SOME EXTENSIONS TO CREDITRISK+: FFT, FFT-PANJER AND POISSON-INAR PROCESS

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FEBRUARY 2007

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A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF APPLIED MATHEMATICS
OF
THE MIDDLE EAST TECHNICAL UNIVERSITY

BY

KAMİL KORHAN NAZLIBEN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
IN
THE DEPARTMENT OF FINANCIAL MATHEMATICS

FEBRUARY 2007

Approval of the Graduate School of Applied Mathematics

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Abstract

SOME EXTENSIONS TO CREDITRISK+: FFT, FFT-PANJER AND POISSON-INAR PROCESS

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February 2007, 68 pages

The various versions of CreditRisk+ have widely been used in the financial industry. We compute the loss distribution under CreditRisk+ model by fast fourier transform technique in order to have faster and more stable results. Moreover, we link the parameters of the model to the exogenously observed variables which could be obtained from the financial markets by the use of Poisson INAR process. It is shown that the estimation of the parameters become available under this setup. This enables us to build a system that allows users to monitor and predict the banks loss characteristics without having specific and current information on banks.

Keywords: Credit risk, CreditRisk+, FFT, INAR Process.

Öz

CREDITRISK+ ÜZERİNE YENİ YAKLAŞIMLAR : FFT, FFT-PANJER VE POISSON-INAR SÜREÇLERİ

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Yüksek Lisans, Finansal Matematik Bölümü

Tez Yöneticisi: Prof. Dr. Hayri KÖREZLİOĞLU

Şubat 2007, 68 sayfa

Bir kredi riski ölçüm yöntemi olan CreditRisk+'ın finans endüstrisinde kullanılmakta olan çeşitli versiyonları bulunmaktadır. Bu çalışmada, daha hızlı ve istikrarlı sonuçlar elde edebilmek için, kredi portföyü kayıp dağılımı "fast fourier transform" (FFT) tekniği ile elde edilmiştir. Bununla beraber, gözlemlenebilen finansal market verileri de modele dahil edilerek, temerrüt riskinin hesabında Poisson INAR prosesinden faydalanılmış ve söz konusu modelin parametreleri hesaplanabilmiştir. Bu model bize kredi riskinin ölçülmesi ve muhtemel risklerin öngörülmesinde önemli kolaylıklar sağlamış, portföy kayıp dağılımının karakteristiğini belirlemede bankalar hakkında spesifik ve güncel bilgiye gerek kalmamaktadır.

Anahtar kelimeler: Kredi riski, CreditRisk+, FFT, INAR Proses.

To my family

Acknowledgements

I would like to thank Prof. Dr. Hayri Körezlioğlu, Assist. Prof. Dr. Kasırga Yıldırak and Assist. Prof. Dr. Ömür Uğur for guiding, motivating and encouraging me throughout this study.

I am grateful to Assoc.Prof. Dr. Azize Hayfavi for her great contributions and for motivating and encouraging me at the beginning of my academic life.

I am also grateful to Dr. Coşkun Küçüközmen for his kindness, valuable suggestions and lectures.

I would also like to thank my parents for their valuable supports. Without their encouragement this thesis would not have been possible.

Finally, I would like to thank all my friends and the members of the Institute of Applied Mathematics.

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Chapter 1

INTRODUCTION

1.1 Introduction

Since the beginning of the 1980's, there has been a dramatic development in credit markets. Considering high amounts of credits in their trading books, quantifying and controlling credit risk has become crucial to all financial institutions. Moreover, the recent developments of new credit products, such as credit derivatives and asset basket credit securities, have made credit markets more complex. Accordingly, some new regulations are required to organize and control these markets. According to Basel II Capital Accord, all financial institutions must measure and manage their own risks by using some quantitative risk measurement techniques[7],[8]. Banks can either develop their internal models or utilize some models from industry. Academic and industrial models are mainly divided into three broad categories: *Merton's Asset Value Models*, *Intensity-Actuarial Based Models*, and *Macro Economic Approaches*.

The most well known Merton based credit risk approach is JP Morgan's CreditMetrics, first published in 1995[18]. The theocratical framework of this model is based on Merton's work on the theory of option pricing[60]. It includes the credit migration analysis event and obligors' credit migrations within a given time horizon. It models the full-forward distribution of any bond or loan portfolio. Moreover, KMV Corporation has developed another model which relies upon "the Expected Default Frequency (EDF)". McKinsey proposes its macro-economic approach, CreditPortfolioView, which is a discrete-time multi-period model using macro-economic variables such as unemployment, the level of interest rates, growth rates in economy, government expenses, foreign exchange rates, etc.

Credit Suisse Financial Products (CSFP) released CreditRisk+ in October 1997 [22]. Unlike Merton's based credit risk models, CreditRisk+ is an analytical portfolio approach, so it does not require high computational effort. The main advantage of this model stems from its simplicity and computational efficiency. Moreover it does not require too many input arguments. Exposure amounts, expected default frequencies and standard deviations of obligors are sufficient information to perform this model. Mainly, it is assumed that the default event is a Poisson process, and the behavior of sectors are independent. This model mainly concentrates on achieving the loss distribution of a whole credit portfolio. In the literature, there are some alternative techniques used to compute the loss function of the portfolio. Although the standard model uses the Panjer recursion technique[63], it has been shown that this technique is numerically unstable in some cases, especially for a large number of sector dependencies. Some techniques show faster, more accurate and more effective computational performance with respect to the standard model. Whereas the original model only uses a probability generating function, it is also possible to incorporate other auxiliary generating functions such as characteristic functions, and cumulative or moment generating functions into the model. In the *Enhanced CreditRisk+ Approach*, proposed by Giese, the moment generating function of the factors are incorporated into the model [39]. Giese demonstrated a new recursion scheme which represents a faster and more accurate computation performance than the standard approach. In addition to this, *Saddle Point Approximation* uses a cumulative generating function which offers a robust and extremely fast alternative to the Panjer recursion technique[58]. Finally, *the Fourier Inversion Technique*, which describes loss in terms of a characteristic function, is easy to implement, and has numerically stable algorithms [62],[55].

In this study, we mainly focus on the implementation of the Fast Fourier Transform (FFT) technique in view of CreditRisk+ framework.

1.2 Literature Review

Among all credit risk models, main challenge is the determination of default probabilities. In the historical method, default probabilities are determined from actual historical data. Rating agencies generally use this approach and construct a Markovian transition probability matrix. If we know the history of credit movements, it is possible to estimate future default probabilities for a given time horizon. In migration analysis, a transition matrix should be developed by using all historical information. However, the main problem is generally lack of data. Today, mainly three fundamental approaches are used to estimating default likelihood. These are the qualitative dependent variable model, discriminant analysis and neural networks.

Generally, the transition matrix and default probabilities of obligors are essential input data for any credit portfolio risk model. In the literature, the first transition matrix was published in 1991 by both Edward Altman and Lucas and

Lonski of Moody's Investors Service[53]. At the beginning of the nineties, there have been several studies of their predictive power and stationarity [1]. In addition to academics, several practitioners have used migration analysis to a have better accounting-based allowance for loan and least-loss estimation [61], [67]. Moreover, these tools have been used to estimate holding period returns [17]. Finally, arbitrage free credit pricing models have been developed by Ginzburg, Maloney and Willner [37], Jarrow, Lando and Turnbull [42] and Das and Tufano [24].

Besides historical models, many rating agencies incorporate their model through accounting, analytical, statistical and option theoretical approaches. They use some financial ratios in order to estimate a firm specific quality, focusing on leverage and coverage ratios and a firms cash flows. One of the best approach is the Debt Rating Criteria proposed by Standard and Poor's. It gives us an alphabetic rating for the firm's credit quality. On the other hand, from a macro-economic perspective, it is stated that default likelihood is correlated with measures of business and credit cycle [34], [45].

Portfolio credit risk models can generally be classified in three groups: *The Asset-Based Models, Intensity-Actuarian Models and Macro-economic approaches.* The asset based method, which sometimes called as the *structural approach*, is based on the valuation of the underlying assets of a firm. Merton's main assumption is that a default event can only occur when the value of a firm's asset is below the value of the debt at expiry [60]. Asset based credit risk models, such as CreditMetrics and KMV's models, are the extensions of Merton's first proposal. This method is sometimes referred to the option theoretical approach, because it is inspired from the Black-Scholes-Merton methodology of option pricing, Black and Cox (1976)[10], Brannan and Schwartz (1977) [12], Longstaff and Schwartz (1995)[52], Briys and de Varenne(1997) [14] and Cathcart and El-Jahel (1998)[16]. In addition, these models are developed by other researches such as Geske (1977)[36], Ho and Singer (1982)[40], Schimko, Tejimo and Deventer (1993) [64] , Zhou(1997) [70], Vasicek(1997)[68], Schimid (2004) [20], Mason and Bhattacharya (1981) [56] and Zhou (1996)[69].

Besides default probabilities, the other important driving factor is the recovery rate. Altman and Kishore(1996) [2] and Carty and Lieberman (1996)[15] focused on econometric studies of recovery rates. Furthermore, the structural approach allows for a study of the optimal capital structure of the firm. The studies were originated by Black and Cox (1976)[10], Anderson, Pan and Sunderesan(1992)[4], Leland (1994)[50], Anderson, Sunderesan and Tychon (1996)[3], Lealand and Toft (1996) [51], Mella-Barral and Tychon (1996) [59], Fon and Sunderesan (1997), [33], Ericsson (2000) [31].

The intensity based method, sometimes called as reduced form model, is based on the default time to the stopping time of some specified hazard rate process. It allows for the modeling of the unpredictable random time of defaults. There are various mathematical results underlying the reduced form approach. Here, research focuses on the characterization of random times in terms of hazard functions, hazard process, as well as the evaluation of conditional probabilities and conditional expectations in terms of these functions and processes. The most cited studies are Dellacherie and Meyer (1978) [26], Davis (1976) [25] , Elliott (1977) [30], Jeulin and Yor (1978) [44], Mazitto and Szpirglas (1979) [57], Bremaud (1981) [13], Artzner and Delbean (1992) [5], Dufie et al (1996) [27], Lando (1998) [49], Kusuoka(1999) [48], Jeanblack and Rutkowski (2000) [9].

Academic work show that there may be some contradictions between these models and real world observations. Therefore, new hybrid term structure models are developed in order to have better estimation of default probabilities, pricing defaultable bonds and other securities [20],[21]. It is thought that these hybrid models are more powerful than the classical models. New studies show that a combination of asset and intensity based models give us more realistic results than classical approaches. For example, Madan and Unal(1998) assume that the stochastic hazard rate is a linear function of the default free short rate and the logarithm of the value of the firm's asset[54].

