

PAIRWISE MULTIPLE COMPARISONS UNDER SHORT-TAILED
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ABSTRACT

PAIRWISE MULTIPLE COMPARISONS UNDER SHORT-TAILED SYMMETRIC DISTRIBUTION

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In this thesis, pairwise multiple comparisons and multiple comparisons with a control are studied when the observations have short-tailed symmetric distributions.

Under non-normality, the testing procedure is given and Huber estimators, trimmed mean with winsorized standard deviation, modified maximum likelihood estimators and ordinary sample mean and sample variance used in this procedure are reviewed.

Finally, robustness properties of the stated estimators are compared with each other and it is shown that the test based on the modified maximum likelihood estimators has better robustness properties under short-tailed symmetric distribution.

Key words: Pairwise multiple comparison, short-tailed symmetric distribution, robust estimators, modified maximum likelihood estimation.

ÖZ

KISA KUYRUKLU SİMETRİK DAĞILIM ALTINDA İKİLİ ÇOKLU KARŞILAŞTIRMALAR

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Bu tezde, gözlemler kısa kuyruklu simetrik dağılıma sahip olduğunda ikili çoklu karşılaştırmalar ve kontrol grubuyla çoklu karşılaştırmalar çalışılmıştır.

Normal olmayan dağılımlar altında test prosedürü verilmiş ve bu prosedürde kullanılan Huber tahmin edicileri, winsorized standart sapma ile budanmış ortalama, uyarlanmış en çok olabilirlik tahmin edicileri ve örneklem ortalaması ve varyansı sunulmuştur.

Son olarak, kısa kuyruklu simetrik dağılım altında belirtilen tahmin edicilerin sağlamlık özellikleri karşılaştırılmış ve uyarlanmış en çok olabilirlik tahmin edicisine dayanan testin daha iyi sağlamlık özelliklerine sahip olduğu gösterilmiştir.

Anahtar Kelimeler: İkili çoklu karşılaştırmalar, kısa kuyruklu simetrik dağılım, sağlam tahmin ediciler, uyarlanmış en çok olabilirlik tahmini.

To My Parents

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CHAPTER 1

INTRODUCTION

The problem of multiple comparisons is encountered in food and drug industries and related administrative and research establishments. The aim of the multiple comparisons is to compare the average effects of the treatment means $\mu_1, \mu_2, \dots, \mu_k$ to determine which treatments are preferable. In comparing more than two treatment means, an ANOVA F-test shows whether the treatment means are significantly different from each other or not, but it does not show which pair of means differ. In this case, pairwise multiple comparison procedures are used.

A control treatment is required when the general effectiveness of the treatments under study is not known, or when the general effectiveness of the treatments is known but is not consistent under all conditions (Neter, Wasserman, Kutner, 1985). In this case, multiple comparison with a control procedure is used.

Most of the pairwise multiple comparisons and the multiple comparison with a control procedure depend on normality assumption. In practice, however, populations have non-normal distributions so frequently. In addition, true underlying distribution can be different from the assumed distribution. Therefore, it is very important to develop estimators of location and the associated variance under non-normality. A number of studies have been carried out for pairwise multiple comparisons. Tukey (1953) proposed T-method which is based on the studentized range distribution. The T-method is used for equal sample sizes and homogeneous variances. Tukey (1953), Kramer (1956), and Duncan (1957) modified this method for unequal sample sizes. Tamhane (1979) and Dunnett

(1980 b) studied the pairwise multiple comparisons when the treatment variances are not equal. A wise choice of the robust estimators was given by Andrews et al. (1972). Gross (1976) used a set of estimators of location with associated variance estimates and examined the robustness of an estimator both by its experimentwise error rates (robustness of validity) and its expected length of confidence interval (robustness of efficiency). Dunnett (1982) compared the use of several robust estimators in a modified T-method for pairwise multiple comparisons with the sample mean and sample variance and with the k-sample rank sum test. The robust estimators used in these studies are not efficient for short-tailed symmetric distributions (Tiku and Vaughan, 1999). However, Akkaya and Tiku (2006) showed that the modified maximum likelihood (MML) estimators are highly efficient and robust for short-tailed symmetric distributions.

The main aim of this thesis is to incorporate these estimates in both pairwise multiple comparisons and pairwise multiple comparisons with a control group and examine the efficiency and robustness of the resulting estimates through a comprehensive simulation study. Furthermore, the robustness properties are examined for the multiple comparisons with a control group when the model has inliers.

The organization of this thesis is as follows: Chapter 1 briefly presents historical review of the pairwise multiple comparisons and multiple comparisons with a control and gives a theoretical background of the techniques used throughout the thesis. In Chapter 2, the robust estimators used for pairwise multiple comparisons and multiple comparisons with a control are reviewed. Also, the modified maximum likelihood estimators of location and scale parameters are given under short-tailed symmetric distributions. Chapter 3 gives the efficiency, power and robustness comparisons of the robust estimators for pairwise multiple comparisons

and multiple comparisons with a control. Finally, a real life application and conclusions are given in Chapter 4.

1.1 Historical Perspective

1.1.1 Multiple Comparisons Under Normality

Consider the one-way fixed effects model

$$y_{ij} = \mu_i + e_{ij} \quad ; i = 1, \dots, k; j = 1, \dots, n_i \quad (1.1.1.1)$$

where y_{ij} is the j^{th} observation of the i^{th} treatment, μ_i is the mean of the i^{th} treatment, e_{ij} are the error terms and n_i is the number of observations in the i^{th} treatment. Here, it is assumed that e_{ij} are distributed independently as $N(0, \sigma^2)$.

Since $E(e_{ij}) = 0$ and $\text{Var}(e_{ij}) = \sigma^2$, $E(Y_{ij}) = \mu_i$ and $\text{Var}(Y_{ij}) = \sigma^2$. Also the observations Y_{ij} are independent and normally distributed since e_{ij} is distributed independently as normal.

The maximum likelihood estimators of μ_i and σ^2 are as follows:

$$\hat{\mu}_i = \bar{y}_i \quad \text{and} \quad \hat{\sigma}^2 = MS_E = \frac{\sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2}{n_T - k} \quad (1.1.1.2)$$

where

$$\bar{y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij} \quad \text{and} \quad n_T = \sum_{i=1}^k n_i .$$

To test the equality of k treatment means

$$H_0 : \mu_1 = \mu_2 = \dots = \mu_k \quad (1.1.1.3)$$

H_1 : At least one μ_i is different from the others

the analysis of variance procedure is used. This procedure partitions the total sum of squares which is the measure of total variability of the y_{ij} observations. The total sum of squares denoted by SS_T is

$$SS_T = \sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{..})^2 \quad (1.1.1.4)$$

where

$$\bar{y}_{..} = \frac{1}{n_T} \sum_{i=1}^k \sum_{j=1}^{n_i} y_{ij} .$$

Total sum of squares can be decomposed as follows:

$$\sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{..})^2 = \sum_{i=1}^k n_i (\bar{y}_i - \bar{y}_{..})^2 + \sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2 \quad (1.1.1.5)$$

The first term on the right is the treatment sum of squares denoted by SS_{Tr} and the second term is the error sum of squares denoted by SS_E . SS_{Tr} is a measure of the extent of differences between treatment means and SS_E is the measure of the

random variation of the observations around the respective treatment sample means.

Under the null hypothesis H_0 given in (1.1.1.3), $\frac{SS_T}{\sigma^2}$, $\frac{SS_{Tr}}{\sigma^2}$, and $\frac{SS_R}{\sigma^2}$ have chi-square distributions with $n_T - 1$, $k - 1$, and $n_T - k$ degrees of freedom, respectively. In addition, according to Cochran's theorem, $\frac{SS_{Tr}}{\sigma^2}$, and $\frac{SS_E}{\sigma^2}$ are independent random variables. Therefore, when the null hypothesis is true, the test statistic

$$F = \frac{SS_{Tr}/(k-1)}{SS_E/(n_T - k)} \quad (1.1.1.6)$$

has central F distribution with $k - 1$ and $n_T - k$ degrees of freedom.

The large values of F lead to rejection of the null hypothesis H_0 . Thus, decision rule is that the null hypothesis is rejected for the level of significance α if F is greater than or equal to the tabulated value $F(1 - \alpha; k - 1, n_T - k)$.

Type I error is the probability of rejecting the null hypothesis H_0 when it is true. Thus, Type I error of the test denoted by α is

$$\alpha = P(F \geq F(1 - \alpha; k - 1, n_T - k)). \quad (1.1.1.7)$$

The power of the test is the probability of rejecting the null hypothesis H_0 when the alternative hypothesis H_1 is true. Under alternative hypothesis H_1 , the test

statistic has non-central F distribution with $k-1$ and $n_T - k$ degrees of freedom and non-centrality parameter λ which is given by

$$\lambda = \frac{1}{\sigma} \sqrt{\frac{\sum_{i=1}^k n_i (\mu_i - \mu_{\cdot})^2}{k}} \quad (1.1.1.8)$$

where

$$\mu_{\cdot} = \frac{\sum_{i=1}^k n_i \mu_i}{n_T}.$$

Hence, the power of the test denoted by $1-\beta$ is

$$1-\beta = P(F \geq F(1-\alpha; k-1, n_T - k | \lambda)). \quad (1.1.1.9)$$

i) Pairwise Multiple Comparisons

If the ANOVA F test leads to rejection of the null hypothesis H_0 given in (1.1.1.3), the pairwise multiple comparisons method are used to determine which pair of treatments differ.

Consider the one-way model given in (1.1.1.1). In testing the pairwise equality of the treatment means

$$H_0 : \mu_i = \mu_j \quad (\text{for } i, j = 1, 2, \dots, k; i \neq j) \quad (1.1.1.10)$$

$$H_1 : \mu_i \neq \mu_j$$

the difference γ between two treatment means

$$\gamma = \mu_i - \mu_j, \quad i, j = 1, 2, \dots, k; i \neq j \quad (1.1.1.11)$$

is estimated. This difference is called as pairwise comparisons. There are $k(k-1)/2$ pairwise differences of treatments.

A point estimator of difference γ is

$$\hat{\gamma} = \bar{y}_i - \bar{y}_j, \quad i, j = 1, 2, \dots, k; i \neq j \quad (1.1.1.12)$$

where \bar{y}_i and \bar{y}_j denote the sample means for the i^{th} and j^{th} treatment, respectively.

$\hat{\gamma}$ is an unbiased estimator of γ :

$$E(\hat{\gamma}) = \mu_i - \mu_j. \quad (1.1.1.13)$$

The variance of $\hat{\gamma}$ is given as follows:

$$V(\hat{\gamma}) = \sigma^2 \left(\frac{1}{n_i} + \frac{1}{n_j} \right). \quad (1.1.1.14)$$

Also, the estimated variance of $\hat{\gamma}$, denoted by $s^2(\hat{\gamma})$, is given by

$$s^2(\hat{\gamma}) = MS_E \left(\frac{1}{n_i} + \frac{1}{n_j} \right). \quad (1.1.1.15)$$

There are six methods developed for the pairwise comparisons:

- The least significant difference (LSD) method
- Duncan's multiple range test
- The Newman-Keuls test
- Tukey's test
- Scheffé method
- Bonferroni method

The Least Significant Difference (LSD) Method

For the one-way model given in (1.1.1.1), LSD method uses the usual t-statistic

$$t = \frac{\bar{y}_i - \bar{y}_j}{\sqrt{MS_E \left(\frac{1}{n_i} - \frac{1}{n_j} \right)}} \quad (1.1.1.16)$$

having t-distribution with $n_T - k$ degrees of freedom.

Here,

$$\bar{y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij} \quad , \quad (1.1.1.17)$$

$$MS_E = \frac{\sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2}{n_T - k} \quad (1.1.1.18)$$

and

$$n_T = \sum_{i=1}^k n_i . \quad (1.1.1.19)$$

Thus, $(1 - \alpha)$ confidence intervals for the difference γ are given by

$$(\bar{y}_i - \bar{y}_j) \pm t_{(1-\alpha/2; n_T - k)} \sqrt{MS_E \left(\frac{1}{n_i} + \frac{1}{n_j} \right)} . \quad (1.1.1.20)$$

To test the hypothesis given in (1.1.1.10), the least significant difference (LSD)

$$LSD = t_{(1-\alpha/2; n_T - k)} \sqrt{MS_E \left(\frac{1}{n_i} + \frac{1}{n_j} \right)} \quad (1.1.1.21)$$

is calculated. If the model is balanced, i.e., $n_1 = n_2 = \dots = n_k = n$, then

$$LSD = t_{(1-\alpha/2; n_T - k)} \sqrt{\frac{2MS_E}{n}} . \quad (1.1.1.22)$$

The null hypothesis H_0 is rejected if $|\bar{y}_i - \bar{y}_j| > LSD$.

Duncan's Multiple Range Test

Multiple range test was developed by Duncan (1955). This test arranges the sample mean for the treatment in an ascending order and calculates the standard error of each sample mean. For the balanced one-way model, the standard error is calculated as follows:

$$S_{\bar{y}_i} = \sqrt{\frac{MS_E}{n}} . \quad (1.1.1.23)$$

For unbalanced one-way model, the harmonic mean n_h

$$n_h = \frac{k}{\sum_{i=1}^k \frac{1}{n_i}} \quad (1.1.1.24)$$

is used instead of n in (1.1.1.23).

To test the hypothesis given in (1.1.1.10), the least significant ranges

$$R_p = r_\alpha(p, f) S_{\bar{y}_i} \quad \text{for } p = 2, 3, \dots, k \quad (1.1.1.25)$$

are calculated. Here $r_\alpha(p, f)$ is the constant where α is the significance level and f is the degrees of freedom for error. The values of $r_\alpha(p, f)$ are given in Duncan (1955).

Then, the observed differences between means are tested, beginning with largest versus smallest, which would be compared with the least significant range R_k . Next, the difference of the largest and the second smallest is computed and compared with the least significant range R_{k-1} . These comparisons are continued until all means have been compared with the largest mean. Finally, the difference of the second largest mean and the smallest is computed and compared against the least significant range R_{k-1} . This process is continued until the differences of all possible $k(k-1)/2$ pairs of means have been considered (Montgomery, 1976).

If difference between two treatment means is greater than the corresponding least significant range, the null hypothesis H_0 is rejected.

The Newman-Keuls Test

This test was devised by Newman (1939). Since new interest in Newman's test was generated by Keuls (1952), the procedure is usually called the Newman-Keuls test (Montgomery, 1976).

The procedure used in the test depends on the studentized range statistic. The studentized range is defined as

$$q = \frac{\bar{y}_{\max} - \bar{y}_{\min}}{\sqrt{\frac{MS_E}{n}}} \quad (1.1.1.26)$$

where \bar{y}_{\max} is the largest sample mean and \bar{y}_{\min} is the smallest sample mean.

To test the hypothesis given in (1.1.1.10), the critical values

$$K_p = q_\alpha(p, f)S_{\bar{y}_i} \quad \text{for } p = 2, 3, \dots, k \quad (1.1.1.27)$$

are calculated. Here, $q_\alpha(p, f)$ represented by the table of studentized range distribution is the upper α percentage point of the studentized range for groups of means of size p , α is the significance level, and f is the degrees of freedom for error.

The comparison procedure for extreme pairs of means in groups of size p is the same as in Duncan's multiple range test.

Tukey's Test

This test was proposed by Tukey (1953). It is based on the studentized range distribution.

To test the hypothesis H_0 , the critical value

$$T_\alpha = q_\alpha(k, f)S_{\bar{y}_i} \quad (1.1.1.28)$$

is calculated. Here, $q_\alpha(k, f)$ is the upper α percentage point of the studentized range, α is the significance level, f is the degrees of freedom for error. In this test, a single critical value is calculated for all comparisons.

If the absolute value of difference between two sample mean is greater than the critical value T_α , the null hypothesis H_0 given in (1.1.1.10) is rejected.

Scheffé Method

When the aim is to test all possible contrasts among the factor treatment means, the Scheffé method of multiple comparisons can be used. Since each difference between two treatment means is a special case of a contrast, this method, also, can be used for pairwise multiple comparisons.

The Scheffé method is exact regardless to the equality of sample sizes. In this method, all contrasts

$$L = \sum c_i \mu_i \quad \text{where} \quad \sum c_i = 0 \quad (1.1.1.29)$$

are estimated.

An unbiased estimator of L is

$$\hat{L} = \sum c_i \bar{y}_i \quad (1.1.1.30)$$

where \bar{y}_i denotes the sample mean for the i^{th} treatment. The estimated variance of \hat{L} , denoted by $s^2(\hat{L})$, is

$$s^2(\hat{L}) = MS_E \sum \frac{c_i^2}{n_i} . \quad (1.1.1.31)$$

The confidence interval for L is given by

$$\hat{L} \mp Ss(\hat{L}) \quad (1.1.1.32)$$

where

$$S = \sqrt{(k-1)F(1-\alpha; k-1, n_T - k)} .$$

The null hypothesis H_0 given in (1.1.1.10) is rejected if $|\hat{L}| > Ss(\hat{L})$.

Bonferroni Method

The Bonferroni method of multiple comparisons is used for the pairwise comparisons of contrasts, or linear combinations of the factor level means.

The Bonferroni method can be applied for both equal and unequal sample sizes.

Let L_i ($i=1, 2, \dots, g$) be the linear combinations of the treatment means. Then the Bonferroni confidence limits for g linear combinations are given by

$$\hat{L}_i \pm Bs(\hat{L}_i) \quad (1.1.1.33)$$

where

$$B = t(1 - \alpha/2g; n_t - r).$$

Since pairwise comparisons are special cases of linear combinations, these confidence limits can be used for testing pairwise differences of the treatment means.

The null hypothesis H_0 given in (1.1.1.10) is rejected if $|\hat{L}_i| > Bs(\hat{L}_i)$.

ii) Multiple Comparisons with a Control

The control group may be a placebo, or a 'standard' treatment, or any other 'specified' treatment. The experimental setting for a control group must be identical to the setting for the other treatments.

To compare the treatment and control means, the difference δ between each treatment mean and the control mean

$$\delta = \mu_i - \mu_k, \quad i = 1, 2, \dots, k-1 \quad (1.1.1.34)$$

is estimated. Here, μ_i is the mean of the i^{th} treatment and μ_k is the mean of the control group.

Balanced One –Way Model

Consider the balanced one-way model

$$y_{ij} = \mu_i + e_{ij} \quad , \quad i = 1, 2, \dots, k; j = 1, 2, \dots, n \quad (1.1.1.35)$$

where y_{ij} is the j^{th} observation for the i^{th} treatment, μ_i is the mean of the i^{th} treatment, and e_{ij} 's are iid normal with mean 0 and variance σ^2 . Here, σ^2 is not known.

The point estimators of μ_i and σ^2 are

$$\hat{\mu}_i = \bar{y}_i \quad \text{and} \quad (1.1.1.36)$$

$$\hat{\sigma}^2 = \text{MSE} = \frac{\sum_{i=1}^k \sum_{j=1}^n (y_{ij} - \bar{y}_i)^2}{k(n-1)} \quad (1.1.1.37)$$

where

$$\bar{y}_i = \frac{1}{n} \sum_{j=1}^n y_{ij} .$$

For the balanced one-way model given in (1.1.1.35), the simultaneous confidence intervals for the difference between each treatment mean and mean of the control group, $\mu_i - \mu_k$ ($i = 1, 2, \dots, k - 1$), are given by Dunnett (1964) as follows:

$$\hat{\mu}_i - \hat{\mu}_k \pm d_\alpha(k-1, v)\hat{\sigma}\sqrt{2/n} \quad (1.1.1.38)$$

where d is the tabulated constant for the joint significance level of α , $k-1$, and $v = k(n-1)$.

The null hypothesis

$$H_0 : \mu_i = \mu_k \quad (i = 1, 2, \dots, k - 1) \quad \text{against} \quad (1.1.1.39)$$

$$H_1 : \mu_i \neq \mu_k$$

is rejected if $|\hat{\mu}_i - \hat{\mu}_k| > d_\alpha(k-1, v)\hat{\sigma}\sqrt{2/n}$ at the level of significance α .

Unbalanced One-Way Model

Consider the unbalanced one-way model

$$y_{ij} = \mu_i + e_{ij}, \quad i = 1, 2, \dots, k; j = 1, 2, \dots, n_i \quad (1.1.1.40)$$

where y_{ij} is the j^{th} observation for the i^{th} treatment, μ_i is the mean of the i^{th} treatment, and e_{ij} 's are iid normal with mean 0 and variance σ^2 . Here, σ^2 is not known.

The point estimators of μ_i and σ^2 are

$$\hat{\mu}_i = \bar{y}_i \quad \text{and} \quad (1.1.1.41)$$

$$\hat{\sigma}^2 = \text{MSE} = \frac{\sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2}{n_T - k} \quad (1.1.1.42)$$

where

$$\bar{y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij} \quad \text{and} \quad n_T = \sum_{i=1}^k n_i .$$

i) Dunnett's Method

For the unbalanced one-way model given in (1.1.1.40), the simultaneous confidence intervals for the difference between treatment and control means, $\mu_i - \mu_k$ ($i = 1, 2, \dots, k-1$), are given by Dunnett (1964) as follows:

$$\hat{\mu}_i - \hat{\mu}_k \pm d_{\alpha}(k-1, v) \hat{\sigma} \sqrt{\frac{1}{n_i} + \frac{1}{n_k}} \quad (1.1.1.43)$$

where d is the tabulated constant for the joint significance level of α , $k-1$, and $v = n_T - k$.

