on the benefits of active suspensions have concentrated on performance improvements obtainable for a particular set of road input, vehicle velocity, and vehicle parameter conditions. In practice, however, vehicles travel over a wide range of road inputs, at a wide range of velocities, and experience various changes in their dynamic behaviors. Therefore, the active suspension should adapt or tune itself in response to the varying conditions.

At this point, it is necessary to give some information on the adaptive control schemes that can be applied to the active suspension systems.

The term "adaptive control" is commonly used by control engineers to indicate that the control process is adapted to the source of the disturbance or perturbation.

Good adaptive control requires correct identification and if the system identification is recursive –that is the plant model is periodically updated on the basis of previous estimates and new data- identification and control may be performed concurrently.

Adaptive control is a technique of applying some system identification technique to obtain a model of the process and its environment from input-output experiments and using this model to design a controller. The parameters of the controller are adjusted during the operation of the plant as the amount of data available for plant identification increases. For a number of simple PID (proportional + integral + derivative) controllers in process control, this is often done manually. However, when the number of parameters is large, and they vary with time, automatic adjustment is needed. The design techniques for adaptive systems are studied and analyzed in theory for *unknown* but *fixed* (i.e. time invariant) plants. In practice, they are applied to *slowly time-varying* and *unknown* plants.

Thus, an adaptive controller is a controller that can modify itself in response to changes in the character of the disturbances and plant dynamics as well. It can also be thought of as a feedback control system in which the feedback-gain can be continuously adjusted according to the changes in the plant parameters and external disturbances. So, an adaptive controller is a controller with adjustable parameters and a mechanism for adjusting the parameters. The adaptive controller is nonlinear because of the adjusting mechanism of the parameters. Two loops can be seen in adaptive controllers. One of these two loops is a normal feedback control system with the plant, and the other is the parameter-adjusting loop. The above information can be summarized in Figure 4 below.

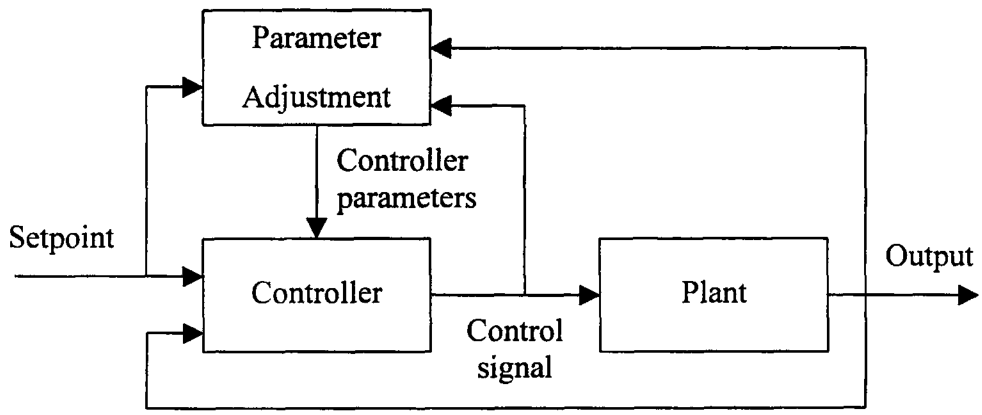

Figure 4. Block diagram of an adaptive system

There are four main classes of adaptive control schemes [43, 44].

Gain Scheduling:

In some adaptive control systems, some measurable variables that are correlated with the changes in plant dynamics or character of disturbances can be found, and thus, these variables can be used for the purpose of adjusting the controller parameters. So, the changes in plant dynamics are recognized and the controller is changed, that is, scheduled to compensate for these changes and thus, this approach is called *gain scheduling.*

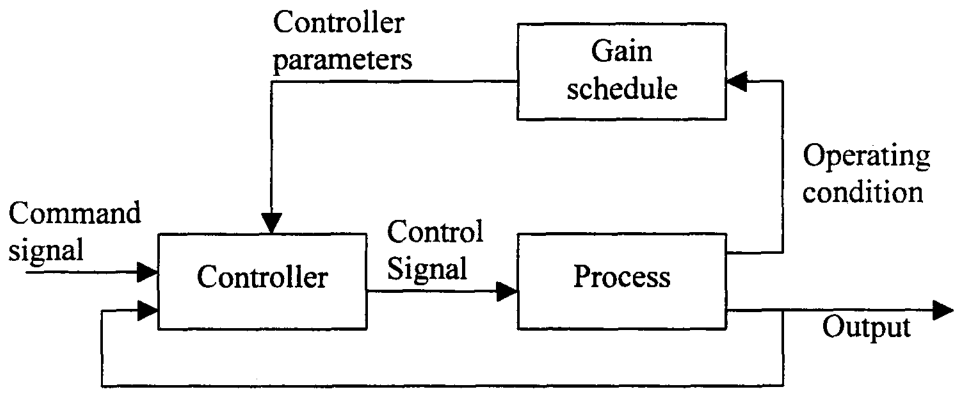

Figure 5. Block diagram of a system with gain scheduling

As shown in Figure 5 above, the adaptive system with gain scheduling has two loops. The inner loop consists of the plant and controller blocks and the outer loop consists of the parameter adjusting mechanism that vary the controller parameters in response to the changing conditions making use of a function or a look-up table for storing and recalling controller parameters. So, gain scheduling can be considered as a mapping from process parameters to controller parameters.

Model Reference Adaptive Control:

The model reference adaptive control system is a system in which the actual output of the controlled process is compared with the reference model output and the parameters of the controller are modified in such a way that the response of the controlled process follows that of the reference model.

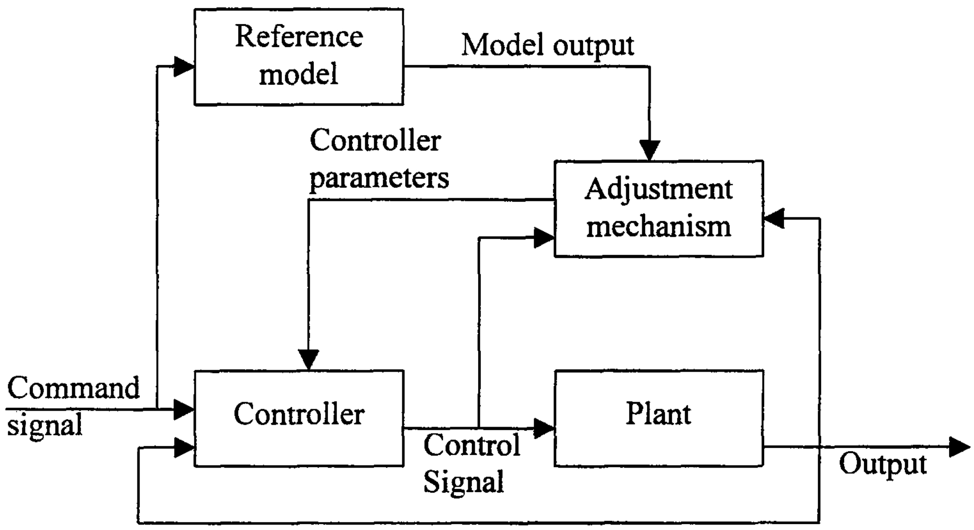

Figure 6. Block diagram of a Model Reference Adaptive System (MRAS)

As shown in Figure 6 above, the controller is composed of two loops. The inner loop is an ordinary feedback loop consisting of plant and controller. The outer loop consists of the adjusting mechanism, which adjusts the parameters of the controller such that the error, which is the difference between the plant output and the model output, is small. The ideal model reference adaptive system is the stable system that takes the error to zero.

Self-tuning Regulators (STR):

In the adaptive control schemes considered up to now, the adjustment rules determine directly how the controller parameters should be updated, so these methods are called *direct methods*. Self-tuning regulators are adaptive controllers in which the *estimates* of the plant parameters are updated and the new controller parameters are obtained from the solution of a special design problem using the *estimated* parameters.

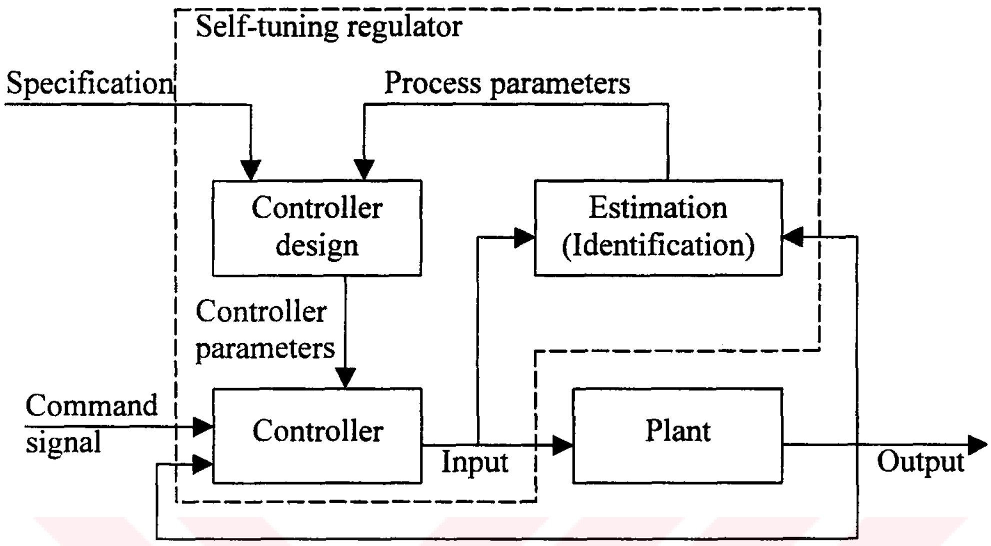

Figure 7. Block diagram of a self-tuning regulator (STR)

As shown in Figure 7 above, the inner loop is composed of the plant and the ordinary feedback controller. The outer loop where the parameters of the controller are adjusted consists of a *recursive parameter estimator* and a controller design process. Self-tuning regulator may be considered as an automation of process modeling and design, in which the process model and the control design are updated recursively at each sampling period.

Rapidly developing computer technology has made the implementation of the adaptive active suspensions an affordable reality, and various adaptive control schemes have been proposed.

In the study of Sachs [16], a hypothetical 2-DOF-vehicle model is assumed to pass over the terrain at a constant velocity. In this study, the objective is to find sets of spring rates and damping coefficients as a function of speed, such that the transmitted acceleration values corresponding to a stochastic road disturbance that is represented by Fourier series, is a minimum without exceeding the design axle clearance more often than 3 times in 1000 working cycles. The measured acceleration values are compared with the previously classified road types by pattern recognition method and the spring rate and damping coefficient values are set to the corresponding optimal values which were calculated before and stored in the on-board microprocessor. The acceleration transmissibilities between the system with fixed suspension parameters and the system with optimized suspension parameters are compared and it is found that the suggested adaptive suspension control leads to successive improvements in ride comfort.

Hac proposed an adaptive scheme for a 2-DOF-vehicle model with active suspension [17]. The objective of this study is to minimize the variance of vehicle body acceleration under inequality constraints imposed on the variance of either tire or suspension deflection. Reconstructed road disturbance, which is presented as the output of a linear filter with white noise input, is compared with the three standard road types that are already represented by empirical rational functions. The optimized control gains that minimize the vehicle body acceleration for the three road types are calculated by linear quadratic stochastic optimal control theory and this data is stored in the on-board computer. After the road type is determined, the corresponding control gain is applied and the vehicle body acceleration is minimized. The performance of the adaptive active suspension system is found to be superior to that of an active system with constant gain factors and to a passive system with adjustable parameters.

Lizell presented a solution to a low power-active suspension system and its control [18]. Improved performance is achieved by continuously sensing the dynamic behavior of the vehicle and actively compensating for the varying disturbances by an adaptive scheme that changes the vehicle damping gains when the road condition interferes with the natural frequency of the vehicle. The adaptation is done by varying the gain matrices according to the frequency content of the wheel motion and its amplitude. The hardware for the system consists of a closed hydraulic leveling and damping system and the required electronics. The closed loop frequency response functions of the vertical acceleration and suspension motion are compared for the passive and adaptive active systems and it is concluded that the ride comfort and the road holding of the adaptive active system are considerably improved.

In the study of Truscott and Wellstead [19], development of a digital adaptive control algorithm for active suspension systems based on optimal regulation methods is described. In this paper the objective is to design an algorithm which will automatically tune at start-up to changed vehicle conditions and adaptively re-tune to changes in both vehicle parameters and also the changes in road conditions. The adaptation of the vehicle is managed by a self-tuning regulator based on generalized minimum variance (GMV) control which is a robust type linear quadratic optimal regulation method. Simulation results demonstrate that the algorithm adapt fairly rapidly and smoothly to both the system and environmental changes which are typical of real driving conditions.

In the study of Sunwoo et al. [20], a model reference adaptive control (MRAC) technique for vehicle active suspensions is presented. The objective is to design a model reference adaptive controller which automatically self tunes the active suspension such that the disturbance and vibration of a vehicle is reduced to a level determined by an ideal conceptual suspension reference model which is the sky-hook damper model in this case. The adaptive controller is based on minimizing the errors between the plant and reference states by the Lypunov stability criterion, which is satisfied by changing appropriate gains adaptively selecting them from a look-up table. It is demonstrated that the adaptability and robustness of the MRAC system overcome the large changes in system parameters and thus, significant improvements are achieved by the active MRAC suspension system over the passive system in terms of a reduction in sprung mass accelerations.

Dukkipati et al. analyzed an adaptive control approach for an active suspension to deal with a nonlinear time varying (NTV) single input single output (SISO) vehicle plant [21]. The objective is to obtain near optimal performance and maintain static equilibrium during parameter variations of a vehicle plant, which involves an active suspension that is controlled by a robust stable adaptive controller with time varying parameters. Adaptive controller based on error augmentation approach of model reference adaptive control with a velocity error term is derived to obtain the optimal performance without a priory knowledge of variations in vehicle plant parameters and sky-hook damper model is chosen as a reference model since it gives an optimal suspension performance. Comparative study of the adaptive active suspension with that of a passive suspension demonstrates that the vehicle plant achieves the optimal performance and adapts to the reference model quickly and smoothly.

In the study of Ismailzadeh and Fahimi [22], an adaptive active suspension for a 7 DOF full car model in which controlled actuators replace dampers in a conventional suspension system is presented. The objective is to obtain a vehicle plant, which follows an optimal reference model irrespective of the changes in vehicle parameters and roadway conditions so that ride comfort is optimally improved. The adaptation to these changes is managed by a model reference adaptive control (MRAC) scheme in which a model that is optimized by using the linear quadratic regulation methods is used as a reference model. It is shown that after 4 seconds the state variables of the active system converges to that of the model and from then on the active system behaves exactly the same as the optimal model.

Karnopp and Margolis analyzed a class of basically passive suspensions the parameters of which can be varied actively in response to various measured signals on the vehicle [23]. The objective is to obtain a control scheme in which the damping ratio and the spring stiffness can be varied according to measured signals such as velocity, mean square acceleration (for the ride comfort), suspension travel (for road holding), brake pressure (for preventing diving that occurs because of load shift from rear to front), steering wheel angular velocity and acceleration (for maintaining roll stability). It is stated that neither adjusting the damper nor the spring stiffness alone will be effective in achieving an improved vibration isolation or road holding for varying roadway inputs, instead both of them should be changed. However, changing spring stiffness requires high power since it is impractical to recover stored energy from the spring. Finally, the means of obtaining variable spring stiffness constants by practical elements, such as air spring systems, are suggested.

In the study of Vallurupalli et al [24], a discrete adaptive control approach with time delays (normally encountered in practical implementation) for an active suspension is presented. A nonlinear time varying (NTV), single input single output (SISO) half car suspension model is analyzed throughout the analysis. The objective is designing a controller, which maintains the static equilibrium irrespective of the dynamic load variations as a disturbance force on the model. A discrete model reference adaptive control (DMRAC) approach in which a linear time invariant (LTI) sky-hook damper model is chosen as the reference model, is used with recursive least square (RLS) estimation technique for estimating the controller parameters. The adaptation is managed by on-line estimation of controller parameter vector, which is required for the particular ride condition. Simulation results show that the suggested adaptive controller performs quite well even in large dynamic variations of the model and that the adaptive controller has the potential for a successful hardware implementation.

Chen and Guenter proposed a self-tuning adaptive control approach for an active suspension, which is composed of an identifier and a controller [25]. The structure of the identifier uses parallel model reference adaptive system to identify the varying parameters of the vehicle. After the identification process is performed, the self tuning control rule is constructed according to the identified parameters and fed forward to the controller which is a linear quadratic regulator optimizing the system. It is demonstrated that the vibration isolation capability of the self-tuning active suspension is superior to that of the optimal active suspension with nominal vehicle parameters.

In the study of Yu and Crolla [26], adaptive control of an active vehicle suspension is shown to provide optimal ride comfort performance over a range of typical operating conditions. The objectives of this study are to examine the benefits available from adaptation, to propose different control strategies and to assess the performance obtainable with these strategies by simulation studies. Road inputs are modeled as Gaussian random processes and classified by using the power spectral density functions. A quarter car model with 2 DOF is taken as the vehicle plant. Linear quadratic regulation (LQG) technique is used to minimize the cost function, which is a weighted sum of mean square values of output variables. Both full state feedback and limited state feedback are considered. To overcome the conflicting requirements of the suspensions on adaptive strategy based on gain scheduling is devised. Thus, the most appropriate set of gains for the prevailing conditions from predetermined sets stored in memory are selected. Three gain scheduling strategies are considered where gains are selected based on wheel acceleration or based on suspension working space and tire displacement. In the simulation results part, these three strategies are compared with each other for various constraints of the vehicle suspension.