In the 1990's, the most complicated financial product was introduced. Credit derivatives, which transfer credit risk exposure and behave as if insurance of the credits. Although the structure of the credit derivatives depends on the counterparties, the main forms are Collateral Debt Obligations (CDO), Collateralised Bond Obligations (CBO), Collateralised Loan Obligations (CLO), Collateral Mortgage Obligations (CMO). In spite of a great amount of trading of credit derivatives in financial markets, there are few articles on the direct pricing of credit derivatives. The most cited articles are Longstaff and Schwartz [52], Das and Tufano [24], Duffie [28], Hull and White [41] and Schonbucher [66].

1.3 Industrial Credit Risk Models

The most famous industrial internal credit risk models are $PortfolioManager^{TM}$ of Moody's KMV, the RiskMetrics Group's $CreditMetrics^{TM}$ [18], Credit Suisse Financial Products' CreditRisk+, and McKinsey's $CreditPortfolioView^{TM}$. All these models are the well known model to measure and quantify credit risk.

Moody's KMV model is a Merton based approach [60]. According to this model, equities behave as a call option on the value of the companies business. Here, default event occurs if the value of the company drops below a critical level. The other Merton based approach CreditMetrics, similar to KMV, concentrates on probabilistic behavior of individual asset returns considering mutual correlations. It uses transition matrix which shows the probabilities of credit ratings at the end of a specified time period. CreditRisk+, which is an actuarial based approach, only uses mean default probabilities and standard deviations. The CreditPortfolioView which uses macroeconomic variables in modeling has completely different principles than others.

CreditMetrics, first published by JP Morgan, mainly generates possible outcomes of market values of companies depending on ratings. This is Credit Migration approach which on the probability of moving from one credit quality to another within a specified time horizon. In this model, the Monte Carlo simulation technique is applied to obtain whole portfolio loss distribution. Therefore, it takes too much computational effort [18].

KMV's model is an option pricing approach which is also based on the asset valuation model. In this model the default process relating to the capital structure of the firm is endogenous. Default can only occur below a critical level of the firm's value.

On the other hand, CreditRisk+, which is an actuarial approach, concentrates on the default event. It considers the average default rates of obligors, and correlations are implied from the model. This is an analytical model which constructs a continuous distribution of default probabilities. CreditRisk+ yields associated risk capital estimates. In contrast with all other credit models, it does not use the

Monte Carlo simulation technique. Therefore, the calculation procedure becomes very fast [22].

McKinsey's CreditPortfolioView, which is an discrete-time multi period model, focuses on the impact of macroeconomic variables on credit portfolio. The main advantage of this product is that the CreditPortfolioView can be easily applied to all market instruments.

To summarize, CreditMetrics and KMV's models are microeconomic casual models of individual default. On the other hand, CreditRisk+ has no assumptions about causality, it only takes default rates into account. Moreover, it has been shown that in spite of all these differences, CreditRisk+ and CreditMetrics fundamentally have similar underlying mathematical structures [38]. In addition, Koyluoglu and Hickman (1998) showed that these four credit risk models have few differences in theory and results produced [47]. Crouhy, Galai and Mark(2000) examined the current credit risk models in a comparative perspective [23].

Chapter 2

PRELIMINARIES

2.1 THE BASICS OF CREDIT RISK

In general, Credit risk models illustrate how much they will loose and what would be the shape of loss, and expected and unexpected loss in a certain period of time. For a credit portfolio, there is a high uncertainty around the expected value and the distribution of outcomes which are heavily skewed. Unlike the market returns, credit return displays non-Gaussian behavior. Unlike Market returns are relatively symmetrical and well approximated by normal distributions, while credit returns are highly skewed and fat tailed. Therefore simple summary statistics of credit portfolio are not helpful to understand the level of riskiness of the portfolio. So the full distribution of a credit portfolio should be illustrated. Generally, industrial credit risk models use analytical or simulation techniques. Percentile levels of the full distributions give us good information about the behavior of tail and possible losses. By applying these techniques, we can evaluate Value at Risk (VaR), Conditional Value at Risk (C-VaR) or Expected Short Fall exposures of the portfolio.

Credit risk models, together with its components, include some important parameters : Default Probabilities (DP), Loss Given Default (LGD) and Exposure at Default (EAD) and maturity. The first and the most significant and difficult step is the determination of the obligors' default probabilities. There are mainly two commonly used approaches to asses default probabilities. The first is estimation of default probabilities from market data. KMV uses this technique relying by upon the concept of Expected Default Frequency (EDF). The other approach is estimating default probabilities from ratings. The rating of a loan gives us the creditworthiness of that company attributed and its future expectation of credit migration. Leading rating agencies use several techniques in order to determine obligors' credit quality. There are several rating systems and each agency use its own rating scale. For instance, Standard and Poor's scale is $AAA, AA, A, BBB, BB, B, C$, while Moody's uses $Aaa, Aa, A, Baa, Ba, B, C$. Obviously, Aaa is the highest and C is the poorest credit quality. The main difficulty in the determination of credit quality is lack of data, so it is very difficult to develop a pure mathematical credit rating model. Today we only have 20-25's years rating data which is insufficient for a historical perspective. Therefore, despite the existence of advanced statistical and mathematical tools and techniques, qualitative methods such as experience and judgement have still premier importance.

The second driving factor is Loss Given Default (LGD) which describes the fraction of the loan's exposure expected to be lost in the case of default. Loss given default is the amount of loss in the case of a default. In other words,

$$Loss\ Given\ Default = 1 - Recovery\ Rate, \tag{2.1.1}$$

where recovery rate is the amount of the promised cash flows recovered in case of default. The estimation of LGD depends on a firm's specific features such as collateral and severity of the bank's claim on the borrowers' assets, etc. Mathematically LGD can be written as the expectation of the severity (SEV).

$$\mathbb{E}[Severity] = LGD = 1 - Recovery\ Rate. \tag{2.1.2}$$

The other driving factor is the Exposure at Default (EAD). Although several models assume the Exposure at Default (EAD) as a constant, one can also model by considering two major parts: Outstanding and Commitments parts. Outstanding refers to the portion of the exposure already drawn by the obligor. If default event occurs, the bank is exposed to the total amount of outstanding. Thus, one can also model exposure at default by considering these parts of the exposures [11].

Here, we give the mathematical definition of loss a variable[11].

Definition 2.1. Let $(\Omega, \mathbb{F}, \mathbb{P})$ be the probability space, with sample space Ω, σ algebra, $\mathbb{F}$ and probability measure $\mathbb{P}$.Let $\mathfrak{L}$ be a loss variable and EAD, LGD and $\mathbb{L}$ such that

$$\mathfrak{L} = EAD \times LGD \times \mathbb{L} \tag{2.1.3}$$

$$\mathbb{L} = \mathbb{I}_D, \mathbb{P}(D) = DP, \tag{2.1.4}$$

where D denotes the default indicator, $\mathbb{P}$(D) is the default probability.

Here we can define the expected and unexpected loss of any obligor.

Definition 2.2. Let $(\Omega, \mathbb{F}, \mathbb{P})$ be the probability space, with sample space Ω, σ algebra, $\mathbb{F}$ measurable, probability measure $\mathbb{P}$. Let EL denote expected loss such that

$$EL = \mathbb{E}[\mathfrak{L}] = EAD \times LGD \times DP. \tag{2.1.5}$$

[11].

Proposition 2.1. Let the severity and the default event be independent, then, the unexpected loss is

$$UL = EADx\sqrt{Var[SEV] \times DP + LGD^2 \times DP(1 - DP)}$$

[11].

2.2 THE POISSON MODEL FOR CREDIT RISK

This section and following sections are mainly inspired from Uwe Schmock lecture notes [65] . Let m denote the number of obligors, and let $(N_1, N_2, ...N_m)$ be a vector of Poisson distributed random Variables with parameters $\lambda_1, ...\lambda_m > 0$. This means that the default event of any obligor has a Poisson distribution. Our convention is 0^0:=1.

For all $n_i \epsilon \mathbb{N}_i$ and $i \epsilon 1, 2...m$

$$\mathbb{P}[N_i = n_i] = \frac{{\lambda_i}^{n_i} e^{-\lambda_i}}{n_i!} \tag{2.2.6}$$

$$N_i \sim Poisson(\lambda_i). \tag{2.2.7}$$

2.2.1 Properties of a Poisson Distribution

Suppose that $N \sim Poisson(\lambda)$, then

$$\mathbb{E}[N] = \sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda_i}}{n!}, \tag{2.2.8}$$

by power series expansion, expectation is

$$\mathbb{E}[N] = \lambda, \tag{2.2.9}$$

for the variance

$$Var[N] = \lambda. \tag{2.2.10}$$

Lemma 2.1. If $N_1, ...N_m$ are independent with $N_i \sim Poisson(\lambda_i)$ for all $i\epsilon 1, ...m$ then

$$N_i := \sum_{i=1}^{m} N_i \sim Poisson(\lambda) \tag{2.2.11}$$

with $\lambda = \lambda_1 + ...\lambda_m$

In addition the probability generating function is given by the formula

$$\psi_N = \mathbb{E}[s^N] = \sum_{n=0}^{\infty} s^n \frac{\lambda^n}{n!} e^{-\lambda} = e^{\lambda s} e^{-\lambda} = e^{\lambda(s-1)}. \tag{2.2.12}$$

Practically, we are interested in $N_i = 0$ and $N_i \neq 0$ values which denote non-default and default events respectively. Parameter λ_i should be calibrated. There are two calibration methods:

(a) If $p_i \epsilon [0, 1)$

$$p_i = \mathbb{P}[N_i \geq 1] = 1 - e^{-\lambda_i} \tag{2.2.13}$$

(b)

$$\lambda_i = \mathbb{E}[N_i] = p_i.$$

The second equation is the approximation of the first with the λ_i is neglected for small λ_i and $n \geq 2$.