The null hypothesis

$$H_0 : \mu_i = \mu_k \quad (i = 1, 2, \dots, k-1) \quad \text{against} \quad (1.1.1.44)$$

$$H_1 : \mu_i \neq \mu_k$$

is rejected if $|\hat{\mu}_i - \hat{\mu}_k| > d_\alpha(k-1, v) \hat{\sigma} \sqrt{\frac{1}{n_i} + \frac{1}{n_k}}$ at the level of significance α .

ii) The Miller-Winer Method

For the unbalanced one-way model given in (1.1.1.40), the Miller-Winer method uses the harmonic mean of the treatment sample sizes, denoted by \tilde{n} ,

$$\tilde{n} = \left(\frac{\frac{1}{n_1} + \dots + \frac{1}{n_{k-1}}}{k-1} \right)^{-1} \quad (1.1.1.45)$$

as the common sample size excluding the control group in the Dunnett's method. Thus, the simultaneous confidence intervals for the differences between treatment and control means are as follows:

$$\hat{\mu}_i - \hat{\mu}_k \pm d_\alpha(k-1, v) \hat{\sigma} \sqrt{\frac{1}{\tilde{n}} + \frac{1}{n_k}} \quad (1.1.1.46)$$

where d is the tabulated constant for the joint significance level of α , $k-1$, and $v = n_T - k$.

The null hypothesis

$$\begin{aligned} H_0 : \mu_i &= \mu_k \quad (i = 1, 2, \dots, k-1) \quad \text{against} & (1.1.1.47) \\ H_1 : \mu_i &\neq \mu_k \end{aligned}$$

is rejected if $|\hat{\mu}_i - \hat{\mu}_k| > d_\alpha(k-1, v)\hat{\sigma}\sqrt{\frac{1}{\tilde{n}} + \frac{1}{n_k}}$ at the level of significance α .

1.1.2 Robustness

Most statistical procedures such as classical t-test and F-test are based on normality assumption. In practice, however, populations have non-normal distributions frequently and assuming normality may lead to wrong statistical inferences. On the other hand, the functional form of the underlying distribution can not be determined exactly. There are several procedures such as graph-plotting techniques and goodness-of-fit tests to provide information about the underlying distribution. These procedures can distinguish normal from extremely non-normal symmetric distributions and extremely skew distributions, however, they are not very successful in distinguishing normal from moderately non-normal distributions. In these situations, robust estimation and hypothesis testing procedures are used.

An estimator is called robust if it is fully efficient (or nearly so) for an assumed distribution but maintains high efficiency for plausible alternatives. A fully efficient estimator is one which is unbiased and its variance is equal to the Cramer-Rao minimum variance bound. A statistical test has been traditionally called robust if both its Type I error and power are not affected much by departures from normality (Tiku and Akkaya, 2004).

In this study, it is assumed that observations are independent and have short-tailed symmetric distribution.

1.2 Theoretical Background

1.2.1 Short-Tailed Symmetric Distribution

The family of short-tailed symmetric distributions (STS) was introduced by Tiku and Vaughan (1999). This family is given by

$$\text{STS}(r, a_1) : f(y) \propto \frac{1}{\sigma} \left\{ 1 + \frac{\lambda}{2r} \left(\frac{y-\mu}{\sigma} \right)^2 \right\}^r \left[1 + \frac{1}{k_1} \left(\frac{y-\mu}{\sigma} \right)^2 \right]^{-p}, \quad -\infty < y < \infty \quad (1.2.1.1)$$

where r is an constant, $\lambda = \frac{r}{r-a_1}$, $a_1 < r$, $k_1 = p - \frac{3}{2}$ and $p > r + \frac{3}{2}$. The kurtosis μ_4/μ_2^2 is less than 3 for all values of p . For $p = \infty$, a special form of $f(y)$ is obtained, i.e.,

$$f(y) = \frac{C_1}{\sqrt{2\pi}\sigma} \left\{ 1 + \frac{\lambda}{2r} \left(\frac{y-\mu}{\sigma} \right)^2 \right\}^r e^{-(y-\mu)^2/2\sigma^2}, \quad -\infty < y < \infty \quad (1.2.1.2)$$

where C_1 is constant which makes

$$\int_{-\infty}^{\infty} f(y) dy = 1.$$

Since

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^{2j} e^{-t^2/2} dt = (2j)!/(2^j j!) \quad (1.2.1.3)$$

the value of C_1 can be calculated easily;

$$C_1 = 1 / \left\{ \sum_{j=0}^r \binom{r}{j} \left(\frac{\lambda}{2r} \right)^j \frac{(2j)!}{2^j (j)!} \right\}. \quad (1.2.1.4)$$

All odd moments of Y are zero because of symmetry. Mean and variance of Y are given by

$$E(Y) = \mu \quad \text{and} \quad (1.2.1.5)$$

$$\text{Var}(Y) = \mu_2 \sigma^2 = C_1 \sum_{j=0}^r \binom{r}{j} \left(\frac{\lambda}{2r} \right)^j \frac{(2(j+1))!}{2^{j+1} (j+1)!} \sigma^2. \quad (1.2.1.6)$$

For $a_1 \leq 0$, the distributions (1.2.1.1) are unimodal but for $a_1 > 0$, they are generally multimodal. For a given r , the kurtosis of the distribution decreases as a_1 increases. It is known that no distribution can have kurtosis less than 1.0 (Pearson and Tiku, 1970). To provide this restriction, a_1 must be less than or equal to $\frac{3r}{4}$.

CHAPTER 2

MULTIPLE COMPARISONS UNDER NON-NORMALITY

In this chapter, the test statistic used in pairwise multiple comparisons and multiple comparisons with a control procedures is given for short-tailed symmetric distribution. To provide robustness under short-tailed symmetric distribution, Huber estimators of location and scale parameters and trimmed mean with winsorized standard deviation are used besides ordinary sample mean and sample variance. In addition to these estimators, the modified maximum likelihood estimators given by Tiku (1967) are used under short-tailed symmetric distribution.

2.1 Testing Procedure

As mentioned in Chapter 1, Tukey (1953) proposed a method called as T-method to make all possible pairwise comparisons. Dunnett (1982) described this method as follows:

Consider the one-way fixed effects analysis of variance model

$$y_{ij} = \mu_i + e_{ij} \quad ; \quad i = 1, \dots, k; \quad j = 1, \dots, n_i \quad (2.1.1)$$

where the errors e_{ij} are iid normal with mean 0 and variance σ_i^2 and μ_i and σ_i^2 are unknown parameters. This method assumes equal sample sizes and variances, i.e., $n_i = n$ and $\sigma_i^2 = \sigma^2$.

The joint confidence intervals for the difference of pairwise treatment means, $\mu_i - \mu_j$ ($i, j = 1, 2, \dots, k; i \neq j$), are given by

$$\bar{y}_i - \bar{y}_j \pm \text{SR}_{\alpha, k, v} \sqrt{2s^2/n} \quad (2.1.2)$$

where \bar{y}_i is the sample mean of the i^{th} treatment, s^2 is an unbiased estimate of σ^2 based on v degrees of freedom and $\text{SR}_{\alpha, k, v}$ is the α -point of studentized range distribution with k normal variates. For the equal sample sizes and equal variances, the joint confidence coefficient is exactly $1 - \alpha$.

For the unequal sample sizes, Tukey (1953) proposed the use of $s^2 \left(\frac{1}{n_i} + \frac{1}{n_j} \right)$

instead of $\frac{2s^2}{n}$ in (2.1.2). Thus, the joint confidence interval for unequal sample sizes are given by

$$\bar{y}_i - \bar{y}_j \pm (\text{SR}_{\alpha, k, v} / \sqrt{2}) \sqrt{s^2 \left(\frac{1}{n_i} + \frac{1}{n_j} \right)}. \quad (2.1.3)$$

Dunnett (1980 a) showed that the joint confidence coefficient is greater than or equal to $1 - \alpha$ for these confidence intervals.

For the case of unequal variances, Dunnett (1980 b) stated that the pooling of individual estimates s_i^2 was inappropriate since the confidence interval given in (2.1.3) were lack of robustness, i.e., the desired joint confidence coefficient couldn't be achieved. To improve robustness, Dunnett (1980 b) proposed the following confidence intervals:

$$\bar{y}_i - \bar{y}_j \pm A_{ij, \alpha, k} \sqrt{\frac{s_i^2}{n_i} + \frac{s_j^2}{n_j}} \quad (2.1.4)$$

where $A_{ij, \alpha, k}$ is chosen to achieve the desired joint confidence coefficient $1 - \alpha$ as closely as possible and s_i^2 is an unbiased estimate of σ_i^2 based on v_i degrees of freedom.

To achieve robustness under non-normality, Dunnett (1982) used the robust estimators of location and scale parameters in the modified T-method which gives the following confidence intervals:

$$\tilde{\mu}_i - \tilde{\mu}_j \pm A_{ij, \alpha, k}^* \sqrt{\frac{\tilde{\sigma}_i^2}{\tilde{n}_i} + \frac{\tilde{\sigma}_j^2}{\tilde{n}_j}} \quad (2.1.5)$$

where $\tilde{\mu}_i$ is the robust estimate of location for the i^{th} treatment, $\tilde{\sigma}_i$ is the corresponding robust estimate of variance, \tilde{n}_i is the effective sample size, $A_{ij, \alpha, k}^*$ is a constant which is chosen to obtain the desired confidence coefficient $1 - \alpha$ as closely as possible.

Under non-normality, $\max |\tilde{t}_{ij}|$ was given by Dunnett (1982) as a test statistic for pairwise multiple comparisons where

$$\tilde{t}_{ij} = \frac{\tilde{\mu}_i - \tilde{\mu}_j - (\mu_i - \mu_j)}{\sqrt{\frac{\tilde{\sigma}_i^2}{\tilde{n}_i} + \frac{\tilde{\sigma}_j^2}{\tilde{n}_j}}}. \quad (2.1.6)$$

Under normality, it is assumed that the distribution of \tilde{t}_{ij} is approximately Student's t with \tilde{v}_{ij} degrees of freedom which is given by

$$\tilde{v}_{ij} = \frac{(\tilde{\sigma}_i^2/\tilde{n}_i + \tilde{\sigma}_j^2/\tilde{n}_j)^2}{\tilde{\sigma}_i^4/\tilde{n}_i^2 v_i + \tilde{\sigma}_j^4/\tilde{n}_j^2 v_j}. \quad (2.1.7)$$

Here v_i is the effective degrees of freedom associated with s_i^2 .

To determine the value of $A_{ij,\alpha,k}^*$, the α -point of the distribution of $\max|\tilde{t}_{ij}|$ is required, however, its distribution is different from the distribution of \tilde{t}_{ij} since $\max|\tilde{t}_{ij}|$ is the largest order statistic. In this study, therefore, the value of $A_{ij,\alpha,k}^*$ is found through simulation so that the true experimentwise error rate α is achieved.

If $\max|\tilde{t}_{ij}| > A_{ij,\alpha,k}^*$, the null hypothesis

$$\begin{aligned} H_0 : \mu_i &= \mu_j \quad (i, j = 1, 2, \dots, k; i \neq j) \quad \text{against} & (2.1.8) \\ H_1 : \mu_i &\neq \mu_j \end{aligned}$$

is rejected at the level of significance α .

The same procedure was used for the multiple comparisons with a control in our study. Thus, the joint confidence intervals for the difference between each treatment mean and the control mean are given by

$$\tilde{\mu}_i - \tilde{\mu}_k \pm A_{ik,\alpha,k}^* \sqrt{\frac{\tilde{\sigma}_i^2}{\tilde{n}_i} + \frac{\tilde{\sigma}_k^2}{\tilde{n}_k}}, \quad i = 1, \dots, k-1 \quad (2.1.9)$$

where $\tilde{\mu}_i$ is the robust estimate of location for the i^{th} treatment, $\tilde{\mu}_k$ is the robust estimate of location for control treatment, $\tilde{\sigma}_i$ and $\tilde{\sigma}_k$ are the corresponding robust estimate of variance, \tilde{n}_i is the effective sample size and $A_{ij, \alpha, k}^*$ is a constant which is chosen to obtain the desired confidence coefficient $1 - \alpha$ as closely as possible.

To compare each treatment mean with a control mean, $\max|\tilde{t}_{ik}|$ ($i = 1, \dots, k - 1$) is used as a test statistic where

$$\tilde{t}_{ik} = \frac{\tilde{\mu}_i - \tilde{\mu}_k - (\mu_i - \mu_k)}{\sqrt{\frac{\tilde{\sigma}_i^2}{\tilde{n}_i} + \frac{\tilde{\sigma}_k^2}{\tilde{n}_k}}}. \quad (2.1.10)$$

If $\max|\tilde{t}_{ik}| > A_{ik, \alpha, k}^*$, the null hypothesis

$$H_0 : \mu_i = \mu_k \quad (i = 1, 2, \dots, k - 1) \quad \text{against} \quad (2.1.11)$$

$$H_1 : \mu_i \neq \mu_k$$

is rejected at the level of significance α .

2.2 Robust Estimators

2.2.1 Huber Estimators

Consider a random sample from a distribution of type

$$\frac{1}{\sigma} f\left(\frac{y - \mu}{\sigma}\right). \quad (2.2.1.1)$$

where μ and σ are the location and scale parameters, respectively.

Huber (1964) proposed a new method to estimate μ assuming in particular that f is symmetric and long-tailed distribution.

The log-likelihood function is

$$\ln L = -n \ln \sigma + \sum_{i=1}^n \ln f\left(\frac{y_i - \mu}{\sigma}\right). \quad (2.2.1.2)$$

If the functional form of f is known, the maximum likelihood estimator of μ is obtained from the equation

$$\frac{\partial \ln L}{\partial \mu} = \frac{1}{\sigma} \sum_{i=1}^n \psi(z_i) = 0 \quad (2.2.1.3)$$

where

$$\psi(z_i) = -\frac{f'(z_i)}{f(z_i)} \quad \text{and} \quad z_i = \frac{y_i - \mu}{\sigma}.$$

By letting $w_i = w_i(z) = \frac{\psi(z_i)}{z_i}$, equation (2.2.1.3) reduces to $\sum_{i=1}^n w_i (y_i - \mu) = 0$.

The solution of the equation gives μ as follows:

$$\mu = \frac{\sum_{i=1}^n w_i y_i}{\sum_{i=1}^n w_i}. \quad (2.2.1.4)$$

Given σ and $\psi(z_i)$, equation (2.2.1.2) may be solved by iteration (Low, 1959). It may also be solved by applying Newton-Raphson's procedure to equation (2.2.1.3) (Gross, 1976).

In practice, however, σ and $\psi(z_i)$ are not known. Therefore, Huber (1964) proposed a function $\psi(z_i)$ as

$$\psi(z) = \begin{cases} z & \text{if } |z| \leq c \\ \text{csgn}(z) & \text{if } |z| > c \end{cases} \quad (2.2.1.5)$$

which is a combination of the normal distribution in the middle with the double-exponential distribution in the tails. Birch and Myers (1982) give 1.345, 1.5, and 2.0 as the popular choice of c values since these choices correspond roughly to 10, 5, and 2.5 percent censoring on either side of a normal sample.

The solution of (2.2.1.3) is referred to as Huber's M-estimator and denoted by $\hat{\mu}_H$.

For unknown σ , $\tilde{\sigma}_0 = \text{mad} = \text{median}|y_i - \text{median}(y_i)|$ is used by Huber (1964, 1977) and Gross (1976, 1977) to estimate σ . However, Huber (1981) and Birch and Myers (1982) suggest to use $\tilde{\sigma}_0/0.6745$ instead of $\tilde{\sigma}_0$ to obtain an asymptotically unbiased estimator of σ in the case of normal distribution.

By using the asymptotic variance of the M-estimator $\hat{\mu}_H$

$$\frac{(1/n)\sigma^2 E(\psi^2(z))}{[E(\psi'(z))]^2}, \quad (2.2.1.6)$$

Huber (1977, 1981) calculated the estimator of scale, $\hat{\sigma}_H$, as

$$\hat{\sigma}_H = \left\{ \frac{n\hat{\sigma}_0^2 \left[\sum_{i=1}^n \psi^2 \left(\frac{y_i - \hat{\mu}_H}{\tilde{\sigma}_0} \right) \right] \right\}^{1/2} \left/ \left[\sum_{i=1}^n \psi' \left(\frac{y_i - \hat{\mu}_H}{\tilde{\sigma}_0} \right) \right]^2 \right\} . \quad (2.2.1.7)$$

When the functional form of f is not known, $\psi(z)$ may be approximated by descending functions. The function which decreases as $|z|$ increases is called as descending function. There are three important descending functions:

1. The wave function (Andrews et al., 1972; Andrews, 1974)

$$\psi(z) = \begin{cases} \sin(z) & \text{if } |z| \leq \pi \\ 0 & \text{if } |z| > \pi \end{cases} \quad (2.2.1.8)$$

2. The bisquare function (Beaton and Tukey, 1974)

$$\psi(z) = \begin{cases} z(1-z^2)^2 & \text{if } |z| \leq 1 \\ 0 & \text{if } |z| > 1 \end{cases} \quad (2.2.1.9)$$

3. The Hampel piecewise linear function (Hampel, 1974)

$$\psi(z) = \text{sgn}(z) \begin{cases} |z| & 0 \leq |z| < a \\ a & a \leq |z| < b \\ \frac{c-|z|}{c-b} & b \leq |z| < c \\ 0 & c \leq |z| \end{cases} . \quad (2.2.1.10)$$

For different values of a, b, and c, different estimators are obtained.

Gross (1976) showed that the wave, bisquare, and Hampel piecewise linear functions were the most efficient descending functions when the adjusting constant h was equal to 2.4, 8.2, and 2.2, respectively. The estimators of location and scale obtained by using these three functions are called as the wave estimator (W24), bisquare estimator (BS82), and Hampel estimator (H22). These estimators are given as follows:

Let $T_0 = \text{median}\{y_i\}$ and $S_0 = \text{median}\{|y_i - T_0|\}$

i) W24 estimator

$$\hat{\mu}_{w24} = T_0 + (hS_0)\tan^{-1}\left[\frac{\sum \sin(z_i)}{\sum \cos(z_i)}\right] \quad (2.2.1.11)$$

and

$$\hat{\sigma}_{w24} = (hS_0)\left[n\frac{\sum \sin^2(z_i)}{(\sum \cos(z_i))^2}\right]^{1/2} \quad (2.2.1.12)$$

where

$$z_i = \frac{y_i - T_0}{hS_0} \quad \text{and} \quad h = 2.4.$$

Here, summations include only those i such that $|z_i| < \pi$.

ii) BS82 estimator

$$\hat{\mu}_{\text{BS82}} = T_0 + (hS_0) \frac{\sum \psi(z_i)}{\sum \psi'(z_i)} \quad (2.2.1.13)$$

and

$$\hat{\sigma}_{\text{BS82}} = (hS_0) \left[n \frac{\sum \psi^2(z_i)}{(\sum \psi'(z_i))^2} \right]^{1/2} \quad (2.2.1.14)$$

where

$$z_i = \frac{y_i - T_0}{hS_0} \quad \text{and} \quad h = 8.2.$$

Here, $\psi(z)$ is the Beaton and Tukey's (1974) bisquare function given in (2.2.1.9) and $\psi'(z)$ is the derivative of $\psi(z)$.

iii) H22 estimator

$$\hat{\mu}_{\text{H22}} = T_0 + \frac{S_0 \sum \psi(z_i)}{\sum \psi'(z_i)} \quad (2.2.1.15)$$

and

$$\hat{\sigma}_{\text{H22}} = \left[\frac{n(S_0)^2 \sum \psi^2(z_i)}{(\sum \psi'(z_i))^2} \right]^{1/2} \quad (2.2.1.16)$$

where

$$z_i = \frac{y_i - T_0}{S_0}.$$

Here, $\psi(z)$ is the Hampel piecewise linear function given in (2.2.1.10) for $a = 2.25$, $b = 3.75$, and $c = 15.0$ and $\psi'(z)$ is the derivative of $\psi(z)$.

For symmetric distributions, $\hat{\mu}_{W24}$, $\hat{\mu}_{B82}$ and $\hat{\mu}_{H22}$ are unbiased and uncorrelated with $\hat{\sigma}_{W24}$, $\hat{\sigma}_{B82}$ and $\hat{\sigma}_{H22}$, respectively (Tiku, Tan, Balakrishnan, 1986).

2.2.2 Trimmed Mean With Winsorized Standard Deviation

Consider a random sample y_1, y_2, \dots, y_n from a distribution with location parameter μ and scale parameter σ . The trimmed mean with winsorized standard deviation is defined by Tukey and McLaughlin (1963) as follows:

$$\hat{\mu}_T = \frac{\sum_{i=r+1}^{n-r} y_{(i)} + (r - \alpha n)(y_{(r)} + y_{(n-r+1)})}{n(1 - 2\alpha)} \quad \text{and} \quad (2.2.2.1)$$

$$\hat{\sigma}_T = \frac{\sum_{i=r+1}^{n-r} (y_{(i)} - \hat{\mu}_T)^2 + r(y_{(r)} - \hat{\mu}_T)^2 + r(y_{(n-r+1)} - \hat{\mu}_T)^2}{n(1 - 2\alpha)^2} \quad (2.2.2.2)$$

where $r = [\alpha n] + 1$ ($[.]$ is the greatest integer function), $y_{(i)}$ is the i^{th} order statistic and α is the trimming proportion.