CHAPTER III

FORMULATION

3.1 DYNAMIC MODEL OF THE VEHICLE

A two-degree-of-freedom linear quarter car model of a vehicle with an active suspension considered here is given in Figure 8.

Here, m_1 represents the mass of the wheel together with the axle (unsprung mass) and m_2 represents the mass of the vehicle body (sprung mass); absolute vertical displacements of these masses are y_1 and y_2 respectively. k_1 denotes the spring stiffness of the passive suspension energy storage element, and c_1 denotes the damping ratio of the passive suspension energy dissipating element (damper). In this model, tire is modeled as a pure spring element and k_2 represents the tire stiffness. In addition to the suspension force supplied by the passive elements (spring and damper), an active element exerts a control force u which is a control variable. The vehicle travels with a constant horizontal velocity V. The model is excited by the road unevenness $y_0(t)$. Note that y_0, y_1 and y_2 are measured from the static equilibrium position.

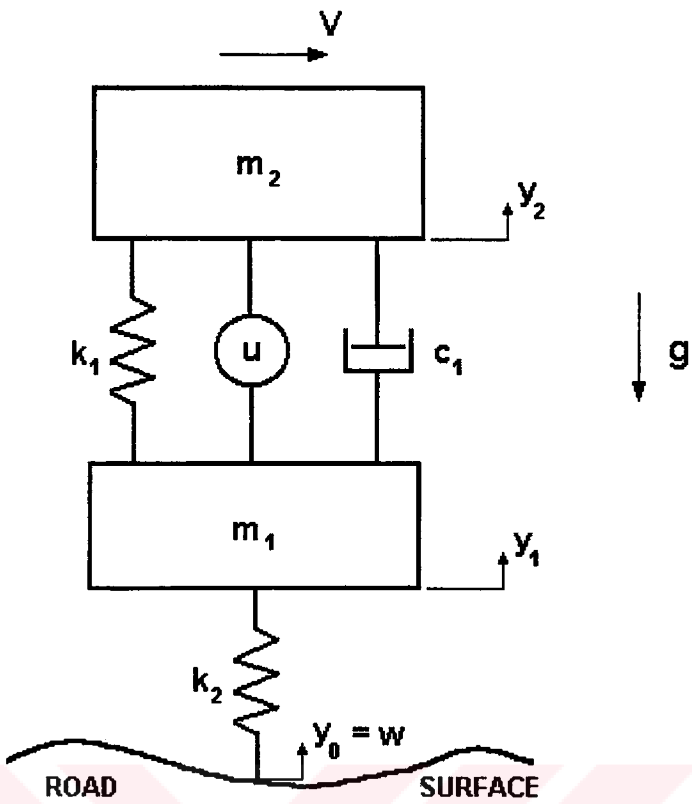

Figure 8. Model of the quarter car vehicle model

The reasons that the passive elements are introduced with the active element as supplementary are the following.

- The passive elements guarantee the continuous operation of the suspension system in case of an active element failure.

- They produce a force supplementary to that produced by the active element and this decreases the amount of force that should be produced by the active element alone. Both the power consumption decreases and the life of the active element increases.

The equations of motion of the system are:

$$m_2\ddot{y}_2 = k_1(y_1 - y_2) + c_1(\dot{y}_1 - \dot{y}_2) + u \tag{3.1}$$

$$m_1\ddot{y}_1 = k_2(y_0 - y_1) - k_1(y_1 - y_2) - c_1(\dot{y}_1 - \dot{y}_2) - u \tag{3.2}$$

The state variables are defined as:

$$x_1 = y_2$$

$$x_2 = \dot{y}_2$$

$$x_3 = y_1$$

$$x_4 = \dot{y}_1$$

$$w = y_0$$

Introducing the state variables into the equations of motion, the state equations are obtained as

$$m_2\dot{x}_2 = k_1(x_3 - x_1) + c_1(x_4 - x_2) + u \tag{3.3}$$

$$m_1\dot{x}_4 = k_2(w - x_3) - k_1(x_3 - x_1) - c_1(x_4 - x_2) - u \tag{3.4}$$

The state space form of the state equations are obtained as:

$$\dot{x}_1 = x_2 \tag{3.5}$$

$$\dot{x}_2 = \frac{-k_1}{m_2}x_1 - \frac{c_1}{m_2}x_2 + \frac{k_1}{m_2}x_3 + \frac{c_1}{m_2}x_4 + \frac{1}{m_2}u \tag{3.6}$$

$$\dot{x}_3 = x_4 \tag{3.7}$$

$$\dot{x}_4 = \frac{k_1}{m_1}x_1 + \frac{c_1}{m_1}x_2 - \frac{k_1 + k_2}{m_1}x_3 - \frac{c_1}{m_1}x_4 + \frac{k_2}{m_1}w - \frac{1}{m_1}u \tag{3.8}$$

If the state space equation is written in matrix form as:

$$\dot{\boldsymbol{x}} = \boldsymbol{F}_x\boldsymbol{x} + \boldsymbol{G}_x\boldsymbol{u} + \boldsymbol{D}_x\boldsymbol{w} \tag{3.9}$$

where $\boldsymbol{x} = [x_1 \quad x_2 \quad x_3 \quad x_4]^T$ is the state vector,

F_x. G_x and D_x are obtained as

$$F_x = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1}{m_2} & \frac{-c_1}{m_2} & \frac{k_1}{m_2} & \frac{c_1}{m_2} \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_1} & \frac{c_1}{m_1} & \frac{-(k_1 + k_2)}{m_1} & \frac{-c_1}{m_1} \end{bmatrix} \tag{3.10}$$

$$G_x = \begin{bmatrix} 0 \\ \frac{1}{m_2} \\ 0 \\ \frac{-1}{m_1} \end{bmatrix} \tag{3.11}$$

$$D_x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_2}{m_1} \end{bmatrix} \tag{3.12}$$

As stated before, in the road surface quality identification process, the road signal is reconstructed from the signals obtained from two accelerometers attached to the vehicle body and axle. Here, Sachs' concept [16] can be used, and the road input can be obtained by combining the equations (3.3) and (3.4) as below.

$$w_{rec} = x_3 + \frac{m_2}{k_2}\dot{x}_2 + \frac{m_1}{k_2}\dot{x}_4 \tag{3.13}$$

where w_{rec} is the reconstructed road surface irregularity vector.

Knowing the vehicle model parameters m_1, m_2, and k_2, and obtaining the acceleration data $\dot{x}_2$ and $\dot{x}_4$ measured on the vehicle body and axle during ride, the road input w is reconstructed by equation (3.13) on line. In practical applications, the result of equation (3.13) should be filtered to diminish the effect of the measure-born noises.

3.2 MODELING AND IDENTIFICATION OF THE ROAD SURFACE

For the purpose of describing the road surface irregularities in the frequency domain, either correlation function or power spectral density function can be used. In this study, power spectral density function will be used to describe the road surface irregularities. and to identify and compare the disturbance energy levels of different road types.

The object of the analysis presented in this section is to model the power spectral density of the road surface irregularities by a rational transfer function, to use this approach in the design of a linear shaping filter with a white noise input, to obtain the power spectral density functions for different road types, and to identify the type of road that the vehicle is traversing on, via comparing the power spectral density of the reconstructed road irregularities w_{rec} with the previously defined road types.

3.2.1 ROAD SURFACE MODELING

For a variety of road and terrain inputs, a good approximation of power spectral density function is given in [10] as

$$S(\omega) = \frac{\sigma^2}{\pi} \cdot \frac{\alpha \cdot V}{\omega^2 + (\alpha \cdot V)^2} \tag{3.14}$$

where ω denotes the angular frequency in rad/s, V the vehicle velocity in m/s, σ^2 the variance of the road irregularities in m^2 and α the spatial cut-off frequency (a coefficient depending on the type of road surface) in m^{-1}.

In this study, 6 different road types whose power spectral density functions are in the form of (3.14), have been considered. The 6 road types have been named as

1. Very Good Asphalt Road
2. Good Asphalt Road
3. Average Asphalt Road
4. Poor Asphalt Road
5. Paved Road
6. Dirt Road (terrain)

σ^2 and α values for the 6 road/terrain types discussed above are given in Table 2 below.

Table 2. Road Surface Type Parameters

Type of road/terrain	$\alpha\ (m^{-1})$	$\sigma^2\ (m^2)$
Very Good Asphalt Road	0.150	$9 \cdot 10^{-6}$
Good Asphalt Road	0.225	$44 \cdot 10^{-6}$
Average Asphalt Road	0.300	$105 \cdot 10^{-6}$
Poor Asphalt Road	0.375	$190 \cdot 10^{-6}$
Paved Road	0.450	$300 \cdot 10^{-6}$
Dirt Road	0.750	$750 \cdot 10^{-6}$

The power spectral density function (3.14) are plotted for the 6 road types at a vehicle velocity of 20 m/s (72 km/h) in Figure 9 below.

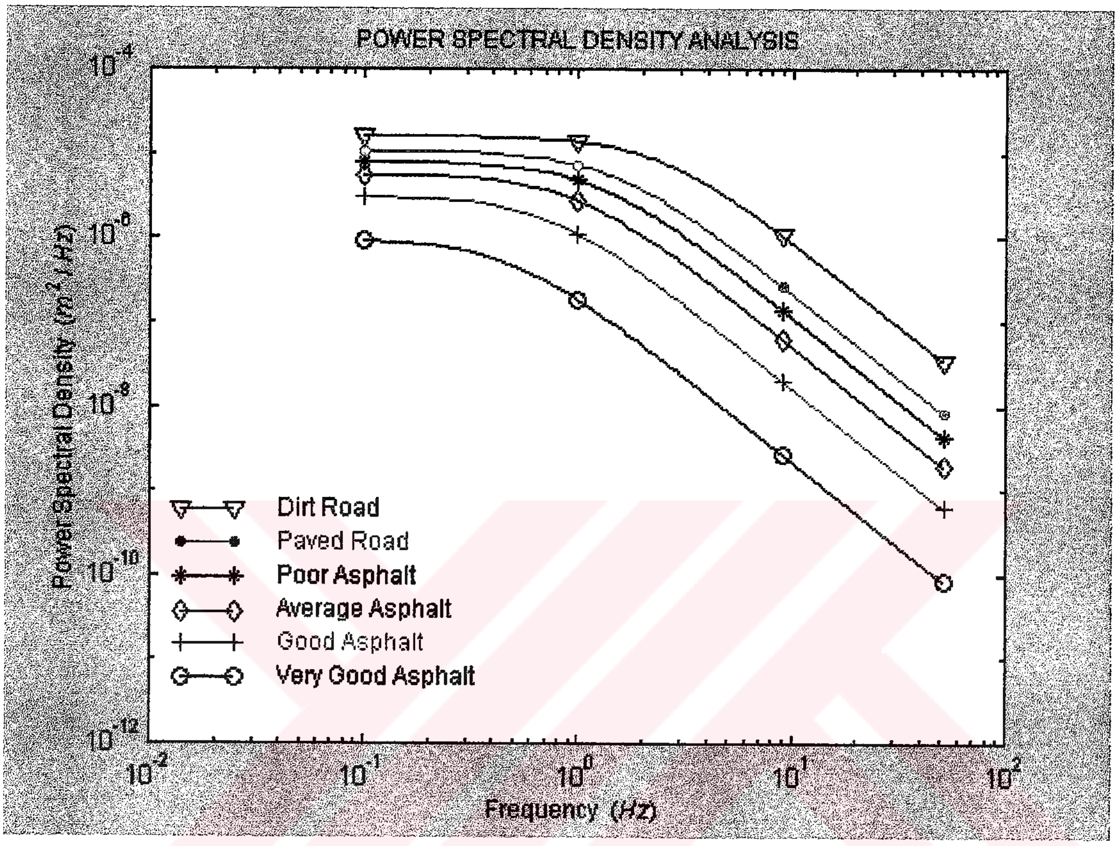

Figure 9. Power spectral density functions for the 6 road/terrain types at a vehicle velocity of 20 m/s

The power spectral density (3.14) of the road irregularities also changes with the vehicle velocity as shown in Figure 10 below.

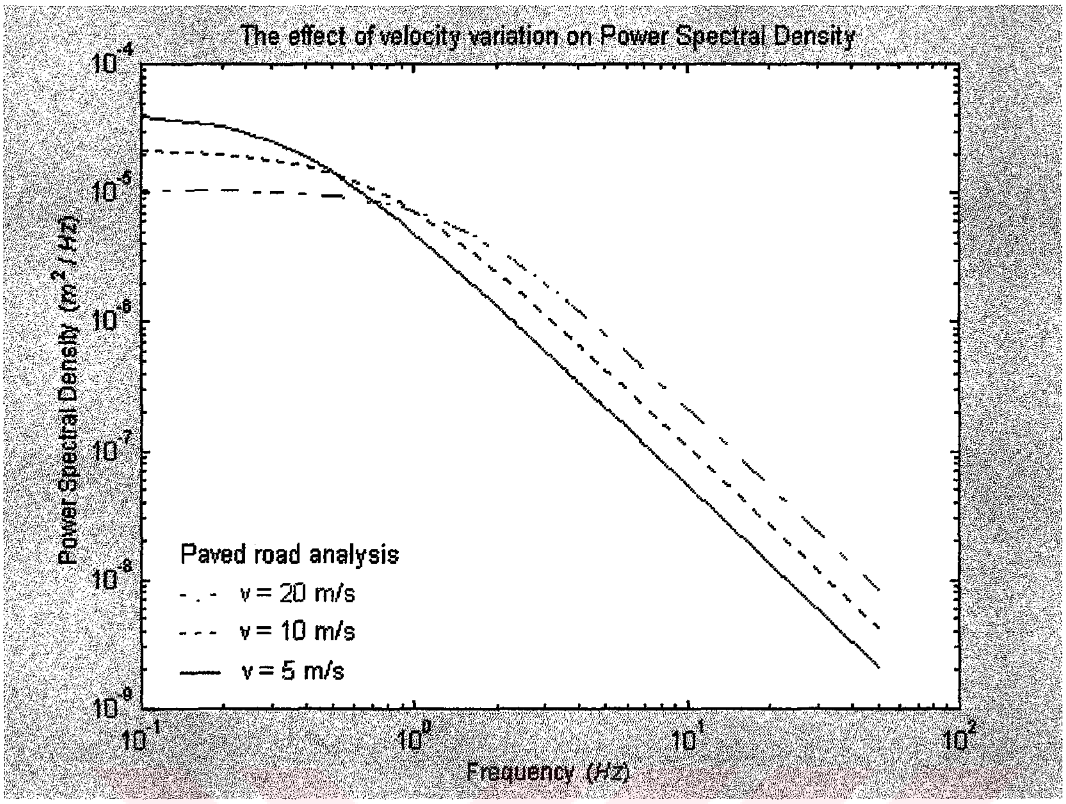

Figure 10. The effect of velocity variation on power spectral density

The stationary Gaussian road irregularity w with spectral density function (3.14) can be treated as a stationary solution of the linear filter of the first order

$$\dot{w}(t) = -\alpha \cdot V \cdot w(t) + \xi(t) \qquad (3.15)$$

where $\xi(t)$ is a white noise process with zero mean and correlation function

$$E[\xi(t) \cdot \xi(t-\tau)] = I \cdot \delta(\tau) \qquad (3.16)$$

as in [45].

$\delta(t)$ is the Dirac delta function defined by

$$\delta(\tau) = 0 \quad \text{if } \tau \neq 0$$
$$\delta(\tau) = \infty \quad \text{if } \tau = 0 \tag{3.17}$$

An important relation concerning the Dirac delta function is

$$\int_{-\infty}^{\infty} f(\tau) \cdot \delta(\tau - T) d\tau = f(\tau = T) \tag{3.18}$$

The autocorrelation function of a stationary random process is defined in [46] as the ensemble average of the product $\xi(t) \cdot \xi(t-\tau)$ where τ is the time separation between the samples.

$$R_x(\tau) = E[\xi(t) \cdot \xi(t-\tau)] \tag{3.19}$$

The power spectral density function of a white noise process is constant. In [46], the autocorrelation function of a white noise process is defined as

$$R_x(\tau) = 2 \cdot \pi \cdot S_0 \cdot \delta(\tau) \tag{3.20}$$

in terms of the constant power spectral density S_0 of the white noise process $\xi(t)$.

If equations (3.16), (3.19) and (3.20) are combined, one obtains

$$R_x(\tau) = 2 \cdot \pi \cdot S_0 \cdot \delta(\tau) = E[\xi(t) \cdot \xi(t-\tau)] = I \cdot \delta(\tau) \tag{3.21}$$

For the filter given by (3.15), $\xi(t)$ is the white noise input and $w(t)$ is the output of the filter, and the relation between the power spectral density functions of the input and output of a system is given in [46] as

$$S_{output}(\omega) = |H(\omega)|^2 \cdot S_{input}(\omega) \tag{3.22}$$

where $H(\omega)$ is the complex frequency response function, which has been, obtained for the filter (3.15) in Appendix A as

$$H(\omega) = \frac{1}{i \cdot \omega + \alpha \cdot V} \tag{3.23}$$

After some manipulations, one can obtain

$$|H(\omega)|^2 = \frac{1}{\omega^2 + (\alpha \cdot V)^2} \tag{3.24}$$

Then from (3.22), the spectral density function of the road irregularities is obtained as

$$S_w(\omega) = \frac{1}{\omega^2 + (\alpha \cdot V)^2} \cdot S_0 \tag{3.25}$$

where $S_w(\omega)$ is the power spectral density function of the road irregularities, and S_0 is the constant power spectral density function of the white noise process.