2.3 THE GENERAL POISSON MIXTURE MODEL

Let us introduce [0,∞)-valued random variables $\Lambda_1, \Lambda_2, ...\Lambda_m$ with joint distribution F, i.e., $\Lambda_1, \Lambda_2, ...\Lambda_m \sim F$. The assumption is

$$\mathbb{P}[N_i = n_i|\Lambda_1, ...\Lambda_m] = \mathbb{P}[N_i = n_i|\Lambda_i] = e^{-\Lambda_i}\frac{\Lambda_i^{n_i}}{n_i!} \qquad (a.s.) \tag{2.3.14}$$

i.e., $N_i \sim Poisson(\lambda)$ given $\Lambda_1, \Lambda_2, ...\Lambda_m$

Therefore $N_1, N_2, ..., N_m$ are independent random variables given $\Lambda_1, ...\Lambda_m$ for all $n_i \epsilon \mathbb{N}_0$ and the conditional independence of $(N_1, ...N_m)$ given $(\Lambda_1, ...\Lambda_m)$ i.e.,

$$\mathbb{P}[N_1 = n_1, ..., N_m = n_m|\Lambda_1, ...\Lambda_m] = \prod_{i=1}^{m} \mathbb{P}[N_i = n_i|\Lambda_1, ..., \Lambda m] = \prod_{i=1}^{m} e^{-\Lambda_i}\frac{\Lambda_i^{n_i}}{n_i!}.$$

Then, we have

$$\mathbb{P}[N_1 = n_1, ..., N_m = n_m] = \mathbb{E}[\prod_{i=1}^{m} e^{-\Lambda_i}\frac{\Lambda_i^{n_i}}{n_i!}] = \int_{[0,\infty)} e^{(\lambda_1+...+\lambda_m)} \prod_{i=1}^{m} \frac{\lambda_i^{n_i}}{n_i!} F(d\lambda_1, ..., d\lambda_m).$$

From these relations ,the expectation and variance are

$$\mathbb{E}[N_i|\Lambda_i] = \Lambda_i \qquad (a.s) \tag{2.3.15}$$

$$Var(N_i|\Lambda_i) = \Lambda_i \tag{2.3.16}$$

$N = \sum_{i=1}^{m} N_i$ is the random variable which represents the total number of default, with the expectation

$$\mathbb{E}[N] = \sum_{i=1}^{m} \mathbb{E}[N_i] = \sum \mathbb{E}[\mathbb{E}[N_i|\Lambda_i]] = \sum_{i=1}^{m} \mathbb{E}[\Lambda_i], \tag{2.3.17}$$

the variance,

$$Var(N) = \sum_{i=1}^{m} Var(N_i) + \sum_{i,j=1;i\neq j} Cov(N_i, N_j), \tag{2.3.18}$$

where

$$Var(N_i) = \mathbb{E}[(N_i - \mathbb{E}[N_i])^2]$$

$$= \mathbb{E}[\mathbb{E}(N_i - \mathbb{E}[N_i|\Lambda_i]] + \mathbb{E}[N_i|\Lambda_i] - \mathbb{E}[N_i])^2|\Lambda_i] + \mathbb{E}[\mathbb{E}[(N_i - \mathbb{E}[N_i|\Lambda_i])^2|\Lambda_i] + \mathbb{E}[(\mathbb{E}(N_i|\Lambda_i)) - \mathbb{E}[N_i]]^2$$

$$Var(N_i) = \mathbb{E}[Var(N_i|\Lambda_i)] + Var[\mathbb{E}[N_i|\Lambda_i]] = E[\Lambda_i] + Var[\Lambda_i] \tag{2.3.19}$$

and, for $i \neq j$

$$Cov(N_i, N_j) = \mathbb{E}[N_i, N_j] - \mathbb{E}[N_i]\mathbb{E}[N_j] = \mathbb{E}[\Lambda_i, \Lambda_j] - \mathbb{E}[\Lambda_i]\mathbb{E}[\Lambda_j] = Cov(\Lambda_i, \Lambda_j) \tag{2.3.20}$$

by using the conditional independence of N_i and N_j

$$\mathbb{E}[N_i, N_j] = \mathbb{E}[\mathbb{E}[N_i N_j|\Lambda_1, ..., \Lambda_m]] = \mathbb{E}[\mathbb{E}[N_i|\Lambda_i]\mathbb{E}[N_j|\Lambda_j]] = \mathbb{E}[\Lambda_i, \Lambda_j]. \tag{2.3.21}$$

2.4 PROBABILITY GENERATING FUNCTIONS

Definition 2.3. For an integer valued random Variable $L : \Omega \to N_0$, the probability generating function is

$$G_L^{(n)} = \mathbb{E}[s^L] = \sum_{n=0}^{\infty} s^n \mathbb{P}[L = n].$$

2.4.1 Some Basic Properties of Probability Generating Functions

- $G_L(0) = \mathbb{P}[L = 0]$
- $G_L(1) = \sum_{n=0}^{\infty} s^n \mathbb{P}[L = n] = 1$
- $G_L^n(0) = n!\mathbb{P}[L = n] \; n\epsilon\mathbb{N}_0$
- $G_L'(s) = \mathbb{E}[Ls^{L-1}]$ and $G_L''(s) = \mathbb{E}[L(L-1)s^{L-2}]$
- $G_L'(1-) = \mathbb{E}[L]$ and $G_L'' = \mathbb{E}[L(L-1)]$.

Therefore,

$G_Y' = \mathbb{E}[Y]$ and

$\sigma_Y^2 = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = G_Y''(1) + G_Y'(1) - G_Y'(1)^2$

$G_L' = G_L(lnG_L)'$ and $G_L''(lnG_L)''(1) + (lnG_L)'(1)$

Theorem 2.1. (Multiplication Theorem) Suppose that $X, Y : \Omega \to \mathbb{N}_0$ are independent, then

$$G_{X+Y}(s) = \mathbb{E}[s^{X+Y}] = \mathbb{E}[s^X]\mathbb{E}[s^Y] = G_X(s).G_Y(s).$$

2.4.2 The General Mixture Model in Terms of A Probability Generating Function

Let N be the total number of defaults, at least for all $s\epsilon\mathbb{C}$ with $|s| \leq 1$,

$$G_N(s) = \mathbb{E}[s^{N_1+...+N_m}] = \mathbb{E}[\mathbb{E}[s^{N_1+...+N_m}|\Lambda_1, ..., \Lambda_m]] = \mathbb{E}[\prod_{i=1}^{m} \mathbb{E}[s^{N_i}|\Lambda_i]] = \mathbb{E}[e^{(\Lambda_1+...+\Lambda_m)(s-1)}].$$

If Λ_i's are independent then,

$$G_N(s) = \prod_{i=1}^{m} \mathbb{E}[e^{\Lambda_i(s-1)}].$$

2.4.3 The Gamma Mixed Poisson Distribution

Suppose that intensity Λ has is a Gamma distribution with parameters $\alpha, \beta > 0$, and we note $\Lambda \sim (\alpha, \beta)$, the density

$$f_\Lambda(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda},$$

where

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx.$$

We note that $\alpha\Gamma(\alpha) = \Gamma(\alpha+1)$, $\Gamma(1) = 1$ and $\Gamma(n) = (n-1)!$ for $n\epsilon\mathbb{N}$.

The expectation and the variance of Gamma Distribution is

$$\mathbb{E}[\Lambda] = \frac{\Gamma(\alpha+1)}{\beta\Gamma(\alpha)} = \frac{\alpha}{\beta}$$

$$\mathbb{E}[\Lambda^2] = \frac{\Gamma(\alpha+2)}{\beta^2\Gamma(\alpha)} = \frac{\alpha(\alpha+1)}{\beta^2}$$

$$Var[\Lambda] = \mathbb{E}[\Lambda^2] - \mathbb{E}[\Lambda]^2 = \frac{\alpha}{\beta^2}$$

2.5 THE DISCRETE FOURIER TRANSFORM

This part is actually inspired from the book "'Fourier Analysis and Its Applications"' written by Gerald B. Folland.

The discrete Fourier transform is a linear mapping that operates on complex N-dimensional vectors in much the same way that the Fourier transform operates on functions on $\mathbb{R}$. This method is actually a numerical approximation of Fourier transform. The continuous Fourier transform is

$$\hat{f}(z) = \int_{-\infty}^{\infty} e^{-ixz} f(x)dx. \tag{2.5.22}$$

We need to apply the Fourier transform to a finite number of algebraic calculations performed on a finite set of data. Therefore, we can replace the integral over (-∞,∞) by the interval of the finite interval. If we take the finite interval as [0,Ω]. $\hat{f}$ values can be constricted from its value at point $2\pi m/\Omega$ where m is an integer. Hence, the equations can be written as

$$\hat{f}(\frac{2\pi m}{\Omega}) = \int_{0}^{\Omega} e^{-2\pi imx/\Omega} f(x)dx \qquad .. \tag{2.5.23}$$

Notice that the choice of the bounded interval can be determined considering the behavior of $\hat{f}$(z).

Replacing the above integral by Riemann Sum,

$$\hat{f}(\frac{2\pi m}{\Omega}) \approx \sum_{n=0}^{N-1} e^{(-2\pi imn/N)} f(\frac{n\Omega}{N})\frac{\Omega}{N}. \tag{2.5.24}$$

We can write that

$$if \quad f(\frac{n\Omega}{N}) = a_n \quad . \tag{2.5.25}$$

Then

$$\hat{f}(\frac{2\pi m}{\Omega}) \approx \frac{\Omega}{N} a_m \tag{2.5.26}$$

for $|m| << N$, where

$$\hat{a}_m = \sum_{n=0}^{N-1} e^{-2\pi imn/N} a_n. \tag{2.5.27}$$

Since $e^{-2\pi im}$=1, the sequence $\hat{a}_m$ is periodic with period N such that $\hat{a}_{M+N} = \hat{a}_n$. So, the information is completely continuous in the finite sequence $\hat{a}_0 ... \hat{a}_{N-1}$.

Finally, the vector $\mathbf{a} \in C^N$ can be transformed into another vector $\hat{a} \in C^N$ Formally, the N-point discrete Fourier Transform is a linear mapping such that

$$F_N : C^N \rightarrow C^N \tag{2.5.28}$$

$$F_N a = \hat{a} \quad . \tag{2.5.29}$$

$$\hat{a}_m = \sum_{n=0}^{N-1} e^{-2\pi imn/N} a_n \qquad (0 \leq m < N). \tag{2.5.30}$$

In addition, the convolution property of the discrete Fourier transform can be written as

$$F_N(a * b) = (\hat{a}_0 \hat{b}_0, ..., \hat{a}_{N-1} \hat{b}_{N-1}), \tag{2.5.31}$$

where the discrete convolution $a * b$ is defined by

$$(a * b)_n = \sum_{k=0}^{N-1} a_k b[n-k]. \tag{2.5.32}$$

In summary, this technique transforms the real sequence of numbers into a sequence of complex numbers having the same dimensions. Therefore, the discrete Fourier transform does not yield completely real sequence.

