# A STUDY ON PRE-SERVICE ELEMENTARY MATHEMATICS TEACHERS' SUBJECT MATTER KNOWLEDGE AND PEDAGOGICAL CONTENT KNOWLEDGE REGARDING THE MULTIPLICATION AND DIVISION OF FRACTIONS

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# ABSTRACT

A STUDY ON PRE-SERVICE ELEMENTARY MATHEMATICS TEACHERS' SUBJECT MATTER KNOWLEDGE AND PEDAGOGICAL CONTENT KNOWLEDGE REGARDING THE MULTIPLICATION AND DIVISION OF FRACTIONS

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The purpose of this study was to examine pre-service mathematics teachers' subject matter knowledge, pedagogical content knowledge, and the relationships between the two on multiplication and division of fractions. For this purpose, pre-service teachers' understanding of key facts, concepts, principles, and proofs, their knowledge on common (mis)conceptions and difficulties held by the elementary students, their strategies of teaching these concepts, and representations they use to reason their understanding on multiplication and division of fractions were examined.

Qualitative case study was performed where; data was collected from the pre-service teachers at the end of the spring semester of 2004-2005. Pre-service teachers were senior students enrolled in a teacher education program at a public university.

Results revealed that pre-service teachers could easily symbolize and solve the basic questions on multiplication and division of fractions. However, in terms of pre-service teachers' interpretation and reasoning of key facts and principles on multiplication and division of fractions, their subject matter knowledge could not be regarded as conceptually deep. Furthermore, although pre-service teachers have strong belief that they should teach multiplication and division of fractions conceptually, where the logical background of the operations is explained, they do not have sufficient knowledge to represent and explain these topics and relationships conceptually.

Keywords: Mathematics education, pre-service elementary mathematics teachers, subject matter knowledge, pedagogical content knowledge, multiplication and division of fractions

# ÖZ

İLKÖĞRETİM MATEMATİK ÖĞRETMEN ADAYLARININ KESİRLERDE ÇARPMA VE BÖLMEYE İLİŞKİN ALAN VE PEDAGOJİK İÇERİK BİLGİLERİ ÜZERİNE BİR ÇALIŞMA

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Bu çalışmanın amacı ilköğretim matematik öğretman adaylarının kesirlerde çarpma ve bölmeye ilişkin alan bilgileri, pedagojik alan bilgileri ve bu bilgiler arasındaki ilişkiyi incelemektir. Diğer bir deyişle, bu çalışmada öğretmen adaylarının kesirlerde çarpma ve bölmeye ilişkin kavram, prensip ve ispatlara yönelik anlamaları, 6 ve 7. sınıf öğrencilerinin bu konularda sahip olabilecekleri kavram ve kavram yanılgıları hakkındaki bilgileri, bu konuların öğretiminde kullandıkları stratejiler ve kesirlerde çarpma ile bölmeyi anlamlandırmalarına yönelik gösterimleri incelenmiştir.

Çalışma 2004-2005 bahar dönemi sonunda, öğretmen adayları ile nitel durum çalışması yapılarak gerçekleştirilmiştir. Öğretmen adayları, bir devlet üniversitesinde, öğretmen yetiştirme programına devam eden son sınıf öğrencileridir.

Çalışmanın sonuçları, öğretmen adaylarının kesirlerde çarpma ve bölmeyle ilgili problemleri kolaylıkla sembolize edip çözebildiklerini göstermiştir. Buna karşın, öğretmen adaylarının bu kavramları yorumlama ve anlamdırmalarındaki alan bilgilerinin yeterince derin olmadığı belirlenmiştir. Bu sonuçlara ek olarak, öğretmen adaylarının kesirlerde çarpma ve bölmeye ilişkin kavramların mantığına vurgu yapılması gerektiğine yönelik inançlarının yüksek olmasına rağmen, bu kavramların açıklama ve gösterimine yönelik bilgilerinin yeterli olmadığı belirlenmiştir.

Anahtar Kelimeler: Matematik eğitimi, ilköğretim öğretmen adayları, konu alan bilgisi, pedagojik içerik bilgisi, kesirlerde çarpma ve bölme

To my mother Seval IŞIKSAL,

a wonderful mathematics teacher,

who makes me love mathematics

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## LIST OF ABBREVIATIONS

EME: Elementary Mathematics Education

LMRTM: Lesh Multiple Representations Transition Model

MDFQ: Multiplication and Division of Frcations Questionnaire

METU: Middle East Technical University

NCTM: National Council of Teachers of Mathematics

PCK: Pedagogical Content Knowledge

PCKg: Pedagogical Content Knowing

PST: Pre-service Teacher

SMK: Subject Matter Knowledge

# CHAPTER I

# INTRODUCTION

"The teacher need not only understand *that* something is so; the teacher must further understand *why* it is so" (Shulman, 1986, p. 9).

Teaching mathematics well is a complex endeavor, and there are no easy recipes for success. Effective teachers must know and understand the mathematics they are teaching, and they must flexibly draw on that knowledge. While challenging and supporting the students, teachers need to understand the gap between what their students know and what they need to learn. In addition, effective teachers should know the concepts and topics that students often have difficulty with, and the ways to clarify students' common misunderstandings (National Council of Teachers of Mathematics [NCTM], 2000). In order to improve instruction, more importance should be given to the understanding of the concepts where NCTM (1989) have called for a decreased emphasis on computation and facts and a greater emphasis on conceptual understanding, problem solving, applications, and communication of ideas.

Fennema, Carpenter, Franke, Levi, Jacobs, and Empson (1996) mentioned that one major way to improve mathematics instruction and learning is to help teachers to understand the mathematical thought processes of their students. They emphasized that knowledge is not static and acquired outside classrooms in workshops, but dynamic and ever growing, and can probably only be acquired in the context of teaching mathematics. Fennema et al., (1996) added that children should not be asked to practice procedures they do not understand, and that the way to find out if they understand is to ask them to explain their thinking.

Educators generally agree that learning occurs when one challenges a student's existing conceptions. Therefore, those calling for improvements in mathematics education state that knowing students' common conceptions and misconceptions about the subject matter is essential for teaching (NCTM, 1989, 1991). Simon and Blume (1994) stated that the teachers' and teacher educators' ability to make useful interpretations of their students' reasoning can be enhanced if they are aware of the reasoning processes and conceptual connections that are important for their students. They also added that, as teacher educators, we must question the extent to which we are preparing teachers for this important aspect of their teaching practice.

It is obvious that one of the essential elements in improving instruction and students' understanding in the mathematics classroom is the role of the teacher. NCTM (1991) emphasized that "Teachers must help every student develop conceptual and procedural understandings of numbers, operations, geometry, measurement, statistics, probability, functions, and algebra and the connections among ideas" (p. 21). Thus, in order to develop conceptual and procedural understanding of the students, teachers should understand the content from both perspectives.

Most of the studies that concentrate on the teacher's role in increasing the effectiveness of instruction emphasized the importance of teacher knowledge on related content areas. Researches emphasized that teachers' content knowledge is a major determinant of mathematics instruction and learning. The importance of teachers' subject matter knowledge and pedagogical knowledge, which are also the main concerns of this study, emphasized by many researchers (Ball, 1990a; Crespo & Nicol, 2006; Fennema & Franke, 1992; Hill, Schilling, & Ball, 2004; Tirosh & Graeber, 1991; Tirosh, 2000).

Even (1990) mentioned that mathematics teachers who have deficiencies in subject matter knowledge are likely to pass their misconceptions and misunderstandings on to those they teach. In contrast, a teacher who has solid mathematical knowledge for teaching is more capable of helping his/her students achieve a meaningful understanding of the subject matter. Therefore, students' understanding of the topic goes beyond procedures; they also comprehend concepts. In order to increase the teachers' knowledge, researchers concentrate on the programs for teacher education since teachers, like their students, are individuals who learn differently and change differently (Fennema et al., 1996; Niess, 2005; Crespo & Nicol, 2006).

Tirosh (2000) stated that a major goal in teacher education programs should be to promote development of pre-service teachers' knowledge of common ways children think about the mathematics topics the teacher will teach. Since many of tomorrow's in-service teachers are today's pre-service teachers, great emphasize is given to the pre-service teachers' knowledge of subject matter knowledge and pedagogical knowledge. She mentioned that the experience acquired during the course of teaching is the main but not the only source of teachers' knowledge of students' common conceptions and misconceptions. Pre-service teachers' own experiences as learners together with their familiarity with relevant developmental and cognitive research could be used to enhance their knowledge of common ways of thinking among children. Tirosh further added that in analyzing pre-service teachers' knowledge of students' ways of thinking about specific mathematical topics, we should take into account the effects of subject matter knowledge and pre-service teachers' beliefs about mathematics, mathematics instruction and about learning mathematics. Tirosh added:

> Knowing that without understanding why in the context of teachers' knowledge of students' ways of thinking is no more meaningful for teachers than such knowledge in the context of mathematic content. Thus, teachers' knowledge of children's alternative conceptions should be linked to their understanding of general and specific sources of such conceptions. (p. 22)

Thus, as stated by Tirosh (2000), teachers' subject matter knowledge is also important in affecting students' conceptions. Similarly, NCTM (1991) acknowledged the role of subject matter knowledge and pedagogical content knowledge in effective mathematics teaching and in preparation of mathematics teachers. It was emphasized that teachers of mathematics should develop their content knowledge, provide multiple perspectives on students, and know to represent mathematics topics, concepts, and procedures.

> Teachers with only superficial knowledge of their subject matter will have little flexibility in their pedagogical choices….by contrast, teachers who have mastered the rich interconnections and multiple forms of knowledge found in a subject area will have the substantive control of the subject needed to develop the kinds of activities and strategies involved in teaching for understanding. (Cohen, McLaughlin, & Talbert, 1993, p. 3)

Based on these ideas, again we could question pre-service teachers' subject and pedagogical knowledge of specific mathematics topics. What teachers know or how they teach could be two essential questions that should be answered before trying to improve our mathematical instruction.

Elementary teachers' knowledge of mathematics gains importance since they are the primary source of supplying formal mathematical training to the students. Therefore, educators should know what skills and knowledge pre-service teachers possess and what skills they lack in order to design curricula. In addition, it is clear that if the conceptions and misconceptions that pre-service teachers possess can be detected during pre-service training, their teaching performance would improve (Johnson, 1998). It's reasonable that the teacher who has a lack of understanding about specific topic will be unable to transfer the correct knowledge to the students, so it's important to pay attention to the teacher's knowledge.

### 1.1 Statement of the Problem

Students, especially in elementary school, have difficulties constructing their own knowledge on specific topics in mathematics, since the topics are abstract for them. Most of the students just memorize the specific rules related to a subject without questioning them.

Operations with fractions are one of these topics. Fractions and rational numbers are considered the most complex mathematical domain in elementary school mathematics. Although many students memorize the rote procedures needed to manipulate the symbols, they soon forget the procedures and thus find it difficult to learn fractions (Mack, 1990).

Multiplicative concepts are one of the topics where conceptual understanding is critical. "Children's and teachers' understanding of multiplicative concepts—multiplication, division, ratio, rational number, and others—is important to their ability to gain mathematical understanding" (Behr, Khoury, Harel, Post, & Lesh, 1997, p. 48). Similarly, McDiarmid and Wilson (1991) stated that understanding of division is a basic conceptual knowledge that is vital in order for students to understand a variety of ideas in mathematics.

Division of fractions is often considered the most mechanical and least understood topic in elementary school (Fendel, 1987; Payne, 1976, Tirosh, 2000). Carpenter et al. (1988) stated that children's success rates on various tasks related to such division are usually very low. Therefore, pre-service teachers' understanding of the topic helps students to understand other mathematical concepts including fractions, decimals, and ratios (McDiarmid & Wilson, 1991). However, studies on mathematical understanding of elementary teachers and pre-service elementary teachers show that most of the teachers have many misconceptions about meaning in mathematics. Teachers carried out mathematical algorithms and rules based on memorization and generally were not prepared to teach the mathematical subject matter entrusted to them (Joyner, 1994; Khoury & Zazkis, 1994, Tirosh, 2000).

Much research has been accomplished on the knowing, learning, and teaching of the multiplicative concepts including multiplication, division, ratio, and rational numbers during the last decade, but much work remains to be done (Behr, Khoury, Harel, Post, & Lesh, 1997). Behr et al. (1992) stated that the elementary school curriculum lacks some basic concepts and principles to relate multiplicative structures necessary for later learning in the upper grades. In addition, multiplicative concepts that are presented in the middle grades are isolated and are not interconnected. They added that there is a lack of problem situations that provide a wide range of experience for children to develop less-constrained models of multiplication and division. According to Behr et al., these deficiencies arise from the lack of analytical understanding of how multiplicative concepts interrelate from theoretical, mathematical, and cognitive perspectives. Graeber, Tirosh, and Glover (1989) stated that teachers were the cornerstone of the learning cycle of students who might affect misconceptions and misunderstandings on multiplication and division. Therefore, efficient strategies were needed for training teachers to examine and control the impact that misconceptions and primitive models have on their thinking and on their student's thinking. In addition, Ball (1990b) stated that the typical eighth-grade textbook introduces the division of fractions by giving the "invert and multiply" rule. Little or no attention is given to the meaning of fractions, and no connection is made between division of whole numbers and division of fractions. Therefore, we can ask the following questions:

Why do teachers prefer to teach multiplication and division of fractions procedurally with little attention to the meaning of the mathematical expressions?

Do teachers have enough knowledge to teach these topics conceptually?

The importance of teachers' knowledge of students' learning has encouraged many researchers to examine pre-service and inservice teachers' subject matter knowledge of division of fractions where studies revealed that teachers' knowledge is largely procedural (Ball, 1990a, 1990b; Graeber, Tirosh, & Glover, 1989; Ma, 1996, 1999; McDiarmid & Wilson, 1991; Tirosh, 2000; Tirosh & Graeber, 1991, Simon, 1993). Changes in the education programs are recommended, but before changing the curriculum, the details of the knowledge of pre-service teachers should be analyzed.

Since pre-service teachers constitute the population of the tomorrow's teachers, the conceptions pre-service teachers hold of fundamental operations should be of concern to teacher educators (Tirosh & Graeber, 1991). This study attempts to illuminate pre-service teachers' understanding of multiplication and division of fractions relative to the meaning, conceptions, and misconceptions that students have. This study also examines the connections between pre-service teachers' subject matter knowledge and pedagogical knowledge with the mathematical models and strategies used in transferring this knowledge to the students.

As with all studies, the focus of this study is narrowed down with respect to the area of interest of content that is investigated. Multiplication and division of fractions were chosen, since this is a mathematical topic that is known to be difficult for students and that need a deeper understanding of subject matter knowledge on the part of the teacher. Ball (1990b) stated that division is a central concept in mathematics at all levels and figures predominantly throughout the K-12 curriculum. Students can learn about rational and irrational numbers' connections among the four operations while studying with division. In addition, pre-service teachers were selected since it is believed that findings of the study in terms of knowledge structures will give valuable implications to policy makers and teacher educators in terms of designing content of the courses in teacher education programs.

Thus, this study aims to answer the following questions:

1. What is the nature of pre-service elementary mathematics teachers' subject matter knowledge about multiplication and division of fractions?

- How do pre-service elementary mathematics teachers construct their understanding of key facts, concepts, principles, and proofs related to multiplication and division of fractions?

2. What is the nature of pre-service mathematics teachers' pedagogical content knowledge about multiplication and division of fractions?

- What do pre-service elementary mathematics teachers know about common conceptions and misconceptions/difficulties held by the elementary ($6^{th}$ and $7^{th}$ grade) students related to multiplication and division of fractions?

- What do pre-service elementary mathematics teachers know about the possible sources of misconceptions/difficulties held by elementary ($6^{th}$ and $7^{th}$ grade) students related to multiplication and division of fractions?

- What kind of strategies do pre-service elementary mathematics teachers use to overcome the misconceptions/difficulties held by elementary ($6^{th}$ and $7^{th}$ grade) students related to multiplication and division of fractions?

- What kind of strategies do pre-service elementary mathematics teachers use to explain/verify the key facts, concepts, principles, and proofs related to multiplication and division of fractions?

- What kind of representations and modeling do pre-service elementary mathematics teachers use to reason their understanding of multiplication and division of fractions?

3. How are pre-service elementary mathematics teachers' subject matter knowledge and pedagogical content knowledge on multiplication and division of fractions interconnected?

- How does pre-service elementary mathematics teachers' subject matter knowledge influence their knowledge of common conceptions and misconceptions/difficulties held by elementary ($6^{th}$ and $7^{th}$ grade) students?

- How does pre-service elementary mathematics teachers' subject matter knowledge influence their knowledge of the possible sources of misconceptions/difficulties held by elementary ($6^{th}$ and $7^{th}$ grade) students related to multiplication and division of fractions?

- How does pre-service elementary mathematics teachers' subject matter knowledge influence the kind of strategies they use to overcome the misconceptions/difficulties held by elementary ($6^{th}$ and $7^{th}$ grade) students?

- How does pre-service elementary mathematics teachers' subject matter knowledge influence the strategies they use to explain/verify the key facts, concepts, principles, and proofs?

- How does pre-service elementary mathematics teachers' subject matter knowledge influence the kind of representations and modeling that they use to reason their understanding of multiplication and division of fractions?

## 1.2 Definition of Important Terms

The research questions consist of several terms that need to be defined constitutively and operationally.

*Pre-service elementary mathematics teachers*

Pre-service elementary teachers are senior students majoring in mathematics education. Pre-service teachers are in the four-year undergraduate teacher education program and they have taken all the courses that teacher education program offer to them. Pre-service elementary mathematics teachers are teacher candidates who are going to teach mathematics from fourth grade to eighth grade in primary and middle schools after their graduations.

*Fraction*

Lamon (1999) stated that a “fraction represent one or more parts of a unit that has been divided into some number of equal-sized pieces” (p. 27).

A fraction is represented by symbols in the form of $\frac{a}{b}$, b ≠ 0, that means *a* parts out *b* parts, each part having the same size. A fraction representing a positive rational number is in the form of $\frac{a}{b}$, such that *a* and *b* are natural numbers.

Therefore, in this study “fraction” represent the positive rational number in the form of $\frac{a}{b}$, such that *a* and *b* are whole numbers.

*Subject Matter Knowledge*

According to Schwab (as cited in Shulman, 1986), subject matter knowledge consists of both substantive and the syntactic structures. Substantive knowledge consists of key facts, concepts, principles, and the explanatory framework of the discipline whereas syntactic structures are rules of evidence and proof that guide inquiry in a discipline.

Ball (1991) proposed another framework for teachers’ subject matter knowledge. She made a distinction between knowledge *of* mathematics and knowledge *about* mathematics. She used the term knowledge of mathematics to denote one’s understanding of mathematics topics, concepts, and procedures, as well as the organizing structures and connections within mathematics. Knowledge about mathematics, in contrast, refers to one’s understanding of the nature and discourse of mathematics—where it comes from, how it changes, how truth is established, and what it means to know and do mathematics.

In this study, subject mater knowledge refers to a pre-service elementary mathematics teacher’s own knowledge regarding key facts, concepts, principles, and proofs regarding expressions concerning multiplication and division of fractions. For this study, by using the term facts, concepts, principles and proofs, I referred specific relations related to multiplication and division operations on fractions. Pre-service teachers were supposed to write expressions, solve questions related to understanding of the basic facts ($\frac{3}{4} \times \frac{4}{3} = 1$) and principles (e.g. $\frac{2}{9} \div \frac{1}{3} = \frac{2 \div 1}{9 \div 3} = \frac{2}{3}$), and reason the underlying proofs on multiplication and division operations regarding expressions "multiply numerators and denominators in multiplication operation" and "invert and multiply in division operation".

*Pedagogical Content Knowledge*

Shulman (1986) defined pedagogical content knowledge as "the ways of representing and formulating the subject that make it comprehensible to others" (p. 9). Shulman also added that it includes "an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons" (p. 9). Grossman (1989) expanded on Shulman's definition of pedagogical content knowledge as including four central components: (1) teachers' overarching conceptions of what it is means to teach a particular subject, (2) teachers' knowledge of curricular materials and curriculum materials, (3) knowledge of students' understanding and potential misunderstandings of a subject area, and (4) teachers' knowledge of strategies and representations for teaching particular topics.

In this study, pedagogical knowledge refers to pre-service teachers' knowledge of common conceptions and misconceptions/difficulties and the possible source of these conceptions and misconceptions that sixth and seventh grade students have on solving multiplication and division of fractions. In addition, it refers to pre-service teachers' knowledge of a variety of representations, models, and strategies useful for teaching multiplication and division of fractions.

*Modeling*

In this research study, the term modeling was used to refer pre-service teachers' representations of their understanding of multiplication and division operations of fractions. For instance, for the multiplication operation, modeling refers to repeated addition, multiplicative compare, area concepts, and Cartesian product and for division operation it refers measurement and partitive modeling. Detailed description for these models was given in the Chapter 2.

*Representations*

Johnson (1998) stated that the "Mathematical model of a number is a representation of the number in a context" (p. 11) and added that a model may consist of concrete items, a verbal story, a picture, or a symbol.

According to Johnson, fractional representation of a rational number consists of the following models:

A concrete model consists of manipulatives that can be arranged to represent a number. For example, for a concrete set of 12 pencils where 5 of them are red and the 7 of them are blue, the part of the red pencils is $\frac{5}{12}$ of the set of pencils.

A verbal model, similar to the concrete model, the words are used to define the set where no manipulative or pictures are used. That is, I have 12 books, 5 are red and 7 are blue, $\frac{5}{12}$ of the books are red.

A pictorial model includes a sketch, picture, or drawing to show the amount of unit that can be quantified. For example, a rectangle may represent a unit and when the rectangle separated into twelve parts, each having the same size, and five parts are shaded, then the shaded portion represents $\frac{5}{12}$ of the rectangle.

A symbolic model is in the $\frac{a}{b}$ form. For example, $\frac{5}{12}$ means five parts out of a unit consisting of 12 parts or of division of 5 by 12.

Behr, Lesh, Post, and Silver (1983) stated that region (area) models, discrete (set) models and number line models are commonly used to represent fractions. The representation system, also known as the Lesh Transition Model, consists of five modes of representations: pictures, written symbols, real-world situations, manipulative aids ands spoken symbols.

In this study, the term representation used to refer to images that pre-service teachers use to explain their understanding of multiplication and division of fractions. Representations involved pictorial representations (number line, area models, discrete models, figures, picture, graph or sketch), manipulatives, written symbols, and real-life examples.

*Strategies*

Skemp (1978) introduced two terms types of understanding "relational understanding" and "instrumental understanding." Skemp stated:

> By the former is meant what I have always meant by understanding, and probably most readers of this article: knowing both what to do and why. Instrumental understanding I would until recently not have regarded as understanding at all. It is what I have in the past described as 'rules without reasons', without realizing that for many pupils and their teachers the possession of such a rule, and ability to use it, was what they meant by 'understanding. (p. 9)

In most recent discussions, the term "relational understanding" has been replaced by the term "conceptual understanding." and the term "instrumental understanding" by the term "procedural understanding." Hiebert and Lefevre (1986) defined conceptual knowledge as "characterized most clearly as knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete piece of information" (p. 3-4) whereas procedural knowledge "… is made up of two distinct parts. One part is composed of the formal language, or symbol representation system, of mathematics. The other part consists of the algorithms, or rules, for completing mathematical tasks" (p. 6).

In this research study, strategies that pre-service elementary mathematics teachers use to overcome the misconceptions/difficulties held by elementary grade students and strategies that pre-service elementary mathematics teachers use to explain/verify the key facts, concepts, principles, and proofs on multiplication and division of fractions were investigated. Thus, pre-service teachers' strategies refers to the methodology, ways, approaches that they used to overcome difficulties and approaches they used to explain/verify the key facts, concepts, principles, and proofs on multiplication and division of fractions. Pre-service teachers' strategies are categorized as conceptual and procedural strategies based on their relations to conceptual understanding and procedural understanding.

### 1.3 Significance of the Study

In recent years, teachers' knowledge of subject matter they teach has attracted increasing attention from policymakers where emphasize is given to the highly qualified teachers. Programs are aimed at providing content-focused professional development intended to improve teachers' content knowledge (Hill, Rowan, & Ball, 2005)

According to NCTM (2000), "Effective teaching requires knowing and understanding mathematics, students as learners, and pedagogical strategies" (p. 17). Mathematics teachers not only need to have sufficient content knowledge of mathematics, but also should have pedagogical knowledge. Teachers need to know why mathematical statements are true, how to represent mathematical ideas in multiple ways, what is involved in an appropriate definition of a term or concept, and methods for appraising and evaluating mathematical methods, representations, or solutions (Hill, Schilling, & Ball, 2004)

Despite many calls for rich content and pedagogical content knowledge for teachers, there is a considerable body of research suggesting that novice teachers often do not possess the content and pedagogical knowledge to teach for understanding in their respective disciplines (Ball & McDiarmid, 1990; Brown & Borko, 1992). Rarely are teachers treated as learners who actively construct understanding, therefore, many high school and college students including pre-service teachers graduate with limited conceptual understanding of the subject matter taught (Ball, 1990a).

Haser (2006) who investigated pre-service and in-service teachers' mathematics related beliefs and perceived effect of middle school mathematics education program on these beliefs, mentioned that cross-sectional data of the sophomore, junior, and senior pre-service teachers' beliefs about the nature of, teaching and learning of mathematics did not differ due to the grade level in the program. She added that courses in elementary mathematics education program were not specifically designed to challenge pre-service teachers, beliefs and did not have strong or long-lasting impact on pre-service teachers' belief. She stated that program lacked the continuity and consistency of the courses that challenges the pedagogical content knowledge of pre-service teaches (Haser, 2006). Teacher educators are no exception. They tend to take pre-service teachers' subject mater knowledge for granted and just provide the techniques and materials for presenting lessons (Ball & Feiman-Nemser, 1988).

The teacher with weak mathematics subject matter knowledge is not well placed to take into account the structure of the subject in planning teaching and learning sequences where weak subject matter knowledge associated with poor classroom teaching. If teachers believe that mathematics is principally a subject of rules and routines which have to be remembered, then their own approach to unfamiliar problems will be constrained, and this may have an impact on their teaching. If teachers lack confidence in their subject matter knowledge, then they may avoid risky situations in the classroom and inhibit in responding to children's unexpected questions (Goulding, Rowland, & Barber, 2002).

Analyses of the research findings revealed the fact that knowledge structures of teachers are important constructs where teachers' mathematical knowledge was significantly related to student achievement (Hill et al., 2005).

Researchers have recognized the need for more in-depth qualitative measures of teachers' conceptual framework of subject matter to enlighten the discussion of teachers' subject matter knowledge, its formation, and its potential impacts on instructional practice (Lederman, Gess-Newsome, & Latz, 1994).

Based on the literature review above, it is believed that more attention should be give to teacher education programs. In addition, in order to develop instructional activities that enhance pre-service teachers' knowledge on particular topics that leads students learning, it is necessary to obtain a better understanding of the nature of mathematical knowledge that pre-service teachers bring to the teacher education program. It is believed that exploration of the subject matter knowledge and pedagogical content knowledge of pre-service teachers concerning multiplication and division of fractions give invaluable implications to the mathematics educators and policy makers in designing the course content of the teacher education programs.

In addition, while several research studies concerning pre-service mathematics teachers' knowledge of multiplication and division of rational numbers have been conducted in various countries including Israel and United States, none has been conducted in Turkey. In addition, besides subject matter knowledge and pedagogical content knowledge, the relationship of both knowledge is also important in increasing the quality of mathematics instruction. In the above studies although subject matter knowledge or pedagogical content knowledge were investigated individually, those topics could not be treated separately. This study also aims to investigate the nature of pre-service teachers' subject matter and pedagogical content knowledge and how pre-service teachers' subject matter knowledge affects their pedagogical knowledge structures.

It's significant to investigate the Turkish pre-service teachers' subject matter knowledge and pedagogical content knowledge in order to inform policy makers whether they should revise or improve teacher education programs with respect to needs of the pre-service teachers. Pre-service teachers, like all high school graduates, are placed in the universities according to their scores in Student Selection Examination (SSE). However, it was argued that this kind of admission is problematic since pre-service teachers' selection as teacher candidates to the education faculties is based on their scores in SSE without any specialties in the teaching profession. That is; pre-service teachers are accepted to the teacher education programs based on their performance on paper and pencil tests. So, among the people whoever gets at least a base score needed for the entrance to the education faculties of universities can be eligible to become a teacher candidate (Binbaşıoğlu, 1995; cf. Çakıroğlu & Çakıroğlu, 2003). After their graduations, these students are going to teach primary and middle school mathematics that construct the basis for higher mathematics. But, can we say that to get higher scores in SSE could be an indicator of being a good teacher? That is, having higher scores in SSE implies that these pre-service teachers have enough subject matter and pedagogical content knowledge to teach mathematics. Thus, we should evaluate pre-service teachers' subject matter knowledge and pedagogical content knowledge before their graduations from the educational faculties without taking into consideration their scores at SSE. Pre-service teachers' explicit statements about operations and even successful calculations can mask their misconceptions (Graeber & Tirosh, 1988) thus, before making such a refinement or changes in the teacher education programs, we should know what pre-service teachers know about a particular subject area and how competent they are in transferring their knowledge to the students.

In addition, this study needs special attention since it tries to improve the awareness of pre-service teachers on their competencies on specific mathematics topics in terms of both learning and teaching. Most pre-service teachers graduate from the universities without realizing their competencies and inadequacies in mathematics that they will actively teach after graduating from the program. By taking part in this study, pre-service teachers can have the chance to rate their subject matter knowledge and pedagogical content knowledge both theoretically and professionally before they will work as active mathematics teachers in primary and middle schools.

In other words, with this research study, I seek to stimulate the pre-service teachers' curiosity and disposition to raise and investigate other such questions of their taken-for-granted and unquestioned mathematical knowledge.

# CHAPTER II

# REVIEW OF THE LITERATURE

The purpose of this study was to examine the pre-service elementary mathematics teachers' subject matter knowledge and pedagogical content knowledge on multiplication and division of fractions. The underlying theories based on teacher knowledge structured the conceptual framework for this study. Even though there is no single unifying theory comprised the subject matter and pedagogical content knowledge, fundamentals from different theories provide an illumination for the theoretical framework for conceptualizing subject matter and pedagogical content knowledge. Theoretical background and related research studies were stated throughout the chapter. In addition, developmental model for pre-service teachers suggested based on the findings of the study was discussed at the end of the chapter. To state differently, in order to better understand the framework for this study, the review of the literature focused on specific areas of investigation based on the research questions namely: conceptual overview of subject matter knowledge and pedagogical content knowledge, studies on subject matter knowledge and pedagogical content knowledge on multiplication and division of fractions, studies on common misconceptions on multiplication and division of fractions, procedural and conceptual teaching strategies, and mode of models and representations.

## 2.1 Conceptual Framework for Types of Teachers' Knowledge

Good teaching demands that teachers should know many things, about teaching; about their students; and about the cultural, political, and social context within which they work (Ball & McDiarmid, 1990). Teachers who have conceptual understanding can answer students' questions about the meaning behind their symbolic manipulations. However, subject matter knowledge is not enough to achieve this goal. Teachers should also transform the content into representations that help students develop understanding (Shulman, 1986).

NCTM (1989) pointed out that teachers who teach for understanding would need to: understand mathematical concepts, structures, procedures, and their relationships; identify and interpret representations of mathematical concepts, structures, and procedures; reason and communicate mathematically; understand the nature of mathematics and the role of mathematics in different cultures, and develop disposition to do mathematics. If teachers have inaccurate information or see the knowledge in narrow ways, they might pass on these ideas to their students. They might fail to challenge students' misconceptions; they might use texts uncritically or change them inappropriately (Ball & McDiarmid, 1990).

Borko, Eisenhart, Brown, Underhill, Jones, & Agard (1992) carried out a project to examine the process of becoming a middle school mathematics teacher by small number of novice teachers throughout their final year of teacher preparation and first year of teaching. Their primary goal was to describe and understand the novice teachers' knowledge, beliefs, thinking, and actions related to the teaching of mathematics over the 2-year course of the study. Researchers stated that they mostly depend on Shulman's (1986) theoretical model of domains of teachers' professional knowledge to develop the knowledge and belief components of the framework. In their project they investigated knowledge and beliefs related to the mathematics, general pedagogy, mathematics-specific pedagogy, mathematics curriculum, learners and learning, elementary school, middle school, learning to teach, teachers as professionals, and self as teacher. In their model (Fig 2.1) double arrow between knowledge and beliefs (Box 1) and classroom thinking and action (Box 2) reflects their interest classroom environment and knowledge. The other purpose of the project was to describe the context for learning to teach created by the novice teachers' university teacher education experiences (Box 3) and their experiences in the public schools (Box 4) where teachers taught for the time of the project. Researchers stated that the university and the public school were two important sources related to learning to teach in most teacher preparation programs, they hypothesized that these two contexts were the major sources of external influence on the process of learning to teach. In the model the teachers' personal histories (Box 5) and the research project itself (Box 6) were expected to be the other sources influencing the process of learning to teach.

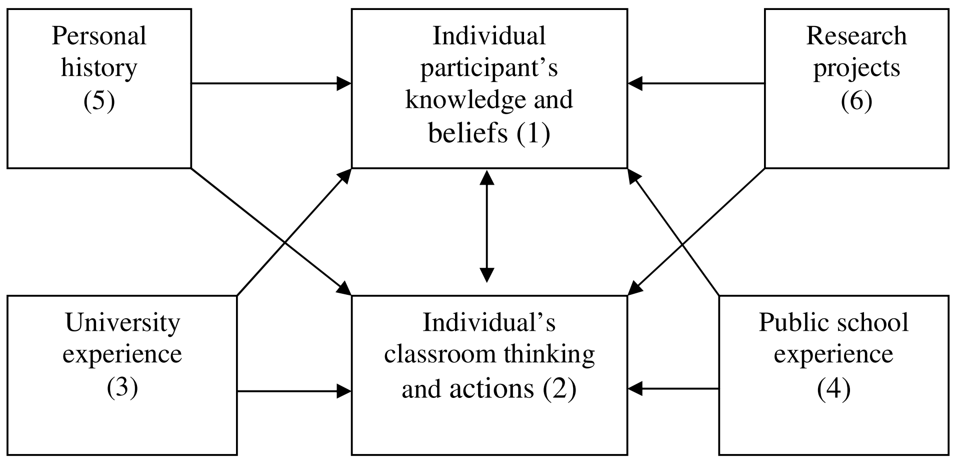



**Figure 2.1.** Borko et al.'s (1992, p. 200) model for becoming a middle school mathematics teacher.

In case of teaching, teachers' thinking is directly affected by their mathematical and pedagogical knowledge. This thinking then contributed to teachers' instructional behavior. If the goal of a mathematics teacher education program is to help teachers implement programs of instruction that increase students' mathematical understanding, then it is reasonable to expect that teacher have, and are continuing to develop, a well connected and extensive knowledge base to support their mathematics teaching (Howald, 1998). There was an assertion that knowledge related to subject matter and pedagogical knowledge was essential component of teachers' professional development (Ball & McDiarmid, 1990). In this research study, one part of Borko et al. (1992)'s model that was pre-service teachers' subject matter knowledge and pedagogical content knowledge was the main concern where; the importance of both knowledge on teaching was accepted by many researchers.

Shulman (1987) stated that knowledge of subject matter for teaching consists of two overlapping domains: subject matter knowledge and pedagogical content knowledge. His conceptualization has been accepted as a framework for most of the researches on teacher knowledge (Ball, 1990a, 1990b; Ball & McDiarmid, 1990; Borko et al., 1992; Shulman & Grossman, 1988). Howald (1998) mentioned that this theoretical model served an examination of the teachers' knowledge and the relationship among the domain knowledge. Content knowledge is important to being inventive and to creating worthwhile opportunities for learning that take into account learners' experiences, interests, and needs (Ball, 2000). Ball stated that we should reexamine what does content knowledge matters for good teaching. She added that subject matter knowledge is commonly known as knowledge that students are to learn. In other words, it was defined as what teachers teach, but inquiry was needed to bring subject matter knowledge and practice in together, which would enable teacher education to be more effective.

Similarly, Schifter and Fosnot (1993) pointed out that in order to teach mathematics for understanding "teachers must have an understanding of the mathematical concepts they are charged with teaching, including a sense of the connections that link these concepts to one another and to relevant physical context" (p. 13). However, there was considerable evidence suggesting that many prospective teachers, both elementary and secondary, did not understand their subjects in depth (McDiarmid et al., 1989). NCTM (1991) acknowledged the role of subject matter knowledge and pedagogical content knowledge in effective mathematics teaching and in preparation of mathematics teachers. It was emphasized that teachers of mathematics should develop their content knowledge, provide multiple perspectives on students, and know how to represent mathematics topics, concepts, and procedures. Thus, the importance of subject matter knowledge and pedagogical content knowledge were two essential components for being effective teachers as stated by many researchers. On the other hand, the definition of these two terms is not as definite as their importance in mathematics teaching. Ma (1996) stated that it was even harder to discriminate between subject content knowledge for teaching and pedagogical content knowledge because these two contents are close to each other. Similarly, Borko and Putnam (1996) stated that it is important to note that any categorization of teacher knowledge and beliefs is somewhat arbitrary. There is no single system for characterizing the organization of teachers' knowledge. All the knowledge is highly related, the categories of teacher knowledge within a system are not discrete entities, and their limits are unclear (Mack, 1990).

Ma (1996) used a method to make it easy to identify two kinds of knowledge. She stated that subject content knowledge deals with "what" is to be taught, and pedagogical content knowledge deals with "how" to teach it. She gave representation as an example and mentioned that the concept it represents belongs to the subject matter knowledge since it is related to what is to be thought. On the other hand, the way of representing the conception, belongs to pedagogical content knowledge since it is related to how to teach it. In this study, subject matter knowledge and pedagogical content knowledge were defined treated separately based on this discrimination.

### 2.1.1 Nature of Subject Matter Knowledge

Teachers' knowledge of subject matter attract increasing attention from policy makers in recent years since more emphasis is given to highly qualified teachers (Hill, Rowan, & Ball, 2005; Crespo & Nicol, 2006). "Subject matter is an essential component of teacher knowledge" (Ball & McDiarmid, 1990, p. 437). To help students learn subject matter involves not only giving the facts and information. The goal of teaching is to help students in their development of intellectual resources that let them inquire (Ball & McDiarmid, 1990).

Ball (1990a) mentioned that in order to improve content that is subject knowledge in teaching, we would need to identify core activities of teaching, such as figuring out what students know; choosing and managing representations of ideas; appraising, selecting, and modifying textbooks; and deciding on alternative courses of action that could provide a view of subject matter as it is used in practice. Similarly, National Council of Teachers of Mathematics (NCTM) reported that teacher possessing the knowledge of the content being taught is the single most important factor in student achievement. Teachers who have strong subject matter knowledge give details in their lesson, link the topic to other topics, ask students many questions, and stray from the textbook (National Council of Teachers of Mathematics, 2000).

In addition to the school subjects such as English, history, and science, mathematics is a field where much current research on teachers' subject matter knowledge has been accomplished (Ma, 1996). In the literature, we could identify three main groups of researchers according to their definitions and studies on subject matter. In this part of the literature review, I discussed these three groups of researchers in terms of their own definitions, similarities, and differences in describing subject matter knowledge (SMK).

The first group of researchers was Shulman and colleagues where subject matter knowledge was defined as knowledge of the key facts, concepts, principles, and explanatory frameworks of a discipline in addition to the rules of evidence used to guide inquiry in the field (Grossman, Wilson & Shulman, 1989). Schwab (as cited in Shulman, 1986) emphasized the distinction between substantive and syntactic structures while defining subject matter knowledge. Substantive structures were the ways in which the ideas, concepts, and facts of a discipline are organized, the key principles, theories, and explanatory frameworks of the discipline. On the other hand, syntactic structures were rules of evidence and proof that guide inquiry in a discipline, in other words the way of representing new knowledge and determining the validity of claims. Shulman and his colleagues, distinguish four categories of subject matter knowledge: knowledge of content (facts, concepts, and procedures), knowledge of substantive structures, knowledge of syntactic structures, and beliefs about the discipline. They argued that all four components influence how teacher choose to teach. Shulman (1986) argued, "teachers must not only be capable of defining for students the accepted truths in a domain. They must also be able to explain why a particular preposition is deemed warranted, why it is worth knowing, and how it relates to other propositions" (p. 9). A key researcher in the second group, Ball (1990b, 1991) made a similar distinction to Shulman in defining subject matter knowledge.

According to Ball (1988, 1991), with her large number of studies on teachers' subject matter knowledge, knowledge needed for teaching includes both knowledge of mathematics and knowledge about mathematics. Knowledge of mathematics was characterized by an explicit conceptual understanding of the principles and meaning underlying mathematical procedures, rules, and definitions whereas, knowledge about mathematics characterized by the nature and communication of mathematics and an understanding of what it means to know and do mathematics.

Ball (1991) showed that understanding of mathematics consists of *substantive knowledge*—knowledge *of* the substance of the domain and ideas about and dispositions *about* the subject. Ball stated:

> …substantive knowledge includes propositional and procedural knowledge of mathematics: understandings of particular topics (e.g., fractions and trigonometry), procedures (e.g., long division and factoring quadratic equations), and concepts (e.g., quadrilaterals and infinity). Mathematical structures and connections, the relationships among these topics, procedures, and concepts, are also part of the substantive knowledge of mathematics. (p. 7)

She pointed out that three criteria characterize the kind of substantive knowledge teachers need. First, teachers' knowledge of concepts and procedures should be correct. For instance, they should be able to draw a rectangle or identify a function. Second, they should understand underlying principles and meanings like each line of a long multiplication problem. Finally, teachers must judge and understand the connections among mathematical ideas like how fractions are related to division (Ball, 1990b). Ball added that substantive knowledge is mostly recognized as subject-matter knowledge by other researchers.

Another critical dimension of subject matter is *knowledge about mathematics* that included understandings about the nature of mathematical knowledge. It included questions like what do mathematicians do? Which ideas were arbitrary or conventional and which were logical? What was the origin of some of the mathematics we use today and how did mathematics change? (Ball, 1990b). According to Ball (1991), knowledge about the nature and discourse of mathematics includes:

> Understandings about the nature of mathematical knowledge and activity: what is entailed in doing mathematics and how truth is established in the domain. What counts as a solution in mathematics? How are solutions justified and conjectures disproved? Which ideas are arbitrary or conventional and which are necessary or logical? Knowledge about mathematics entails understanding the role of mathematical tools and accepted knowledge in the pursuit of new ideas, generalizations, and procedures. (p. 7)

She also pointed out that these aspects of knowledge were hardly mentioned in the curriculum or in college. Mathematics students rarely learn about the evolution of mathematical ideas or ways of thinking. In addition, Ball (1990b) stated that teachers should understand the subject in depth to be able to represent it in appropriate and multiple ways like story problems, pictures, situations, and concrete materials. They should understand the subject flexibly enough to interpret and judge students' ideas.

Being knowledgeable on teaching includes not only being able to perform step-by-step algorithms, but also to talk about mathematics, make judgments, and reason certain relationships and procedures. Teaching involves being able to explain why the procedures work, as well as to relate particular ideas or procedures to others within mathematics. However, the standard school mathematics curriculum to which most prospective teachers have been subjected, treats mathematics as a collection of discrete bits of procedural knowledge (Ball, 1991).

The last group, Leinhardt, Putnam, Stein, & Baxer (1991) have a broader definition of subject matter knowledge. In their early work, they gave the following definition:

> Subject matter knowledge includes concepts, algorithmic operations, the connections among different algorithmic procedures, the subset of the number system being drawn upon, the understanding of classes of student errors, and curriculum presentation. Subject matter knowledge supports lesson structure and acts as a resource in the selection of examples, formulation of explanations, and demonstrations. (Leinhardt & Smith, 1985, p. 247)

The first three components of this definition also were emphasized by Shulman and his colleagues and by Ball. However, the last two components, the understanding of classes of student errors and curriculum presentation, were not classified as subject matter knowledge by Shulman and his colleagues and by Ball. These were included in Shulman's definition of pedagogical content knowledge. In Leinhardt, Putnam, Stein, and Baxter's (1991), latest work, the definition of subject matter was even broader, including knowledge of what students bring to the learning situation (Leinhardt et al., 1991).

In this study, it was aimed to study pre-service teachers' subject matter knowledge on multiplication and division of fractions. That is; how they understand and reason the basic operations, procedures, and proofs on given operations were examined. Thus, in this study by using the term *subject matter,* I have combined Shulman's and Ball's subject matter knowledge definitions where; how pre-service teachers construct their understanding of key facts, concepts, principles, and proofs on multiplication and division of fractions were examined.

After the analysis of the theoretical background of subject matter knowledge, now it's time to turn our attention to the pedagogical content knowledge, the meaning and related theories.