If equations (3.14) and (3.25) are equated to each other, one obtains

$$\frac{\sigma^2}{\pi} \cdot \frac{\alpha \cdot V}{\omega^2 + (\alpha \cdot V)^2} = \frac{S_0}{\omega^2 + (\alpha \cdot V)^2} \tag{3.26}$$

From (3.26), S_0 is obtained as

$$S_0 = \frac{\sigma^2 \cdot \alpha \cdot V}{\pi} \tag{3.27}$$

If (3.27) is substituted into (3.21), the intensity of the white noise process I is obtained as

$$I = 2 \cdot \alpha \cdot V \cdot \sigma^2 \qquad (3.28)$$

Here, note that I is a constant value since the filter input $\xi(t)$ is a white noise process as stated in [45].

3.2.2 ROAD SURFACE IDENTIFICATION

As summary up to here, white noise with a constant power spectral density S_0, and intensity $2 \cdot \alpha \cdot V \cdot \sigma^2$ is passed through the filter of (3.15), and the road surface irregularities $w(t)$ is obtained. Then, knowing that the control input $u(t)$ is initially zero, $w(t)$ is substituted into the state-space equation (3.9), and the state vector is obtained. Finally having calculated the states, the road surface irregularity input is reconstructed from (3.13) as $w_{rec}(t)$.

In the road identification process, the power spectral density function of the reconstructed road surface $w_{rec}(t)$ is calculated and named as $S_{wrec}(\omega)$. Then, the final step is to compare $S_{wrec}(\omega)$ with the power spectral density functions (3.14) calculated for the 6 road/terrain types defined before, and to identify the type of road on which the vehicle is moving.

The comparison and road type identification are done by the help of a special matrix called Road Type Comparison Matrix or RTC matrix in short. RTC matrix is a 1x6 matrix with elements given below:

$$RTC = \left[\left|\overline{S_{wrec} - S1}\right| \quad \left|\overline{S_{wrec} - S2}\right| \quad \left|\overline{S_{wrec} - S3}\right| \quad \left|\overline{S_{wrec} - S4}\right| \quad \left|\overline{S_{wrec} - S5}\right| \quad \left|\overline{S_{wrec} - S6}\right| \right] \tag{3.29}$$

The elements of the RTC matrix are composed of the absolute values of the mean of the differences of power spectral density functions of the reconstructed road surface irregularities and power spectral density functions (3.14) calculated for the 6 road/terrain types defined before

So, here, $S1$, $S2$, $S3$, $S4$, $S5$, $S6$ are the power spectral density functions of the very good asphalt road, good asphalt road, average asphalt road, poor asphalt road, paved road, and dirt road respectively, and the upper score represents the symbol for the averaging operation that gives the mean values.

After the elements of the RTC matrix are calculated, the element with the smallest value (it should be noted that all the elements of the RTC matrix are positive values because of the absolute value operator.) determines the road type on which the vehicle is moving. As an example, if the minimum element of the RTC matrix is the first element, then the power spectral density of the reconstructed road input S_{wrec} is nearer to the power spectral density of "Very Good Asphalt" road $S1$, therefore the road type is determined to be of type "Very

Good Asphalt". The same analysis is conducted if the road type is different. So, the road type identification process is finalized.

3.3 OPTIMAL CONTROL OF THE ACTIVE SUSPENSION

3.3.1 INTRODUCTION

Linear quadratic optimal control theory is extensively used for the optimization problems, since this method provides a compact analytical solution, i.e. full-state feedback control, with relatively low computational time and the stability of the system is guaranteed. Since quadratic regulators consider and feed back all system states, they are very advantageous when compared to any classical controller. The disadvantages of quadratic regulators are the necessity to measure or estimate all state variables and limited choice of performance index that has to be a quadratic functional of state and control variables.

In this section stochastic optimal linear control theory is used in the optimization process and a full-state feedback gain with constant elements are obtained. In optimization problems with quadratic performance indices and linear constraints such as state-space systems, a linear control law, which is suitable for implementation in engineering problems, is obtained. Thus, in this study, the performance index is chosen to be quadratic and composed of the ensemble average of a weighted sum of the mean square values of the sprung mass acceleration, suspension travel, road holding, and control forces. Finally, the average behavior of the optimally controlled system is obtained.

Before proceeding with the types of quadratic regulators [47, 48], it is here worth to make a definition of the *regulator*. The regulator is an optimal control algorithm in which a linear state-feedback control input is produced to bring the nonzero plant output or any of its derivatives to the zero state. In other words, the problem is to apply a control to take the plant from a non-zero state to the zero state. This problem may typically occur where the plant is subjected to unwanted disturbances that perturb its output.

3.3.2 FINITE TIME (HORIZON) REGULATOR PROBLEM

Finite time regulator problem is given below.

Consider the system

$$\dot{x}(t) = F(t)\cdot x(t) + G(t)\cdot u(t) \qquad \text{with } x(t_0) \text{ given} \tag{3.30}$$

with the entries of $F(t)$, $G(t)$ assumed continuous. Let the matrices $Q(t)$ and $R(t)$ have continuous entries, be symmetric, and be nonnegative and positive definite, respectively. Let P_f be a nonnegative definite symmetric matrix. Let the performance index be defined as

$$J(x(t_0), u(\bullet), t_0) = \int_{t_0}^{T} \left(u^T(t)R(t)u(t) + x^T(t)Q(t)x(t)\right)dt + x^T(T)P_f x(T) \tag{3.31}$$

Here, $(\bullet)$ means that the corresponding function has only one variable.

Define the minimization problem as the task of finding an optimal control $u^*(t)$, $t \in [t_0, t]$, minimizing J and finding the associated optimum performance index $J^*(x(t), t)$.

The problem is called finite time or finite horizon regulator problem, since T is assumed to have a finite value.

The optimal control $u^*(t)$ that minimizes (3.31) is found as

$$u^*(t) = -R^{-1}(t)G^T(t)P(t)x(t) \tag{3.32}$$

where $P(t)$ is the solution of the nonlinear differential equation, named the matrix Riccati equation, given below.

$$-\dot{P}(t) = P(t)F(t) + F^T(t)P(t) - P(t)G(t)R^{-1}(t)G^T(t)P(t) + Q(t) \tag{3.33}$$

The boundary condition of the matrix Riccati equation is

$$P(T) = P_f \tag{3.34}$$

3.3.3 INFINITE TIME (HORIZON) REGULATOR PROBLEM

Infinite time regulator problem is given below.

Consider the system

$$\dot{x}(t) = F(t) \cdot x(t) + G(t) \cdot u(t) \qquad \text{with } x(t_0) \text{ given} \tag{3.30}$$

with the entries of $F(t)$, $G(t)$ assumed continuous. Let the matrices $Q(t)$ and $R(t)$ have continuous entries, be symmetric, and be nonnegative and positive definite, respectively. Define the performance index as

$$J(x(t_0),u(\bullet),t_0) = \int_{t_0}^{\infty} \left(u^T(t)R(t)u(t) + x^T(t)Q(t)x(t)\right)dt \qquad (3.35)$$

Define the minimization problem as the task of finding an optimal control $u^*(t)$, $t \geq t_0$, minimizing J and finding the associated optimum performance index $J^*(x(t),t)$.

The problem is called infinite time or infinite horizon regulator problem, since T is infinite as shown in the performance index (3.35).

Here, it is of vital importance to state that with the finite-time regulator problem, the optimal performance index J^* is always finite; however this may not be so in the infinite-time regulator problem. In the infinite-time regulator problem, system (3.30) should be completely controllable for every time t in order to obtain a finite optimal performance index J^*.

Here, it is important to note that if a system is completely controllable at time t, it means that there exists for every state variable $x(t)$ a control $u(t)$ and a time t_2 such that $u(t)$ transfers $x(t)$ to the zero state at time t_2.

Now, if the system (3.30) is assumed to be completely controllable, we can proceed with the solution of the infinite-time regulator problem.

Let $P(t,T)$ be the solution of the matrix Riccati equation below

$$-\dot{P}(t) = P(t)F(t) + F^T(t)P(t) - P(t)G(t)R^{-1}(t)G^T(t)P(t) + Q(t) \qquad (3.33)$$

with boundary condition $P(T,T) = 0$. Then $\lim_{T\to\infty} P(t,T) = \overline{P}(t)$ exists for all t and is a solution of (3.33). The corresponding optimal control is then

$$u^*(t) = -R^{-1}(t)G^T(t)\overline{P}(t)x(t) \qquad (3.36)$$

where $\overline{P}(t)$ is the solution of the matrix Riccati equation, given below.

$$-\dot{\overline{P}}(t) = \overline{P}(t)F(t) + F^T(t)\overline{P}(t) - \overline{P}(t)G(t)R^{-1}(t)G^T(t)\overline{P}(t) + Q(t) \qquad (3.37)$$

3.3.4 TIME-INVARIANT REGULATOR PROBLEM

For finite-time regulator problems, a time-invariant control law of the form $u(t) = K^T \cdot x(t)$, where K is a constant, cannot be established, i.e. there are no T, R, and Q which will give a time-invariant control law when F and G are constant, unless the matrix P_f takes on certain special values. However, if the regulator problem is infinite-time, a time-invariant control law can be established. So, if a time-invariant control law is to be obtained, the optimization problem should be infinite-time regulator problem.

Time-invariant regulator problem is given on the next page.

Consider the system

$$\dot{x}(t) = F \cdot x(t) + G \cdot u(t) \qquad \text{with } x(t_0) \text{ given} \tag{3.38}$$

where F and G are constant. The matrices R and Q are also constant, and positive definite and non-negative definite respectively. Define the performance index as

$$J(x(t_0), u(\bullet), t_0) = \int_{t_0}^{\infty} \left(u^T(t) R u(t) + x^T(t) Q x(t) \right) dt \tag{3.39}$$

We are going to define the minimization problem as the task of finding an optimal control $u^*(t)$, $t \geq t_0$, minimizing J and finding the associated optimum performance index $J^*(x(t), t)$. Here, it should be noticed that T is infinite.

As stated in [47], for the time-invariant regulator problem to be solved, system (18) should be completely controllable or stabilizible. The definitions of controllability and stabilizibility have been given in Appendix B. If it is assumed that system (3.38) is completely controllable and stabilizible, the solution of the time-invariant regulator problem is as follows.

Let $P(t,T)$ be the solution of the matrix-Riccati equation

$$-\dot{P}(t) = P(t)F + F^T P(t) - P(t) G R^{-1} G^T P(t) + Q \tag{3.40}$$

with initial condition $P(T,T) = 0$. Then $\lim_{T \to \infty} P(t,T) = \overline{P}$ exists and is constant. The corresponding optimal control input is then,

$$u^*(t) = -R^{-1}G^T\overline{P}x(t) \tag{3.41}$$

where the constant $\overline{P}$ satisfies the matrix Riccati equation given for the time-invariant regulator problem below.

$$\overline{P}F + F^T\overline{P} - \overline{P}GR^{-1}G^T\overline{P} + Q = 0 \tag{3.42}$$

3.3.5 STOCHASTIC OPTIMAL LINEAR CONTROL THEORY

Having studied the deterministic control problem, the types of quadratic regulators in particular, stochastic optimal control theory, which is accepted as the synthesis of the feedback controllers, which are optimum in the ensemble average sense, in the presence of random disturbances, will be considered in this section.

The stochastic optimal linear control problem is that of designing an optimal controller for a linear system disturbed by Gaussian white noise, where the performance index is quadratic, the initial conditions are random, but perfect knowledge of the system states are available [45].

Let the state space system to be controlled be given below as

$$\dot{x} = F(t)\cdot x + G(t)\cdot u + D(t)\cdot \xi(t) \tag{3.43}$$

where x is the state vector, u is the control input, and ξ is the Gaussian white noise process with zero mean and a covariance function given as

$$E[\xi(t)\cdot\xi^T(t-\tau)] = I\cdot\delta(\tau) \tag{3.16}$$

$$E[\xi(t)] = \bar{\xi}(t) = 0 \tag{3.44}$$

$$E[x(t_0)] = 0 \tag{3.45}$$

$$E[x(t_0)x^T(t_0)] = X_0 \tag{3.46}$$

Let the performance index be the ensemble average of a quadratic form like below:

$$J = E\left\{\left(x^T P_f x\right)_{t=t_f} + \int_{t_0}^{t_f}\left(x^T Qx + u^T Ru\right)dt\right\} \tag{3.47}$$

where P_f and $Q(t)$ are positive-semidefinite matrices and $R(t)$ is a positive-definite matrix. Clearly, here, the average value (3.47) will be minimized.

Here, $\xi(t)$ represents Gaussian white noise disturbances with zero mean and short correlation times compared to the characteristic times of the system. Thus, it is impossible to predict $\xi(\tau)$ for $\tau > t$, even with perfect knowledge of the state for $\tau < t$. Clearly, then, the optimal controller is identical to the deterministic controller of Section 3.3.2 [45]. Then, the corresponding optimal control is given by

$$u^*(t) = -R^{-1}(t)G^T(t)P(t)x(t) \tag{3.32}$$

where $P(t)$ is the solution of the matrix Riccati equation, given below.

$$-\dot{P}(t) = P(t)F(t) + F^T(t)P(t) - P(t)G(t)R^{-1}(t)G^T(t)P(t) + Q(t) \qquad (3.33)$$

The boundary condition of the matrix Riccati equation is

$$P(t_f) = P_f \qquad (3.48)$$

Thus, the optimal control problem of a linear time-variant system with a quadratic performance index has been solved.

Combining the results obtained in Sections 3.3.2, 3.3.3 and 3.3.4 with the results obtained in this section, one can easily conclude that for a linear time-invariant system

$$\dot{x} = F \cdot x + G \cdot u + D \cdot \xi(t) \qquad (3.49)$$

where ξ is the Gaussian white noise process with zero mean and covariance function

$$E\left[\xi(t) \cdot \xi^T(t-\tau)\right] = I \cdot \delta(\tau) \qquad (3.16)$$

the optimal control that minimizes the performance index

$$J = E\left\{\int_{t_0}^{\infty}\left(x^T Q x + u^T R u\right)dt\right\} \qquad (3.50)$$

where P_f and $Q(t)$ are positive-semidefinite matrices and $R(t)$ is a positive-definite matrix, is given as

$$u^*(t) = -R^{-1}G^T P x(t) \qquad (3.51)$$

where the constant P satisfies the matrix Riccati equation given for the time-invariant regulator problem below.

$$PF + F^T P - PGR^{-1}G^T P + Q = 0 \tag{3.52}$$

3.3.6 COMBINING FILTER EQUATION WITH STATE EQUATIONS

In Section 3.3.5, stochastic optimal controller has been designed for a linear system disturbed by Gaussian white noise. The vehicle model that will be analyzed throughout this study leads to system (3.9) where the disturbance is not white noise, but the output of a linear filter of the first order with white noise input instead. However, in order apply the results obtained in Section 3.3.5 to the vehicle model. the system equation should have a white noise disturbance. This difficulty can be circumvented by combining the state equations

$$\dot{x}_1 = x_2 \tag{3.5}$$

$$\dot{x}_2 = \frac{-k_1}{m_2}x_1 - \frac{c_1}{m_2}x_2 + \frac{k_1}{m_2}x_3 + \frac{c_1}{m_2}x_4 + \frac{1}{m_2}u \tag{3.6}$$

$$\dot{x}_3 = x_4 \tag{3.7}$$

$$\dot{x}_4 = \frac{k_1}{m_1}x_1 + \frac{c_1}{m_1}x_2 - \frac{k_1 + k_2}{m_1}x_3 - \frac{c_1}{m_1}x_4 + \frac{k_2}{m_1}w - \frac{1}{m_1}u \tag{3.8}$$

with the filter equation

$$\dot{w} = -\alpha \cdot V \cdot w + \xi(t) \tag{3.15}$$

where $\xi(t)$ is a stationary white noise process with zero mean and covariance function

$$E[\xi(t) \cdot \xi(t-\tau)] = I \cdot \delta(\tau) \tag{3.16}$$

with

$$I = 2 \cdot \alpha \cdot V \cdot \sigma^2 \tag{3.28}$$

as derived before.

By introducing the notation

$$F_w = -\alpha \cdot V \tag{3.53}$$

$$D_w = 1 \tag{3.54}$$

the shaping filter (3.15) can be incorporated with the system equations yielding a widened state-space equation with the white noise input as given below.

$$\dot{\hat{x}}(t) = \begin{bmatrix} F_x & D_x \\ 0 & F_w \end{bmatrix} \cdot \hat{x}(t) + \begin{bmatrix} G_x \\ 0 \end{bmatrix} \cdot u(t) + \begin{bmatrix} 0 \\ D_w \end{bmatrix} \cdot \xi(t) \tag{3.55}$$

Here, the vector $\hat{x}$ is a widened state vector (the variable w describing the road disturbance is added as the 5th state, so one can use $w = x_5$) as given below.

$$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ w \end{bmatrix} \tag{3.56}$$

Consequently, state-space equation of the combined system becomes

$$\dot{\hat{x}} = F \cdot \hat{x} + G \cdot u + D \cdot \xi \tag{3.57}$$

where

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{-k_1}{m_2} & \frac{-c_1}{m_2} & \frac{k_1}{m_2} & \frac{c_1}{m_2} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{k_1}{m_1} & \frac{c_1}{m_1} & \frac{-(k_1+k_2)}{m_1} & \frac{-c_1}{m_1} & \frac{k_2}{m_1} \\ 0 & 0 & 0 & 0 & -\alpha \cdot V \end{bmatrix} \tag{3.58}$$

$$G = \begin{bmatrix} 0 \\ \frac{1}{m_2} \\ 0 \\ \frac{-1}{m_1} \\ 0 \end{bmatrix} \tag{3.59}$$

$$D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.60}$$

The results derived for stochastic optimal linear control of the systems disturbed by white noise in Section 3.3.6 can now be used for the optimal control of the vehicle active suspension.