Chapter 3

CREDITRISK+

3.1 THE STANDARD MODEL

CreditRisk+ was introduced by Credit Suisse Financial Products(CSFP) in October 1997, and created by Tom Wilde of CSFP. Essentially, the success of this actuarian based model comes from its simplicity, since it only focuses on the default event of an obligor. The standard model is an analytical portfolio approach, so the calculation of loss distribution does not require much computational effort. Moreover, it demonstrates tails of distribution more explicitly than any other simulation based approach. On the other hand, the main disadvantage is that the Poisson approximation in CreditRisk+ requires expected default probabilities to be small. Thus, the loss calculation and the risk contributions may have some error for riskier markets.

3.1.1 The Mathematical Foundations

Before illustrating the mathematical details of the model, we should note that this section is based on the technical document of the original model [22].

Consider a portfolio consisting of N obligor. It is assumed that each obligor has a default probability for a specified time horizon, generally one year. Let

p_A denote the default probability of obligor A, and the probability generating function can be written in terms of an auxiliary variable z by

$$F(z) = \sum_{n=0}^{\infty} \mathbb{P}(n \ \ default) z^n. \tag{3.1.1}$$

The main assumption of CreditRisk+ was the Poissonian default event, thus we know that

$$\mathbb{P}(n \ \ default) = \frac{e^{-\mu}\mu^n}{n!}. \tag{3.1.2}$$

As it is seen from the equation, the distribution has only one parameter μ that represents the expected number of default. The generating function is

$$F(z) = \sum_{n=0}^{\infty} \frac{e^{-\mu}\mu^n}{n!} z^n. \tag{3.1.3}$$

In order to reduce computational burden CreditRisk+ uses exposure bands instead of real exposures. In other words, there are certain levels of integer valued amounts which are called as bands are taken into the model. According to the model, each exposure is rounded to the nearest integer number and the nearest band. More explicitly, each obligor credit amount is sent to the nearest state(band). Then it evaluates the loss distribution from these exposure bands. This procedure dramatically reduce computational burden, although the output is less accurate. The portfolio can be divided into m exposure bands represented by the index j where $1 \leq j \leq m$. According to the original model, $\nu_j, \varepsilon_j, \mu_j$ are common exposure, expected loss and expected number of defaults in exposure band j respectively. Then, the expected loss in terms of the probability default events can be written as

$$\varepsilon_j = \nu_j \times \mu_j, \tag{3.1.4}$$

$$\mu_j = \frac{\varepsilon_j}{\nu_j} = \sum_{A:\nu_A=\nu_j} \frac{\varepsilon_a}{\nu_A}. \tag{3.1.5}$$

In addition, let μ be the total expected number of default events in one year, then

$$\mu = \sum_{j=1}^{m} \mu_j = \sum_{j=1}^{m} \frac{\varepsilon_j}{\nu_j} \tag{3.1.6}$$

The model does not use fixed default rates. But for the simplest case, we can assume that default rate is fixed. Let G(z) denote the probability generating function of losses depending on a certain default rate. It can be expressed as in the multiplies of unit L of exposure

$$G(z) = \sum_{n=0}^{\infty} \mathbb{P}(AggregateLosses = n \times L) z^n \tag{3.1.7}$$

Since the exposures are assumed to be independent, the exposure bands are also independent. Therefore,

$$G(z) = \prod_{i=1}^{m} G_i(z). \tag{3.1.8}$$

The following formula represents the probability generating function for the j^{th} band, i.e.

$$G_j = \sum_{n=0} \mathbb{P}(n \ \ defaults) z^{n\nu_j} = \sum_{n=0}^{\infty} \frac{e^{-\mu_j}\mu_j^n}{n!} = e^{-\mu_j+\mu_j z^{\nu_j}}. \tag{3.1.9}$$

In other words, in case an n default occur in j^{th} band of the portfolio, the characteristic function of loss can be expressed by this formula.

Thus,

$$G(z) = \prod_{j=1}^{m} e^{-\mu_j + \mu_j z^{\nu_j}} = e^{-\sum_{j=1}^{m} + \sum_{j=1}^{m} \mu_j z^{\nu_j}}. \tag{3.1.10}$$

Above formula is the probability generating function for losses of the portfolio for default losses with fixed default rate.

From this initial point of view, it is possible to develop an actual generating function of loss by incorporating default rate uncertainty and sector analysis. The model assumes that the default rate is random variable, and it depends on sector characteristics. CreditRisk+ divides the portfolio into sectors represented as $S_k : 1 \leq k \leq n$. Further notations can be illustrated below.

χ_k: Random variable representing the mean number of defaults

μ_k: The long term average number of defaults

σ_k: The standard deviation of χ_k

Therefore,

$$\mu_k = \sum_{j=1}^{m(k)} \frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}}. \tag{3.1.11}$$

One can estimate the standard deviation of sectors σ_k from the set σ_A of obligor standard deviations by an averaging process. The model states that

$$\chi_A = \frac{\varepsilon_A}{\nu_A} \frac{\chi_k}{\mu_k}. \tag{3.1.12}$$

In particular, we have

$$\sum_A \sigma_A = \sum_A \frac{\varepsilon_A \sigma_k}{\nu_A \mu_k} = \frac{\sigma_k}{\mu_k} \sum_A \frac{\varepsilon_A}{\nu_A} = \sigma_k. \tag{3.1.13}$$

The key assumption of CreditRisk+ is that χ_k is a Gamma distributed random variable with mean μ_k and standard deviation σ_k. In the previous chapter we emphasized some features of Gamma distribution.

For the k^{th} sector, the conditional probability generating function given χ_k is

$$F_k(z) = \sum_{n=0}^{\infty} \mathbb{P}(n \ defaults) z^n = \sum_{n=0}^{\infty} z^n \int_{x=0}^{\infty} \mathbb{P}(ndefault|\chi_k) f(x) dx. \quad (3.1.14)$$

where F is the probability density of k.

$$F_k = \int_{x=0}^{\infty} e^{x(z-1)} f(x) dx. \quad (3.1.15)$$

Replacing f by the Gamma distribution density we get

$$F_k = \int_{x=0}^{\infty} e^{x(z-1)} \frac{e^{-\frac{-x}{\beta}} x^{\alpha-1}}{\beta^{\alpha}\Gamma(\alpha)} dx. \quad (3.1.16)$$

$$F_k(z) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_{y=0}^{\infty} (\frac{y}{\beta^{-1}+1-z})^{\alpha-1} e^{-y} \frac{dy}{\beta^{-1}+1-z}$$

$$\frac{\Gamma(\alpha)}{\beta^{\alpha}\Gamma(\alpha)(1+\beta^{-1}-z)^{\alpha}} = \frac{1}{\beta^{\alpha}(1+\beta^{-1}-z)^{\alpha}}, \quad (3.1.17)$$

where $\alpha_k = \frac{\mu_k^2}{\sigma_k^2}$ and $\beta_k = \sigma_k^2/\mu_k$

According to the technical document, the probability generating function of the distribution of default event is

$$F_k(z) = \frac{1}{\beta^{\alpha}(1+\beta^{-1}-z)^{\alpha}}.$$

Equally,

$$F_k = (\frac{1-p_k}{1-p_k z})^{\alpha_k},$$

where

$$p_k = \frac{\beta_k}{1+\beta_k}.$$

This result states that the distribution of default events can be identified as the probability density of the negative binomial distribution.

By analogy, the portfolio loss distribution is simply

$$G(z) = \sum_{n=0}^{\infty} \mathbb{P}(AggregateLosses = n \times L) z^n.$$

Since the sectors are independent,

$$G(z) = \prod_{k=1}^{n} G_k(z) bosluk 1 \leq k \leq n,$$

Let's define $P_k(z)$ as

$$P_k(z) = \frac{\sum_{j=1}^{m(k)} (\frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}}) z^{\nu_j^{(k)}}}{\sum_{j=1}^{m(k)} (\frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}})} = \frac{1}{\mu_k} \sum_{j=1}^{m(k)} (\frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}}) z^{\nu_j^{(k)}}.$$

Notice that it is possible to write the following equation, i.e.,

$$\mathrm{e}^{\sum_A \chi_A + \sum_A \chi_A z^{v_A}} = e^{\sum_A (z_A^v - 1)} = e^{\frac{\chi_k}{\mu_k} \sum_A \frac{\epsilon_A}{\nu_A}(z_A^\nu - 1)} = e^{\chi_k (P_k(z)-1)}.$$

(3.1.18)

Using this equation we can write that

$$G_k(z) = \sum_{n=0}^{\infty} z^n \int_{\chi_k=0}^{\infty} \mathbb{P}(Loss\ \ of\ \ nL|\chi_k) f_k(x) dx_k \tag{3.1.19}$$

$$G_k(z) = \int_{x_k=0}^{\infty} e^{\sum_A x_A(z^{v_A}-1)} f_k(x_k)dx_k = \int_{x_k=0}^{\infty} e^{x_k(P_k(z)-1)} f_k(x_k)dx_k. \tag{3.1.20}$$

Then replacing $P_k(z)$ into the equation we get

$$G_k(z) = F_k(P_k(z)). \tag{3.1.21}$$

Thus, final result becomes

$$G(z) = \prod_{k=1}^{n} G_k(z) = \prod_{k=1}^{n} \left(\frac{1-p_k}{1-\frac{p_k}{\mu_k}\sum_{j=1}^{m(k)} \frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}} z^{\nu_j^{(k)}}}\right)^{\alpha_k}. \tag{3.1.22}$$

This formula is the final result of probability generating function of the CreditRisk+ which gives us complete information on the portfolio loss distribution. The original model uses the Panjer Recursion technique to demonstrate the actual loss distribution.

3.1.2 The Panjer Recursion Technique

Although we obtain the analytical formula, it is not easy to compute the probability generating function of an entire credit portfolio in practice. The standard model uses the Panjer recursion technique which is briefly explained below.