Huber (1964) proved that an asymptotically unbiased estimator of the variance of an M-estimator is obtained by using the winsorized sample variance and not the trimmed sample variance.

2.2.3 Modified Maximum Likelihood Estimators

Consider the one-way fixed effects analysis of variance model,

$$y_{ij} = \mu_i + e_{ij} \quad ; \quad i = 1, \dots, k; \quad j = 1, \dots, n_i$$

where the e_{ij} are independent and have short-tailed symmetric distribution with parameters r and a_1 .

The Fisher likelihood function is

$$L(a_1, r) = \frac{C_1^N}{(\sqrt{2\pi}\sigma)^N} \prod_{i=1}^k \prod_{j=1}^{n_i} \left\{ \left(1 + \frac{\lambda}{2r} z_{ij}^2 \right)^r e^{-z_{ij}^2/2} \right\} \quad (2.2.3.1)$$

where

$$z_{ij} = \frac{y_{ij} - \mu_i}{\sigma} \quad \text{and} \quad N = \sum_{i=1}^k n_i .$$

Thus, the likelihood equations for estimating μ and σ are

$$\frac{\partial \ln L}{\partial \mu_i} = -\frac{\lambda}{\sigma} \sum_{i=1}^k \sum_{j=1}^{n_i} g(z_{ij}) + \frac{1}{\sigma} \sum_{i=1}^k \sum_{j=1}^{n_i} z_{ij} = 0 \quad \text{and} \quad (2.2.3.2)$$

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{N}{\sigma} - \frac{\lambda}{\sigma} \sum_{i=1}^k \sum_{j=1}^{n_i} z_{ij} g(z_{ij}) + \frac{1}{\sigma} \sum_{i=1}^k \sum_{j=1}^{n_i} z_{ij}^2 = 0$$

where

$$g(z_{ij}) = \frac{z_{ij}}{1 + \frac{\lambda}{2r} z_{ij}^2}.$$

Likelihood equations given above have no explicit solutions since they include non-linear function $g(z_{ij})$. They can be solved by iteration, however, it is difficult to make any analytical study of the resulting maximum likelihood estimators (MLE), especially for small samples. Therefore, method of modified maximum likelihood (MML) developed by Tiku (1967) is used to find the explicit solutions.

In this method, the likelihood equations given in (2.2.3.2) are expressed in terms of the ordered variates $z_{i(j)} = (y_{i(j)} - \mu_i)/\sigma$ ($1 \leq i \leq k, 1 \leq j \leq n_i$). Since summations are invariant to ordering, the likelihood equations are written as follows:

$$\frac{\partial \ln L}{\partial \mu_i} = -\frac{\lambda}{\sigma} \sum_{i=1}^k \sum_{j=1}^{n_i} g(z_{i(j)}) + \frac{1}{\sigma} \sum_{i=1}^k \sum_{j=1}^{n_i} z_{i(j)} = 0 \quad \text{and} \quad (2.2.3.3)$$

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{N}{\sigma} - \frac{\lambda}{\sigma} \sum_{i=1}^k \sum_{j=1}^{n_i} z_{i(j)} g(z_{i(j)}) + \frac{1}{\sigma} \sum_{i=1}^k \sum_{j=1}^{n_i} z_{i(j)}^2 = 0.$$

The linear approximations for the non-linear function $g(z_{i(j)})$ is given by

$$g(z_{i(j)}) \cong \alpha_{ij} + \gamma_{ij} z_{i(j)} \quad ; \quad 1 \leq i \leq k, 1 \leq j \leq n_i. \quad (2.2.3.4)$$

By using first two terms of a Taylor series expansion of $g(z_{i(j)})$ around $t_{i(j)} = E(z_{i(j)})$, the coefficients α_{ij} and γ_{ij} are obtained as follows:

$$\alpha_{ij} = \frac{\frac{\lambda}{r} t_{i(j)}^3}{\left(1 + \frac{\lambda}{2r} t_{i(j)}^2\right)^2} \quad \text{and} \quad (2.2.3.5)$$

$$\gamma_{ij} = \frac{1 - \frac{\lambda}{2r} t_{i(j)}^2}{\left(1 + \frac{\lambda}{2r} t_{i(j)}^2\right)^2}. \quad (2.2.3.6)$$

The exact values of $t_{i(j)}$ are not available, but their appropriate values can be obtained from the equation

$$\int_{-\infty}^{t_{i(j)}} f(z) dz = \frac{j}{n_i + 1}; \quad 1 \leq i \leq k, \quad 1 \leq j \leq n_i. \quad (2.2.3.7)$$

In our study, an algorithm given by Dr. M. Q. Islam was used to evaluate $t_{i(j)}$.

Replacing $g(z_{i(j)})$ by the linear approximation given in (2.2.3.4), the modified likelihood equations are obtained as follows:

$$\frac{\partial \ln L}{\partial \mu} \cong \frac{\partial \ln L^*}{\partial \mu} = \frac{1}{\sigma} \sum_{i=1}^k \sum_{j=1}^{n_i} z_{i(j)} - \frac{\lambda}{\sigma} \sum_{i=1}^k \sum_{j=1}^{n_i} (\alpha_{ij} + \gamma_{ij} z_{i(j)}) = 0 \quad \text{and} \quad (2.2.3.8)$$

$$\frac{\partial \ln L}{\partial \sigma} \cong \frac{\partial \ln L^*}{\partial \sigma} = -\frac{N}{\sigma} + \frac{1}{\sigma} \sum_{i=1}^k \sum_{j=1}^{n_i} z_{i(j)}^2 - \frac{\lambda}{\sigma} \sum_{i=1}^k \sum_{j=1}^{n_i} z_{i(j)} (\alpha_{ij} + \gamma_{ij} z_{i(j)}) = 0.$$

The solutions of these equations give the following MMLE (Akkaya and Tiku, 2006),

$$\hat{\mu}_i = \frac{1}{m_i} \sum_{j=1}^{n_i} \beta_{ij} y_{i(j)} \quad \text{and} \quad (2.2.3.9)$$

$$\hat{\sigma} = \frac{-\lambda B + \sqrt{(\lambda B)^2 + 4NC}}{2\sqrt{N(N-1)}} \quad (2.2.3.10)$$

where

$$\beta_{ij} = 1 - \lambda \gamma_{ij}, \quad m_i = \sum_{j=1}^{n_i} \beta_{ij}, \quad B = \sum_{i=1}^k \sum_{j=1}^{n_i} \alpha_{ij} (y_{i(j)} - \hat{\mu}_i) \quad \text{and}$$

$$C = \sum_{i=1}^k \sum_{j=1}^{n_i} \beta_{ij} (y_{i(j)} - \hat{\mu}_i)^2.$$

Here, $\lambda \leq 1$, $\beta_{ij} > 0$ ($1 \leq i \leq k$, $1 \leq j \leq n_i$) and $C > 0$. Consequently, $\hat{\sigma}$ is real and positive.

For $\lambda > 1$, however, some of the coefficients β_{ij} in the middle can be negative. In this case, $\hat{\sigma}$ may not be real and positive. To prevent this, the coefficients α_{ij} , γ_{ij} and β_{ij} are replaced by α_{ij}^* , γ_{ij}^* and β_{ij}^* , respectively. They are given by

$$\alpha_{ij}^* = \frac{\frac{\lambda}{r} t_{i(j)}^3 + \left(1 - \frac{1}{\lambda}\right) t_{i(j)}}{\left\{1 + \frac{\lambda}{2r} t_{i(j)}^2\right\}^2} \quad \text{and} \quad (2.2.3.11)$$

$$\gamma_{ij}^* = \frac{\frac{1}{\lambda} - \frac{\lambda}{2r} t_{i(j)}^2}{\left\{1 + \frac{\lambda}{2r} t_{i(j)}^2\right\}^2}, \quad (2.2.3.12)$$

$$\beta_{ij}^* = 1 - \lambda \gamma_{ij}^* \quad (2.2.3.13)$$

where $\lambda > 1$.

For $\lambda \leq 1$ and $\lambda > 1$, $\hat{\mu}_i$ and $\hat{\sigma}$ have the same asymptotic properties since

$$\alpha_{ij} + \gamma_{ij} z_{i(j)} \cong \alpha_{ij}^* + \gamma_{ij}^* z_{i(j)}. \quad (2.2.3.14)$$

For $\lambda = 1$, $\alpha_{ij} = \alpha_{ij}^*$ and $\beta_{ij} = \beta_{ij}^*$. Also $t_{i(j)} = -t_{i(n-j+1)}$ because of symmetry. Thus,

$$\alpha_{ij} = -\alpha_{i(n-j+1)}, \quad (2.2.3.15)$$

$$\alpha_{ij}^* = -\alpha_{i(n-j+1)}^*,$$

$$\gamma_{ij} = \gamma_{i(n-j+1)} \quad \text{and}$$

$$\gamma_{ij}^* = \gamma_{i(n-j+1)}^*.$$

Since MML estimators are equivalent to ML estimators asymptotically, they are asymptotically unbiased and efficient (Tiku, 1970; Bhattacharyya, 1985; Tiku and Suresh, 1992 and Vaughan, 1992). Also, $\hat{\mu}_i$ is unbiased and $\text{Cov}(\hat{\mu}_i, \hat{\sigma}) = 0$ because of symmetry.

Fisher information matrix is given by

$$\mathbf{I} = \begin{bmatrix} -E\left(\frac{\partial^2 \ln L}{\partial \mu^2}\right) & -E\left(\frac{\partial^2 \ln L}{\partial \mu \partial \sigma}\right) \\ -E\left(\frac{\partial^2 \ln L}{\partial \mu \partial \sigma}\right) & -E\left(\frac{\partial^2 \ln L}{\partial \sigma^2}\right) \end{bmatrix}. \quad (2.2.3.16)$$

By using the inverse of the Fisher information matrix, the asymptotic covariance matrix of $\hat{\mu}$ and $\hat{\sigma}$ is obtained as follows:

$$\text{Cov}(\hat{\mu}, \hat{\sigma}) \cong \frac{\sigma^2}{n} \begin{bmatrix} 1/D & 0 \\ 0 & 1/D^* \end{bmatrix} \quad (2.2.3.17)$$

Here,

$$D = 1 - \lambda E(0, 1) + \frac{\lambda^2}{r} E(1, 2) \quad \text{and} \quad (2.2.3.18)$$

$$D^* = 2 - 3\lambda E(1, 1) + \frac{\lambda^2}{r} E(2, 2) \quad (2.2.3.19)$$

where

$$E(u, v) = C \sum_{j=1}^{r-v} \binom{r-v}{j} \left(\frac{\lambda}{2r}\right)^j \frac{(2(u+j))!}{2^{u+j}(j+1)!}.$$

Under some very general regularity conditions, the MMLE are fully efficient for large n (Bhattacharyya, 1985; Vaughan and Tiku, 2000).

2.2.4 Sample Mean and Sample Variance

Consider a random sample y_1, y_2, \dots, y_n from a distribution with location parameter μ and scale parameter σ . Commonly used estimators of μ and σ are sample mean, \bar{y} , and sample variance, s^2 , respectively:

$$\bar{y} = \frac{\sum_{i=1}^n y_i}{n} \quad \text{and} \quad s^2 = \frac{\sum_{i=1}^n (y_i - \bar{y})^2}{n-1}.$$

The use of these estimators for pairwise multiple comparisons or multiple comparisons with a control is called as MEAN method (Dunnett, 1982). Under normality, these estimators have very good robustness properties but not for short-tailed symmetric distribution.

CHAPTER 3

POWER COMPARISONS OF ROBUST TEST

In this chapter, robust estimators besides ordinary sample mean and sample variance (MEAN method) used in pairwise multiple comparisons and multiple comparisons with a control are compared with each other under short-tailed symmetric family of distributions. As the estimators of location with the associated variance estimates, Wave estimator with adjusting constant $h = 24$ (W24), trimmed mean with %10 trimming (TM10), the modified maximum likelihood (MML) estimator and ordinary sample mean are used in our study. These estimators are compared in terms of their efficiency and robustness properties. Power of the test statistics computed by using these estimators are also compared.

3.1 Power Comparisons of the Robust Test for Pairwise Differences of Treatment Means

3.1.1 Efficiency Comparisons

As mentioned in Chapter 2, the MML estimators are asymptotically fully efficient under some very general regularity conditions for large n , i.e., they are asymptotically unbiased and their variance is equal to Cramer-Rao minimum variance bound.

The minimum variance bounds for mean and variance are given by

$$\text{MVB}(\mu) = -1/E\left(\frac{\partial^2 \ln L}{\partial \mu^2}\right) = \frac{\sigma^2}{nD} \quad (3.1.1.1)$$

and

$$\text{MVB}(\sigma) = -1/E\left(\frac{\partial^2 \ln L}{\partial \sigma^2}\right) = \frac{\sigma^2}{nD^*}, \quad (3.1.1.2)$$

where D and D^* are given in (2.2.3.18) and (2.2.3.19), respectively.

To examine the efficiency properties of the estimators used in pairwise multiple comparisons under short-tailed symmetric distribution, the means and variances of the maximum difference $\max(\tilde{\mu}_i - \tilde{\mu}_j)$ ($i, j = 1, 2, \dots, k; i \neq j$) are simulated based on 100,000/n Monte Carlo runs.

Given in Table 3.1 are the simulated means of the maximum difference $\max(\tilde{\mu}_i - \tilde{\mu}_j)$ ($i, j = 1, 2, \dots, k; i \neq j$). Since the test statistic used for pairwise multiple comparisons is the largest order statistic, its distribution is not known. Thus, to decide whether W24, TM10 and MML estimators are unbiased or not, the simulated means of the max difference obtained by using these estimators are compared with the simulated means of the max difference obtained by using ordinary sample mean and sample variance which are known as unbiased estimators. Table 3.1 shows that most of the simulated means of $\max(\tilde{\mu}_i - \tilde{\mu}_j)$ ($i, j = 1, 2, \dots, k; i \neq j$) obtained by using the robust estimators are very close to simulated means of the one obtained by using the sample mean and sample variance and bias is negligible. Thus, the simulated variances of the maximum difference $\max(\tilde{\mu}_i - \tilde{\mu}_j)$ ($i, j = 1, 2, \dots, k; i \neq j$) are compared in this study.

Table 3.1 Means of $\max(\tilde{\mu}_i - \tilde{\mu}_j)$ ($i, j = 1, 2, \dots, k; i \neq j$), $r = 2$

		$a_1 :$	-0.5	0.0	0.5	1.0	1.5
k=2	n=10	MML	0.4706	0.4858	0.5031	0.5411	0.6714
		TM10	0.5113	0.5433	0.5905	0.6634	0.7789
		W24	0.5167	0.5495	0.5991	0.6828	0.8542
		MEAN	0.4819	0.5070	0.5430	0.5972	0.6821
	n=20	MML	0.3236	0.3316	0.3430	0.3682	0.4517
		TM10	0.3582	0.3809	0.4142	0.4651	0.5444
		W24	0.3528	0.3736	0.4040	0.4532	0.5496
		MEAN	0.3348	0.3523	0.3773	0.4148	0.4725
k=4	n=10	MML	0.8499	0.8773	0.9092	0.9800	1.2179
		TM10	0.9282	0.9864	1.0720	1.2042	1.4134
		W24	0.9363	0.9961	1.0870	1.2412	1.5568
		MEAN	0.8724	0.9181	0.9836	1.0823	1.2359
	n=20	MML	0.6000	0.6153	0.6360	0.6830	0.8420
		TM10	0.6646	0.7068	0.7688	0.8636	1.0111
		W24	0.6534	0.6922	0.7495	0.8422	1.0252
		MEAN	0.6212	0.6537	0.7001	0.7700	0.8780

Table 3.2 gives the simulated variances of the maximum difference $\max(\tilde{\mu}_i - \tilde{\mu}_j)$ ($i, j = 1, 2, \dots, k; i \neq j$). The table indicates that the variances obtained by using MML estimators are smaller than the variances obtained by using other estimators. This implies that MML estimators are more efficient than others under short-tailed symmetric distribution for pairwise multiple comparisons.

Table 3.2 Variances of $\max(\tilde{\mu}_i - \tilde{\mu}_j)$ ($i, j = 1, 2, \dots, k; i \neq j$), $r = 2$

		$a_1 :$	-0.5	0.0	0.5	1.0	1.5
k=2	n=10	MML	0.1277	0.1367	0.1477	0.1736	0.2634
		TM10	0.1519	0.1711	0.2011	0.2523	0.3455
		W24	0.1571	0.1782	0.2133	0.2815	0.4508
		MEAN	0.1345	0.1488	0.1705	0.2059	0.2671
	n=20	MML	0.0599	0.0633	0.0677	0.0785	0.1218
		TM10	0.0746	0.0843	0.0996	0.1255	0.1715
		W24	0.0719	0.0808	0.0950	0.1214	0.1892
		MEAN	0.0643	0.0712	0.0817	0.0990	0.1289
k=4	n=10	MML	0.1325	0.1435	0.1566	0.1860	0.2740
		TM10	0.1557	0.1747	0.2043	0.2546	0.3454
		W24	0.1633	0.1858	0.2239	0.3004	0.4959
		MEAN	0.1378	0.1522	0.1740	0.2095	0.2708
	n=20	MML	0.0644	0.0684	0.0735	0.0863	0.1336
		TM10	0.0788	0.0888	0.1046	0.1313	0.1799
		W24	0.0770	0.0868	0.1030	0.1347	0.2252
		MEAN	0.0685	0.0758	0.0868	0.1049	0.1365

3. 1. 2 Power Comparisons

As explained above and from Table 3.2, we see that the maximum difference $\max(\tilde{\mu}_i - \tilde{\mu}_j)$ ($i, j = 1, 2, \dots, k; i \neq j$) obtained by using MML estimators have smaller variances for various values of n and k . Thus, it is expected that the power of the test statistic which uses the MML estimators will be the highest for pairwise multiple comparisons under short-tailed symmetric distribution.

The simulated power values of the tests obtained by using W24, TM10, MML estimators besides sample mean and sample variance are given in Table 3.3 for various values of $d = \mu_i - \mu_j$ ($i, j = 1, 2, \dots, k; i \neq j$). For $d = 0$, the power reduces to Type I error which is assumed as 0.05 in this study.

Table 3.3 Power of the test for pairwise multiple comparisons, $\alpha = 0.05$,
 $n = 10$, $k = 4$ and $r = 2$

a_1	d	MML	TM10	W24	MEAN
-0.5	0.0	0.0500	0.0500	0.0500	0.0500
	0.5	0.1839	0.1232	0.1174	0.1446
	0.7	0.3244	0.1977	0.1849	0.2458
	0.9	0.4982	0.3072	0.2830	0.3946
	1.0	0.5986	0.3774	0.3469	0.4705
	1.1	0.6940	0.4468	0.4176	0.5523
	1.2	0.7794	0.5162	0.4900	0.6343
	1.3	0.8555	0.5926	0.5596	0.7123
	1.4	0.9147	0.6633	0.6319	0.7876
	1.5	0.9580	0.7296	0.7022	0.8485
	1.6	0.9852	0.7924	0.7693	0.9005
	1.7	0.9945	0.8453	0.8312	0.9378
	1.9	0.9996	0.9268	0.9155	0.9813
2.0	1.0000	0.9529	0.9474	0.9909	
0.0	0.0	0.0500	0.0500	0.0500	0.0500
	0.5	0.1692	0.1133	0.1068	0.1336
	0.7	0.2995	0.1763	0.1640	0.2223
	0.9	0.4628	0.2711	0.2452	0.3513
	1.0	0.5548	0.3278	0.2962	0.4261
	1.1	0.6510	0.3913	0.3560	0.5005
	1.2	0.7414	0.4590	0.4234	0.5779
	1.3	0.8189	0.5230	0.4900	0.6535
	1.4	0.8809	0.5932	0.5552	0.7324
	1.5	0.9358	0.6605	0.6250	0.8009
	1.6	0.9731	0.7230	0.6893	0.8554
	1.7	0.9897	0.7828	0.7537	0.9044

Table 3.3 (Continued)

a_1	d	MML	TM10	W24	MEAN
0.0	1.9	0.9989	0.8808	0.8640	0.9669
	2.0	0.9999	0.9176	0.9021	0.9806
	2.1	1.0000	0.9431	0.9329	0.9907
0.5	0.0	0.0500	0.0500	0.0500	0.0500
	0.7	0.2750	0.1512	0.1327	0.1954
	1.0	0.5101	0.2688	0.2306	0.3674
	1.1	0.6045	0.3193	0.2726	0.4363
	1.2	0.6930	0.3769	0.3214	0.5053
	1.3	0.7761	0.4394	0.3747	0.5788
	1.4	0.8466	0.4974	0.4349	0.6490
	1.5	0.9047	0.5571	0.4952	0.7209
	1.6	0.9483	0.6233	0.5568	0.7862
	1.7	0.9794	0.6828	0.6171	0.8433
	1.9	0.9974	0.7919	0.7373	0.9274
	2.0	0.9995	0.8416	0.7913	0.9565
2.2	1.0000	0.9166	0.8819	0.9863	
1.0	0.0	0.0500	0.0500	0.0500	0.0500
	0.6	0.1794	0.0998	0.0874	0.1319
	0.8	0.2955	0.1440	0.1177	0.2009
	0.9	0.3628	0.1695	0.1350	0.2426
	1.0	0.4400	0.2011	0.1577	0.2941
	1.1	0.5223	0.2368	0.1828	0.3521
	1.2	0.6040	0.2778	0.2115	0.4170
	1.4	0.7611	0.3744	0.2761	0.5421
	1.6	0.8898	0.4787	0.3594	0.6737
	1.8	0.9690	0.5879	0.4583	0.7934
	2.0	0.9944	0.6983	0.5632	0.8911
	2.2	0.9998	0.7950	0.6669	0.9519
2.4	1.0000	0.8774	0.7690	0.9838	
2.6	1.0000	0.9352	0.8530	0.9964	
1.5	0.0	0.0500	0.0500	0.0500	0.0501
	0.7	0.1773	0.0933	0.0745	0.1285
	0.9	0.2614	0.1267	0.0910	0.1836
	1.0	0.3160	0.1465	0.1013	0.2198
	1.2	0.4431	0.1940	0.1277	0.2994
	1.4	0.5753	0.2496	0.1536	0.4059
	1.6	0.7001	0.3251	0.1860	0.5214
	1.8	0.8165	0.4103	0.2249	0.6330

Table 3.3 (Continued)

a_1	d	MML	TM10	W24	MEAN
1.5	2.0	0.9078	0.4994	0.2667	0.7434
	2.2	0.9657	0.5954	0.3153	0.8404
	2.4	0.9936	0.6868	0.3699	0.9172
	2.6	0.9990	0.7764	0.4313	0.9620
	2.8	0.9999	0.8523	0.5027	0.9880
	3.0	0.9999	0.9149	0.5766	0.9975
	3.2	1.0000	0.9533	0.6534	0.9994

Table 3.3 indicates that all Type I errors are equal to $\alpha = 0.05$. For various values of $d = \mu_i - \mu_j$, power values of the test for the pairwise multiple comparisons are higher and converge to 1.0 faster when MML estimators are used in testing procedure. For the large sample sizes, it is expected that the rate of convergence will be higher. Figure 3.1 gives the graphs of the power curves for various values of parameter a_1 .