3.3.7 PERFORMANCE INDEX DEFINITION

The active suspension system is optimized with respect to ride comfort, working space of the suspension (suspension travel), road holding, and the actuator power requirement. The quadratic performance index is chosen to be the ensemble average of a weighted sum of the mean square values of sprung mass acceleration, suspension travel, road holding and control forces. Thus, the performance index includes the following parts:

- $J_1 = E\left(\ddot{y}_2^2\right)$, the mean square acceleration of the passenger compartment, which is a measure of ride comfort and should be kept small on rough roads

- $\rho_1 J_2 = \rho_1 E\left[\left(y_2 - y_1\right)^2\right]$, the mean square value of the relative displacement between the body and the wheel i.e. suspension travel, weighted by the constant ρ_1. This term limits the space required for the suspension and should be kept small on smooth roads and at high velocities to improve pitch and roll (cornering) stability.

- $\rho_2 J_3 = \rho_2 E\left[(y_1 - y_0)^2\right]$, the mean square value of the relative displacement between the wheel and the road surface, weighted by the constant ρ_2. It is a measure of road holding and it is preferable to keep this term small both to avoid losing contact between the wheel and the road and to avoid applying impulsive forces to the system.

- $\rho_3 J_4 = \rho_3 E(u^2)$, mean square value of the actuator force weighted by the constant ρ_3. This term should be minimized in order minimize the cost of the suspension.

So, the performance index is taken as:

$$J = E\left\{ \int_{t_0}^{T} \left[\ddot{y}_2^2 + \rho_1 (y_2 - y_1)^2 + \rho_2 (y_1 - y_0)^2 + \rho_3 u^2 \right] dt \right\} \tag{3.61}$$

The performance index (3.61) can be written in terms of state variables as below.

$$J = E\left\{ \int_{t_0}^{T} \left[\begin{array}{l} \left(\frac{-k_1}{m_2} x_1 - \frac{c_1}{m_2} x_2 + \frac{k_1}{m_2} x_3 + \frac{c_1}{m_2} x_4 + \frac{1}{m_2} u \right)^2 + \\ \qquad + \rho_1 (x_1 - x_3)^2 + \rho_2 (x_3 - x_5)^2 + \rho_3 u^2 \end{array} \right] dt \right\} \tag{3.62}$$

The expression (3.62) can after some certain manipulations, be written in the following form.

$$J = E\left\{ \int_{t_0}^{T} \begin{bmatrix} \hat{x}^T & u \end{bmatrix} \begin{bmatrix} A & N \\ N^T & B \end{bmatrix} \begin{bmatrix} \hat{x} \\ u \end{bmatrix} dt \right\} \tag{3.63}$$

At steady state, the performance index (3.63) can be converted to the following performance index.

$$J = \lim_{T \to \infty} \frac{1}{T} E\left\{ \int_{0}^{T} \begin{bmatrix} \hat{x}^T(t) & u^T(t) \end{bmatrix} \begin{bmatrix} A & N \\ N^T & B \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ u(t) \end{bmatrix} dt \right\} \tag{3.64}$$

where matrices A and B are symmetric, A is non-negative definite and B is positive definite with elements

$$A = \begin{bmatrix} \frac{k_1^2}{m_2^2} + \rho_1 & \frac{k_1 c_1}{m_2^2} & -\left(\frac{k_1^2}{m_2^2} + \rho_1\right) & -\frac{k_1 c_1}{m_2^2} & 0 \\ \frac{k_1 c_1}{m_2^2} & \frac{c_1^2}{m_2} & -\frac{k_1 c_1}{m_2^2} & -\frac{c_1^2}{m_2} & 0 \\ -\left(\frac{k_1^2}{m_2^2} + \rho_1\right) & -\frac{k_1 c_1}{m_2^2} & \frac{k_1^2}{m_2^2} + \rho_1 + \rho_2 & \frac{k_1 c_1}{m_2^2} & -\rho_2 \\ -\frac{k_1 c_1}{m_2^2} & -\frac{c_1^2}{m_2} & \frac{k_1 c_1}{m_2^2} & \frac{c_1^2}{m_2} & 0 \\ 0 & 0 & -\rho_2 & 0 & \rho_2 \end{bmatrix} \tag{3.65}$$

$$N = \begin{bmatrix} -\dfrac{k_1}{m_2^2} \\ -\dfrac{c_1}{m_2^2} \\ \dfrac{k_1}{m_2^2} \\ \dfrac{c_1}{m_2^2} \\ 0 \end{bmatrix} \tag{3.66}$$

$$B = \frac{1}{m_2^2} + \rho_3 \tag{3.67}$$

The performance index (3.64) is the ensemble-averaged form of the standard performance index with cross-product terms given in [45]. Obtaining the optimal control minimizing a quadratic performance index with cross-product terms and with linear constraints, e.g. state-space systems, has been given in Appendix C.

3.3.8 OPTIMIZATION PROBLEM DEFINITION

The optimization problem of this study can be stated as follows:

For a system governed by

$$\dot{\hat{x}} = F \cdot \hat{x} + G \cdot u + D \cdot \xi \tag{3.57}$$

where F, G and D are constant and $\xi(t)$ is a white noise process with zero mean and covariance function

$$E[\xi(t)\cdot\xi(t-\tau)] = I\cdot\delta(\tau) \tag{3.16}$$

with

$$I = 2\cdot\alpha\cdot V\cdot\sigma^2, \tag{3.28}$$

the optimal control vector $u^*(t)$ which minimizes the performance index

$$J = \lim_{T\to\infty}\frac{1}{T}E\left\{\int_0^T\begin{bmatrix}\hat{x}^T(t) & u^T(t)\end{bmatrix}\begin{bmatrix}A & N\\ N^T & B\end{bmatrix}\begin{bmatrix}\hat{x}(t)\\ u(t)\end{bmatrix}dt\right\} \tag{3.64}$$

is obtained by combining the results (3.51), (3.52), (C.10) and (C.11) as

$$u^*(t) = -K\cdot\hat{x}(t) \tag{3.68}$$

where

$$K = R^{-1}\left[G^T P + N^T\right] \tag{3.69}$$

and P is the solution of the matrix Riccati equation

$$P\left(F - GR^{-1}N^T\right) + \left(F - GR^{-1}N^T\right)^T P - PGR^{-1}G^T P + \left(A - NR^{-1}N^T\right) = 0 \tag{3.70}$$

However, here, a difficulty arises, because the algorithm that MATLAB® uses to solve the matrix Riccati equation requires the system (3.57) to be completely controllable. In the case of a system with a stochastic input, this property means physically that variances of all state vector components can be reduced by introducing a control vector. Unfortunately, this is not the case in this study, where the last state vector component, which describes the exciting process associated with the road, cannot be changed by applying a control force. So system (3.57) is not completely controllable and equation (3.70) cannot be solved by the standard algorithms. To circumvent this difficulty F, G, N, A, P and K matrices are separated into controllable and non-controllable parts, and these parts are manipulated and solved separately. This procedure has been explained in Appendix D in detail.

As explained in Appendix D in detail, the optimal control for the active suspension with the uncontrollable state is obtained as

$$u^*(t) = -K \cdot \hat{x}(t) \qquad (3.68)$$

where

$$K = R^{-1}\left[\left(G_x^T P_{xx} + N_x^T\right) \mid G_x^T P_{xw}\right] \qquad (3.71)$$

3.3.9 AVERAGE BEHAVIOUR OF THE SYSTEM

It is often of interest to determine the average behavior of the optimally controlled system. If the system is

$$\dot{x} = F(t) \cdot x + G(t) \cdot u + D(t) \cdot \xi(t) \qquad (3.43)$$

where $\xi(t)$ is a stationary white noise process with zero mean and covariance function

$$E[\xi(t)\cdot\xi(t-\tau)] = I\cdot\delta(\tau) \tag{3.16}$$

we know that the optimal control that minimizes the performance index (3.47) is

$$u^*(t) = -K(t)x(t) \tag{3.32}$$

where

$$K(t) = R^{-1}(t)G^T(t)P(t)$$

Then, the optimal system is

$$\dot{x}(t) = [F(t) - G(t)\cdot K(t)]x(t) + D(t)\cdot w(t) \tag{3.72}$$

Let

$$X(t) = E[x(t)\cdot x^T(t)] \tag{3.73}$$

be the mean square value of $x(t)$, we get, from Appendix E,

$$\dot{X} = (F - G\cdot K)X + X(F - G\cdot K)^T + D\cdot I\cdot D^T \tag{3.74}$$

and $X(t_0) = X_0$. In Appendix E, the derivation of (3.74) has been given in detail.

If the system to be controlled and the process noise are stationary (i.e. F, G and I are constant) and the matrices Q and R of the performance index are constant, the controller also becomes stationary, i.e. P and K become constant. The mean-square value of the state, X, becomes constant and can be found by setting $\dot{X} = 0$ in (3.74) as below.

$$(F - G \cdot K)X + X(F - G \cdot K)^T + D \cdot I \cdot D^T = 0 \tag{3.75}$$

Equation (3.75) allows us to predict the mean-square histories of the state variables [45], which is of importance in calculating the parts of the quadratic performance index (3.61) analyzed throughout this study.

By using the result (3.75), the parts of the performance index can be easily found as follows:

Using the equations (3.57), (3.58), (3.59) and (3.60), one can write

$$\dot{x}_2 = \sum_{i=1}^{5} \left(F(2,i) \cdot x_i \right) + G(2) \cdot u \tag{3.76}$$

We know that the optimal control minimizing the performance index (3.64) is

$$u^*(t) = -K \cdot \hat{x}(t) \tag{3.68}$$

Then, one can write combining (3.76) and (3.68) that

$$\begin{aligned} \dot{x}_2 &= \sum_{i=1}^{5} \left[F(2,i) \cdot x_i - G(2)K(i)x_i \right] \\ &= \sum_{i=1}^{5} \left\{ \left[F(2,i) - G(2)K(i) \right] \cdot x_i \right\} \end{aligned} \tag{3.77}$$

From Section 3.3.7, it is known that the first part of the performance index is

$$J_1 = E(\ddot{y}_2^2) = E(\dot{x}_2^2) \tag{3.78}$$

After some manipulations, the expression (3.78) can be written in the following form:

$$J_1 = \sum_{i=1}^{5}\sum_{j=1}^{5}\{[F(2,i)-G(2)K(i)][F(2,j)-G(2)K(j)]\cdot E[x_i x_j]\} \tag{3.79}$$

It is known from (3.73) that the covariance matrix X is defined as

$$X(t) = E[x(t)\cdot x^T(t)]$$

So, expression (3.79) can be written as

$$J_1 = \sum_{i=1}^{5}\sum_{j=1}^{5}\{[F(2,i)-G(2)K(i)][F(2,j)-G(2)K(j)]\cdot X(i,j)\} \tag{3.80}$$

Therefore, the first part of the performance index, J_1, has been found in terms of the covariance matrix X of the states which is found from the Lyapunov equation (3.75).

In Section 3.3.7, the second part of the performance index, J_2, has been defined as

$$J_2 = E[(y_2 - y_1)^2] = E[(x_1 - x_3)^2] = E[x_1^2] - 2E[x_1 x_3] + E[x_3^2] \tag{3.81}$$

Thus, J_2 can be written in terms of the covariance matrix X of the states as

$$J_2 = X(1,1) - 2X(1,3) + X(3,3) \tag{3.82}$$

Similarly, the third part of the performance index J_3 is found from:

$$J_3 = E\left[(y_1 - y_0)^2\right] = E\left[(x_3 - x_5)^2\right] = E\left[x_3^2\right] - 2E\left[x_3 x_5\right] + E\left[x_5^2\right] \tag{3.83}$$

In terms of the covariance matrix,

$$J_3 = X(3,3) - 2X(3,5) + X(5,5) \tag{3.84}$$

Using the result of (3.68), one can write the fourth part of performance index, J_4, as

$$J_4 = E\left(u^2\right) = E\left[(-K \cdot \hat{x})^2\right] \tag{3.85}$$

Finally, in terms of the covariance matrix, one obtains

$$J_4 = \sum_{i=1}^{5} \sum_{j=1}^{5} \left[K(i)K(j)X(i,j)\right] \tag{3.86}$$

CHAPTER IV

IMPLEMENTATION ON MATLAB®

4.1 USER INTERFACE WINDOWS

4.1.1 INTRODUCTION

As stated before, MATLAB® software package is used throughout this study. Besides the calculations throughout the analysis made by MATLAB®, a user interface environment in which the user can specify the conditions that the vehicle operates and obtain the results visually in frequency and time domains. Also, in the user interface environment, the user can make a road scenario consisting of two road types on which the vehicle moves continuously and see the changes when the vehicle leaves one of the roads and pass to the other one. The user also can see the dynamic simulation of the vehicle for specified conditions. The dynamic simulation environment has been done in Simulink environment, which is a simulation package of MATLAB®.

User interface environment is formed by 5 control interface windows and 1 Simulink window.

4.1.2 START-UP WINDOW

The first user interface window has been named "Adaptive Control of Active Vehicle Suspensions" which is the start-up window seen in the Figure 11 below. In the start-up window, the user can switch to the Road Data Input Window, Frequency Domain Analysis Window, Time Domain Analysis Window, Road Scenario Window and also determine the mode of adaptation. The road identification is seen visually in this window, i.e. the PSD plots of the 6 standard road types and the given road excitation is seen in this window. The PSD plots can also be printed.

4.1.3 ROAD DATA INPUT WINDOW

In the 'Road Data Input Window', the user can either select a predefined road type or input an arbitrary road type and velocity as seen below in Figure 12.

The user can input an arbitrary road type by inserting the data of cut-off frequency and variance. By the slider named 'Filtering Time', the user can specify the solution time for the shaping filter (3.15) and thus the number of road excitation data. As the filtering time increases, the number of road excitation data increases and the power spectral density analysis of the road excitation gives smoother results and the deviation from the analytical curves decreases.

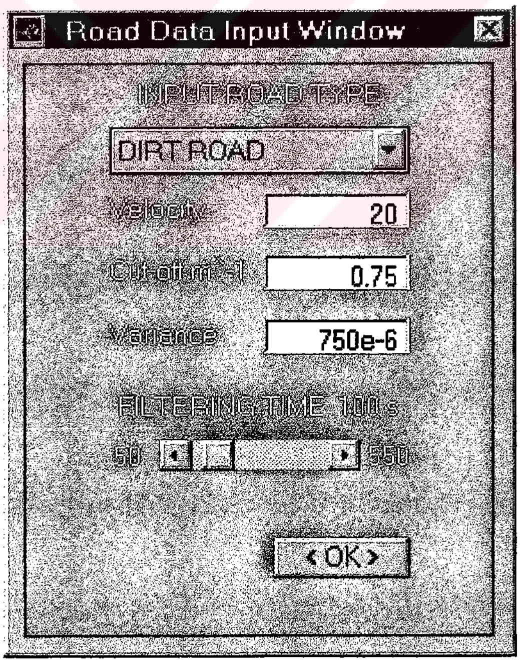

Figure 12. Road Data Input Window

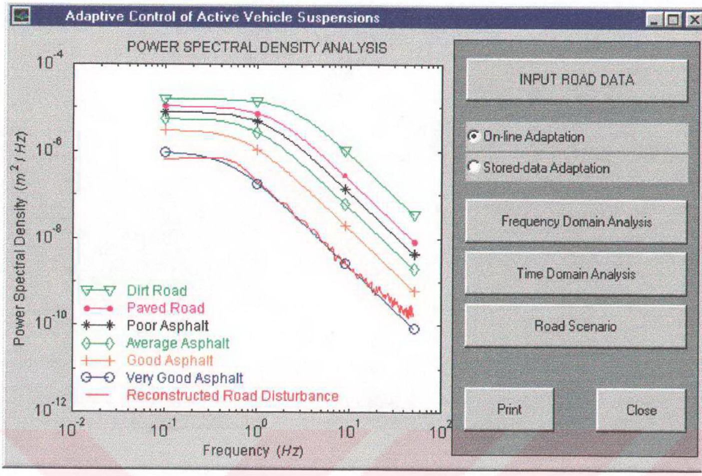

Figure 11. Start-up Window

In the start-up window, the user can specify two modes of adaptation, namely, 'On-line Adaptation' and 'Stored Data Adaptation'. If the user chooses On-line Adaptation mode, then the program calculates the optimal control vector passing through all the procedures as explained in the case study in Chapter 4. However, if the user chooses Stored Data Adaptation mode, then the program uses previously defined control gain matrices for different road types and emphasis modes. So, in Stored Data Adaptation mode, the program skips the steps of calculating the control gain matrix, but rather uses the stored control gain matrix data. This shortens the time needed for the analysis and is very important in fast-changing road conditions and velocities.

4.1.3 ROAD DATA INPUT WINDOW

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The user can input an arbitrary road type by inserting the data of cut-off frequency and variance. By the slider named 'Filtering Time', the user can specify the solution time for the shaping filter (3.15) and thus the number of road excitation data. As the filtering time increases, the number of road excitation data increases and the power spectral density analysis of the road excitation gives smoother results and the deviation from the analytical curves decreases.