In general, the power series

$$G(z) = \sum_{n=0}^{\infty} A_n z^n$$

by taking the differentials of both sides,

$$\frac{d(logG(z))}{dz} = \frac{1}{G(z)}\frac{dG(z)}{dz} = \frac{A(z)}{B(z)} = \frac{a_0 + a_1 z^1 + ... + a_r z^r}{b_0 + b_1 z^1 + ... + b_s z^s}.$$

The recursion relation is

$$A_{n+1} = \frac{1}{b_0(n+1)}[\sum_{i=0}^{min(r,n)} a_i A_{n-i} - \sum_{j=0}^{min(s-1,n-1)} b_{j+1}(n-j)A_{n-j}].$$

Combining with previous equations

$$\frac{A(z)}{B(z)} = \sum_{k=1}^{n} \frac{\frac{p_k \alpha_k}{\mu_k} \sum_{j=1}^{m(k)} \varepsilon_j^{(k)} z^{v_j^{(k)}-1}}{1 - \frac{p_k}{\mu_k} \sum_{j=1}^{m(k)} \frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}} z^{\nu_j^{(k)}}}.$$

3.1.3 General Sector Analysis

CreditRisk+ assumes that a portfolio is divided into sectors which are mutually independent. The model incorporates systematic factors for each obligor. An obligor may be influenced from several sectors instead of only one. Let θ_{Ak}

denote the weighted sector influence of the obligor A for sector k, such that

$$\theta_{Ak} : \sum_{k=1}^{n} \theta_{Ak} = 1. \tag{3.1.23}$$

In other words, it states how much sector k influences obligor A. According to the model

$$\sum_{k=1}^{n} \chi_k (P_k(z) - 1) = \sum_{A,k} \theta_{Ak} \frac{\chi_k}{\mu_k} \frac{\varepsilon_A}{\nu_A} (z^{\nu_A} - 1), \tag{3.1.24}$$

where

$$\chi_A = \frac{\varepsilon_A}{\nu_A} \sum_{k=1}^{n} \theta_{Ak} \frac{\chi_k}{\mu_k} \tag{3.1.25}$$

$$\mu_k = \sum_{A} \theta_{Ak} \frac{\varepsilon_A}{\nu_A}. \tag{3.1.26}$$

From this perspective, one can evaluate sector mean and standard deviations from obligors. The mean of sector k is the sum of the contribution of each obligor, such that

$$\mu_k = \sum_{A} \theta_{Ak} \mu_A. \tag{3.1.27}$$

It is possible to write

$$\frac{\sigma_k}{\mu_k} = \frac{\sum_A \theta_{Ak} \mu_A \frac{\sigma_A}{\mu_A}}{\sum_A \theta_{Ak} \mu_A}, \tag{3.1.28}$$

thus

$$\sigma_k = \sum_{A} \theta_{Ak} \sigma_A. \tag{3.1.29}$$

The specific factor is itself a portfolio composed of a large number of sub-sectors. For the specific factor, the standard deviation can be set at zero.

3.1.4 Risk Contributions and Pairwise Correlation

The risk contribution of an obligor can be expressed as the marginal effect of its exposure on the standard deviation of the distribution of credit losses. In other words, it gives us obligors' degree of riskiness in the portfolio. Mathematically, it is the sensitivity of the standard deviation of the portfolio with respect to the obligors exposures.

$$RC_A = E_A \frac{\partial \sigma}{\partial E_A} \tag{3.1.30}$$

equally

$$RC_A = \frac{E_A}{2\sigma} \frac{\partial \sigma^2}{\partial E_A}. \tag{3.1.31}$$

Our assumption is that the sum of the risk contributions gives us the standard deviation of the portfolio, i.e.

$$\sum_A RC_A = \frac{1}{2\sigma} \sum_A E_A \frac{\partial \sigma^2}{\partial E_A} = \frac{2\sigma^2}{2\sigma} = \sigma \tag{3.1.32}$$

$$\sum_A RC_A = \sigma. \tag{3.1.33}$$

The original document states the final result as

$$RC_A = \frac{E_A \mu_A}{\sigma} [E_A + \sum_k (\frac{\sigma_k}{\mu_k})^2 \varepsilon_k \theta_{Ak}], \tag{3.1.34}$$

where

$$\sigma^2 = \sum_{k=1}^{n} \varepsilon_k^2 (\frac{\sigma_k}{\mu_k})^2 + \sum_A \varepsilon_A \nu_A \tag{3.1.35}$$

These are the formula required to calculate an obligor's risk contributions. The derivation of this formula is illustrated below according to the original technical document [22].

Remember that

$$G(z) = F(P(z)).$$

Let us denote E(z,x) as the default event probability generating function conditional on mean x, and let μ_E and σ^2 denote the mean and the variance of function E. In addition, the pairs (μ_f, σ_f^2), (μ_F, σ_F^2) and (μ_G, σ_G^2) also represent the mean and the variance of functions f(x), F(Z) and G(z). We note that $\mu_F = \mu_k$ and $\mu_G = \epsilon_k$. We can write the default event probability generating function F(z) as follows:

$$F(z) = \int_x E(z,x) f(x) dx. \tag{3.1.36}$$

Since E(z,x) is the generating function for the Poisson distribution, by definition

$$\mu_E(x) = x \qquad \sigma_E^2 = \mu_E. \tag{3.1.37}$$

We can also represent mean (μ) for other generating functions such that

$$\mu_E = \frac{dE}{dz}(1), \qquad \mu_F = \frac{dF}{dz}(1), \qquad \mu_G = \frac{dG}{dz}(1) \tag{3.1.38}$$

$$\sigma_E^2 + \mu_E^2 = \frac{d^2E}{dz^2}(1) + \frac{dE}{dz}(1) \tag{3.1.39}$$

$$\sigma_F^2 + \mu_F^2 = \frac{d^2F}{dz^2}(1) + \frac{dF}{dz}(1) \tag{3.1.40}$$

$$\sigma_G^2 + \mu_G^2 = \frac{d^2G}{dz^2}(1) + \frac{dG}{dz}(1), \tag{3.1.41}$$

from the definition,

$$\mu_E(x) = x \tag{3.1.42}$$

Since E(z,x) is the generating function of a Poisson distribution,

$$\sigma_E^2 = \mu_E \tag{3.1.43}$$

using these equations,

$$\mu_F = \int_x \mu_E(x) f(x) dx = \int_x x f(x) dx = \mu_f \tag{3.1.44}$$

and,

$$\sigma_F^2 + \mu_F^2 = \int_x (\sigma_E^2 + \mu_E^2) f(x) dx = \int_x (\mu_E + \mu_E^2) f(x) dx = \mu_f + \sigma_f^2 + \mu_f^2. \tag{3.1.45}$$

Thus,

$$\sigma_F^2 = \mu_f + \sigma_f^2. \tag{3.1.46}$$

Here, we have evaluated mean and variance formula of the distribution of default events. By using these equations, we can derive the equations for an entire distribution of portfolio.

Let us take the first and second derivative of G(z) w.r.t. z.

$$\frac{dG}{dz}(z) = \frac{dF}{dz}(P(z)) \frac{dP}{dz} \tag{3.1.47}$$

$$\frac{d^2G}{dz^2}(z) = \frac{d^2F}{dz^2}(z)(P(z))[\frac{dP}{dz}]^2 + \frac{dF}{dz}(P(z)) \frac{d^2P(z)}{dz^2}. \tag{3.1.48}$$

So

$$\sigma_G^2 = \frac{d^2}{dz^2}(1) + \frac{dG}{dz}(1) - \mu_G^2. \quad (3.1.49)$$

Then,

$$\sigma_G^2 = \frac{d^2F}{dz^2}P(1)\frac{dP}{dz}(1)^2 + \frac{dF}{dz}(P(1))\frac{d^2P}{dz^2}(1) \quad (3.1.50)$$

$$+\frac{dF}{dz}(P(1))\frac{dP}{dz}(1) - [\frac{dF}{dz}P(1)\frac{dP}{dz}(1)]^2 \quad (3.1.51)$$

Where, P(1)=1,

$$\frac{dP}{dz}(1) = \frac{1}{\mu_k}\sum_A \theta_{Ak}\epsilon_A = \frac{\epsilon_k}{\mu_k} \quad (3.1.52)$$

$$\frac{d^2P}{dz^2}(1) = \frac{1}{\mu_k}\sum_A \theta_{Ak}\epsilon_A(\nu_A - 1). \quad (3.1.53)$$

Substituting these equations, we can get the variance formula of G(z), i.e.,

$$\sigma_G^2 = (\sigma_k^2 + \mu_k^2)(\frac{\epsilon_k}{\mu_k})^2 + \sum_A \theta_{Ak}\epsilon_A\nu_A - \epsilon_k^2 \quad (3.1.54)$$

$$= \sigma_k^2[\frac{\epsilon_k}{\mu_k}]^2 + \sum_A \theta_{Ak}\epsilon_A\nu_A \quad (3.1.55)$$

More explicitly, the standard deviation of the actual loss distribution of the portfolio is

$$\sigma^2 = \sum_{k=1}^{n} \epsilon_k^2(\frac{\sigma_k}{\mu_k})^2 + \sum_A \epsilon_A\nu_A. \quad (3.1.56)$$

In addition, the CreditRisk+ model illustrates the pairwise correlation between default events. The correlation ρ between default of two obligors A and B can be written as

$$\rho_{A,B} = \rho(\mathbb{I}_A, \mathbb{I}_B), \quad (3.1.57)$$

where $\mathbb{I}_A, \mathbb{I}_B$ are the default indicators. The standard expression of the pairwise correlation is

$$\rho_{A,B} = \frac{\mu_{AB} - \mu_A\mu_B}{(\mu_A - \mu_A^2)^{1/2}(\mu_B - \mu_B^2)^{1/2}}. \qquad (3.1.58)$$

The final result is

$$\rho_{AB} = (\mu_A\mu_B)^{1/2}\sum_{k=1}^{n}\theta_{Ak}\theta_{Bk}(\frac{\sigma_k}{\mu_k})^2. \qquad (3.1.59)$$

Accordingly, as it is seen from the equation, the correlations can be calculated from sector statistics. This is an implied method which does not require direct calculation of pairwise correlation.

3.2 RISK MEASURES AND CAPITAL ALLOCATION IN CREDITRISK+

3.2.1 Value at Risk (VaR) and Expected Shortfall (ES)

In credit industry, value at risk framework is one of the most commonly used risk measurement framework. However there are many publications illustrating deficiencies of the VaR approach. The main reason for this is that VaR is not a coherent risk measure. In order to be a coherent risk measure, four conditions should be satisfied: subadditivity, monotonicity, positive homogeneity, translation invariance [32]. VaR is a risk measure which does not have the subadditivity property.

Formally VaR can be defined for a $\mathbb{P}$ probability measure and some confidence interval α.

$$VaR_\alpha(X) = inf\{x \geq 0|\mathbb{P}[X \leq x] \geq \alpha\}, \qquad (3.2.60)$$

where X is a α-quantile of a loss random variable.