#### 2.1.2 Nature of Pedagogical Content Knowledge

A number of studies had suggested that, in general, teachers with greater subject matter knowledge emphasize the conceptual, problem-solving, and inquiry aspects of their own subject matter. Teachers could explaine why certain procedures work and some not, and address the relationship among concepts (Borko & Putnam, 1996). On the other hand, less knowledgeable teachers tend to emphasize facts, rules, and procedures, and they strictly depend on their lesson plan. Pedagogical Content Knowledge (PCK) differentiated expert teachers in a subject area from the subject area experts. The PCK concerns how teachers related their subject matter knowledge (what they know about what they teach) to their pedagogical knowledge (what they know about teaching) and how subject matter knowledge is related to the process of pedagogical reasoning (Shulman, 1987).

Cochran, DeRuiter, and King (1993) stated that PCK was directly tied to the subject matter concepts but it is much more than just subject matter knowledge. As in subject matter knowledge, there are different views in the literature related to the definition of pedagogical content knowledge. In this part, some views along with their similarities and differences were discussed.

Shulman (1986) mentioned three types of content knowledge: (a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge. He defined content knowledge as the amount and organization of knowledge per se in the mind of the teacher. Content knowledge requires going beyond the knowledge of facts or concepts of a domain that requires understanding of the structures underlying facts, algorithms and concepts.

Shulman defined pedagogical content knowledge as subject matter knowledge for teaching. According to Shulman, pedagogical content knowledge includes:

> The most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations-in a word, the ways of representing and formulating the subject that make it comprehensible to other….it also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons.(p. 9)

Lastly, Shulman defined curricular knowledge as the full range of programs designed for the teaching specific subject to a given grade level, the variety of instructional materials related to those programs, and the set of characteristics that supply both the indications and contradictions for the use of particular curriculum materials in particular circumstances.

Grossman (1989, 1990) expanded Shulman's definition and stated that pedagogical content knowledge includes four main components: (a) an overarching conception of what it means to teach a particular subject, (b) knowledge of instructional strategies and representations for teaching particular topics,(c) knowledge of students' understanding and potential misunderstandings of a subject area, and (d) knowledge of curriculum and curricular materials. The first component was teacher's overarching conception of the purposes for teaching a subject matter. In other words, what she knows and believes about the nature of the subject and what was important for students to learn. Therefore, this was the basis for judgments about classroom objectives, appropriate instructional strategies, textbooks, curricular materials, evaluation of students. Grossman's second category was related to knowledge of students; understandings and potential misunderstandings of a subject area. Knowledge and beliefs about how students learn in a particular content domain were included in this category. The importance of this component was that it focused on specific content in the given subject area. For example, learners' preconceptions, misconceptions about negative numbers were included into this category.

Grossman's third category was knowledge of curriculum and curricular materials. This component includes familiarity with the range textbooks and other instructional materials that were available for teaching particular topics. This category also included knowledge of how the topics and ideas in a subject were organized within a grade level or across the kinder garden through grade 12 curriculum. The last category was the teachers' knowledge of strategies and representations for teaching particular topics. This category also extensively addressed in Shulman's definition of pedagogical content knowledge. The representations include the models, examples, metaphors, simulations, demonstrations, and illustrations that teachers use to increase student understandings.

On the other hand, Cochran et al., (1993) proposed a modification of PCK based on a constructivist view of learning and its application to teaching and teacher preparation in addition to other researchers. They used the term pedagogical content knowing (PCKg) as a constructivist version of PCK in which they emphasized its dynamic nature.

Lerman (1989) emphasized that in the constructivist perspective, knowledge is actively created by the knower and not passively received in an unmodified way from the outside environment. For constructivist educators, knowing is created rather than transferred, and teachers must understand how students construct and use their understandings (Steffe, 1991; von Glasersfeld, 1989). In addition, since each student construct his or her own understanding, the more a teacher understands about each student's understanding, the more effective teaching is likely to be (Reynolds, 1992).

Cochran et al., (1993) proposed that enhancement of pedagogical content knowing requires the modeling and sharing of teaching decisions and strategies with students by education and subject area faculty. University faculty must understand the nature of PCKg to foster its development in teacher education students and to enrich their own teaching. They emphasized the central role that teachers' understanding of their students plays in teaching. Cochran et al. (1993) defined pedagogical content knowing as " a teacher's integrated understanding of four components of pedagogy, subject matter content, student characteristics, and the environmental context of learning" (p. 266). Researchers added that PCKg development is continual and as PCKg increases, it enables teachers to use their understanding to create teaching strategies for teaching specific content that enables specific students to construct useful understandings in a given context. Cochran et al., (1993) added more components to pedagogical content knowledge compared to Shulman's and Grossman's definitions. The first additional component was teacher's understanding of students, which included students' abilities, and learning strategies, ages and developmental levels, attitudes, motivations, and prior conceptions of the subject they are learning. The other additional components of teacher understanding that contribute to pedagogical content knowing are teachers' understandings of the social, political, cultural and physical environmental contexts that shape the teaching and learning process. They stated,

> From the constructive perspective, the teacher's understanding of these two aspects provides the basis for teaching because learning is created by the student, not the teacher, with the students' understandings and the learning setting forming the contexts for that learning. Moreover, it is the students who decide whether or not the understanding constructed in the classroom is viable. (p. 26)

Cochran et al.'s, (1993) developmental model of PCKg for teacher preparation was given in Figure 2.2. The model included four components of understanding pedagogy, subject matter, students, and the environmental context. The expanding circles showed the changes in a pre-service teacher's understanding in each of the four components because development in each starts with limited knowledge and expand through activities and experiences. They explained the dark arrows and expanding core of the model as the growth in pedagogical content knowing. The overlapping circles indicate the integration of the four components of PCKg, which theoretically become so integrated that they no longer can be considered as separate. They further asserted that, the four components should not be acquired first and then somehow put together, but rather preparation programs should promote integration by having teachers simultaneously experience the PCKg components. Researchers also added that although the circles in the model were symmetric, the development of all components of teacher understanding, and the relative contributions of four components of PCKg vary during pre-service programs. Depending on the order of the course, and field experiences, the four components may unevenly develop. They also pointed out that PCK develops over time as a result of experience in a real classroom environment.

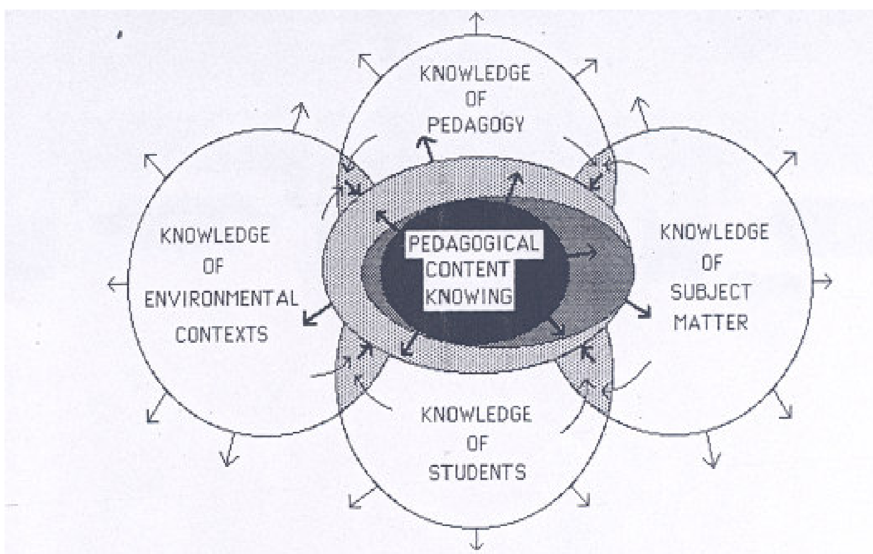



**Figure 2.2.** Cochran et al.'s (1993, p.268) model of Pedagogical Content Knowing.

When discussing pedagogical content knowledge, many researchers emphasized the role of representations in teaching. Indeed Shulman's definition of pedagogical content knowledge focuses primarily on "ways of representing and formulating the subject that make it comprehensible for others" (Shulman, 1986, p. 9). Borko and Putnam (1996) stated that despite the importance of this factor there was limited research evidence concerning its role in novice teachers' learning to teach. One reason for this may be the difficulty of discriminating between pedagogical content knowledge of instructional strategies and representations of subject matter knowledge both in theory and in practice. Mostly researchers who indicated limited subject matter knowledge viewed it as evidence for limited pedagogical content knowledge of instructional representations, and vice versa. For example, in the literature, there are examples that reveal novice teachers' inability to produce appropriate representations for division of fractions (Ball, 1990a, 1990b, Borko et al., 1992; National Center for Research on Teacher Education, 1991). In these studies, Ball discussed the results as pre-service teachers' inadequate knowledge of subject matter whereas; Borko et al. (1992) interpreted the results as limited pedagogical content knowledge and subject matter knowledge.

In her study, Ball (1990) reported that a teacher possessing pedagogical content knowledge provided learners with multiple approaches to learning. In mathematics, a teacher with pedagogical content knowledge used manipulatives and was ready for possible student misconceptions. In addition, Borko and Putnam (1996) stated that teachers who spent several years in classrooms have acquired considerable general pedagogical knowledge in addition to subject matter knowledge.

In this study, in terms of pedagogical content knowledge, Shulman's (1986) and Grossman's (1989) definitions were used where; pre-service teachers' knowledge on common conceptions and misconceptions held by the elementary students, their knowledge on the possible sources of these conceptions and misconceptions, the strategies that pre-service teachers used to overcome these misconceptions, the representations that pre-service teachers used to reason their understanding, and the strategies that pre-service teachers used to explain the key facts, concepts, principles and proofs on multiplication and division of fractions were investigated.

After concerning related theories based on the subject matter knowledge and pedagogical content knowledge in general, in the next section focus is turned to studies based on teachers' knowledge on multiplication and division of fractions.

## 2.2 Pre-service Teachers' Knowledge on Multiplication and Division of Fractions

In recent years, pre-service teachers' subject matter knowledge and their knowledge of students' conceptions have been the focus of many studies. In this part of the literature review, studies on teachers' subject matter knowledge and pedagogical content knowledge on division of fractions were mentioned.

Mack (1990) stated that fractions and rational numbers are considered as the most complex mathematical domains in elementary school mathematics. Although many students understand the rote algorithm needed to manipulate the symbols, they soon forget the procedures and thus find it difficult to learn operations on fractions and rational numbers. Many students' understanding of fractions is characterized by knowledge of rote procedures, that are often incorrect, rather than by the concepts underlying the procedures (Behr, Lesh, Post, & Silver, 1983). Behr et al., (1983) pointed out that students' understanding of rational numbers depends on first developing a broad conception of rational numbers and then progressing through a sequence of topics within each strand of rational numbers that are based on mathematical prerequisites.

Behr, Harel, Post, and Lesh (1992) pointed out that learning fractions concepts remains a serious obstacle in the mathematical development of children. Numerous questions about how to facilitate children's construction of rational number knowledge remain unanswered even if clearly formulated. They suggested that teachers must define the experiences that children need in order to develop their rational number understanding.

Elementary teachers' should understand fraction operation conceptually in order to help pupils to develop ideas on division and mathematical concepts regarding fractions and ratios (McDiarmid & Wilson, 1991). However, previous studies showed that teachers' understanding was limited and replete with misunderstandings about fraction concepts, procedures an operations involving multiplication and division (Azim, 1995; Ball, 1990; Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992; Eisenhart, Borko, Underhill, Brown, Jones, & Agard, 1993; Leinhardt & Smith, 1985; Post, Harel, Behr, & Lesh, 1991; Simon & Blume, 1994; Tirosh, 2000).

One of the key operations on fractions is multiplication where; "Multiplicative reasoning is one aspect of quantitative reasoning, which Thompson defines as thinking about a situation as a set of quantities and relationships between them" (Simon & Blume, 1994, p. 473). Behr et al. (1992) stated that the elementary school curriculum lacked some basic concepts and principles to relate multiplicative structures necessary for later learning in upper grades. In addition, the multiplicative concepts that are presented in the middle grades are isolated and are not interconnected as in division. They added that there is a lack of problem situations that provide a wide range of experience for children to develop less-constrained models of multiplication and division. According to Behr et al., these deficiencies arise from the lack of analytical understanding of how multiplicative concepts interrelate from theoretical, mathematical, and cognitive perspectives.

In a research study, Simon and Blume (1994) investigated the quantitative reasoning involved in understanding the evaluation of the area of a rectangular region as a multiplicative relationship between the lengths of the sides. Researchers were interested in mathematical understandings of learners and the reasoning processes in which they engage. Twenty-six prospective elementary teachers who were in their junior year at the university were participated in the study. Researchers stated that although pre-service teachers responded to area-of-rectangle problems by multiplying, their choice of multiplication was often because of the learned formula or procedure, rather than a solid conceptual link between their understandings of multiplication and their understandings of area. Results revealed that prospective elementary teachers did not have a well-developed concept of why the relationship of the length and width of a rectangle to its area is appropriately modeled by multiplication. Researchers added that analysis of data from a whole-class teaching experiment resulted in the development of a description of the quantitative reasoning where teachers' engagement with the problems in the instruction helps them to develop more complete understandings of the constitution of area units. Similarly, Azim (1995) who investigated pre-service elementary teachers' reasoning about multiplication and multiplication with fractions in an elementary mathematics method course mentioned that fifty-six percent of the pre-service teachers entered the method course without a way to reason about multiplication with fraction less than one.

In another research study, Behr et al. (1992) stated that children and some adults like middle grades teachers have misconceptions about many multiplicative concepts that result from deficiencies in the curricular experiences provided in school. The common intuitive rules "like multiplication always makes bigger and division smaller" arise because of the curricular overemphasis on multiplication and division of whole numbers. In the absence of counter experience or higher level of mathematics, this misconception prolongs to the adult life.

Division, another key operation on rational numbers, is one of the most important topics in elementary school mathematics, yet it is often considered the most mechanical and least understood (Fendel, 1987; Payne, 1976). Carpenter, Lindquist, Brown, Kouba, Silver, and Swafford (1988) stated that children's success rates on various tasks related to such division are usually very low. Ball (1990b) mentioned that division is a central concept in mathematics at all levels and figures highly throughout the K-12 curriculum. While working division, students can learn about rational and irrational numbers, place values, the relationship among the four operations, as well as the limits and power of relating mathematics to real life. In addition, understanding division is crucial conceptual knowledge in order to understand a variety of ideas in mathematics.

Researchers emphasized that when asked by a student why you get a bigger value when dividing fractions, teacher's most typical response was to ignore the question or simply told the student to invert and multiple (Ball, 1990, 1991; Tirosh, 2000). In addition, fractions were often taught using procedures instead of allowing students to experience multiple ways of manipulating operations using fractions. Ball (1991) stated that prospective teachers did not connect the concept of division across the different context like division of fractions, division by zero, and division in algebra. They treat each division as a separate topic without any relationship to the others and cite a particular procedure or rule for each of them. Thus, in examining teacher knowledge, teachers' knowledge of particular concepts across different contexts and from different perspectives should be considered. Similarly, Tirosh (2000) stated that most participants knew how to divide fractions but could not explain the procedure. The prospective teachers were unaware of the major sources of students' incorrect responses in this domain. From an instructional aspect, teachers lack a deep understanding of fractions, which inhibits them or puts them at a disadvantage in using multiple strategies in their instruction (Mack, 1990).

In her cross-cultural study between American and Chinese elementary school teachers, Ma (1996, 1999) asked teachers how they would go about solving the problem "$1\frac{3}{4} \div \frac{1}{2}$" and what would be a good story problem for the given division of fractions. Ma reported that only 43% of American elementary teachers completed their computations and reached the correct answer but all the Chinese teachers completed the question and reached the correct solution. American teachers reported various representations on division of fractions based on their pedagogical knowledge. However, because of their incomplete subject matter knowledge these representations were not proper. Most of the American teachers created several stories with misconceptions. On the other hand, Chinese teachers represented the concept correctly using three different models of division: measurement (or quotitive), partitive, and product and factors while creating their story problems on the given division of fraction problem. Ma (1996, 1999) stated that when teachers guide students toward the understanding of fractions, they should be ready with multiple ways of guiding learners to understand the division of fractions. Teachers should understand how hard this topic is and should provide the details and the basic knowledge to students so that those who believe that division always makes a number smaller can understand that division does not always make a number smaller. She found that it was common to confuse dividing by a unit fraction with dividing by the whole number denominator, that is, dividing by one-half was often confused with dividing by two. In addition, it was common to confuse division by a fraction with multiplication by a fraction such as, dividing by one-half and multiplying by one-half. In some cases, there was also confusion about dividing by one-half, multiplying by one-half and dividing by two. Ma (1996, 1999) stated that division is the most complicated of the four operations. Fractions are mostly considered the most complex numbers in elementary school mathematics. Thus, division by fractions can be thought as the most complicated operation with the most complex numbers. She generated a model based on the conceptions that teachers mentioned during the discussion on meaning of division of fractions where she emphasized the importance of multiplication while teaching division concepts. Model contains the meaning of multiplication by whole numbers, the conception of division as the inverse operation of multiplication, the models of division with whole numbers, the meaning of multiplication with fractions, the concept of a fraction, the concept of a unit, etc. The relationship among these concepts was identified in the model given in Figure 2.3. Ma added that one's understanding of the meaning of division by fractions supported by three conceptions: the conception of division as the inverse operation of multiplication, conception of the models of division with whole numbers, and that of multiplication with fractions that are the cornerstone in understanding the meaning of division of fractions.

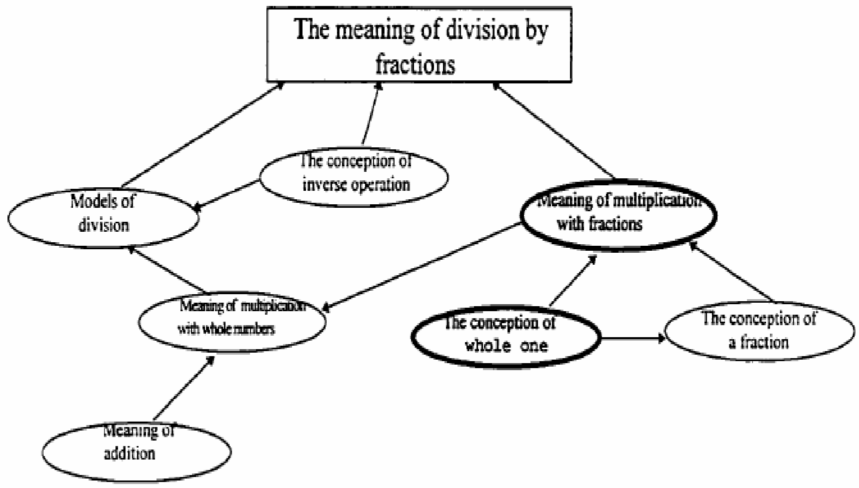



**Figure 2.3.** Ma's (1996, p. 153) model on knowledge package for understanding the meaning of division by fractions

Tirosh and Graeber (1991) conducted a study to determine the relative effect of problem type and common misconceptions on pre-service teachers' ways of thinking about division. The participants, 80 female pre-service elementary teachers from elementary education majors in a large university in the southeastern United States, were given two paper and pencil instruments related to writing an expression for word problems and writing division word problems. Word problems include both partitive and measurement models. Some of the division problems for each type included data that conformed to the primitive model. An example item for the partitive problem is "It takes 5.25 meters of ribbon to wrap 3 packages of the same size. How many meters of ribbon are required to wrap one of these packages?" and for the measurement model, "You prepared 5.25 liters of punch. You have punch bowls that hold 3 liters. How many punch bowls can you fill with the prepared punch?". However, problems also included data that violated the "dividend must be greater than the divisor" constraint common to both primitive models. Examples for partitive and measurement models in this category are "It takes 3.25 meters of ribbon to wrap 5 packages of the same size. How many meters of ribbon are required to wrap one of these packages?" and "You prepared 3.25 liters of punch. You have a punch bowl that holds 5 liters. How much of the punch bowl can you fill with the prepared punch?" respectively. For the writing division word problem instrument, four division expression (6 ÷ 3, 2 ÷ 6, 4 ÷ 0.5, and 0.5 ÷ 4) were presented to the pre-service teachers to write a word problem for each. Results indicated that pre-service teachers were more successful in writing expressions for partitive type word problems than for measurement problems. Interviews strengthened the belief that the majority of the pre-service teachers have access only to the partitive interpretation of division since, when asked to interpret the expression 6 ÷ 2, they almost always suggested a partitive explanation. When the teachers were asked to offer another interpretation for the expression, most of them were unable to answer without many clues or prompts from the interviewers. Results also revealed that pre-service teachers were more successful with word problems that did not challenge common misconceptions than with word problems that challenged the misconceptions. Their expressions showed that pre-service teachers were affected by the constraint of both primitive models, "the divisor must be smaller than the dividend." Also pre-service teachers mentioned that the primitive partitive division model dominated their thinking even when they solved measurement type problems. Pre-service teachers who wrote incorrect expressions for the primitive models commonly voiced the misconceptions "Division always make smaller" and " The divisor must be a whole number" that are the constrains of the primitive partitive model. Researchers also stated that the effect of problem type and conformity to misconceptions appeared to vary with the type of task being attempted. Pre-service teachers' success was more affected by conformity to the misconceptions than by the problem type in writing expression for a given word problem. On the other hand, regarding writing word problems for the given expressions, pre-service teachers' success was more affected by the compatibility of the expressions with the partitive interpretation of division than by their adherence to the limitations imposed by the primitive models. As suggested, pre-service teachers tend to give a partitive interpretation when attaching meaning to the division expression.

Tirosh (2000) carried out a research study to describe prospective elementary teachers' own subject matter knowledge and their knowledge of students' conceptions of division of fractions. A class of 30 prospective elementary teachers where all were women in their second year of a four-year teacher education program in an Israeli State Teachers' College participated in a study. All teachers completed a questionnaire designed to assess their subject matter knowledge and pedagogical content knowledge of rational numbers, and then each was interviewed. During the entire academic year, prospective teachers participated in a mathematics methods course designed to develop their subject matter knowledge and pedagogical content knowledge. A diagnostic questionnaire to measure the subject matter knowledge and knowledge of students' conception of rational numbers was administered to the prospective teachers at the beginning of the course. The questionnaire include two division-of-fractions items where for the first item participants were request to (a) calculate the expressions like $\frac{1}{4} \div 4$; $\frac{1}{4} \div \frac{3}{5}$, (b) list common mistakes students in seventh grade may make after finishing their studies of fractions, and (c) describe possible sources for each of these mistakes. Similarly, for the second item students were supposed to (a) write an expression to solve the problem without calculating the expression, (b) write common incorrect responses, and (c) describe sources of these incorrect responses for the given word problems related to rational numbers. An example of the word problem is "A five-meter-long stick was divided into 15 equal sticks. What is the length of each stick?" (p. 9). The results from the pretest revealed that 5 of the 30 prospective teachers gave incorrect response to some of the division-by-fraction expressions for item 1. Tirosh was interested in the common difficulties that children experience with division of fractions (knowing that) and to what teachers attribute these difficulties (knowing why). In other words, Tirosh was interested in the pedagogical content knowledge of pre-service teachers in addition to their subject matter knowledge. In terms of knowing *that,* in other words listing common errors in division expression involving fractions, results showed that vast majority of the prospective teachers had this knowledge. In terms of knowing *why*, in other words understanding the possible sources of specific students' reactions to this item, results showed most teachers who participated in the study attributed these mistakes to algorithmic errors, and only a few suggested both algorithmic and intuitive sources of students' incorrect responses. Stated differently, prospective teachers believed that the steps of the algorithm are memorized, and if the steps are forgotten, it is not possible for students to reconstruct them through mathematical inquiry. The possibility of performing the division without using the standard algorithm was not considered. Analysis of responses to the word problems, the part (a) to item 2, revealed that all but one participant provided the correct response and in terms of knowing *that*, most of the prospective teachers prefer using division operations instead of multiplication or having students try to divide the big number by the small one. For example, the expression $15 \div 5$ is stated as a common error for the world problem mentioned above. Regarding knowing *why*, it was mentioned that intuitive beliefs like children's tendencies to attribute properties of operations with natural numbers to fractions was the possible source for incorrect responses since students are used to dividing the big number by the small one when they are learning natural numbers. The other reason that was mentioned by the prospective teachers was that students tend to think that "multiplication makes bigger and division makes smaller". In addition, some prospective teachers mentioned general reading comprehension difficulties as sources of incorrect responses, thinking children would not read the questions carefully.

Although operations with fractions have been in the elementary and middle school curriculum for many years, finding examples of practical problems that illustrate the usefulness of division with fractions and mixed numbers was not easy. Most of the real-world applications of rational numbers involve decimal numerals, but examples of division with fractions and mixed numbers are often obviously contrived. In addition, many teachers and prospective teachers have difficulty constructing examples and concrete models for the operation of division with fractions (Borko et al., 1992). Borko et al. (1992) analyzed several points of a classroom lesson where a student teacher was unsuccessful in providing a conceptually based justification for the standard division-of-fraction algorithm. The primary goals of the study were to describe novice teachers' developing knowledge, beliefs, thinking, and actions related to the teaching of mathematics, to understand the interdependence and mutual influence of these components of teaching and learning to teach and impact of teacher education on the process of learning to teach. In their article they concentrate on single student teacher's knowledge on division of fractions and investigate the student teacher's beliefs about good mathematics teaching, her knowledge related to division of fractions, her beliefs about learning to teach and the treatment of division of fractions in the method course that she took. Analysis revealed that the student teacher's conceptions were similar to the current views of effective mathematics teaching like good mathematics teaching school include making mathematics relevant and meaningful for students, making mathematics relevant for students required to incorporate into their lessons, carrying out mathematics activities that students enjoy. However, she has limited knowledge on instructional representations for division of fractions and limited knowledge on what students understand about the topic. Also the mathematics method course did not necessitate student teacher to reconsider her knowledge base, to confront the contradictions between her knowledge base or to reassess her beliefs about how she would learn to teach. Based on these findings researchers suggested that mathematics education programs should reconsider how they provide subject matter knowledge and opportunities to teach it, and whether and how they challenge student teachers' existing beliefs. Also it was stated that prospective teachers should take the advantage of opportunities that are provided them in teaching learning circumstances.

As stated above, one of the purposes of this research study was to investigate the pedagogical content knowledge of pre-service teachers in terms of common (mis)conceptions/difficulties held by the elementary grade level students, possible sources of these misconceptions/difficulties, and strategies and representations that pre-service teachers used to reason their understanding on multiplication and division of fractions. Thus, after stating the studies related to multiplication and division of fractions, more specific issues related to the pedagogical content knowledge that is conceptual framework and related literature on mistakes and strategies were discussed accordingly.

### 2.3 Studies Related to Students' Conceptions and Misconceptions/difficulties about Multiplication and Division of Fractions

When we reviewed the literature related to the multiplication and division of fractions, there were three main categories of mistakes that were carried out by children. These could be categorized under the headings algorithmically based mistakes, intuitively based mistakes, and mistakes based on formal knowledge. Studies showed that category of algorithmically based mistakes arose because of the rote memorization of the algorithm. When an algorithm was viewed as a meaningless series of steps, students might skip some steps or change them and result in errors. The most common error in this category was inverting the dividend instead of the divisor or inverting both the dividend and the divisor before multiplying numerators and denominators while dividing two fractions (Ashlock, 1990). On the other hand, intuitively based mistakes result from intuitions held about division. In Barash & Klein's (1996) diagnostic test, intuitive is referred as "ability to identify the adequate operations for solving multiplication and division word-problems" (p. 36) where; intuitive errors stem from the intuitions held about operations. In the literature, basic intuitions on multiplication and division of fractions were stated as follows: for the multiplication problem, the product is always bigger than one, the product is always equal or bigger than both factors, in a division problem, the dividend is always bigger than the divisor, the dividend is always bigger than the quotient, the quotient must be integer, and divisor must be a whole number (Ashlock, 1990; Barash & Klein,1996; Fischbein, 1987; Graeber, Tirosh, & Glover, 1989; Tirosh, 2000).

Students tend to over generalize properties of operations with whole numbers to fractions and to interpret division primarily using a primitive, partitive model of division. In this model of division an object is divided into a number of equal parts. The primitive, partitive model of division imposes the following: (a) the divisor must be a whole number; (b) the divisor must be less than the dividend; and (c) the quotient must be less than the dividend. The predominance of this model limited children's and prospective teachers' abilities to correctly solve the division problems involving fractions (Fischbein, Deri, Nello, & Marino, 1985; Graeber, Tirosh, & Glover, 1989). Students stated that "one can not divide a small number by a large number because it is impossible to share less among more" (Tirosh, Fischbein, Graeber, & Wilson, 1993, p. 18). A more detailed description of Fischbein's theory of primitive model was given below.

Fischbein et al. (1985) suggested a theory based on children's performance on both multiplication and division word problems. In particular, they identified students' ability in choice of single operation word problems without carrying out any calculation.

The primitive model that they proposed for multiplication is repeated addition. According to the model, 3 children having 4 oranges each, conceptualized as 4 oranges + 4 oranges + 4 oranges, and the answer can be calculated as repeated addition. However, Fishbein et al. (1985) emphasized that for a situation that can be modified to this model, multiplier must be an integer, and no restriction applies to the multiplicand. Model also proposed that result is always larger than the multiplicand. The primitive models that Fishbein et al., (1985) proposed for the division are namely partition—sharing into equal sub-collections or sub-quantities—and quotation—determining how many sub-collection or quantity were proposed. If the problem is partitive division, then the divisor must be an integer and smaller than the dividend. On the other hand, if the problem is quotitive division, the model only proposes that the divisor must be smaller than the dividend.

Fishbein et al., (1985) suggested that partition is the original intuitive primitive model for division and quotitive model is acquired later through instruction. When students have been asked to write word problems, they prefer to write partitive word problems over quotitive problems. Similarly, Graeber and

Tirosh (1988) stated that pre-service elementary teachers prefer to use partitive models to quotitive models. Fischbein (1987) stated that children's early experience with multiplication and division, in and out of school, largely limited to situations involving discrete objects and restricted to integer domain. When new classes of situations emerged-beyond integers-because of the deep rootness of earlier conceptualizations of primitive models caused problems.

Greer (1992) stated that students prefer to choose division instead of multiplication when the operand is less than 1, since they have familiar misconceptions "multiplication always makes bigger and division smaller, and division is always carried out by dividing large number by the smaller one". Generalization of multiplication and division beyond the integer domain is difficult and requires a major conceptual restructuring.

The last mistake mentioned in the literature was based on formal knowledge related to the inadequate knowledge on the properties of operations. Errors can have various sources; for example, Hart (1981) reported that students think that division is commutative as multiplication and consequently argue that $1 \div \frac{1}{2} = \frac{1}{2}$ because $1 \div \frac{1}{2} = \frac{1}{2} \div 1 = \frac{1}{2}$. These errors resulted from inadequate formal knowledge or initiative beliefs on division where; the dividend should always be greater than the divisor could yield $\frac{1}{4} \div \frac{1}{2} = \frac{1}{2} \div \frac{1}{4} = 2$ or even originated from a bug in algorithm like $\frac{1}{4} \div \frac{1}{2} = \frac{1}{2} \div \frac{1}{4} = 2$ (Hart, 1981; Tirosh, 2000).

In a research study, Graeber, Tirosh, and Glover (1989) investigated whether pre-service elementary teachers had misconceptions such as "the divisor must be a whole number" or "multiplication always makes bigger and division always makes smaller" by determining whether pre-service teachers selected the correct operation when they were presented with problems having data that conflict with the implicit rules of the primitive behavioral models of multiplication and division. A test with 12 multiplication and 14 division problems was administered to 129 female college students in early elementary education at a large university in the southern United States. Interviews were conducted to obtain more information about the conceptions the pre-service teachers held and reasoning they used. Thirty-three pre-service teachers who had given incorrect answers to one or more of the eight most commonly missed problems were selected for interviews. Results from the written test indicated that 39% of the pre-service teachers answered four or more of the 13 multiplication or division problems incorrectly. Moreover, every interviewee gave evidence of holding some misconceptions. With regard to multiplication, the written work also provided that the common misconception "multiplication always makes bigger and division always makes smaller" was held by pre-service teachers. More than 25% of the sample incorrectly wrote a division expression as appropriate to the solution for multiplication problems. Results from the interviews also revealed that pre-service teachers' reasoning about multiplication problems with decimal operands involved an overgeneralization of procedures used with unit fractions. With respect to division, the data indicated that the problems that violated constraints of the primitive division models proved more difficult. The majority of the incorrect responses to problems in which the whole-number divisor was larger than the dividend ($5 \div 15$) were expressions that reversed the roles of the divisor and the dividend. During the interviews 22 of the 33 interviewees claimed that in division the larger number should be divided by the smaller number and four of the interviewees claimed that, "it is impossible to divide a smaller number by a bigger number". Results revealed that problems that involved the division of a decimal by a large whole number ($3.25 \div 5$) were easier than those with whole-number divisors greater than the whole-number dividend. Four interviewees who reversed the whole-number dividends and divisors in problems were asked about their correct response to $3.25 \div 5$. One of the interviewee reported that she first wrote an expression like $5 \div 3.25$ but since she confronted by decimal divisor she interchanged the divisor and dividend. Thus, results revealed that some pre-service teachers answer these problems correctly only because they avoid using a decimal divisor. This shows that belief "the divisor must be a whole number" seem to have greater strength on some pre-service teachers. Researchers also concluded that a substantial number of pre-service teachers had difficulty selecting the correct operation to solve multiplication and division word problems involving positive decimal factors less than one. Interviews indicated that some of the pre-service teachers held explicit misbelieve about the operations. Also, other pre-service teachers were influenced by implicit, unconscious, and primitive intuitive models for the operations.

Tirosh (2000) pointed out that pre-service teachers who were aware of children's tendencies to attribute properties of operations with whole numbers to operations with fractions used this knowledge to describe incorrect responses and to describe possible sources for children's errors. On the other hand, pre-service teachers who were unaware, have tendency to attribute the incorrect responses to algorithmic or reading comprehension difficulties. Based on these findings, Tirosh developed a methods course with several activities to enhance prospective teachers' subject matter knowledge and pedagogical content knowledge on division of fractions. She tried to encourage prospective teachers to examine their own understanding of the standard operations with fractions and also to draw their attention to the differences between "knowing how to perform" and "explaining why a certain operation is performed in a certain way." Activities were designed to encourage prospective teachers to view students' why type questions as essential components of meaningful instruction and to increase their awareness of possible reactions of students to the algorithms for addition and division of fractions (Tirosh, 2000). Tirosh (2000) concluded that, before the course, most prospective teachers only mentioned algorithmically-based mistakes or reading-comprehension difficulties as possible sources of incorrect responses to division of fraction. However, at the end of the course, most of the participants were familiar with various sources of incorrect responses where data suggest that some components of teachers' knowledge of students' thinking process can be acquired in teacher preparation programs.

> In particular, prospective teachers were aware of students' tendencies to attribute properties of division of natural numbers to division of fractions; constraint that the primitive, partitive, intuitive model of division imposed on the operation of division, and of related intuitively based mistakes. Moreover, they pursue and examine various explanations of the standard division-of-fractions algorithm in light of both their accessibility by children and their possible long-term effects on students' mathematical conceptions and ways of thinking. (Tirosh, 2000, p. 21)

Since today's pre-service teachers were tomorrow's teachers, the learning and teaching cycle might effect misconceptions and misunderstandings about multiplication and division. Consequently, effective strategies were needed for training teachers to examine and control the impact that misconceptions and primitive models have on their thinking and their student's thinking (Graeber, Tirosh, and Glover, 1989)

Researchers also stated that teacher education programs should attempt to make pre-service teachers aware of the common, sometimes erroneous, cognitive processes used by students in dividing fractions and the effects of using of such processes. In addition to the misconceptions, the one dimension of pedagogical content knowledge, strategies that pre-service teachers used to explain their understanding were discussed in the following section as a component of pedagogical content knowledge.

### 2.4 Procedural and Conceptual Strategies on Teaching Concepts in Mathematics

Hiebert and Lefevre (1986) stated that conceptual knowledge was characterized most clearly as knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network. Conceptual knowledge cannot be an isolated piece of knowledge.

It will be conceptual if the relationship is recognized among the other knowledge and conceptual knowledge is increase by the construction of relationships between pieces of information. Hiebert & Lefevre mentioned that connection process can occur between two pieces of information that already have been stored in memory or between an existing piece of knowledge and one that is newly learned. Conceptual knowledge also grows in creation of relationships between exiting knowledge and new information just entering the system. Conceptual knowledge directly related to the "meaningful learning" and "understanding" where new mathematical information is connected appropriately to existing knowledge. Hiebert & Lefevre (1986) stated that assimilation is the heart of the conceptual knowledge where new material becomes part of an existing network. They emphasized the importance of levels in conceptual knowledge where the relationship between pieces of mathematical knowledge can be established. The first level is primary level where relationship connecting the information is constructed at the same level with the information itself. In the primary level, the connection between two pieces of information tied to just specific context and the relationship is no more abstract than the information it is connecting. Abstractness increases as knowledge become free from specific context. On the other hand, at reflective level, higher and more abstract pieces of information were connected. In this level, relationships are less tied to specific contexts and common features of different looking pieces of knowledge tie together. Subject required stepping back and reflecting on the information being connected where the learner can see much more mathematical relationships on various contexts.

In the literature, many studies emphasize the importance of using conceptual strategies while teaching however; it is known that teachers who do not conceptually understand the content are unlikely to teach conceptually (Stoddart, Connell, Stofflett, & Peck, 1993). Thus, before using effective strategies that give importance to students' understandings, teachers should first question their own knowledge of concepts.

On the other hand, Hiebert & Lefevre, (1986) stated that procedural knowledge has two distinct parts. One part was composed of the formal language, or symbol representation system, of mathematics. In this part, people are familiar with the symbols used to represent mathematical ideas and they are aware of syntactic rules for writing symbols in an accepted form. At more advanced levels of mathematics, this knowledge included knowledge of the syntactic configurations of formal proofs but only the style in which proof statement written, not the content and logic of proofs. The second part of procedural knowledge consisted of rules, algorithms, or procedures used to solve mathematical tasks. These are step-by-step instructions that were related to how to complete tasks. Here the important thing was procedures carried out in a predetermined linear sequence. Procedural knowledge also included strategies for solving problems including concrete objects, visual diagrams, or other entities. However, it was stated that the biggest difference between procedural and conceptual knowledge is that "primary relationship in procedural knowledge is *after,* which is used to sequence sub procedures and super procedures linearly. In contrast, conceptual knowledge is saturated with relationships of many kinds" (p. 8).

> Not all knowledge can be usefully described as either conceptual or procedural. Some knowledge seems to be a little of both, and some knowledge seems to be neither. Nevertheless, we believe that it is possible to distinguish between the two types of knowledge and that such a distinction provides a way of interpreting the learning process that helps us better understand students' failures and successes. (Hiebert & Lefevre, 1986, p. 3)

Hiebert & Lefevre, (1986) also differentiated between meaningful learning and rote learning where they related meaningful learning to conceptual knowledge. Conceptual knowledge must be learned meaningfully but procedures may or may not be learned with meaning. Procedures that are learned by meaning are linked to conceptual knowledge. On the other hand, in rote learning, knowledge was absent in relationship and it was loosely tied to the context in which it was learned. Facts and prepositions learned by rote stored in memory as isolated pieces of information, not linked with any conceptual network and procedures could be learnt by rote.

> Mathematical knowledge, in its fullest sense, includes significant, fundamental relationships between conceptual and procedural knowledge. Students are not fully competent in mathematics if either kind of knowledge is deficient or if they both have been acquire but remain separate entities. When concepts and procedures are not connected, students may have a good intuitive feel for mathematics but not solve the problems, or they may generate answers but not understand what they are doing. (Hiebert & Lefevre, 1986, p. 9)

Hiebert & Lefevre, (1986) added that linking conceptual knowledge with symbols would create a meaningful representation system and linking conceptual knowledge with rules, algorithms, or procedures reduced the number of procedures that should be remembered and increase the appropriateness of recalling the right procedure to use effectively. So, being competent in mathematics means knowing concepts, symbols and procedures, and knowing how they are related. Hiebert & Lefevre, (1986) stated that during the earliest year or as children enter school, conceptual knowledge and procedural knowledge were closely related. If students connect the symbols with conceptually based referents, the symbols get meanings and become powerful tools on communicating mathematical reasoning. However, many students seem to learn symbols as meaningless signs and symbols are separated from the conceptual knowledge that they are supposed to link. In addition, as students moved through elementary and junior high school, conceptual knowledge and procedural knowledge continued to develop separately and the focus of instruction remained procedural. This is why many students from elementary school through college could perform successfully during the paper and pencil problems but lacks the conceptual understandings.

Lamon (1993) categorized informal strategies that children use for solving ratio and proportion problems. She classifies avoiding, visual or additive, pattern building strategies as nonconstructive strategies that is procedural and preproportional reasoning, qualitative proportional reasoning, and quantitative proportional reasoning as constructive strategies, that is conceptual. By avoiding, she stated that there is no serious interaction with the problem. Visual or additive strategies involve trial and error or responses without reasons. Using oral or written patterns without understanding numerical relationships was categorized under the pattern building strategies. On the other hand, constructive strategies like preproportional reasoning involve intuitive, sense-making activities like pictures, charts, modeling and manipulating. Likewise, qualitative proportional reasoning involves use of ratios as a unit and understanding of some numerical relationships, and lastly quantitative proportional reasoning involves algebraic symbols to represent proportions with full understanding of functional and scalar relationships. Lamon (1993) added that before proceeding to the traditional symbolism and the cross-multiply-and-divide algorithm for solving proportions, students can be given the time to explore multiplicative situations and relationships of additive and relative perspectives. Lamon further emphasized that instruction may begin with associated-set problems presented in a concrete pictorial or manipulative-based context and could develop a richer sense of knowledge with well-chunked measures. Problem situations involving stretchers and shrinkers might be delayed until students have had time to develop multiplicative reasoning. Lamon added that teachers might encourage the unitizing process by posing problems that allow for multiple solution strategies.

Similarly, Lubinski and Fox (1998) mentioned that current reform documents in mathematics education recommend that teachers should help students develop both conceptual and procedural understandings. However, teachers often do not possess the in-depth mathematical reasoning necessary to accomplish this goal. Inadequate mathematical competency of both the students and the teachers causes students to revert to rote-learned procedural knowledge when under pressure to complete tasks (Tall, 1995).

In this study pre-service teachers' strategies used to overcome the difficulties held by elementary grade students and strategies that pre-service teachers reason their understanding on multiplication and division of fractions were analyzed in terms of conceptual and procedural strategies. In the following section, one more dimensions of pedagogical content knowledge that is pre-service teachers' representations on multiplication and division were explored.

### 2.5 Models for Multiplication and Division of Whole Numbers and Fractions

One of the purposes of this research study was to examine pre-service teachers' representations to reason their understandings on multiplication and division of fractions. Thus, in this part of literature, multiplication and division models and representations on multiplication and division of fractions were stated.

In Vergnaud's (1988) analysis of multiplicative structures he used the term "the conceptual field of multiplicative structures" that he uses multiplication and division within larger context. According to Vergnaud, several mathematical concepts—linear and n-linear functions, vector spaces, dimensional analysis, fractions, ratio, rate, rational numbers, multiplication and division—can be analyzed as simple and multiple proportion problems where one usually needs to multiply or divide.

Vergnaud pointed out that to understand the conceptual field takes a long time. His main classes or problems related to multiplicative structures are termed *isomorphism of measures*, *product of measures*, *and multiple proportions*. By isomorphism Vergnaud (1988) referred to all situations where there is a direct proportion between two measure spaces. The quantities within each measure space may be integers, fractions, or decimals. Thus, multiplication and division problems involving two equal ratios, where one of them is one, are categorized in this measure. (eg. A boat moves 13.9 meters in 3.3 seconds. What is its average speed in meters per second?). The second category stated by Vergnaud is product of measures where two measure spaces are mapped onto a third. Vergnaud use cartesian products and rectangular area as schematic representation of product of measure for both multiplication and division problems. ( A heater uses 3.3 kilowatts per hour. For how long can it be used on 13.9 kilowatt-hours of electricity?). Vergnaud used the following problem for the multiple-proportion problem: A family of 4 person wants to spend 13 days at a resort. The cost per person per day is $35. What will be the total cost of the holiday be? Where; the problem contained more than one class composing simpler problems.

Azim (1995) used the term model in two aspects. One of them is to write a word problem that models multiplication, and to model multiplication expression using physical representations or concepts. Azim (1995) stated that when multiplication with whole numbers is transferred to fractions, multiplication must be reconstructed to accommodate the fractional quantities involved. Understanding multiplication involving fractions requires a re-conceptualization of multiplication to accommodate working with fractional quantities (Greer, 1992; 1994; Graeber & Tirosh, 1998).