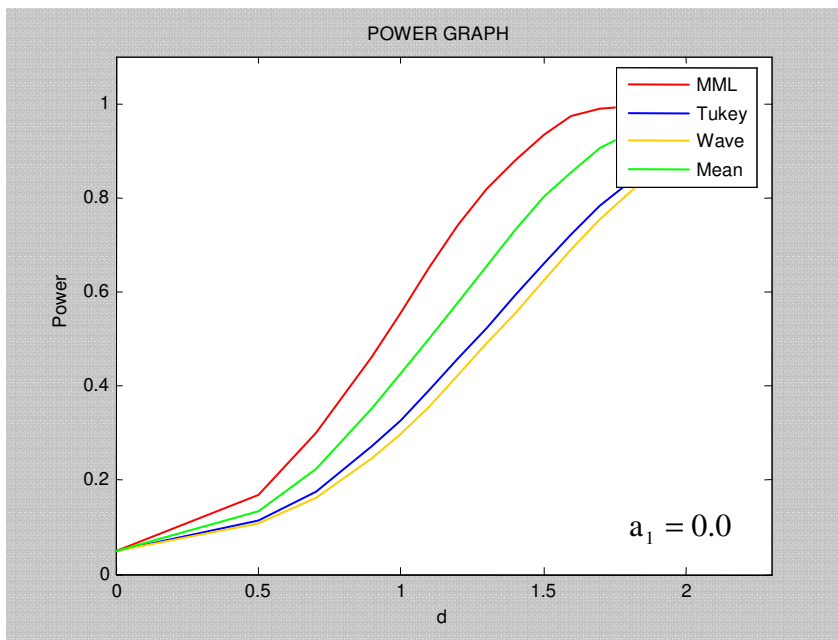
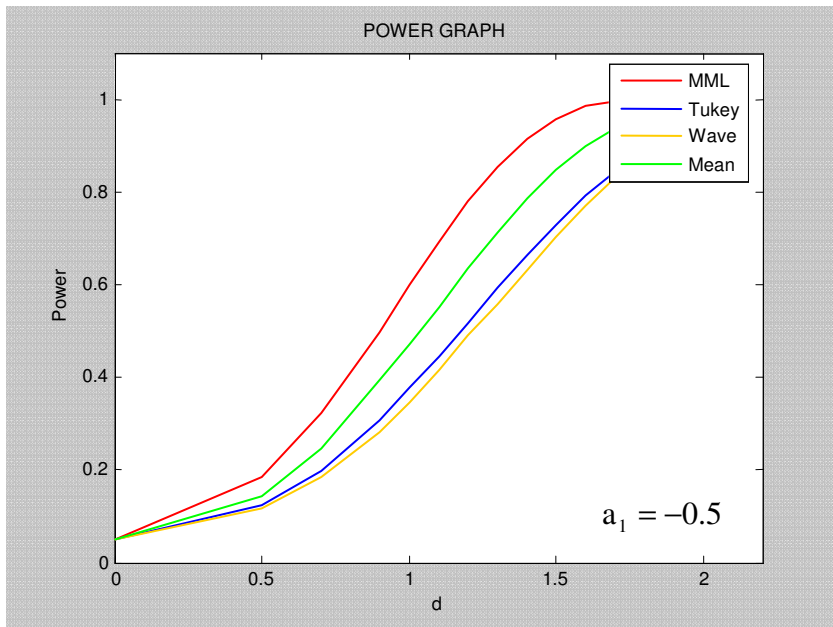


Figure 3.1 Power graphs of the test for pairwise multiple comparisons, $n = 10$, $k = 4$ and $r = 2$

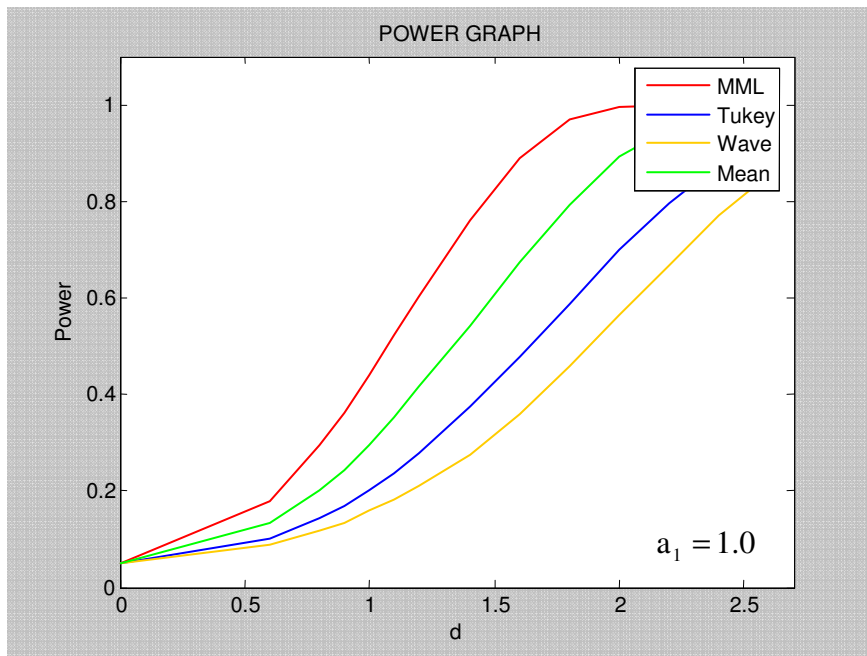
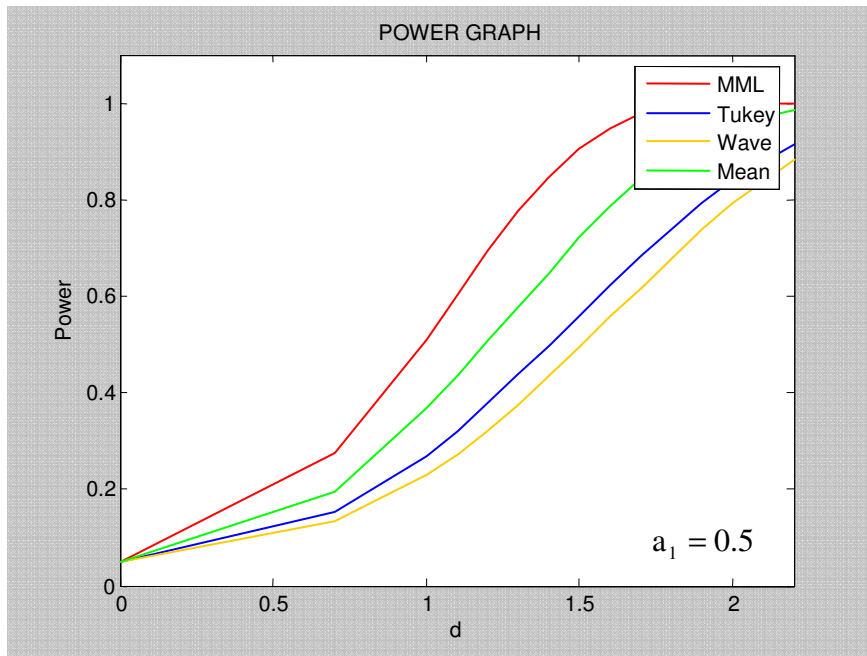


Figure 3.1 (Continued)

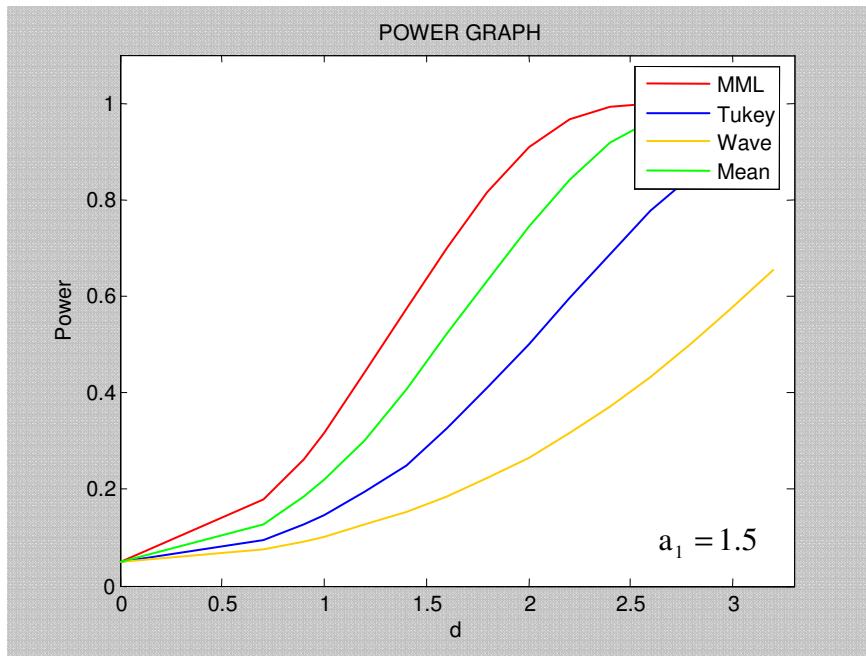


Figure 3.1 (Continued)

3.3 Robustness Comparisons

Since our aim is to obtain robust estimators for pairwise multiple comparisons under short-tailed symmetric distribution, robustness properties of the estimators are also examined: The short-tailed symmetric distribution with parameter $r=2$ and $a_1 = 0$ is taken as the assumed distribution and following symmetric short-tailed alternatives are considered:

(a) The Tukey lambda-family which is generated as

$$z = (u^l - (1-u)^l)/l \quad (-1 < z < 1), u \text{ being uniform } (0, 1). \quad (3.3.1)$$

In our study we take $l=1.45$ in which case the kurtosis $\beta_2^* = \mu_4/\mu_2^2$ is equal to 1.75 (Joiner and Rosenblatt, 1971). The mean of the distribution is zero and its variance is

$$\mu_2 = \frac{2}{l^2(1+2l)} \left[1 - \frac{1}{2} \Gamma^2(l) / \Gamma(2l) \right]. \quad (3.3.2)$$

(b) The symmetric short-tailed family introduced by Tiku and Vaughan (1999):

$$f(z) = C_1 \left\{ 1 + \frac{\lambda}{2r} z^2 \right\}^2 \varphi(z), \quad -\infty < z < \infty \quad (3.3.3)$$

where $\lambda = r/(r - a_1)$, $d < r$ and $r \geq 2$ is an integer, $\varphi(z)$ is the pdf of standard normal distribution. In this study, we take r and a_1 as 2 and 1, respectively. The mean of the distribution is zero and the variance is

$$\mu_2 = \left[\sum_{j=0}^r \binom{r}{j} \left(\frac{\lambda}{2r} \right)^j \frac{(2(j+1))!}{2^{j+1}(j+1)!} \right] / \left[\sum_{j=0}^r \binom{r}{j} \left(\frac{\lambda}{2r} \right)^j \frac{(2j)!}{2^j(j)!} \right]. \quad (3.3.4)$$

(c) The symmetric short-tailed family with cdf:

$$\begin{aligned} F(z) &= 2^{k-1} z^k, & 0 < z < 0.5 \quad (k > 1) \\ &= 1 - 2^{k-1} (1-z)^k, & 0.5 < z < 1. \end{aligned} \quad (3.3.5)$$

In our study, $k = 1.5$ (Dudewicz and Meulen, 1981) in which case the kurtosis $\beta_2^* = \mu_4/\mu_2^2$ is equal to 2.123. The mean of the distribution is 1/2 and the variance is

$$\mu_2 = 1/2(k+1)(k+2). \quad (3.3.6)$$

(d) The symmetric short-tailed family with cdf:

$$F(z) = 0.5 - 2^{k-1}(0.5 - z)^k, \quad 0 < z < 0.5 \quad (k > 1) \quad (3.3.7)$$

$$= 0.5 + 2^{k-1}(z - 0.5)^k, \quad 0.5 < z < 1$$

where k is taken as 2.0 (Dudewicz and Meulen, 1981) in this study. For this value of k , the kurtosis $\beta_2^* = \mu_4/\mu_2^2$ is equal to 1.330. The mean of the distribution is $1/2$ and the variance is

$$\mu_2 = k/4(k+2). \quad (3.3.8)$$

Random samples of size n from the alternative distributions (a)-(d) given above were generated by using $[100,000/n]$ Monte Carlo runs and assumed that they were taken from the short-tailed symmetric distribution with parameters $r=2$ and $a_1=0.0$. By computing the power of the test, we examined the sensitivity of the test to the departures from the assumed distributions.

For the alternative distribution (a), Table 3.4 gives the power values of the test for various values of $d = \mu_i - \mu_j$ ($i, j = 1, 2, \dots, k; i \neq j$). For $d = 0$, power reduces to Type I error which is assumed to be equal to 0.05. The graphs of the power values given in Table 3.4 are illustrated in Figure 3.2. The table and the corresponding figure show that the test used for pairwise multiple comparisons has the highest power values and smallest Type I error when the MML estimators are used in testing procedure. Power of the test obtained by using MML estimators converge to

1.0 faster. For the large sample sizes, the rate of convergence is higher. Thus, it is obvious that MML estimators have better robustness properties.

Table 3.4 Power values of the test under the alternative distribution (a),
 $k = 4$, $r = 2$ and $a_1 = 0.0$

	d	MML	TM10	W24	MEAN
n=10	0.00	0.0471	0.0505	0.0492	0.0521
	0.15	0.2223	0.1024	0.0802	0.1374
	0.20	0.3703	0.1511	0.1101	0.2136
	0.25	0.5614	0.2088	0.1531	0.3113
	0.30	0.7437	0.2820	0.1961	0.4322
	0.35	0.8910	0.3794	0.2644	0.5657
	0.40	0.9769	0.4613	0.3307	0.6882
	0.45	0.9984	0.5702	0.4173	0.8061
	0.50	0.9999	0.6875	0.5275	0.9082
	0.55	1.0000	0.7803	0.6333	0.9610
	0.60	1.0000	0.8666	0.7400	0.9896
	0.70	1.0000	0.9649	0.9068	0.9999
0.80	1.0000	0.9965	0.9819	1.0000	
n=20	0.00	0.0474	0.0554	0.0476	0.0482
	0.12	0.3214	0.1622	0.1454	0.1950
	0.15	0.4600	0.2092	0.1994	0.2676
	0.17	0.5846	0.2670	0.2510	0.3456
	0.20	0.7452	0.3384	0.3282	0.4518
	0.23	0.8806	0.4350	0.4296	0.5736
	0.25	0.9348	0.4954	0.5056	0.6486
	0.30	0.9982	0.6688	0.6844	0.8368
	0.35	1.0000	0.8280	0.8532	0.9530
	0.40	1.0000	0.9314	0.9554	0.9952

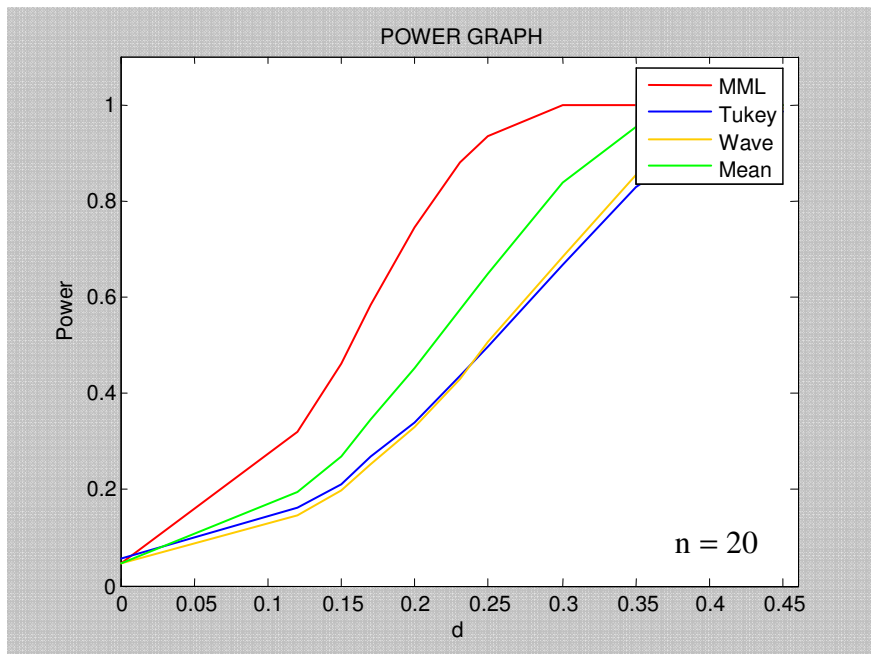
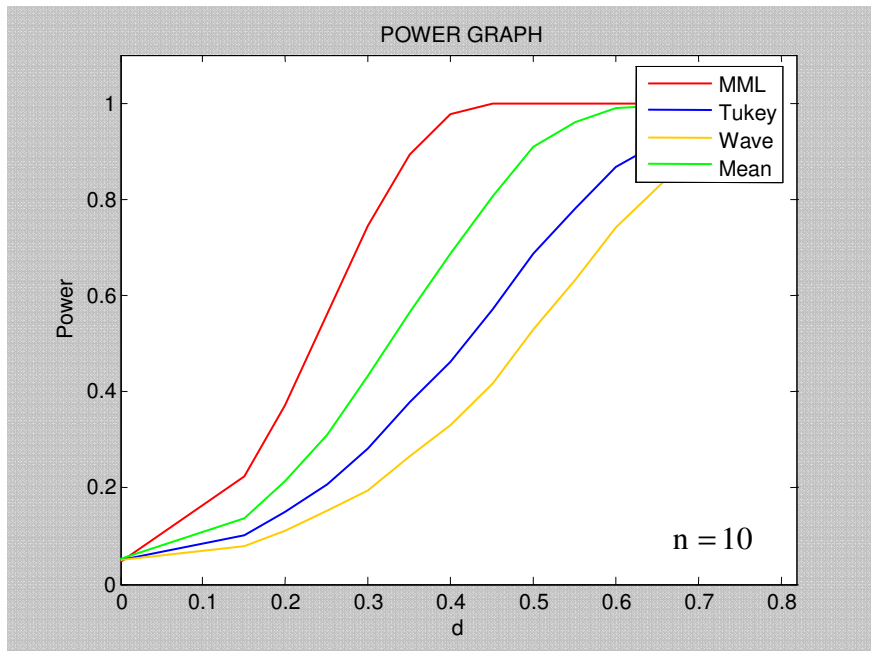


Figure 3.2 Power graphs of the test under the alternative distribution (a)

$$k = 4, r = 2 \text{ and } a_1 = 0.0$$

Table 3.5 gives the power values of the test under the alternative distribution (b) for various values of $d = \mu_i - \mu_j$ ($i, j = 1, 2, \dots, k; i \neq j$). For $d = 0$, power reduces to Type I error. Figure 3.3 shows the graphs of power values given in this table. It is seen that all Type I errors are equal to presumed $\alpha = 0.05$ and the test given for pairwise multiple comparisons has the highest power when the MML estimators are used to obtain the test statistic. In addition, the powers computed by using MML estimators converge to 1.0 faster. Hence, robustness properties of MML estimators are better than other estimators.