On the other hand, the expected short fall (ES) (or tail conditional expectation) is a coherent risk measure which focuses on the behavior of the tails of the distribution. Formally, expected shortfall with respect to α confidence level can be defined as

$$ES_\alpha(X) = \mathbb{E}[X|X \geq VaR_\alpha(X)]. \tag{3.2.61}$$

In other words, the expected shortfall mainly focuses on expected loss beyond a critical point c in the tail. This is the $c = VaR_\alpha(X)$ critical loss threshold with respect to α confidence level.

In the context of an expected shortfall, the economic capital is

$$EC_{ES}(c) = \mathbb{E}[X|X \geq c] - \mathbb{E}[X], \tag{3.2.62}$$

where

$$c = VaR_\alpha(X). \tag{3.2.63}$$

Chapter 4

FAST FOURIER TRANSORM TECHNIQUE IN CREDITRISK+

In the FFT approach, in order to compute loss distribution, we use a characteristic function instead of a probability generating function. The main advantage of this technique stems from its computational efficiency for larger portfolios having several sectoral dependencies. Moreover, this algorithm is numerically more stable and faster than the standard model which is based on the Panjer recursion technique.

4.1 The Algorithhm of FFT in CreditRisk+

In general, this technique transforms a purely real sequence into a complex plane. In CreditRisk+, we deal with a discrete probability generating function, so it is possible to apply the FFT technique in order to obtain the characteristic function of distribution. Generally, a simple form of the characteristic function can be written as

$$\varphi_x(z) = \mathbb{E}[e^{izx}].$$

Let us denote the probability generating function as G(z), and the relation can be expressed as follows.

$$\varphi_x(z) = FG_x(z).$$

Therefore,

$$\varphi_x(z) = G_x(e^{-iz}).$$

The coefficient vector of the probability generating function, in other words the probability vector, gives us sufficient information about the shape of the loss. If we know the probability generating functions of the obligors or their exposure bands, it is possible to construct a portfolio loss distribution by the FFT technique. Assuming that we have N independent sectors, by incorporating the systematic risks of the obligors, the portfolio can be divided into N+1 sub-portfolios considering corresponding sector weights. For each sub-portfolio, discrete characteristic functions can be evaluated from the obligors' probability vector. The convolution property helps us to achieve portfolio loss distribution combining the characteristic functions of each obligor and the sub-portfolios.

As mentioned, the original model uses exposure bands and a basic loss unit in order to have a faster and more powerful computational algorithm. But, the higher computational speed decreases the accuracy of the model. On the other hand, in order to obtain higher accuracy, one should not take into account large exposure bands and basic loss unit. In addition, this approach needs powerful computers and may take too much time. In this study, we prefer using exposure bands and basic loss unit. The basic steps of the algorithm is briefly explained below.

- **Data calibration**

As a first step, one should determine a unit of exposure L and group the exposures in the portfolio into bands. After dividing the portfolio into m exposure bands, the total exposures and expected loss of each band should be evaluated. Denoting ν_j and ε_j to be exposures and expected loss for j^th band and μ_j denotes the number of expected loss, where $1 \leq j \leq m$, the following relation should be satisfied:

$$\varepsilon_j = \nu_j \times \mu_j.$$

Then, assuming that the default event is a poissonian process, the probability vector of each band should be obtained from the following relation.

$$\mathbb{P}(n \ defaults) = \frac{\mu_j e^{-\mu_j}}{n!}$$

for $n = 1, 2, ...2^r$, $j = 1, 2, ..., m$ where r is an integer.

- **Dividing the portfolio into N+1 sub-portfolios:**

  Assuming that we have N independent sectors, and considering corresponding sector weights, the portfolio should be divided into N+1 sub-sectors by incorporating the systematic risks. The algorithm should be applied to each generated sub-portfolio which are assumed to be independent from each other. It is possible to obtain probability generating functions and the probability vector (coefficients of the PGF's) for each exposure band of that sub-sectors. Note that r is an integer which should be sufficiently large to have a better and more accurate result.

- **Obtaining Probability Generating Functions of Sub-portfolios**

Let f_j^s denote a probability vector of j^{th} band for the sector s. By using the FFT technique, its characteristic function can be written as

$$\varphi_j^s = FFT(f_j^s).$$

By using a convolution property, the characteristic function of the sector s becomes

$$\varphi^s = \prod_{j=1}^{m} [\varphi_j^s].$$

Then the inverse fast fourier transform gives us the probability generating function of the sector s,

$$G^s = IFFT(\varphi^s).$$

- **Calculating Portfolio Loss Distribution**

  At the final step, we can evaluate the probability generating function of the entire portfolio. Combining all generating functions, the characteristic function of the portfolio is written as

$$\varphi = \prod_{s=0}^{N} (\varphi^s).$$

Then, the inverse fast fourier transform gives us the final probability vector

$$G = IFFT(\varphi).$$

The probability vector G contains sufficient information about loss distribution of our credit portfolio. Since we are dealing with the complex numbers in the Fourier space, the final generating function also have imaginary coefficients. Therefore, one should only take the real part of the final probability vector.

4.2 The Mixed Model: The Combination of Fast Fourier(FFT) and Panjer Recursion Technique

Alternatively, we can also combine the FFT technique and Panjer's algorithm in a single model. We propose that the fast fourier transform can be applied to the sector's probability generating functions which are obtained from Panjer's algorithm. In other words, the distribution of sub-portfolios can be obtained by Panjer recursion technique, and it is thought that the FFT algorithm can be applied to these marginal probability generating functions. In the standard model, the computational effort increases exponentially as the number of sectors increases. Therefore, in order to build faster algorithm, FFT technique can be applied to the sub-portfolios. We think that this technique have faster computational power, and yields more accurate distribution functions than the original version of the CreditRisk+.

Similar to the previous algorithm, the portfolio should be divided into sub-portfolios corresponding to the sector dependencies. We should apply Panjer Recursion algorithm for each sub-portfolio as stated in the original document. Thus, in this case, we would have several marginal portfolios having single sector.

According to the original technical study, the recursion relation is

$$A_{n+1} = \frac{1}{b_0(n+1)} \left[\sum_{i=0}^{min(r,n)} a_i A_{n-i} - \sum_{j=0}^{min(s-1,n-1)} b_{j+1}(n-j) A_{n-j} \right],$$

where A_n denotes the probability of amount of losses (n x unit loss) and, a and b are the coefficients of the polynomials A and B respectively.

$$\frac{A(z)}{B(z)} = \frac{\frac{p\alpha}{\mu} \sum_{j=1}^{m} \varepsilon_j z^{v_j - 1}}{1 - \frac{p}{\mu} \sum_{j=1}^{m} \frac{\varepsilon_j}{\nu_j} z^{\nu_j}}$$

where ϵ_j and ν_j are expected loss and total exposure respectively at j^{th} band, and p,α , μ are distribution parameters. In order to find coefficients a and b, we need to develop an algorithm which evaluates the coefficients of the polynomials accurately and fast. Although building the algorithm is easy for one sector portfolio, it is very hard to compute it in the multi-dimensional case. Accordingly, the FFT technique will give us a significant advantage in the computation for large sector dependencies portfolios. Similar to the previous algorithm, the discrete probability generating functions of the sub-portfolios are transformed into Fourier space to obtain their characteristic functions. We should apply same procedure as explained in the previous algorithm, so the loss distribution of the whole portfolio can be obtained. The MATLAB software can be preferred to perform this algorithm to for easy and fast computation.

4.3 Applications

The purpose of this section is to demonstrate the application of Fourier CreditRisk+ approach to an hypothetical portfolio. Moreover, we also illustrate and compare pure FFT, Panjer-FFT based approaches and standard CreditRisk+ model. The portfolio consisting of international treasury bonds varying credit quality and size of exposures. The sovereign bond portfolio consists of t-bonds of Argentina, Belgium, Belie, Brazil, Bulgaria, Canada, Czech Republic, Chile, Dominican, Ecuador, Estonia, France, India, Israel, Indonesia, Italy, Japan, Korea, Kuwait, Pakistan, Paraguay, Russia, Romania, Spain, Turkey, Ukraine, UK, USA, Mexico and Venezuelan. Notice that the exposure amounts are net recoveries. The details of the portfolio is illustrated at the table-1.

It should be noted that the credit rating data and corresponding default rates' and default rates volatilities are obtained from Standard and Poor's data source. In this application, we assume that each obligor are allocated to three different sectors with specific weights. during the application countries can be considered as sectors. Moreover obligors depend on only one systematic factor.

The essential goal is to obtain some specific information about riskiness of the credit portfolio. Specifically, we need to know percentiles of loss, full loss distribution and risk contributions of each obligors. In this part, we also demonstrate Value-at-Risk and Expected-Shortfall exposures of the portfolio. The sector weights are chart 4.2.

Countries	Exposures	Ratings	Default Rate	Std. Deviation
Argentina	9.560.000	B	15,00	7,50
Belgium	16.570.000	AA	3,00	1,50
Brazil	7.430.000	BB	10,00	5,00
Bulgaria	13.450.000	BBB	7,50	3,75
Czech Rep.	3.480.000	A	5,00	2,50
Chile	17.870.000	AA	3,00	1,50
Dominican	600.000	B	15,00	7,50
Ecuador	7.190.000	CCC	30,00	15,00
Estonia	19.590.000	A	5,00	2,50
France	15.700.000	AAA	1,50	0,75
India	14.170.000	BB	10,00	5,00
Indonesia	4.500.000	BB	10,00	5,00
Italy	8.700.000	A	5,00	2,50
Japan	13.920.000	AA	3,00	1,50
Korea	12.530.000	A	5,00	2,50
Mexico	18.640.000	A	5,00	2,50
Pakistan	7.430.000	BB	10,00	5,00
Paraguay	6.510.000	B	15,00	7,50
Russia	9.400.000	A	5,00	2,50
Romania	8.330.000	BBB	7,50	3,75
Spain	14.700.000	AAA	1,50	0,75
Turkey	6.300.000	BB	10,00	5,00
Ukraine	1.590.000	BB	10,00	5,00
UK	15.430.000	AAA	1,50	0,75
Venezuelan	8.560.000	BB	10,00	5,00

Table 4.1: *Sovereign Bond Portfolio Example, Standard and Poor's*

Country	Specific	Sector A	Sector B	Sector C
Argentina	50	30	10	10
Belgium	25	25	25	25
Brazil	75	5	10	10
Bulgaria	50	10	10	30
Czech Rep.	25	10	10	55
Chile	25	25	20	30
Dominican	25	25	25	25
Ecuador	75	10	5	10
Estonia	50	10	10	30
France	50	20	10	20
India	25	25	25	25
Indonesia	25	25	30	20
Italy	75	10	5	10
Japan	50	20	10	20
Korea	50	10	10	30
Mexico	75	5	10	10
Pakistan	25	10	10	55
Paraguay	25	25	20	30
Russia	75	10	5	10
Romania	25	25	20	30
Spain	50	20	10	20
Turkey	75	10	10	5
Ukraine	50	30	10	10
UK	25	25	20	30
Venezuelan	50	10	10	30

Table 4.2: *Sector Weights*

As it is emphasized previously that the loss distribution any credit portfolio is expected to be highly skewed. Therefore, summary statistics are not helpful to better understand the risk. So we need to know possible losses of each percentile level. In addition, for some spcific percentile levels, we also demonstrate Value at Risk (VaR) and Expected Shortfall (ES) exposures and risk contributions of the obligors.