Azim (1995) categorized multiplication models under four headings. Namely, repeated addition, multiplicative compare, area concepts, and Cartesian product. The most common concept of multiplication is repeated addition like 3 x 4 can be interpreted as the sum of three fours: 4 + 4 + 4. On the other hand, multiplicative compare is another way of multiplication where the given amounts are compared. The multiplication expression 3 x 4, for example can be modeled as a distance that is 3 times as long as a distance of 4 or a quantity that is 3 times as great as a quantity of 4. This concept is sometimes referred to as 'multiples of a quantity" (Van de Walle, 2005). The other model was Cartesian product where; pairing of elements from two sets usually described in real world context. The multiplication of two numbers, gives the total number of possible pairings between elements of the two sets. For instance, the Cartesian product of 3 pairs of socks with 4 pairs of shoes gives 12 possible socks-shoe pairings. Lastly, multiplication can be modeled by using rectangular area consisting of three rows, with 4 small squares in each row or the area of the rectangle can be interpreted as the product of the two length measures, width and length, of the rectangle (fig. 2.4). Azim (1995) added that repeated addition, multiplicative compare, and area concepts can also be applied to fractions. On the other hand, Cartesian product requires discrete, or whole, quantities for each factor. In order to interpret fraction multiplication expressions using the whole number models, first the invariance properties of multiplication is constructed. Multiplication must be reconceptualize as an invariant operation across fractions and whole numbers where the product can be less than, equal to, or greater than the original factors.

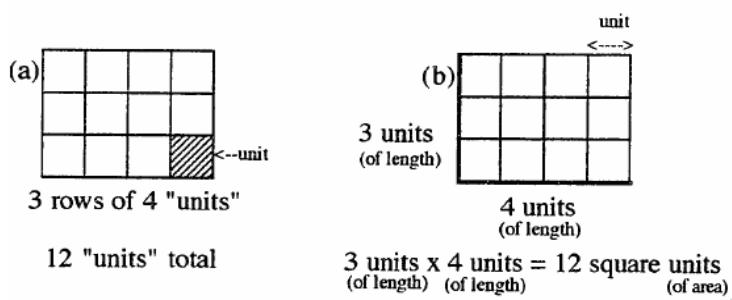
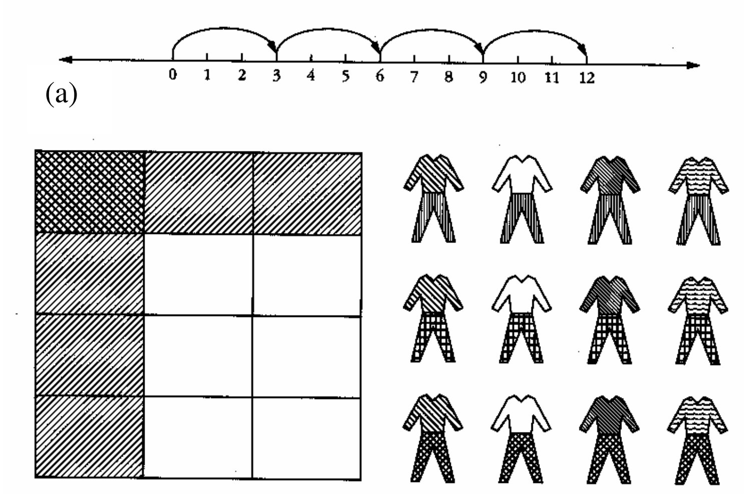



**Figure 2.4.** Area models for multiplication of whole numbers (Azim, 1995, p. 67)

Similarly, Greer (1992) mentioned that multiplication and division of positive integers and rational numbers might be considered relatively easy from a mathematical point of view. However, researchers revealed the psychological complexity behind the mathematical simplicity. Indeed, complexity arises not just from the computational point of view, but also in terms of how they modeled situations. Greer stated several models for division and multiplication. Some of the models, which are mentioned in his article, are number-line model for representing either multiplication or quotitive division (Fig. 2.5a), traditional representation for the product of two fractions ($\frac{1}{3} x \frac{1}{4} = \frac{1}{12}$) (Fig. 2.5b) , pictorial representation of Cartesian product (Fig. 2.5c).




(b) (c)

**Figure 2.5.** Models for multiplication and division (Greer, 1992, p. 281)

On the other hand, Ball (1990b) stated that division has to do with forming groups, and two kinds of models are possible. The first model is forming groups of a certain size, which is also known as the measurement model. In this model, the critical question is “How many groups of that size can be formed?” The second model is forming a certain number of groups, which is also known as the partitive model. In this model, the problem is to determine the size of each group. Burton & Knifong (1983) explain the two modes of division by using the example of twelve divided by four. The example 12 ÷ 4 can be viewed as 12 objects from which we can form subsets of four objects each and ask, “How many such subsets are to be formed to use up all the elements?” to refer a measurement, since the larger set of 12 is being measured to see how many sets of four make up 12. In addition, 12 ÷ 4 can also be explained as there is a set of 12 objects which must be divided into four matching sets, leading to the question, “ How many elements will there be in each set?” to explain the partition, since the divisor four tells how many subsets (partitions) are to be formed from the larger set of 12. The explanation of measurement and partitive models are explained in Figure 2.6.

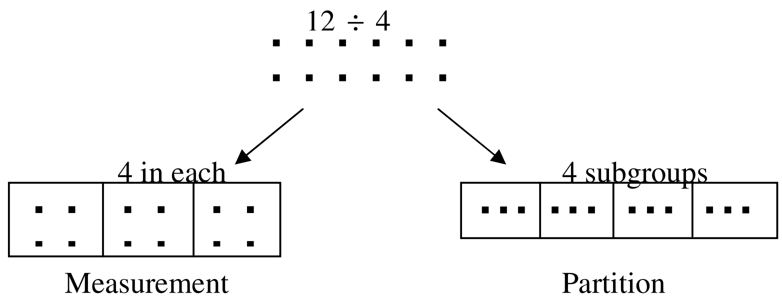



**Figure 2.6.** Measurement and partitive models of division (Burton & Knifong, 1983, p. 466)

In her study, Lamon (1996) analyzed students' partitioning strategies from grades four through eight. She described the strategies where four pizzas were distributed among three people. Her first strategy was preserved-piece strategy where each person is to receive more than one unit of the total quantity being shared, the student marks and cuts only the piece that requires cutting and leaves the 1-unit unmarked. Preserved-piece strategy is given in Figure 2.7.

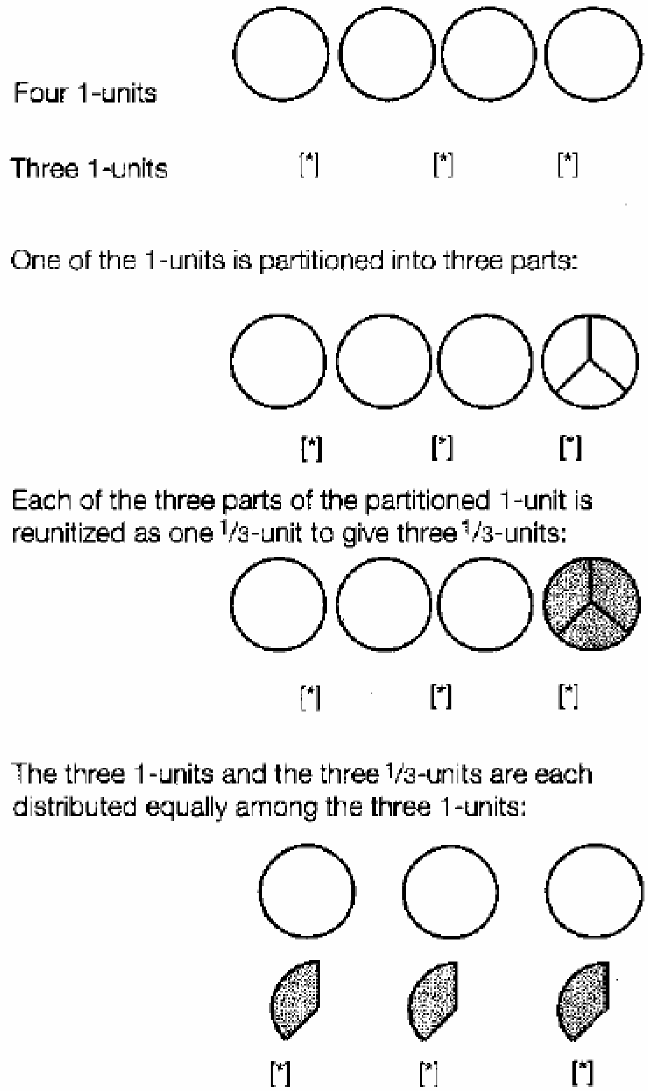



**Figure 2.7.** Preserved-piece strategies for division (Lamon, 1996, p. 175)

Her second strategy given in Figure 2.8 was mark all strategy where all pieces are marked, even those that will remain inact, but only the pieces(s) that require cutting will be cut.

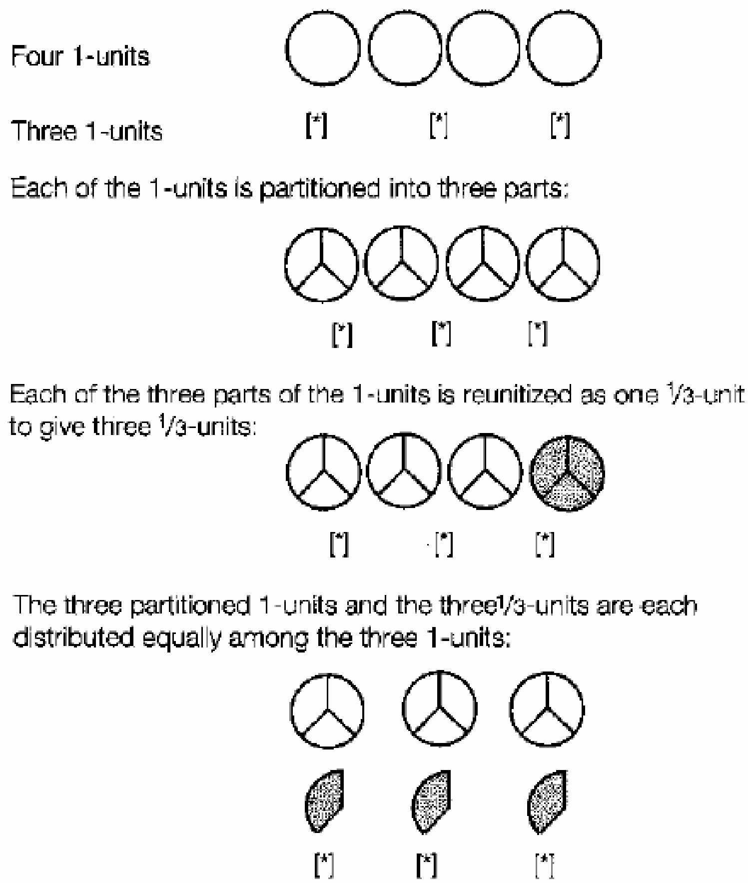



**Figure 2.8.** Mark all strategy of division (Lamon, 1996, p. 176)

Her last strategy was distribution strategy where all pieces of the whole are marked and cut, and smaller pieces are distributed. Distribution strategy is given in Figure 2.9.

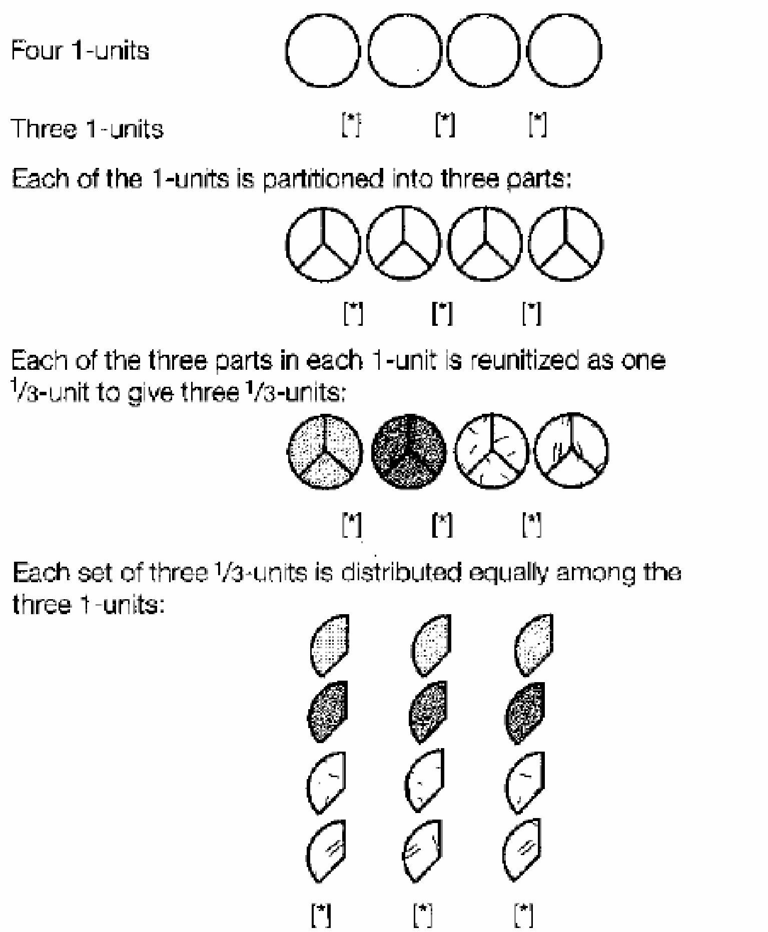



**Figure 2.9.** Distributive strategy (Lamon, 1996, p. 177)

In her study, Ma (1996, 1999) mentioned three different model for the division of fractions. For example, $1\frac{3}{4} \div \frac{1}{2}$ might represent:

- $1\frac{3}{4}\, feet \div \frac{1}{2}\, feet = \frac{7}{2}$ (measurement model)
- $1\frac{3}{4}\, feet \div \frac{1}{2} = \frac{7}{2}\, feet$ (partitive model)
- $1\frac{3}{4}\, squarefeet \div \frac{1}{2}\, feet = \frac{7}{2}\, feet$ (product and factors)

Ma (1999) defined the measurement model for the problem as "Finding How Many $\frac{1}{2}s$ there are in $1\frac{3}{4}$" or "Finding How Many Times $1\frac{3}{4}$ is of $\frac{1}{2}$" (p. 72). Ma stated that sixteen stories were generated by the teachers to illustrate the ideas related to the measurement model of division. Partitive model of Division is defined as "Finding a number such that $\frac{1}{2}$ of it is $1\frac{3}{4}$" (p. 74). Ma reported that among more than 80 story problems representing the meaning of $1\frac{3}{4} \div \frac{1}{2}$, 62 stories represented the partitive model of division by fractions. The third model that is factors and products is defined as "Finding a factor that multiplied by $\frac{1}{2}$ will make $1\frac{3}{4}$. Three teachers use this model in generating their story problems.

As mentioned above the term modeling used in different context in different studies. In this research study, the term model is specifically used for multiplication and division operations. By using the term models of division, I referred to the measurement (or quotitive), partitive models similar to Ma's (1996, 1999) definition. On the other hand, by using the term multiplication models, I referred to the repeated addition, multiplicative compare, area concepts, and Cartesian product models similar to Azim's (1995) categorization. After stating the specific definition for modeling of multiplication and division of fractions, now it is time to turn our attention to the representations of multiplication and division of fractions.

### 2.6 Mode for Representations of Mathematical Concepts

Curriculum reform movement in Turkey emphasized the importance of developing students' abilities in problem solving and communication through multiple representations (MEB, 2005) where; the emphasis was only on the symbolic part of algebraic concepts particularly in algebra courses (MEB, 2002). In order to develop fractional understanding, children should practice the use of multiple representations (Pape & Tchoshanov, 2001). However, results of the examination of middle grade students' abilities in translating among representations of fractions were low due to the limited conceptual understanding on the concept of fractions (Kurt, 2006). Kieren, Nelson, & Smith (1985) highlighted the need for children to build a deep understanding of fractions by using variety of concrete and pictorial models. Majority of students entering elementary and secondary pre-service teacher education programs are not able to select or generate appropriate representations for division of fractions (Ball, 1990b).

The importance of representations used for defining mathematical expressions draws most researchers' attention. Researchers emphasized that representations are cornerstones in both teachers' subject matter knowledge and pedagogical content knowledge (Shulman, 1986; McDiarmid, Ball, & Anderson, 1989). McDiarmid et al., (1989) stated that instructional representations are central to the task of teaching subject matter. "To develop, select, and use appropriate representations, teachers must understand the content they are representing, the ways of thinking and knowing associated with this content, and the pupils they are teaching" (p. 198). Likewise, Ball (1990b) pointed out that teachers should understand the subject in depth to be able to represent it in appropriate and multiple ways like story problems, pictures, situations, and concrete materials. Similarly, Shulman (1986) mentioned that generating a representation to make the topic teachable relies on one's pedagogical content knowledge. However, teachers' subject matter knowledge of a topic is one of the major concerns in building pedagogical content knowledge. Before teaching something, one first has to understand the topic.

Shulman (1986) also noted that pedagogical content knowledge includes the most useful forms of representations of ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations, the ways of representations of the subject that make it comprehensible to others. It also consisted of an understanding of what makes the learning of specific topics easy or difficult.

Behr, Lesh, Post, Silver (1983) proposed a model for the representation of rational numbers. In their project, the major focus is the role of the manipulative materials in assistance and use of rational-number concepts as the children's understanding moves from concrete to abstract. The project also hypothesized the ability to make transition among and within several modes of representation in order to make ideas meaningful to the learners. According to Behr et al. (1983), different materials are useful for modeling different real world situations or different rational-number sub-constructs, and different materials may be useful at different points in the development of rational-number concepts. For example, paper folding may be good for representing part-whole relationship or equivalent fractions but may be misleading for addition of fractions.

In their model, Behr et al., (1983) stated that Lesh re-conceptualized Bruner's enactive mode, partitioned Bruner's iconic mode into manipulative materials and static figural models (i.e., pictures), and similarly partitioned Bruner's symbolic mode into spoken language and written symbols. In addition, in Lesh's model the system of representation was interpreted as interactive rather than linear, and translation within and between modes was given emphasis. Lesh, Post, & Behr (1987) identified five distinct modes of representations in case of mathematical learning and problem solving: (1) real-word situations- where knowledge is organized from real life; (2) manipulatives-like fraction bars, Cuisenaire rods (3) pictures or diagrams-like number lines, region, discrete models (4) spoken symbols-can be everyday language (5) written symbols-specialized sentence and phrases (Lesh, et al. 1987, p. 38). Lesh Multiple Representations Transition Model (LMRTM) was given in Figure 2.10.

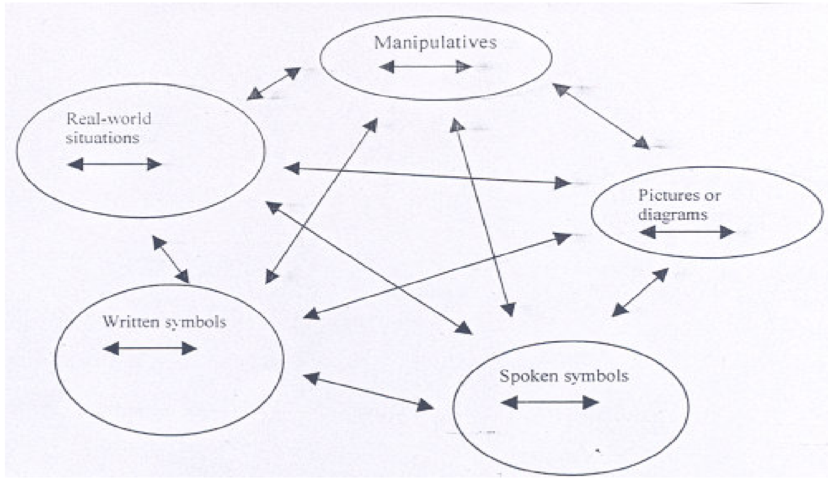



**Figure 2.10.** Lesh et al.'s (1987, p. 38) Multiple Representations Transition Model (LMRTM)

In the Figure, in addition to the five distinct types of representational modes, translation among modes and transformations within them were also important (Ainsworth, Bibby, & Wood, 2002; Behr et al., 1983, Cramer, 2003; Lesh et al. 1987). Thus, translation among representations aim to require students to establish a relationship from one representational system to another keeping the meaning same. According to the model, young learners do not work in a single representational mode while solving a problem. They might think about one part of the problem in one way of representation, such as in a concrete way, but think about another part by using another way, such as written symbol procedures. This model emphasized that realistic mathematical problems are usually solved by translating from the real situation to some system of representation, transforming within the representational system to suggest some solutions, and then translating the result back to the real world. The model also emphasized that many problems are solved using several representational modes. For example, pictures or concrete materials may be used as an intermediary between a real situation and written symbols. The researchers also added that some problems involved more than one mode at the start. They give an example that real addition situations involving fractions, two items to be added, may not be always be two written symbols, or two spoken symbols; they may be one pizza and one written symbol.

Behr et al., (1983) stated that geometric regions especially understanding of the notion of area, set of discrete objects, and the number line are the models most commonly used to represent fractions in the elementary and junior high school. However, according to Leinhardt and Smith (1985) teachers lack a deep understanding of fractions, which puts them at a disadvantage in using multiple strategies in their instructions. Teachers often taught division of fractions simply by inverting and multiplying the fractions instead of allowing students to experience multiple ways of manipulating operations involving fractions. McDiarmid et al., (1989) stated that regardless of how the knowledge is labeled, there is evidence that novice teachers lack adequate knowledge of powerful representations for teaching.

Sharp and Adams (2002) examined the thinking of children who had the opportunity to construct personal knowledge about division of fractions. The authors based this study on a teaching experiment design and used relevant contexts/situations to foster students' development of knowledge. Participants were a group of mixed-ability, 5th-grade mathematics students. They used pictures, symbols, and words to resolve situations and communicate their solutions. The authors analyzed the solutions to describe the students' constructions of division-of-fractions concepts and procedures. All strategies that the students used represented some manifestation of conceptual knowledge about addition and subtraction of fractions and a definition of division of fractions. Some students developed formal symbolic procedures, and others developed pictorial procedures; none invented an invert-and-multiply procedure. Through the window of constructivism, this study allowed the authors to glimpse children's constructions of knowledge and provided alternatives to the traditional view of the expected procedure (invert and multiply) that children should learn for division of fractions. Akkuş-Çıkla (2004) stated that mathematics educators, who seek alternative pedagogical instructions in their mathematics classes, should focus on using multiple-representation based environments where students directed to develop algebraic thinking through conceptual understanding. In addition, in her study, she emphasized the significant effect of multiple representation-based instruction on students' algebra performance compared to the conventional teaching.

In this research study, pre-service teachers' representations used to reason their understanding of multiplication and division of fractions was examined. The representation model for the study was adapted from Lesh model with slight modifications. As Bright, Behr, Post, & Wachsmuth (1998) emphasized that number line model differs from other pictorial models (e.g. regions, discrete objects). That is; number line is totally continuous and requires an integration of visual and symbolic representations where the other representations do not need such integration. In addition, length represented the unit and number line suggested not only repetition of the unit but also subdivisions of all iterated units (Bright et al., 1998). Thus, in this study the parts of the pictorial models were treated separately as individual mode of representations. In addition, the representation system of this research study involves pictorial representations in terms of rectangular area models, pie charts, number line, discrete objects and figural models; symbolic representations involving both verbal and written symbols; manipulatives and real life problems involving word problems.

### 2.7 Summary of the Literature Review

If teacher's role is to help the learner achieve understanding of the subject matter, teachers must obviously receive training and instruction that prepare them to teach. Understanding of content and pedagogy is powerfully influenced by teachers' own experiences as students. Teaching and learning in high schools were limited to a traditional pattern where; faculty treat students as passive recipients of knowledge presented primarily through lecture and textbooks (Boyer, 1987). However, teacher is the only person who makes the final decision about what to teach, when to teach and how to teach (Leinhardt & Smith, 1985). Thus, teachers' knowledge on content and transformation of this content to the students is the heart of effective mathematics classrooms as stated by many researchers. NCTM (1989) pointed out that teachers should need to understand mathematical concepts, identify mathematical relationships, and communicate mathematically. If teachers do not have adequate knowledge they might pass on these ideas to their students. However, as stated above, many researchers have examined pre-service and in-service teachers' knowledge on multiplication and division of fractions (Ball, 1990a, 1990b; Graeber, Tirosh, & Glover, 1989; Leinhardt & Smith, 1985; Ma, 1999; Mack, 1993; McDiarmid & Wilson, 1991; Simon, 1993; Tirosh & Graeber, 1989;) and their results revealed that pre-service and in-service teachers' subject matter knowledge was largely procedural (Frykholm & Glasson, 2005). The results showed that most of the explanations were weak and there was no connection between concepts and procedures.

According to researchers, teachers must know the subject matter thoroughly so that they can present the topic in a more clear and challenging way. If the teacher has a deep understanding of a subject matter, he can represent the topic in multiple ways such as with examples, exercises, demonstrations, metaphors, and activities that can be grasped easily by students (Frykholm & Glasson, 2005; Johston & Ahtee, 2006). Effective teaching is depends highly on both content knowledge and pedagogical content knowledge, on how well one understands the subject matter and on how well one understands ways of transforming subject matter into pedagogically powerful representations (Akkuş-Çıkla, 2004; Crespo & Nicol, 2006; Niess, 2005). However, as stated in the significant of the study and literature part, there were few studies focusing on pre-service teachers' subject matter knowledge and pedagogical content knowledge. Especially, in Turkey, there is no such an investigation on senior pre-service teachers' SMK and PCK and their relationships. In an attempt of examining pre-service teachers' knowledge structures before their graduation from teacher education program is believed to give valuable insights to both policy makers and mathematics educators in terms of understand and develop the nature of this knowledge structures.

Thus, in this research study, my aim was to investigate pre-service teachers' nature of subject matter and pedagogical content knowledge in a specific context.

# CHAPTER III

# RESEARCH METHODOLOGY

The purpose of this study was to examine the pre-service elementary mathematics teachers' subject matter knowledge and pedagogical content knowledge about multiplication and division of fractions. Pre-service teachers' understanding of key facts, concepts, principles, and proofs related to multiplication and division of fractions was one of the concerns of this study. The other focuses were to examine the pre-service teachers' knowledge on common conceptions and misconceptions/difficulties held by elementary students, the possible sources of these conceptions and misconceptions, and the strategies that pre-service teachers would use to overcome these misconceptions. This study also aimed to investigate the representations that pre-service teachers used to reason their understanding and the strategies that pre-service teachers used to explain the key facts, concepts, principles and proofs on multiplication and division of fractions.

In this chapter, the method of inquiry was described in detail. The related issues concerning the context in which the study took place, the participants of the study, the data collection techniques that were used, the procedures of data collection and data analysis were described. In addition, the issues related to the quality of the study were addressed at the end of the chapter.

## 3.1 Conceptual Overview

In order to examine the pre-service elementary mathematics teachers' subject matter knowledge and pedagogical content knowledge in case of multiplication and division of fractions, qualitative research methodology was used to support methodological perspective and findings of the research study.

Denzin and Lincoln (2000) defined qualitative research as a field of inquiry in its own right. Complex, interrelated terms, concepts, and assumptions comprise the qualitative research. Qualitative researchers emphasized the socially constructed nature of reality, and the close relationship between the researcher and the topic studied. Qualitative research uses a naturalistic approach that seeks to understand phenomena in context-specific settings, where "researcher does not attempt to manipulate the phenomenon of interest" (Patton, 2002, p. 39).

Merriam (1998) stated that qualitative researchers are interested in understanding the meaning people have constructed, that is, how they make sense of their world and the experiences they have in the world. Qualitative researcher focus on process and qualitative study is rich in description. Words and pictures are commonly used instead of numbers. Merriam categorized qualitative research methodologies under five headings: basic or generic qualitative study; ethnography, phenomenology, grounded theory, and case study. She added that five methodologies often can work in conjunction with each other. The qualitative design used for this study was a case study. "Qualitative case study is characterized by researchers spending extended time, on site, personally in contact with activities and operations of the case, reflecting, revising meanings of what is going on" (Stake, 2000, p. 445).

When I examined the definition of the qualitative case study, I found slightly different definitions in the literature. Yin (1994) differentiated the case study from other methods like experiments, history, and survey by comparing the characteristics of the related methodologies. Yin (1994) stated:

> A case study is an empirical inquiry that investigates a contemporary phenomenon within its real-life context, especially when the boundaries between phenomenon and context are not clearly evident…Case study inquiry copes with the technically distinctive situation in which there will be many more variables of interest than data points, and as one result relies on multiple sources of evidence, with data needing to converge in a triangulating fashion, and as another result benefits from the prior

development of theoretical propositions to guide data collection and analysis. (p. 13)

Similarly, Merriam (1998) defined qualitative study as "intensive, holistic description and analysis of a single instance, phenomenon, or social unit" (p. 21). Merriam emphasized the importance of the case as *a thing*, *a single entity*, and *a unit that has boundaries*. Stake (1995) mentioned that "case study is the study of the particularity and complexity of a single case, coming to understand its activity within important circumstances" (p. 6).

Smith (1978) stated that basic characteristics of case study that differentiates it from other qualitative research methodologies is that it is intensive descriptions and analysis of a single unit or bounded system, such as individual, event, group, program. Similarly, Sanders (1981) mentioned that "case studies help us to understand processes of events, projects, and programs and to discover context characteristics that will shed light on an issue or object" (p. 44).

From the definitions, it could be deduced that the most important property of the case study is the "unit" or the object the study. In addition, object of the common point is that it is specific, unique and bounded system (Smith, 1978; Merriam, 1989; Stake, 2000; Yin, 1994, 2003). Thus, in case studies researcher should bound the case, conceptualize the object of the study.

Merriam (1998) stated that case studies give a change of examining complex social units consisting of multiple variables in understanding the phenomenon. A case study design employed to gain an in-depth understanding of the situation and meaning for those involved. Stake (2000) added that case study is not a methodological choice but a choice of what is to be studied. Case is a "bounded system" where the coherence and sequence are important. One should identify that certain characteristics are within the system, within the boundaries of case, and others are outside. In addition, Stake (1995) stated that boundness and behavior patterns are important in defining the case. A case study is both a process of inquiry about the case and the product of that inquiry. Moreover, Stake (2000) emphasized that in case study, there is something to describe and interpret. Stake further added that the purpose of case study is not representing the world, but to represent the case. Thus, the purpose of case study is not representing the world, but to represent the case. The use of case study to practitioners and policy makers is in its extension of experience.

My research study could be characterized from Merriam and Stake's point of view. The aim was to "gain an in-depth understanding of the situation and meaning for those who are involved" (Merriam 1998, p. 19) and particularly interested in analyzing the subject mater and pedagogical content knowledge of pre-service teachers. Stake (2000) named the case studies into three categories. He named intrinsic case study where case itself is of interest. The purpose of intrinsic case study is not to understand the some abstract construct, or not to build a theory but to take place because of intrinsic interest. He called instrumental case study if a particular case is studies to provide insight into as issue or to redraw a generalization. Case is in secondary interest where it play essential role in understanding of something else. Researcher examined the case, context in depth, to trail an external interest. Stake, used the term collective case study where; the researcher may jointly study a number of cases in order to examine a phenomenon or context. Similarly, Merriam (1998) categorized case study with respect to overall intent of the study. Namely, these categories are descriptive, interpretive and evaluative case studies. She stated descriptive case study in education concentrates on detailed account of the phenomenon under study. They are useful in presenting basic information on the topic they are studied. On the other hand, interpretive case studies contain thick rich descriptions. These descriptions are used to develop conceptual categories to support theoretical assumptions held by before the data gathering. However, evaluative case studies involved description, explanation and judgment. Thus, I could say that this study was interpretive and instrumental case study since the purpose was to provide an insight and get rich and thick description about the pre-service mathematics teachers' subject matter knowledge and pedagogical content knowledge.

### 3.2 The Design of the Study

In order to investigate the pre-service teachers' subject matter knowledge and pedagogical content knowledge on multiplication and division of fractions, senior pre-service elementary mathematics teachers from the Middle East Technical University in a four-year teacher education program in the department of elementary mathematics were selected. Basically, I was interested in the nature of the subject matter and pedagogical content knowledge. Thus, whether pre-service teachers have enough knowledge after their graduation from the education program directed me to focus on the case for senior pre-service teachers.

Yin (1994, 2003) stated that in a case study design there are situations where it was impossible to separate the phenomenon's variables from their context. Yin emphasized that in such a design, same case study may involve more than one unit of analysis. Thus, in a single design the attention was given to the subunits. Yin's model for the embedded case study design, where single-case design embedded multiple units of analysis is given in Figure 3.1. Yin (2003) emphasized that in a case study, the boundaries and unit of analysis should be defined in order to be informative on the research design.

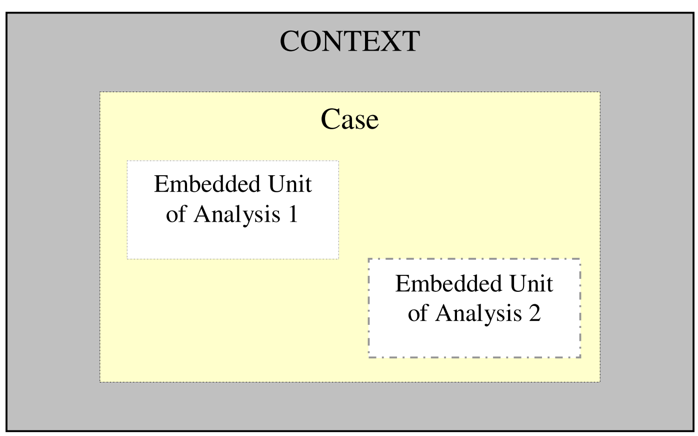



**Figure 3.1.** Embedded case study design: Single-case design embedded multiple units of analysis (Yin, 2003, p.40)

In this study, the context that was elementary mathematics education program, pre-service teachers and their subject matter and pedagogical content knowledge were thought all together. In this study, senior pre-service teachers constitute the “case” of the study. Subject matter knowledge and pedagogical content knowledge of pre-service teachers’ were embedded “unit of analysis’ one and two respectively. In Figure 3.2, the model for elementary mathematics education program, pre-service teachers and their subject matter and pedagogical content knowledge were given. Since the aim of the study was to investigate the subject matter knowledge, pedagogical content knowledge, and assumed relationship between subject matter and pedagogical content knowledge, the model also emphasized the intersection that is relationship between these knowledge. In the following sections, I gave detailed information on pre-service program and pre-service teachers enrolled in the research study.

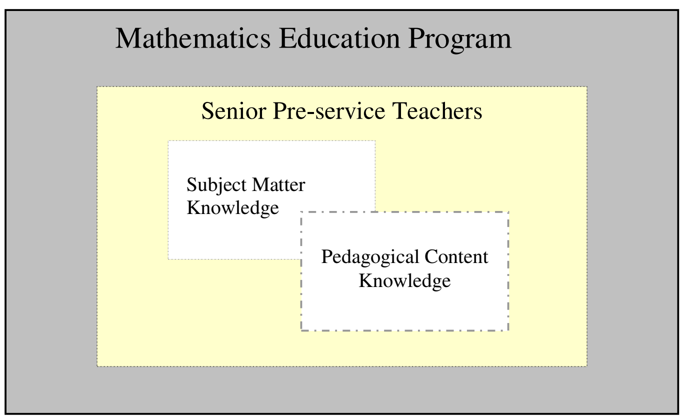



**Figure 3.2.** Embedded case study design of the research: Single-case design embedded two units of analysis

### 3.2.1 The Elementary Mathematics Education Program

In this part, the context of the study that was the elementary mathematics education program was described. In order to graduate from the Elementary Mathematics Education (EME) Program at Middle East Technical University, pre-service teachers take mathematics and mathematics education courses, as well as physics, chemistry, English, Turkish, history, statistics and general educational science courses. The EME program emphasizes high order skills and professional development of the pre-service teachers. The graduates of the program are qualified as mathematics teachers in elementary schools from grade 1 to grade 8 (Middle East Technical University, 2003). The EME program mainly focuses on mathematics and science courses in the first and second years followed by the mathematics teaching courses in the third and fourth years. The program includes nine courses from the Department of Mathematics, four courses from the Department of Educational Sciences and 12 courses from the Department of Elementary Education. There are a total of four courses in chemistry, biology and physics content to support the minor degree in the Elementary Science Education.

Pre-service mathematics teachers engage in mathematics teaching and learning process mostly during their teaching practice courses and teaching method courses. School experience and teaching practice courses are offered at the second, seventh, and eighth semesters. The first school experience course is based mostly on observation of the classroom without involving active teaching. However, second school experience and teaching practice courses are generally based on both observation and practice. Pre-service teachers are expected to be actively involved in teaching and learning process during those courses. Methods of Science and Mathematics Teaching course offered in the sixth semester of the program and Methods of Mathematics teaching offered in the seventh semester of the program with the School experience II. The courses offered by the program are given in Table 3.1 (Middle East Technical University, 2003).

**Table 3.1.** Courses taken by the pre-service mathematics teachers

| FIRST YEAR | |
|---|---|
| First Semester | Second Semester |
| MATH 111 Fundamentals of Mathematics<br>MATH 119 Calculus with Analytic Geometry<br>PHYS 181 Basic Physics I<br>ENG 101 Development of Reading and Writing Skills I<br>EDS 119 Introduction to Teaching Profession<br>IS 100 Introduction to Information Technologies and Applications | MATH 112 Introductory Discrete Mathematics<br>MATH 120 Calculus for Functions of Several Variables<br>PHYS 182 Basic Physics II<br>ENG 101 Development of Reading and Writing Skills II<br>ELE 132 School Experience I |
| SECOND YEAR | |
| Third Semester<br>MATH 115 Analytical Geometry<br>MATH 201 Elementary Geometry<br>CHEM 283 Introductory General Chemistry<br>EDS 221 Development and Learning<br>ENG 211 Academic Oral Presentation Skills<br>HIST 2201 Principles of Kemal Atatürk I | Fourth Semester<br>MATH 116 Basic Algebraic Basic Algebraic Structures<br>MATH 219 Introduction to Differential Equations<br>BIO 106 General Biology<br>ELE 224 Instructional Planning and Evaluation<br>ELE 300 Computer Applications in Education<br>HIST 2202 Principles of Kemal Atatürk II |
| THIRD YEAR | |
| Fifth Semester<br>MATH 260 Linear Algebra<br>ELE 317 Instructional Development and Media in Mathematics<br>ELE 331 Laboratory Applications in Science I<br>TURK 305 Oral Communication<br>Elective I<br>Elective II | Sixth Semester<br>ELE 240 Probability and Statistics<br>ELE 332 Laboratory Applications in Science II<br>ELE 336 Methods of Science and Mathematic Teaching<br>EDS 304 Classroom Management<br>TURK 306 Written Communication<br>Elective III |
| FOURTH YEAR | |
| Seventh Semester<br>ELE 437 School Experience II<br>ELE 443 Methods of Mathematics Teaching<br>ENG 311 Advanced Communication Skills<br>Elective IV<br>Elective V | Eight Semester<br>ELE 420 Practice Teaching in Elementary Education<br>ELE 448 Textbook Analysis in Mathematics Education<br>EDS 424 Guidance<br>Elective VI |

### 3.2.2 Participants of the Study

In this part, senior pre-service teachers that are the “case” of the study were described. In a research study, once the research questions were identified, the next task was the selection of the unit of analysis, the participants. Since the generalization in statistical concern was not the goal of qualitative research, non-probabilistic sampling was the method of choice (Merriam, 1998). The most common form of non-probability sampling, purposeful sampling was used in this study. Purposive sampling was based on the assumption that the investigator wanted to discover, understand, and gain insight and therefore must select a sample from which the most can be learned (Merriam, 1998). For the qualitative study, we used purposive sampling where cases were selected from who we feel we can learn most, we can most access, and one can spend the most time with potential for learning. Purposive sampling is a different and sometimes superior that the representativeness (Stake, 2000). Patton (1987) stated that power of purposive sampling lies in selecting information rich cases in order to get in depth information and in purposive sampling the important thing is to determine the selection criteria that is the interest of the study (Merriam, 1998).

In this research study, it was important for the researcher to investigate pre-service teachers' subject matter knowledge and pedagogical content knowledge. Thus, rather than using the quantitative approach, selecting a random sample from a large group of pre-service teachers, researcher chose to use purposive sampling.

The present study has two phases for the sampling procedure. In the first phase, twenty eight senior pre-service teachers from the Middle East Technical University (METU) in a four-year teacher education program in the department of elementary mathematics education took the Multiplication and Division of Fractions Questionnaire (MDFQ). I purposively selected pre-service teachers from METU based on certain criteria. The first task was to decide on the existence for more than five years where the university has potentials of educating highly qualified teachers. In addition, I should easily able to access to the participants to carry out deep investigation since I should spend most of time with the participants. Thus, the accessibility of the sample gained importance. After deciding on the university, senior teachers who were believed to be the ones who had the highest potential of obtaining deep information on subject matter knowledge and pedagogical content knowledge were selected as the potential subjects. The underlying rationale for choosing senior pre-service teachers was their experience in the undergraduate program. The main purpose of this study was to examine the pre-service elementary mathematics teachers' subject matter knowledge and pedagogical content knowledge in case of multiplication and division of fractions. Senior students who were participated in the first phase of the study had already completed all the courses offered by the teacher education program where; they were the potential participants in order to have deep insight in what sort of knowledge, thought, understanding, and experiences were critical in understanding the conceptions of pre-service teachers on multiplication and division of fractions. This is why researcher, by using purposeful sampling technique, preferred to study with senior pre-service teachers.

Thus, in order to get deep insight about the pre-service teachers' understanding of key facts, concepts, principles, proofs on multiplication and division of fractions, their knowledge on common conceptions/misconceptions and possible sources of these conceptions/misconceptions, and the strategies that pre-service teachers use to overcome these misconceptions, researcher decided to concentrate on senior pre-service teachers. In addition, senior students had already completed teaching mathematics courses; where they had the opportunity to study the elementary and middle school curriculum that we might assume that they had some degree of understanding about the teaching process and had experiences in schooling. At this stage, total (N) 28 senior pre-service teachers participated in the study. Some demographic characteristics of the 28 senior pre-service teachers' such as gender, general attitude toward teaching, and their confidence in mathematics are given in Table 3.2. Pre-service teachers' general attitude toward teaching was rated by using likert type scale; very like, like, undecided, and don't like. Similarly, confidence in mathematics was rated by using a four-point scale as very high, high, medium, and low. From the table we could deduce those participants' preferences for being mathematics teacher and their confidence in mathematics teaching could be regarded as high.

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**Table 3.2.** The number, gender, general attitude toward teaching, and confidence in mathematics teaching

| | General attitude toward teach. | | | | | Confidence in math. Teach | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | V.like | Like | Undec | Don't Like | N | V. High | High | Med | Low | N |
| Senior Female | 11 | 8 | - | - | 19 | 9 | 8 | 2 | 0 | 19 |
| Senior Male | 3 | 6 | - | - | 9 | 3 | 6 | - | - | 9 |
| Total | 14 | 14 | | | 28 | 12 | 14 | 2 | | 28 |

After implementing MDFQ, all the senior students were asked whether they were voluntary to participate in the semi-structured interviews. Subsets of the senior teachers were participated in interview that was designed to get additional information about their subject matter and pedagogical content knowledge. In that phase, seventeen senior (13 female, 4 male) pre-service teachers were willing to participate in interview. The sampling procedure and participants of the study is summarized in the Figure 3.3.

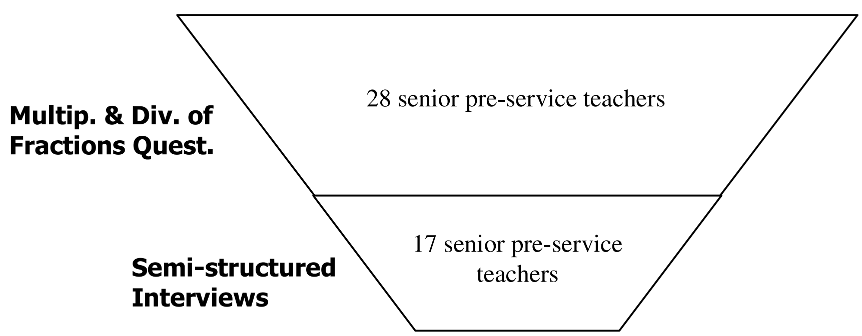



**Figure 3.3.** Sampling procedure and participants of the study

From 17 senior pre-service teachers, ten graduated from Anatolian Teacher Education High School, five from Anatolian High school, and two from high school. By the time of data collected, these pre-service teachers had completed all the courses that the EME program offers. As an elective course nine pre-service teachers took problem solving in mathematics course, one pre-service teacher took distance education course, three took project, four took geometer sketchpad course, and three took research methodologies course. Most of the pre-service teachers had teaching experience apart from school experience and practice teaching courses offered by the program. Among them, six worked in test preparation centers; thirteen of them gave private courses throughout their university education, and 6 of them worked voluntarily as a mathematics teacher in specific organizations. In addition, nine teachers worked with elementary school students range from first to fifth graders. Sixteen pre-service teachers worked with the middle school students, ten with high school students, and nine of them with the students who were preparing for the University Entrance Examination. Analysis of the research study was based on those 17 pre-service teachers.