Table 3.5 Power values of the test under the alternative distribution (b),
 $k = 4, r = 2$ and $a_1 = 0.0$

	d	MML	TM10	W24	MEAN
n=10	0.0	0.0500	0.0500	0.0500	0.0500
	0.7	0.2528	0.1217	0.1023	0.1634
	0.9	0.3982	0.1695	0.1350	0.2426
	1.1	0.5635	0.2368	0.1828	0.3521
	1.3	0.7420	0.3215	0.2424	0.4804
	1.5	0.8790	0.4271	0.3148	0.6070
	1.7	0.9693	0.5297	0.4077	0.7363
	1.9	0.9955	0.6436	0.5111	0.8477
	2.1	0.9999	0.7436	0.6147	0.9250
	2.2	1.0000	0.7950	0.6669	0.9519
n=20	0.0	0.0500	0.0500	0.0500	0.0500
	0.5	0.2952	0.1598	0.1462	0.1984
	0.6	0.4054	0.2132	0.1984	0.2612
	0.7	0.5328	0.2702	0.2550	0.3444
	0.8	0.6650	0.3408	0.3206	0.4356
	0.9	0.7798	0.4118	0.3924	0.5334
	1.0	0.8786	0.5006	0.4790	0.6330
	1.1	0.9446	0.5844	0.5698	0.7316
	1.2	0.9812	0.6780	0.6628	0.8214

Table 3.5 (Continued)

	d	MML	TM10	W24	MEAN
n=20	1.3	0.9960	0.7596	0.7534	0.8886
	1.4	0.9998	0.8336	0.8296	0.9402
	1.5	1.0000	0.8914	0.8894	0.9714

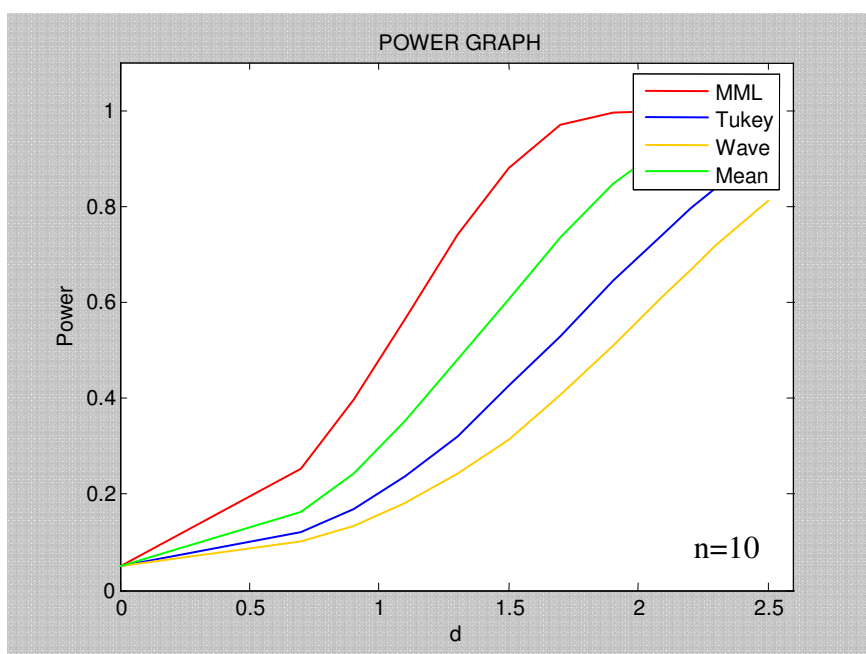


Figure 3.3 Power graphs of the test under the alternative distribution (b)

$k = 4$, $r = 2$ and $a_1 = 0.0$

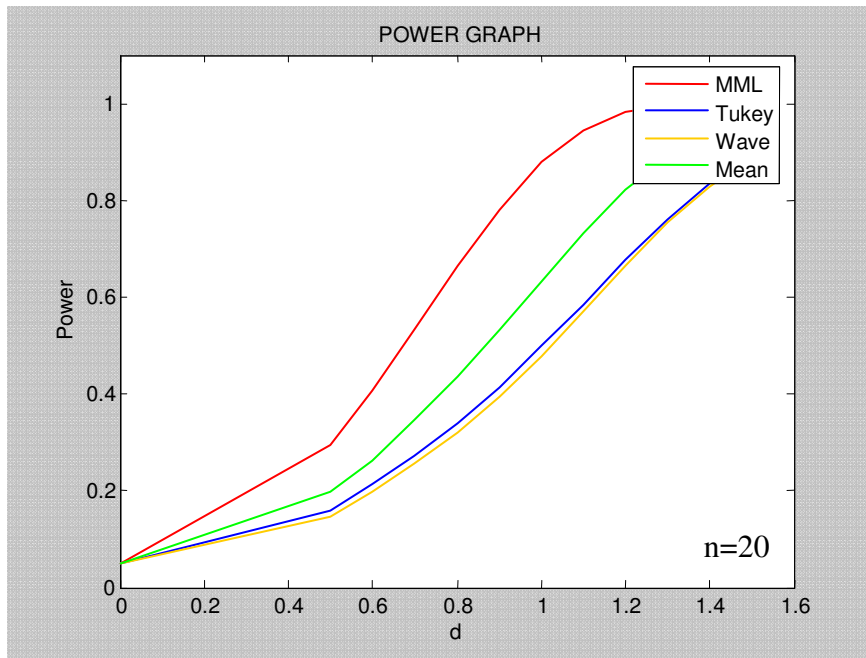


Figure 3.3 (Continued)

Given in Table 3.6 are the power values of the test under the alternative distribution (c) for various values of $d = \mu_i - \mu_j$ ($i, j = 1, 2, \dots, k; i \neq j$). The power graphs are given in Figure 3.3. The table and figure both show that the MML estimators have better robustness properties since the test used for pairwise multiple comparisons has higher power values and smaller Type I errors when the MML estimators are used in testing procedure.

Table 3.6 Power values of the test under the alternative distribution (c),

$k = 4$, $r = 2$ and $a_1 = 0.0$

	d	MML	TM10	W24	MEAN
n=10	0.00	0.0485	0.0524	0.0544	0.0493
	0.10	0.2540	0.1411	0.1326	0.1638
	0.15	0.5062	0.2532	0.2389	0.3158
	0.18	0.6851	0.3562	0.3405	0.4533
	0.21	0.8257	0.4574	0.4390	0.5827
	0.24	0.9325	0.5800	0.5655	0.7155
	0.27	0.9849	0.6932	0.6814	0.8326
	0.30	0.9983	0.7917	0.7831	0.9193
	0.33	1.0000	0.8815	0.8789	0.9670
	0.40	1.0000	0.9782	0.9822	0.9989
n=20	0.00	0.0516	0.0428	0.0554	0.0526
	0.07	0.2570	0.1408	0.1688	0.1902
	0.10	0.4582	0.2428	0.2884	0.3350
	0.11	0.5598	0.2956	0.3558	0.4138
	0.13	0.7164	0.4034	0.4808	0.5556
	0.15	0.8460	0.5166	0.5988	0.6790
	0.17	0.9448	0.6500	0.7402	0.8196
	0.19	0.9882	0.7612	0.8472	0.9130
	0.21	0.9978	0.8414	0.9126	0.9596
	0.23	0.9998	0.9184	0.9646	0.9890
	0.25	1.0000	0.9650	0.9892	0.9980

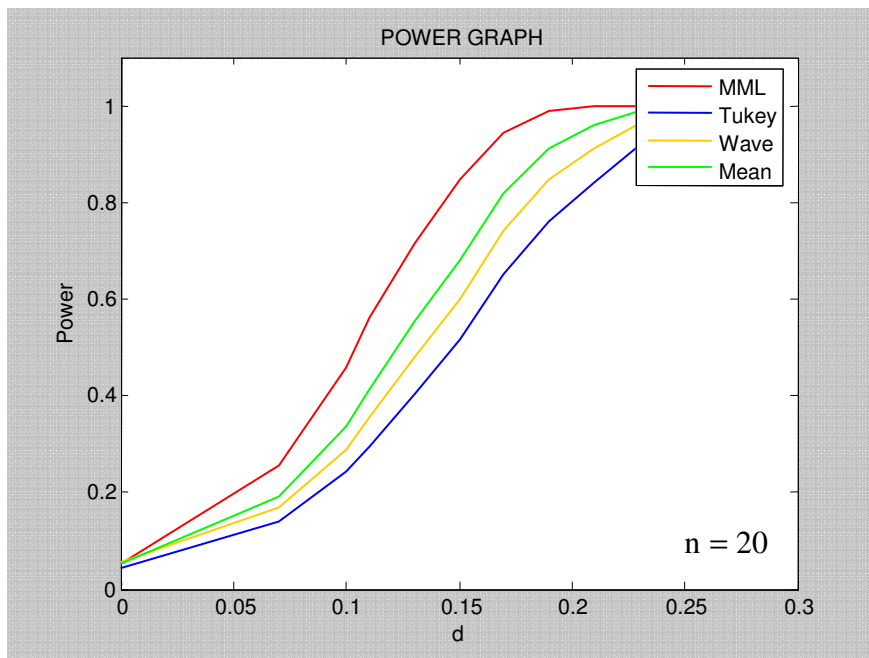
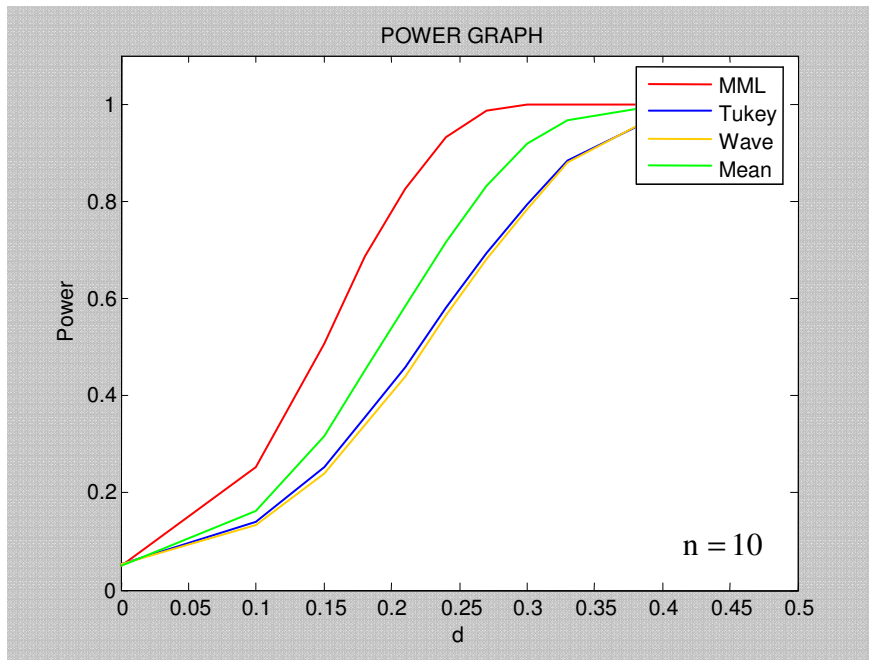


Figure 3.4 Power graphs of the test under the alternative distribution (c)

$k = 4$, $r = 2$ and $a_1 = 0.0$

Given in Table 3.7 are the power values of the test for various values of $d = \mu_i - \mu_j$ ($i, j = 1, 2, \dots, k; i \neq j$) when the distribution given by (d) is taken as the alternative distribution in pairwise multiple comparisons. Figure 3.5 shows the graphs of these power values. From the following table and Figure 3.5, it is seen that power values given for the test are higher and converge to 1.0 faster when the MML estimators are used. This implies that MML estimators have better robustness properties than the other estimators under the alternative distribution (d).

Table 3.7 Power values of the test under the alternative distribution (d),
 $k = 4, r = 2$ and $a_1 = 0.0$

	d	MML	TM10	W24	MEAN
n=10	0.00	0.0491	0.0487	0.0519	0.0473
	0.20	0.5861	0.1419	0.0649	0.2397
	0.25	0.8591	0.1907	0.0670	0.3429
	0.30	0.9897	0.2774	0.0890	0.4916
	0.35	0.9997	0.3708	0.1016	0.6436
	0.40	1.0000	0.4853	0.1294	0.7855
	0.45	1.0000	0.5964	0.1625	0.9040
	0.50	1.0000	0.7210	0.2001	0.9717
	0.55	1.0000	0.8294	0.2453	0.9956
	0.60	1.0000	0.9146	0.2812	0.9998
	0.70	1.0000	0.9915	0.3584	1.0000
	0.80	1.0000	0.9999	0.4468	1.0000
	0.90	1.0000	1.0000	0.5401	1.0000
	1.00	1.0000	1.0000	0.6637	1.0000
1.10	1.0000	1.0000	0.8157	1.0000	
1.20	1.0000	1.0000	0.9390	1.0000	

Table 3.7 (Continued)

	d	MML	TM10	W24	MEAN
n=20	0.00	0.0494	0.0528	0.0566	0.0574
	0.11	0.5262	0.1430	0.0752	0.2142
	0.15	0.8888	0.2424	0.1018	0.3664
	0.17	0.9750	0.2930	0.1136	0.4456
	0.20	0.9990	0.3988	0.1466	0.5866
	0.22	1.0000	0.4464	0.1532	0.6680
	0.25	1.0000	0.5644	0.1908	0.8008
	0.28	1.0000	0.6950	0.2296	0.9078
	0.30	1.0000	0.7624	0.2788	0.9526
	0.33	1.0000	0.8430	0.3414	0.9888
	0.35	1.0000	0.9080	0.3844	0.9978
	0.40	1.0000	0.9786	0.5322	0.9998
	0.45	1.0000	0.9990	0.7288	1.0000

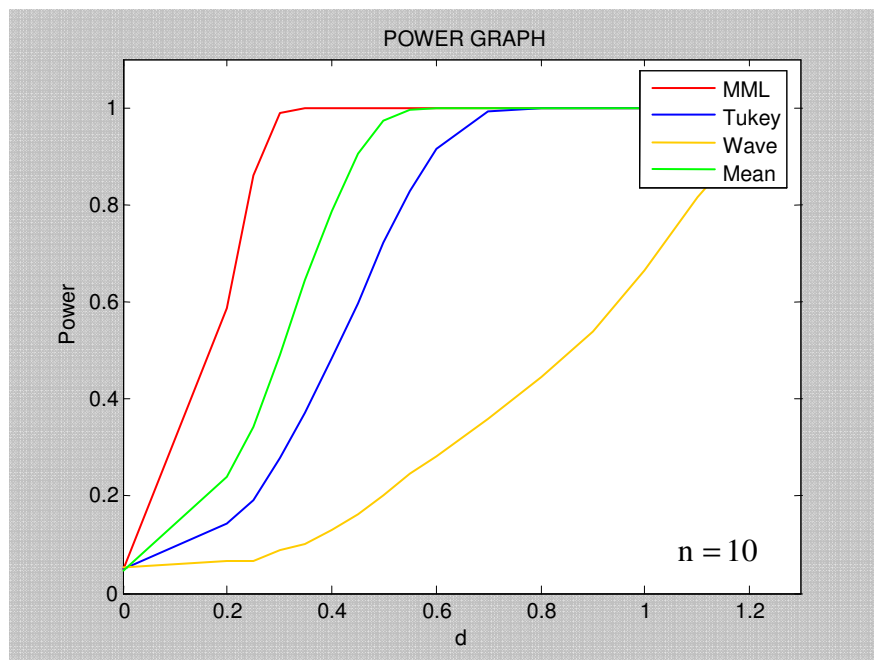


Figure 3.5 Power graphs of the test under the alternative distribution (d)

$k = 4, r = 2$ and $a_1 = 0.0$

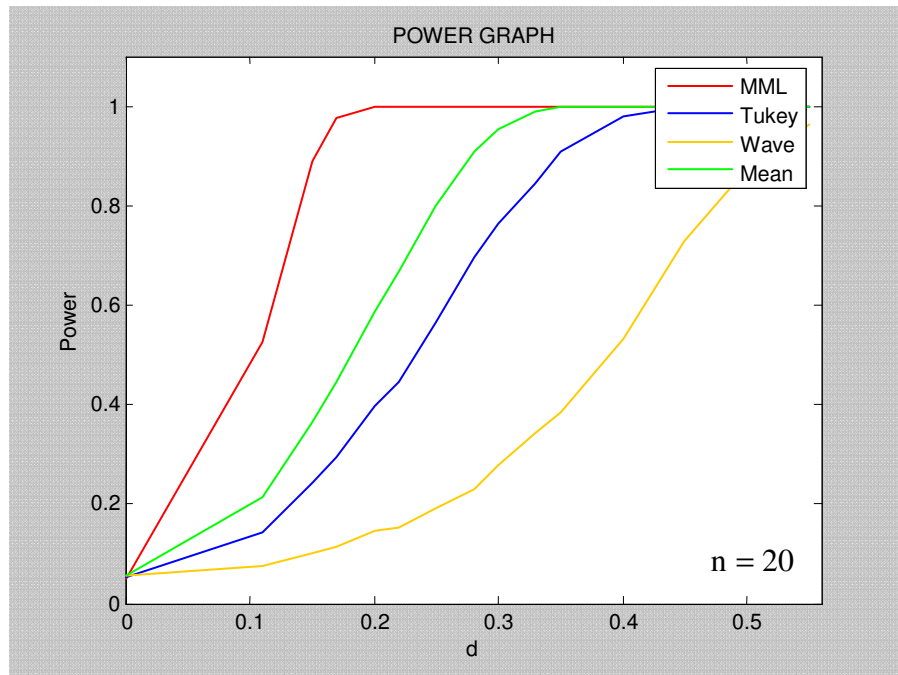


Figure 3.5 (Continued)

3.2 Power Comparisons of Robust Tests for the Pairwise Differences Between the Control and Treatment Means

3.2.1 Power Comparisons

To examine the power of the test obtained by using W24, TM10 and MML estimators besides the sample mean and sample variance, the procedure developed for pairwise multiple comparisons is followed for multiple comparisons with a control.

100,000/n Monte Carlo runs are used to generate random samples from short-tailed symmetric distribution. The power values of the test for multiple comparisons with

a control is given in Table 3.8 for various values of $d = \mu_i - \mu_k$ ($i = 1, 2, \dots, k - 1$) and graphs of power values are shown in Figure 3.6. For $d = 0$, power reduces to Type I error. Table 3.8 indicates that all Type I errors obtained by using the W24, TM10 and MML estimators in addition to sample mean and sample variance are equal to presumed $\alpha = 0.05$. Figure 3.6 shows that the power values given for the test are higher and converge to 1.0 faster when the MML estimators are used in testing procedures. For the large sample sizes, it is expected that the rate of convergence will be higher.

Table 3.8 Power of the test for multiple comparisons with a control,
 $n = 10, k = 4, r = 2$ and $\alpha = 0.05$

d	MML	TM10	W24	MEAN
0.0	0.0500	0.0500	0.0500	0.0500
0.5	0.3230	0.0815	0.1109	0.1645
0.6	0.4398	0.1213	0.1573	0.2442
0.7	0.5676	0.1823	0.2165	0.3334
0.8	0.6911	0.2486	0.2930	0.4409
0.9	0.7961	0.3367	0.3810	0.5571
1.0	0.8721	0.4328	0.4810	0.6661
1.1	0.9268	0.5396	0.5833	0.7585
1.2	0.9641	0.6406	0.6836	0.8460
1.3	0.9857	0.7330	0.7670	0.9108
1.4	0.9942	0.8154	0.8478	0.9509
1.5	0.9981	0.8821	0.9083	0.9749
1.6	0.9994	0.9336	0.9493	0.9889
1.9	1.0000	0.9910	0.9938	0.9997

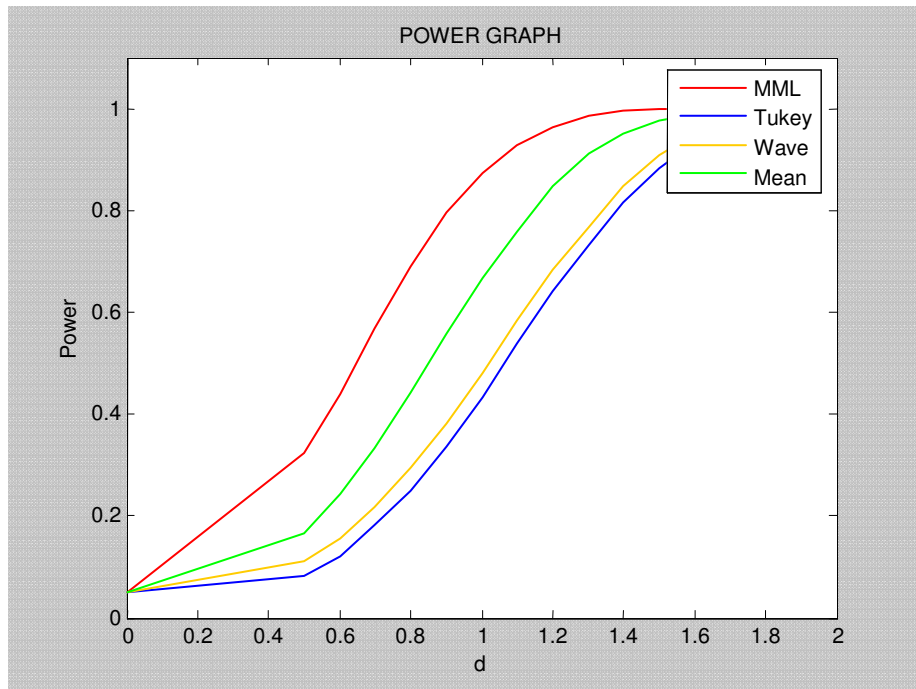


Figure 3.6 Power graph for multiple comparisons with a control, $n = 10$, $k = 4$, $r = 2$ and $a_1 = 0.0$

3. 2. 2 Robustness against Inlier Model

To examine the robustness properties of W24, TM10 and MML estimators besides the sample mean and sample variance for multiple comparisons with a control under short-tailed symmetric distribution, an inlier model is used as the alternative distribution.