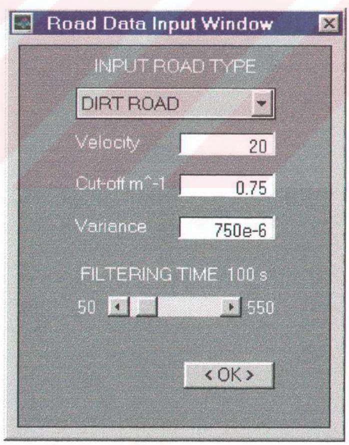

Figure 12. Road Data Input Window

4.1.4 FREQUENCY DOMAIN ANALYSIS WINDOW

One of the windows in which the performance results for the specified road input can be observed is the Frequency Domain Analysis Window named "Vehicle Performance Analysis in Frequency Domain". In this window which is seen in the Figure 13, the user can see the results of active control in frequency domain in terms of Sprung Mass Acceleration, Sprung Mass Velocity, Suspension Travel, Tire Deflection, Unsprung Mass Acceleration and Unsprung Mass Velocity.

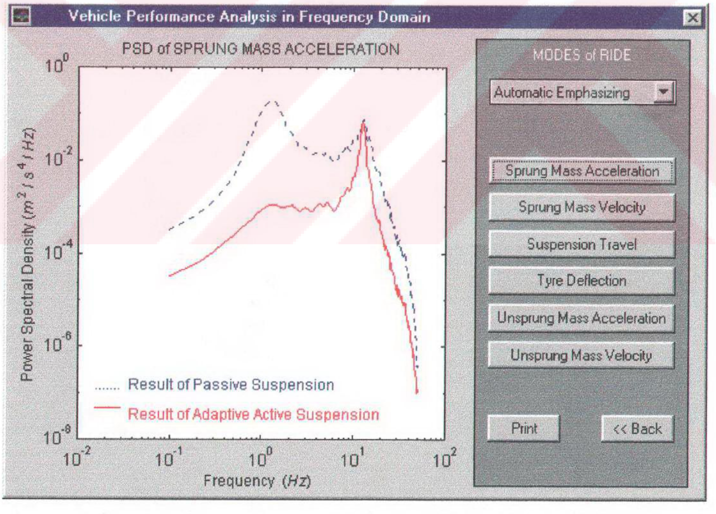

Figure 13. Frequency Domain Analysis Window

As shown in the Figure 13 above, the Frequency Domain Analysis Window includes a pull-down menu, which consists of 'Automatic Emphasize', 'Emphasize Ride Comfort' and 'Emphasize Road Holding' modes. The gain scheduling adaptive control algorithm of the program requires that the default mode of this menu be 'Automatic Emphasize' mode. At this mode, the program automatically adjusts the values of the weighing coefficients according to the Table 3. However, the other modes enable the user to see the results in either 'Emphasize Ride Comfort' mode or 'Emphasize Road Holding' mode as desired.

4.1.5 TIME DOMAIN ANALYSIS WINDOW

By the Time Domain Analysis Window named "Vehicle Performance Analysis in Time Domain", the user can see the performance results in time domain and also switch to the Dynamic Simulation Environment as shown in Figure 14 below. In this window, the user can see the results of active control in time domain in terms of Sprung Mass Acceleration, Sprung Mass Velocity, Suspension Travel, Tire Deflection, Unsprung Mass Acceleration and Unsprung Mass Velocity.

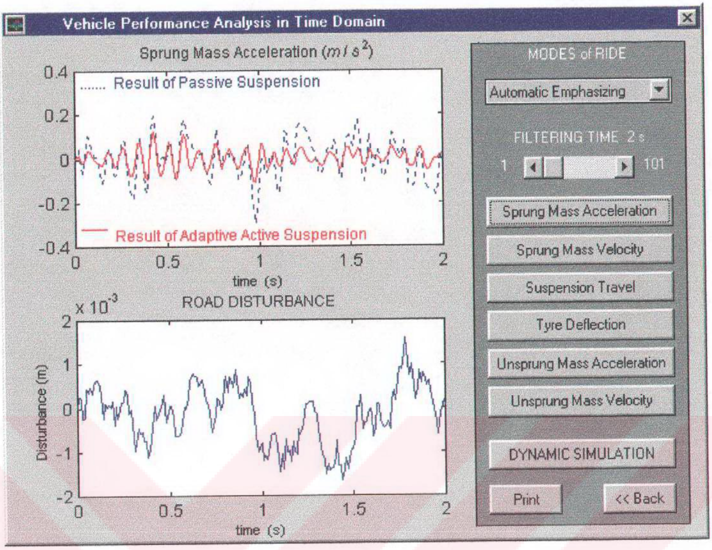

Figure 14. Time Domain Analysis Window

The user can select any mode of emphasis as discussed in Section 4.1.4 in this window. By the slider named 'Filtering Time', the user can re-specify the solution time of the filter and thus the amount of road excitation data. The major purpose of establishing a 'Filtering Time' slider in this window is facilitating the user with the chance of adjusting the density of the plots. As the amount of processing data increases, the plots get denser and a difficulty of observing the performance visually occurs.

4.1.6 ROAD SCENARIO WINDOW

By the Road Scenario Window seen in Figure 15 below, the user can choose two successive road types on which the vehicle moves continuously at a velocity defined by the user. The user also determines the time of travel by a slider. The program divides the time of travel into 4 regions. In the first region, the program only identifies the road type and there is no control at this step. At the beginning of the second quarter, the program identifies the road type and applies the corresponding control input during this quarter. The program continues the identification process during each quarter without regarding whether there is a control input or not. In the third quarter the road type changes but since the program can sense this change only at the end of the third quarter, the program still applies the control of the first road type in this region. However, at the end of the third quarter, the program identifies the change in the road excitation and applies the corresponding control in the last region. The delay of identification occurs from the fact that the algorithm of identification is not recursive. By this window, the results in time domain can be observed in terms of ride comfort (sprung mass acceleration) and road holding (suspension travel and tire deflection).

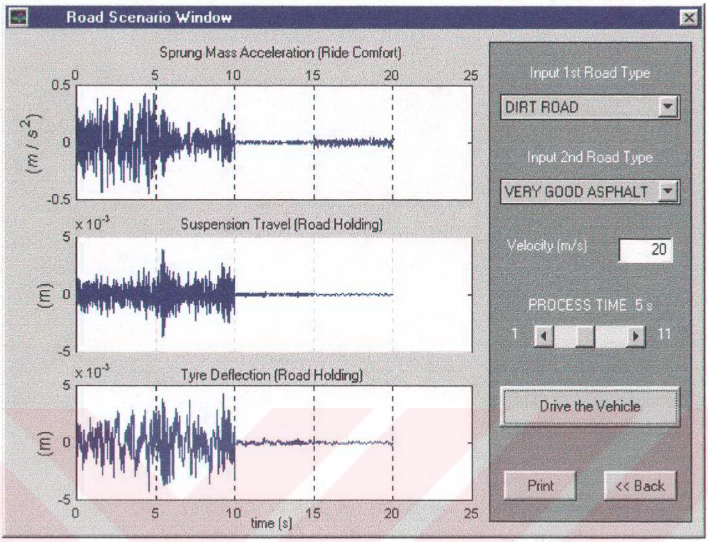

Figure 15. Road Scenario Window

As shown in Figure 15 above, the vehicle first moves on a dirt road where ride comfort is emphasized. Up to 5 seconds the vehicle identifies the type of road and applies adaptive control emphasizing ride comfort. Thus, after 5 seconds, the sprung mass acceleration decreases at the expense of the increase in suspension travel and tire deflection as shown in Figure 15. At the 10^{th} second the vehicle passes to the very good asphalt road. The vehicle identifies the asphalt road at the 15^{th} second and from then on, road holding is emphasized and suspension travel and tire deflection are decreased at the expense of increasing sprung mass acceleration.

4.1.6 DYNAMIC SIMULATION ENVIRONMENT

Simulink Package enables the user to see the results obtained in the Time Domain Analysis Window in a dynamic simulation environment through oscilloscopes dynamically in different time ranges. The dynamic simulation of a dirt road data and minimized sprung mass accelerations can be seen in Figures 16 and 17 respectively.

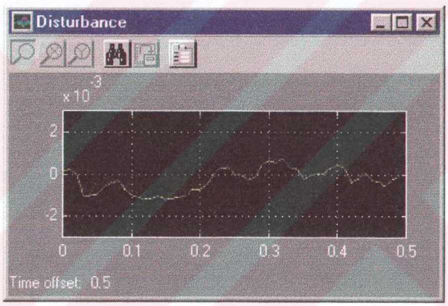

Figure 16. Dirt Road Dynamic Simulation Oscilloscope

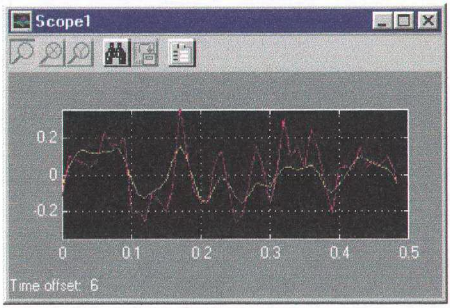

Figure 17. Sprung Mass Acceleration Dynamic Simulation Oscilloscope

4.2 OBTAINING WHITE NOISE WITH THE DESIRED PSD LEVEL

The theoretical continuous time white noise is a process with infinite correlation at zero separation time and zero correlation at other separation time values. Theoretical continuous time white noise has an infinite variance and constant power spectral density level, and cannot be obtained practically [49]. Theoretical continuous time white noise cannot be Gaussian, since a Gaussian process has a finite mean square value.

It is a fact that continuous time processes are much more difficult to analyze than the discrete time processes. Discrete time white noise can be defined as a process, which consists of a sequence of uncorrelated stochastic variables [49]. Thus, discrete time white noise process has a finite autocorrelation value only if the separation increment is zero i.e. when the sequence is correlated with itself.

In this study, white noise has been obtained by the command "randn(n)" which produces random numbers with a Gaussian distribution, unit variance and zero mean. The random data obtained by the 'randn' command exhibits a discrete time white noise character. The reason is that the computer produces uncorrelated random data for different seeds. So, the random data produced by the computer has finite autocorrelation function only at zero separation time. However, as the number of random data increases, the computer begins to repeat the old data and the random data begins to deviate from the white noise character after a certain period. Because of this fact, the power spectral density of the random data is constant up to a certain frequency beyond which it deviates from the constant value.

Now that the random data given by MATLAB® 'randn' function can be taken as a discrete time band-limited normally (Gaussian) distributed white noise of unity variance and with zero mean, the next step is to obtain white noise process the variance of which is determined by the road excitation and velocity data.

It is known that variance σ^2 of a random data is defined as the mean of the square of the deviation of the random variable, say, x from its mean level $E[x]$ or by some manipulations

$$\sigma^2 = E[x^2] - (E[x])^2 \tag{4.1}$$

So, for a process with zero mean value, the variance equals the mean square value. Then, for a process with zero mean,

$$\sigma^2 = E\left[x^2\right] \tag{4.2}$$

The mean square value is related to the power spectral density as [46]

$$E\left[x^2\right] = \int_{-\infty}^{\infty} S(\omega)d\omega \tag{4.3}$$

Thus, the mean square value is the area under the PSD versus frequency curve.

Then, the fact that theoretical continuous time white noise process has an infinite variance can be proved by combining (4.2) and (4.3) as

$$\sigma^2 = E\left[x^2\right] = \int_{-\infty}^{\infty} S_0 d\omega = \infty \tag{4.4}$$

where S_0 is the constant PSD level.

Thus, for a band-limited white noise with a frequency band of unity, mean square value is equal to the constant PSD level S_0 and

$$E\left[x^2\right] = S_0 \tag{4.5}$$

The algorithm that MATLAB® uses to calculate the power spectral density accepts that the average power spectral density value equals the mean square value. Thus, Equation (4.5) is also valid for MATLAB®.

As a consequence of the above discussion, if a zero mean random data with a constant power spectral density level S_0 is desired in MATLAB®, the mean square value of this data should be equal to S_0. Then, from (4.2), one can conclude that the data should have a variance equal to S_0.

It is known that to obtain a zero mean random data $z(k)$ with a variance of σ^2 from another zero mean random data $x(k)$ with a unity variance, random data $x(k)$ should be multiplied by σ. Thus, $z(k) = \sigma \cdot x(k)$ has a variance of σ^2.

In this study, 'randn' function gives a discrete time white noise with a unity variance and zero mean as explained before. However, it is desired to obtain a white noise with a PSD level given by (3.27) as

$$S_0 = \frac{\sigma^2 \cdot \alpha \cdot V}{\pi}$$

Thus, the white noise produced by 'randn' function should be multiplied by the square root of (3.27) so that the obtained white noise has a PSD level of S_0.

As a consequence the desired white noise is obtained in MATLAB® as

$$\textit{Desired White Noise} = \sqrt{\frac{\sigma^2 \cdot \alpha \cdot V}{\pi}} \cdot randn(N)$$

CHAPTER V

CASE STUDIES

5.1 VEHICLE PARAMETERS

All the results given in this chapter have been obtained for the vehicle having the following parameters:

$$m_1 = 100\,\text{kg}$$

$$m_2 = 1000\ \text{kg}$$

$$k_1 = 60 \cdot 10^3\ \text{N/m}$$

$$k_2 = 6 \cdot 10^5\ \text{N/m}$$

$$c_1 = 3 \cdot 10^3\ \text{Ns/m}$$

Road excitation parameters have been taken from Table 2.

5.2 ADAPTIVE CONTROL OF THE ACTIVE SUSPENSION

In this study, gain scheduling adaptive control method has been used to control the active suspension.

As given in Section 3.3.7, the active suspension system is optimized with respect to ride comfort, suspension travel, road holding, and the actuator power requirement as given in equation (3.61).

$$J = E\left\{ \int_{t_0}^{T} \left[\ddot{y}_2^2 + \rho_1 (y_2 - y_1)^2 + \rho_2 (y_1 - y_0)^2 + \rho_3 u^2 \right] dt \right\} \tag{3.61}$$

The main concept of adaptive control of the active suspension is the decision of emphasizing whether ride comfort or road holding, and the amount of this emphasis according to the type of the road and vehicle velocity. The actuator power given by the 4th term, J_4, in the performance index (3.61) should be kept as small as possible for minimizing the cost of the active suspension. However, keeping J_4 too small gives an almost passive system where the active control has no effect; keeping J_4 too high on the other hand gives an active system with high performance but at the expense of increasing the power requirement of the active suspension and therefore increasing the cost of the system. As a result, the weighing coefficient ρ_3 should be selected accordingly. In (3.61), the weighing coefficient of the ride comfort term J_1 is 1 and the emphasis on ride comfort and road holding is managed by adjusting the values of weighing coefficients of suspension travel ρ_1 and road holding term ρ_2. Therefore, gain scheduling adaptive control algorithm considered in this study schedules the weighing coefficients according to the road type and vehicle velocity. For each set of weighing coefficients, different control gain matrices are found and the controller is redesigned as the road type or vehicle velocity changes.

To sum up the discussion above, performance of the suspension depends on the numerical values of the weighing coefficients ρ_1, ρ_2 and ρ_3 in the performance index. This dependence can be determined numerically for the vehicle model by computing the control gain matrix (3.71) and the steady state average behavior of the system (3.75) for various weighing constants. So, in this part the covariance matrix has been calculated by solving (3.75), then the performance index terms (J_1, J_2, J_3, J_4) have been found from (3.80), (3.82), (3.84), and (3.86).

In Figures 18 and 19 below, the dependence of all performance index parts (J_1, J_2, J_3, J_4) and the performance index itself (J) upon the weighing coefficient ρ_3 as it changes from 10^{-1} to 10^{-10}. Vehicle velocity has been taken as $V = 30$ m/s throughout this analysis.

In Figure 18, the weighing coefficients of suspension travel and road holding terms have been taken as $\rho_1 = 10^5$ and $\rho_2 = 10^6$. For $\rho_3 = 10^{-7}$, it can be seen that suspension travel term J_2 and road holding term J_3 reach their minimum values while ride comfort term J_1 reaches its maximum value. Therefore for $\rho_1 = 10^5$ and $\rho_2 = 10^6$, road holding is emphasized over ride comfort and these two values with $\rho_3 = 10^{-7}$ can be used when the vehicle is on smooth road with a relatively high velocity where low tire deflection and high pitch and roll stability are required.

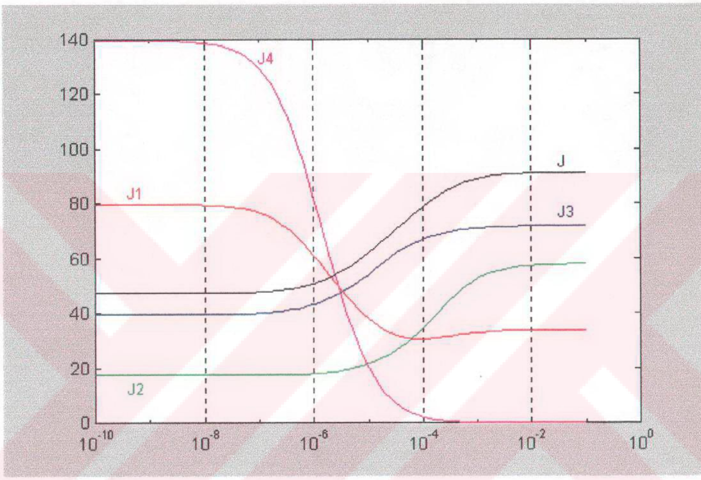

Figure 18. Performance index and its terms versus ρ_3 when $\rho_1 = 10^5$ and

$\rho_2 = 10^6$

In Figure 19, the weighing coefficients of suspension travel and road holding terms have been taken as $\rho_1 = 10^2$ and $\rho_2 = 10^3$. For $\rho_3 = 10^{-8}$, it can be seen that ride comfort term J_1 reaches its near-minimum value while suspension travel term J_2 and road holding term J_3 are increased considerably.