### 3.3 Data Collection

Data is collected from the pre-service teachers enrolled in a elementary mathematics teacher education program at Middle East Technical University at the end of the spring semester of 2004-2005 academic year. A schedule indicating the order of events conducted for the data collection is given in Table 3.3. Details about the each parts of the design are explained in the sections that follow.

In order to get deep information from the pre-service teachers, different data collection procedures were used. Creswell (1998) referred this type of data collection as 'multiple source of information'. The questionnaires and interviews were utilized to get information.

**Table 3.3.** Timeline for data collection

| Date | Events |
|---|---|
| May 2004- April 2005 | Development of the data collection tools (questionnaire, interview protocol) |
| May 2005-June 2005 | Pilot study of the instruments and last version of data collection tools |
| June 2005-August 2005 | Data collection-Implementation |

Since the primary purpose of this study was to understand conceptions, situations and to develop some theoretical constructs on these conceptions, the method for obtaining research data was qualitative as stated before. The questionnaire and semi-structured interviews enabled me to investigate what sort of knowledge, thought, understanding, and experiences were critical in understanding the conceptions of pre-service teachers on multiplication and division of fractions. As I collected data, I always tried to ask myself why events or facts appear as they do which also helps me while analyzing my data. Following sections give further information about the measuring tools.

## 3.4 Data Sources

This study investigated the pre-service teachers' subject matter knowledge and pedagogical content knowledge on multiplication and division of fractions. To gather and triangulate information, following data collection tools were used: 1) A questionnaire on pre-service teachers' subject matter knowledge and pedagogical content knowledge; 2) interviews following the questionnaire on multiplication and division of fractions.

### 3.4.1 The MDFQ Questionnaire

In order to understand the subject matter knowledge and pedagogical content knowledge of pre-service teachers on multiplication and division of fractions, Multiplication and Division of Fractions Questionnaire (MDFQ) was developed by the researcher based on instruments used by several researchers (Ball, 1990a; Ma, 1996, 1999; Schifter, 1998; Simon, 1993; Tirosh, 2000). The questionnaire focused on assessing pre-service teachers' subject matter knowledge and pedagogical content knowledge on multiplication and division of fractions.

The questionnaire consisted of two types of questions. There were 10 open-ended questions with sub-dimensions for each question. The open-ended structured questions prevent a distortion of the result due to the chance factors. The first type of questions was designed to measure the pre-service teachers' subject matter knowledge of multiplication and division of fractions. The items were prepared to measure the pre-service teachers' understanding of key facts, concepts, principles, and proofs on multiplication and division of fractions. The objectives for the subject matter knowledge questions were designed based on the concepts covered in elementary and middle school mathematics curriculum. The objectives for the subject matter knowledge questions are given in Appendix A. In order to increase the face validity of the questionnaire, objectives were also grouped according to the levels based on the research questions asked. A table of specification for objectives and questionnaire items based on this classification is given in Appendix C. Subject matter knowledge was categorized as pre-service teachers' knowledge on basic operations, verbalizing expressions, basic facts, principles and proofs on multiplication and division of fractions. The Turkish version of the questionnaire items were provided in Appendix B.

The second type of the questions was designed to measure the pre-service teachers' pedagogical content knowledge of multiplication and division of fractions. Items were prepared to measure the pre-service teachers' knowledge on common conceptions and misconceptions held by the elementary students, the possible sources of these conceptions and misconceptions, and the strategies that pre-service teachers use to overcome these misconceptions. The objectives of the pedagogical content knowledge items were categorized into students' conceptions, representations, and strategies. The objectives for the pedagogical content knowledge items were given in Appendix A. A table of specification for objectives and questionnaire items were given in Appendix C. In a table of specification, pedagogical content knowledge is categorized as knowledge of students' conceptions, representations, and strategies.

Content validity for the questionnaire was established by having the questionnaire reviewed by the two mathematics education professionals, prior to administration, to determine if the items were consistent with the stated objectives. Ninety percent agreement was reached by the reviewers in the first round. Then, questionnaire was revised until hundred percent agreement was reach among the reviewers. A detailed description of the items in the questionnaire was given below.

The first and the third questions were prepared by the researcher in order to understand the nature of the pre-service mathematics teachers' knowledge on key facts, concepts and principles on multiplication and division of fractions. In these questions, pre-service teachers' knowledge on common conceptions and misconceptions/difficulties that sixth and seventh grade students held, the source of these misconceptions were also investigated. In addition, strategies that pre-service teachers used to explain their conceptions on multiplication and division of fractions were examined. These questions were prepared based on the purpose of the research study.

The first question:

Mert has 7 chocolate bars. He decided to give one-third of these chocolates to his close friend Emre. How many chocolates will Emre get?

**a**. Write a mathematical expression for the problem

**b**. Find the answer to the problem

**c**. List two common mistakes students in sixth or seventh grade may make while performing (a) and/or (b)

**d.** Describe possible sources for each of these mistakes depending on students' thinking

**e**. How will you overcome these difficulties?

**f**. Use a representation/model to solve (explain) the problem to your students.

The third question:

Elif bought a bottle of milk. She gave $\frac{1}{2}$ of it, which was $1\frac{3}{4}$ lt to her grandmother. How much did the bottle of milk originally contain?

**a**. Write a mathematical expression for the problem

**b**. Find the answer to the problem

**c**. List two common mistakes students in sixth or seventh grade may make while performing (a) and/or (b)

**d.** Describe possible sources for each of these mistakes depending on students' thinking

**e**. How will you overcome these difficulties?

**f**. Use a representation/model to solve (explain) the problem to your students.

The second question was taken and adapted from Tirosh (2000) which aimed to understand the pre-service teachers' conceptions on partitive and quotitive division. After the necessary permission from the author has been taken, necessary revisions and additions are made in order to understand the nature of the subject matter knowledge and pedagogical content knowledge of pre-service teachers.

The second question:

For each of the following word problems **(a)** Write an expression that will solve the problem (do not calculate the expression), **(b)** List two common mistakes students in sixth or seventh grade may make, **(c)** Describe possible sources for each of these mistakes **(d)** How will you overcome these difficulties? **(e)** Use a representation/model to solve (explain) the problem to your students.

**(i)** Four friends bought $\frac{1}{4}$ kilogram of sweets and shared it equally. How much sweet did each person get?

**(ii)** Four kilograms of cheese were packed in packages of $\frac{1}{4}$ kilogram each. How many packages were needed to pack all the cheese?

The fourth question was prepared by the researcher in order to understand the nature of the pre-service mathematics teachers' knowledge on key facts, concepts and principles on multiplication of fractions. In this question investigation of pre-service teachers' subject matter knowledge and pedagogical content knowledge based on the research questions also aimed.

The fourth question:

Consider the following multiplication of fraction problems and answer the following questions for each of them.

**a**. $\frac{2}{3}\times\frac{3}{5}$ **b**. $1\frac{1}{2}\times\frac{1}{3}$,

**i**. List two common mistakes students in sixth or seventh grade may make while performing the operations

**ii.** Describe possible sources for each of these mistakes depending on students' thinking

**iii**. How will you overcome these difficulties?

**iv.** Use a representation/model to solve the problem to your students.

The fifth question was adapted from several studies (Ball, 1990; Simon, 1993; Ma, 1996, 1999; Schifter, 1998) where; the representations used by the pre-service teachers to reason their understanding of multiplication and division of fractions was investigated. In addition, the strategies that pre-service teachers use to explain the given operations were also investigated.

The fifth question:

For the expressions:

**a.** $11\frac{1}{2}\times\frac{1}{4}$ **b**. $6\frac{2}{3}\div\frac{5}{6}$

**(i)** Write a story problem or describe a real world situation for each of the following expressions where the computation would solve the problem.

**(ii)** Which strategy will you use to solve (explain) each of these problems to your students?

The sixth and the ninth questions were designed by the researcher to investigate the understanding of the pre-service teachers' conceptions on multiplication and division of fractions. Again, the strategies used by the pre-service teachers in reasoning their understanding were investigated.

The sixth question:

For the expression $\frac{2}{3} \div \frac{1}{2}$, Ceren said that "$\frac{2}{3} \div \frac{1}{2} = \frac{3}{2} \times \frac{1}{2}$", Cenk said that "$\frac{2}{3} \div \frac{1}{2} = \frac{3}{2} \times \frac{2}{1}$" and Eda said "$\frac{2}{3} \div \frac{1}{2} = \frac{2}{1} \times \frac{2}{3}$".

**a.** Who is right? (Please explain your answer) What might each of the students be thinking?

Ceren…….. Cenk…….... Eda……..

**b.** Which strategy will you use to solve (explain) the equation $\frac{2}{3} \div \frac{1}{2} = \frac{2}{3} \times \frac{2}{1}$ to your students?

The ninth question:

How would you explain the following operations to your students and why?

**a.** $\frac{3}{4} \times \frac{4}{3} = 1$ **b.** $1 \div \frac{2}{3} = \frac{3}{2}$ **c.** $\frac{2}{3} \div \frac{1}{3} = 2$ ?

The seventh and eighth questions were taken from the questionnaires used by Tirosh (2000) and Singmuang (2002). Again, pre-service teachers' conceptions on multiplication and division of fractions and strategies used to reason their understandings were investigated.

The seventh question:

Berk argues that he prefers to divide fractions in a way similar to multiplication. For instance $\frac{2}{9} \div \frac{1}{3} = \frac{2 \div 1}{9 \div 3} = \frac{2}{3}$,

**a**. Would you accept Emre's proposal? Why?

**b**. How will you use to explain the solution to Emre?

The Eighth question:

Tuğçe argues that distributive law can be used to calculate $1\frac{3}{4} \div \frac{1}{2}$.
She suggests the following:

$$
\begin{aligned}
1\frac{3}{4} \div \frac{1}{2} &= (1+\frac{3}{4}) \div \frac{1}{2} \\
&= (1 \div \frac{1}{2}) + (\frac{3}{4} \div \frac{1}{2}) \\
&= 2 + 1\frac{1}{2} \\
&= 3\frac{1}{2}
\end{aligned}
$$

**a**. Would you accept Tuğçe's proposal? Why?

**b.** How will you explain the solution to Tuğçe?

The tenth question was taken and adapted from Tirosh (2000) and Singmuang (2002). In this question, how pre-service teachers construct their understanding on principles, and proofs on multiplication and division of fractions and the strategies used by the pre-service teachers in reasoning their understandings were investigated.

**a. (i)** Why does the "multiply numerator of first fraction by numerator of the second fraction and denominator of the first fraction by denominator of the second fraction" rule work for the multiplication of rational numbers?

(Why $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$, where a, b, c and d are integers and b and d not zero)

**(ii)** Which strategy will you use to solve (explain) the proof to your students?

**b. (i)** Why does the "invert and multiply" rule work for the division of rational numbers?
(Why $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$, where a, b, c and d are integers and b, c, and d not zero)

**(ii)** Which strategy will you use to solve (explain) the proof to your students?

The questionnaire was piloted before used in the study. The purpose of the pilot study was to help me refine my data collection plans in terms of both content of data and procedures to be followed (Yin, 2003). The main criteria for selecting the pilot case were the access and convenience of the pilot cases. With the help of pilot study, I had considerable insight into the basic issues and final case study protocol before beginning the actual study.

Pilot study provided space for respondents to make criticism and recommendations for improving the questionnaire. Pilot study was included 31 freshman pre-service teachers, a sample of individuals from the context which the researcher drew respondents. Students were asked to state in their words what they understand from the questions, and if there were some unclear points for them. The questionnaire revised and retested by using all the suggestions until there was an agreement between researcher and on the final version.

The final version of the MDFQ questionnaire was administered to 28 senior pre-service teachers enrolled in the teacher education program. Pre-service teachers were informed verbally and in writing that their identities would remain confidential at all times, with pseudonyms being used in all reports related to the study. They were also informed that they would have the option of withdrawing their consent at any point during the study. The MDFQ was administered to the pre-service teachers at their regular course hours, and all the pre-service teachers who attended to the course on that day volunteered to take the questionnaire.

### 3.4.2 Interview of Pre-service Teachers

Interviewing is an essential data collection procedure for this study since we can not observe feelings, thoughts of the participants and their interpretation of the world around them without a face to face interaction. Interviewing is often the major source of qualitative data needed for understanding the study (Merriam, 1998), and are one of the most important sources of information for the case studies (Yin, 2003). Yin categorized interviews under three headings: open-ended interviews, focused interviews, and more structured survey-like interviews. In this study, focus interview is used where; semi-structured interview protocol containing both open ended questions and certain set of questions were used. Interview questions were prepared to collaborate that certain facts were established on the nature of pre-service teachers' subject matter and pedagogical content knowledge.

After administering the MDFQ to senior pre-service teachers, interviews were conducted to obtain a more complete picture of the pre-service teachers' subject matter and pedagogical content knowledge. In addition to the MDFQ questionnaire, data from the interviews provided information for the teachers' explanations on their knowledge.

Semi-structured interviews were conducted with 17 senior pre-service teachers who were voluntarily participated in the second part of the data collection. The interview consisted of three parts: (1) Background questions about pre-service teachers and some related questions on subject matter knowledge and pedagogical content knowledge that did not appear on questionnaire, (2) Questions on general knowledge on multiplication and division of fractions, and (3) Questions based on the review of the responses to the MDFQ questionnaire. All the parts of the interview were prepared to get deep information on both subject matter knowledge and pedagogical content knowledge of pre-service teachers. The entire guide interview was given in Appendix D.

Two mathematics education professor were asked to determine the face validity of the interview questions. These experts were asked to determine whether the interview questions were matched the research questions and the goal of the study. Experts were also asked to determine whether the questions were leading or biased. There is a 95% agreement among the mathematics education professors on interview questions. Since one of the purposes of the interview questions was to understand the pre-service teachers' answers to the questionnaire, the interview questions were piloted. For the pilot study, three pre-service teachers who would not participate in the actual study were interviewed before the actual interview protocol had been constructed. Pilot interviews were important in trying to figure out the questions since questions which were confusing, need rewording, and yield useless data could be identified during pilot studies (Merriam, 1998).

During the pilot interviews, participants were asked whether the interview questions were clear or not and if there are any suggestions to modify the interview questions to give more insight to the research study. Information from the pre-service teachers led me to construct a final version of the interview protocol. All the interviews were audio, and videotaped, recorded and transcribed.

**3.5 Data Analysis**

Data collection and analyzing is simultaneous activity in qualitative research and data analysis in a case study research is aimed to provide intensive and holistic description of the case (Merriam, 1998; Yin, 2003). Case study research is a challenging experience for the researcher because of the absence of routine formulas (Yin, 2003, p. 57).

The role of the researcher is to make sense of data through interviews, observations, fieldworks, and other documents along the interpretation of the findings (Merriam, 1998). Merriam added that data analysis is a complex procedure consisting moving back and forth between concrete bits of data, abstract concepts, between inductive and deductive reasoning. Data analysis consisted of examining, categorizing, tabulating, testing, or otherwise recombining both quantitative and qualitative evidence to address the initial propositions of a study (Yin, 2003, p. 109).

Merriam (1998) categorized qualitative data analysis under six categories: ethnographic analysis, narrative analysis, phenomenological analysis, the constant comparative method, content analysis and analytic induction. In this study, in order to provide intensive and holistic description of the case constant comparative method is used. The relationship among categories was used to develop a model rather than a theory.

The constant comparative method of data analysis that was commonly used in all qualitative studies (Merriam, 1998) was developed by Glaser and Strauss (1967). In this method the researcher begins with a particular occasion and compares it with another occasion in the same or another set of data. These comparisons lead the tentative categories which are then compared with each other as well. That is this method involves comparing one segment of data with another to determine similarities and differences (Merriam, 1998) and then data was grouped under similar dimensions. This dimension was tentatively given a name and then it became a category. The overall goal of the analysis was to seek patterns in data where the important point is that researchers should focus on describing the study in sufficient detail so that reader can make comparison (Stake, 2000).

Glaser and Strauss (1967) proposed several steps in order to analyze data with the constant comparative method. In the first step of this method which was creating categories and codes, “the analyst starts by coding each incident in his data into many categories of analysis as much as possible, as categories emerge or as the data emerge that fit an existing category” (Glaser & Strauss, 1967, p.

105). In this method, in general, the researcher started the analysis of the data that could be interview transcript, observation, or document. While the researcher reading the first data set, he developed a list of concepts (units of data) under different categories. Before putting the concept under specific category, he needed to compare the concept with the other concepts under different categories. Then, the concepts moved to the category where they shared the most familiar characteristics. At the end of the analysis of the first data set, researcher had certain categories and list of concepts under these categories and researcher moved to the second data set. He carried out the same procedure with this new data set and at the end researcher came up with similar or new categories compared to the first data set. Then, she compared the list of concepts under the given categories in both data set and tried to get extensive list of concepts for the fix data. He continued the same procedure after finishing the coding of each new data set.

After certain categories and the concepts determining the properties of the categories emerge, researcher used three approaches to name her categories. Merriam, (1998) mentioned that names of the category comes from three different sources: researcher, participant and literature. Similarly, Glaser and Strauss (1967) stated that researcher could construct the names for the categories based on her experiences with the data. On the other hand, researcher could give names to the categories based on the framework or words from the participants' statements. Lastly, researcher give the names to the category by using literature or coding from previously related studies.

In the second step of the constant comparative method of data analysis, "categories are related to their subcategories to form more precise and complete explanations about phenomena" (Strauss & Corbin, 1998, p. 124). At this stage, researcher started to integrate the categories based on their properties. Researcher started to integrate the categories when the categories were saturated. That was when the coding procedure brought to end producing the concepts for the categories. On the other hand, new categories might still emerge from the data. In the final step of the analysis, researcher discovered consistency among the categories and within their properties. Relationship and patterns among the categories gave light to the researcher to formulate the theory based on the data (Glaser & Strauss, 1967).

In this study, where an attempt to produce an in-depth description of pre-service teachers' subject matter knowledge and pedagogical content knowledge on multiplication and division of fractions, questionnaires and the semi-structure interviews of 17 pre-service teachers were analyzed. I started analyzing the questionnaire, and tried to make coding based on the statements of the participants and related literature framework. I compared the incidents in the same data set and tried to generate the categories based on these comparisons. After comparing the concepts (incidents or unit of data) within the given data set, I compared these concepts with the other participants' questionnaires and after finishing the comparison questionnaires with each other, I extended my categories and the concepts underlying these categories. This list outlines the primitive classification system of my data reflecting the recurring patterns and these recurring patterns become categories or themes. Analysis process involved searching through data and recording words, phrases for the patterns on pre-service teachers' subject matter knowledge and pedagogical content knowledge. These words, phrases are then used as coding categories to synthesize and organize the data.

When I was coding the given data set, I had trouble to place the codes under related categories. Sometimes, I hesitated whether the given concepts should move to the existing category or I should create new category to relate the given concepts. At this time, I always wrote some memos next to the concepts which I thought during that time and in later stage of the analysis these memos help me to construct the relationship among the concepts and categories. I continued to compare the concepts with each other until the categories were saturated. That is when I could not find and new categories or concepts while reading the data set. I sometimes used verbatim of participants in constructing category name. Sometimes, I constructed the names based on my understanding about the given data set but in other occasions I take them from the literature based on the framework.

After finishing analysis of all the questionnaires, I started to integrate the categories with their subcategories that share similar characteristics. I followed the same strategy while analyzing the interview transcripts as coding the questionnaires. Final categories obtained from the questionnaires, were compared with the categories obtained from the interview transcripts. While comparing the categories obtained from questionnaires with the categories obtained from interviews, I sometimes construct new categories and subcategories. On the other hand, sometimes I deleted same categories that are less informative or can be a subset of other category. I also display set of categories in the form of chart or table in order to see the relationships in coherence.

Finally, the whole data set condensed into the several categories and subcategories and this approach helped me to develop a model for the nature of the subject matter knowledge and pedagogical content knowledge of pre-service teachers. After explaining the method of analysis of data, following sections give information on studies carried out to increase the quality of the research study.

### 3.6 Trustworthiness

Patton (2002) stated that validity and reliability are two important concepts that any researcher should consider while designing a study, analyzing results, and judging the quality of the study.

Joppe (2000) defined validity in quantitative study as **"**whether the researcher truly measures that which it was intended to measure or how truthful the research results are" (p. 1). In addition, Joppe (2000) defined reliability in quantitative study as "…The extent to which results is consistent over time and an accurate representation of the total population under study" (p. 1). Similarly, Yin (1994, 2003) discussed four tests to judge the quality of the case designs. These tests are construct validity, internal validity, external validity, and reliability. However, reliability and validity issues were perceived differently by some qualitative researchers where; they taught that these concepts expressed in quantitative research areas is inadequate. For instance, Glesne and Peshkin (1992) mentioned the question of replicability in the results does not concern of qualitative research. However, accuracy (Winter, 2000) credibility, and transferability (Hoepf, 1997) give insight evaluation in qualitative research. Thus, many qualitative researchers preferred to use different terminology instead of using the terms validity and reliability (Golafshani, 2003; Shenton, 2004).

Ensuring trustworthiness in qualitative research projects is important in judging the quality of study (Lincoln & Guba, 1985; Seale, 1999; Stenbacka, 2001). Lincoln and Guba (1985) identified credibility, transferability, dependability, and confirmability as indicators of trustworthiness in qualitative studies. Shenton (2004) by addressing the similar concepts, connected these terms with the ones used in quantitative research as: credibility (in preference to internal validity), transferability (in preference to external validity/generalisability); dependability (in preference to reliability); and confirmability (in preference to objectivity).

Literature review showed that although reliability and validity in quantitative research treated separately, they were not apart in qualitative research and different researchers used different terminologies (Creswell & Miller, 2000; Hoepf, 1997; Lincoln & Guba, 1985, Seale, 1999; Stenbacka, 2001, Patton, 2002; Winter, 2000) to refer the terms validity and reliability. Thus, instead of using the term validity and reliability, I preferred to use the term trustworthiness and certain criteria for judging the quality of research study was described below.

Lincoln and Guba (1985) argued that ensuring credibility is one of the most important factors in establishing trustworthiness. According to Merriam (1998), qualitative investigator's equivalent concept, i.e. credibility, deals with the question "How congruent are the findings with reality? Are investigators observing or measuring what they think they are measuring?" (p. 201). Merriam (1998) suggested six basic strategies to enhance internal validity under six headings: Triangulation-using multiple sources, multiple investigators, or multiple methods, member checks, long-term observation, peer examination, participatory or collaborative modes of research and research's biases. Similarly, Yin (1994, 2003) suggested the strategies: pattern-matching, explanation-building, addressing rival explanations and using logic models during the data analysis to overcome the problem of internal validity.

Triangulation, one method to establish credibility, is a procedure where researchers search for convergence among multiple and different sources of information to form themes or categories in a study (Creswell & Miller, 2000). "Triangulation strengthens a study by combining methods. This can mean using several kinds of methods or data, including using both quantitative and qualitative approaches" (Patton, 2002, p. 247). "Triangulation has been generally considered a process of using multiple perceptions to clarify meaning, verifying the repeatability of an observation or interpretation" (Stake, 2000, p. 443). There are four types of triangulation: data triangulation (the use of a variety of data sources in a study), investigator triangulation (the use of several different researchers or evaluators), theory triangulation (the use of multiple perspectives to interpret a single set of data, and methodological triangulation (the use of multiple methods to study a single problem or program) (Denzin, 1978; Patton, 1987, 2002). Yin (1994, 2003) stated that when you really triangulated, the data facts of the case study have been supported by more than a single source. By this way, validity has been established since multiple sources of data provide multiple measures of the same phenomenon.

In this study, I used data triangulation, investigator triangulation, and method triangulation for increasing the credibility of the study. I used 17 senior pre-service teachers that is more than one individual as a source of data and I used second coder for the analysis of data. In addition, I used different types of data collection tools including both questionnaires and interviews from the methodological perspective (Denzin, 1978. In addition to triangulation, I also used member checking, where I had the participants viewed the raw data (questionnaires and transcriptions) and wanted them to comment on their accuracy. To state differently, I gave their questionnaire to them again and during the interviews, I asked them whether they agreed what they had written or was there anything that they wanted to change or add. In addition, after finishing the transcription of the interviews, I showed them to the participants being studied and wanted them to read the whole transcription and stated whether they concur or not. I also used low inference descriptors, where I always tried to use the phrases that were very close to the participants' wordings and verbatim in reporting the analysis of the research findings. Thus, these were the evidences on increasing the credibility of the given case study. In addition to these strategies, frequent debriefing sessions, peer scrutiny of the research, and examination of previous research findings were the other strategies for increasing the credibility (Shenton, 2004). In my frequent meeting sessions with my supervisor and dissertation committee, my vision widened as others bring to bear their experiences, perceptions, and comments throughout the research. In such collaborative sessions, through discussions, I had a chance to test my developing ideas and interpretations, and probing from my committee helped me to recognize my own biases and preferences. Similarly, peer examination that is asking colleagues to comment on the findings was another strategy used to increase credibility. I welcomed any feedbacks that were given during the research study given by academics that are qualified in qualitative research. In our face to face meetings, their interpretations made me refine my methods, and develop greater explanation of the research design and strength my arguments in light of comments made.

In addition, a doctoral student from the Elementary Science and Mathematics Education Program at METU was recruited for data analysis in order to have consensus of findings and reduce the researcher bias. The second coder was trained about the issues that were search on subject matter knowledge and pedagogical content knowledge of pre-service teachers on multiplication and division of fractions. I also explained the data analysis framework of the study to her and both of us analyzed the questionnaires and interview transcripts together to concur on patterns. During the data coding both coders tried to identify the patterns and themes to increase the quality of the research study. Both coders analyzed the transcribed data with pseudonym names for the participants in order to eliminate the bias for the credibility of the research study. Comparison of my codes with her codes also gave evidence for inter reliability. Additionally, I examined the previous research (Shenton, 2004) findings to assess the degree to which the research study findings' were congruent with those of past studies. These studies also helped in refining the codes and themes from data that were described in analysis part.

Lincoln and Guba (1985) stressed the close ties between credibility and dependability, arguing that, in practice a demonstration of the former goes some distance in ensuring the later. Thus, the strategies that were stated above also helped to increase the reliability that is dependability of the research study. Additional strategies were mentioned in the following parts.

Merriam (1998) stated that reliability refers to the extend in which research findings can be replicated. Lincoln and Guba (1985) suggested thinking about "dependability" or "consistency" of the results obtained from the data instead of using the term reliability. By this way, rather than trying to find the same result, a researcher wishes outsiders to concur that, given the data collected, the results make sense that they are consistent and dependable. Thus, the reliability refers to not finding the similar results but whether the results are consistent with the data collected. In addition, Yin (1994) believed that researcher carrying out a case study should develop a formal, presentable database, so that other investigators can review the evidence directly. By this way, the reliability of the entire case study is also increase.

In issue of reliability, the later investigator followed the same procedures as given by the early investigator and conducted the same case study over again and gets the same results (Yin, 1994, 2003). The goal of the reliability in qualitative studies is to decrease the errors and bias in the study. Yin suggested using case study protocol while data collected and develop case study database. By this way, investigator can make sure that someone else can follow the same procedures and get the same result. Emphasize in case studies is that, it is doing the same case over again, not replicating the results of one case to another case.

Lincoln and Guba (1985) stated that the notion of reliability with regard to instrumentation can be applied to the qualitative case studies similar to its meaning to traditional research. As researcher defines instrument and use statistical techniques to have evidence on reliability human instruments can become more reliable through training and practice. In addition, reliability of documents and persons can be assessed through various techniques and triangulations. Similarly, Merriam (1998) stated that there are several techniques to be sure that results are dependable. These are the investigator's position, triangulation, and audit trail. From other point of view, Lincoln and Guba (1985) mentioned that a demonstration of credibility is usually sufficient to establish the dependability. Thus, using multiple methods of data collection and analysis, as well as validity triangulation also increases the dependability of the study. Researcher also should describe in detail how data were collected, how categories were derived, and how decisions made through out the research study in order to increase dependability (Merriam, 1998).

In this study using multiple data collection and analysis methods to ensure the consistency of results and strategies during the data collection and analysis increases both credibility and dependability of the research study. In addition, case study protocol was prepared where; the questionnaire, interview questions as well as procedures and general rules to be followed in research design were discussed. Additionally, in the methodology part, the detail of the data collection and data collection tools were described in order to increase the dependability. In addition, I presented a section called researcher role and bias where; I stated potential problems, how these might affect the research, and what strategies I used to address the potential problems. Thus, by clearly defining the investigator position (Merriam, 1998) helped to increase the credibility and dependability of the research study.

According to Yin (1994, 2003) external validity, the qualitative investigator's equivalent concept, i.e. transferability of the case study is related to the generalization of the findings beyond the given case studies. External validity is concern with the extent to which the findings of one study can be applied to other situations (Merriam, 1998). However, in qualitative research, a single case or small nonrandom sample was selected in order to understand the context in depth not to find what was true across the population. Merriam added that, one who views external validity in terms of traditional views can use many cases to study the same phenomenon. In multicases or cross-case analysis, the use of predetermined questions and specific procedures for coding and analyzing can increase the external validity, generalizability in traditional way (Miles & Huberman, 1994; Yin, 1994, 2003). Yin stated that external validity is the major problem in case studies because samples and universe where the generalization is made is not the concern of case studies. In case studies the major aim of the investigator is not to make statistical generalizations but to have analytical generalizations where; the researcher is try to generalize the particular set of results to some broader theories. He suggested to usage of theory in single case study or replication logic in multiple case studies in research designs.

In this study, in terms of transferability, I prepared a case study protocol where the questionnaire related to the subject matter and pedagogical content knowledge of pre-service teachers was prepared before hand. In addition, semi structure interview protocol was prepared to have consistency among the cases. The specific procedures for coding and analyzing the data also increased the external validity of the case study. Additionally, in this study, I tried to provide a thick description on the case that was pre-service elementary teachers and on findings in order to associate my findings with the readers in an effective way. During the data collection and analysis, I always shared my findings with my supervisor and qualified academics where, I always had a chance to discuss and revised my findings and gave new directions to the research study. In addition, my purpose was to have in-depth understanding of the pre-service teachers' subject matter knowledge and pedagogical content knowledge, thus the generalization of the findings to all pre-service teachers was not concern of this study. However, the findings of this study could easily be shared with the pre-service teachers from different universities having similar characteristics to further understand the nature of the subject matter and pedagogical content knowledge of pre-service teachers.

The concept comfirmability, the last construct of trustworthiness, is the qualitative investigator's comparable concern to objectivity. Researcher should ensure that findings are the result of the experiences and ideas of the informants, rather than characteristics and preferences of the researcher (Shenton, 2004). Shenton also emphasized the role of triangulation in promoting such confirmability and reduce the effect of investigator bias. Similarly, Miles and Huberman (1994) mentioned that a key criterion for confirmability is the extent to which the research admits his or her bias. Thus, detailed methodological descriptions like how data were collected, how categories were derived, and how preliminary theories were supported by the data should be clearly described for the confirmability of the research study. In addition, reflective commentary of the researcher for monitoring her own developing constructions, and effectiveness of the techniques used through the study should clearly be stated (Miles & Huberman, 1994; Shenton, 2004).

In this study, triangulation to reduce the researcher bias, in-depth methodological description, admission of bias, peer debriefing, and presence of second coder were the evidences for the comfirmability of the study. In addition, throughout the study, I wrote personal diaries on my reflections and judgments of methodological decisions. I recorded my impressions for each data collections, patterns appearing to emerge and models generated from the data in order to increase the confirmability of the research study.

### 3.6.1 Researcher Role and Bias

In qualitative research, researcher is the primary instrument for gathering and analyzing data (Merriam, 1998). Since qualitative research is open ended and less structured that quantitative research, researcher is the major stone and she finds what she wants to find (Johnson, 1997). Johnson stated that the key strategy to understand research bias is reflexivity where researcher actively engages in critical self reflection on his potential bias that can affect the research process and research results. Through reflexivity, researchers monitor and attempt to control their biases (Johnson, 1997). Thus, my role was so important in increasing the trustworthiness of the research study.

During the study, as a researcher, I was not only involved in the process; but also I was standing as an outside observer. Since I have been research assistant at the same department, I had a strong relationship with the pre-service teachers. I was teaching assistant and supervisor of some of the senior pre-service teachers throughout their participation in the program. This is why; I feel that I was involved in the process since I personally know them. This was an advantage for me, because when I explained the purpose of my study, almost all of the pre-service teachers were willing to participate in my study and share their knowledge and experiences objectively. On the other hand, in order to reduce the effect of being a supervisor and research assistant I followed several strategies.

At the end of the spring semester of 2004-2005 academic years, I asked the instructors for permission to participate in the last 15 minutes of the courses that pre-service teachers have attended. I clarified my presence at the course and explained the purpose of my dissertation. I also explained what they are supposed to do if they were voluntary to involve in the study. I also made sure that, their participation to this study was not obligatory, they were free to volunteer. All the answers to the questionnaire and the following interviews were confidential and I was the only person who had access to the data. I analyzed the data with pseudonym names for the participants in order to eliminate the bias. None of their professors or teaching assistant saw the data, and answers they provided did not affect any grade or their progress in the program.

Almost the entire senior pre-service teachers were voluntary to take in the questionnaire for the next class hour. Then, I interviewed senior pre-service teachers who were voluntarily participated in the semi-structure interviews. In order to make them feel comfortable during the interviews, I tried to ensure the confidentiality of the data and I repeated several times that there appeared no correct answers for the questions. Since most of the questions need arithmetic solutions and representations, I also took permission to videotape their answers in order to understand their thought process clearly. The participants were interviewed when they felt comfortable in terms of context, place, and timing, so that their answers would not be in a rush. During the interviews, I tried to express what I understood from the participants' responses, I asked whether they agree on their answers given to the questionnaire. If the response were not clear I want them to explain it again. I also made sure them to understand that I was one of the outsiders and I just want to understand their thought process. I always tried to ask them whether I understood their point correctly or not. If they said that my wordings or interpretations were wrong, I asked them to correct my wordings. Their wordings also so important for me since verbatim transcription of recorded interviews provided database for further analysis.

# CHAPTER IV

# RESULTS

This chapter summarized the findings of the research study under two main sections and related subsections. Each section dealt with one of the research questions. In the first section, the nature of pre-service teachers' subject matter knowledge about multiplication and division of fractions was analyzed. In the second section, the nature of the pre-service teachers' pedagogical content knowledge was explored. Second section was further subdivided into three headings: pre-service teachers' knowledge on students' common conceptions and sources of these common conceptions, pre-service teachers' strategies to overcome students' difficulties and to explain the key concepts, and pre-service teachers' representations on multiplication and division of fractions. In addition, in this section, the third research question that is, the relationship of pre-service teachers' subject matter knowledge and pedagogical content knowledge was examined.

For each section, a general strategy was followed based on the constant comparative method of data analysis. Firstly, the saturated codes and categories derived from literature, from other studies, from the framework of participants' statements, and from my experiences with the data, were identified. After the identification of the consistency among the categories and properties within these categories, themes from the recurring patterns were stated based on the analysis of the Multiplication and Division of Fractions Questionnaire (MDFQ). Then, these categories and themes were compared and contrasted with the categories and themes obtained from the interview and video transcripts. I explained the main categories from the questionnaire, interview, and video transcripts while presenting the findings of the analysis and then I gave detailed information for each finding while providing excerpts from the participants' responses.

## 4.1 The Nature of Pre-service Teachers' Subject Matter Knowledge

As stated above, this study attempt to provide enlightenment of pre-service teachers' understanding of multiplication and division of fractions relative to the meaning, conceptions, and misconceptions/difficulties that elementary (sixth and seventh grade) students have. In addition, the sources of these (mis)conceptions/difficulties, and strategies that pre-service teachers used to overcome these misconceptions were investigated. This study also examined the connections of pre-service teachers' subject matter knowledge and pedagogical knowledge with the mathematical models and strategies used in transferring this knowledge to the students. In this section, the analysis of the nature of pre-service elementary mathematics teachers' subject matter knowledge about multiplication and division of fractions were presented. More specifically, how do pre-service elementary mathematics teachers construct their understanding of key facts, concepts, principles, and proofs on multiplication and division of fractions were examined.

As given in literature part, Grossman, Wilson and Shulman (1989) defined subject matter knowledge as knowledge of the key facts, concepts, principles, and explanatory frameworks of a discipline in addition to the rules of evidence used to guide inquiry in the field. Similarly, Ball (1991) related her definition of understanding of mathematics that is knowledge of the ideas about the subject with Shulman and his colleagues' definition of subject matter knowledge and added that knowledge of mathematics includes understanding of particular topics, procedures, concepts, mathematical structures and connections, and the relationships among these topics, procedures, and concepts.

In this study, pre-service teachers' subject matter knowledge showed variety depending on the questions being asked. By using the term subject matter knowledge, I referred to pre-service elementary mathematics teacher's own knowledge, regarding key facts, concepts, principles, reasoning and proofs on multiplication and division of fractions. Thus, analysis of subject matter knowledge of pre-service teachers referred to the investigation of teachers' knowledge on how they symbolize and solve the basic operations on multiplication and division of fractions, how they verbalize the given multiplication and division operations and expressions, and how they interpret and reason the key facts and principles on multiplication and division of fractions. In addition, pre-service teachers' knowledge on proof of the basic facts involving multiplication and division of fractions were investigated accordingly.

Pre-service teachers' own experiences as learners and their knowledge on key facts, concepts, principles, and proofs on multiplication and division of fractions were searched systematically. The analysis was based on available literature, participants' statements, and my own experiences with the data.

Results were presented based on the subject matter knowledge of pre-service teachers involving their symbolization and solution to the basic operations, their verbalizations of the given multiplication and division operations and expressions, their interpretations of principles, and their knowledge on proof of basic facts. Categories were formed by using the recurring patterns and themes obtained from the investigation of the Multiplication and Division of Fractions Questionnaire, interview and video transcripts of 17 pre-service teachers. The themes and recurring patterns were compared and contrasted with each consecutive participant and also with the answers to other data collecting instrument.

Analysis of the subject matter knowledge of pre-service teachers starts with the investigation of teachers' knowledge on their symbolization and solution to the basic operations on multiplication and division of fractions. In MDFQ pre-service teachers were asked to write a mathematical expression for the given word problems and then find the answers to the given multiplication and division problems. Pre-service teachers were specifically asked the following questions:

Q1: Mert has 7 chocolates. He decided to give one-third of these chocolates to his close friend Emre. How many chocolates will Emre get?

Q2a: Four friends bought $\frac{1}{4}$ kilogram of sweets and shared it equally. How much sweet did each person get?

Q2b: Four kilograms of cheese were packed in packages of $\frac{1}{4}$ kilogram each. How many packages were needed to pack all the cheese?

Q3: Elif bought a bottle of milk. She gave $\frac{1}{2}$ of it, which was $1\frac{3}{4}lt$ to her grandmother. How much did the bottle of milk originally contain?

The analysis of the questionnaire, interview, and video transcripts revealed the fact that all the pre-service teachers were correctly symbolized and solved the given problems. Pre-service teachers' symbolizations and solutions to the problems are only one dimension of their subject matter knowledge. Since the given questions involve multiplication and division of fractions, pre-service teachers had no problem while solving them as it was expected.

In addition to the symbolization and finding solution to the given multiplication and division problems, pre-service teachers were asked to define multiplication and division operations and to explain relationship between multiplication of whole numbers with multiplication of fractions and division with whole numbers and division with fractions. When pre-service teachers were asked to define multiplication operations, all of the pre-service teachers agreed that multiplication is repeated addition. For instance:

> Participant 1: "I think multiplication operation should be explained by using the addition operations. That is the relationship between two operations should be taught. Like 3 x 4 means three times four that is combination of three fours. I mean in combining three groups of four it is same thing with multiplication. It's something like quick addition (SMKQ1-P1)."

On the other hand, when pre-service teachers were asked if multiplication operation could be generalized to the fractions, most of them confused and hesitated to give direct answer. Almost half of the pre-service teachers stated that multiplication of fractions was too complex to explain. They stated that the multiplication of two whole numbers could be generalized to the situation where we multiply one fraction with whole number but they can not generalize this situation to the multiplication of two fractions. For instance:

> Participant 4: "…in fractions the situation is completely different. In fractions you are taking some parts from the whole. For instance, to multiply with $\frac{1}{2}$ is something to divide into two. It has no relation with whole numbers. In two times $\frac{1}{3}$, you add two $\frac{1}{3}$s as in addition of whole numbers. It's repeated addition. But think about $\frac{1}{2}\times\frac{1}{3}$. It is repeating of $\frac{1}{3}$ with $\frac{1}{2}$ times. May be we can use the same wording while expressing the meaning but it's to hard and complex to show and explain this operation. I mean when you multiply two fractions you confused what you are doing. It's not meaningful for you (SMKQ4-P4)."

> Participant 7: " Yes, multiplication is same for all numbers. Think about $\frac{1}{2}\times\frac{2}{3}$. It's group of $\frac{1}{2}$s and you are taking $\frac{2}{3}$ of it. One second I confuse. One second I should work with simpler examples. Two times $\frac{1}{2}$ means repeated addition. We add $\frac{1}{2}$ with $\frac{1}{2}$. But when we multiply two fractions…Um sorry but I believe that it's something like repeated addition but now I realize that it is not addition. Sorry teacher I can not explain (SMKQ4-P7)."

On the other hand, some of the pre-service teachers related division operation with multiplication with fraction. For instance:

Participant 13: "I can relate multiplication of whole numbers to multiplication of fractions. If you define 4 times 3 as addition of three with four times then multiplication of $\frac{1}{2}$ with $\frac{1}{8}$ is same as taking half of $\frac{1}{8}$. It's same as division. If we move from simple to complex, 1 times $\frac{1}{8}$ is $\frac{1}{8}$, two times is addition of $\frac{1}{8}$ with $\frac{1}{8}$ and multiplication of $\frac{1}{2}$ with $\frac{1}{8}$ is a kind of division such as taking half of the $\frac{1}{8}$ (SMKQ4-P13)."

Participant 2: "Um multiplication of fractions. I have never thought about that. Lets think about $\frac{1}{2}$ times $\frac{1}{2}$. It means taking half of the $\frac{1}{2}$. It's something related to division. I mean this time we are dividing instead of adding. Think about $\frac{1}{2}$ times $\frac{3}{4}$. It is division of $\frac{3}{4}$ into two pieces. Thus while we are working with whole numbers it is same as repeated addition but here it is related to division (SMKQ4-P2)."

Thus, analysis of the questionnaire, interview and video transcripts revealed that most of the pre-service teachers had confusions in defining the meaning of the multiplication operation on fractions. On the other hand, some of the pre-service teachers tried to relate multiplication operation with division and make some connections.

In case of the investigation of answers to the meaning of the division operation, results of the pre-service teachers were divided into two. Most of the pre-service teachers used measurement modeling of division while defining the division operations. For instance:

> Participant 13: "….division is inverse operation of multiplication….Like 4 over 2 means how many twos are there in four. I mean how many multiple of twos are there in four. Similarly, ten over two means how many twos are there in ten. That is think about the group of twos, then how many this groups are there in ten (SMKQ2-P13)."

On the other hand, few of the pre-service teachers preferred to use partitive model. Like;

> Participant 4: "Division means to separate into pieces. Think about that I have 3 apples and I want to share these 3 apples among three people. I will give one apple to each person which is sharing. I mean I will give one apple to first person, one apple to second person and one apple to third person. Division is distribution of something equally. Think about 6 over 2. It means divide six pieces between two people and think about how much each person gets? It's equally sharing and at the end finding the amount that each person gets (SMKQ2-P13)."

In addition, pre-service teachers were asked about the relationship of division operation in whole numbers with the division in fractions. As in the case of multiplication, some of the pre-service teachers had difficulty while explaining the division of two fractions. For example:

> Participant 8: "I have to think about division of fractions. I think to show the division operation is really hard. When I think about the definition I mean it is not same as in whole numbers. I mean I do not know what does it means to divide fraction with another fraction (SMKQ4-P8)."

On the other hand, most of the pre-service teachers preferred to use measurement modeling of division while they are explaining the division of two fractions. For instance:

> Participant 15: "Um, division is same for all numbers. Think about four over two. It means how many twos are there in four. If we are working with rational numbers the logic is same. Whatever number we have, we think about how many second numbers are there in the first number. For example, $\frac{3}{4}$ divided by $\frac{1}{3}$. It means how many $\frac{1}{3}$s are there in $\frac{3}{4}$ (SMKQ4-P15)."

In addition to given models, only two of the pre-service teachers could explain the two meanings of division on fractions. For instance:

> Participant 3: "…Think about $2 \div \frac{1}{2}$. We can not separate pieces of $\frac{1}{2}$s. In order to divide into pieces it should be whole number. So, we have to think about group numbers. I have 2 apples and I want to divide into half apples. So, I think about how many $\frac{1}{2}$s are there in 2. Division has two meanings. We either divide into pieces or took the amount of each piece or we divide into groups and took the group number. While dividing two fractions or while dividing a whole number and fraction I will use the second definition. I mean while doing the operation $2 \div \frac{1}{2}$ it is not meaningful to divide between $\frac{1}{2}$ people this is why we have to think about how many $\frac{1}{2}$s. While explaining the meaning I realize this distinction. Ohh…this is great (SMKQ4-P3)."