Inliers are erroneous observations which are located closer to the mean opposite to outliers. To generate inliers, Akkaya and Tiku (2005) proposed a mechanism which replace a proportion of smallest or largest order statistics in a random sample

y_1, y_2, \dots, y_n from $N(\mu, \sigma^2)$ by $\bar{y} + (-\delta)^i \sigma$, $|\delta| \leq 1$. Thus, the displaced observations get located closer to \bar{y} and are erroneous, and located within $\bar{y} \pm \sigma$.

To preserve the symmetry and make the resulting sample have kurtosis less than 3, equal proportion r/n of smallest and largest order statistics are replaced.

Thus, random samples of size n are generated from the inlier model given above. For the control group, the random sample from the short-tailed symmetric distribution with parameters $r = 2$ and $a_1 = 0.0$, are generated.

Table 3.9 gives the power values of the test for various values of $d = \mu_i - \mu_k$ ($i = 1, 2, \dots, k - 1$). Figure 3.7 illustrates the graphs of power values in Table 3.9.

Similarly, it is clear that the test used for multiple comparisons with a control has the highest power values and smallest Type I error when the MML estimators are used in testing procedure. Also, powers obtained by using MML estimators converge to 1.0 faster. Thus, under inlier model the MML estimators have better robustness properties than other estimators

Table 3.9 Power of the test for multiple comparisons with a control against inliers, $n = 10$, $k = 4$, $r = 2$ and $a_1 = 0.0$

d	MML	TM10	W24	MEAN
0.0	0.0475	0.0506	0.0529	0.0487
0.4	0.3145	0.0688	0.1443	0.1779
0.5	0.4502	0.1243	0.2381	0.2824
0.6	0.5960	0.2086	0.3657	0.4239
0.7	0.7390	0.3279	0.5133	0.5795
0.8	0.8537	0.4844	0.6744	0.7302
0.9	0.9181	0.6261	0.7973	0.8403
1.0	0.9673	0.7679	0.8881	0.9201
1.1	0.9891	0.8751	0.9539	0.9709
1.2	0.9972	0.9382	0.9823	0.9899
1.3	0.9996	0.9779	0.9958	0.9977
1.5	1.0000	0.9982	0.9997	0.9999

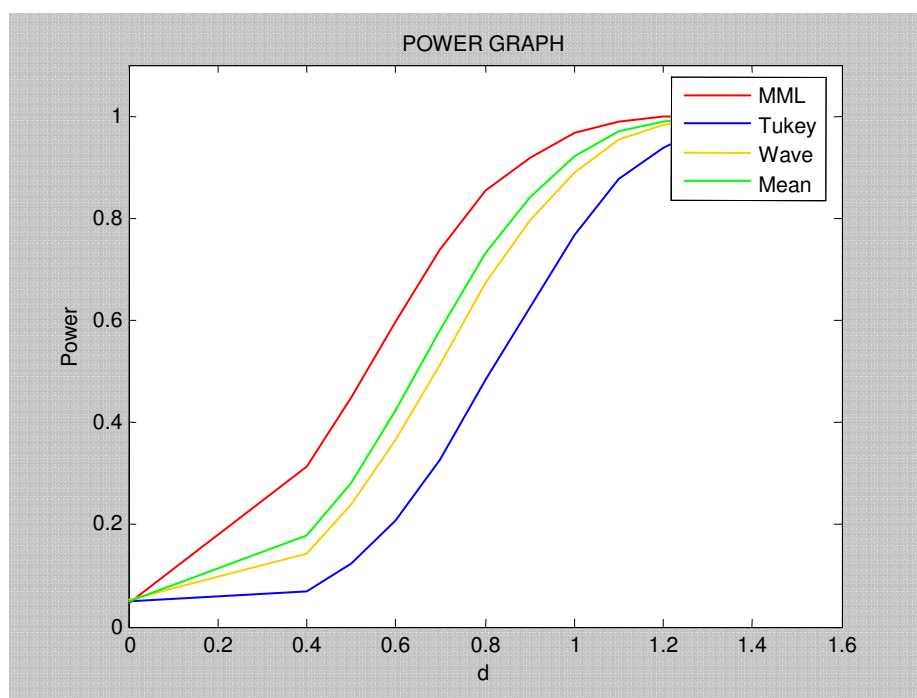


Figure 3.7 Power graph for multiple comparisons with a control against inliers, $n = 10$, $k = 4$, $r = 2$ and $a_1 = 0.0$

CHAPTER 4

APPLICATIONS AND CONCLUSIONS

4.1 Applications

Example 4.1: Draper and Smith (1981) gives the number of finger taps per minute for the male college students receiving different doses of caffeine (0, 100 and 200 ml.). This data is reproduced in Hand et al. (1994, p. 40):

0 ml caffeine (Treatment I)	100 ml caffeine (Treatment II)	200 ml caffeine (Treatment III)
242	248	246
245	246	248
244	245	250
248	247	252
247	248	248
248	250	250
242	247	246
244	246	248
246	243	245
242	244	250

Since our aim is to compare treatment means, we must determine the plausible distribution. To locate the plausible distribution, the Q-Q plots and kurtosis values of the treatments are used. Kurtosis values of treatment I, II, and III are 1.5791, 2.3874 and 1.9902, respectively. These values are smaller than 3 which implies that these treatments may come from short-tailed symmetric distribution with parameters $r = 4$ and $a_1 = 1.5$. The Q-Q plots of treatment I, treatment II and

treatment III for short-tailed symmetric distribution are given in Figure 4.1, 4.2 and 4.3, respectively.

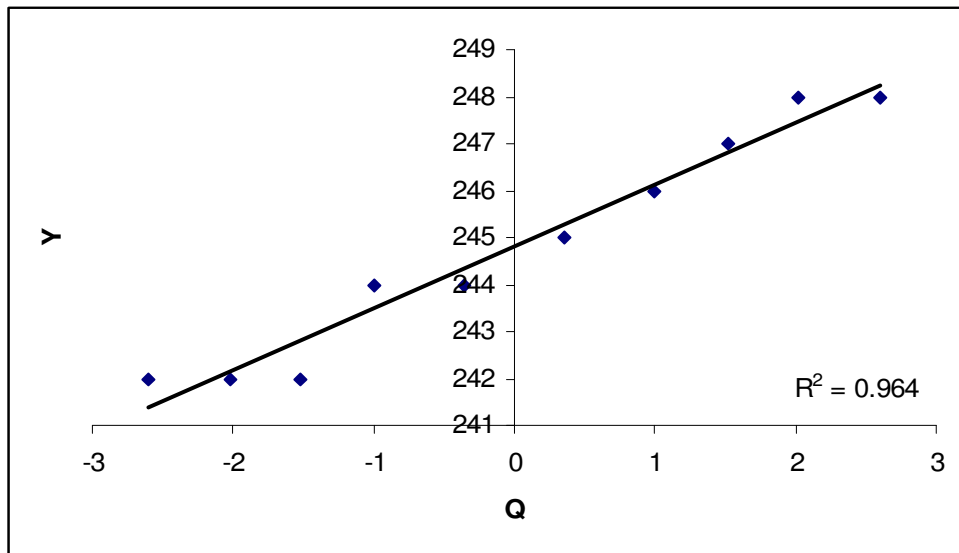


Figure 4.1 Q-Q Plot of 0 ml Caffeine

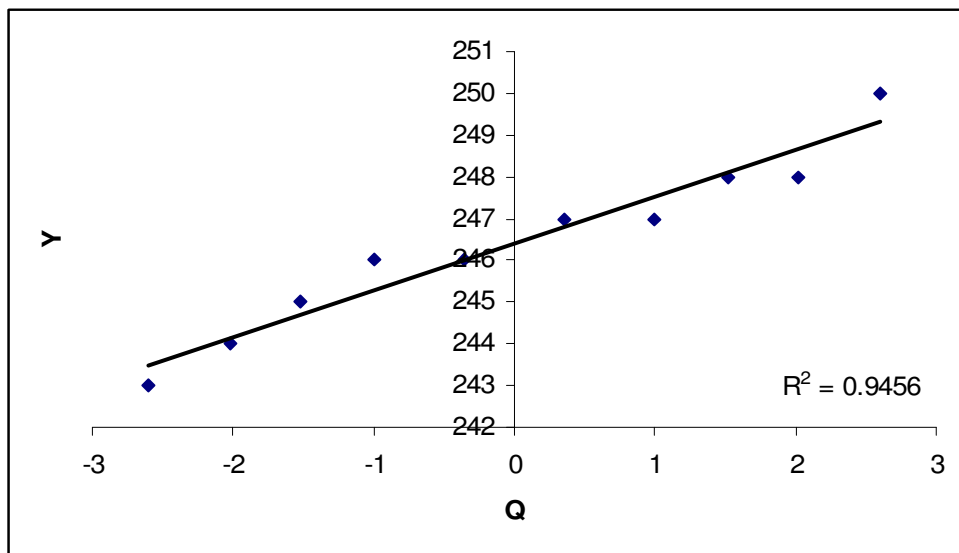


Figure 4.2 Q-Q Plot of 100 ml Caffeine

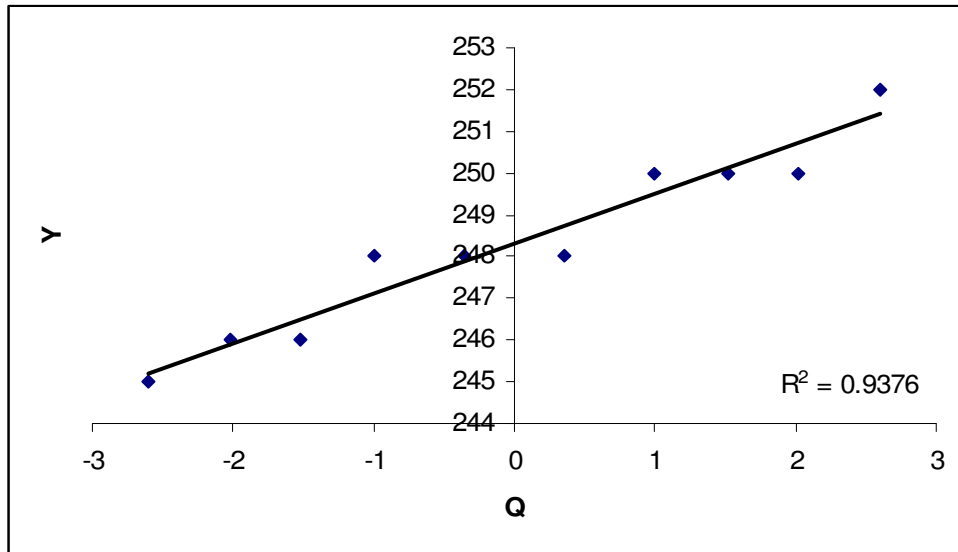


Figure 4.3 Q-Q Plot of 200 ml Caffeine

Figure 4.1, 4.2 and 4.3 show that these data sets have approximately symmetric distribution with short tails.

The estimates and their standard errors are given as follows:

MML	TM10	W24	MEAN
$\hat{\mu}_1 = 244.872$	$\hat{\mu}_1 = 244.750$	$\hat{\mu}_1 = 244.795$	$\hat{\mu}_1 = 244.800$
$SE(\hat{\mu}_1) = \pm 0.426$	$SE(\hat{\mu}_1) = \pm 1.159$	$SE(\hat{\mu}_1) = \pm 0.749$	$SE(\hat{\mu}_1) = \pm 0.757$
$\hat{\mu}_2 = 246.356$	$\hat{\mu}_2 = 246.375$	$\hat{\mu}_2 = 246.402$	$\hat{\mu}_2 = 246.400$
$SE(\hat{\mu}_2) = \pm 0.394$	$SE(\hat{\mu}_2) = \pm 0.759$	$SE(\hat{\mu}_2) = \pm 0.641$	$SE(\hat{\mu}_2) = \pm 0.653$
$\hat{\mu}_3 = 248.279$	$\hat{\mu}_3 = 248.250$	$\hat{\mu}_3 = 248.296$	$\hat{\mu}_3 = 248.300$
$SE(\hat{\mu}_3) = \pm 0.409$	$SE(\hat{\mu}_3) = \pm 0.848$	$SE(\hat{\mu}_3) = \pm 0.693$	$SE(\hat{\mu}_3) = \pm 0.699$

To test the null hypothesis

$$H_0 : \mu_i = \mu_j$$

$$H_1 : \mu_i \neq \mu_j$$

the test statistic is calculated by using the following robust estimators:

	MML	TM10	W24	MEAN
$\max \tilde{t}_{ij} :$	5.800	2.436	3.431	3.395

Since the distribution of the test statistic is not known, we simulated the p-values to compare the robust estimators. They are given as follows.

	MML	TM10	W24	MEAN
p-value:	0.0067	0.0372	0.0479	0.0110

The p-values obtained by using the robust estimators are less than $\alpha = 0.05$. Thus, we reject the null hypothesis H_0 . However, the p-value obtained by using MML estimators is smaller which implies that MML estimators are more efficient.

Example 4.2 Till (1974) gives sets of salinity values (parts per thousand) for three separate water masses. The data reproduced in Hand et al. (1994, p. 201):

Treatment I	Treatment II	Treatment III
37.54	40.14	39.04
37.01	40.80	39.21
36.71	39.76	39.05
37.03	39.70	38.24
37.32	40.79	38.53
37.01	40.44	38.71
37.03	39.79	38.89
37.70	39.38	38.66
37.36		38.51
36.75		40.08
37.45		
38.85		

The Q-Q plots of treatment I, II and III for normal distribution are given in Figure 4.4, 4.5 and 4.6, respectively.

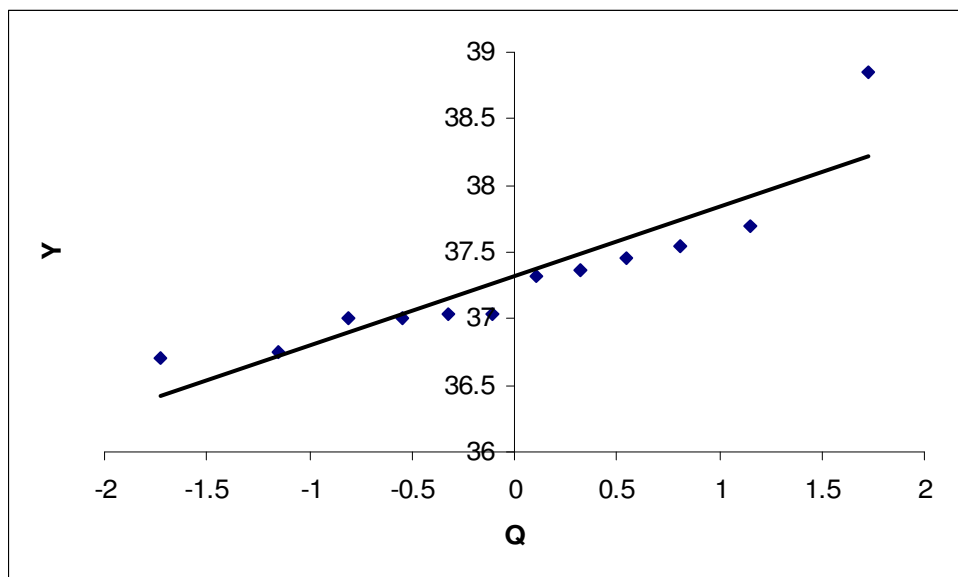


Figure 4.4 Q-Q Plot of Treatment I

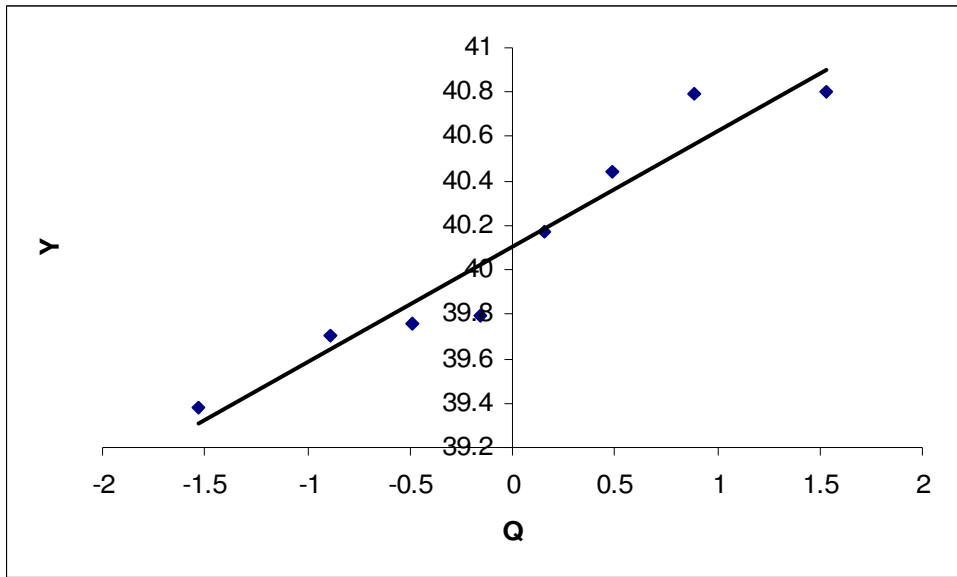


Figure 4.5 Q-Q Plot of Treatment II

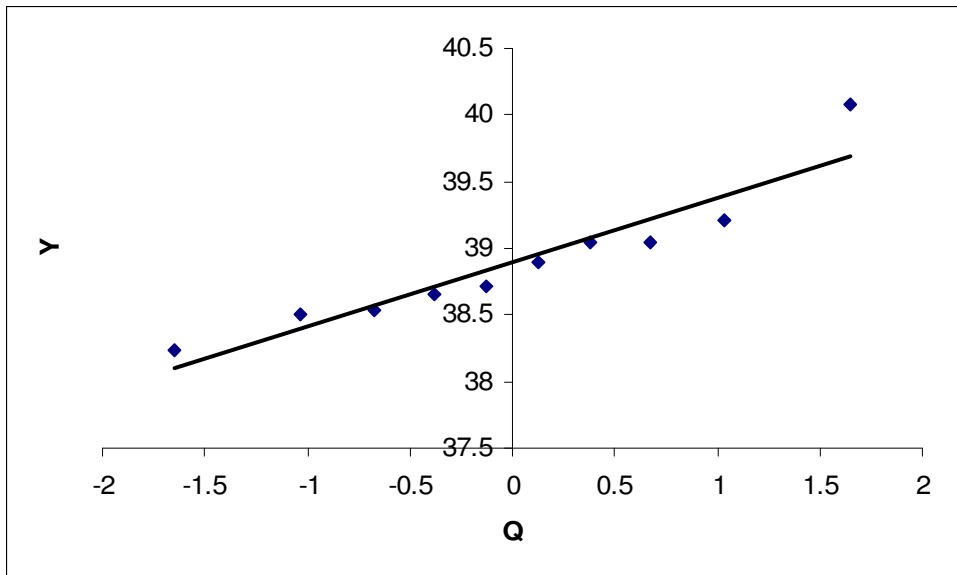


Figure 4.6 Q-Q Plot of Treatment III

The Q-Q plots of the three data sets show that the observations 38.85 in treatment I and 40.08 in treatment II are outliers and thus, these observations are eliminated.

Kurtosis values of treatment I, II, and III were computed as 1.8860, 1.6262 and 1.9877, respectively. Q-Q plots and kurtosis values show that these data sets have short-tailed symmetric distribution with parameters $r = 4$ and $a_1 = 2$.