Therefore for $\rho_1 = 10^2$ and $\rho_2 = 10^3$, ride comfort is emphasized over road holding and these two values with $\rho_3 = 10^{-8}$ can be used when the vehicle is on rough road with a relatively low velocity where ride comfort is required.

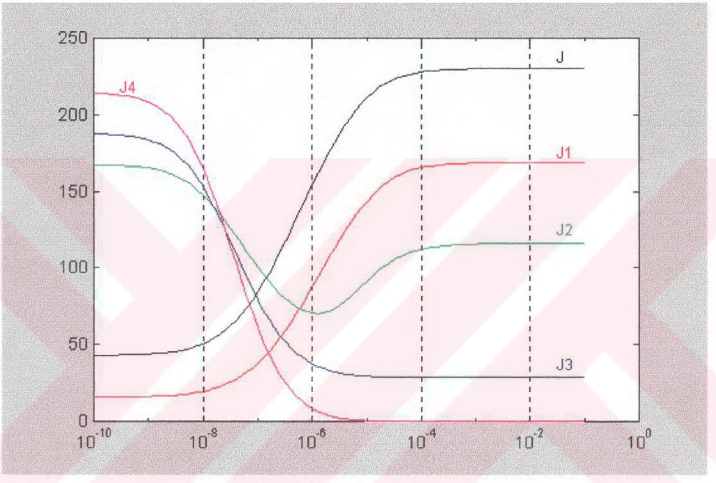

Figure 19. Performance index and its terms versus ρ_3 when $\rho_1 = 10^2$ and $\rho_2 = 10^3$

According to the results obtained above, the weighing coefficients ρ_1, ρ_2 and ρ_3 have been defined for the 6 road types, which have been considered in this study, in Table 3 given below.

Table 3. Weighing Coefficients according to the road types

Type of road/terrain	ρ_1	ρ_2	ρ_3
Very Good Asphalt Road	10^5	10^6	10^{-7}
Good Asphalt Road	$10^{4.4}$	$10^{5.4}$	10^{-7}
Average Asphalt Road	$10^{3.8}$	$10^{4.8}$	10^{-7}
Poor Asphalt Road	$10^{3.2}$	$10^{4.2}$	10^{-7}
Paved Road	$10^{2.6}$	$10^{3.6}$	10^{-8}
Dirt Road	10^2	10^3	10^{-8}

It can be seen form Figures 18 and 19 that for certain values of ρ_3, all the performance index part parts become stationary; therefore a further increase of active control force does not reduce the performance index. Figure 18 shows that the use of active suspension with $\rho_1 = 10^5$ and $\rho_2 = 10^6$ can reduce the tire deflection and the suspension travel 3 and 1.8 times respectively, in comparison to those of a passive system, but at the expense of increasing the ride comfort measure 2 times. Similarly, Figure 19 shows that the use of active suspension with $\rho_1 = 10^2$ and $\rho_2 = 10^3$ can improve ride comfort 8 times in comparison to those of a passive system, however the measures of tire deflection and suspension travel are increased 1.5 and 6 times respectively.

5.3 CASE STUDIES

In this section, case studies that exhibit the performance of the vehicle with the active suspension in different road excitation types.

According to the given vehicle parameter data given in Section 5.1, the matrices (3.10), (3.11), (3.12) that do not change with the changing velocity and road spectra are obtained as follows.

$$F_x = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -60 & -3 & 60 & 3 \\ 0 & 0 & 0 & 1 \\ 600 & 30 & -6600 & -30 \end{bmatrix} \quad G_x = \begin{bmatrix} 0 \\ 0.001 \\ 0 \\ -0.01 \end{bmatrix} \quad D_x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6000 \end{bmatrix}$$

Now let the vehicle move on a paved road with parameters $\alpha = 0.45\ m^{-1}$, and $\sigma^2 = 300 \cdot 10^{-6}\ m^2$ as given in Table 2, with a velocity of $V = 20$ m/s (72 km/h).

As stated in Sections 3.2.1 and 3.2.2, white noise $\xi(t)$ with a constant power spectral density S_0, and intensity I is passed through the filter of (3.15), and the road surface irregularities $w(t)$ is obtained where S_0 is given by 3.27 and I is given by (3.28) as below.

$$S_0 = \frac{\sigma^2 \cdot \alpha \cdot V}{\pi} \quad \text{and} \quad I = 2 \cdot \alpha \cdot V \cdot \sigma^2$$

So, for a paved road input, the constant power spectral density level S_0 of the white noise process $\xi(t)$ has been calculated as below.

$$S_0 = \frac{300 \cdot 10^{-6} \cdot 0.45 \cdot 20}{\pi} = 8.6 \cdot 10^{-4}$$

White noise input with a constant power spectral density level $8.6 \cdot 10^{-4}$ has been obtained by MATLAB® and the PSD has been plotted as shown in the Figure 20 below. The method of obtaining white noise signals with a desired power spectral density level has been explained in Chapter IV.

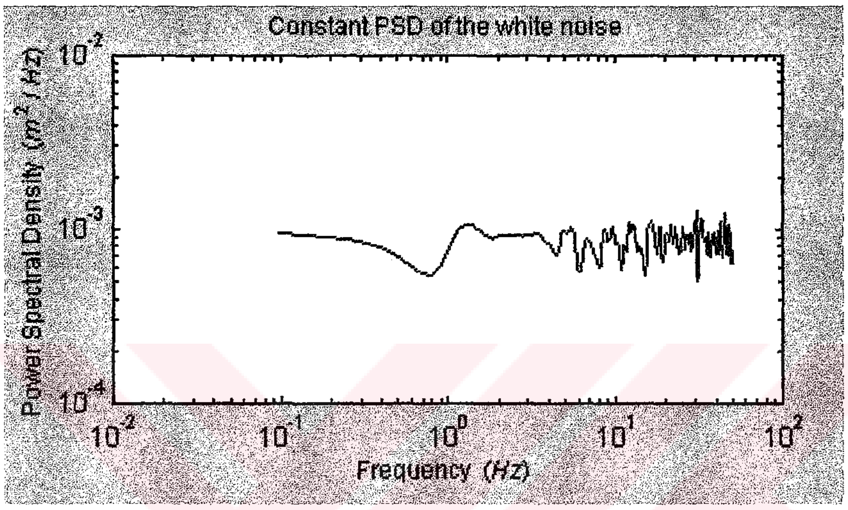

Figure 20. Constant PSD level of the white noise

Now that the desired white noise signal has been obtained, the road excitation $w(t)$ can be calculated form (3.15) as

$$\dot{w}(t) = -0.45 \cdot 20 \cdot w(t) + \xi(t)$$

Knowing the road excitation $w(t)$, one can obtain the reconstructed road excitation $w_{rec}(t)$ from (3.13) as

$$w_{rec} = x_3 + 1.67 \cdot 10^{-3} \dot{x}_2 + 1.67 \cdot 10^{-4} \dot{x}_4$$

Then, the power spectral density $S_{wrec}(t)$ of the reconstructed road excitation $w_{rec}(t)$ is obtained and compared with the power spectral density functions (3.14) calculated for the 6 road/terrain types defined before for the purpose of road type identification. This comparison is managed by the help of the RTC matrix defined in Section 3.2.2. The RTC matrix has been obtained for this case study as

$$RTC = 10^{-6} \cdot [0.453 \quad 0.402 \quad 0.311 \quad 0.184 \quad 0.019 \quad 0.647]$$

As shown in the RTC matrix above, the minimum element is the 4th one, which corresponds to the Paved Road. Consequently, the road type has been identified to be of paved type, which can also be seen in the Figure 21 below.

For the paved road, the combined system equation has been obtained from (3.57) as below

$$\dot{\hat{x}} = F \cdot \hat{x} + G \cdot u + D \cdot \xi \tag{3.57}$$

where

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -60 & -3 & 60 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 600 & 30 & -6600 & -30 & 6000 \\ 0 & 0 & 0 & 0 & -9 \end{bmatrix} \qquad G = \begin{bmatrix} 0 \\ 0.001 \\ 0 \\ -0.01 \\ 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

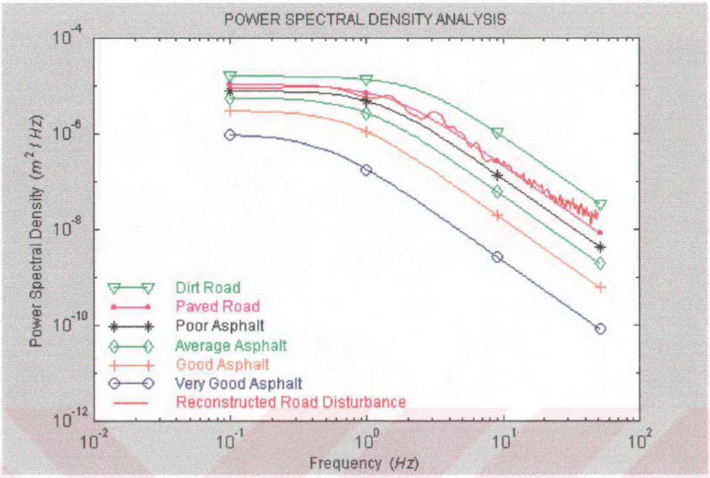

Figure 21. Paved road identification by PSD analysis

Now that the road type has been identified, the active suspension system can be optimized and the results in frequency and time domains can be obtained.

The road type is paved which is a rough road and the aim should be to ensure low variance of vehicle body acceleration which means an improved ride comfort and a relatively soft stiffness and low damping ratio of the suspension are required. Therefore, on a paved road ride comfort is emphasized over road holding and the aim is to minimize the sprung mass acceleration variance.

Accordingly, the weighing coefficients are selected from Table 3 as $\rho_1 = 10^{2.6}$, $\rho_2 = 10^{3.6}$ and $\rho_3 = 10^{-8}$. The performance index that should be minimized has been given by (3.64) as

$$J = \lim_{T \to \infty} \frac{1}{T} E\left\{ \int_0^T \begin{bmatrix} \hat{x}^T(t) & u^T(t) \end{bmatrix} \begin{bmatrix} A & N \\ N^T & B \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ u(t) \end{bmatrix} dt \right\} \tag{3.64}$$

where

$$A = \begin{bmatrix} 3998 & 180 & -3998 & -180 & 0 \\ 180 & 9 & -180 & -9 & 0 \\ -3998 & -180 & 7979 & 180 & -3981 \\ -180 & -9 & 180 & 9 & 0 \\ 0 & 0 & -3981 & 0 & 3981 \end{bmatrix}$$

$$N = \begin{bmatrix} -0.06 \\ -0.003 \\ 0.06 \\ 0.003 \\ 0 \end{bmatrix} \quad \text{and} \quad B = 1.01 \cdot 10^{-6}$$

As explained in Section 3.3.8, the algorithm that MATLAB® software uses in solving the optimal control problems requires that the system (3.57) be completely controllable. So, the complete controllability condition given in Appendix B should be checked. In Appendix B, the system (B.1) is completely controllable if

$$rank\begin{bmatrix} G & FG & \cdots & F^{n-1}G \end{bmatrix} = n$$

For the case study, after the matrices F and G are substituted, it has been found that

$$rank\begin{bmatrix} 0 & 0.001 & -0.033 & -0.43 & 187 \\ 0.001 & -0.033 & -0.43 & 187 & -8814 \\ 0 & -0.01 & 0.33 & 56 & -3856 \\ -0.01 & 0.33 & 56 & -3856 & -246113 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 4$$

Since the rank is not equal to 5, the system (3.57) is not completely controllable and the procedure given in Appendix D should be conducted.

Then, the control gain matrix of (3.71) is obtained as

$$K = \begin{bmatrix} -39268 & 3293 & 50332 & 2104 & -1473 \end{bmatrix}$$

and the optimum control $u^*(t)$, which minimizes the performance index (3.64), is obtained as

$$u^*(t) = -K \cdot \hat{x}(t)$$

Now that the optimal control that minimizes the performance index (3.64) emphasizing ride comfort over road holding has been found, the reduction in the sprung mass acceleration can be seen in frequency domain. The power spectral density of sprung mass acceleration of active and passive systems can be seen in Figure 22 below.

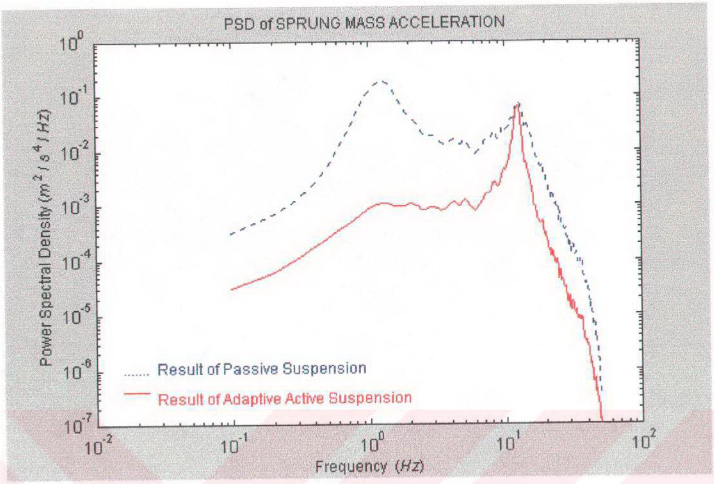

Figure 22. PSD functions of sprung mass acceleration

It is clearly seen in Figure 22 that the sprung mass acceleration has been considerably reduced around the sprung mass resonant frequency (body bounce frequency) by the adaptive control of the active suspension compared to that of the passive suspension. The passive system shows sprung mass resonance around 1 Hz, and this resonance has been eliminated by the active system. In Figure 22, it is seen that with control characteristics optimized for ride comfort, there is no significant change in response at the unsprung mass resonant frequency (wheel hop frequency) near 10 Hz. This is because of the fact that for the suspension to exert control forces which will reduce unsprung mass motions, these forces must be reacted against the sprung mass, thus increasing the ride vibrations at this region. The improvement of the ride comfort can also be seen in time domain as below in Figure 23.

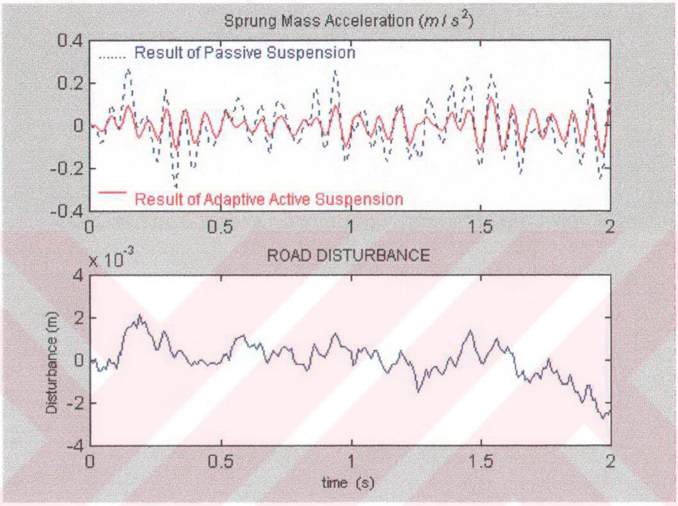

Figure 23. Reduction of sprung mass acceleration and road excitation in time domain

Ride comfort is improved at the expense of decreasing road-holding performance. Knowing that the handling is affected by system response at the wheel hop frequency, the decrease in road holding performance can be seen in Figures 24 and 25 in terms of suspension travel and tire deflection, respectively.

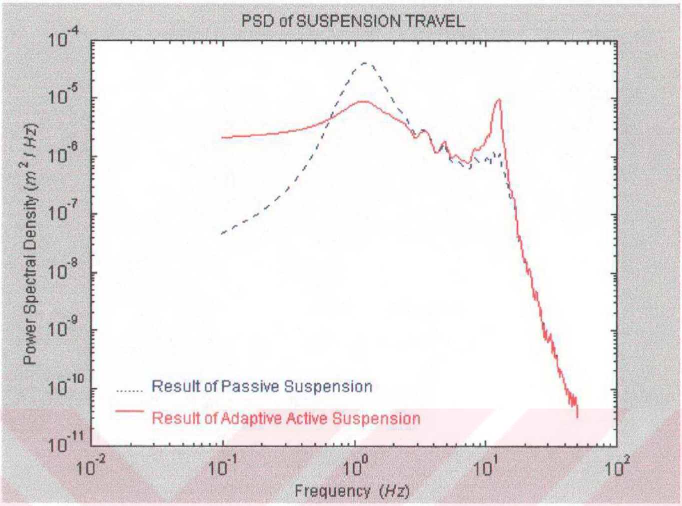

Figure 24. The decrease in road holding at the wheel hop frequency in terms of suspension travel

The same analysis can be conducted for a vehicle moving on a 'Very Good Asphalt Road' with parameters $\alpha = 0.15\ m^{-1}$, and $\sigma^2 = 9 \cdot 10^{-6}\ m^2$ as given in Table 2, with a velocity of $V = 30$ m/s (108 km/h). The road disturbance obtained as the output of filter (3.15) with white noise input, is compared with the 6 road/terrain types and the road type is identified as shown in Figure 26.