Thus, analysis of the subject matter knowledge on the multiplication and division operations on whole numbers and fractions revealed the fact that most of the pre-service teachers have difficulty while they were constructing relationship of multiplication and division operations with whole numbers and with fractions.

Pre-service teachers stated their inadequacy while defining and representing the multiplication and division of fractions compared to the multiplication and division of whole numbers.

In addition to the knowledge on basic operations, the following multiplication ($11\frac{1}{2} \times \frac{1}{4}$) and division ($6\frac{2}{3} \div \frac{5}{6}$) operations were given to the pre-service teachers. Pre-service teachers were asked to create a word problem which describes the given multiplication and division expressions.

Analysis of the verbalization of the given multiplication of simple fraction with the mixed fraction expression revealed that 9 of the 17 pre-service teachers construct the sentence where the part of the given amount was investigated along with the proportional relationship. For instance:

> Participant 4: “My mother bought 11.5 meter fabric to sew a cloth for me. If the one-fourth of the fabric is faulty then what is the amount of faulty fabric? (SMKQ5-P4).

> Participant 16: “My mother only used $\frac{1}{4}$ of the 11.5 kg rice to cook for me. What is the amount of rice that my mother used? (SMKQ5-P16).”

On the other hand, six of the pre-service teachers preferred to use the relationship between multiplication and division operation while they were verbalizing the given expression on multiplication of two fractions. All of these pre-service teachers used partitive division model while they were constructing the verbal expression for the multiplication of fractions. For instance:

> Participant 8: “There is a stick with 11.5 meter long. If I want to divide the given stick into four equal parts then how long will be the each piece? (SMKQ5-P8).”

> Participant 9: "I have eleven whole and one half apples. I want to share these apples among four people equally. How much apple will each person gets? (SMKQ5-P9).

In addition to these verbalizations, two of the pre-service teachers stated that they could not meaningfully verbalize the given multiplication expressions by using daily life examples. They stated that since the given numbers were not whole numbers, it's not easy to verbalize the given fractional expressions.

Thus, analysis of pre-service teachers' verbalization of the given expression of multiplication operations revealed that most of the pre-service teachers correctly construct a real life problem by using the relationship between multiplication and division operations and partitive model of division.

In terms of verbalization of the given division expression, results revealed that 12 of the pre-service teachers used quotitive model division model while verbalizing the given division expression. For instance:

> Participant 1: "I have $6\frac{2}{3}lt$ bowl of water and I want to pour this amount of water into bowls with volume of $\frac{5}{6}lt$ each. How many bowls do I need? (SMKQ5-P1)."

> Participant 8: "I have a garden that has $6\frac{2}{3}m$ circumference. I want to plant trees where the gap between two successive trees will be $\frac{5}{6}m$. How many trees will be planted around the garden? (SMKQ5-P8)."

In addition to quotitive model of division, three pre-service teachers preferred to use the relationship between multiplication and division operations while verbalizing the given division expression. For instance:

Participant 6: “A salesman sold $\frac{5}{6}$ of the total amount of the milk that he has. If the rest of the milk is $6\frac{2}{3}lt$ then what is the amount of the total milk at the very beginning? (SMKQ5-P6).”

Here, participant 6 verbalized the given division of fraction expression by thinking the equation a x $\frac{5}{6} = 6\frac{2}{3}$ which leads to a division operation to find the value of a.

On the other hand, as in the multiplication operation, two of the pre-service teachers mentioned that they could not verbalize the given division expression since it was not easy to create a word problem by using the fractions.

In addition to the knowledge on basic operations, and verbalization of the given operations, pre-service teachers were also asked to interpret and reason the key facts and principles on multiplication and division of fractions. More specifically, they were asked the following three questions:

Q6a. For the expression “$\frac{2}{3} \div \frac{1}{2}$”, Ceren said that “$\frac{2}{3} \div \frac{1}{2} = \frac{3}{2} \times \frac{1}{2}$”, Cenk said that “$\frac{2}{3} \div \frac{1}{2} = \frac{3}{2} \times \frac{2}{1}$” and Eda said “$\frac{2}{3} \div \frac{1}{2} = \frac{2}{1} \times \frac{2}{3}$”. Who is right? (Please explain your answer) What might each of the students be thinking?

Q7a. Berk argues that he prefers to divide fractions in a way similar to multiplication. For instance; $\frac{2}{9} \div \frac{1}{3} = \frac{2 \div 1}{9 \div 3} = \frac{2}{3}$, Would you accept Berk’s proposal? Why?

Q8a. Tuğçe argues that distributive law can be used to calculate $1\frac{3}{4} \div \frac{1}{2}$. She suggests the following:

$= 1\frac{3}{4} \div \frac{1}{2} = (1 + \frac{3}{4}) \div \frac{1}{2}$

$= (1 \div \frac{1}{2}) + (\frac{3}{4} \div \frac{1}{2})$

$= 2 + 1\frac{1}{2}$

$= 3\frac{1}{2}$

Would you accept Tuğçe's proposal? Why?

For the first question, all of the pre-service teachers stated that only Eda's answer was right. Pre-service teachers agreed that Ceren and Cenk had some misconceptions on multiplication and division of fraction operations that is why they carried out the operation incorrectly. Although pre-service teachers agreed that Eda's answer was right, they could not give the detail explanation for their reasoning. To state differently, eleven pre-service teachers mentioned that Eda was right since both sides of the equation that is; result of division operation and multiplication operation was equal to each other. In addition, pre-service teachers stated that Eda knew that in division operation we invert and multiply the second fraction with the first one. Here, Eda inverted the second fraction and changed the place of the both fractions before multiplying two fractions. Similarly, Ceren and Cenk's answer was incorrect since they have confusion on which fraction should be inverted. Pre-service teachers also stated that Ceren and Cenk forgot to invert the second fraction but since Eda understood the topic clearly she remember to invert the second fraction. For instance:

Eda did the same thing and then interchange the place of the first fraction with the second one that was acceptable. On the other hand, Ceren remembered that in the division operation we should invert one of the fractions, but since she did not learn the topic completely she perform the operation incorrectly as Cenk (SMKQ6-P10)."

Five of the pre-service teachers emphasized that Eda; use the commutative property of multiplication in addition to the division operation.

Again those pre-service teachers stated that according to the rule of the division we should invert the second fraction, but we could also combine the properties of division and multiplication operation and change the places of the fractions before multiplying them. In addition, one of the pre-service teachers stated that Eda performed the division operation within the numerators and denominators separately. For instance,

> Participant 5: "I can accept the Eda's solution. Because Eda divide the numerator of the first fraction with the second one and similarly, she divide the denominator of the first fraction with the denominator of the second fraction. Then, she invert and multiply the denominators with the numerators ($\frac{2 \div 1}{3 \div 2} = \frac{\frac{2}{1}}{\frac{3}{2}} = \frac{2}{1} \times \frac{2}{3}$) which gives the correct solution. On the other hand, Ceren thinks that she should invert the first fraction instead of second one and Cenk has misconception of inverting both fractions (SMKQ6-P5)."

In the second question, pre-service teachers were asked whether they can perform the division operation by dividing numerator of the first fraction with the numerator of the second fraction and denominator of the first fraction with the denominator of the second fraction.

The analysis of the questionnaire, interview and video transcripts revealed that 8 of the pre-service teachers accepted the solution based on their trial with different numbers. Pre-service teachers mentioned that when they invert and multiply the given division operations, result is the same with the given statement where the numerators and denominators are divided with each other. They stated that this was the first time that they come up with such a solution. This is why they hesitated to decide whether the result is true or not but they added that they used different numbers to test the correctness of the result. At the end of such trials, these pre-service teachers decided to accept the solution. However, these pre-service teachers were not sure whether they could make generalization based on this implication since they did not know the logic of the formula or how to reason their understanding. In addition, they stated that such a rule might result in confusion on the operations on division of fractions since it was a little bit complex. For instance:

> Participant 1: "I think the result is true. Yes..It is true I have never met such a solution before but I tried it with other numbers and it is really working. I can not say the reason because I do not know the logic but I think it's a little bit confusing. I mean students have already known the rule that is; in division we invert and multiply the second fraction with the first one and thus there is no need to confuse their mind. By this way, we can mix them up. I believe that solution is true but there is no need to teach this solution to students (SMKQ7-P1)."

Among the 17 pre-service teachers, four of them stated that they could accept the solution that is we could divide the numerators and denominators of two fractions respectively since division and multiplication are inverse operations.

> Participant 14: "I accept the solution. I mean why not. Division and multiplication are inverse operations. We can multiply the numerators and denominators while we are doing multiplication. So, while dividing two fractions we can divide numerators with numerators and denominators with denominators. It's really meaningful and result is true but I do not know the logic (SMKQ7-P14)."

In addition, three of the pre-service teachers mentioned that they could accept the solution conditionally. In other words, the solution was correct but the result could not be generalized to other questions. They also gave examples for the counter examples where we can not generalize the rule. For instance:

Participant 2: "I believe that the result is true. I tried with different numbers and result is true but it depends on the given fractions. I mean if numerators are multiples of each other and denominators are multiples of each other we can say that the rule is valid. It's like proportional relationship between numerators and between denominators. Here, the numerators that are 2 and 1 are multiplies of each other. Similarly, denominators that are 9 and 3 are also related that is why the result is true. But, think about $\frac{3}{8} \div \frac{2}{5}$. They are not the multiples of each other. Thus, $\frac{3}{8} \div \frac{2}{5} = \frac{3 \div 2}{8 \div 5} = \frac{3}{2} \div \frac{8}{5}$ where; we can not use the rule (SMKQ7-P2)."

On the other hand, one of the pre-serviced teachers stated that he could not accept such a statement. He stated that the result is correct just by coincident and he never met such a solution for division of fractions.

In line with the given statement, pre-service teachers were also asked another statement where the division operation was distributed over addition. For this question, in terms of the correctness of the statement, pre-service teachers were divided into two. Twelve pre-service teachers agreed that the given statement solved correctly where; division operation can be distributed over addition as in multiplication operation. For instance:

Participant 11: "I accept this solution. Division means to invert and multiply thus we can apply all the properties of multiplication to the division. I mean here in the first step $1\frac{3}{4} \div \frac{1}{2} = (1+\frac{3}{4}) \div \frac{1}{2}$, we can think that it is multiplication of $(1+\frac{3}{4}) \times 2$ since to divide with $\frac{1}{2}$ means to multiply with 2. Thus, second step, $(1 \div \frac{1}{2}) + (\frac{3}{4} \div \frac{1}{2})$ is same as $(1 \times 2) + (\frac{3}{4} \times 2)$ where multiplication is distributed over addition. Thus, the result will be $2 + 1\frac{1}{2}$ which is equal to $3\frac{1}{2}$ (SMKQ8-P11)."

Pre-service teachers also emphasized that before they checked the operations on the right hand side of the equation, they carried out the operation by inverting and multiplying the second fraction. Then, they realized that result was same on either side of the equation which increased their beliefs on the correctness of the given statement. For instance:

> Participant 16: "The given operations are correct. When I first come up with such a question, I checked the answer and see that when I invert and multiply the second fraction the result is same when I distributed division over addition operation. Then, I wrote the operation by using division bar and since the denominator is common for all the expressions in numerator, I distributed denominator for each numerator. The equation will be $\frac{(1+\frac{3}{4})}{\frac{1}{2}} = \frac{1}{\frac{1}{2}} + \frac{\frac{3}{4}}{\frac{1}{2}} = \quad 2 + 1\frac{1}{2} = 3\frac{1}{2}$ which gives the correct solution (SMKQ8-P16)."

On the other hand, five of the pre-service teachers stated that they were really confused with the answer since they knew that division could not be distributed over addition. They stated that division was not multiplication, we could distribute multiplication over addition but this rule was not true for the division operation. For instance:

> Participant 4: "I did not accept this solution. I know that there is no distributive property of division over addition. But, when I check the result, answer is true but I have never seen such a property. I mean, I do not know how this could be but I do not think it is just by coincidence. I tried it by using other numbers and still the answer is true but I can not generalize this rule because I do not know any distributive property of division (SMKQ8-P4)."

Some of these pre-service teachers stated that they were shocked when the answer is same when they invert and multiply the second fraction and they can not give any explanation for their reasoning. In addition, one of the pre-service teachers realized the correctness of the given operations while he was trying to find the counter example for the given expression.

> Participant 13: "Here, there is a misconception. In multiplication operation, we can distribute multiplication over addition but here the students perform the same operation for the division. In other words, student generalizes the distributive property of multiplication over addition to the division over addition and that's why they made a mistake. We can also disprove the given statement by giving simpler example like $(4 + 2) \div 2$. This is equal to $(4 \div 2)$ + $(2 \div 2)$ which equals 3. But, when we first carried out the operation inside the parenthesis the result will be 6, and solution is three. Ohhh answer is same. One minute I have to think. I have to check another example..(1 minute ). Teacher may I change my answer. The operation is correct. I mean, I have never seen such property. I mean how can division be distributed over addition. But, I tried it with many examples and the result is true. I mean…here, I learn that division can be distributed over addition ohh…(SMKQ8-P13)."

Thus, some of pre-service teachers had some confusion on given statement since they have not seen such an expression before. On the other hand, some of them related the division and multiplication operations and generalized the properties of multiplication to the properties of division as stated above.

In the tenth question on the MDFQ, pre-service teachers were asked to prove the expressions "in multiplication of two fractions, we multiply the numerator of the first fraction with the numerator of the second fraction and denominator of the first fraction with the denominator of the second fraction" ($\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$, a, b, c, d $\in Z$ where; b, d $\neq 0$) and "in division of two fractions, we invert and multiply the second fraction with the first one"

($\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$, a, b, c, d $\in Z$ where; b, c, d $\neq 0$). In the analysis of the questionnaire, interview, and video transcripts for the proof of the multiplication of two fractions, eight pre-service teachers stated that they used numbers instead of symbols and by using these specific examples, they made generalization on the given formula. For instance:

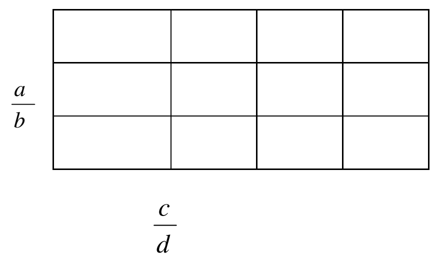
> Participant 3: "I will assume that d=1, then the left hand side will be $\frac{a}{b} \times c$. Then, by giving specific numbers to the symbols, I will show that $\frac{a}{b} \times c$ is c times $\frac{a}{b}$ that is equal to $\frac{c \times a}{b}$ which is the right hand side of the equation. (SMKQ10-P3)."

Similarly, eight pre-service teachers preferred to use numbers instead of symbols and equate both sides of the equation. Although they believed that this was not a real mathematical proof, this kind of exemplification could lead the students to understand the proof of the given expressions. For instance:

> Participant 10 : "I will use examples for each occasions. Like, $\frac{1}{3} \times \frac{1}{3}$. What does $\frac{1}{3} \times \frac{1}{3}$ mean. It means I have one-third of the given cake. To have $\frac{1}{3}$ of the given piece of the cake means to divide this piece into three where we divided the total amount of the cake into 9 pieces and took one. Thus, $\frac{1}{3} \times \frac{1}{3}$ is equal to the $\frac{1}{9}$ where we multiply the numerator with the numerator and denominator with the denominator. I will also use other specific examples to prove the equality (SMKQ10-P10)."

On the other hand, five of the pre-service teachers stated that they would use rectangular area model with transparencies in order to prove the multiplication of the two fractions. For instance:

Participant 16: “I will show the intersection area of two fractions $\frac{a}{b}$ and $\frac{c}{d}$ by using the transparencies that are represented by the rectangular area. The length of the rectangle is $\frac{c}{d}$ and width is $\frac{a}{b}$. The area of this rectangle will be $b \times d$ and the shaded area that is intersection area is $a \times c$. Thus, the shaded area will be the $a \times c$ part of the whole area $b \times d$ which can be represented by $\frac{a \times c}{b \times d}$ (SMKQ10-P16).”




In addition to these attempts to prove the multiplication of the two fractions, four pre-service teachers stated that they did not have any idea on how to prove the multiplication of two fractions and they prefer to teach it as the mathematical rule.

For the proof of the division of two fractions, four pre-service teachers used numbers instead of symbols and by using these specific examples, they made generalization on the invert and multiply formula of the division of fractions.

Participant 3: “As in the multiplication, I will assume that b=1, then the left hand side will be $a \div \frac{c}{d}$. Then by giving specific numbers to the given symbols, I will show that $a \div \frac{c}{d}$ is equal to $ax\frac{d}{c}$. Similarly, I will do this by assuming d=1, where $\frac{a}{b} \div c$ is equal to $\frac{a}{b} \times \frac{1}{c}$. Thus, by using the specific examples, I can make generalization for the formula (SMKQ10-P3).”

Similarly, the rest of the pre-service teachers preferred to use numbers instead of symbols and equate both sides of the equation. According to pre-service teachers, it was easier to use numbers instead of symbols in equating mathematical equations. In addition, they stated that to perform mathematical proofs by using symbols was a challenging task although the given operations were so basic. They stated that they would try the formula by using three or four examples and then generalized the formula. On the other hand, six of the 17 pre-service teachers used the definition of the division in their attempt of proving the division of two fractions. For instance;

> Participant 13: "I will use the definition of the division and used some manipulative to prove this definition. For instance: $\frac{1}{2} \div \frac{1}{4}$ means how many $\frac{1}{4}$'s are there in $\frac{1}{2}$. Thus, I will draw the figure ($\frac{1}{2}$) and ($\frac{1}{4}$) and then showed that two times $\frac{1}{4}$ is equal to $\frac{1}{2}$. Thus, by using such examples, I will make generalizations of the formula $\frac{a}{b} \div \frac{c}{d}$ like how many $\frac{c}{d}$'s are there in $\frac{a}{b}$ (SMKQ10-P13)."

Similarly, one of the pre-service teachers used figures and symbols in order to prove the division of two fractions.

> Participant 5: "For the given division problem $\frac{a}{b} \div \frac{c}{d}$ I will symbolize it as $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{1}{\frac{c}{d}} = \frac{a}{b} \times \frac{d}{c}$. I will explain that $\frac{1}{\frac{c}{d}}$ means how many $\frac{c}{d}$'s are there in one whole. I will show that the answer is $\frac{d}{c}$ by using the figures. Then I made the same generalization for the $\frac{a}{b} \div \frac{c}{d}$ (SMKQ10-P5)."

In addition to these attempts to prove the division of two fractions, five pre-service teachers stated that they did not know how to prove the division of two fractions and just gave it as the mathematical rule like invert and multiply the second fraction.

In the interview protocol pre-service teachers were asked the characteristics of the good mathematics teacher. Results revealed that fifteen of the 17 pre-service teachers emphasized that knowing subject matter in depth is one of the important characteristics of good mathematics teacher. In addition, pre-service teachers stated that teacher should know the topic in detail in order to transfer it to students. For instance:

> Participant 2: "Subject matter knowledge is so important. Teacher should know even the smallest detail related to the topic since students can ask different questions. Teacher should control everything in the class. There can be something that is unexpected thus teacher should have perfect subject matter knowledge to handle these problems (SMKQ13-P2)."

> Participant 6: "I think subject matter knowledge is so important since a person can not explain anything if he does know it. Some of our friends complain that why do we have to take the differential equation or linear algebra courses in the university since we are not going to use them in our teaching. I believe that we should understand what mathematics is and we should understand the background and details in order to explain the things easily. Students do not need to know the background but teacher should. Teacher should know where each expression comes from and what the logic is. Teacher should have ten things in order to teach just one thing (SMKQ13-P6)."

On the other hand, when pre-service teachers were asked to judge their subject matter knowledge on multiplication and division of fractions, eight of the pre-service teachers stated that up to now they think that they have enough knowledge. But with this research study, they realized that they do not have deep knowledge on multiplication and division of fractions. For example:

> Participant 9: "I do not think that I have enough knowledge until I see that kind of questions. I mean, if we mean the traditional methodologies, like invert and multiply the second fraction in division operations, I have enough knowledge but if we mean the logic of the operations, I do not think that I have enough knowledge. I know the solutions of the problem, but I can only apply one methodology and that is why I do not know the other methodologies. I do not know the reason. May be nobody teach me or I do not search for the other approaches. I have to study hard (SMKQ13-P9)."

> Participant 17: "As you see I do not have enough knowledge on the logic of multiplication and division of fractions. I think that I have clear picture on my mind on some topics but not for the others. For instance, I know how to multiply two fractions but what do we mean by multiplication of two fractions. Since I do not know the logic, I can not explain it to my students. Similarly, in multiplication of fractions, we used the intersection area of two fractions but I do not know what the relationship of intersection area with the multiplication of two fractions is (SMKQ13-P17)."

The analysis of subject matter knowledge of pre-service teachers showed that pre-service teachers could easily symbolize and solve the basic operations on multiplication and division of fractions. In addition, most of the pre-service teachers could verbalize the given multiplication and division expressions by using daily life examples with constructing the relationship between multiplication and division models.

On the other hand, in terms of the definition of basic concepts such as multiplication and division of fractions and in terms of pre-service teachers' interpretation and reasoning of key facts and principles, pre-service teachers' subject matter knowledge can not be regarded as conceptually deep.

Based on the analysis, I could say that, while dealing with the multiplication and division expressions, pre-service teachers stated that they were not used to solve the multiplication and division operations in that given manner. In addition, they had problems while reasoning their answers on the correctness of the given expression. Additionally, pre-service teachers did not have enough subject matter knowledge on proof of the basic facts involving multiplication and division of fractions where, most of the pre-service teachers preferred to use numbers instead of symbols in proving the given expressions. Interview transcripts also revealed that some of the pre-service teachers thought that they have enough knowledge to judge the multiplication and division operations. However, when they saw such questions they changed their minds and accepted that their knowledge was not deep enough to reason their understanding. The reasons for this could be that pre-service teachers could easily multiply and divide the given multiplication and division expressions since from their elementary school years. Thus, they might not question the reason behind these rules since they thought that they are simple and so basic. However, results of the data revealed that pre-service teachers' subject matter knowledge was not deep enough to reason the multiplication and division operations on fractions conceptually.

In the next part of the analysis, pre-service teachers' nature of pedagogical knowledge relationship between subject matter knowledge and pedagogical content knowledge were examined.

### 4.2 The Nature of Pre-service Teachers' Pedagogical Content Knowledge

After analysis of subject matter knowledge of pre-service teachers on multiplication and division of fractions, in this section pre-service teachers' pedagogical content knowledge in terms of pre-service teachers' knowledge on students' common conceptions and on sources of these common conceptions, pre-service teachers' strategies to overcome students' difficulties and to verify the key concepts, and pre-service teachers' representations on multiplication and division of fractions were investigated. In addition, in this section, the third research question that is; the relationship of pre-service teachers' subject matter knowledge and pedagogical content knowledge was analyzed. More specifically, how do pre-service elementary mathematics teachers' understanding of key facts, concepts, principles, and proofs influence their knowledge on students' common conceptions, sources of these common conceptions, and their strategies to overcome students' difficulties were analyzed. In addition, the influence of subject matter knowledge on strategies that pre-service teachers used to explain the key facts, and the representations that they used on multiplication and division of fractions were examined accordingly.

### 4.2.1 Pre-service Teachers' Knowledge on Students' Common (Mis)conceptions/Difficulties and Their Knowledge on the Sources of These Common (Mis)conceptions/Difficulties

In this section, pre-service elementary mathematics teachers' knowledge about common conceptions and misconceptions/difficulties held by the elementary ($6^{th}$ and $7^{th}$ grade) students and pre-service teachers' knowledge about the possible sources of these conceptions and misconceptions/difficulties on multiplication and division of fractions were investigated. In other words, in this section the first two sub-problems of the second research question and the first two sub-problems of the third research question were answered.

Pre-service teachers' own experiences as learners and their knowledge on students' cognitive thinking process were searched systematically. Based on the analysis of the data, pre-service teachers' knowledge on students' conceptions is grouped under four headings namely: algorithmically based mistakes, intuitively based mistakes, mistakes based on formal knowledge on fractions, and misunderstanding of the problem. In addition to the mistakes, possible sources of these conceptions and misconceptions/difficulties are also discussed under these four headings. These headings are based on available literature, participants' statements, and my own experiences with the data.

Pre-service teachers emphasized various conceptions and misconceptions/difficulties that elementary grade level students could perform while they are studying multiplication and division of fractions. By using the term (mis)conceptions/difficulties, I referred to the students' (mis)construction of their own knowledge, (mis)understanding on given terms, concepts, operations and students difficulties while solving the given problems. Thus, analysis of knowledge on students' common (mis)conceptions/difficulties refers to the pre-service teachers' perception of children's mistakes while they are performing multiplication and division of fractions. Based on these conceptions and misconceptions/difficulties, teachers' knowledge on the possible sources of these mistakes was also discussed accordingly.

Results were presented under the subheadings of the categorization of the errors. To state differently, categories were formed by using the recurring patterns and themes obtained from the investigation of the Multiplication and Division of Fractions Questionnaire, interview and video transcripts of 17 pre-service teachers. In MDFQ, pre-service teachers were asked to write two errors that elementary grade students might do while performing multiplication and division of fractions problems. More specifically, problems on finding fractional part of a whole number, fraction over whole number, whole number over fraction, fraction over fraction, and fractional part of the fraction respectively. Based on the analysis of data, the number of pre-service teachers within each categorization was also stated depending on their statements for the students' conceptions and misconceptions/difficulties. Within this analysis, the sources of students' conceptions and misconceptions/difficulties were also discussed. Additionally, summary of the pre-service teachers' knowledge on common conceptions and misconceptions/difficulties that students have and the possible sources of these mistakes were presented at the end of the section.

Mathematical knowledge is embedded in a set of connections among algorithmic, intuitive and formal dimensions of knowledge (Fischbein, 1987). In their study, Barash & Klein (1996) examined the theoretical framework for analyzing seventh grade students' knowledge of rational numbers. They presented the picture of the algorithmic, intuitive and formal dimensions of knowledge and their interactions. In their diagnostic test, algorithmic dimension is referred as "ability to compute with fractions and with decimal numbers, and the capability to explain the rationale behind the various algorithmic aspects" (p. 36).

Based on the literature review and the analysis of the data, algorithmically based mistakes comprised one of the dimensions of pre-service teachers' knowledge on students' common conceptions and misconceptions/difficulties in this research study. By using the term algorithmically based mistake, two types of errors emerged from the data originated from the errors made operationally. The first one was carrying out inappropriate operations like performing division operation instead of multiplication. The second one was misapplication of basic rules like inverting both dividend and divisor while dividing two fractions or finding common denominator for the given fractions while carrying out multiplication. When we considered the first question on the MDFQ, that was finding the fractional part of the whole number, eleven pre-service teachers stated that elementary school students would perform algorithmically based mistakes stemmed from their inadequate knowledge on basic operations. For instance, pre-service teachers mentioned that for this question instead of using the operation $7\times\frac{1}{3}$ or $7\div 3$, students perform $7\div\frac{1}{3}, 7\times\frac{2}{3}$, $7-3$, and $7\times\frac{3}{1}$ which would show their misunderstanding on basic operations. In terms of possible sources of these (mis)conceptions/difficulties, pre-service teachers emphasized the importance of the formal knowledge on multiplication, division, addition and subtraction. According to the pre-service teachers, students lacked a deep knowledge of what does multiplication or division means that yield them to use inappropriate operation. For instance:

Participant 7: "Students will perform $7 \div \frac{1}{3}$ since they do not know which operation should be performed in order to find one-third of seven (SQ1-P7)."

Besides, two of the pre-service teachers emphasized the errors based on the applications of the algorithm and they stated that the main source for these mistakes was the rote memorization of the algorithm. According to them, students performed the operation incorrectly since they viewed algorithm as a meaningless series of steps and they might forget or change these steps in a way that lead errors (Tirosh, 2000). For this question, pre-service teachers emphasized the errors like involving multiplication operation. They stated that while students calculate the fractional part of the whole, they would multiply the whole number with the denominator instead of numerator or they multiply both numerator and denominator with the given whole number. For instance:

Participant 11: "For the given multiplication operation like $a \times \frac{b}{c}$ students perform the operation like this $a \times \frac{b}{c} = \frac{ab}{ac}$ or $a \times \frac{b}{c} = \frac{b}{ac}$ and the sources of these mistakes stemmed from students' inadequate knowledge on properties of multiplication operation in fractions (CSQ1-P11)."

Thus, pre-service teachers stated that students' rote memorization on steps while performing basic operations and their inadequate knowledge on basic operations lead them to make algorithmically based mistakes. Similar to this, in the second question of MDFQ pre-service teachers were asked the following questions: "four friends bought $\frac{1}{4}$ kilogram of sweets and shared it equally. How much sweet did each person get?", and "Four kilograms of cheese were packed in packages of $\frac{1}{4}$ kilogram each. How many packages were needed to pack all the cheese?" where; the division of fraction by the whole number and division of whole number by the fraction were investigated. In these questions, sixteen pre-service teachers emphasized the importance of algorithmically based mistakes resulted from both rote-memorization and inadequate knowledge on four operations. Twelve pre-service teachers mentioned that students prefer to use multiplication, subtraction, and addition instead of division operation. Specifically, pre-service teachers mentioned that for the first question, instead of using $\frac{1}{4} \div 4$, students could use operations like $4 \div \frac{1}{4}$, $\frac{1}{4} \times 4$, $4 \div 4$, or $4 - \frac{1}{4}$ since students do not have enough knowledge to discuss what does to share mean. Thus, pre-service teachers emphasized the lack of knowledge on operations that could be the source of the algorithmically based mistakes. On the other hand, seven pre-service teachers stated the errors based on the rote memorization of the algorithm. For instance:

> Participant 11: "For the given division problems students can perform like this $\frac{a}{b} \div c = \frac{a}{b} \times \frac{c}{1} = \frac{a.c}{b}$ or $\frac{a}{b} \div c = a \times \frac{c}{b} = \frac{a.c}{b}$. In addition, for the $a \div \frac{b}{c}$ students can perform like $a \div \frac{b}{c} = \frac{a \div b}{a \div c}$ and these are because of the lack of inadequate knowledge on division. Since they do not know the meaning of sharing to share mean and the order of operations, they confuse how to perform division (CQ2-P11)."
>
> Participant 3: "For the division like $\frac{1}{4} \div 4$, pre-service teachers can make simplification like in multiplication $\frac{1}{4} \div 4 = 1$ (CQ2-P3)."

Thus, in terms of algorithmically based mistakes pre-service teachers emphasized the role of misinterpretation of the algorithm and performing incorrect operations. They stated that for the given division problem common misconceptions include: inverting the dividend instead of the divisor, inverting both the dividend and the divisor, and simplifying the numerator and denominator as in multiplication. In addition, pre-service teachers stated that basic sources for the algorithmically based mistakes stemmed from the students' rote memorization of the algorithm procedurally and their inadequate formal knowledge on four operations.

In the analysis of the third question that is "Elif bought a bottle of milk. She gave $\frac{1}{2}$ of it, which was $1\frac{3}{4}lt$ to her grandmother. How much did the bottle of milk originally contain", sixteen pre-service teachers stated the usage of algorithmically based mistakes among the students. Among them 13 pre-service teachers emphasized the importance of the lack of formal knowledge on four operations as the source of algorithmically based mistakes. For the given question that involves division of two fractions, pre-service teachers emphasized the usage of addition, subtraction, and multiplication operation instead of division operation. For instance;

> Participant 1: "For the given question, students can perform $1\frac{3}{4}\div 2$ or $1\frac{3}{4}\times\frac{1}{2}$ instead of $1\frac{3}{4}\div\frac{1}{2}$ or $1\frac{3}{4}\times 2$, because they confuse on the multiplication and division operation (CSQ3-P1)."

Thus, pre-service teachers stated that elementary school students could perform the operation incorrectly based on their inadequate knowledge on four operations. On the other hand, eleven pre-service teachers emphasized the importance of rote memorization of algorithm while performing multiplication and division of fractions as a source of these misconceptions. For the given question, errors depending on the rote memorization of the algorithm could be stated as follows: inverting the dividend instead of the divisor ($1\frac{3}{4}\div\frac{1}{2}= \frac{4}{7}\div\frac{1}{2}$), inverting both the dividend and the divisor ($1\frac{3}{4}\div\frac{1}{2}=\frac{4}{7}\div\frac{2}{1}$), multiplying the numerator and denominator like in multiplication ($\frac{7}{4} \div \frac{1}{2} = \frac{7}{8}$), multiplying the whole part of the mixed fraction in multiplication ($1\frac{3}{4} \times 2 = 2\frac{3}{4}$), multiplying both the numerator and denominator in mixed fractions ($1\frac{3}{4} \times 2 = 1\frac{6}{4}$), and adding both numerators and denominators ($1\frac{3}{4} \times 2 = \frac{7}{4} + \frac{7}{4} = \frac{14}{8}$). Thus, analysis revealed that the importance of the rote memorization and inadequate formal knowledge on basic operations as a source of misconceptions while performing division of two fractions.

When we consider the multiplication of two fractions as given in the fourth question of MDFQ, almost all of the pre-service teachers stated the influence of algorithmically based mistakes on students' conceptions and misconceptions/difficulties. The common conceptions identified by the pre-service teachers include multiplication of numerator and denominator by ignoring the whole part of the mixed fraction ($1\frac{1}{2} \times \frac{1}{3} = 1\frac{1}{6}$), finding common factors like in addition of fractions and then perform multiplication ($\frac{2}{3} \times \frac{3}{5} = \frac{10}{15} \times \frac{9}{15} = \frac{90}{15}$), inverting the second multiplier like in division ($1\frac{1}{2} \times \frac{1}{3} = \frac{3}{2} \times \frac{3}{1} = \frac{9}{2}$), and adding the numerators and denominators ($\frac{2}{3} \times \frac{3}{5} = \frac{2+3}{3+5}$). Most of the pre-service teachers emphasized the inadequacy of formal knowledge in converting mixed fractions into improper fraction as a main source for the algorithmically based mistakes in multiplication of fractions. In addition, pre-service teachers added that students memorized the basic rules while performing four operations; thus it was obvious that they confused operations and multiply fractions as if they divide or add them.

Thus, analysis of the data revealed the importance of teachers' knowledge on algorithmically based mistakes held by the elementary school students in multiplication and division of fractions.

In addition to the algorithmically based mistakes, intuitively based mistakes performed by the elementary school students were also popular in pre-service teachers' answers. These errors result from intuitions held about the operations where; students tend to overgeneralize the properties of operations of whole numbers to the fractions (Tirosh, 2000).

In the first question of the questionnaire where; the fractional part ($\frac{1}{3}$) of the whole number (7) was investigated, four of the pre-service teachers emphasized that students would say that the answer was two since seven could not be divided by three completely. Pre-service teachers mentioned that since the students would think that the quotient should be whole number, based on the primitive model of division, they would not deal with the remainder and they just said that result should be a whole number that is two. In addition, the analysis revealed that in case of division of fraction by the whole number and division of whole number by the fraction, the effect of overgeneralization of properties of whole numbers to fractions and the misinterpretation of primitive division model was also popular. Pre-service teachers mentioned that students would have the conceptions that in a division problem, divisor must be a whole number. Thus, in a given question where students were expected to divide four by one-fourth, pre-service teachers stated that students perform the operation incorrectly since to divide by the fraction is not meaningful for them. For instance:

> Participant 4: "In the second part of the second question, it's given that students will package four kilograms of cheese into packages of $\frac{1}{4}$ kilogram each. Here, $\frac{1}{4}$ is not a whole number so to put in a one-fourth package may be ridiculous to them (CQ2-P4)."

Pre-service teachers, who emphasized the importance of the intuitively based mistakes, emphasized the overgeneralization of the properties of whole numbers to fractions and students misinterpretation of division and multiplication as primitive models. In other words, according to the primitive model of division, dividend is always bigger than the divisor, thus students' intuitions about division lead them erroneous results. For instance:

> Participant 6: "For the given sharing problem where; $\frac{1}{4}$ will be shared among 4 people, students will perform $4 \div \frac{1}{4}$, in other words, they switch the places of dividend and the divisor since the dividend is less than the divisor. These are because of the students' overgeneralizations of the properties of division operation to the fractions. In addition, teacher can be the one of the sources of this mistake. If the teachers do not explain the properties of the division properly students can have difficulties in performing operations (CQ2-P6)."

For the rest of the questionnaire where; the multiplication of two fractions were investigated, pre-service teachers did not mention the importance of intuitively based mistakes result from intuitions about multiplication of fractions.

Investigation of teachers' knowledge on elementary grade students' conceptions and misconceptions/difficulties on multiplication and division of fractions revealed another dimension of errors result from the students' formal knowledge on fractions. Tirosh (2000) emphasized that errors under this category are due to both limited conceptions of the notion of fractions and inadequate knowledge related to the properties of the operations and result lead incorrect performance. Pre-service teachers knowledge on students' mistakes based on the formal knowledge on fractions are discussed below.

Data analysis revealed that pre-service teachers emphasized the importance of formal knowledge on fractional knowledge while performing four operations on fractions.

In a given problem, where the fractional part of the whole number is investigated, three pre-service teachers stated that students could not perform the multiplication operation ($7\times\frac{1}{3}$) since they do not know what does $\frac{1}{3}$ mean or how to find one-third of the given whole number. For instance;

> Participant 7: "In this question, students are really confused since they do not understand how to divide a given whole into pieces. They do not know what does $\frac{a}{b}$ mean, and how to express it (CQ1-P7)."

Similarly, for the given sharing problem where; $\frac{1}{4}$ kilogram of sweets shared equally among the four people, according to pre-service teachers, students may have difficulties since they do not know what does one-fourth means. Teachers also added that students' informal knowledge on the concept of 'whole' can lead them to make errors. For instance;

> Participant 4: "Students may confuse one-fourth with the four. I mean they can confuse the one whole and one-fourth. Firstly, we should ask them what does one-fourth mean (CQ2-P4)."

From other point of view;

> Participant 13: "For the given sharing problem students can say that four over one-fourth is equal to one-fourth. I mean since they have informal knowledge on one whole, they can stated that instead of four students, one student represents one whole and then they divide one by four and say that result is one-fourth (CQ2-P13)."

In terms of given multiplication and division problem of mix fractions ($1\frac{3}{4}\div 2$, $1\frac{3}{4}\times\frac{1}{2}$), pre-service teachers emphasized the students' misconceptions on their formal knowledge on converting mix fractions into improper fractions. They added that since students could not perform the conversion successfully, they can not multiply or divide two fractions correctly in the second step. One of the pre-service teacher also added that students could have informal knowledge on the concept of 'half' and this lead them to perform the operation incorrectly. For instance:

> Participant 15: "If students do not have the formal knowledge like to divide by two is the same as multiplying by one-over two, they can not perform the operation correctly. In addition, they can confuse division by two with the division by $\frac{1}{2}$ which leads errors in computation (CS3-P15)."

Pre-service teachers mentioned that mistakes that were related to the formal knowledge on the concept of fractions and operations of fractions were stemmed from the inadequate formal knowledge of the properties of the four operations and limited conceptions of students' notion of fractions.

Pre-service teachers also added the importance of teacher factor and education system as the sources of algorithmically based mistakes, intuitively based mistakes, and mistakes based on formal knowledge. Pre-service-teachers stated that teacher should have enough subject matter knowledge in order to teach the topic properly. In addition, the education system should not direct the students to memorize the given rules but should make them to understand the logic behind the algorithmic operations.

In addition to the algorithmically based mistakes, intuitively based mistakes, and mistakes based on formal knowledge on fractions, pre-service teachers' emphasized another categorization of mistakes that is misunderstanding of the problem by the elementary school students. To state differently, some pre-service teachers stated that students can make errors since they do not understand the problem, what is being given and what is being asked to them and perform the operation incorrectly like misconstructing the equation from the given word problem.

Pre-service teachers explained various sources in misunderstanding of the problem that lead incorrect solutions. Pre-service teachers' knowledge on possible sources of students' conceptions are as follows: lack of care, lack of adequacy in mathematical knowledge, and lack of self-efficacy. Pre-service teachers stated that students do not read the question carefully and then they do not understand what is being given and what is being asked and they perform the operation incorrectly. In addition, pre-service teachers stated that some of the students are not competent in using mathematical language effectively. For instance, students could not understand what does one-third mean or in general what does to find $\frac{a}{b}$ of the given whole number. In addition, pre-service teachers stated that based on the inadequate mathematical knowledge, students might not convert the given word problem into symbolic representation.

Pre-service teachers stated that mathematics anxiety or lack of self-efficacy can be another source for the misunderstanding of the problem. Since most of the students have mathematics anxiety, before reading the question they feel as if they can not understand the question and perform the operation incorrectly.

Analysis of the data also revealed the effect of the subject matter knowledge of pre-service teachers on their knowledge on common (mis)conceptions held by the sixth and seventh grade students and on pre-service teachers' knowledge of the possible sources of these (mis)conceptions. Pre-service teachers' own knowledge on key facts and concepts, more specifically their own definition of multiplication and division interconnected with their knowledge on students' (mis)conceptions and possible sources of these (mis)conceptions. While analyzing the nature of the subject matter knowledge of pre-service teachers, results showed that pre-service teachers could describe and give examples on multiplication and division operations properly. In line with this, analysis revealed that pre-service teachers emphasized the importance of algorithmically based mistakes performed by the elementary grade level students in calculating fractional part of a whole number, fraction over whole number, whole number over fraction, fraction over fraction, and fractional part of the fraction. Thus, similar to above, I could say that pre-service teachers' competencies in performing algorithms influenced their knowledge of common (mis)conceptions held by the students.

On the other hand, most of the pre-service teachers hesitated on generalizing the properties of multiplication operation with whole numbers to the multiplication with fractions. Most of the pre-service teachers stated that multiplication of fractions with whole numbers is same as the multiplication of two whole numbers. Both of them are kind of repeated addition. However, in case of the multiplication of fractions, most of the pre-service teachers have confusion like; it is multiplication operation but how they are going to partially add one fraction to another that is relate addition with multiplication. Pre-service teachers further added that they have never thought about the logic of multiplication of two fractions, it's kind of division operation but the meaning is more complex than they thought. In case of the division, pre-service teachers used both partitive and measurement modeling of division while they are explaining the meaning of division of two whole numbers. On the other hand, most of the pre-service teachers used measurement modeling of division but emphasized the complexity of division on fractions. Additionally, pre-service teachers added that it was not easy to divide small fractions with larger one even if they used measurement modeling. Thus, results revealed that pre-service teachers' confusions on definition of multiplication and division operations reflected to their knowledge on students' conceptions where they dedicated the same difficulties to the students.

In Figure 4.1, the summary of the pre-service elementary teachers' knowledge on common conceptions and misconceptions/difficulties held by the elementary school students and possible sources for these misconceptions/difficulties while performing multiplication and division of fractions is given. Here, it could be summarized that pre-service teachers categorized the misconceptions/difficulties under four headings: algorithmically based mistakes, intuitively based mistakes, mistakes based on formal knowledge, and misunderstanding on problem. For the algorithmically based mistakes, students' rote memorization of basic rules and their inadequate knowledge on four operations are thought to be the main sources for these mistakes. Pre-service teachers also stated that students' misinterpretation of division and multiplication as primitive models were the sources for intuitively based mistakes. On the other hand, some pre-service teachers believed that inadequacy of formal knowledge on four operations and students' limited conceptions of notion of fractions lead them to make mistakes based on formal knowledge on fractions. In addition, according to the teachers' lack of care, lack of mathematical knowledge, lack of mathematical language, and lack of self-efficacy are the main sources for the misunderstanding of the problem that leads students to make errors.

In this section, pre-service elementary teachers' knowledge on common conceptions and misconceptions/difficulties held by the elementary school students and possible sources for these misconceptions/difficulties while performing multiplication and division of fractions were given. In addition, the influence of subject matter knowledge of pre-service teachers on their knowledge of common conceptions and misconceptions/difficulties held by the elementary grade students and influence of subject matter knowledge on pre-service teachers' knowledge on possible sources of these (mis)conceptions/difficulties were investigated. In the next part of the analysis, strategies that pre-service teachers used to overcome the misconceptions and difficulties held by the sixth and seventh grade students on multiplication and division of fractions were investigated. The strategies that pre-service teachers used to explain/verify the key facts, concepts, principles, and proofs on multiplication and division of fractions were discussed accordingly. In addition, the influence of subject matter knowledge of pre-service teachers on these strategies was analyzed.