The estimates and their standard errors are as follows:

MML	TM10	W24	MEAN
$\hat{\mu}_1 = 37.189$	$\hat{\mu}_1 = 37.167$	$\hat{\mu}_1 = 37.172$	$\hat{\mu}_1 = 37.174$
$SE(\hat{\mu}_1) = \pm 0.059$	$SE(\hat{\mu}_1) = \pm 0.116$	$SE(\hat{\mu}_1) = \pm 0.105$	$SE(\hat{\mu}_1) = \pm 0.097$
$\hat{\mu}_2 = 40.134$	$\hat{\mu}_2 = 40.107$	$\hat{\mu}_2 = 40.098$	$\hat{\mu}_2 = 40.104$
$SE(\hat{\mu}_2) = \pm 0.112$	$SE(\hat{\mu}_2) = \pm 0.254$	$SE(\hat{\mu}_2) = \pm 0.193$	$SE(\hat{\mu}_2) = \pm 0.188$
$\hat{\mu}_3 = 38.764$	$\hat{\mu}_3 = 38.769$	$\hat{\mu}_3 = 38.763$	$\hat{\mu}_3 = 38.760$
$SE(\hat{\mu}_3) = \pm 0.064$	$SE(\hat{\mu}_3) = \pm 0.139$	$SE(\hat{\mu}_3) = \pm 0.107$	$SE(\hat{\mu}_3) = \pm 0.104$

To test the null hypothesis

$$H_0 : \mu_i = \mu_j$$

$$H_1 : \mu_i \neq \mu_j$$

the test statistic is calculated by using the following robust estimators:

	MML	TM10	W24	MEAN
$\max \tilde{t}_{ij} :$	23.228	10.288	13.318	13.873

Since the distribution of the test statistic is not known, the simulated p-values are given as follows:

	MML	TM10	W24	MEAN
p-value:	0.0	0.0	0.0001	0.0

The p-values obtained by using the robust estimators are less than $\alpha = 0.05$. Thus, we reject the null hypothesis H_0 .

4.2 Summary and Conclusions

The theoretical background for the pairwise multiple comparisons and multiple comparisons with a control are given and the treatment means of the one-way fixed effect analysis of variance model are estimated by considering several estimation methods.

By using robust estimators of location and scale parameters which are the modified maximum likelihood estimators, trimmed mean with winsorized standard deviation, wave estimators and sample mean and sample variance, a test statistic is obtained for pairwise multiple comparisons and multiple comparisons with a control under short-tailed symmetric distribution.

A simulation study is carried out to compare the efficiency of the related estimators and power and robustness properties of the test statistics are examined.

On the basis of this study, the following conclusions can be stated:

- 1) MML estimators are more efficient and the power of the test statistic for pairwise multiple comparisons and multiple comparisons with a control are higher when the MML estimators are used in the test procedure.
- 2) The power values obtained by using MML estimators converge to 1.0 faster.
- 3) For the large sample sizes, the rate of convergence is higher.
- 4) MML estimators are asymptotically MVB (minimum variance bound) since $\frac{\partial \ln L}{\partial \theta}$ is asymptotically equivalent to $\frac{\partial \ln L^*}{\partial \theta}$ where θ is any parameter.
- 5) MML estimators are robust.
- 6) In general, mean method gives the second best estimator among all others.

REFERENCES

Akkaya, A. D. and Tiku, M. L. (2005). Robust estimation and hypothesis testing under short-tailedness and inliers. *Test*, 14 (1), 129-150.

Akkaya, A. D. and Tiku, M. L. (2006). Short-tailed distributions and inliers. *Test* (in press).

Andrews, D. F., Bickel, P. J., Hampel, F. R., Huber, P. J., Rogers, W. H., and Tukey, J. W. (1972). *Robust Estimates of Location*. Princeton University Press: Princeton.

Andrews, D. F. (1974). A robust method for multiple linear regression. *Technometrics*, 16, 523-531.

Beaton, A. E. and Tukey, J. W. (1974). The fitting of power series, meaning polynomials, illustrated on band-spectroscopic data. *Technometrics*, 16, 147-186.

Bhattacharyya, G. K. (1985). The asymptotics of maximum likelihood and related estimators based on Type II censored data. *J. Amer. Statist. Assoc.*, 80, 398-404.

Birch, J. B. and Myers, R. H. (1982). Robust analysis of covariance. *Biometrics*, 38, 699-713.

Draper, N. R. and Smith, H. (1981). *Applied Regression Analysis*. Second edition. Wiley: New York.

Dudewicz, E. and Meulen, E. V. (1981). Entropy based tests of uniformity. *J. Amer. Statist. Assoc.*, 76, 967-974.

Duncan, D. B. (1955). Multiple range and multiple F tests. *Biometrics*, 11, 1-42.

Duncan, D. B. (1957). Multiple range tests for correlated and heteroscedastic means. *Biometrics*, 13, 164-176.

Dunnett, C. W. (1964). New tables for multiple comparisons with a control. *Biometrics*, 20, 482-491.

Dunnett, C. W. (1980 a). Pairwise multiple comparisons in the homogeneous variance, unequal sample size case. *J. Amer. Statist. Assoc.*, 75, 789-795.

Dunnett, C. W. (1980 b). Pairwise multiple comparisons in the unequal variance case. *J. Amer. Statist. Assoc.*, 75, 796-800.

Dunnett, C. W. (1982). Robust multiple comparisons. *Commun. Statist. –Theor. Meth.*, 11 (22), 2611-2629

Gross, A. M. (1976). Confidence interval robustness with long tailed symmetric distributions. *J. Amer. Statist. Assoc.*, 71, 409-416.

Gross, A. M. (1977). Confidence intervals for bisquare regression estimates. *J. Amer. Statist. Assoc.*, 72, 341-354.

Hampel, F. R. (1974). The influence curve and its role in robust estimation. *J. Amer. Statist. Assoc.*, 62, 1179-1186.

Hand, D. J., Daly, F., Lunn, A. D., McConway, K. J. And Ostrowski, E. (1994). *Small Data Sets*. Chapman & Hall: New York.

Huber, P. J. (1964). Robust estimation of a location parameter. *Ann. Math. Statist.*, 35, 73-101.

Huber, P. J. (1977). *Robust Statistical Procedures*. Regional Conference Series in Applied Mathematics, 27. Soc. Industr. Appl. Math., Philadelphia.

Huber, P. J. (1981). *Robust Statistics*. Wiley: New York.

Joiner, B. L. and Rosenblatt, J. R. (1971). Some properties of the range in samples from Tukey's symmetric lambda distributions. *J. Amer. Statist. Assoc.*, 66, 394-399.

Keuls, M. (1952). The use of the studentized range in connection with an analysis of variance. *Euphytica*, 1, 112-122.

Kramer, C. Y. (1956). Extension of multiple range tests to group means with unequal numbers of replications. *Biometrics*, 13, 13-18.

Low, B. B. (1959). *Mathematics*. Neill and Co: Edinburgh.

Montgomery, D. C. (1976). *Design and Analysis of Experiments* (1984) Second Edition. John Wiley and Sons. Inc.

Neter, J., Wasserman, W., and Kutner, M. H. (1985). *Applied Statistical Models*. Richard D. Irwin, Inc.

Newman, D. (1939). The distribution of the range in samples from a normal population, expressed in terms of an independent estimate of standard deviation. *Biometrika*, 31, 20-30.

Pearson, E. S. and Tiku, M. L. (1970). Some notes on the relationship between the distributions of central and non-central F. *Biometrika* 57, 175-179.

Tamhane, A. C. (1979). A comparison of procedures for multiple comparisons. *J. Amer. Statist. Assoc.*, 74, 471-480.

Tiku, M. L. (1967). Estimating the mean and standard deviation from censored normal samples. *Biometrika*, 54, 155-165.

Tiku, M. L. (1970). Monte Carlo study of some simple estimators in censored normal samples. *Biometrika*, 57, 207-211.

Tiku, M. L., Tan, W. Y., and Balakrishnan, N. (1986). *Robust Inference*. Marcel Dekker: New York.

Tiku, M. L. and Suresh, R. P. (1992). A new method of estimation for location and scale parameters. *J. Stat. Plan. Inf.*, 30, 281-292.

Tiku, M. L. and Vaughan, D. C. (1999). A family of short-tailed symmetric distributions. Technical Report: McMaster University: Canada.

Tiku, M. L. and Akkaya, A. D. (2004). *Robust Estimation and Hypothesis Testing*. New Age International Publishers (Wiley Eastern): New Delhi.

Till, R. (1974). *Statistical Methods For the Earth Scientist*. MacMillan: London.

Tukey, J. W. (1953). *The Problem of Multiple Comparisons*. Princeton University, Department of Mathematics: Princeton.

Tukey, J. W. and McLaughlin, D. H. (1963). Less vulnerable confidence and significance procedures for location based on a single sample: Trimming/Winsorization I. *Sankhya*, Ser. A, 25, 331-352.

Vaughan, D. C. (1992). On the Tiku-Suresh method of estimation. *Commun. Stat. Theory Meth.*, 21, 451-469.

Vaughan, D. C. and Tiku, M. L. (2000). Estimation and hypothesis testing for a non-normal bivariate distribution with applications. *Journal of Mathematical and Computer Modelling*, 32, 53-67.

APPENDIX A

SIMULATED CRITICAL VALUES

Table A.1 Simulated critical values to obtain the power of the test for pairwise multiple comparisons, $n = 10$, $k = 4$, $r = 2$ and $\alpha = 0.05$

a_1	MML	TM10	W24	MEAN
-0.5	2.5631	2.5432	3.381	2.8855
0.0	2.6086	2.5534	3.4273	2.8916
0.5	2.7199	2.5749	3.5654	2.9009
1.0	3.1586	2.6114	3.8667	2.9154
1.5	4.8352	2.6305	4.6949	2.9446

Table A.2 Simulated critical values to obtain power values of the test under the alternative distribution (a), $k = 4$, $r = 2$, $a_1 = 0.0$ and $\alpha = 0.05$

n	MML	TM10	W24	MEAN
10	0.6418	2.5998	3.9637	2.9069
20	0.6084	2.3656	2.9830	2.7151

Table A.3 Simulated critical values to obtain power values of the test under the alternative distribution (b), $k = 4$, $r = 2$, $a_1 = 0.0$ and $\alpha = 0.05$

n	MML	TM10	W24	MEAN
10	2.7941	2.6114	3.8667	2.9154
20	2.6554	2.4069	3.0148	2.7252

Table A.4 Simulated critical values to obtain Power values of the test under the alternative distribution (c), $k = 4$, $r = 2$, $a_1 = 0.0$ and $\alpha = 0.05$

n	MML	TM10	W24	MEAN
10	0.4184	2.5629	3.4210	2.9448
20	0.4105	2.4687	2.8560	2.7142

Table A.4 Simulated critical values to obtain power values of the test under the alternative distribution (d), $k = 4$, $r = 2$, $a_1 = 0.0$ and $\alpha = 0.05$

n	MML	TM10	W24	MEAN
10	0.4623	2.6743	7.2285	3.0057
20	0.3875	2.3820	4.3895	2.6860

Table A.5 Simulated critical values to obtain the power of the test for multiple comparisons with a control, $n = 10$, $k = 4$, $r = 2$, $a_1 = 0.0$, $\alpha = 0.05$

MML	TM10	W24	MEAN
1.7092	1.4384	2.0018	1.7244

Table A.6 Simulated critical values to obtain the power of the test for multiple comparisons with a control against inliers, $n = 10$, $k = 4$, $r = 2$, $a_1 = 0.0$ and $\alpha = 0.05$

MML	TM10	W24	MEAN
1.4343	1.2151	1.5860	1.5041

APPENDIX B

COMPUTER PROGRAM FOR PAIRWISE MULTIPLE COMPARISONS

USE NUMERICAL_LIBRARIES

dimension alf(600),bet(600),e(600)

dimension w(600),v(600),w1(600),xa(600),ra(600),ae(600)

dimension y1(600),bet1(600),t(600),samp(100,100),y(100,100)

dimension sum1(100),yvar(100),sammean(100),varsam(100),yt(100)

dimension ybar(100),sigma(100),tt(100,100),ts(10000)

dimension z(100,100),yo(100,100)

dimension amu_mml(100),amu_wave(100),amu_tukey(100)

dimension s_square(100),sigma_square_mml(100)

dimension sigma_square_wave(100),sigma_square_tukey(100)

dimension tt1(100),tt2(100),tt3(100),tt4(100)

dimension ttt1(100),ttt2(100),ttt3(100),ttt4(100)

dimension aa(10000),bb(10000),cc(10000),ee(10000)

dimension aaa(10000),bbb(10000),ccc(10000),eee(10000)

dimension ts1(10000),ts2(10000),ts3(10000),ts4(10000)

dimension tts1(10000),tts2(10000),tts3(10000),tts4(10000)

dimension T01(100),T0(100),T02(100),t_t(100),ss(100)

dimension S0(100),S01(100),S02(100)

dimension ff(100),ss1(100),ss2(100),sigma_square_mml1(100)

dimension amu_mml1(100),amu_tukey1(100),amu_wave1(100)

dimension power(4),icount(4),ybar1(100)

integer r,b1

real ld

open(unit=1,file='result1')

3333 read*,nn,k,r,a1,dd

d_d=1.0-(2.0/(2.0-a1))*((1.0-1.0/(2.0*(2.0-a1)))/(1.0+1.0/
+(2.0-a1)+3.0*1.0/(4.0*((2.0-a1)**2))))

nb=k*(k-1)/2

mm=100000/nn

ld=r/(r-a1)

```

s1=0.0
sde=0.0
do 20 j=0,r
s1=s1+binom(r,j)*((ld/(2.*r))**j)*(fac(2*j)/((2**j)*fac(j)))
sde=sde+binom(r,j)*((ld/(2.*r))**j)*(fac(2*(j+1))/((2**(j+1))*
+fac(j+1)))
20 continue
c=1./s1
sde=sde*c
sd=sqrt(sde)

c COMPUTING t(i) VALUES
n1=int(nn/2.+5)
do 10 i=1,nn
if(i.gt.n1) then
t(i)=-t(nn-i+1)
goto 10
endif
q=i/(nn+1.)
call cdf(r,ld,c,q,t(i))
10 continue

c GENERATING RANDOM DEVIATES
if(ld.gt.1.0) go to 900
do 11 i=1,nn
alf(i)=((ld/r)*t(i)**3)/((1.0+(ld/(2.0*r))*t(i)**2)**2)
bet1(i)=(1.0-(ld/(2.0*r))*t(i)**2)/((1.0+(ld/(2.0*r))*t(i)**2)**2)
bet(i)=1.0-ld*bet1(i)
11 continue
go to 790
900 do 16 i=1,nn
h=1.0+(ld/(2.0*r))*t(i)**2
alf(i)=(((ld/r)*t(i)**3)+(1.0-1.0/ld)*t(i))/(h**2)
bet1(i)=(1.0/ld-(ld/(2.0*r))*t(i)**2)/(h**2)
bet(i)=1.0-ld*bet1(i)
16 continue
790 sumbet=0.0
do 110 i=1,nn
110 sumbet=sumbet+bet(i)
em=sumbet
do 100 l=1,mm

c GENERATION OF e(i)
do 1088 j=1,k

```

```

do 108 i=1,nn
9  call random(u)
   if(u.lt.0.001.or.u.gt.0.999) goto 9
   call cdf(r,ld,c,u,e(i))
   y(i,j)=e(i)
108  continue
1088 continue

c COMPUTING ORDERED e(i)
do 80000 j=1,k
do 90000 i=1,nn
yt(i)=y(i,j)
90000 continue
call SVRGN(nn,Yt,Y1)
do 1 i=1,nn
1  yo(i,j)=y1(i)
80000 continue

c MML METHOD
do 6000 j=1,k
amu_mml(j)=0.0
do 300 i=1,nn
amu_mml(j)=amu_mml(j)+bet(i)*yo(i,j)
300  continue
amu_mml(j)=amu_mml(j)/em
6000 continue
do 311 j=1,k
be=0.0
c1=0.0
do 310 i=1,nn
be=be+alf(i)*yo(i,j)
c1 =c1+((yo(i,j)-amu_mml(j))**2)*bet(i)
310  continue
bn=(nn*1.0)*(nn*1.0-1.0)
sigma_square_mml1(j)=(-ld*be+sqrt((ld*be)**2+4.0*(nn*1.0)*c1))**2
sigma_square_mml(j)=sigma_square_mml1(j)/((2.0*sqrt(bn))**2)
311  continue
ii=0
do 977 i=1,k
do 988 j=1,k
if (j.le.i) go to 988
ii=ii+1
tt1(ii)=abs((amu_mml(i)-amu_mml(j))/sqrt(2.0/((nn*1.0)*d_d)))
ttt1(ii)=abs(((amu_mml(i)-amu_mml(j))-dd)/sqrt(2.0/((nn*1.0)*

```

```

+d_d)))
988  continue
977  continue
      call SVRGN(nb,tt1,ts1)
      call SVRGN(nb,ttt1,tts1)
      aa(l)=ts1(nb)
      aaa(l)=tts1(nb)

c    TM10 METHOD
      rr=int(nn*0.1)+1
      do 302 j=1,k
        t_t(j)=0.0
        do 402 i=(rr+1),(nn-rr)
          t_t(j)=t_t(j)+yo(i,j)
402  continue
          amu_tukey(j)=(t_t(j)+(rr*1.0-0.1*(nn*1.0))*(yo(rr,j)+
+yo(nn-rr+1,j)))/((nn*1.0)*(1.0-2.0*0.1))
          ss(j)=0
          do 4002 i=(rr+1),(nn-rr)
            ss(j)=ss(j)+(yo(i,j)-amu_tukey(j))**2
4002 continue
            sigma_square_tukey(j)=(ss(j)+(rr*1.0)*(yo(rr,j)-amu_tukey(j))**2+
+(rr*1.0)*(yo((nn-rr+1,j)-amu_tukey(j))**2)/
+((nn*1.0)*((1.0-2.0*0.10)**2))
302  continue
        ii=0
        do 9771 i=1,k
          do 9881 j=1,k
            if (j.le.i) go to 9881
            ii=ii+1
            tt2(ii)=abs((amu_tukey(i)-amu_tukey(j))/
+sqrt(sigma_square_tukey(i)/(nn*1.0-2.0*rr)+sigma_square_tukey(j)/
+(nn*1.0-2.0*rr)))
            ttt2(ii)=abs(((amu_tukey(i)-amu_tukey(j))-dd)/
+sqrt(sigma_square_tukey(i)/(nn*1.0-2.0*rr)+sigma_square_tukey(j)/
+(nn*1.0-2.0*rr)))
9881  continue
9771  continue
      call SVRGN(nb,tt2,ts2)
      call SVRGN(nb,ttt2,tts2)
      bb(l)=ts2(nb)
      bbb(l)=tts2(nb)

```



```

c   WAVE METHOD
   hh=2.4
   do 222 j=1,k
   do 111 i=1,nn
   T01(i)=y(i,j)
111  continue
   call SVRGN(nn,T01,T02)
   aa1=nn/2
   bb1=aa1+1
   T0(j)=(T02(aa1)+T02(bb1))/2.0
222  continue
   do 2222 j=1,k
   do 1111 i=1,nn
   S01(i)=abs(y(i,j)-T0(j))
1111 continue
   call SVRGN(nn,S01,S02)
   S0(j)=(S02(aa1)+S02(bb1))/2.0
2222 continue
   do 229 j=1,k
   do 400 i=1,nn
   z(i,j)=(y(i,j)-T0(j))/(hh*S0(j))
400  continue
229  continue
   do 301 j=1,k
   ff(j)=0.0
   ss1(j)=0.0
   ss2(j)=0.0
   pi=22.0/7.0
   do 401 i=1,nn
   if (abs(z(i,j)).lt.(pi)) go to 228
   go to 401
228  ss1(j)=ss1(j)+sin(z(i,j))
   ff(j)=ff(j)+cos(z(i,j))
   ss2(j)=ss2(j)+(sin(z(i,j)))**2
401  continue
   amu_wave(j)=T0(j)+(hh*S0(j))*atan(ss1(j)/ff(j))
   sigma_square_wave(j)=((hh*S0(j))**2)*(nn*1.0)*ss2(j)/((ff(j))**2)
301  continue
   ii=0
   do 9772 i=1,k
   do 9882 j=1,k
   if (j.le.i) go to 9882
   ii=ii+1
   tt3(ii)=abs((amu_wave(i)-amu_wave(j))/sqrt(sigma_square_wave(i)/

```

```

+(nn*1.0)+sigma_square_wave(j)/(nn*1.0)))
  ttt3(ii)=abs(((amu_wave(i)-amu_wave(j))-dd)/
+sqrt(sigma_square_wave(i)/(nn*1.0)+sigma_square_wave(j)/(nn*1.0)))
9882  continue
9772  continue
      call SVRGN(nb,tt3,ts3)
      call SVRGN(nb,ttt3,tts3)
      cc(l)=ts3(nb)
      ccc(l)=tts3(nb)

c    MEAN METHOD
      do 1444 j=1,k
      sum1(j)=0.0
      yvar(j)=0.0
      do 144 i=1,nn
      sum1(j)=sum1(j)+y(i,j)
      yvar(j)=yvar(j)+y(i,j)**2
144  continue
1444 continue
      do 8888 j=1,k
      ybar(j)=sum1(j)/(nn*1.0)
      varsam(j)=yvar(j)-(nn*1.0)*ybar(j)**2
      s_square(j)=(varsam(j)/(nn*1.0-1.0))
8888 continue
      ii=0
      do 9773 i=1,k
      do 9883 j=1,k
      if (j.le.i) go to 9883
      ii=ii+1
      tt4(ii)=abs((ybar(i)-ybar(j))/sqrt(s_square(i)/
+(nn*1.0)+s_square(j)/(nn*1.0)))
      ttt4(ii)=abs(((ybar(i)-ybar(j))-dd)/
+sqrt(s_square(i)/(nn*1.0)+s_square(j)/(nn*1.0)))
9883  continue
9773  continue
      call SVRGN(nb,tt4,ts4)
      call SVRGN(nb,ttt4,tts4)
      ee(l)=ts4(nb)
      eee(l)=tts4(nb)
100  continue

c    COMPUTING CRITICAL VALUES
      call cv(mm,aa,p1)
      call cv(mm,bb,p2)