In this application, we apply three different algorithm: The Standard CreditRisk+ model, the Pure-FFT and FFT-Panjer Technique. For each model, statistical characteristics are demonstrated.

The key point of these models is the calibration of the data set. For each algorithm, we apply some common techniques reduce the computational burden. The most significant point is that the real exposures are divided by a unit exposure L_0 and they are rounded to the nearest integer numbers. Notice that each integer valued new exposures are denoted as exposure bands and their corresponding default rates and default rates volatilities. This procedure is required to have faster algorithm. In this application, we choose L_0 unit money as a unit exposure. If we reduce this value, the accuracy of the model increase but the computational procedure takes much longer time. In order to have quick and dirty result, one can choose a higher unit exposure.

As it is explained in previous part, the standard model use Panjer recursive algorithm to produce generating function of the portfolio loss. The input parameters of the recursive scheme are evaluated from the characteristics of exposure bands and sector influence (weights) on obligors. On the other hand, the pure-FFT technique, we have primitive but very fast algorithm. Unlike the standard model, he behavior of sectors are not included into the model. Here, we apply FFT technique for each sub-portfolio which represents corresponding sectors. The another approach, FFT-Panjer, mixes the standard model and fast Fourier transform technique. This new approach allow us to incorporate several sectors into the model. It should be noted that in standard CreditRisk+ increasing number of sectors increase computational burden exponentially. However, in FFT-Panjer model, we apply FFT technique for each sub-portfolios' generating functions which are evaluated from Panjer's algorithm. As we demonstrated below, the statistics are very close the standard model.

For each model, we develop the computer programs in MATLAB software package. The output is illustrated in the charts

Aggregate Exposure	252.150.000 unit currency
Expected Loss	16.044.250 unit currency

	The Standard Model	FFT-Panjer	Pure FFT
Standard Deviation	13.309.316	13.230.806	13.023.691

Table 4.3: *Standard Deviations*

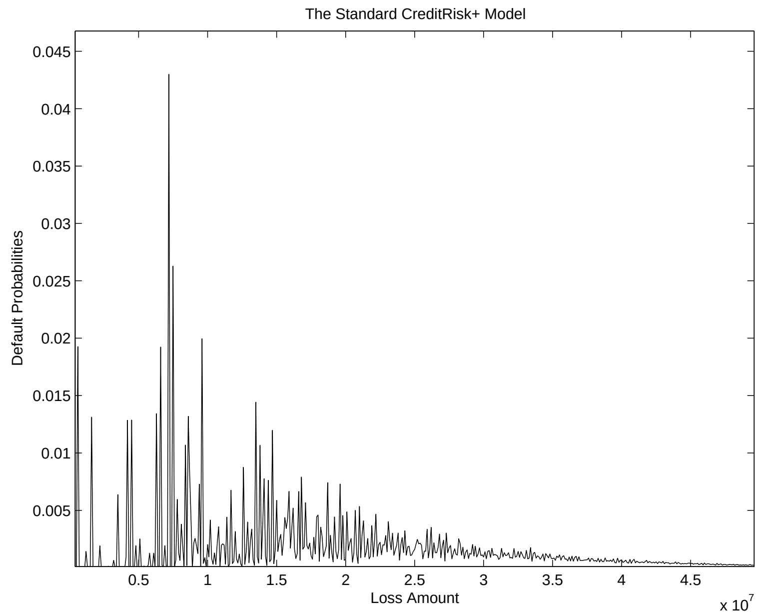

Figure 4.1: *Loss Distribution (The Standard CreditRisk+ Approach)*

Percentile VaR	Standard Model	FFT-Panjer	Pure-FFT
50	14100000	14100000	14100000
75	23700000	23700000	23600000
95	41200000	41200000	40700000
97.50	47600000	47600000	47000000
99	55600000	55600000	54700000
99.50	61400000	61400000	60200000
99.75	66900000	66800000	65600000
99.90	74000000	73800000	72200000

Table 4.4: *Value at Risk Exposures*

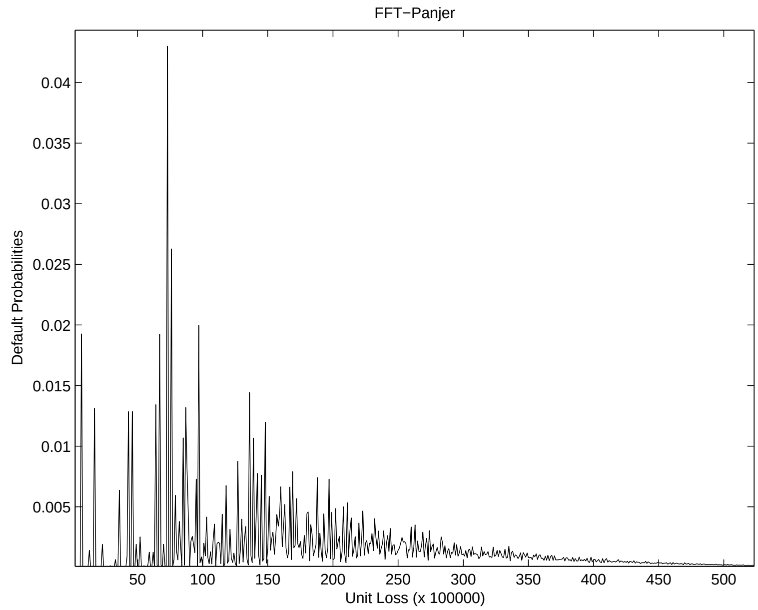

Figure 4.2: *Loss Distribution (The FFT-Panjer Model)*

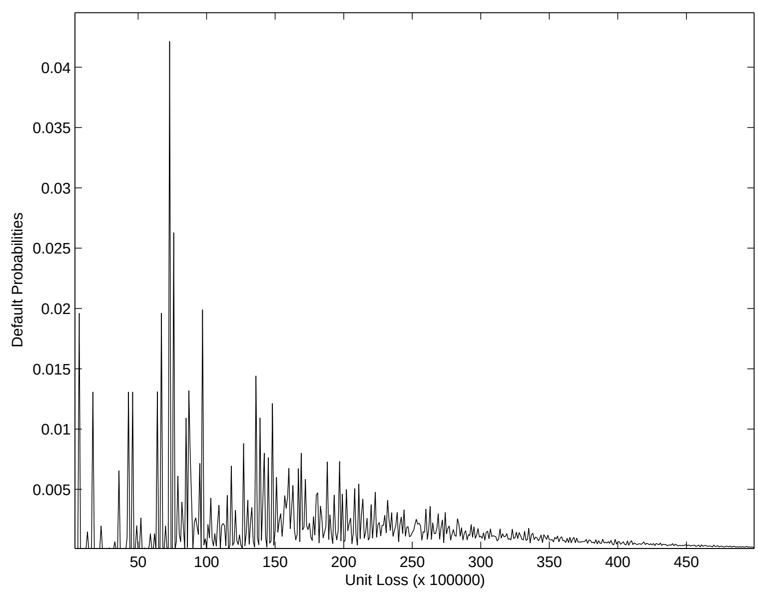

Figure 4.3: *Loss Distribution (The Pure FFT)*

Percentile VaR	Standard Model	FFT-Panjer	Pure-FFT
50	26398114	26398077	26184465
75	34444481	34444468	34103238
95	50717375	50717372	49735752
97.50	57480790	57480788	56184366
99	67711707	67711706	65330267
99.50	78090063	78090061	73879915
99.75	93674455	93273618	86388871
99.90	135271683	133482269	115765681

Table 4.5: *Expected Shortfall Risk Exposures*

According tho the results obtained, we can say that the outcomes are very close to each other for each model. Especially, the standard model and FFT-Panjer technique produce very similar results. The distribution outcome is less riskier in pure FFT model than other models. On the other hand, the standard CreditRisk+ algorithm represent the riskiest model. In addition, it is observed that depending on the numbers of sectors, FFT-Panjer algorithm is faster than the standard model in CPU time. But both these two models produce negligibly close outcomes. Therefore, one can rather choose FFT-Panjer algorithm especially for the large number of sectoral dependent portfolios.

Country	Unit Amount	Risk Percentiles
Argentina	4663355	0.0839
Belgium	2418294	0.0435
Brazil	2046989	0.0368
Bulgaria	4180300	0.0752
Czech Rep.	336254	0.0060
Chile	2767414	0.0498
Dominican	112209	0.0020
Ecuador	5752578	0.1035
Estonia	5394618	0.0971
France	1087942	0.0195
India	864718	0.0155
Indonesia	955043	0.0171
Italy	1305694	0.0235
Japan	1779216	0.0320
Korea	2473173	0.0445
Mexico	2117820	0.0381
Pakistan	2578404	0.0464
Paraguay	1484965	0.0267
Russia	1896607	0.0341
Romania	968906	0.0174
Spain	1547281	0.0278
Turkey	229411	0.0041
Ukraine	1071331	0.0192
UK	4914158	0.0884
Venezuelan	2600092	0.0468

Table 4.6: *Risk Contributions at 99th percentile (The Standard Model)*

Country	Unit Amount	Risk Percentiles
Argentina	4663416	0.0839
Belgium	2418330	0.0435
Brazil	2047014	0.0368
Bulgaria	4180359	0.0752
Czech Rep.	336257	0.0060
Chile	2767456	0.0498
Dominican	112210	0.0020
Ecuador	5752646	0.1035
Estonia	5394701	0.0971
France	1087958	0.0195
India	864727	0.0155
Indonesia	955052	0.0171
Italy	1305710	0.0235
Japan	1779242	0.0320
Korea	2473208	0.0445
Mexico	2117846	0.0381
Pakistan	2578434	0.0464
Paraguay	1484984	0.0267
Russia	1896631	0.0341
Romania	968920	0.0174
Spain	1547298	0.0278
Turkey	229413	0.0041
Ukraine	171347	0.0192
UK	4914233	0.0884
Venezuelan	2600125	0.0468