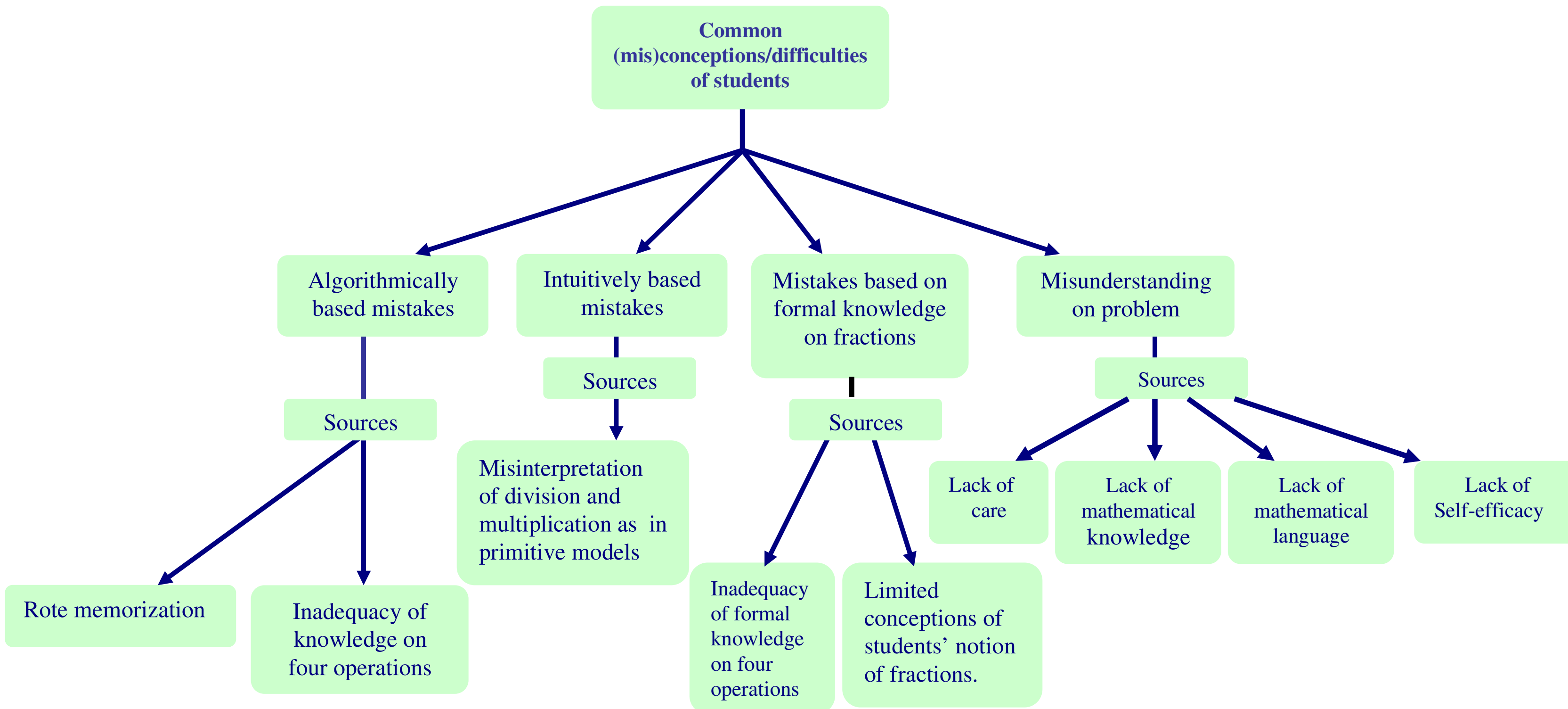



Figure 4.1. Summary of the (mis)conceptions/difficulties and sources of these (mis)conceptions/difficulties

### 4.2.2 Pre-service Teachers' Strategies to Overcome Students' (Mis)conceptions/Difficulties and Their Strategies to Explain the Key Concepts

In this section, strategies that pre-service teachers used to overcome the misconceptions and difficulties held by the sixth and seventh grade students on multiplication and division of fractions were investigated. The strategies that pre-service teachers used to explain/verify the key facts, concepts, principles, and proofs on multiplication and division of fractions were also discussed. In addition, the influence of pre-service elementary mathematics teachers' subject matter knowledge on strategies they use to overcome the misconceptions/difficulties held by elementary students and influence on strategies that they used to explain/verify the key facts, concepts, principles, and proofs were discussed accordingly.

Pre-service teachers' own experiences and knowledge as mathematics teachers were researched systematically. Pre-service teachers suggested various strategies that they can use to overcome the misconceptions and difficulties held by the sixth and seventh grade students while working on multiplication and division problems. By using the term strategies, I referred to the approaches, methodologies that pre-service teachers plan to use when their sixth and seventh grade students face with difficulties and strategies that pre-service teachers used to explain/verify the basic concepts in multiplication and division of fractions. The summary of the strategies suggested were presented at the end of the section.

Holt (1967) stated that learning is enhanced if students were asked the followings: (1) state the information in your own words, (2) give examples of it, (3) recognize it in various guises and circumstances, (4) see connections between it and other facts or ideas, (5) make use of it in various ways, (6) foresee some of its consequences, (7) state its opposite or converse. Hiebert & Lefevre (1986) added that linking conceptual knowledge with symbols will create a meaningful representation system and linking conceptual knowledge with rules, algorithms, or procedures reduces the number of procedures that should be remembered and increase the appropriateness of recalling the right procedure to use effectively.

Similarly, Lamon (1993) mentioned that teachers might encourage the unitizing process by posing problems that allow for multiple solution strategies. In the literature, many studies emphasize the importance of using conceptual strategies while teaching. However, it is known that teachers who do not conceptually understand the content are unlikely to teach conceptually (Stoddart, Connell, Stofflett, & Peck, 1993). Thus, before using effective strategies that give importance to students' understandings, teachers should first question their own knowledge of concepts. Conceptual knowledge cannot be an isolated piece of knowledge. It will be conceptual if the relationship is recognized among the other knowledge and conceptual knowledge is increase by the construction of relationships between pieces of information. Conceptual knowledge directly related to the "meaningful learning" and "understanding" where new mathematical information is connected appropriately to existing knowledge. On the other hand, in the procedural knowledge the important thing is procedures that are carried out in a predetermined linear sequence (Hiebert & Lefevre, 1986).

The analysis of data showed that strategies used by the pre-service teachers to overcome the difficulties that students held can be grouped under three headings: strategies based on teaching methodologies, strategies based on formal knowledge on fractions, and strategies based on psychological constructs. Strategies based on teaching methodologies consist of using multiple representations (e.g. verbal expressions, figures, and graphics), using different teaching methodologies (e.g. problem solving), emphasizing on drill and practice, and directly focusing on question. On the other hand, strategies based on the formal knowledge consist of meaning of concepts, logical relationship among operations, alternative solutions to problems, and knowledge on misconceptions. Lastly, strategies based on psychological constructs involved increasing self-efficacy of students.

These categories were formed by using the recurring patterns and themes obtained from the investigation of the Multiplication and Division of Fractions Questionnaire, interview and video transcripts of 17 pre-service teachers. In

MDFQ pre-service teachers were asked to write strategies that they plan to use to overcome the difficulties or misconceptions that sixth and seventh grade students have while studying multiplication and division of fractions. Pre-service teachers' questionnaire, interview and video transcripts revealed that in order to overcome the misconceptions and difficulties held by students, teachers should use conceptual strategies where; teachers needs both subject matter knowledge and pedagogical competencies.

Using multiple representations is one of the strategies offered by the pre-service teachers. Most of the pre-service teachers stated that they can use figures, verbal expressions, visual materials, and daily life examples to overcome the difficulties on multiplication and division of fractions. Similarly, Lamon (1993) emphasized that instruction may begin with associated-set problems presented in a concrete pictorial or manipulative-based context and then could develop a richer sense of knowledge. Pre-service teachers also added that they can also make a transition among multiple representations in order to explain the concepts more clearly. For instance, they mentioned that they can teach how to transfer verbal expressions or word problems into symbolic expressions. Then, with the help of the mathematical language, teacher can overcome the difficulties held by the elementary grade level students while studying multiplication and division of fractions. For instance:

> Participant 1: "I think that I can overcome the misunderstandings by using the figures. I can give examples from materials that they are familiar with their daily lives. I also teach how to solve a verbal expression when it's written in mathematical language. I think this will really work (STQ2-P1)."

> Participant 3: "I can make a model for the question. For example, for the second question, I bring a cheese of 4kg and also distribute packages each of one-fourth kilograms. Then, I can ask how many packages I need to

put this 4 kilogram cheese. It's better to use concrete materials to reduce the misunderstandings (STQ2-P3)."

In addition to multiple representations, using different teaching methods is another strategy suggested by the pre-service teachers in order to overcome the difficulties. Four of the pre-service teachers emphasized that problems should be solved step by step as in Polya's problem solving strategies. Teacher's clear descriptions of question like what is being given and what is being asked can enable the students to understand the question that leads correct solution. For instance:

> Participant 13: "We should avoid the memorization of the rules. I mean we should not say like, 'always multiply with numerator and divide with denominator'. We should use manipulative to make the concept understandable. We should reinforce the students to solve the problems according to Polya's steps (what is given, asked,…) (STQ2-P13)."

> Participant 3: "In a give question we can let the students to discover their own mistakes. We can make them use materials and create a situation according to the given question. If students can realize their mistakes, then they are more able to understand the topic and reduce their misunderstandings on the given topic (STQ1-P3)."

Using drill and practice in mathematics classes is another strategy suggested by three of the pre-service teachers. Pre-service teachers stated that they should solve many questions in order to overcome difficulties and misconceptions held by students. Teacher should ask different questions on given topic and also should suggest alternative solutions to the given questions. In other words, pre-service teachers emphasized the importance of drill and practice while overcoming the difficulties held by the elementary grade level students. For instance:

Participant 13: "While solving the questions we should focus on different types of questions. In addition, we should develop suitable model for answering this new kind of questions and teach them to the students (STQ4-P13)."

The other strategy given by the pre-service teacher is about directly focusing on the question that is being asked. Pre-service teachers stated that they should make students to read questions carefully until they understand it completely. For instance:

Participant 2: "Students perform the operations incorrectly while they were solving the questions. This is because of not reading the give question carefully. Thus, we should make the students to read the question carefully still they understand what is being given and asked to them (STQ1-P2)."

Participant 14: "We should make the students to have a habit like to read the question carefully. In addition, while reading the question they should imagine the situation in order to solve the question. By this way, we can overcome the misunderstandings held by the elementary grade students (STQ3-P14)."

In addition to the strategies based on teaching methodologies such as using multiple representations, using different teaching methods, using drill and practice, and focusing on understanding of question itself, most of the pre-service teachers emphasized the importance of strategies based on formal knowledge in order to overcome the difficulties and misconceptions held by the elementary school students. Pre-service teachers mentioned various strategies based on the formal knowledge on operations on fractions. Thirteen of the pre-service teachers agreed that teachers should know the concept very well and make the students understand the concepts first before letting them to solve the related questions.

Teachers stated that before proceeding to the traditional symbolism –in multiplication, multiply numerators with numerators and denominators with denominators and in division invert the second fraction and multiply- for solving the given questions, we should focus on the meaning of the concept. Pre-service teachers also emphasized that mathematics teachers should first clearly explain what the fraction concept is, what the multiplication, division, addition, and subtraction operations mean. Teachers should focus on the relationship between fraction and basic operations on fractions. In addition, how these operations can be related to each others, and how operations on fractions can be generalized to other topics in mathematics. In addition, pre-service teachers believed that if teachers can explain the logic under the concept then students can be avoided from misunderstandings and misconceptions. For instance:

> Participant 8: "Firstly, we should express the fraction concept clearly. Like what does one-third mean to the students. For the second question, teacher should explain the division operation on fractions more clearly and deeply before solving the question. Teachers can use manipulative, materials to make it more understandable and clear (STQ-P8)."

> Participant 11: "For the first question in the questionnaire, teacher should first explain the logic of multiplication. He can either use figures or materials to do this. Then, teacher should not only focus on the rules of multiplication but also explain the logic of these rules. This is also true for the rest of the questionnaire. Like in the second question, teacher should explain the logic of division by using the materials and manipulatives (STQ-P11)."

Pre-service teachers also emphasized the explanation of the logic of the operations while performing them. For example:

Participant 7: "For the third question in the questionnaire, students can form the equation like $1\frac{3}{4}a = \frac{1}{2}$. Thus, while solving the equation teacher should express the logic of why we invert the fraction while transferring it to the other side of the equation (STQ3-P7)."

Participant 2: "Performing operations on addition, subtraction, multiplication and division should be emphasized clearly. Students should understand that they should find the common denominator while adding and subtracting the fractions but not in multiplication and division operations. Similarly, as given in the fourth question of the questionnaire, students should know that they will convert the mix fractions into improper fractions before carrying out the multiplication and division operations. Teacher should emphasize that if students convert the mix fraction into improper fraction they can easily perform the operations. Teacher should also explain that the whole part of the mix fraction is also one of the parts of the fraction thus we should involve it into multiplication operation by converting it into improper fraction (STQ-P2)."

In addition to the importance of formal knowledge on operations on fractions, two of the pre-service teachers emphasized the importance of making students to explain the reasons of the steps that they are carrying out while performing these operations. In other words, pre-service teachers stated that students can be given extra time to explore multiplicative relationships and related perspectives. According to the pre-service teachers, students should be able to explain each step that they are carried out while solving the given problems. In addition, if students could reason their steps, they will not have difficulties while performing the related operations. For instance:

Participant 3: "In order to overcome the difficulties, we can ask the student to explain what he did. While students are performing the

> operation, we can ask them what and why they did it like this. By this way, we can also understand whether they know the logic of operations that they performed (STQ4-P3)."

Apart from these strategies, one of the pre-service teachers mentioned that teacher should inform the students about the misconceptions/difficulties on related topics. If the students aware of the misconceptions/difficulties and sources of these misconceptions/difficulties, they can get rid of making the same mistakes while studying the multiplication and division of fractions.

The importance of the sequence of the concepts like beginning from easy examples and moved to the harder ones or beginning from concrete ideas and moved to the abstract ides was also stated by the two of the pre-service teachers in order to overcome the misconceptions and difficulties held by the students. In addition, pre-service teachers added that teachers should perform the operations in reverse order like used multiplication in checking division operation and properties of specific operations like commutative property in multiplication operation should be emphasized to the students. For instance:

> Participant 9: "For example, in the second question, students can have difficulties while performing the division operation since they are not working with whole numbers. I think, before asking such a question teachers should give the numbers that are completely divided and then perform the operations by using the fractions (STQ2-P9)."

The last strategy mentioned by the pre-service teachers was related to the affective domains. Pre-service teachers stated that in order to overcome the misconceptions/difficulties that students have, teachers should also concentrated on the needs of the students besides improving their cognitive skills. Students could also perform the operations incorrectly because of their inadequate self-concepts or high level of anxiety besides their low level of subject matter. That's why some pre-service teachers stated that teachers should on the psychological constructs involving mathematics anxiety and self-efficacy in order to overcome students' misconceptions/difficulties. For instance:

> Participant 14: "We can overcome the misconceptions that students have by helping them to increase their self-concept. This is related to the support supply by the teacher. She can motivate them to perform the operations and have the correct results (STQ1-P14)."

The summary of the analysis of pre-service teachers' strategies to overcome the misconceptions and difficulties that students have was given in Figure 4.5. In addition, participants who used these specific strategies were identified under each category in the figure.

Additionally, the analysis showed that subject matter knowledge of pre-service teachers influenced the strategies that they used to overcome the misconceptions/difficulties held by elementary students. That is, almost all of the participants stated that in order to overcome the misconceptions/difficulties held by the students, they will use manipulatives and materials to visualize the concept and make it clear. They added that they will try to give the logic of the operations with the help of these materials and by this way, students will have no difficulties while performing the operations. In other words, the analysis of the questionnaire, interview, and video transcripts revealed that pre-service teachers with various degree of subject matter knowledge agreed on using concrete materials and emphasizing the logic of operations instead of stating the rules directly. That is, even the pre-service teachers who do not have clear conceptual understanding of key facts, concepts, principles, and proofs on multiplication and division of fractions, stated conceptual strategies to overcome the misconceptions/difficulties. For instance, participant 6 had some problems while explaining the logic of multiplication and division rules. He stated that multiplication and division of fractions are not easy topics and he could not teach the meaning of these concepts now. Thus, I could say that even pre-service teachers subject matter knowledge on multiplication and division of fractions can not be regarded as deep enough; they could suggest various strategies to overcome the misconceptions/difficulties held by the elementary sixth and seventh grade students. For instance:

> Participant 6: "It's important to teach the meaning of rules. I mean, I should clearly explain the invert and multiply rule in division of fractions. But, most probably I will teach the logic of operations. I will use transparencies, figures, manipulatives, and posters. Think about multiplication of $\frac{2}{3}$ with $\frac{1}{2}$. I will use the transparencies and show the common area. I mean I will visualize the operations. As I stated before, I'm not ready to do all these now. I'm not ready to teach the topic to overcome the misconceptions, but I will study. This topic is too hard to teach but I will improve myself (STQ7-P6)."

In addition to the strategies used to overcome the possible misconceptions and misunderstandings held by the students, pre-service teachers were also asked to state the strategies that they use to explain/verify the key facts, concepts, principles, and proofs on multiplication and division of fractions. Pre-service teachers specifically asked to state the strategies that they would use to explain while multiplying (e.g. $11\frac{1}{2}\times\frac{1}{4}$), and dividing (e.g. $6\frac{2}{3}\div\frac{5}{6}$) two fractions. Pre-service teachers were also asked how they would explain the expressions involving multiplication (e.g. $\frac{3}{4}\times\frac{4}{3}=1$) and division (e.g. $\frac{2}{3}\div\frac{1}{2}=\frac{2}{3}\times\frac{2}{1}$) of fractions, the key facts and relations, and proof of multiplication operation ($\frac{a}{b}\times\frac{c}{d}=\frac{a\times c}{b\times d}$, where; a, b, c, d $\in Z$ and b, d$\neq 0$) and division operation ($\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}$, where; a, b, c, d $\in Z$ and b, c, d$\neq 0$).

Based on the analysis of data, strategies that pre-service teachers used to explain/verify the key facts, concepts, principles, and proofs on multiplication and division of fractions can be grouped under two headings: conceptual strategies and procedural strategies. Hiebert & Lefevre (1986) mentioned that linking conceptual knowledge with symbols will create a meaningful representation system and linking conceptual knowledge with rules, algorithms, or procedures reduces the number of procedures that should be remembered and increase the appropriateness of recalling the right procedure to use effectively.

For the given multiplication and division problems where pre-service teachers were asked to explain how they would explain these operations to their students, fourteen pre-service teachers among 17 suggested the strategies that may be categorized as the characteristics of the conceptual strategies.

Almost all of the pre-service teachers agreed that the logic of the operations on multiplication and division should be clearly stated to the students. The rules for the algorithms should also be emphasized after explaining the logic of operations. For instance:

> Participant 12: “I explain that multiplication is used in determining the amount of the given whole. For example, multiplication of 15 by 2 is taking the 2 times of 15 that is 30. But, to take multiple does not mean that to multiply it by whole number. Fractions can also be the multiple and multiplication of number with fraction can also be treated as multiple of fraction. Thus, we can multiply fraction by the whole number or by the other fraction. Similarly, for the division, I explain that to divide one number with another is searching the amount of number in that given whole. Like, $6\frac{2}{3} \div \frac{5}{6}$ is finding how many $\frac{5}{6}$s are there in $6\frac{2}{3}$. Multiplication and division are inverse operations, this is why we invert and multiply in division operations. Like; 5-2 is 5 + (-2) where; we perform the inverse operation. Thus, $30 \div 6 = \frac{30}{6} = 30 \times \frac{1}{6} = 5$ (STQ5-P12).”

Pre-service teachers who suggested conceptual strategies stated that they would use concrete pictorial figures or manipulatives to make the multiplication and division operation more understandable for the students. For instance, participant 1 used the figures to express the multiplication of two fractions. First, he showed the multiplication of whole numbers, whole numbers with fractions, simple fraction with another simple fraction, and simple fraction with mix fractions which lead him to make generalization. He used the logic of part of the part while multiplying two fractions. As stated in figure below, for the given multiplication operation $\frac{3}{2}\times\frac{1}{3}$, he first draw $\frac{3}{2}$ and then divide this amount by three and then take one part that is $\frac{1}{2}$.

Participant 1:

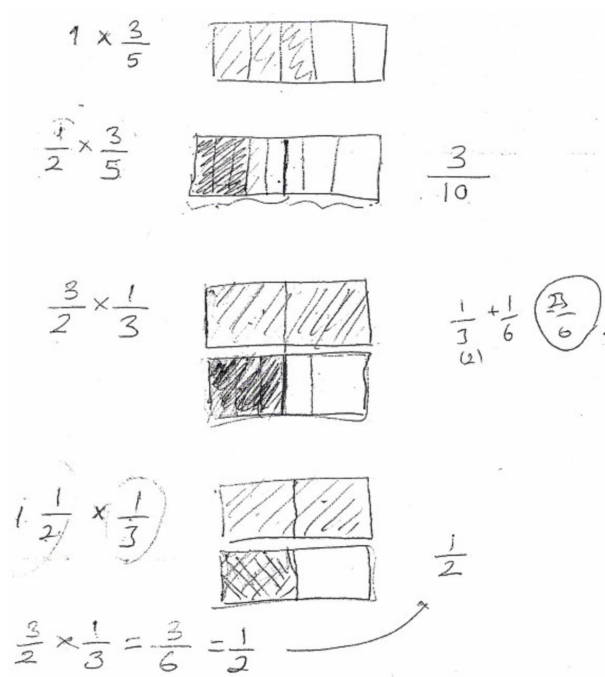



**Figure 4.2.** Multiplication of two fractions (participant 1)

Similarly, in order to express the division operation, eleven of the pre-service teachers preferred to use figures to explain the logic. For instance, participant 7 drew rectangular area model to represent the first fraction and then try to find the number of second fraction in first fraction. More specifically, in order to divide $\frac{1}{4}$ by $\frac{1}{2}$, he first showed $\frac{1}{4}$ by the rectangle and then search $\frac{1}{2}$ in that given $\frac{1}{4}$. Similarly, in order to divide $\frac{1}{2}$ by $\frac{1}{3}$, he first represent $\frac{1}{2}$ and then search $\frac{1}{3}$ in it.

Participant 7:

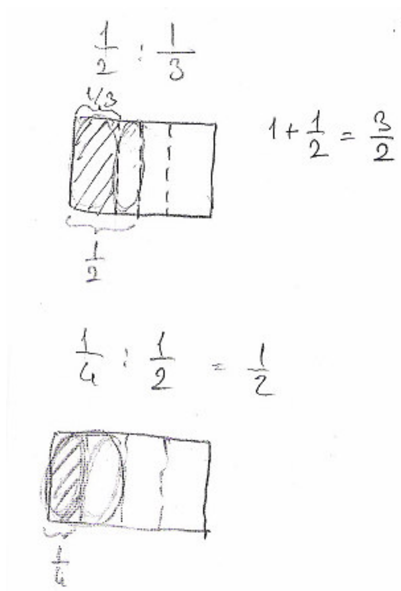



**Figure 4.3.** Division of two fractions (participant 7).

In addition, pre-service teachers suggested using daily life examples to relate the operations with the concepts. For instance:

Participant 8: "For the given division problem $6\frac{2}{3} \div \frac{5}{6}$, I draw the figures for the given real life problem. Division of $6\frac{2}{3}$ by $\frac{5}{6}$ can be thought like a problem: I have a garden with perimeter of $6\frac{2}{3}$ m. I want to plant a tree across the perimeter of this garden with $\frac{5}{6}$ m gap for each consecutive tree. Thus, in order to cover whole perimeter of the garden, how many trees do I need? By using this daily life problem, I can draw a garden and then explain that $\frac{5}{6}$ m gap means to divide the perimeter with $\frac{5}{6}$ (STQ5-P8)."

Thus, the analysis of the questionnaire, interview, and video transcripts revealed the fact that all of the pre-service teachers wanted to use manipulatives, materials while teaching the basic operations in multiplication and division of fractions although few of them could do it conceptually. They stated that to emphasize the meaning of operations is more important than application of rules. As a teacher, they firstly tried to express the meaning of operations by using suitable materials and then give the general rule. Thus, I could deduce that even if pre-service teachers have enough competencies or not in explaining the multiplication and division operation on fractions, they all suggested to use conceptual strategies in their real classroom applications in order to teach the given concepts. That is; pre-service teachers' level of subject matter knowledge regardless of high or low positively influence the strategies that they use to explain basic concepts on multiplication and division of fractions. Most of the pre-service teachers insisted on using conceptual strategies on both overcoming the difficulties held by the elementary student and explaining the basic operations on multiplication and division of fractions. For instance:

> Participant 15: "As I did above, I will connect the multiplication and division operations in whole numbers and fractions. I'm not going to directly give the rule of multiplication and division. If I give the rule, students will memorize and forget everything…I will know the characteristics of my students, thus while teaching this topic I prepare materials according to their subject matter knowledge. I will use simpler

examples and then make generalizations based on these examples (STQ7-P15)."

Participant 17: "Students should not have any question in their minds related to the multiplication and division. I will use materials; I mean I will use visual materials to explain the logic of the operations. With the help of the materials, I will relate multiplication and division operations on fractions. I will ask how many $\frac{1}{2}$s are there in 2, and how you can relate it to whole numbers. I will make students to think about these relationships. But, firstly I will have to construct this relationship in my mind. I will study (STQ7-P17)."

In addition to the multiplication and division operations on fractions, pre-service teachers were asked which strategies they would use to explain the invert and multiply rule (e.g. $\frac{2}{3} \div \frac{1}{2} = \frac{2}{3} \times \frac{2}{1}$) to their students. Ten of the pre-service teachers among 17, emphasized that they will explain the invert and multiply rule like how many $\frac{1}{2}$s are there in $\frac{2}{3}$. As stated above, pre-service teachers preferred to use manipulatives and figures in order to explain the invert and multiply rule. For instance:

Participant 13: "We should explain the definition of division. Before performing the operation, we should explain the meaning of division. Like, what do we mean by 4 over 2. It means how many twos are there in four. Later, I applied the same logic to the two fractions. The $\frac{2}{3} \div \frac{1}{2}$ means how many $\frac{1}{2}$s are there in $\frac{2}{3}$. I will also use manipulatives or figures in order to show this operation (STQ6-P13)."

Thus, analysis of the interview and video transcripts revealed that most of the pre-service teachers are able to explain the logic of division of fractions to their students without directly using invert and multiply rule. On the other hand, they are not successful enough to connect the division operation with multiplication operation. In other words, while performing division operation on fractions they could not explain why they are using multiplication operation (invert and multiply). They performed the division operation successfully by finding how many second fractions are there in the first one and compare the result with the right hand side where; the second fraction is invert and multiplied. In addition, pre-service teachers preferred to use many examples and then make generalizations based on these given examples that can be all categorized under procedural strategies. Thus, results revealed that participants can explain the logic of division but few of them can relate the multiplication and division operations in fractions. Only two of the pre-service teachers related the multiplication and division operations to each other while explaining the invert and multiply rule. That is:

> Participant 6: "For example, $\frac{2}{3} \div \frac{1}{2}$ means to divide $\frac{2}{3}$ into two. That is, after dividing the fraction into two, we will have two pieces that is we multiply the fraction by two. This is why we invert and multiply the second fraction. Multiplication and division are inverse operations. Division means the number of the pieces that can be found by multiplication (STQ6-P6)."

> Participant 13: "$\frac{1}{2} \div \frac{1}{4}$, how many $\frac{1}{4}$s are there in $\frac{1}{2}$. My purpose is to fill $\frac{1}{2}$ of the whole with the given pieces ($\frac{1}{4}$). For this reason, I should do multiplication in order to fill. I should increase the amount of the given fraction and fill the empty places. Thus, by using multiplication I can increase the given amount, I should multiply by four in order to find

how many $\frac{1}{4}$s are there in $\frac{1}{2}$. That is, I can use four pieces of $\frac{1}{2}$ in order to find number of $\frac{1}{4}$s in $\frac{1}{2}$. How many pieces should I use to fill the $\frac{1}{2}$ by using pieces of $\frac{1}{4}$s (STQ6-P13)."

In addition to these strategies, most of the pre-service teachers said that they will use easy numbers like two whole numbers that can be divisible completely and then they generalize this rule to the fractions without working with fractions. That is, pre-service teachers' inadequate subject matter knowledge on explaining why we invert and multiply while dividing fractions lead them to prefer procedural strategies while teaching division of fractions. Similarly, some of the pre-service teachers stated that they can not find meaningful explanation for this relation and they just gave the formula to the students that can also be categorized under procedural strategies.

The analysis of the strategies that are used to prove the expressions ($\frac{a}{b}\times\frac{c}{d}=\frac{a\times c}{b\times d}$, a, b, c, d $\in Z$ where; b, d $\neq 0$ and $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}$, a, b, c, d $\in Z$ where; b, c, d $\neq 0$) revealed the fact that few of the pre-service teachers had strategies that could be categorized as conceptual. For instance:

> Participant 7: "I will use the transparencies in order to proof the given expressions. I will show it with specific examples. For instance, ($\frac{2}{3}\times\frac{1}{2}$) I will show the first fraction that is $\frac{2}{3}$, and then took the $\frac{1}{2}$ of this shaded region ($\frac{2}{3}$). The intersecting area will give the multiplication of fractions. Because multiplication means that you think the first fraction as the whole and then you will take the part of it that is second fraction. Then, I will generalize this rule to the multiplication of all fractions. I will do the same thing for the division. I will show that division means is to search

for the first fraction in the second one. Thus, by using the examples I can make generalizations for all fractions (STQ10-P7). That is:

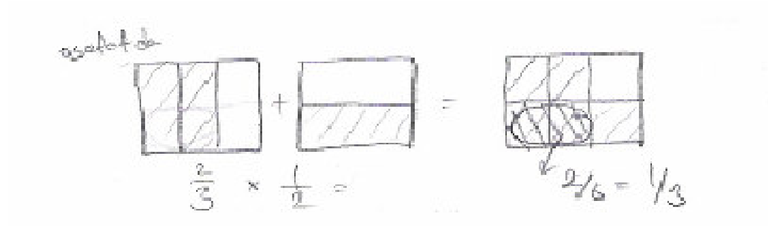



**Figure 4.4**. Multiplication of fractions by using transparencies (participant 7).

On the other hand, ten of the pre-service teachers mentioned that they had no idea on proving the multiplication and division rules and preferred to use various examples in order to prove the given multiplication and division expression. They stated that by using numbers instead of expressions *a, b, c,* and *d,* they can show the equality of both sides of the equation and hence generalize this as the rule. For instance:

> Participant 14: “I mean it’s not easy to prove or explain such operations to students. I think first of all, I will use divisible numbers. I will select some easy numbers such as 6 and 3. I would tell them to divide them and explain the rule that is trying to find how many 3s are there in 6. Then, by using such examples, I will make generalizations. I mean I will use easy examples to explain the logic of the operations and then teach them the rules and say that this is the short way what we did (STQ7-P14).”

The influence of the subject matter knowledge of pre-service teachers on the strategies that they use to explain basic principles and proofs on multiplication and division of fractions was obvious. As stated before, most of the pre-service teachers preferred to use numbers instead of symbols in proving the given multiplicative expressions since they did not have enough subject matter knowledge in proving the given expressions. On the other hand, these pre-service teachers agreed on using figures, manuipulatives, and materials to explain the given expression and operations conceptually. In other words, pre-service teachers have strong belief in using materials for expressing the logic of conceptual background given operations without having enough subject knowledge. They believed that by using manipulatives, and materials they can teach the logical background of the given operations and then after teaching the concepts conceptually they will make generalizations and state the general rules. That is, for easy and basic operations, pre-service teachers preferred to use conceptual strategies, on the other hand for the complex and difficult tasks, they preferred procedural ones. For instance:

> Participant 13: “We should take the attention of students first. Students should know that they have to pass through the certain process in order to understand the topic. Students should know what does multiplication and division means, how they are related to the operations on fractions. Students should know why we multiply the numerator with numerator and denominator with denominator in multiplication. Similarly, students should know why we invert and multiply in division. We should teach the basic elements. Thus, we should use materials, manipulatives in teaching those concepts meaningfully. I do not know how, but I will learn. After teaching the topic I will make generalizations and then use this generalization (STQ7-P13).”

The summary of the strategies used by the pre-service teachers to explain the concepts in multiplication and division of fractions under the categorization of conceptual and procedural strategies is given in Figure 4.6. In addition, from the analysis of the data, it was clear that pre-service teachers have strong belief that they should teach these topics conceptually where; the logical background of the operations should be explained to the students. Pre-service teachers also added that during this process, the usage of materials, figures and manipulatives is important although they do not have enough competencies and knowledge on the meaning of basic facts and concepts. In addition, most of the pre-service teachers stated that if it is the first time that they are going to teach the concepts related to multiplication and division of fractions, they preferred to use the transparencies, figures and manipulatives. In addition, they will try to explain the meaning of the concept and relationships among the concepts even they do not have enough subject matter knowledge. On the other hand, they stated that after explaining the topic that is if it is not the first time that they are explaining the multiplication and division of fractions, they preferred to use the rules for both multiplication and division of fractions.

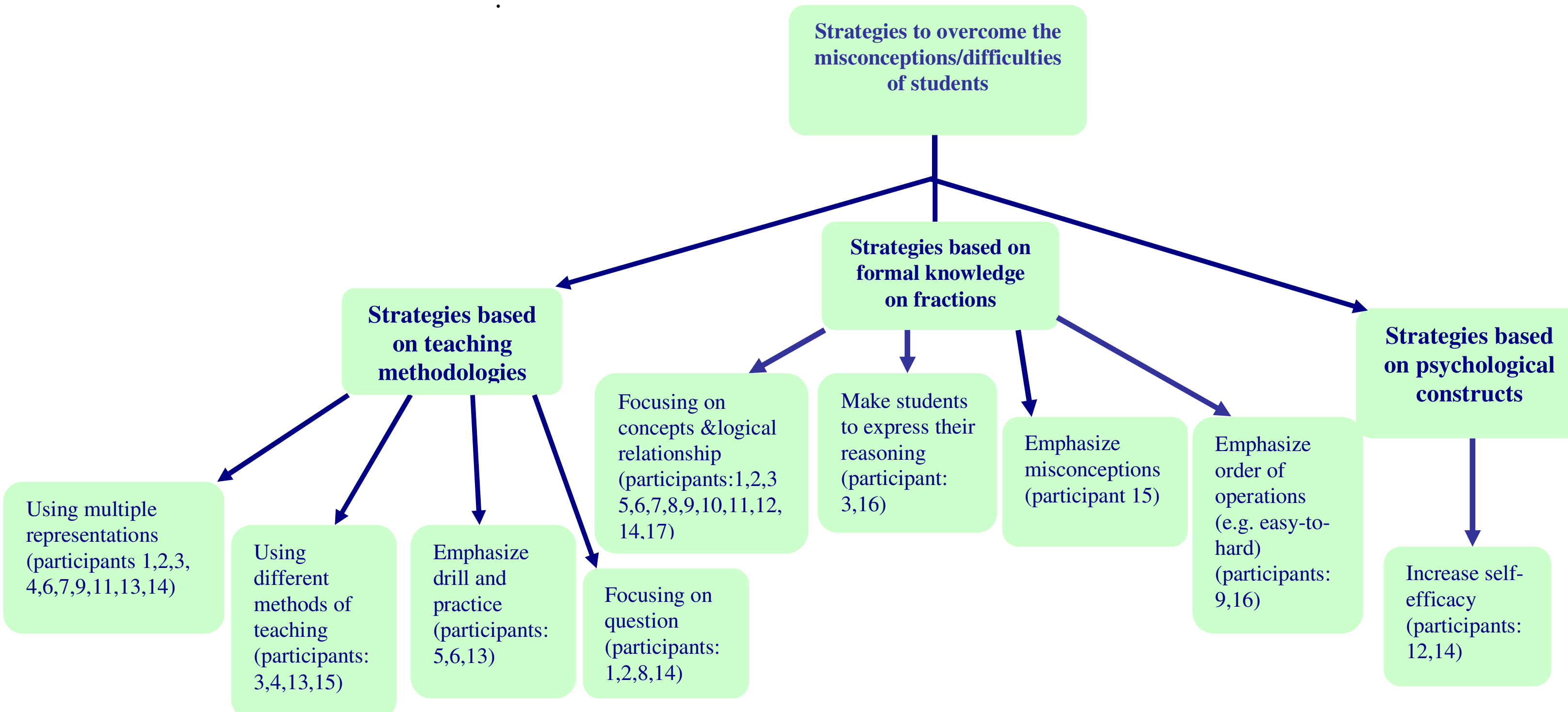



**Figure 4.5.** Summary of the strategies used by the pre-service teachers to overcome the misconceptions/difficulties of students

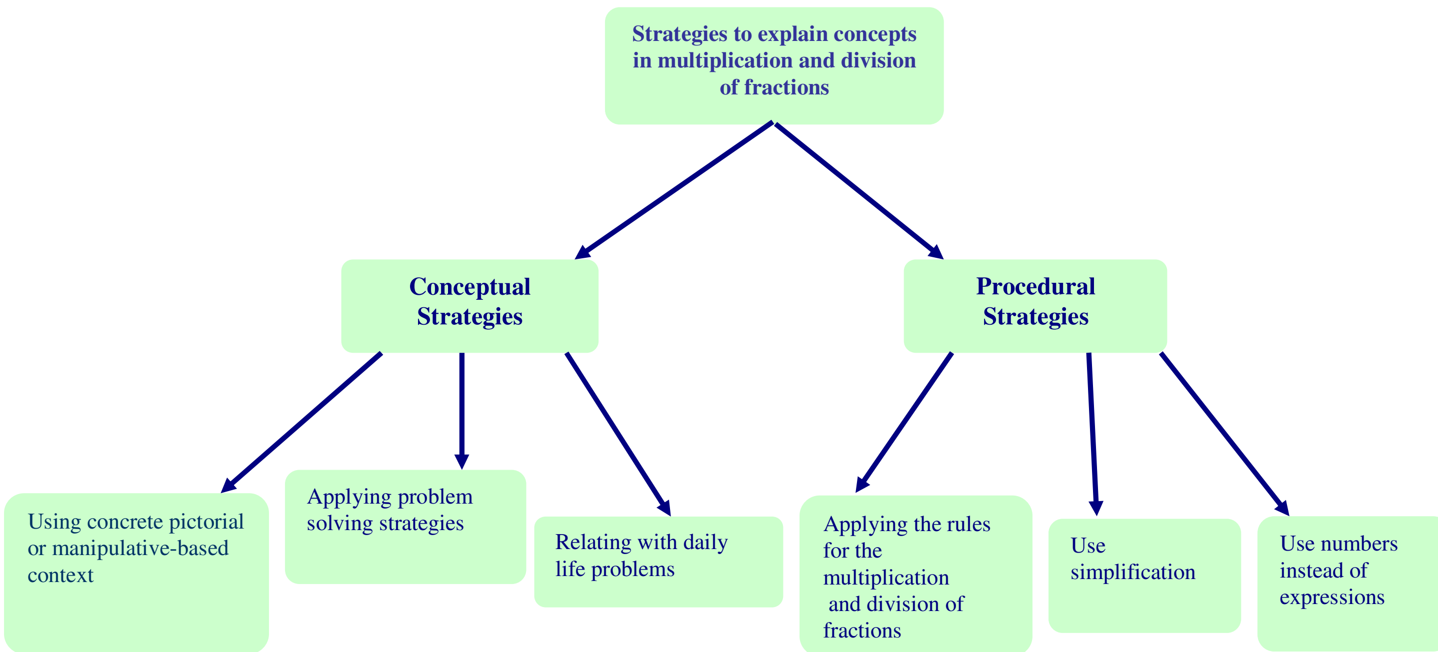



**Figure 4.6.** Summary of the strategies used by the pre-service teachers to explain the concepts in multiplication and division of fractions

### 4.2.3 Pre-service Teachers' Representations on Multiplication and Division of Fractions

In this section, research question that is representations (modeling) used by the pre-service elementary mathematics teachers to reason their understanding of multiplication and division of fractions was investigated. In addition, the influence of pre-service elementary mathematics teachers' subject matter knowledge on the kind of representations they use to reason their understanding of multiplication and division of fractions were discussed accordingly. From the analysis of the data stated above, this section was presented under the headings; "representation of fractional part of a whole number", "representation of the fraction over whole number", "representation of the whole number over fraction", "representation of the fraction over fraction" and, "representation of the fractional part of the fraction", based on the literature review, categories from other studies, from the framework of participants' statements, and from my experiences with the data. Pre-service teachers used various representations and models while presenting their understanding of multiplication and division of fractions. By using the term model, I referred to the two terms used by Ball (1990b): measurement (quotitive) division model that is forming groups of a certain size and partitive model that means forming a certain number of groups. Besides measurement and partitive modeling, four different classes of multiplicative structures; repeated addition, multiplicative comparison, cartesian products and product of measures (Greer, 1992; Azim, 1995) were used as naming the models for representations. In addition to the modeling, pre-service teachers used multiple representations while making transitions (Ainsworth et al. 2002; Post, Behr, Lesh, 1991) from one representation to another or within the same representation. In this study, pre-service teachers used representations written symbols, pictures and diagrams, word problems concerning real life situations as described by Lesh et al., (1987). In terms of pictures or diagrams, pre-service teachers used area models including pie charts, rectangles, and discrete object models that are categorized as pictorial representations by Lesh, Landau, and Hamilton (1983).

The analysis of the results was grouped under multiplication and division models and then representations within each model were discussed. Results were given as the representations within the given division and multiplication models. To state differently, categories were formed by using the models and representations were categorized within the models. At the end of the each section, summary of the models and representations used within the section are given. Thus, based on the recurring patterns and themes pre-service teachers' representations of five different cases are explained in the subsequent sections.

#### 4.2.3.1 Representation of Fractional Part of a Whole Number

In this part, representation of fractional part of the whole number was investigated. In the MDFQ, the first question was related to the representations of one-third of seven chocolates that is categorized as the modeling of fractional part (one-third) of the whole number (seven). Since the question is given as word problem, we may consider it as asking a translation from word problem to any other representation. Pre-service teachers used various representations within different models including written symbols, pictures and diagrams including pie charts, rectangles, and word problems concerning real life situations in order to represent the one-third of seven chocolates. In terms of modeling, pre-service teachers used division models as measurement modeling "How many 3's are in 7", and partitioning modeling "Find a number such that 3 of it is 7" to represent one-third of seven chocolates. On the other hand, some pre-service teachers used multiplicative compare (distributive property) where; the one-third of seven chocolates was determined by taking one-third of each sub-unit and then summing the parts to determine the whole. Here results are given as representations within the given division and multiplication models.

The analysis of the MDFQ questionnaire revealed that four pre-service teachers among 17 used measurement modeling in their explanation of one-third of seven where they grouped 3 chocolates to represent one whole and then they divide the last piece into three as seen in Figure 4.7.

Participant 8:

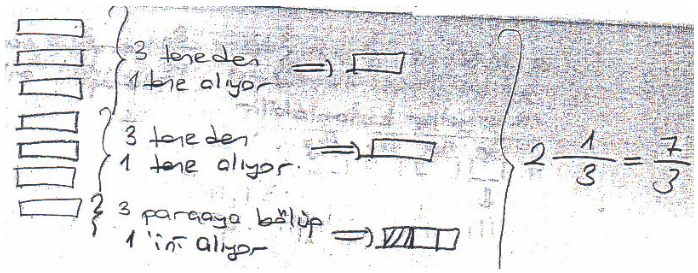



**Figure 4.7.** Measurement modeling of one-third of seven (participant 8)

These four pre-service teachers who prefer measurement modeling to represent fractional part of the whole used rectangular area representation and symbolic representation such as multiplication ($7 \times \frac{1}{3}$) or division ($\frac{7}{3}$) to represent one third of seven as seen in Figure 4.8. Each chocolate was represented as a rectangular bar and then last bar is further divided into three to find the group number.

Participant 11:

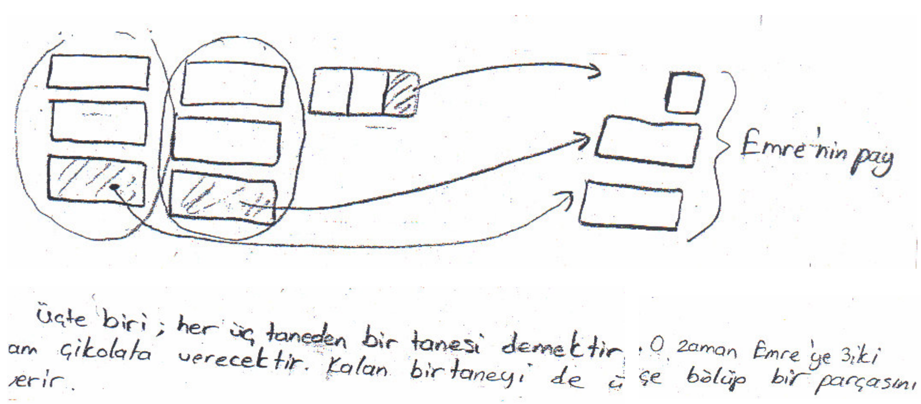



**Figure 4.8.** Rectangular area representation of measurement modeling (participant 11).

While analyzing pre-service teachers' representations of fractional part of the whole number, interview and video transcripts revealed the relationship between the modeling that pre-service teachers used to represent the fractional part of the whole and their definitions of division. That is pre-service teachers' subject matter knowledge on basic concepts effect their representations where; pre-service teachers who preferred to use measurement division model defined division as the number of groups of three in seven. For example:

> Participant 8: "Division means separating into pieces. For example, 10 ÷2 means grouping by two's and finding the number of groups. In other words, dividing one number by another number is trying to find how many group of second number is in the first number (RQ1-P8)."