```

```

call cv(mm,cc,p3)
call cv(mm,ee,p4)

c COMPUTING POWER VALUES
icount(1)=0
do 777 j=1,mm
if (aaa(j).ge.p1) go to 777
icount(1)=icount(1)+1
777 continue
icount(2)=0
do 778 j=1,mm
if (bbb(j).ge.p1) go to 778
icount(2)=icount(2)+1
778 continue
icount(3)=0
do 779 j=1,mm
if (ccc(j).ge.p1) go to 779
icount(3)=icount(3)+1
779 continue
icount(4)=0
do 780 j=1,mm
if (eee(j).ge.p1) go to 780
icount(4)=icount(4)+1
780 continue
power(1)=1.0-(icount(1)*1.0/(mm*1.0))
power(2)=1.0-(icount(2)*1.0/(mm*1.0))
power(3)=1.0-(icount(3)*1.0/(mm*1.0))
power(4)=1.0-(icount(4)*1.0/(mm*1.0))

Open(2,File="power.txt")
print*,power
write(2,*) ,power
332 end

c SUBROUTINE FOR CRITICAL VALUES
subroutine cv(mm,qq,q1)
dimension acount1(40001),pp1(40001),pp2(40001),pps1(40001)
dimension pps2(40001),iper1(40001),iper2(40001)
dimension SR(40001),qq(10000)
SR(1)=1.0
do 112 i=2,40001
SR(i)=SR(i-1)+0.0001
112 continue
do 113 i=1,40001

```

```

account11=0
do 114 j=1,mm
if (qq(j).lt.SR(i)) go to 114
account11=account11+1
114 continue
account1(i)=(account11*1.0)/(mm*1.0)
113 continue
i1=0
i2=0
do 115 i=1,40001
if (account1(i).gt.0.05) go to 116
i1=i1+1
pp1(i1)=account1(i)
iperml(i1)=i
go to 115
116 i2=i2+1
pp2(i2)=account1(i)
iperml2(i2)=i
115 continue
if (i1.eq.0) go to 120
if (i2.eq.0) go to 122
call SVRGP(i1,pp1,pps1,iperml)
call SVRGP(i2,pp2,pps2,iperml2)
ai1=iperml(i1)
ai2=iperml2(1)
if ((abs(pps1(i1)-0.05)).le.(abs(pps2(1)-0.05))) go to 119
q1=SR(ai2)
go to 124
119 q1=SR(ai1)
go to 124
120 do 121 i=1,i2
iperml2(i)=i
121 continue
call SVRGP(i2,pp2,pps2,iperml2)
ai2=iperml2(1)
q1=SR(ai2)
go to 124
122 do 123 i=1,i1
iperml(i)=i
123 continue
call SVRGP(i1,pp1,pps1,iperml)
ai1=iperml(i1)
q1=SR(ai1)
124 end

```

```

c  SUBROUTINE FOR t(i) VALUES OF STS
  subroutine cdf(r,ldt,ct,ux,xm)
  integer r
  real ldt
  pi=22./7.
  eps=1.e-6
  xl=-10; xu=10
22  xm=(xl+xu)/2.
  del=1
  if(xm.le.0.) del=-1
  u=xm*xm/2.
  g=0
  do 32 j=0,r
  g=g+BINOM(r,j)*((ldt/r)**j)*(gamma(j+0.5)+del*gami(j+0.5,u))
32  continue
  f=ct*g/(2.*sqrt(pi))
  if(abs(f-ux).lt.eps.or.abs(xl-xu).lt.eps) return
  if(f.gt.ux) then
                                xu=xm
  else
                                xl=xm
  endif
  goto 22
end

```

APPENDIX C

COMPUTER PROGRAM FOR MULTIPLE COMPARISONS WITH A CONTROL

```
USE NUMERICAL_LIBRARIES
dimension alf(600),bet(600),e(600)
dimension w(600),v(600),w1(600),xa(600),ra(600),ae(600)
dimension y1(600),bet1(600),t(600),samp(100,100),y(100,100)
dimension sum1(100),yvar(100),sammean(100),varsam(100),yt(100)
dimension ybar(100),sigma(100),tt(100,100),ts(10000)
dimension z(100,100),yo(100,100)
dimension amu_mml(100),amu_wave(100),amu_tukey(100)
dimension s_square(100),sigma_square_mml(100)
dimension sigma_square_wave(100),sigma_square_tukey(100)
dimension tt1(100),tt2(100),tt3(100),tt4(100)
dimension ttt1(100),ttt2(100),ttt3(100),ttt4(100)
dimension aa(10000),bb(10000),cc(10000),ee(10000)
dimension aaa(10000),bbb(10000),ccc(10000),eee(10000)
dimension ts1(10000),ts2(10000),ts3(10000),ts4(10000)
dimension tts1(10000),tts2(10000),tts3(10000),tts4(10000)
dimension T01(100),T0(100),T02(100),t_t(100),ss(100)
dimension S0(100),S01(100),S02(100)
dimension ff(100),ss1(100),ss2(100),sigma_square_mml1(100)
dimension amu_mml1(100),amu_tukey1(100),amu_wave1(100)
dimension power(4),icount(4),ybar1(100)
dimension yc(100),yoc(100),ec(100),S01c(100),S02c(100),zc(100)
integer r,b1
real ld
```

```
open(unit=1,file='result1')
```

```
3333 read*,nn,k,r,a1,dd
```

```
  d_d=1.0-(2.0/(2.0-a1))*((1.0-1.0/(2.0*(2.0-a1)))/(1.0+1.0/
+(2.0-a1)+3.0*1.0/(4.0*((2.0-a1)**2))))
  nb=k*(k-1)/2
  mm=100000/nn
```

```

ld=r/(r-a1)
s1=0.0
sde=0.0
do 20 j=0,r
s1=s1+binom(r,j)*((ld/(2.*r))**j)*(fac(2*j)/((2**j)*fac(j)))
sde=sde+binom(r,j)*((ld/(2.*r))**j)*(fac(2*(j+1))/((2**(j+1))*
+fac(j+1)))
20 continue
c=1./s1
sde=sde*c
sd=sqrt(sde)

c COMPUTING t(i) VALUES
n1=int(nn/2.+5)
do 10 i=1,nn
if(i.gt.n1) then
t(i)=-t(nn-i+1)
goto 10
endif
q=i/(nn+1.)
call cdf(r,ld,c,q,t(i))
10 continue

c GENERATING RANDOM DEVIATES
if(ld.gt.1.0) go to 900
do 11 i=1,nn
alf(i)=(ld/r)*t(i)**3/((1.0+(ld/(2.0*r))*t(i)**2)**2)
bet1(i)=(1.0-(ld/(2.0*r))*t(i)**2)/((1.0+(ld/(2.0*r))*t(i)**2)**2)
bet(i)=1.0-ld*bet1(i)
11 continue
go to 790
900 do 16 i=1,nn
h=1.0+(ld/(2.0*r))*t(i)**2
alf(i)=(((ld/r)*t(i)**3)+(1.0-1.0/ld)*t(i))/(h**2)
bet1(i)=(1.0/ld-(ld/(2.0*r))*t(i)**2)/(h**2)
bet(i)=1.0-ld*bet1(i)
16 continue
790 sumbet=0.0
do 110 i=1,nn
110 sumbet=sumbet+bet(i)
em=sumbet

c GENERATION OF e(i) FOR CONTROL GROUP
do 107 i=1,nn

```

```

8      call random(uu)
      if(uu.lt.0.001.or.uu.gt.0.999) goto 8
      call cdf(r,ld,c,uu,ec(i))
      yc(i)=ec(i)
107    continue
      call SVRGN(nn,yc,yoc)

c      MML METHOD FOR CONTROL GROUP
      amu_mmlc=0.0
      do 299 i=1,nn
      amu_mmlc=amu_mmlc+bet(i)*yoc(i)
299    continue
      amu_mmlc=amu_mmlc/em
      b_c=0.0
      c_c=0.0
      do 309 i=1,nn
      b_c=b_c+alf(i)*yoc(i)
      c_c=c_c+((yoc(i)-amu_mmlc)**2)*bet(i)
309    continue
      bn=(nn*1.0)*(nn*1.0-1.0)
      sigma_squarec1=(-ld*b_c+sqrt((ld*b_c)**2+4.0*(nn*1.0)*c_c))**2
      sigma_squarec=sigma_squarec1/((2.0*sqrt(bn))**2)

c      TUKEY METHOD FOR CONTROL GROUP
      rr=int(nn*0.1)+1
      t_tc=0.0
      do 501 i=(rr+1),(nn-rr)
      t_tc=t_tc+yoc(i)
501    continue
      amu_tukeyc=(t_tc+(rr*1.0-0.1*(nn*1.0))*(yoc(rr)+
+ yoc(nn-rr+1)))/((nn*1.0)*(1.0-2.0*0.1))
      ssc=0
      do 4001 i=(rr+1),(nn-rr)
      ssc=ssc+(yoc(i)-amu_tukeyc)**2
4001   continue
      sigma_square_tukeyc=(ssc+(rr*1.0)*(yoc(rr)-amu_tukeyc)**2+
+ (rr*1.0)*(yoc(nn-rr+1)-amu_tukeyc)**2)/
+ ((nn*1.0)*((1.0-2.0*0.10)**2))

c      WAVE METHOD FOR CONTROL GROUP
      hh=2.4
      aa1=nn/2
      bb1=aa1+1
      T0c=(yoc(aa1)+yoc(bb1))/2.0

```



```

do 1110 i=1,nn
S01c(i)=abs(yc(i)-T0c)
1110 continue
call SVRGN(nn,S01c,S02c)
S0c=(S02c(aa1)+S02c(bb1))/2.0
do 399 i=1,nn
zc(i)=(yc(i)-T0c)/(hh*S0c)
399 continue
ffc=0.0
ss1c=0.0
ss2c=0.0
pi=22.0/7.0
do 500 i=1,nn
if (abs(zc(i)).lt.(pi)) go to 227
go to 500
227 ss1c=ss1c+sin(zc(i))
ffc=ffc+cos(zc(i))
ss2c=ss2c+(sin(zc(i)))**2
500 continue
amu_wavec=T0c+(hh*S0c)*atan(ss1c/ffc)
sigma_square_wavec=((hh*S0c)**2)*(nn*1.0)*ss2c/((ffc)**2)

c MEAN METHOD FOR CONTROL GROUP
sum1c=0.0
yvarc=0.0
do 143 i=1,nn
sum1c=sum1c+yc(i)
yvarc=yvarc+yc(i)**2
143 continue
ybarc=sum1c/(nn*1.0)
varsamc=yvarc-(nn*1.0)*ybarc**2
s_squarec=(varsamc/(nn*1.0-1.0))
do 100 l=1,mm

c GENERATION OF e(i)
do 1088 j=1,k
do 108 i=1,nn
9 call random(u)
if(u.lt.0.001.or.u.gt.0.999) goto 9
call cdf(r,ld,c,u,e(i))
y(i,j)=e(i)
108 continue
1088 continue
nrr=int(0.5+0.1*nn)

```

```

do 1078 j=1,k
do 10788 i=1,nn
if (i.gt.nrr) go to 10788
y(i,j)=y(i,j)*4.0
10788 continue
1078 continue
do 80000 j=1,k
do 90000 i=1,nn
yt(i)=y(i,j)
90000 continue

c COMPUTING ORDERED e(i)
call SVRGN(nn,Yt,Y1)
do 1 i=1,nn
1 yo(i,j)=y1(i)
80000 continue

c MML METHOD
do 6000 j=1,k
amu_mml(j)=0.0
do 300 i=1,nn
amu_mml(j)=amu_mml(j)+bet(i)*yo(i,j)
300 continue
amu_mml(j)=amu_mml(j)/em
6000 continue
do 311 j=1,k
be=0.0
c1=0.0
do 310 i=1,nn
be=be+alf(i)*yo(i,j)
c1 =c1+((yo(i,j)-amu_mml(j))**2)*bet(i)
310 continue
bn=(nn*1.0)*(nn*1.0-1.0)
sigma_square_mml1(j)=(-ld*be+sqrt((ld*be)**2+4.0*(nn*1.0)*c1))**2
sigma_square_mml(j)=sigma_square_mml1(j)/((2.0*sqrt(bn))**2)
311 continue
do 977 i=1,k
tt1(i)=abs((amu_mml(i)-amu_mmlc)/sqrt(2.0/((nn*1.0)*d_d)))
ttt1(i)=abs(((amu_mml(i)-amu_mmlc)-dd)/sqrt(2.0/((nn*1.0)*
+d_d)))
977 continue
call SVRGN(k,tt1,ts1)
call SVRGN(k,ttt1,tts1)
aa(l)=ts1(k)

```

```

aaa(1)=tts1(k)

c   TUKEY METHOD
    rr=int(nn*0.1)+1
    do 302 j=1,k
      t_t(j)=0.0
      do 402 i=(rr+1),(nn-rr)
        t_t(j)=t_t(j)+yo(i,j)
402  continue
      amu_tukey(j)=(t_t(j)+(rr*1.0-0.1*(nn*1.0))*(yo(rr,j)+
+yo(nn-rr+1,j)))/((nn*1.0)*(1.0-2.0*0.1))
      ss(j)=0
      do 4002 i=(rr+1),(nn-rr)
        ss(j)=ss(j)+(yo(i,j)-amu_tukey(j))**2
4002  continue
      sigma_square_tukey(j)=(ss(j)+(rr*1.0)*(yo(rr,j)-amu_tukey(j))**2+
+(rr*1.0)*(yo((nn-rr+1),j)-amu_tukey(j))**2)/
+((nn*1.0)*((1.0-2.0*0.10)**2))
302  continue
      do 9771 i=1,k
        tt2(i)=abs((amu_tukey(i)-amu_tukeyc)/
+sqrt(sigma_square_tukey(i)/(nn*1.0-2.0*rr)+sigma_square_tukeyc/
+(nn*1.0-2.0*rr)))
        ttt2(i)=abs(((amu_tukey(i)-amu_tukeyc)-dd)/
+sqrt(sigma_square_tukey(i)/(nn*1.0-2.0*rr)+sigma_square_tukeyc/
+(nn*1.0-2.0*rr)))
9771  continue
      call SVRGN(k,tt2,ts2)
      call SVRGN(k,ttt2,tts2)
      bb(1)=ts2(k)
      bbb(1)=tts2(k)

c   WAVE METHOD
    hh=2.4
    do 222 j=1,k
      do 111 i=1,nn
        T01(i)=y(i,j)
111  continue
      call SVRGN(nn,T01,T02)
      aa1=nn/2
      bb1=aa1+1
      T0(j)=(T02(aa1)+T02(bb1))/2.0
222  continue
      do 2222 j=1,k

```

```

do 1111 i=1,nn
S01(i)=abs(y(i,j)-T0(j))
1111 continue
call SVRGN(nn,S01,S02)
S0(j)=(S02(aa1)+S02(bb1))/2.0
2222 continue
do 229 j=1,k
do 400 i=1,nn
z(i,j)=(y(i,j)-T0(j))/(hh*S0(j))
400 continue
229 continue
do 301 j=1,k
ff(j)=0.0
ss1(j)=0.0
ss2(j)=0.0
pi=22.0/7.0
do 401 i=1,nn
if (abs(z(i,j)).lt.(pi)) go to 228
go to 401
228 ss1(j)=ss1(j)+sin(z(i,j))
ff(j)=ff(j)+cos(z(i,j))
ss2(j)=ss2(j)+(sin(z(i,j)))**2
401 continue
amu_wave(j)=T0(j)+(hh*S0(j))*atan(ss1(j)/ff(j))
sigma_square_wave(j)=((hh*S0(j))**2)*(nn*1.0)*ss2(j)/((ff(j))**2)
301 continue
do 9772 i=1,k
tt3(i)=abs((amu_wave(i)-amu_wavec)/sqrt(sigma_square_wave(i)/
+(nn*1.0)+sigma_square_wavec/(nn*1.0)))
tts3(i)=abs(((amu_wave(i)-amu_wavec)-dd)/
+sqrt(sigma_square_wave(i)/(nn*1.0)+sigma_square_wavec/(nn*1.0)))
9772 continue
call SVRGN(k,tt3,ts3)
call SVRGN(k,tts3,tts3)
cc(l)=ts3(k)
ccc(l)=tts3(k)

c MEAN METHOD
do 1444 j=1,k
sum1(j)=0.0
yvar(j)=0.0
do 144 i=1,nn
sum1(j)=sum1(j)+y(i,j)
yvar(j)=yvar(j)+y(i,j)**2

```

```

144  continue
1444 continue
      do 8888 j=1,k
      ybar(j)=sum1(j)/(nn*1.0)
      varsam(j)=yvar(j)-(nn*1.0)*ybar(j)**2
      s_square(j)=(varsam(j)/(nn*1.0-1.0))
8888  continue
      do 9773 i=1,k
      tt4(i)=abs((ybar(i)-ybarc)/sqrt(s_square(i)/
+(nn*1.0)+s_squarec/(nn*1.0)))
      ttt4(i)=abs(((ybar(i)-ybarc)-dd)/
+sqrt(s_square(i)/(nn*1.0)+s_squarec/(nn*1.0)))
9773  continue
      call SVRGN(k,tt4,ts4)
      call SVRGN(k,ttt4,tts4)
      ee(l)=ts4(k)
      eee(l)=tts4(k)
100  continue

      c  COMPUTING CRITICAL VALUES
      call cv(mm,aa,p1)
      call cv(mm,bb,p2)
      call cv(mm,cc,p3)
      call cv(mm,ee,p4)

      c  COMPUTING POWER VALUES
      icount(1)=0
      do 777 j=1,mm
      if (aaa(j).ge.1.709218) go to 777
      icount(1)=icount(1)+1
777  continue
      icount(2)=0
      do 778 j=1,mm
      if (bbb(j).ge.1.438373) go to 778
      icount(2)=icount(2)+1
778  continue
      icount(3)=0
      do 779 j=1,mm
      if (ccc(j).ge.2.001764) go to 779
      icount(3)=icount(3)+1
779  continue
      icount(4)=0
      do 780 j=1,mm
      if (eee(j).ge.1.724420) go to 780

```

```

780   icount(4)=icount(4)+1
      continue
      power(1)=1.0-(icount(1)*1.0/(mm*1.0))
      power(2)=1.0-(icount(2)*1.0/(mm*1.0))
      power(3)=1.0-(icount(3)*1.0/(mm*1.0))
      power(4)=1.0-(icount(4)*1.0/(mm*1.0))
      Open(2,File="power.txt")
      print*,power
      write(2,*,)power
332   end

c   SUBROUTINE FOR CRITICAL VALUES
      subroutine cv(mm,qq,q1)
      dimension acount1(40001),pp1(40001),pp2(40001),pps1(40001)
      dimension pps2(40001),iper1(40001),iper2(40001)
      dimension SR(40001),qq(10000)
      SR(1)=1.0
      do 112 i=2,40001
      SR(i)=SR(i-1)+0.0001
112   continue
      do 113 i=1,40001
      acount11=0
      do 114 j=1,mm
      if (qq(j).lt.SR(i)) go to 114
      acount11=acount11+1
114   continue
      acount1(i)=(acount11*1.0)/(mm*1.0)
113   continue
      i1=0
      i2=0
      do 115 i=1,40001
      if (acount1(i).gt.0.05) go to 116
      i1=i1+1
      pp1(i1)=acount1(i)
      iper1(i1)=i
      go to 115
116   i2=i2+1
      pp2(i2)=acount1(i)
      iper2(i2)=i
115   continue
      if (i1.eq.0) go to 120
      if (i2.eq.0) go to 122
      call SVRGP(i1,pp1,pps1,iper1)
      call SVRGP(i2,pp2,pps2,iper2)

```

```

ai1=iperml(i1)
ai2=iperml2(1)
if ((abs(pps1(i1)-0.05)).le.(abs(pps2(1)-0.05))) go to 119
q1=SR(ai2)
go to 124
119 q1=SR(ai1)
go to 124
120 do 121 i=1,i2
iperml2(i)=i
121 continue
call SVRGP(i2,pp2,pps2,iperml2)
ai2=iperml2(1)
q1=SR(ai2)
go to 124
122 do 123 i=1,i1
iperml(i)=i
123 continue
call SVRGP(i1,pp1,pps1,iperml)
ai1=iperml(i1)
q1=SR(ai1)
124 end

c SUBROUTINE FOR t(i) VALUES OF STS
subroutine cdf(r,ldt,ct,ux,xm)
integer r
real ldt
pi=22./7.
eps=1.e-6
xl=-10; xu=10
22 xm=(xl+xu)/2.
del=1
if(xm.le.0.) del=-1
u=xm*xm/2.
g=0
do 32 j=0,r
g=g+BINOM(r,j)*((ldt/r)**j)*(gamma(j+0.5)+del*gami(j+0.5,u))
32 continue
f=ct*g/(2.*sqrt(pi))
if(abs(f-ux).lt.eps.or.abs(xl-xu).lt.eps) return
if(f.gt.ux) then
xu=xm
else
xl=xm
endif
endif

```

```
goto 22  
end
```