Table 4.7: *Risk Contributions at 99th percentile (FFT-Panjer)*

Country	Unit Amount	Risk Percentiles
Argentina	1755019	0.0713
Belgium	971700	0.0394
Brazil	878703	0.0356
Bulgaria	1952245	0.0793
Czech Rep.	255149	0.010
Chile	1197844	0.0486
Dominican Rep.	93705	0.0038
Ecuador	2515974	0.1022
Estonia	2297592	0.0933
France	404674	0.0164
India	523292	0.0212
Indonesia	544544	0.0221
Italy	521534	0.0211
Japan	689225	0.028
Korea	1176031	0.0477
Mexico	1468882	0.0596
Pakistan	1444779	0.0586
Paraguay	570981	0.0231
Russia	996803	0.0404
Romania	368866	0.0149
Spain	675416	0.0274
Turkey	166103	0.0067
Ukraine	439109	0.0178
UK	1333695	0.0541
Venezuelan	1370588	0.0556

Table 4.8: *Risk Contributions at 99th percentile (The Pure FFT Model)*

Chapter 5

INTEGER AUTOREGRESSIVE APPROACH TO CREDITRISK+

The original CreditRisk+ model does not consider how to estimate expected default frequencies. As input data, it only takes obligors' exposures, expected default probabilities and their standard deviations from corresponding rating matrices. Moreover, the default events in the past are not included in the model. In this work, we propose a dynamic credit risk modeling including the past default observations together with some specific macro economic variables such as inflation, interest rates etc. Our new econometric approach is simply based on an *integer-valued autoregressive process*(INAR). We give below the definition of the INAR process.

5.0.1 Integer Autoregressive(INAR) Process

Before giving the formal definition of INAR process, we should define what a *thinning* operator is. According to [46], the definition and properties are stated below.

Definition 5.0.1. (*Thinning Operator*)Let X be a non-negative integer random variable and $\alpha \in [0,1]$. Then, the *thinning*, denoted by $\circ$, can be defined as

$$\alpha \circ X = \sum_{i=1}^{X} Y_i, \tag{5.0.1}$$

where Y_i is a sequence of independent and identically distributed Bernoulli random variables which is independent of X with success probability α.

The properties of thinning operator,

- $\beta \circ (\alpha \circ X) = (\beta\alpha) \circ X$
- $\mathbb{E}[\alpha \circ X|X] = \alpha X$
- $\mathbb{E}[\alpha \circ X] = \alpha \mathbb{E}[X]]$
- $VAR[\alpha \circ X|X] = \alpha(1-\alpha)X$
- $VAR[\alpha \circ X] = \alpha^2 VAR[X] + \alpha(1-\alpha)\mathbb{E}[X]$.

The moving average representation,

$$X_t = \alpha \circ X_{t-1} + \epsilon_t = \alpha \circ (\alpha \circ X_{t-2} + \epsilon_{t-1}) + \epsilon_t \tag{5.0.2}$$

Finally,

$$X_t = \sum_{j=0}^{\infty} \alpha^j \circ \epsilon_{t-j} \tag{5.0.3}$$

The expectation and the variance of the process can be written as

$$\mathbb{E}[X_t] = \alpha\mathbb{E}[X_{t-1}] + \mu = \alpha^t \mathbb{E}[X_0] + \mu \sum_{j=0}^{t-1} \alpha^j \tag{5.0.4}$$

$$VAR[X_t] = \alpha^2 VAR[X_{t-1}] + \alpha(1-\alpha)\mathbb{E}[X_{t-1}] + \sigma^2 \tag{5.0.5}$$

$$VAR[X_t] = \alpha^2 VAR[X_0] + (1-\alpha)\sum_{j=1}^{t} \alpha^{2j-1}\mathbb{E}X_{t-j} + \sigma^2 \sum_{j=1}^{t} \alpha^2(j-1) \quad (5.0.6)$$

5.0.2 Parameter Estimation of Poisson INAR(1)

It can be assumed that ϵ_t is a sequence of independent and identically distributed poisson random variable with mean μ. The INAR(1) process was

$$X_t = \alpha \circ X_{t-1} + \epsilon_t \ , \quad (5.0.7)$$

and the conditional expectation and the variance of the INAR(1) process is

$$\mathbb{E}[X_t|X_{t-1}] = \alpha X_{t-1} + \mu \quad (5.0.8)$$

$$VAR[X_t|X_{t-1}] = \alpha(1-\alpha)X_{t-1} + \mu \quad (5.0.9)$$

Mainly, there are two methods to estimate the parameters α and μ. The first approach is based on method of moments which yields

$$\widehat{\alpha} = \sum_{t=0}^{N-1} \frac{(X_t - \overline{X})(X_{t+1} - \overline{X})}{\sum_{t=0}^{N}(X_t - \overline{X})^2} \quad (5.0.10)$$

$$\widehat{\mu} = \frac{\sum_{t=1}^{N} \overline{\epsilon_t}}{N} \quad (5.0.11)$$

where $\widehat{\epsilon}_t = \widehat{X}_t - \widehat{\alpha}X_{t-1}$ for t=1,2,...,N. On the other hand, an alternative estimation method is conditional least square estimation.

$$\min_{\alpha,\mu} \sum_{t=1}^{N} (X_t - \alpha X_{t-1} - \mu)^2 \quad (5.0.12)$$

5.0.3 Regression Analysis of Integer Autoregressive Models

In this model, we can express the parameters α and μ by using some explanatory variables. Suppose $Y_{1,t}$ and $Y_{2,t}$ are the observations at time t that may have impact on banks' systematic and idiosyncratic behaviors. The information flow from institutions to the authorities are usually lagged. This causes the authorities could not take immediate action on the unwilling cases. Linking the bank specific information to the information available to authorities on time, thus, is very important. It is possible to define α such that

$$\alpha = \frac{1}{1 + exp(-\Gamma' Y_{1,t-1})} \tag{5.0.13}$$

where Γ is the vector of coefficients. Let ϵ be a sequence of Poisson random variables such that

$$\mathbb{E}[\epsilon_t] = exp(\beta' Y_{2,t-1}), \tag{5.0.14}$$

where β is the vector of coefficients. where Y_t is a covariate process. The conditional expectation is

$$\mathbb{E}[X_t|Y_{t-1}, X_{t-1}] = \frac{1}{1 + exp(-\Gamma' Y_{1,t-1})} X_{t-1} + exp(\beta' Y_{2,t-1}). \tag{5.0.15}$$

The variance,

$$VAR[X_t|Y_{t-1}, X_{t-1}] = \frac{exp(-\Gamma' Y_{1,t-1})}{(1 + exp(-\Gamma' Y_{1,t-1}))^2} X_{t-1} + exp(\beta' Y_{2,t-1}). \tag{5.0.16}$$

Accordingly, we may estimate α and β by taking

$$\min_{\alpha,\beta}\left\{X_t - \frac{1}{1+exp(-\Gamma' Y_{1,t-1})}X_{t-1} - exp(\beta' Y_{2,t-1})\right\}^2. \quad (5.0.17)$$

5.0.4 A Simple Application of Poisson INAR Process

In this part, we illustrate an hypothetical example of Integer Auto Regressive Process. It is assumed that we have sufficient information about past default events for 10-years horizon. Then for each category we can illustrate approximate default rates for each year. The key point is that how can we estimate today's default rates for each rating category by using INAR model. The hypothetical observations are shown below.

Year	AAA	AA	A	BBB	BB	B	CCC
1	0.148	0.0128	0.0283	0.0416	0.0750	0.1474	0.3126
2	0.0143	0.0171	0.0301	0.0548	0.0802	0.1739	0.2756
3	0.0080	0.0165	0.0317	0.0455	0.0761	0.1445	0.2401
4	0.0161	0.0161	0.0333	0.0541	0.0805	0.1159	0.3483
5	0.0183	0.0145	0.0395	0.0418	0.0740	0.1502	0.2680
6	0.0150	0.0150	0.0400	0.0506	0.0750	0.1516	0.2995
7	0.0130	0.0173	0.0380	0.490	0.1202	0.1571	0.3002
8	0.0144	0.0155	0.0347	0.0463	0.0851	0.1612	0.2888
9	0.0120	0.0180	0.0302	0.0535	0.0950	0.1788	0.3178
10	0.0010	0.020	0.0291	0.0610	0.2110	0.1377	0.4021

Table 5.1: *Observations of Default Events in 10-year period of time*

For a large number of data set, it is possible to apply poisson INAR process. According to the data set we can estimate the alpha parameters of corresponding rating categories by using conditional least square technique or method of moments. Here, we would rather to apply least square techniques to INAR(1) process. In order to estimate this parameters and corresponding default rates, we develop a MATLAB code. The output of parameters are given below

Ratings	α
AAA	0.3819
AA	0.0435
A	0.1269
BBB	0.0080
BB	0.0134
B	0.0148
CCC	0.01231

Table 5.2: *The Estimations of* α parameters for rating categories

Ratings	Default Rates
AAA	0.0153
AA	0.0168
A	0.0336
BBB	0.0505
BB	0.1028
B	0.1520
CCC	0.3049

Table 5.3: *The Estimated Default Probabilities*

Accordingly, new default probabilities can be calculated for each rating category. In addition, it is possible to incorporate some explanatory observations into these models. Using some specific information, one can calculate new parameters and so new expected default rates. For instance, a bank authority can incorporate some macroeconomic observations into the model. Hence, the instantaneous riskiness of the portfolio can be obtained by these new default rates. Our suggestion is that together with historical observations, instantaneous macroeconomic events can also be incorporated into CreditRisk+ model. Because the default frequencies are the most significant input parameters of the CreditRisk+ model, it is believed that this approach increase the accuracy of the credit risk measurement, especially for understanding daily riskiness of a credit portfolio.

Chapter 6

CONCLUSION

In this study, we mainly focus on developing CreditRisk+ model considering past observations and current macro economic, financial and bank specific conditions by Poisson-INAR model. Before explaining our econometric CreditRisk+ approach, we introduce fast fourier transformation(FFT) technique as an alternative to the standard model. In addition, we represent a new alternative computational technique which incorporates Panjer Recursion and FFT. We show that under FFT and mixed approaches CreditRisk+ performs better in terms of CPU time and numerical stability. Moreover, it is easier to implement. Furthermore, it is observed that FFT-Panjer and the standard model yield nearly same results, however FFT-Panjer model is very fast in terms of CPU time while number of sectors increases. Moreover, we also show that one can calibrate the parameters of the Poisson-INAR model to fit the observations.

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