> Participant 11: "Division is repeated subtraction. For instance, six over two is subtracting two's until you find zero. So, how many times you subtracted two gives the result which is three (RQ2-P11)."

Besides measurement modeling, five of the pre-service teachers used partitioning modeling where; they tried to find how many chocolates each person will get if he distributes seven chocolates among three people. For instance;

Participant 16:

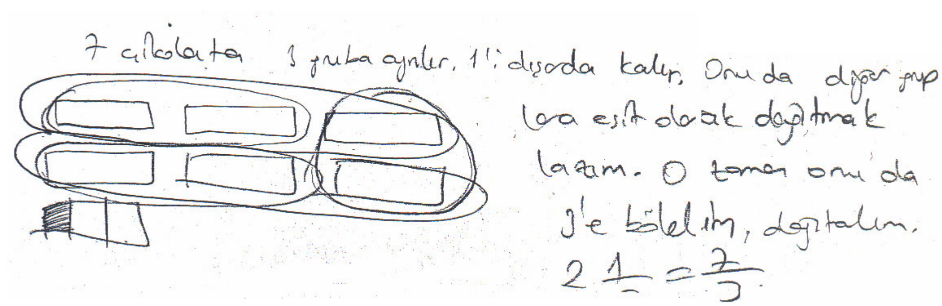



**Figure 4.9.** Partitioning modeling of one-third of seven (participant 16).

Pre-service teachers who used this modeling tried to find the number of chocolates that each person gets where; they shared the seven chocolates among three people. These pre-service teachers used symbolic representations and set models, rectangular area, and pie chart as pictorial representations in order to denote one-third of seven. As stated in Figure 4.9, participant 16 distributed one chocolate to each person and then divided the last piece into three and shared it among the three people again. Then, each person will get $2\frac{1}{3}$.

Analysis of interview and video transcripts of the pre-service teachers who used partitioning modeling revealed the contradiction of representations that pre-service teachers used to define one-third of seven and their definition of division. That is; pre-service teachers own subject matter knowledge on definition of multiplication and division is different from their own representations. These pre-service teachers used the logic of partitive division where; they used the number of elements in each group while carrying out division process but all of the five pre-service teachers defined division as the repeated subtraction or as the number of groups of three in seven. That shows that pre-service teachers used partitive modeling in their representations of part of whole but they used the definition of measurement modeling or repeated addition modeling of multiplication while defining division. For example:

> Participant 13: “Division is a little bit complex. It’s opposite operation of multiplication. For example, four over two means how many twos are there in four. In other words, how many times 2 is equal to 4. Ten over two means how many twos are there in ten (RQ1-P13).”

> Participant 6: “Division is repeated subtraction. For example, four over two means subtracting twos from four one by one and the number of subtraction gives the result. I have some confusion of defining terms since I’m not used to (RQ1-P6).”

Thus, I can say that analysis of the data revealed the consistency of the pre-service teachers’ definition of the division with their representation of the division by measurement modeling; on the other hand, there is a mismatch between the definition of the division and the presentation of this division in usage of partitive modeling.

Among the teachers who used measurement modeling or partitive modeling, one of the pre-service teacher used both measurement and partitioning modeling with discrete model representation to show the one-third of seven chocolates as shown in Figure 4.10.

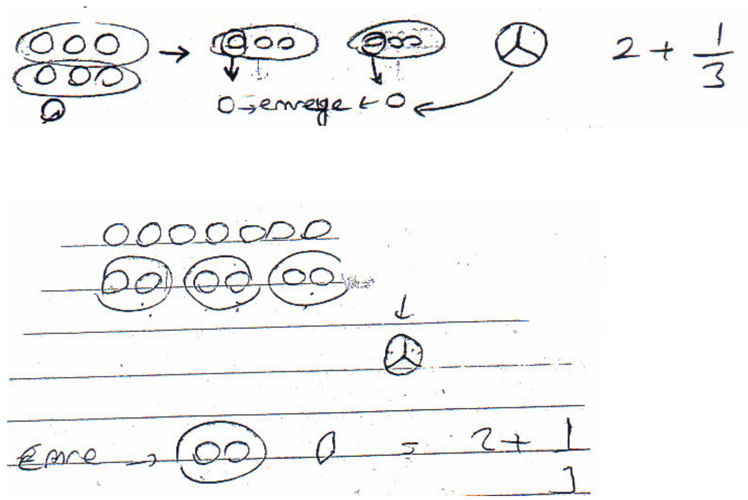


**Figure 4.10.** Partitive and measurement modeling of one-third of seven (participant 3)

Participant 3, who used both partitive and measurement modeling of division, defined the division by using the partitive modeling that is:

> Participant 3: “Division is separating into pieces. Like 6 ÷ 3 means there are 6 apples and you want to share these among three people. How any apples does each person get? They got two. But it should not be a division without remainder. In general in division you separate the large number into groups of smaller number and the number in each group gives the result (RQ1-P3).”

In addition to the measurement and partitioning modeling of division, seven of the pre-service teachers used distributive property (multiplicative compare) of multiplication over addition to where; they used seven times one-third to define the solution of the given problem. For instance; participant 4, divide each chocolates into three and then added them up to show one-third of seven. He draw one-third of seven chocolates and then verbally explained that if he add those one-third pieces they will represent seven times one-third.

Participant 4:

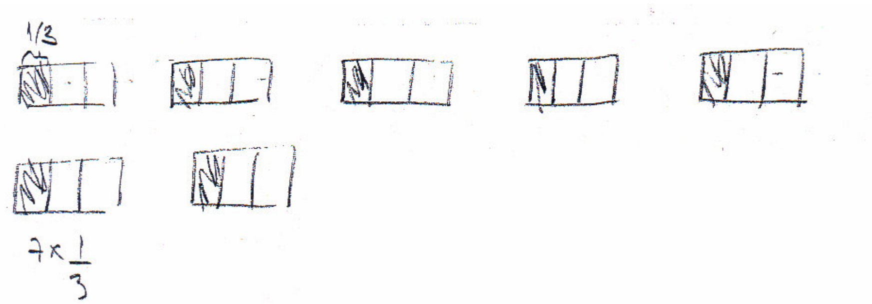



**Figure 4.11.** Distributive modeling of multiplication of one-third of seven (participant 4)

Pre-service teachers who performed distributive modeling used rectangular area and symbolic representations to denote the one-third of seven. From other point of view, the analysis of interview and video transcripts showed that those pre-service teachers who used distributive property define division by using both quotitive model and partitive model definitions. Some of them defined division as repeated subtraction until getting zero and some of them defined it as sharing and finding the amount that each participant gets and define multiplication as repeated addition. On the other hand, among these pre-service teachers two of them agreed that division has more than one meaning and the definition depends on the problem. For instance:

> Participant 1: "Division is like separate whole into pieces. Grouping based on the pieces that we divide. For example, I have ten things and I want to divide this into two. It means I'm going to have two groups and the number in each group gives the result. If I divide into 2 it means there

will be five in each group and $10 \div 2$ is equal to 5… But the meaning changes based on the problem, I can also thing how many twos are there in ten. It depends on question you can use either of them (RQ1-P1).”

Here, it can be summarized that pre-service teachers used measurement modeling, partitive modeling, and multiplicative compare modeling while representing the fractional part of the whole. In addition, within these modeling they used pictorial representations like rectangular area and discrete models to present their understandings. In addition, data revealed the effect of pre-service teachers’ subject matter knowledge on their representations of multiplication and division operations.

#### 4.2.3.2 Representation of the Fraction over Whole Number

In the MDFQ pre-service teachers were asked to represent fraction divided by a whole number. The following question was given to them: “four friends bought $\frac{1}{4}$ kilogram of sweets and shared it equally. How much sweet did each person get?”.

All the seventeen pre-service teachers used partitioning division modeling while dividing the part ($\frac{1}{4}$) by the bigger whole number (4). For instance; participant 14, as shown in Figure 4.12, divided the whole into four and represented it as one-fourth. Then, one-fourth is further divided into four to distribute it among the four people and the amount that each person gets that is one-sixteenth gives the result.

Participant 14:

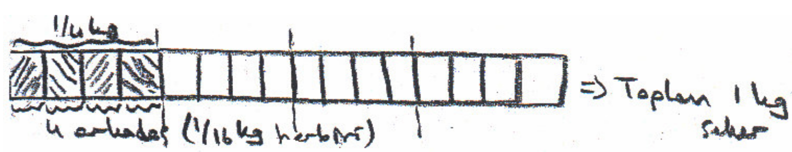



**Figure 4.12.** Partitive division modeling of fraction over whole number (participant 14).

Within the partition modeling, 15 pre-service teachers used rectangular area representation similar to participant 14, and one used word problem to represent the division of one fourth by four. This participants' usage of word problem as real life application is as follows;

Participant 9: "Four friends brought half of the $\frac{1}{2}$ of sugar and shared among them. How much does each of them get? (RQ1-P9)."

In addition to the rectangular area representation and word problem, one pre-service teacher used pie chart as pictorial representation for the fraction over whole number as given in Figure 4.13.

Participant 15:

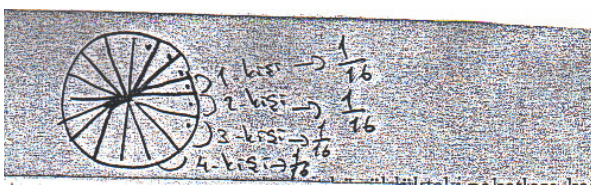



**Figure 4.13.** Pie chart representation of fraction over whole number (participant 15).

During the interviews pre-service teachers were asked whether there is a relationship between division of whole numbers and division of fractions. In this question, fraction over whole number, pre-service teachers divided the whole into four and then they re-divide the one-fourth into four in order to describe the division of fraction by the whole number. Pre-service agreed that if we divided fraction by the whole number it's similar as separating into parts. In other words, it is separating the numerator into the pieces described by the denominator. It's like a sharing (partitive division) and there is no confusion as long as the denominator is whole number. For instance:

Participant 14: “ …we can do the same thing in fractions. For example $\frac{3}{5} \div 2$. What does this mean?. Take the $\frac{3}{5}$ as fraction then directly divide this fraction into two and take the ratio of this piece to the whole (RQ2a-P14).”

On the other hand, pre-service teachers agreed that it is easier to represent if fraction that is larger number is divided by the smaller one. They stated that if the smaller number is divided by the larger one, it’s not easy to explain the meaning and even to represent it. For example:

Participant 17: “ Two over $\frac{1}{2}$ ….that means how many $\frac{1}{2}$s are there in 2. Yes its okey. If we go backwards I mean $\frac{1}{2}$: 2 means how many two’s in $\frac{1}{2}$. Let me draw. Draw $\frac{1}{2}$ then try to find twos in that $\frac{1}{2}$. Ups..it is not the same thing I confuse. Since 2 is larger than $\frac{1}{2}$ it’s difficult..I mean search for smaller number in larger number is easier but it’s not easy to find larger number in smaller one (RQ2a-P17).”

Participant 3: “ In fractions, if we divide larger number by the smaller one there is no problem, it’s like dividing into pieces but the opposite is too hard. I mean, in fractions to divide 2 by $\frac{1}{2}$ is so easy but to divide $\frac{1}{2}$ by 2 is not (RQ2a-P3).”

As we understood from the findings, pre-service teachers used partitive modeling of division while dividing the fraction by whole number. Within the partitive modeling, they used pie chart model, rectangular area model, and word model under the pictorial representation. In addition, the effect of pre-service teachers' subject matter knowledge on basic definitions of multiplication and division operations on their representations of fraction over whole number was obvious. Pre-service teachers reflected their misconceptions on primitive models of multiplication and division operations, where dividend should be larger than divisor, to their representations.

#### 4.2.3.3 Representation of the Whole Number over Fraction

Apart from the division of fraction by larger whole number, pre-service teachers were asked to find a representation of a whole number divided by a fraction. The following question was given to them: "Four kilograms of cheese were packed in packages of $\frac{1}{4}$ kilogram each. How many packages were needed to pack all the cheese?"

Similar to the situation above in this case, all the pre-service teachers used measurement division modeling while dividing the larger whole number (4) by the fractional part ($\frac{1}{4}$). For instance; participant 7 firstly searched for number of one-fourths in one whole and then generalized this to the four wholes.

Participant 7:

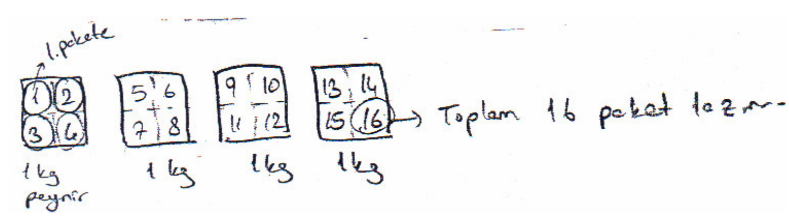



**Figure 4.14.** Measurement modeling of whole number over fraction (participant 7).

Within the measurement modeling, 16 pre-service teachers used rectangular area with symbolic representations to represent the division of four by one-fourth. For instance:

Participant 2:

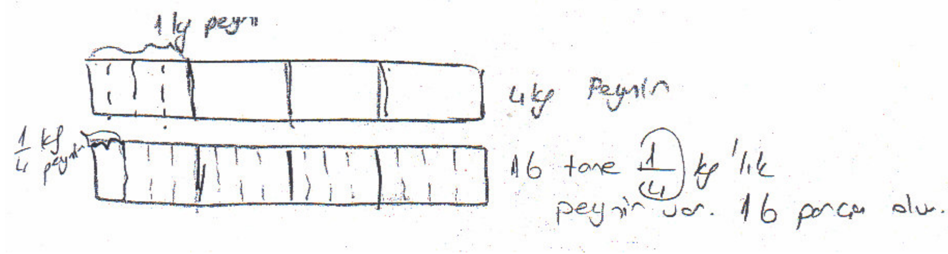



**Figure 4.15.** Rectangular area representation of measurement modeling of whole number over fraction (participant 2).

In addition, only one pre-service teacher used word problem with symbolic representation in order to denote the quotitive measurement. Participant 17, searched for the number of the one-fourths in a given four in order to present his understanding of whole number over fraction representation.

> Participant 17: “First of all, if I distributed the 1 kg cheese into $\frac{1}{4}$ kg’s packages there will be four packages. If we take 4 kg cheese then 4x4 will be 16 packages.” (RQ2a-P17)

Similar to the division of fraction with whole number, interview and video transcripts revealed that pre-service teachers have agreement on division of larger number by smaller one. According to them, if they divide larger number by the smaller one, it’s like finding the number of groups of smaller size in the given larger number. On the other hand, division is not easy to represent even not meaningful if they divide the smaller number with the bigger one as stated by participant seventeen.

> Participant 17: “Um for example, what does $2\div\frac{1}{2}$ mean?. Switch to the whole number. Like $6\div3$, it means how many threes are there in six.

Thus, $2 \div \frac{1}{2}$ means how many $\frac{1}{2}$s are there in 2. …There is no problem if we divide the larger number with the smaller one." (RQ2a-P17-2)

Thus, while presenting the whole number over fraction relationship, pre-service teachers used measurement modeling of division while dividing the whole number by fraction. Within the measurement modeling, they used rectangular area model and word model under the pictorial representation. In addition, similar to the above findings, pre-service teachers' knowledge on primitive models of division effect their representations for the whole number over fraction.

#### 4.2.3.4 Representation of the Fraction over Fraction

Pre-service teachers were given the followings question; "Elif bought a bottle of milk. She gave $\frac{1}{2}$ of it, which was $1\frac{3}{4}$ $\lambda t$ to her grandmother. How much did the bottle of milk originally contain?". In this question, participants used a strategy based on the question: 'if half of the milk is $1\frac{3}{4}$ $\lambda t$ then, what is the total amount?" Thus, participants teacher used multiplication operation ($1\frac{3}{4}$ x 2) or repeated addition ($1\frac{3}{4}+1\frac{3}{4}$) in order to find the original amount of milk that Elif.

In this question, the main purpose was to emphasize the division of two fractions and analyzed pre-service teachers' representations of division of two fractions and that is why the analysis categorized under the name fraction over fraction. But, all participants focus on multiplication operation where they prefer to multiply by two, instead of dividing by $\frac{1}{2}$.

Analysis showed that five of the pre-service teachers used rectangular area model representation with symbolic notation to represent the multiplication of $1\frac{3}{4}$ with two as stated by participant 16.

Participant 16:

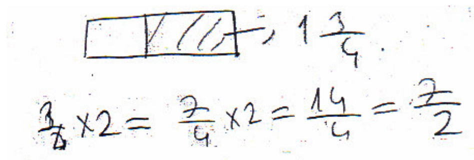



**Figure 4.16.** Rectangular area model representation of fraction over fraction (participant 16).

On the other hand, apart from multiplication by two, eight of the pre-service teachers used rectangular area with symbols to represent the repeated addition ($1\frac{3}{4} + 1\frac{3}{4}$) as stated in Figure 4.17.

Participant 14:

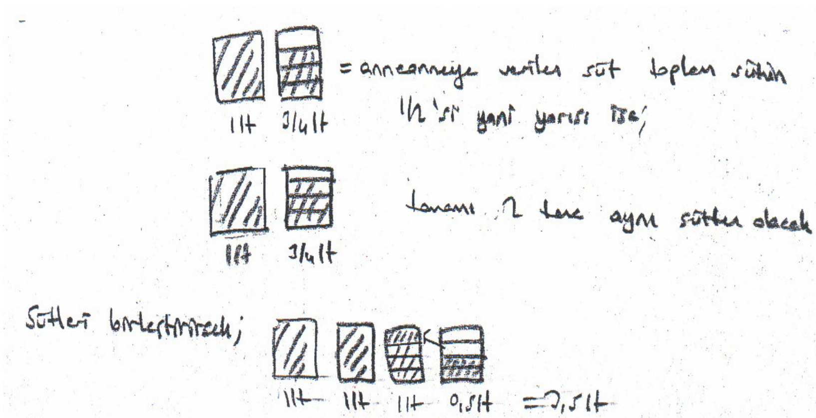



**Figure 4.17.** Repeated addition modeling of the fraction over fraction (participant 14).

Among the pre-service teachers only one used pie chart model to represent the repeated addition. For instance:

Participant 6:

**Figure 4.18.** Pie chart modeling of the repeated addition of fraction over fraction (participant 6).

Apart from the rectangular area model, and pie chart model, three pre-service teachers used figures to represent the repeated addition. For instance;

Participant 10:

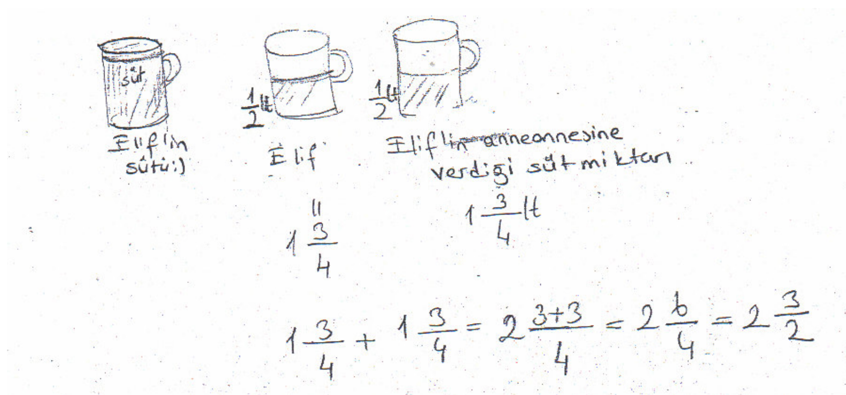



**Figure 4.19.** Figural modeling of the repeated addition of fraction over fraction (participant 10).

Besides multiplicative models for the given questions, none of the pre-service teachers used the logic of partitive and quotitive division models ($1\frac{3}{4} \div \frac{1}{2}$) to find the original amount of milk that Elif has. Interview and video transcripts also revealed the relation that all pre-service teachers move from opposite direction that is; find the double of the given part ($1\frac{3}{4}$) instead of search for the halves in the whole in order to find the original amount. During the interviews, pre-service teachers were asked whether they express the solution by using other symbolization but pre-service teachers insists on using multiplication operation. In line with this, analysis of the subject matter knowledge showed that pre-service teachers had difficulties while explaining and representing the division of two fractions. Apart from the pre-service teachers who mentioned that they could not establish any relationship between division of whole numbers and division of fractions, pre-service teachers generalized the measurement modeling of division of whole numbers to the fractions. They said that division of fractions is same as the division of whole numbers where we are trying to find how many group of second fraction are there in the first one. For instance:

> Participant 13: " Um it means same thing, $4 \div 2$ means how many twos are there in four. We can combine the groups of twos and try to find four. But, for example $\frac{1}{2} \div \frac{1}{4}$ means how many quarters are there in a half. Thus, the logic is same (RQ3-P13)".

On the other hand, results revealed that most of the pre-service teachers who make generalizations on measurement modeling of division both in whole numbers and fractions used examples of fractions where larger fraction is divided by smaller one as stated by participant 13. When they confronted with the opposite situation, in other words, division of smaller fraction with the larger one, they had difficulties in making connections with division of whole numbers. They had problems especially when they try to figure out the division of smaller number with the larger one.

In addition, some pre-service teachers made generalizations from whole numbers to fractions and represent them successfully. For instance:

> Participant 2: " Um now what does it mean $\frac{1}{4} \div \frac{1}{2}$ Umm first I draw $\frac{1}{4}$ and then in the same whole I draw $\frac{1}{2}$ and I said how many $\frac{1}{2}$s are there

in $\frac{1}{4}$? In other words since $\frac{1}{4}$ is half of the $\frac{1}{2}$, there are $\frac{1}{2}$ pieces of $\frac{1}{4}$ in $\frac{1}{2}$. This means I can make same generalizations (RQ3-P2).

Participant 9: " $10 \div \frac{1}{2}$ two means how many $\frac{1}{2}$ s are there in ten. That is how many halve pieces are there in ten pieces. Thus, that means there are 20 halves in the ten pieces. $10 \div \frac{1}{2}$ is 20. Um if we divide fraction by fraction one second, like $\frac{1}{4} \div \frac{1}{2}$ means Um how many $\frac{1}{2}$s are there in $\frac{1}{4}$s. I mean let me draw Um $\frac{1}{2}$ yes indeed the answer is half. There are $\frac{1}{2}$ pieces of $\frac{1}{2}$ in $\frac{1}{4}$. That means I can make generalization to the fractions (RQ3-P9)."

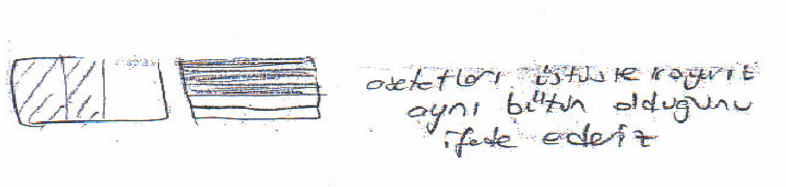
Thus, pre-service teachers used multiplicative models such as repeated addition and multiplicative compare while dividing the fraction by another fraction. Within the multiplicative structures, they used pie chart model, rectangular area model, and figural models under the pictorial representation. In addition, as stated before pre-service teachers' inadequate knowledge on meaning of division of fractions influence them to transfer this knowledge to the problems and to their representations where; they preferred to use multiplication instead of division.

#### 4.2.3.5 Representation of the Fractional Part of the Fraction

Pre-service teachers were also asked to represent the multiplication expression of $\frac{2}{3} \times \frac{3}{5}$ and $1\frac{1}{2} \times \frac{1}{3}$.

Analysis of data revealed that pre-service teachers used two main conceptions while representing the multiplication of two fractions. The first one is the area concept of multiplication where; a square, representing $\frac{2}{3}$ of the given whole is put onto other square representing the fraction $\frac{3}{5}$. Then, intersection area of the two squares represented the multiplication of two fractions. Among the 17 pre-service teachers, six used common area to represent and explain the multiplication of fractions. All of the six teachers used square area representation in presenting the multiplication of fractions as stated in figure 4.20. For instance:

Participant 2 :




:

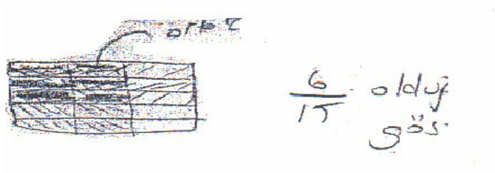



**Figure 4.20.** Intersection area that represents the multiplication of fractions (participant 2).

However, most of these pre-service teachers, who showed that the intersecting area represents the multiplication of fractions, can not explain why this is so. They stated that they just learned that the intersection gives the multiplication and they never thought about the underlying reason. For instance, as stated above participant 2 said that it is a rule that is intersecting area gives the multiplication and she did not question why it is so. On the other hand, two of the pre-service teachers stated the reason why intersection area represents the multiplication of the area. In other words, they mentioned that they can think one side of the rectangle that is represented by the first fraction and the other side the second fraction. Then, the area of the intersecting rectangle gives you the multiplication of two fractions where; each fraction represents the one side of the rectangle.

Apart from the area models of multiplication of fractions, seven teachers used part of the part relationship in their conceptions of multiplication of fractions. Firstly, they showed the part of the whole such as $\frac{2}{3}$ of the whole unit. Then, they accepted the shaded region ($\frac{2}{3}$) as new a whole and they find the part ($\frac{3}{5}$) of this new whole. They stated that fractional part of the fraction, as taking the initial whole as referent point represents the multiplication of two fractions. As stated in Figure 4.21, participant 7 showed the two-thirds of the whole and then he took this whole as the new referent point for finding the three-fifths of this whole. Then, the fractional part of the initial whole gives the result of the multiplication of two fractions.

Participant 7:

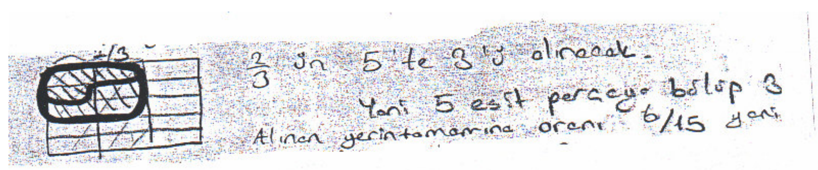



**Figure 4.21.** Area representation of the fractional part of the fraction (participant 7)

Analysis of interview and video transcripts also revealed the fact that pre-service teachers who described the multiplication of fractions as taking the fractional part of the fraction more easily represented and described the logic that they supported. They mention that they divided the whole into parts and then behave this part as a new whole and subdivide it further which gives the part of the part. Thus, compared to the division of two fractions or division of whole numbers to fractions, pre-service teachers give stronger evidence on their assertions on the conception of multiplication of two fractions when they use taking fractional part of the given other fraction. On the other hand, subject matter analysis showed that most of the pre-service teachers accepted the intersection area of two fractions as the multiplication of two fractions without reasoning.

Among the seven of these pre-service teachers who preferred to take the part of the part in order to find fractional part of the fraction, six of them used rectangular area model to represent the multiplication of fractions. Only one pre-service teacher used pie chart in representing $1\frac{1}{2}\times\frac{1}{3}$ as given in Figure 4.22.

Participant 6:

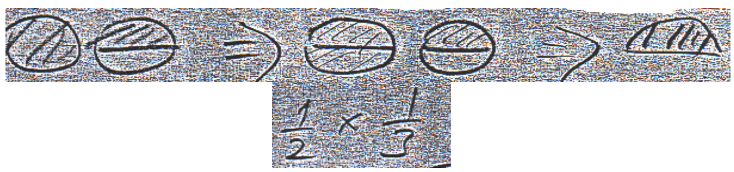



**Figure 4.22.** Pie chart model for the multiplication of two fractions (participant 6).

Apart from those findings, two of the pre-service teachers successfully used both area representation of fractions and the part of the part approaches while presenting the logic of multiplication of fractions as given in Figure 4.23.

Participant 14:

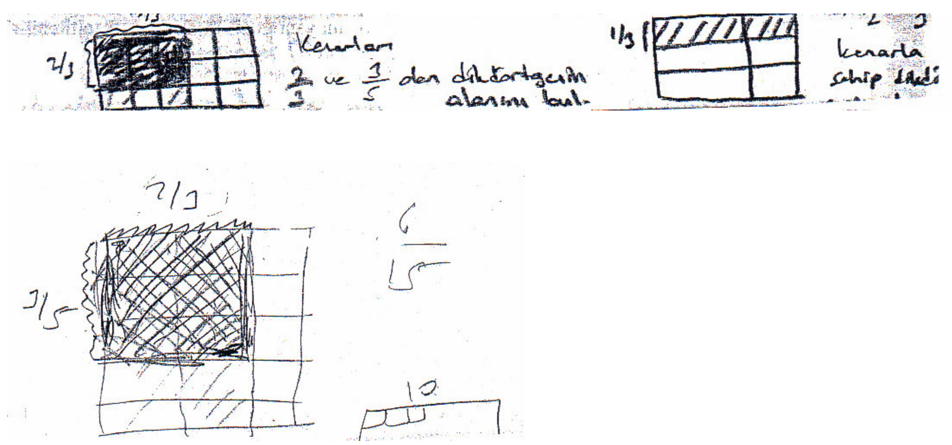



**Figure 4.23.** Fractional part and common area for the multiplication of fractions (participant 14).

Apart from these representations, two pre-service teachers did not use any representations in modeling multiplication of fractions.

In the interview protocol, pre-service teachers further asked to define the multiplication of fractions and if there is any relationship between multiplication of whole numbers and multiplication of fractions. As stated in the subject matter knowledge part, all of the pre-service teachers participated in the study, defined the multiplication operation as repeated addition. On the other hand, they said that there should be a relation between multiplication of whole number and fraction but they have difficulty in describing it. They said that they can make generalization as repeated addition if there are two whole numbers or there are one whole number and one fraction. But, the situation is completely different when they confronted with two fractions. For instance:

> Participant 3: "I think we can generalize the multiplication to the fractions but I can not explain my generalization. Like $\frac{1}{2}$ x $\frac{1}{3}$ is equal to $\frac{1}{6}$. Um like in whole numbers I can not say that I added as much as I multiplied but I think the logic is similar. More simply think about 2 x $\frac{1}{2}$ it means addition of two $\frac{1}{2}$ …..if we have one fraction and one whole number the logic is same as repeated addition. On the other hand, $\frac{1}{2}$ x $\frac{1}{2}$ Um in deed that is half of the half …it's meaningful to take the half of the half but where is the addition then☺….In deed there is no addition (thinking) I do not know how we can add in this situation. I think there is no addition in multiplication of fractions (RQ4-P3)."

On the other hand, some of the pre-service teachers make connection between multiplication of two fractions and division. They stated that multiplication of fraction, that is taking the part of the part, is to divide the part into smaller part. That is why we get smaller number when we multiply two fractions and get bigger number when multiply the two whole number.

> Participant 2: " Um multiplying two fractions is a little bit different. I have never have thought about that. One second…. for example $\frac{1}{2}$ x $\frac{1}{2}$ inn it means taking half of the $\frac{1}{2}$. It's not like addition, it's like division. Think that we are dividing by the second fraction. Then, $\frac{1}{2}$ x $\frac{1}{2}$ means taking the half of the $\frac{1}{2}$ that is $\frac{1}{4}$. Think about $\frac{1}{2}$ x $\frac{3}{4}$ that means taking $\frac{1}{2}$ of the $\frac{3}{4}$ that is dividing $\frac{3}{4}$ into two that is $\frac{3}{8}$…I mean with integers it means repeated addition but here the situation is different there are two fractions so it's like division (RQ4-P2)."

As we understood from the findings, pre-service teachers used multiplicative models such as area concept and multiplicative comparison while multiplying the fraction by another fraction. Within the multiplicative structures, they used pie chart model, and rectangular area model, and figural models under the pictorial representation. In addition, we should stated the influence of the subject matter knowledge of pre-service teachers on their representations of multiplication of two fractions where; most of the pre-service teachers have problems on both defining and representing multiplication of two fractions.

In Table 4.1, summary of the models and representations that pre-service teachers use in their reasoning of multiplication and division expressions were given

**Table 4.1.** Summary of representations of multiplication and division expressions

| Multiplication & Division Expressions | Models | Representations |
|---|---|---|
| Fractional part of the whole number ($7 \times \frac{1}{3}$) | Measurement, Partitive, Multiplicative Compare | Pictorial (rectangular area) representations |
| Fraction over whole number ($\frac{1}{4} \div 4$) | Partitive modeling | Pictorial (rectangular area, pie chart) and real life (word problems) representations |
| Whole number over fraction ($4 \div \frac{1}{4}$) | Measurement modeling | Pictorial (rectangular area), Real life (word problems) representations |
| Fraction over fraction | Repeated Addition Multiplicative Compare | Pictorial (pie chart, figural models, rectangular area) representations |
| Fractional part of fraction | Area model of multiplication Multiplicative Comparison | Pictorial (pie chart, rectangular area) representations |

# CHAPTER V

# CONCLUSION, DISCUSSION, AND IMPLICATION

The purpose of this study was to examine pre-service elementary mathematics teachers' subject matter knowledge and pedagogical content knowledge on multiplication and division of fractions. This chapter addressed conclusion and discussion of the research findings and implications for the further researcher studies. In other words, the important points mentioned in the analysis part reviewed and discussed with references to previous studies in the literature. Recommendations for the mathematics teacher educators and implications for further studies were stated in addition to the limitations of the research study. In addition, based on the findings, a structured model for pre-service teachers' professional development based on multiplication and division of fractions was stated at the end of the chapter.

Conclusion of the research findings were discussed under two main sections based on the research questions. In the first section, the nature of pre-service teachers' subject matter knowledge about multiplication and division of fractions was discussed with references to the previous studies. In the second part, the nature of the pre-service teachers' pedagogical content knowledge was discussed in terms of pre-service teachers' knowledge on students' common conceptions and sources of these common conceptions, pre-service teachers' strategies to overcome students' difficulties and to explain the key concepts, principles and pre-service teachers' representations on multiplication and division of fractions.

## 5.1 The Nature of Pre-service Teachers' Subject Matter Knowledge

Symbolization and finding solution to the given multiplication and division expressions constituted only one part of the subject matter dimension of the research study. For the given expressions, data analysis revealed the fact that all pre-service teachers were correctly symbolized and solved the given problems. As it was expected, pre-service teachers had no difficulty since the given expressions were all related to the basic operations on multiplication and division operations on fractions. In such questions, besides content knowledge on four operations, where pre-service teachers had to think about logical relationships among the multiplication and division and other operations, there was no extra challenge on reasoning operations on fractions conceptually. In other words, pre-service teachers without deep content knowledge on multiplication and division problems could easily solve the given questions. Since pre-service mathematics teachers constituted the subject of the given study, they had enough knowledge on symbolizing and solving the given multiplication and division operations correctly as it was expected from them. Pre-service teachers connected their knowledge on multiplication and division operations with the given questions and did not have any hesitation in choosing the correct operation. For instance, in the third question, participant 2 directly used multiplication in order to find the total amount of milk.

> Participant 2: "Here Elif, gave half of her milk to her grandmother and the amount that she gave was $1\frac{3}{4}\lambda t$. That means if the half of it is $1\frac{3}{4}\lambda t$ then what is the total amount? I should take one more half that means I should add $1\frac{3}{4}$ with $1\frac{3}{4}$ in order to find the total amount or I should multiply $1\frac{3}{4}$ by two. In both cases, I will use multiplication to find the whole (DQ3-P2)."

On the other hand, when pre-service teachers were directly asked what does multiplication or division means or how they relate multiplication and division operations on whole numbers to the multiplication and division on fractions, I could not say that pre-service teachers' knowledge was deep. Results revealed that pre-service teachers had difficulties while defining the multiplication of two fractions (Azim, 1995; Graeber et al., 1989). Pre-service teachers has unique constructions that they had already developed on multiplication of whole numbers. Although pre-service teachers could easily construct the relationship between multiplication of two whole numbers and multiplication of a fraction and a whole number in terms of repeated addition, they could not transfer this knowledge into multiplication of fractions. On the other hand, pre-service teachers could easily solve the given multiplication questions. Thus, results did not support the assumption that claims participants who can multiply whole numbers or fractions have adequate conceptual understanding on fractions (Azim, 1995). This situation is also true for the division of fractions where; pre-service teachers had difficulties while defining the division of two fractions (Ball, 1990; Leinhardt & Smith, 1985; Ma, 1999; McDiarmid & Wilson, 1991; Post, Harel, Behr, & Lesh, 1991; Tirosh, 2000). Compared to the multiplication of fractions where more than half of the pre-service teachers had difficulty in constructing the meaningful explanation and could not connect the relationships of multiplication of whole numbers with multiplication of fractions, most of the pre-service teachers had successfully combined division operation on whole numbers and fractions by adapting the measurement modeling of division of whole numbers to the fractions. Contrary to the research studies where; pre-service elementary teachers preferred to use partitive models to quotitive models while performing operations (Fishbein et al., 1985; Graeber & Tirosh, 1988), pre-service teachers in this study, preferred to use measurement modeling instead of partitive while explaining the division of two fractions. But, pre-service teachers gave specific examples where dividend is larger than divisor while explaining the division of two fractions that shows the effect of primitive model of division (Fischbein et al., 1985; Graeber et al., 1989) on pre-service teachers' knowledge structures. For instance:

> Participant 3: " We are trying to find the number of second fraction in the first one. I mean 2 over $\frac{1}{2}$. Here we are trying to find how many $\frac{1}{2}$ s are

there in 2 and the answer is 4 since there are 4 halves in 2. But the reverse operation is hard. I mean, to divide 2 by $\frac{1}{2}$ is easy but it is not easy to divide $\frac{1}{2}$ by 2. I mean if first one is larger than the second fractions or if we divide integers it is easy to perform and teach those operations but if we divide fractions specifically if we divide smaller fraction with the larger, it is confusing (DI4-P3)."

Thus, results revealed the fact that although pre-service teachers could symbolize and solve multiplication and division problems related to fractions, their reasoning on explaining the meaning of those operations were low. Here, I could easily deduce that pre-service teachers do not put specific emphasis on the conceptual understanding of the operations since they can easily perform the operations without such an understanding. Most of the pre-service teachers had rarely experienced richly connected ideas and concepts in mathematics as learners (Frykholm & Glasson, 2005). Here, pre-service teachers' adequacy in performing the operations on multiplication and division of fractions correctly inhibits their reasoning on the meaning of operations. That is; pre-service teachers' mastery in procedural knowledge did not allow them to reason the operations conceptually. But, in order to use effective strategies that enhance students learning, teachers should first question their own knowledge of concepts (Stoddart et al., 1993). Since teachers need to know what is involved in an appropriate definition of a term or concept and how to represent these concepts (Hill et al., 2004), results challenges the teacher educators on designing the opportunities for the mathematical understanding of pre-service teachers at teacher education courses.

In terms of verbalization of the given multiplication and division expressions, contrary to Graeber and Tirosh (1988) where they stated that pre-service elementary teachers prefer to use partitive model of division, twelve pre-service teachers used quotitive model of division. In other words, pre-service teachers produced real life problems by using quotitive division model when verbalizing the given division operation $(6\frac{2}{3} \div \frac{5}{6})$ which is same as their definition of division of fractions described above. Thus, primitive model of partitive division where the divisor must be a whole number could affect pre-service teachers' verbalization of division expression that lead them to use quotitive model of division (Fischbein et al., 1985; Graeber et al., 1989).

In a given questions, where pre-service teachers were asked to interpret the meaning of division of two fractions, division of numerators and division of denominators for the given division operation and distributive property of division on addition, pre-service teachers' lack of conceptual knowledge was obvious. In other words, although pre-service teachers could decide whether the given statements were correct or not, they could not reason the relationships conceptually.

Pre-service teachers appreciated that the concepts of multiplication and division of fractions were something that consists of set of facts or a bit of procedural knowledge to be mastered. For instance, for the seventh question where pre-service teachers asked to reason whether numerators and denominators could be divided while performing the division operation of two fractions, most of the participants hesitated in giving answer since they have never seen such a relationship before. In addition, they preferred to check the correctness of the given operations just by checking the results that they obtained by inverting the second fraction and multiplying with the first one. Thus, pre-service teachers preferred to perform the given operations as usual manner since they were unable to go beyond describing the steps of algorithm given in the question (Ball, 1990; Borko et al., 1992). The unexpected result of this operation aroused pre-service teachers' attention and curiosity on working with other examples and they expressed great surprise after their unsuccessful attempts to find counterexamples that fail the given relationship (Tirosh, 2000). When asked only three of the pre-service teachers mentioned that the operation is more difficult when there is a fraction in numerator or in denominator or in both. But, realizing this condition was not enough for the pre-service teachers to generalize this rule even to the whole numbers that are multiple of each. As emphasized above, since pre-service teachers could easily solve the given division problems by using “invert divisor and multiply” algorithm (Singmuang, 2002), they do not need any other alternative solution strategy for the given division operation. Thus, to be competent in performing an operation by using one strategy inhibited their further investigations and reasoning on the relationships.

The above situation was also valid for the eighth question where the distributive property of division over addition was investigated. Checking the answer by using the invert and multiply rule increased the curiosity of pre-service teachers on trying the correctness of the relationship on other examples. Thus, apart from five participants, who were really confused since they did not see such an expression before and this relationship contradicted with their previous knowledge that division can not distributed over addition, most of the pre-service teachers accepted the solution. These pre-service teachers, after checking the correctness of the result by invert and multiply rule, tried to reason the answer by relating the properties of multiplication to the properties of division.

Analysis of the results also revealed the fact that pre-service teachers’ subject matter knowledge on proof of basic facts on multiplication and division of fractions was also not strong enough. Although five of the pre-service teachers tried to use transparencies in order to prove the multiplication of two fractions by using area models, their reasoning were not conceptual. Some teachers’ attempt on using symbols in proving a given statements were not complete and their inadequate subject matter knowledge direct them to use specific examples to make generalizations. Results revealed the deep gap in knowledge structures of pre-service teachers on proofs that is to be able to define the operation is not enough to prove the given statement.

In the past two decades, teachers’ knowledge of mathematics has become an object of concern (Hill, Schilling, & Ball, 2004). However, little progress has been made toward a consensus on the question of what teachers need to know (Davis & Simmt, 2006). Subject matter knowledge needed for teaching is not a watered down version of formal mathematics, but a serious and demanding area of mathematical work (Ball & Bass, 2003). Findings of this research study revealed pre-service teachers' inadequacy of giving meaningful explanations for the given multiplication and division of fractions problems. Thus, mentioning about the possible sources of these limited conceptions and suggesting some strategies for the development of subject matter knowledge and pedagogical content knowledge of pre-service teachers believed to increase the coherency of the discussion and give chance to implications for further studies.

Having got higher scores on Students Selection Examination (SSE), or working in Test Preparation Centers (TPS) during their enrollment in teacher education program could be important factors on pre-service teachers' intuitions on their knowledge, where they thought that their subject matter knowledge on multiplication and division of fractions were high. For instance, at TPSs or SSE since the main point is to get competency in solving the given multiple choice questions in a specific time period, no specific attention was given to the understanding of the meaning of concepts. At TPSs, teachers just show the shortest way on solving the questions and obviously this is not the one that contains logical relationship. Thus, being qualified in solving those questions could made pre-service teachers believe in their competencies on solving multiplication and division of fractions. But, solving the given questions procedurally were not an indicator of the subject matter knowledge that is rich in relations. Similar results were also stated by Haser (2006) where, she stated that pre-service teachers mostly attended teacher centered test preparation centers or private tutoring during their enrollment in teacher education program and these informal experiences had a large impact on their beliefs since they lack the opportunities to teaching experiences that they have learnt in method courses.