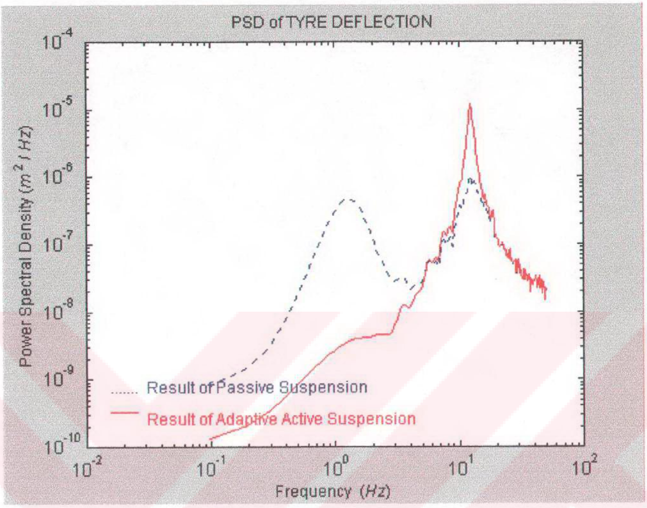

Figure 25. The decrease in road holding at the wheel hop frequency in terms of suspension travel

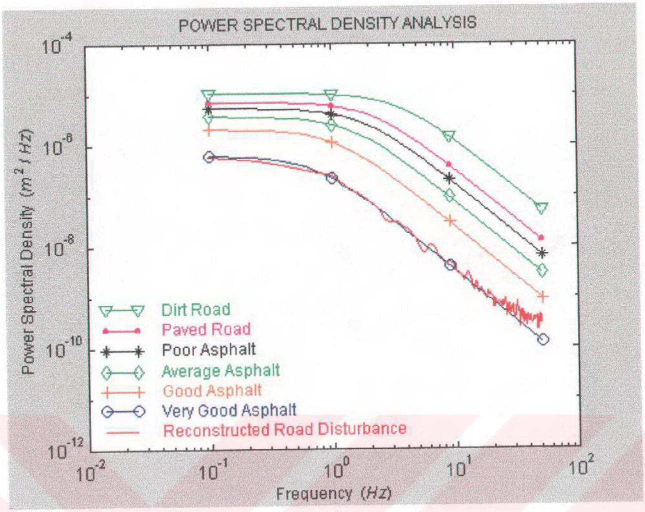

Figure 26. Very good asphalt road identification by PSD analysis

Now that the road type has been identified, the active suspension system can be optimized and the results in frequency and time domains can be obtained.

The road type is very good asphalt which is a smooth road and the aim should be to ensure low variance of suspension travel and tire deflection which means an improved road holding and pitch and roll stability, and a relatively high stiffness and high damping ratio of the suspension are required. Therefore, on a very good asphalt road, road holding is emphasized over ride comfort and the aim is to minimize the tire deflection and suspension travel variances.

Accordingly, the weighing coefficients are selected from Table 3 as $\rho_1 = 10^5$, $\rho_2 = 10^6$ and $\rho_3 = 10^{-7}$, and the results in frequency domain can be seen in Figures 27 and 28 in terms of suspension travel and tire deflection variances respectively.

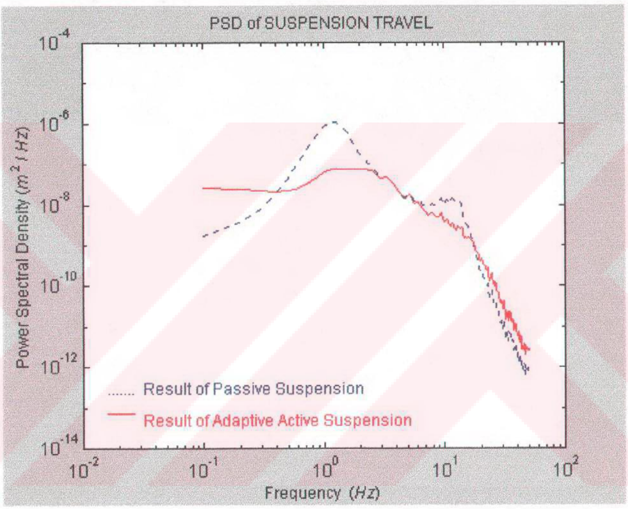

Figure 27. The improvement in road holding at the wheel hop frequency in terms of suspension travel

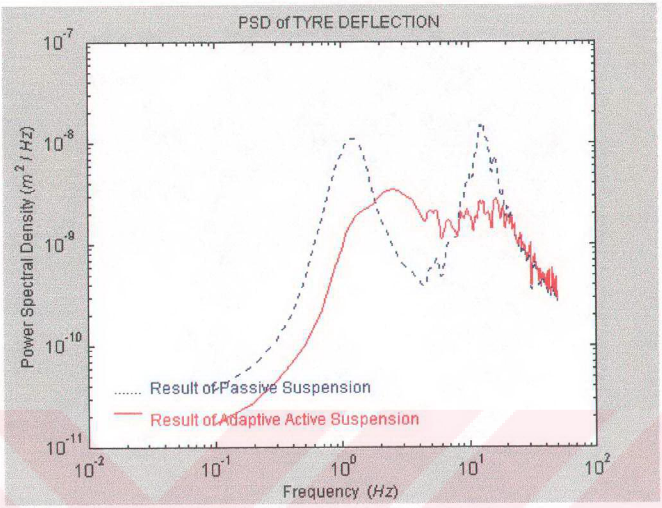

Figure 28. The improvement in road holding at the wheel hop frequency in terms of tire deflection

It is clearly seen in Figures 27 and 28 that the suspension travel and tire deflection have been considerably reduced around the unsprung mass resonant frequency (wheel hop frequency) by the adaptive control of the active suspension compared to that of the passive suspension. The passive system shows unsprung mass resonance around 10 Hz, and this resonance has been eliminated by the active system. The improvement of the road holding can also be seen in time domain as below in Figures 29 and 30 in terms of suspension travel and tire deflection.

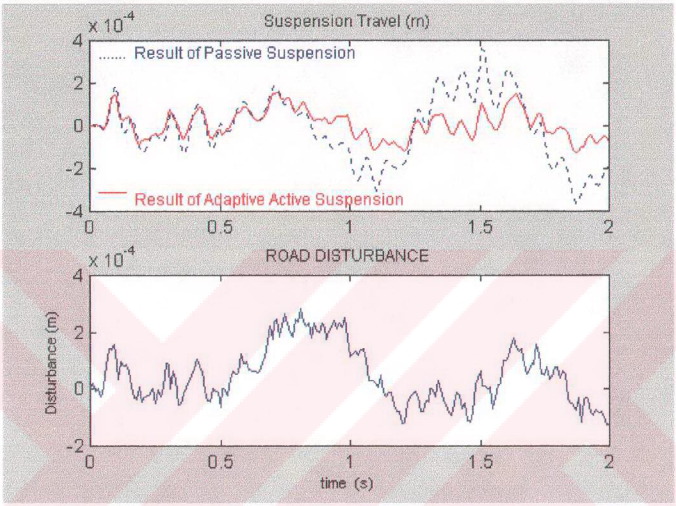

Figure 29. The improvement in road holding in time domain in terms of suspension travel

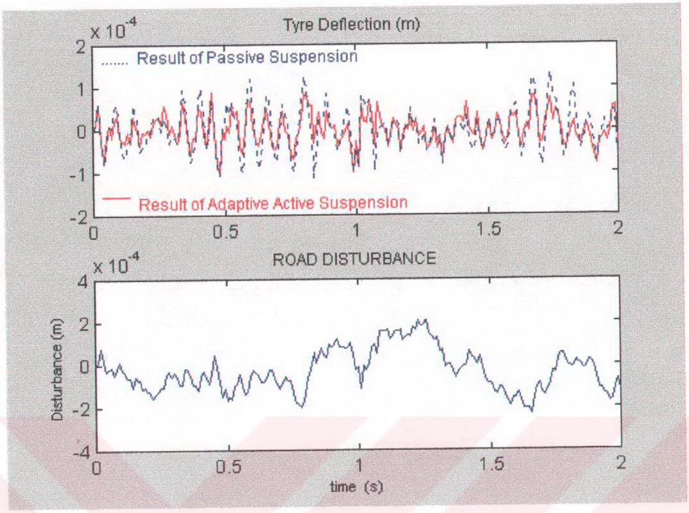

Figure 30. The improvement in road holding in time domain in terms of tire deflection

Road holding is improved at the expense of decreasing ride comfort performance. Knowing that ride comfort is affected by system response around the frequencies 1-5 Hz, the decrease in ride comfort performance can be seen in Figure 31 in terms of sprung mass acceleration.

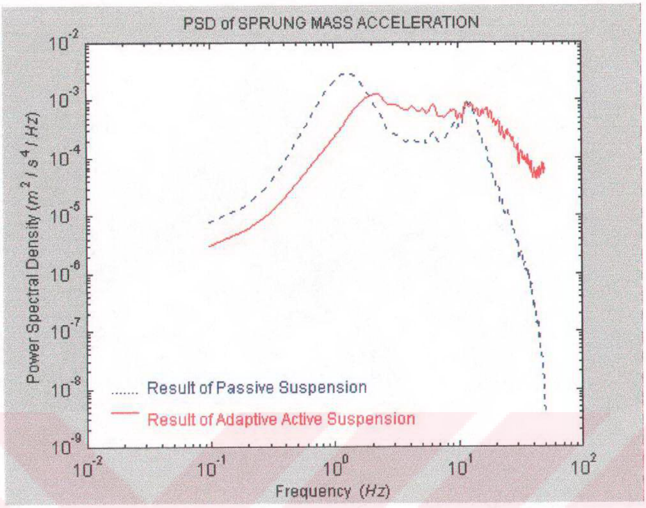

Figure 31. The decrease in ride comfort in terms of sprung mass acceleration

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE STUDIES

Stochastic optimal control theory, as used in this study, is a powerful tool for the analysis of active vehicle suspensions. Using infinite-horizon quadratic performance index provides a compact analytical solution with constant control gain matrices and relatively low design and computational time. Thus required optimal values can easily be reached. However, the use of optimal control is not always adequate. Particularly in cases of varying parameters and/or inputs, the implementation of adaptive control strategies is required for continuity of optimal operation.

In this study, it has been shown that considerable improvement can be obtained in both road holding and ride comfort performances on varying road surfaces and at different vehicle speeds, when compared to those of the passive suspension. By the use of adaptive control of active vehicle suspension, tire deflection and suspension travel values have been decreased considerably on smooth roads where the vehicle moves with high velocities. Similarly, body acceleration values have been minimized and ride comfort has been improved on rough roads where the vehicle velocity is relatively low.

There are two major effects that decrease the performance of the adaptive control implementation in this study.

- The dynamics of the actuator has been neglected, but it can influence the dynamics of the whole system considerably, especially in the high frequency region.

- Complete measurements have been assumed, but in practice the measurement of the states is not always complete and measurement errors are inevitable. As a consequence, Kalman filters may become necessary for such systems.

In this study, the adaptation procedure has been oriented towards the compensation of changes in excitation characteristics and vehicle velocity rather than the changes in vehicle model parameters. A further study would be to design a recursive adaptive controller that would compensate for the changes in vehicle parameters as well. In such a controller the controller parameters should be updated recursively. The adaptive controller, for such a purpose, can be of Model Reference or Self-Tuning type.

The identification algorithm for the purpose of road type identification requires that the vehicle move on the road to be identified for a certain time until the measurements are completed. A further study would be to design a recursive road type identification algorithm, which would introduce a smooth and fast transition in the response of the vehicle. A further study can be done on preview control of the active suspensions in which the road excitation is viewed beforehand and the suspension system is prepared for the changes in road characteristics.

In this study, steady state response of the vehicle has been studied and the vehicle velocity has been assumed to be constant on each road type. A further study would be to analyze the non-stationary response of the vehicle with variable velocities and to control the active suspension of such a suspension.

As observed in the results from the theoretical analysis of Chapter 5, adaptive control of active vehicle suspensions can provide considerable improvements in ride comfort and road holding. However, although it is beyond the scope of this study, it is worth to mention here that the physical application is likely to introduce a number of difficulties and sacrifices from the attainable improvements. Measurement of the road disturbance by accelerometers is a cheap way of identifying the type of road –since it prevents the requirement of sonar, laser, or image processing equipment; it requires the vehicle to traverse across the road for a minimum amount of time. This, of course, increases the reaction time of the suspension system in case of alterations of the road types. Also, to provide the coherence between the electronics and mechanical parts would probably be difficult because of the nonlinearity formed by the tolerances that always exist in the mechanical joints or by the leakage in the hydraulic system.

The developments in the microprocessor and thus computer technology enable the development of the adaptively controlled active suspensions. The usage limitations of the adaptive control applications in terms of price and size of the processors are being eliminated day by day. In the future, it is hoped that broader developments will be done in this field, and adaptive active vehicle suspensions will get the chance of being applied to even mid-class passenger vehicles.

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APPENDICES

A. COMPLEX FREQUENCY RESPONSE FUNCTION

The complex frequency response $H(\omega)$ of a first order linear filter given by (3.15) as

$$\dot{w}(t) = -\alpha \cdot V \cdot w(t) + \xi(t)$$

is obtained by the following procedure.

Let $\xi(t) = e^{i \cdot \omega \cdot t}$ (A.1)

where ω is the frequency in rad/s

If $H(\omega)$ is the frequency response function between the input $\xi = e^{i \cdot \omega \cdot t}$ and the output w, one can write

$$w = H(\omega) \cdot e^{i \cdot \omega \cdot t} \quad \text{(A.2)}$$

By differentiating both sides of the equation, one obtains

$$\dot{w} = i \cdot \omega \cdot H(\omega) \cdot e^{i \cdot \omega \cdot t} \quad \text{(A.3)}$$

Substituting A.3 into the filter equation, one obtains

$$i \cdot \omega \cdot H(\omega) \cdot e^{i \cdot \omega \cdot t} = -\alpha \cdot V \cdot H(\omega) \cdot e^{i \cdot \omega \cdot t} + e^{i \cdot \omega \cdot t} \quad \text{(A.4)}$$

and, thus

$$i \cdot \omega \cdot H(\omega) = -\alpha \cdot V \cdot H(\omega) + 1 \tag{A.5}$$

So, the complex frequency response function $H(\omega)$ is obtained as

$$H(\omega) = \frac{1}{i \cdot \omega + \alpha \cdot V} \tag{A.6}$$

B. CONDITIONS FOR COMPLETE CONTROLLABILITY AND STABILIZIBILITY

Consider the continuous-time system

$$\dot{x} = Fx + Gu \tag{B.1}$$

where x : state vector n-vector

u : control signal (scalar)

F : $n \times n$ matrix

G : $n \times 1$ matrix

The system described by (B.1) is said to be state controllable at $t = t_0$ if it is possible to construct an unconstrained control signal that will transfer an initial state to any final state in a finite time interval $t_0 \le t \le t_1$. If every state is controllable, then the system is said to be completely state controllable.

The system (B.1) is completely controllable if the following condition holds:

$$rank\begin{bmatrix} G & FG & \cdots & F^{n-1}G \end{bmatrix} = n \tag{B.2}$$

The system (B.1) is completely stabilizible if there exists a coordinate basis change such that

$$F = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix} \qquad G = \begin{bmatrix} G_1 \\ 0 \end{bmatrix}$$

with the pair $[F_{11}, \quad G_1]$ completely controllable, and, if F_{22} has nontrivial (nonzero) dimension, $\operatorname{Re} \lambda_i(F_{22}) < 0$ for all i.

C. PERFORMANCE INDICES WITH CROSS PRODUCT TERMS

The cross product terms that exist in some performance indices arise when power into a system is minimized. In this study, since the performance index parts are composed of squares of sprung mass acceleration, suspension travel, road holding and control forces, cross product terms like $2x_1N_1u$ appear in the performance index.

In this part, time-invariant regulator problem with performance index having cross-product terms will be solved.

Consider the system

$$\dot{x}(t) = F \cdot x(t) + G \cdot u(t) \qquad \text{with } x(t_0) \text{ given} \tag{C.1}$$

where F and G are constant. Define the performance index as

$$J(x(t_0), u(\bullet), t_0) = \int_{t_0}^{\infty} \begin{bmatrix} x^T & u^T \end{bmatrix} \cdot \begin{bmatrix} A & N \\ N^T & R \end{bmatrix} \cdot \begin{bmatrix} x \\ u \end{bmatrix} dt \tag{C.2}$$

After some manipulations, this performance index can also be written as

$$J(x(t_0), u(\bullet), t_0) = \int_{t_0}^{\infty} \left\{ x^T Ax + u^T Ru + 2x^T Nu \right\} dt \tag{C.3}$$

If we manipulate the inner side of the parenthesis, we can obtain

$$x^T Ax + u^T Ru + 2x^T Nu = \left(u + R^{-1}N^T x\right)^T R\left(u + R^{-1}N^T x\right) + x^T\left(A - R^{-1}N^T\right)x \qquad \text{(C.4)}$$

Let's make the definition

$$u_1 = u + R^{-1}N^T x \qquad \text{(C.5)}$$

Then, the original system (C.1) becomes equivalent to

$$\dot{x}(t) = \left(F - GR^{-1}N^T\right)\cdot x(t) + G \cdot u_1(t) \qquad \text{with } x(t_0) \text{ given} \qquad \text{(C.6)}$$

and the original performance index (C.3) becomes equivalent to

$$J(x(t_0), u(\bullet), t_0) = \int_{t_0}^{\infty} \left\{u_1{}^T Ru_1 + x^T\left(A - NR^{-1}N^T\right)x\right\}dt \qquad \text{(C.7)}$$

where R is positive definite.