Thus, at that point attention is turned to teacher education programs where deep knowledge on concepts and relationships need to be introduced. Pre-service teachers' subject matter knowledge can not be assumed to be sufficiently comprehensive and articulated for teaching. Teacher education programs should explicitly consider topics included in elementary or secondary curriculum, such as division of fractions in order to prepare them to be effective teachers (Ball, 1990a; Even & Tirosh, 1995; Singmuang, 2002). Teacher education programs should supply opportunities to exercise in thinking through underlying features and process of mathematical situations. Similarly, ongoing research on teaching, on students' learning, and on the mathematical demands of high-qualified instruction can contribute to increasing precision of understanding of the role of teacher knowledge in teaching (Hill et al., 2005). Thus, specific environments should be created in order to give a chance to pre-service teachers to discuss, interpret and share their ideas to enhance their knowledge structures. Courses should be designed to support the conceptual understanding of the mathematical concepts. For instance, in their study Davis and Simmt (2006) assessed teachers' mathematical knowledge by asking a question 'what is multiplication?' Researchers stated that almost everyone answered as 'repeated addition' and when they encountered a question 'and what else?' teachers were hesitated to answer. On the other hand, when researcher asked teachers to share their responses or explain to others, they observed that teachers' mathematics-for-teaching is much more sophisticated than these sorts of initial responses might suggest (Davis & Simmt, 2006). Thus, offering such courses where teachers can easily communicate, interact, discuss, interpret, and share their ideas could one of the solution strategies to increase subject matter knowledge of pre-service teachers. Davis and Simmt (2006) stated:

> If learners were more explicitly aware of the images and metaphors that are invoked in multiplicative situations, it was agreed, it is much more likely that they will appreciate that the concept of multiplication is something more than a set of facts to be memorized or a bit of procedural knowledge to be mastered. Further, the grouped discussed, having explicit access to such metaphorical underpinnings would likely enable students to better understand why multiplication is useful in such a diversity of context. (p. 302)

Johnston & Ahtee (2006) emphasized that pre-service teachers should consider the whole picture, develop generalized ideas and understand principles before developing pedagogical content knowledge to enable effective learning of pupils. Poor content knowledge could prevent the student teachers from concentrating on students' thinking. My direction now changed to another dimension of the research study where the nature of pedagogical content knowledge of pre-service teachers was discussed.

### 5.2 The Nature of Pre-service Teachers' Pedagogical Content Knowledge

Subject matter knowledge of primary teachers has been an interest of considerable time; however, more recent time there has been a shift to pedagogical knowledge especially to pedagogical content knowledge (Johnston & Ahtee, 2006). In this part, nature of pre-service teachers' pedagogical content knowledge was discussed. In other words, research findings in terms of pre-service teachers' knowledge on students' common conceptions, sources of these conceptions, pre-service teachers' strategies to overcome students' difficulties, pre-service teachers' strategies to reason their understandings, and pre-service teachers' representations on multiplication and division of fractions were discussed.

Research findings revealed that pre-service teachers' knowledge on common conceptions and difficulties that elementary grade level students might have could be grouped under four headings: algorithmically based mistakes, intuitively based mistakes, mistakes based on formal knowledge on fractions, and misunderstanding on problem. Pre-service teachers stated that rote memorization and inadequate knowledge on four operations could be the main sources for the algorithmically based mistakes. On the other hand, students' conceptions on primitive models were stated as the main source for the intuitively based mistakes. Inadequate formal knowledge and limited conceptions on the notion of fractions were identified as two important sources for the mistakes based on formal knowledge on fractions. Lastly, lack of care, lack of mathematical knowledge, lack of mathematical language, and lack of self efficacy were emphasized as the sources for the misunderstanding of the problem.

Similar findings were also stated in the literature where category of algorithmically based mistakes arouses because of rote memorization of the algorithm since students perceived algorithm as a meaningless series of steps and they might forget or change steps which leads them to make errors (Ashlock, 1990; Tirosh, 2000). Researchers stated that most common errors was inverting the dividend instead of divisor or inverting both the dividend and the divisor before multiplying two fractions while dividing two fractions (Ashlock, 1990). In this research study in addition to the above findings, pre-service teachers emphasized the errors like in division operation students can multiply the numerators and denominators, and in multiplication operation, students could multiply the whole part of the mixed fraction with the second fraction, multiply both numerator and denominator in mixed fractions, add both numerators while multiplying mixed fractions, or students could find common factors as in addition operation.

It's worth noting that, pre-service teachers suggested various algorithmic mistakes and they emphasized the importance of conceptual understanding of the operations since those mistakes arose from rote memorization and inadequate knowledge on four operations. As stated in the subject matter knowledge part, pre-service teachers performed all operations correctly but when they faced with questions that need conceptual thinking, they were stuck. All the pre-service teachers mentioned that since they did not have deep conceptual knowledge on multiplication and division operations, they could not explain their reasoning. In parallel to these findings, I can say that pre-service teachers dedicated their own inadequate subject matter knowledge to the students and emphasized that since students do not have enough conceptual understanding of operations, they made algorithmic mistakes. These findings were also true for the intuitively based mistakes which are labeled as other categorization of the students' conceptions suggested by pre-service teachers.

Pre-service teachers stated that intuitively based mistakes were also popular among elementary school students where students could overgeneralize the properties of natural numbers to fractions. Analysis of the data revealed intuitive beliefs on multiplication and division operations as: in a division problem quotient should be whole number where dividend should be completely divided by divisor, divisor must be a whole number, and dividend is always bigger than the divisor (Barash & Klein, 1996; Fischbein, 1987; Graeber et al., 1989; Tirosh, 2000). Pre-service teachers emphasized primitive models of multiplication and division as the sources of the intuitively based mistakes. Pre-service teachers' tendency to attribute intuitive based mistakes to the generalization of the properties of whole numbers to the fractions was also emphasized by Tirosh (2000) which is an evidence of pre-service teachers' awareness on this discrimination among whole numbers and fractions.

Analysis also revealed the effect of pre-service teachers' subject matter knowledge on their knowledge of common (mis)conceptions and difficulties that students held. For instance, as stated, most of the pre-service teachers have confusion on dividing smaller number with larger one which is the indication of the effect of primitive model on pre-service teachers' knowledge structures (Fishbein et al., 1985). Pre-service teachers' hesitations in their descriptions of multiplication and division operations on fractions, their attempt but discourage to generalize the multiplication and division definitions of whole numbers to fractions, make them to dedicate the same difficulties to the students. Thus, I could say that pre-service teachers own limited conceptions on multiplication and division operations on fractions influenced their knowledge on students' common conceptions and sources of these conceptions where; pre-service teachers expected students to have the same intuitions that they had.

In addition to the algorithmically and intuitively based mistakes, pre-service teachers emphasized possible student mistakes based on formal knowledge and suggested that inadequate conceptual knowledge of fractions could be the sources for these mistakes. This was one of the interesting results of the research findings since it revealed pre-service teachers' awareness on the importance of the concept ‘whole’ where pre-service teachers emphasized the importance of clarifying the ‘whole’ while solving multiplication and division operations on fractions.

Lastly, pre-service teachers emphasized the difficulties that students may be confronted while they were reading questions. They mentioned that if students could not understand what was given and being asked s/he could not solved the problem. Students’ lack of care, lack of mathematical knowledge, lack of mathematical language and lack of self-efficacy were stated as the sources for this difficulty. In addition to the inadequacy on formal knowledge of fractions, to suggest some psychological constructs as the sources of the difficulties were also one of the interesting results of the research study. Pre-service teachers’ suggestions to improve students’ self-efficacy, and lower their mathematics anxiety could be attributed to their teaching initiatives in their real classroom experiences.

In addition to the findings above, pre-service teachers emphasized the teacher factor as one of the sources for all mistakes. They stated that if teacher do not have enough competencies on the given subject area then s/he is the major source for the difficulties since teacher is responsible to overcome the difficulties held by the students. In line with these findings, pre-service teachers suggested various strategies that can be used to overcome students’ misconceptions and difficulties on multiplication and division of fractions. They suggested strategies included teaching methodologies such as use of multiple representations (e.g. verbal expressions, figures, and graphics), using different teaching methodologies (e.g. problem solving), emphasizing on drill and practice, and teaching how to focus directly on the questions. In addition to these approaches, pre-service teachers suggested strategies based on the formal knowledge emphasizing meaning of concepts, logical relationship among operations, alternative solutions to problems, and knowledge on misconceptions. Additionally, pre-service teachers suggested some psychological constructs such as increasing self-efficacy of students in order to overcome the difficulties that students have on multiplication and division of fractions.

The analysis of the results could be discussed from two perspectives. Firstly, it is interesting that almost all of the pre-service teachers emphasized the usage of conceptual strategies in order to overcome such difficulties. In other words, pre-service teachers agreed that students could perform algorithmically based mistakes, intuitively based mistakes, and mistakes based on formal knowledge since they do not have deep conceptions on operations and hence operations on fractions. They believed that if students have chance to learn meaning of basic concepts and relationships, they can transfer this knowledge to their experiences and thus could avoid from difficulties stated above. Thus, pre-service teachers emphasized that teacher role is so important here. Teacher should use various strategies in order to make students understand the topic. At this point, the effect of teaching method courses was obvious on pre-service teachers' suggestions on using different strategies since in teaching method courses, they learnt various teaching strategies. Pre-service teachers' suggestions consist of multiple dimensions including strategies for students with different abilities as well. Pre-service teachers suggested visualization through concrete materials, or using examples or models from daily life in order to make students familiar with the concept. They suggested using problem solving strategies to make students understand what was given and asked and by this way they tried to avoid memorization of rules on given operations. Pre-service teachers planned to create learning based classrooms where they make students express their reasoning behind their calculations. They emphasized the importance of misconceptions and they stated that they will also emphasize the difficulties that students confronted while working on problems when they become a teacher. In addition, pre-service teachers emphasized that teachers should focus on students' needs and they should increase confidence and efficacy beliefs of students in addition to cognitive skills. That is, teacher should not only focus on teaching concepts to the students but also take into consideration of students' needs in their classroom practices. Thus, these were all well-defined suggestions that give some implications about pre-service teachers' further teaching practices when they became a teacher. On the other hand, as stated in the analysis part, there is real discrepancy between what pre-service teachers plan to do when they became a teacher and what they did.

In the MDFQ pre-service teachers were asked to state the strategies that they would use to explain/verify the key facts, concepts, principles and proofs on multiplication and division of fractions. That is, how their suggestions of teaching those topics match with their teaching was examined. Similar to the case stated above, almost all of the participants emphasized that in order to teach the concepts effectively, teachers should use manipulatives to visualize the concepts and they should explain the logic of the operations. By this way, students will have no difficulties while performing the operations and meaningful learning will take place. But, results revealed that pre-service teachers do not have conceptual understanding of multiplication and division of fractions that they emphasized teachers should have. For instance, while defining the meaning of multiplication and division of fractions, only four of the pre-service teachers could explain the meaning by using suitable figures. However, when asked, almost all of the pre-service teachers suggested using conceptual strategies, involving using manipulatives and figures, while teaching the meaning of multiplication and division of fractions.

Findings also suggested that as questions become more complicated in MDFQ pre-service teachers' suggestions for using conceptual strategies replaced by procedural ones. When they were asked questions like why we invert and multiply in division operation, or why intersection area (area model of multiplication) represented multiplication of two fractions, or how to prove multiplication and division expressions, more than half of the pre-service teachers were not able to give adequate explanation. Although they insisted on using conceptual strategies to explain the basic concepts, they moved to procedural strategies when they confronted with questions that needs exploring relationships and proof of statements. Only few of the pre-service teachers insisted on using conceptual strategies where their subject matter knowledge and suggested strategies were all in coherence. The rest of the pre-service teachers whom comprised the three-fourths of the participants preferred using numbers, or application of rules for explaining relationships. These pre-service teachers mentioned that they would prefer to use conceptual strategies while introducing the topic that is, they would use materials, manipulatives or other objects in reasoning the relationships by using easy examples that lead them generalizations. Then, when students understand the topic, there would be no more need to explain the topics conceptually and they would use procedural strategies. For instance,

> Participant 16: "This should be clear first. When I said multiplication, I should look for students' understanding for the meaning of this operation. What they understand, how this operation make sense to them. It is similar for the division operation. Then, I will use materials, prepare activities. I mean, I prepare manipulatives and teach the logic of the operations and then after they understand the topic, I will teach the rule for the given expression. I should teach the formula also because they can not spend their time in solving all the questions by using the logic. But as I told, I will teach the formula after teaching the concepts with materials and figures. Now, I have to do a research on teaching multiplication and division of fractions. As you see from my answers, I have not enough competencies in proving the given expressions (STQ7-P16)."

Still being a student in the teacher education program may be one of the factors on the discrimination between pre-service teachers' suggestions what should be done and what they did. Being a teacher candidate that is having completed all the courses that program offers or being able to solve given problems correctly might increase pre-service teachers' confidence in suggesting different methodologies and strategies while teaching these topics. On the other hand, although pre-service teachers took practice teaching courses, where they had a chance to teach specific topics at primary schools, none of the pre-service teachers taught multiplication and division of fractions during these practices. Thus, when pre-service teachers really confronted by real situation, that is, in this study they were directly asked how to teach this topics, they realized that they were not ready to teach the topics conceptually where being competent in mathematics means knowing concepts, symbols, and procedures, and knowing how they are related (Hiebert & Lefevre, 1986). Thus, pre-service teachers' assertion that teacher should prepare materials and teach the topic conceptually does not mean that they will do so. Analysis revealed that pre-service teachers' inadequate subject matter knowledge directly affected their strategies in reasoning their understandings which was highlighted in the literature. It is known that teachers who do not conceptually understand the content are unlikely to teach conceptually (Stoddart et al., 1993).

Based on the findings and discussion above, it could be easily implied that teacher education programs should familiarize pre-service teachers with various, and sometimes erroneous, common types of cognitive process and how they may lead to various ways of thinking (Tirosh, 2000). In addition to pedagogically rich courses where pre-service teachers learn various teaching methodologies, emphasis should be given to the concepts and relationships among those concepts. In such courses, if pre-service teachers have chance to discuss the meaning of concepts, relationships, common conceptions and difficulties that elementary grade level students have, they can easily answer to unexpected questions from students, and can create more learning based classroom environment during their real practices as they suggested.

Pre-service teachers' practices on their representations of multiplication and division of fractions supported the findings described above. In some occasions, pre-service teachers' representations were in line with their definitions but sometimes there were discrepancies between what was being said and what was being represented. For instance, four pre-service teachers used measurement modeling while representing fractional part of the whole number and these four teachers defined division operation as finding the number of groups or repeated addition. On the other hand, five pre-service teachers who defined division as repeated subtraction or group number used partitive model division while representing fractional part of the whole and they were hesitated while defining and representing the division operation. It was interesting that only two pre-service teachers could explain, relate and represent both measurement and partitive modeling of division.

In addition, results revealed the limited conceptions of pre-service teachers on their representations of multiplication and division of fractions. All the pre-service teachers who used partitive, quotitive model of division or distributive property of multiplication drew rectangular area model in their representations of fractional part of the whole. Similarly, when pre-service teachers divided fraction by the whole number, all of them used partitive model of division. On the other hand, when pre-service teachers were asked to represent whole number over fraction, contrary to above situation, all pre-service teachers used measurement modeling. Since, divisor was not a whole number, pre-service teachers preferred to use measurement modeling of division instead of sharing. In addition, apart from one pre-service teacher, all of the pre-service teachers used rectangular area representation. There was no variety in pre-service teachers' modeling, where they directly focused on partitive or measurement modeling without questioning the relationship, and they mostly drew rectangular area representation that they were familiar since elementary school.

In addition, analysis of the modeling on multiplication and division of two fractions revealed the fact that pre-service teachers have limited conceptions on their representations based on their limited conceptions on subject matter knowledge. In case of multiplication of two fractions, six of the pre-service teacher preferred to use intersection of area modeling that they were familiar from their teaching method courses. However, most of the pre-service teachers could not explain why interesting area gives multiplication of two fractions. That is, although pre-service teachers could represent the multiplication of fractions by using models, they could not explain why it is so. On the other hand, seven of the pre-service teachers used multiplicative compare modeling and could meaningfully connect this relation to the multiplication of fractions compared to the area model.

Representations of concepts are corner stones of our mathematics classrooms (Akkuş-Çıkla, 2004; Ball, 1990; McDiarmid, Ball, & Anderson, 1989; Kurt, 2006; Shulman, 1986). But, as stated above, in order to use appropriate representations, teachers should have deep knowledge on the content that they are teaching (McDiarmid et al., 1989). Without having an adequate subject matter knowledge, the development of one's pedagogical content knowledge of a topic cannot be developed sufficiently (Ma, 1996, 1999). Representations are crucial for understanding mathematical concepts (Lesh et al., 1987). Based on findings above, I could easily deduce that pre-service teachers' limited conceptions on conceptual understanding of multiplication and division of fractions limited their representations in reasoning their understandings on these operations. Pre-service teachers preferred to use the models that they were familiar from elementary school and they had difficulty in representing the given operations since they were not use to present the given expression by using different representational models (Kurt, 2006).

Findings of this research study extremely recommend the reconstruction of the courses offered to the pre-service teachers. In order to develop teachers who have rich subject and pedagogical content knowledge, educators should offer courses that familiarize pre-service teachers with concepts, relationships, and multiple conceptual strategies. It is believed that mathematics educators, who seek alternative pedagogical instructions in their mathematics classes, should focus on using multiple-representation based environments (Akkuş-Çıkla, 2004).

In addition to those discussions, there is one more point that should be noted. During the data collection, pre-service teachers spent remarkable time in order to complete questionnaires and interviews. Data analysis revealed that pre-service teachers' duration in the study had great influences on their knowledge structures. Most of the time, they find the correct answer or representation while they were answering the given questions or they find the correct representation by tying and error. Thus, data revealed the evidence that to be involved in the process, spend more time on working with questions or judging ideas by themselves developed knowledge structures of pre-service teachers.

**5.3 Implications for Mathematics Teacher Educators**

In this study, pre-service teachers' nature of subject matter knowledge, pedagogical content knowledge, and relationship between these knowledge structures were investigated within the specific teacher education program context. The analysis of the research data revealed the importance of knowledge structures of pre-service teachers on their professional development processes. That is, pre-service teachers' subject matter knowledge and pedagogical content knowledge play important role in their teaching practices as teachers.

Research findings of the study supported the argument that pre-service teachers should have well-formed or stable subject matter or pedagogical content knowledge structures (Lederman, Gess-Newsome, & Latz, 1994). Thus, findings of this study both support and challenge recent policy initiatives where the efforts to improve teacher education programs in terms of content knowledge and professional development. However, there are contrary arguments like teacher education programs do not prepare pre-service teachers to the classroom realities (Wideen, Mayer-Smith, & Moon, 1998) and research findings continue to suggest that increasing academic coursework in science and mathematics "will not guarantee that teachers have the specific kind of subject matter knowledge needed for teaching" (Floden, 1993, p. 2). Teacher preparation programs expect pre-service teachers to develop both a depth and breath in the content knowledge in mathematics but whether their pieces of knowledge are interconnected in a manner that supports them in translating the knowledge and understanding to learners is unknown as they begin their study of learning to teach their subject (Niess, 2005). Thus, results point to the challenges teacher educators face when designing opportunities for pre-service teachers to extend their mathematical understanding during teacher preparation courses (Crespo & Nicol, 2006).

Deep subject knowledge and stimulating teaching methods were recognized as important aspects of effective teaching (Simon, 2000). Thus, emphasize should be given to the teacher education programs in order to have competent teachers in both their subject areas and pedagogical initiatives. That is, in order to have highly interconnected subject matter structures in pre-service teachers, subject-specific pedagogy courses where pre-service teachers learn how to teach specific subjects must be integrated to the teacher education program (Lederman, Gess-Newsome, & Latz, 1994; Crespo & Nicol, 2006). In other words, mathematics education program should offer content-pedagogy rich courses to the pre-service teachers related to the mathematics and mathematics education in order to have qualified teachers. By saying content-pedagogy rich, I mean the courses where pre-service teachers have chance to develop their competencies on subject area and on content pedagogy as stated above. In those courses, pre-service teachers should have opportunities to share their ideas, communicate with their peers, discuss and interpret concepts and relationships among these concepts, struggle with definitions and how to teach those definitions, acquire knowledge on students' thinking and involved in teaching practices.

Practice teaching courses and method should be lengthened and should include continuous opportunities for pre-service teachers to teach several issues. Pre-service teachers required to take additional coursework so that their knowledge on mathematics would be sufficient to promote their teaching (Frykholm & Glasson, 2005). On the other hand, as stated above, integration of the subject matter courses or subject specific courses is not sufficient. Teacher educators should focus on enhancing the quality of these courses where opportunities should be supplied to the pre-service teachers to develop their competencies both on subject area and on content pedagogy where helping pre-service teachers develop such understanding during teacher preparation is challenge (Crespo & Nicol, 2006). Mathematics teacher educators are constantly faced with the question of how to help pre-service teachers develop a deeper understanding of mathematics while also learning about teaching and learning (Crespo & Nicol, 2006). Thus, an agreement arouse where, teacher education programs should include subject matter course work that examines contents relevant to the students. Such courses should be challenging, require critical exploration and should be accompanied by required course work in curriculum or methods of teaching. Such courses then deliberately develop pedagogical content knowledge or pre-service teachers (Marks, 1990). Ma (1999) proposed that teachers develop a profound understanding of elementary mathematics by doing mathematics themselves, learning from colleagues and examining curricular materials. Similarly, Frykholm and Glasson (2005) suggested “By focusing on PCK, the potential problems inherent in the student teachers’ deficiencies in content knowledge dissipated as they collaborated, shared ideas, and help each other with fundamental concepts and procedures” (p. 138). In addition, good explanations like well-reasoned arguments can be justified by using (a) examples that confirm/contradict given arguments, (b) deductive logic that tests hypothetical, and false premise, or (c) analogy or alternate representation that illustrates or proves a more general case (Weston, 2000).

Analyzing prospective teachers’ knowledge of students’ way of thinking about specific mathematical topics should take into account the effects of subject matter knowledge and prospective teachers’ belief about mathematics, about mathematics instruction, about learning mathematics, and about learning other relevant factors (Tirosh, 2000). In addition to subject matter knowledge and pedagogical content knowledge, the importance of pre-service teachers’ attitudes and beliefs toward mathematics and teaching of mathematics on their professional development were mentioned in the literature (Borko et al, 1992; Borko & Putnam, 1996; Haser, 2006; Shulman, 1986). That is; a well qualified teacher should have deep subject matter and pedagogical content knowledge where; he well knows the concepts, the relationships among concepts and how to present these concepts to the students effectively. In addition to those knowledge structures, pre-service teachers should have positive attitudes and beliefs toward mathematics and teaching profession in order to have positive effect on their students learning. Similarly, as stated in the analysis part, interview transcripts revealed the fact those pre-service teachers’ attitudes and beliefs toward mathematics and teaching of mathematics effect their reasoning of basic relationships, their knowledge on students' conceptions and sources of these conceptions, and the way of representing of the given subject although it was not the main concern of this study. Thus, in an attempt to suggesting a framework on professional development of pre-service in a specified context, pre-service teachers' attitudes and beliefs should also be emphasized in addition to their knowledge structures. That is, teacher education programs should concentrate on forming positive attitudes, and focusing on subject matter knowledge and pedagogical content knowledge in a meaningful way. Thus, teacher educators should take into account pre-service teachers' attitudes, and beliefs while developing content and pedagogy rich courses (Haser, 2006).

In addition, as stated by Johnston and Ahtee (2006), teacher education programs should particularly concentrate on helping student teachers to teach difficult concepts in a meaningful way, considering primary students' misconceptions and suggesting strategies for changing them. In addition, in order to do this effectively, student teachers should have good subject matter knowledge where teachers with stringer mathematics background were most successful in helping student to understand the mathematics they were studying (Mewborn, 2000). This study revealed that although multiplication and division of fractions has already been studied by the pre-service teachers during their own school years, their subject matter knowledge on multiplication and division of fractions were not deep enough to reason their understanding and judge the relationships. Pre-service teachers' understanding of concepts on multiplication and division of fractions were founded more on rule-based and flawed reasoning than on well-reasoned mathematical explanations and that they lacked the experience and inclination to understand or appreciate different ideas and approaches to this topic (Crespo & Nicol, 2006). As an evidence of this study, teacher education programs did not address the most fundamental concepts- multiplication and division of fractions, for instance- where little attention was given to the significance of each basic idea where pre-service teachers might never have been given an opportunity to see the connections among the concepts. Thus, environment can be supplied to the pre-service teachers where they can extend their ideas by gaining insight into appropriate and multiple kinds of explanations. Courses can be modified in order to consider the need for multiple representations in mathematics in order to increase pre-service teachers' knowledge on multiple representations (Akkuş-Çıkla, 2004). Pre-service teachers should articulate profound understanding of not just mathematical concepts, but the way in which mathematical concepts were developed and taught (Davis & Simmt, 2006). By this way pre-service teachers assumed to have more conceptually based understanding of the topic than rule based.

In addition, faculties of education and in the academic disciplines would be well advised to soften the traditional distinction between content and methods. Many mathematics professors asserts that their course is a mathematics course not an education course, whereas many education professors maintain that their methods of teaching course is not intended to teach mathematics. Mathematics course should focus on content, but in order to achieve its objective, it must incorporate aspects of curriculum and method. Similarly, students in mathematics method course must wrestle with the mathematical ideas when appropriate (Marks, 1990). The combined courses must be taught by professors who were competent on both subject matter and pedagogical knowledge. Thus, by this way, pre-service teachers may learn to plan and implement instructional strategies so that their future students will achieve mathematical understanding of concepts (Johnston & Ahtee, 2006).

Parallel with the implications for the mathematics teacher educators, findings of this research study open doors for the investigations of the new topics in mathematics education. Recommendations for the further research studies and limitations of the given study were mentioned in the following section.

### 5.4 Recommendations for Further Research Studies

This research study focused on the subject matter knowledge and pedagogical content knowledge of pre-service teachers on multiplication and division of fractions. As stated above, findings believed to suggest valuable implications for the mathematics educators. Based on the analysis of the data, several suggestions for related research studies were identified.

This study provides support for policy initiatives designed to improve teachers' mathematical and pedagogical knowledge. Although this was a case study focusing on only one teacher education program, analysis provided a troublesome picture of pre-service mathematics education in Turkey. Thus, further researches should be carried out in mathematics education to identify the components of subject matter and pedagogical content knowledge of pre-service teachers and how these knowledge structures developed at various stages of the teacher education program. In addition, during the enrollment to the program, what teachers learn from such professional development should be evaluated to improve teaching. Thus, there should be continuous evaluation of pre-service teachers' knowledge structures and beliefs throughout their involvement in teacher education program to monitor their professional development.

A well-formed pedagogy knowledge structure should not be expected without actual experience with real students. Thus, it may necessary to provide increased opportunities for pre-service teachers to conduct systematic classroom observations and practices (Good & Brophy, 1991). A research study could be conducted where pre-service teachers can be monitored during their practice teaching and the effectiveness of teaching method courses on pre-service teachers' teaching experiences could be examined. That is; research study could be carried out to investigate the effectiveness of courses in teacher education program on pre-service teachers' subject matter knowledge and pedagogical content knowledge during their teaching practices with elementary grade level students.

Findings also suggested need for further studies on possible effects of multiple investigative experiences on the development of pre-service teachers' mathematical understanding and attitudes (Crespo & Nicol, 2006). Beliefs about the nature of mathematics may be tied up with subject matter knowledge in the way in which teachers approach mathematical situations. That is, in addition to knowledge structures of pre-service teachers, relationship of this knowledge with other domains including attitudes and beliefs can also be investigated.

Effectiveness of instruction cannot be assessed without students' learning being measured (Hill et al., 2005). Thus, further studies need to be done to explore how subject matter knowledge and pedagogical content knowledge of in-service teachers affect students' learning in various topics and subject areas. Similarly, pre-service teachers and in-service teachers' knowledge structures can be examined from various perspectives in order to give insights to the teacher educators. The continuum of developmental process of knowledge structures of pre-service and in-service teachers' beginning from early stages of teacher education program and during their classroom practices could be examined and the effect of experience or other related factors on these constructs could be examined. In addition, further investigation should be done on pre-service and in-service teachers' disposition of their mathematical ideas and how these ideas are related to their teaching of mathematics. Pre-service and in-service teachers with such disposition may be well suited to teaching mathematics in ways that promote students' understanding.

Since this is a qualitative case study, further quantitative research studies could be performed on evaluating the knowledge structures of pre-service and in-service teachers on various topics in mathematics involving multiplication and division of fractions. With these quantitative studies, researchers could also have a chance to generalize the findings of the research study to the broader context having similar characteristics.

## 5.5 Limitations of the Study

Study is limited by three major factors: representativeness of the pre-service teachers, researcher position, and content areas selected to be studied. The main limitation of the study was the representativeness. As stated in the methodology part, senior pre-service were asked to participate in semi-structured interviews after they completed the given questionnaire. Among twenty eight pre-service teachers, seventeen of them participated in second part of the data collection. Thus, the participants of the study did not constitute all of the pre-service teachers enrolled in mathematics education program.

The other limitation of the study is the position of the researcher. Issues related to the researcher role and bias was discussed in the methodology part. In addition, this study mainly focused on the subject matter knowledge and pedagogical content knowledge of pre-service teachers on multiplication and division of fractions. Thus, the other related constructs involving attitudes and beliefs were not the concern of this study. In addition, I can not generalize the nature of pre-service teachers' subject matter knowledge and pedagogical content knowledge to other topics in mathematics and other disciplines since the main issue was multiplication and division of fractions.

Lastly, data is limited to questionnaire and interviews where results were not confirmed with their practices. Further research studies related with practice were suggested in the recommendation part.

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# APPENDIX A

## Objectives for SMK and PCK

Objectives related to pre-service teachers' subject matter knowledge and pedagogical content knowledge on multiplication and division of fractions.

***Subject Matter Knowledge***

Pre-service teachers will be able to:

1. Write a symbolic expression for the given word problems involving multiplication of fractions.
2. Write a symbolic expression for the given word problems involving division of fractions.
3. Write a symbolic expression in terms of fraction multiplication that is equal to the given division of fraction expression.
4. Write a symbolic expression in terms of fraction division that is equal to the given division of fraction expression.
5. Write a word problem corresponding to the given symbolic expression involving multiplication of fractions.
6. Write a word problem (including partitive or measurement models) corresponding to the given symbolic expression involving division of fractions.
7. Find the solution to the given problems involving multiplication of fractions.
8. Find the solution to the given problems involving division of fractions.

9. Generate representations (figures, pictures, etc. ) to solve (explain) the given multiplication of fraction expression/word problem for the students.

10. Generate representations (figures, pictures, etc. ) to solve (explain) the given division of fraction expression/word problem for the students.

11. Construct proof for the rule "multiply numerators and denominators" on multiplication of rational numbers.

12. Construct proof for the rule "invert and multiply" on division of rational numbers.

***Pedagogical Content Knowledge***

Pre-service teachers will be able to:

1. Identify students' preconceptions, misconceptions, and difficulties in multiplication of fractions.

2. Identify students' preconceptions, misconceptions, and difficulties in division of fractions.

3. Describe possible sources of students' preconceptions, misconceptions, and difficulties in multiplication of fractions.

4. Describe possible sources of students' preconceptions, misconceptions, and difficulties in division of fractions.

5. Generate strategies to overcome students' difficulties/misconceptions on multiplication of fractions.

6. Generate strategies to overcome the students' difficulties/misconceptions on division of fractions.

7. Generate strategies for explaining the given problems/expressions/proofs involving multiplication of fraction.

8. Generate strategies for explaining the given problems/expressions/proofs involving division of fraction.

## APPENDIX B

### Turkish Version of Multiplication and Division of Fractions Questionnaire (MDFQ)

Aşağıdaki soruların tümüne cevap veriniz.

1. Mert'in 7 adet çikolatası vardır. Mert çikolatalarının üçte birini arkadaşı Emre'ye vermeye karar verdiğine göre, Emre ne kadar çikolata alacaktır?

a. Probleme uygun matematiksel ifadeyi sembol kullanarak yazınız.

b. Problemin çözümünü bulunuz.

c. İlköğretim 6. veya 7. sınıf öğrencisinin (a) ve (b) şıklarını cevaplarken yapabileceği iki matematiksel hatayı yazınız.

(i) (ii)

d. (c) kısmındaki hataların her birinin kaynağı ne olabilir? Açıklayınız.

(i) (ii)

e. Bu matematiksel hatalar/kavram yanılgıları sizce nasıl düzeltilebilir?

(i) (ii)

f. İşleminizi bir gösterim/model kullanarak yapınız.

2. Aşağıdaki her bir problem için (a) Probleme uygun matematiksel ifadeyi sembol kullanarak yazınız (sonucu hesaplamayınız) (b) İlköğretim 6. veya 7. sınıf öğrencisinin problemi cevaplarken yapabileceği iki matematiksel hatayı yazınız (c) Bu matematiksel hataların herbirinin kaynağı ne olabilir? Açıklayınız. (d) Bu hatalar/kavram yanılgıları sizce nasıl düzeltilebilir? (e) İşleminizi bir gösterim/model kullanarak yapınız.

(i) Dört arkadaş, $\frac{1}{4}$ kg şeker alıp kendi aralarında eşit olarak paylaşmışlardır. Her biri kaç kg şeker almıştır?

a)
b) (i) (ii)
c) (i) (ii)
d) (i) (ii)
e)

(ii) Dört kilogram peynir, eşit büyüklükteki paketlere konacaktır. Her paket $\frac{1}{4}$ kg peynir aldığına göre, peynirin tamamını yerleştirmek için kaç paket kullanmak gerekir?

a)
b) (i) (ii)
c) (i) (ii)
d) (i) (ii)
e)

3. Elif, aldığı sütün $\frac{1}{2}$sini anneannesine vermiştir. Anneannesine verdiği süt $1\frac{3}{4}$ litre olduğuna göre, Elif ilk başta kaç litre süt almıştır?

a. Probleme uygun matematiksel ifadeyi sembol kullanarak yazınız.

b. Problemin çözümünü bulunuz.

c. İlköğretim 6. veya 7. sınıf öğrencisinin (a) ve (b) şıklarını cevaplarken yapabileceği iki matematiksel hatayı yazınız.

(i)

(ii)

d. (c) kısmındaki hataların her birinin kaynağı ne olabilir? Açıklayınız.

(i)

(ii)

e. Bu matematiksel hatalar/kavram yanılgıları sizce nasıl düzeltilebilir?

(i)

(ii)

f. İşleminizi bir gösterim/model kullanarak yapınız.

4. Aşağıda verilen kesirlerde çarpma işlemlerini düşünerek, herbir ifade için ilgili soruları cevaplandırınız.

I. $\frac{2}{3}\times\frac{3}{5}$ II. $1\frac{1}{2}\times\frac{1}{3}$,

a. İlköğretim 6. veya 7. sınıf öğrencisinin (I) ve (II) ifadelerini çözerken yapabileceği iki matematiksel hatayı yazınız.

(i) (ii)

b. Bu matematiksel hataların her birinin kaynağı ne olabilir? Açıklayınız.

(i) (ii)

c. Bu hatalar/kavram yanılgıları sizce nasıl düzeltilebilir?

(i) (ii)

d. İşleminizi bir gösterim/model kullanarak yapınız.

5. (i) Aşağıdaki her bir matematiksel ifadeyi anlatan bir hikaye veya günlük yaşam problemi yazınız.

a. $11\frac{1}{2} \times \frac{1}{4}$ b. $6\frac{2}{3} \div \frac{5}{6}$

a) b)

(ii) Öğrencilerinize bu ifadelerin çözümünü açıklarken hangi stratejileri kullanırsınız?

a) b)

6. $\frac{2}{3} \div \frac{1}{2}$ ifadesini, Ceren "$\frac{2}{3} \div \frac{1}{2} = \frac{3}{2} \times \frac{1}{2}$" şeklinde, Cenk "$\frac{2}{3} \div \frac{1}{2} = \frac{3}{2} \times \frac{2}{1}$" şeklinde, Eda ise "$\frac{2}{3} \div \frac{1}{2} = \frac{2}{1} \times \frac{2}{3}$" şeklinde ifade etmiştir.

a. Hangi öğrencinin/öğrencilerin yazdığı kabul edilebilir? Cevabınızı açıklayınız.

Her öğrenci ne düşünüyor olmalı?

Ceren………… Cenk…………. Eda……………

b. $\frac{2}{3} \div \frac{1}{2} = \frac{2}{3} \times \frac{2}{1}$ eşitliğini öğrencilerinize nasil açıklarsınız?

7. Berk kesirlerde bölme işleminin de kesirlerde çarpma işlemi gibi yapılabileceğini iddia ediyor. Berk'e göre, $\frac{2}{9} \div \frac{1}{3} = \frac{2 \div 1}{9 \div 3} = \frac{2}{3}$.

a. Berk'in açıklamasını kabul ediyor musunuz? Nedenleriyle açıklayınız?

b. İşlemle ilgili Berk'e nasıl bir açıklama yaparsınız?

8. Tuğçe dağılma özelliğinin, bölme işlemi yaparken de kullanılabileceğini iddia ediyor. Tuğçe'ye göre: $1\frac{3}{4} \div \frac{1}{2} = (1+\frac{3}{4}) \div \frac{1}{2}$

$$= (1 \div \frac{1}{2}) + (\frac{3}{4} \div \frac{1}{2})$$
$$= 2 + 1\frac{1}{2}$$
$$= 3\frac{1}{2}$$

a. Tuğçe'nin açıklamasını kabul ediyor musunuz? Nedenleriyle açıklayınız?

b. İşlemle ilgili Tuğçe'ye nasıl bir açıklama yaparsınız?

9. Aşağıdaki kesirlerde çarpma ve bölme işlemleriyle ilgili eşitliklerin nedenlerini öğrencilerinize nasıl açıklarsınız?

a. $\frac{3}{4} \times \frac{4}{3} = 1$ b. $1 \div \frac{2}{3} = \frac{3}{2}$ c. $\frac{2}{3} \div \frac{1}{3} = 2$?

a.

b.

c.

10. a (i). Kesirlerde çapma işlemi yaparken, “1. kesrin payı 2. kesrin payı ile, 1. kesrin paydası ise 2. kesrin paydası ile çarpılır” ifadesinin doğruluğunu nasıl ispatlarsınız?

( $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$, a, b, c, d $\in Z$ ve b, d $\neq 0$ )

(ii) İspatınızı öğrencilerinize hangi stratejileri kullanarak açıklarsınız?

10. b (i) Kesirlerde bölme işlemi yapılırken, “ İkinci kesir ters çevirilip çarpılır” ifadesini nasıl ispatlarsınız? ( $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$, a, b, c, d $\in Z$ ve b, c, d $\neq 0$ )

(ii) İspatınızı öğrencilerinize hangi stratejileri kullanarak açıklarsınız?

## APPENDIX C

### Table of Specification

A table of specifications for Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK) on multiplication and division of fractions are presented in the following table. Subject matter knowledge is categorized as knowledge on basic operations, verbalizing expressions, basic facts, principles and proofs on multiplication and division of fractions. On the other hand, Pedagogical Content Knowledge is categorized as knowledge of students' conceptions, representations, and strategies.

| Obj. | Item # | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | SMK<br>basic ope. | SMK<br>Verb. Expr. | SMK<br>Basic facts | SMK<br>Prin. | SMK<br>proofs | PCK<br>KSC | PCK<br>Rep. | PCK<br>Str. |
| 1 | 1a | | | | | 1c, 6a, 4(i,ii)a | | |
| 2 | 2(i,ii)a, 3a | | | | | 2(i,ii)b, 3c, 6a | | |
| 3 | | | 6a | | | 1d, 4(i,ii)b | | |
| 4 | | | | 7a, 8a | | 2(i,ii)c, 3d | | |
| 5 | | 5(i)a | | | | | | 1e, 4(i,ii)c |
| 6 | | 5(i)b | | | | | | 2(i,ii)d, 3e |
| 7 | 1b | | | | | | 1f, 4(i,ii)d | |
| 8 | 3b | | | | | | 2(i,ii)e, 3f | |
| 9 | | | | | 10(i)a | | | 5(ii)a, 6b, 9a, 10(i)b |
| 10 | | | | | 10(ii)a | | | 5(ii)b, 6b, 7b, 8b, 9b, 9c, 10(ii)b |

# APPENDIX D

## Turkish Version of Interview Questions

### Birinci Bölüm (Genel Bilgiler)

1. Bugüne kadar almış olduğun eğitimden biraz bahseder misin?

- Hangi tür Ortaokul/Liseden mezun oldun?
- Şimdiye kadar universitede almış olduğun matematik/matematik eğitimi dersleri nelerdir?

2. Şimdiye kadar herhangi bir öğretmenlik tecrüben oldu mu? Nerde? Ne zaman? Ne kadar süre? Hangi sınıf seviyesi?

- Özel ders, dershane, eğitim topluluğu, vakıflar...

3. Iyi bir matematik öğretimi sence nasıl olmalı?

- Iyi bir matematik öğretmeni hangi özelliklere sahip olmalı?
- Ileride iyi bir öğretmen olabilecegini düşünüyormusun? neden?

4. Iyi bir matematik öğretmeni hangi bilgi ve becerilere sahip olmalı?

- Konu/alan bilgisi, öğretmenlik deneyimi, pedagojik bilgi, geri bildirim verme yönünden?

5. Konu/alan bilgisinin matematik öğretimindeki rolü hakkında ne düşünüyorsun?
- sence bir kişinin konu/alan bilgisi nasıl gelişir?

6. Pedagojik bilginin (Pedagojik alan bilgisinin) matematik öğretimindeki rolü hakkında ne düşünüyorsun?

- sence bir kişinin pedagojik içerik bilgisi nasıl gelisir?

7. Sence bu bilgilerden (konu/alan bilgisi, pedagojik alan bilgisi) biri diğerinden daha önemli olabilir mi? neden?

8. Sence Ilköğretim matematik öğretmenliği programından kazanmış olduğun bilgi (konu/alan bilgisi, pedagojik içerik bilgisi) ve beceriler senin ilerideki öğretmenlik hayatın için yeterlimi?

A. Program hangi yönlerden sence yeterli/yeterli degil?

B. Bir öğretmen kendi bilgi ve becerisini (konu/alan bilgisi, pedagojik içerik bilgisi) nasıl geliştirmeli?

**İkinci bölüm (Kesirlerin bölme ve çarpmasına ilişkin genel bilgiler)**

1. Çarpma işlemini nasıl tanımlarsın?

2. Bölme işlemini nasıl tanımlarsın?

3. Sence doğal sayılardala çarpma işlemi ile kesirlerde çarpma işlemi ile ilişkilendirilebilirmi? Hangi yönlerden? Ornek?

4. Sence doğal sayılardala bölme işlemi ile kesirlerde bölme işlemi ile ilişkilendirilebilirmi? Hangi yönlerden? Ornek?

5. Ilkokuldayken/ortaokuldayken kesirlerde çarpma ve bölme işlemini nasıl öğrenmiştin?

- Öğretmenin hangi method ve stratejileri kullanmıştı?

6. Sence kesirlerde çarpma/bölme işlemini öğretmek kolay mı/ zor mu? Neden?

7. Öğrencilerinin kesirlerde çarpma/bölme ile ilgili matematiksel anlamalarını nasıl geliştirmeyi düşünüyorsun?

- Kesirlerde çarpma/bölme işlemini öğrencilerine daha anlamlı nasıl yapabilirsin?

8. Konu bitiminde (kesirlerde çarpma/bölme) öğrencilerinizin hangi bilgi ve becerilere sahip olmasını istersin?

9. Öğrencilerinin bu bilgi ve becerilere sahip olup olmadığından nasıl emin olabilirisin?

10. Sence sahip olduğun bilgi (konu/alan bilgisi, pedagojik içerik bilgisi) beceri kesirlerde çarpma ve bölme işlemini öğretmek için yeterli mi?

- Hangi yönlerden yeterli hangi yönlerden degil?
- Bir öğretmen kesirlerde çarpma ve bölme işlemi ile ilgili bilgi ve becerilerini nasıl geliştirebilir?

**Uçüncü bölüm**

Konu Alan Bilgisi ve Pedagojik Içerik Bilgisi Testi sorularının üzerinden tekrar geçildi. Teste anlaşılmayan sorular tekrar öğrencilere soruldu.

# CURRICULUM VITAE

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## EDUCATION

| | |
|---|---|
| **2002 Fall – 2006 Spring** | Middle East Technical University<br>Ankara, TURKEY<br>Ph.D Student, Dept. of Secondary Science and Mathematics Education<br>(Passed Ph.D Qualification Examination in June 2004) |
| **2004 Fall- 2005 Spring** | Research Scholar<br>The University of Georgia<br>Mathematics Education, Aderhold Hall<br>Athens, GA |
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Thesis Title: "The Effect of Spreadsheet and Dynamic Geometry Software on the Mathematics Achievement and Mathematics Self-Efficacy of 7th Grade students"

| | |
|---|---|
| **1996 Fall- 2000 Spring** | Middle East Technical University<br>Ankara, TURKEY<br>B. S., Mathematics Education |

## WORKING EXPERIENCE

| | |
|---|---|
| **2002 Fall- Present** | Research Assistant<br>Middle East Technical University<br>Faculty of Education<br>Department of Elementary Education<br>(In division of Mathematics Education) |
| **2003 Fall- 2004 Spring** | Content developer in the international project “Integrated Curriculum for Secondary Schools”, sponsored by Ministry of Education Malaysia |
| **2001 Spring- 2002 Spring** | Mathematics teacher<br>Bilkent Private School in Ankara |
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## PUBLICATION

Işıksal, M. & Aşkar, P. (2003). Elektronik Tablolama ve Dinamik Geometri Yazılımını Kullanarak Çalışma Yapraklarını Geliştirilmesi. *İlköğretim-Online*. (http://ilkogretim-online.org.tr).

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