For the same initial states, the trajectories of both systems (C.1) and (C.6) are the same. Also the values of the two performance indices (C.3) and (C.7) are the same. So, the following results can be obtained from the above discussion.

- The optimal controls u_1^* and u^* are related by

$$u_1^* = u^* + R^{-1}N^T x \qquad \text{(C.8)}$$

- The optimal performance indices for the two problems are the same.

- The closed loop trajectories are the same. That is to say, if the optimal control for the system (C.6) is $u_1^* = -K_1 x$ and the optimal control for the system (C.1) is $u^* = -Kx$, one obtains $F - GK = F - GR^{-1}N^T - GK_1$.

Then for the system (C.6), the optimal control is given by

$$u_1^*(t) = -R^{-1}G^T Px(t) \tag{C.9}$$

where P is the solution of the matrix Riccatti equation given below.

$$P\left(F - GR^{-1}N^T\right) + \left(F - GR^{-1}N^T\right)^T P - PGR^{-1}G^T P + \left(A - NR^{-1}N^T\right) = 0 \tag{C.10}$$

So, the optimal control for the time-invariant regulator problem with performance index having cross-product terms is found by combining equations (C.8) and (C.9) as

$$u^*(t) = -R^{-1}\left[G^T P + N^T\right]x(t) \tag{C.11}$$

In a more compact form, the solution of the **infinite time, time-invariant regulator problem with performance index having cross-product terms** is given below

Consider the system below

$$\dot{x}(t) = F \cdot x(t) + G \cdot u(t) \qquad \text{with } x(t_0) \text{ given} \tag{C.1}$$

where F and G are constant and system (C.1) is completely controllable. Define the performance index as

$$J(x(t_0),u(\bullet),t_0) = \int_{t_0}^{\infty} \begin{bmatrix} x^T & u^T \end{bmatrix} \cdot \begin{bmatrix} A & N \\ N^T & R \end{bmatrix} \cdot \begin{bmatrix} x \\ u \end{bmatrix} dt \quad \text{(C.2)}$$

The constant control law for this problem is obtained as

$$u^*(t) = -R^{-1}\left[G^T P + N^T\right]x(t) \quad \text{(C.11)}$$

where P is the solution of the matrix Riccatti equation

$$P\left(F - GR^{-1}N^T\right) + \left(F - GR^{-1}N^T\right)^T P - PGR^{-1}G^T P + \left(A - NR^{-1}N^T\right) = 0 \quad \text{(C.10)}$$

D. MATRIX SEPARATION TO SOLVE CONTROLLABILITY PROBLEM

General algorithms for solving the matrix Riccati equations require the state-space systems be completely controllable. In this study, the state equations representing the vehicle model and the road excitation filter are not completely controllable. To overcome this difficulty and separate the controllable part of the system from the uncontrollable part, first one should partition the matrices F, G and D as below.

$$F = \begin{bmatrix} F_x & D_x \\ 0 & F_w \end{bmatrix} \qquad G = \begin{bmatrix} G_x \\ 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ D_w \end{bmatrix} \tag{D.1}$$

Performance index is chosen as

$$J = \int_{t_0}^{\infty} \begin{bmatrix} x^T & u^T \end{bmatrix} \cdot \begin{bmatrix} A & N \\ N^T & R \end{bmatrix} \cdot \begin{bmatrix} x \\ u \end{bmatrix} dt \tag{D.2}$$

or,

$$J = \int_{t_0}^{\infty} \left\{ x^T A x + u^T R u + 2 x^T N u \right\} dt \tag{D.3}$$

The steady state matrix Riccati equation is

$$P\left(F-GR^{-1}N^{T}\right)+\left(F-GR^{-1}N^{T}\right)^{T}P-PGR^{-1}G^{T}P+\left(A-NR^{-1}N^{T}\right)=0 \quad \text{(D.4)}$$

Then the constant control law is

$$u^{*}(t)=-R^{-1}\left[G^{T}P+N^{T}\right]x(t) \quad \text{(D.5)}$$

Partition the 5×5 A and P matrices into four matrices of dimensions 4×4, 4×1 and 1×1 as below.

$$A=\begin{bmatrix} A_{xx} & A_{xw} \\ A_{xw}^{T} & A_{ww} \end{bmatrix} \qquad P=\begin{bmatrix} P_{xx} & P_{xw} \\ P_{xw}^{T} & P_{ww} \end{bmatrix} \quad \text{(D.6)}$$

Also, partition 5×1 N matrix into two matrices of dimensions 4×1 and 1×1 as below.

$$N=\begin{bmatrix} N_{x} \\ 0 \end{bmatrix} \quad \text{(D.7)}$$

If (D.1), (D.6) and (D.7) are inserted into the matrix Riccati equation (D.4), one obtains

$$P\left(F-GR^{-1}N^{T}\right)=\begin{bmatrix} P_{xx} & P_{xw} \\ P_{xw}^{T} & P_{ww} \end{bmatrix}\cdot\left\{\begin{bmatrix} F_{x} & D_{x} \\ 0 & F_{w} \end{bmatrix}-\begin{bmatrix} G_{x} \\ 0 \end{bmatrix}\cdot R^{-1}\cdot\begin{bmatrix} N_{x}^{T} & 0 \end{bmatrix}\right\}$$

$$=\begin{bmatrix} P_{xx}\left(F_{x}-G_{x}R^{-1}N_{x}^{T}\right) & P_{xx}D_{x}+P_{xw}F_{w} \\ P_{xw}^{T}\left(F_{x}-G_{x}R^{-1}N_{x}^{T}\right) & P_{xw}^{T}D_{x}+P_{ww}F_{w} \end{bmatrix} \quad \text{(D.8)}$$

Multiplying by P from left, one obtains

$$\left(F-GR^{-1}N^T\right)^T P=\left\{\begin{bmatrix}F_x & D_x\\ 0 & F_w\end{bmatrix}-\begin{bmatrix}G_x\\ 0\end{bmatrix}\cdot R^{-1}\cdot\begin{bmatrix}N_x^T & 0\end{bmatrix}\right\}^T\cdot\begin{bmatrix}P_{xx} & P_{xw}\\ P_{xw}^T & P_{ww}\end{bmatrix}$$

$$=\begin{bmatrix}\left(F_x-G_xR^{-1}N_x^T\right)^T P_{xx} & \left(F_x-G_xR^{-1}N_x^T\right)^T P_{xw}\\ D_x^T P_{xx}+F_w P_{xw}^T & D_x^T P_{xw}+F_w P_{ww}\end{bmatrix} \tag{D.9}$$

$$PGR^{-1}G^T P=\begin{bmatrix}P_{xx} & P_{xw}\\ P_{xw}^T & P_{ww}\end{bmatrix}\cdot\begin{bmatrix}G_x\\ 0\end{bmatrix}\cdot R^{-1}\cdot\begin{bmatrix}G_x^T & 0\end{bmatrix}\cdot\begin{bmatrix}P_{xx} & P_{xw}\\ P_{xw}^T & P_{ww}\end{bmatrix}$$

$$=\begin{bmatrix}P_{xx}G_xR^{-1}G_x^T P_{xx} & P_{xx}G_xR^{-1}G_x^T P_{xw}\\ P_{xw}^T G_xR^{-1}G_x^T P_{xx} & P_{xw}^T G_xR^{-1}G_x^T P_{xw}\end{bmatrix} \tag{D.10}$$

$$A-NR^{-1}N^T=\begin{bmatrix}A_{xx} & A_{xw}\\ A_{xw}^T & A_{ww}\end{bmatrix}-\begin{bmatrix}N_x\\ 0\end{bmatrix}\cdot R^{-1}\cdot\begin{bmatrix}N_x^T & 0\end{bmatrix}$$

$$=\begin{bmatrix}A_{xx}-N_xR^{-1}N_x^T & A_{xw}\\ A_{xw}^T & A_{ww}\end{bmatrix} \tag{D.11}$$

When (D.8), (D.9), (D.10) and (D.11) are put into the matrix Riccati equation (D.4), one obtains

$$-P_{xx}\left(F_x-G_xR^{-1}N_x^T\right)-\left(F_x-G_xR^{-1}N_x^T\right)^T P_{xx}+P_{xx}G_xR^{-1}G_x^T P_{xx} -\left(A_{xx}-N_xR^{-1}N_x^T\right)=0 \tag{D.12}$$

$$-P_{xw}F_w-\left[\left(F_x-G_xR^{-1}N_x^T\right)^T-P_{xx}G_xR^{-1}G_x^T\right]P_{xw}-P_{xx}D_x-A_{xw}=0 \tag{D.13}$$

$$-F_w^T P_{xw}^T-P_{xw}^T\left[\left(F_x-G_xR^{-1}N_x^T\right)-G_xR^{-1}G_x^T P_{xx}\right]-D_x^T P_{xx}-A_{xw}^T=0 \tag{D.14}$$

$$P_{ww}F_w+F_w^T P_{ww}+\left(P_{xw}^T D_x+D_x^T P_{xw}+A_{ww}-P_{xw}^T G_xR^{-1}G_x^T P_{xw}\right)=0 \tag{D.15}$$

Here (D.12) is a matrix Riccati equation with unknown matrix P_{xx}, (D.13) and (D.14) are linear matrix equations with unknown matrices P_{xw} and P_{xw}^T, and finally, (D.15) is a Liapunov equation with unknown matrix P_{ww}

After P_{xx}, P_{xw} and P_{ww} are solved from the equations (D.12) to (D.15), they are inserted into the control law (D.5) as below.

$$u^* = -R^{-1} \cdot \left\{ \begin{bmatrix} G_x^T & 0 \end{bmatrix} \cdot \begin{bmatrix} P_{xx} & P_{xw} \\ P_{xw}^T & P_{ww} \end{bmatrix} + \begin{bmatrix} N_x^T & 0 \end{bmatrix} \right\} \cdot \begin{Bmatrix} x \\ w \end{Bmatrix} \tag{D.16}$$

Then,

$$u^* = -R^{-1} \cdot \begin{bmatrix} G_x^T P_{xx} + N_x^T & G_x^T P_{xw} \end{bmatrix} \cdot \begin{Bmatrix} x \\ w \end{Bmatrix} \tag{D.17}$$

or,

$$u^* = -R^{-1} \cdot \{ \left[G_x^T P_{xx} + N_x^T \right] \cdot x + \left[G_x^T P_{xw} \right] \cdot w \tag{D.18}$$

Since $u^* = -Kx$, the control gain matrix K is found to be

$$K = R^{-1} \left[\left(G_x^T P_{xx} + N_x^T \right) | \, G_x^T P_{xw} \right] \tag{D.19}$$

E. COVARIANCE RESPONSE OF A SYSTEM DRIVEN BY WHITE NOISE

Consider the system

$$\dot{x}(t) = F(t) \cdot x(t) + G(t) \cdot w(t) \tag{E.1}$$

where $w(t)$ is a Gaussian purely random process, with

$$E[w(t)] = \overline{w}(t) \tag{E.2}$$

The mean value vector of the states is denoted by

$$\overline{x}(t) = E[x(t)] \tag{E.3}$$

and the covariance matrix of the states is denoted by

$$X(t) = E\{[x(t) - \overline{x}(t)][x(t) - \overline{x}(t)]^T \tag{E.4}$$

The mean value of $x(t)$ is determined by considering the state equation as (by taking the ensemble average):

$$\frac{d}{dt}[\overline{x}(t)] = F(t) \cdot \overline{x}(t) + G(t) \cdot \overline{w}(t) \tag{E.5}$$

Subtracting (E.5) from (E.1) and postmultiplying the result by $[x(t)-\bar{x}(t)]^T$ yields

$$\left\{\frac{d}{dt}[x(t)-\bar{x}(t)]\right\}[x(t)-\bar{x}(t)]^T = F(t)\cdot[x(t)-\bar{x}(t)][x(t)-\bar{x}(t)]^T + + G(t)\cdot[w(t)-\bar{w}(t)][x(t)-\bar{x}(t)]^T \tag{E.6}$$

Adding the transpose of (E.6) to (E.6) gives:

$$\begin{aligned}\frac{d}{dt}\left\{[x(t)-\bar{x}(t)][x(t)-\bar{x}(t)]^T\right\} = {} & F(t)\cdot[x(t)-\bar{x}(t)][x(t)-\bar{x}(t)]^T + \\ & [x(t)-\bar{x}(t)][x(t)-\bar{x}(t)]^T F^T(t) + \\ & + G(t)\cdot[w(t)-\bar{w}(t)][x(t)-\bar{x}(t)]^T + \\ & + [x(t)-\bar{x}(t)][w(t)-\bar{w}(t)]^T G^T(t)\end{aligned} \tag{E.7}$$

Taking the expected value (i.e. the ensemble average) of (E.7) and using the definition of $X(t)$ above (E.4),

$$\begin{aligned}\dot{X} = F(t)\cdot X + X\cdot F^T(t) + G(t)\cdot E\left\{[w(t)-\bar{w}(t)][x(t)-\bar{x}(t)]^T\right\} + \\ + E\left\{[x(t)-\bar{x}(t)][w(t)-\bar{w}(t)]^T\right\} G^T(t)\end{aligned} \tag{E.8}$$

Using the transition matrix $\Phi(t,\tau)$ of the linear dynamic system (E.1), we know that,

$$x(t)-\bar{x}(t) = \Phi(t,t_0)[x(t_0)-\bar{x}(t_0)] + \int_{t_0}^{t}\Phi(t,\tau)G(\tau)[w(\tau)-\bar{w}(\tau)]d\tau \tag{E.9}$$

Assume that

$$E\left\{\left[x(t_0)-\bar{x}(t_0)\right]\left[w(t)-\overline{w}(t)\right]^T\right\}=0 \tag{E.10}$$

That is, random deviations in initial conditions are uncorrelated with the random fluctuations of the forcing function.

Postmultiplying (E.9) by $\left[w(t)-\overline{w}(t)\right]^T$, taking the expected value of the result, and using (E.10), we get

$$E\left\{\left[x(t)-\bar{x}(t)\right]\left[w(t)-\overline{w}(t)\right]^T\right\}=\int_{t_0}^{t}\Phi(t,\tau)G(\tau)E\left\{\left[w(\tau)-\overline{w}(\tau)\right]\left[w(t)-\overline{w}(t)\right]^T\right\}d\tau \tag{E.11}$$

Examining (E.8) and (E.11), it is apparent that the only knowledge of $w(t)-\overline{w}(t)$ needed to determine the covariance matrix $X(t)$ is a weighted integral of its autocorrelation:

$$E\left\{\left[w(t)-\overline{w}(t)\right]\left[w(\tau)-\overline{w}(\tau)\right]^T\right. \tag{E.12}$$

Now the purely random process must be considered as a limiting case of a random process with a short correlation time. Hence, we assume a simple correlation function for (E.12), which will allow us to perform this limit operation:

$$E\left\{\left[w(t)-\overline{w}(t)\right]\left[w(\tau)-\overline{w}(\tau)\right]^T\right\}=\gamma(t)\cdot\exp\left(-\frac{|t-\tau|}{T}\right) \tag{E.13}$$

where T is a constant, very small compared to characteristic times of the transition matrix $\Phi(t,\tau)$ and $\gamma(t)$ is the covariance of $w(t)$; i.e.,

$$\gamma(t) = E\{[w(t) - \overline{w}(t)][w(t) - \overline{w}(t)]^T \tag{E.14}$$

Substituting (E.13) into (E.11), we may approximate $\Phi(t,\tau) \cong I$: unit matrix and replace t_0 by $-\infty$, since the correlation dies out rapidly as $|t-\tau|$ increases beyond T:

$$\begin{aligned} E\{[x(t) - \overline{x}(t)][w(t) - \overline{w}(t)]^T\} &\cong G(t)\cdot\gamma(t)\int_{-\infty}^{t}\exp\left(-\frac{|t-\tau|}{T}\right)d\tau \\ &= T\cdot G(t)\cdot\gamma(t) \end{aligned} \tag{E.15}$$

Using (E.15) and its transpose in (E.8), we have:

$$\dot{X} = F(t)\cdot X + X\cdot F^T + G(t)\cdot I(t)\cdot G^T(t) \quad \text{with } X(t_0) \text{ given} \tag{E.16}$$

where

$$I(t) = 2\cdot T\cdot\gamma(t) \tag{E.17}$$

Thus, $I(t)$ is a nonnegative definite matrix representing the integral of the correlation of the Gaussian purely random process $w(t)$.

As far as (E.16) is concerned, we could write (E.13) as:

$$E\{[w(t) - \overline{w}(t)][w(\tau) - \overline{w}(\tau)]^T\} = I(t)\cdot\delta(t-\tau) \tag{E.18}$$

where $\delta(t-\tau)$ is the Dirac delta function.

Comparing (E.18) with (E.13), the quantity $I(t)$ may be regarded as,

$$I(t) = \lim_{T \to 0} \{2 \cdot T \cdot \gamma(t)\} \qquad \text{(E.19)}$$

where T is the correlation time of the random process and $\gamma(t)$ is the covariance of the random process. Clearly, we have

$$\gamma(t) \to \infty \quad \text{as} \quad T \to 0$$

in such a way that the limit (E.19) is finite.

Thus, the Gaussian purely random process has a very large covariance and very short correlation time.

When I is a constant, $w(t)$ is called a stationary purely random process. The Fourier transform of the correlation (E.18) with respect to $(t - \tau)$ is simply I; that is the spectrum is white. For this reason, the Gaussian purely random process is frequently called white noise.