ASSET PRICING MODELS: STOCHASTIC VOLATILITY AND INFORMATION-BASED APPROACHES

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ABSTRACT

ASSET PRICING MODELS: STOCHASTIC VOLATILITY AND INFORMATION-BASED APPROACHES

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We present two option pricing models, both different from the classical Black-Scholes-Merton model. The first model, suggested by Heston, considers the case where the asset price volatility is stochastic. For this model we study the asset price process and give in detail the derivation of the European call option price process. The second model, suggested by Brody-Hughston-Macrina, describes the observation of certain information about the claim perturbed by a noise represented by a Brownian bridge. Here we also study in detail the properties of this noisy information process and give the derivations of both asset price dynamics and the European call option price process.

Keywords: Option Pricing, Stochastic Volatility, Characteristic Function Method, Incomplete Information, Change of Measure.

ÖZ

FİNANSAL VARLIKLARIN FİYATLAMA MODELLERİ: STOKASTİK VOLATİLİTE VE BİLGİYE DAYANAN YAKLAŞIMLAR

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Bu çalışmada Black-Scholes-Merton modeline alternatif olarak düşünülebilecek iki opsiyon fiyatlama modeli incelenmiştir. Heston tarafından önerilen ilk model opsiyonun üzerine yazıldığı finansal varlığın volatilitesinin stokastik dinamiğe sahip olduğu varsayımı altında opsiyonun fiyatı için analitik çözüm önermiştir. Model ayrıntılı bir şekilde incelenerek, opsiyon fiyatlama modelinin çıkarılışı orjinal çalışmada verilmeyen gerekli ispatlar verilerek sunulmuştur. Literatürde Brody-Hughston-Macrina modeli olarak anılan ikinci modelde ise, piyasalarda yatırımcıların finansal varlığın gelecekteki getirilerine dair doğru bilgiye tam erişimi olmadığı varsayılmıştır. Yatırımcıların erişebildiği bilgiyi, doğru bilginin bir kısmının Brownian köprü ile gürültülenmiş bir yapıda olduğunu varsayarak varlıkların fiyat süreci dinamikleri çıkarılmıştır. Bu varsayım ve bulunan fiyat süreci dinamikleri temel alınarak opsiyon fiyat formülasyonu verilmiştir. Bu modelde de gerekli teoremlerin ve çıkarımların ispatları orjinal çalışmada verilmeyenlerle birlikte ayrıntılı bir şekilde verilerek hem spot piyasadaki varlıklar hem de türev ürün için fiyat dinamikleri sunulmuştur.

Anahtar Kelimeler: Opsiyon fiyatlama, Stokastik Volatilite, Karakteristik Fonksiyon Metodu, Ölçü Değişimi, Arbitraj.

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CHAPTER 1

INTRODUCTION

The *Black-Scholes-Merton* model has been discussed in terms of its advantages and disadvantages in theory and in applications. Obviously, the model is a pioneering work in the area. However, it is a known and experienced fact that the assumptions, that enable the model to have a *closed-form* solution, cause the model not to fit the real data obtained from the market (cf, e.g., [10]). Among these assumptions, the one on constant volatility has been the most discussed and criticized one. Over the years, there have been many alternative models offered to solve this and other drawbacks of the model.

In this work, we mainly concentrate on two different strong approaches conducted by *Heston* [11] and *Brody-Hughston-Macrina* [4]. In the former one, the model is assumed to have stochastic volatility and obtains a "so-called" closed-form solution for the price of the European call option. Moreover, by assuming that the volatility and the underlying price have a non-zero correlation, it captures many properties of the financial data, which the Black-Scholes-Merton model does not. Although the model can be considered as the most popular alternative to the Black-Scholes-Merton pricing model, its reliability is questionable since the assumptions on the underlying asset price and the volatility dynamics display an *"ad-hoc"* nature. The dynamics of the asset price and volatility seem to remain as assumptions without a convincing intuition behind them.

In the latter approach suggested by Brody-Hughston-Macrina (BHM), instead of *pre-specified* dynamics of the price and volatility, the asset price dynamics is derived by adopting a more realistic approach towards market structure. The model is established under the assumption of *incomplete information* in the market. By specifying a model for the structure of the information circulating in the market, the model is motivated by the fact that asset prices are specified by expectations on the future cash flows given the information circulating in the market. Without assuming any dynamic model for asset prices, it is seen that the derived asset price dynamics under the assumption of this information structure naturally has stochastic volatility, which gives a different explanation to the nature of volatility. In fact, according to the model, volatility of volatility is found to be stochastic.

The aim of this study is to review these two pricing models in detail. The second chapter presents the derivation of the option pricing formula suggested in Heston's study [11] step-by-step. The way of how the characteristic function method is used for derivation of option pricing formula is presented. Moreover, the derivation and the solution of the partial differential equation satisfied by the probabilities in the option pricing formula is given explicitly. In the third chapter, we start by giving the motivation of the incomplete information model suggested by Brody-Hughston-Macrina. The dynamics of the asset price process for a single-dividend paying risky asset given in the study [4] is derived explicitly. Furthermore, the change of measure technique used to derive the conditional probability density is presented in detail by giving the derivation of the dynamics of the Radon-

Nikodym Process. In [4], the option pricing formulae were given by not taking the filtration into consideration, meaning that the formulae were obtained for time 0. The fourth chapter gives the derivations and the proofs of the theorems used in derivations of option pricing formulae in the study of the BHM [4]. Additionally, we give the complete derivation for the option price formulae for an arbitrary time t where we use an approach inspired by the work of Rutkowski-Yu [21]. Finally, the conclusion follows.

CHAPTER 2

HESTON'S APPROACH

The main drawback of the Black-Scholes-Merton model is that it assumes that the underlying asset price volatility remains constant over time. However, various econometric and numerical studies show that asset prices do not exhibit constant volatility. Therefore, at first, the main object may be to extend this model to capture this qualitative feature of the financial data. The Heston's stochastic volatility model [11] can be seen as one of the most popular models in the literature. In this chapter, Heston's asset pricing model is presented. The technique to derive a closed form solution to the option pricing problem under the assumption of stochastic volatility is given in detail. The model also captures the correlation between spot asset price and its volatility. The solution technique suggested in [11] is based on characteristic functions and, thus, it has a wide range of applicability.

2.1 Asset Price Dynamics

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be our probability space and W_t^1 and W_t^2 be correlated Brownian motions on this probability space. It is assumed that the underlying risky asset price at time t satisfies the following stochastic differential equation

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^1 \tag{2.1.1}$$

and its volatility follows an Ornstein-Uhlenbeck process given by

$$d\sqrt{v_t} = -\beta\sqrt{v_t}dt + \delta dW_t^2, \tag{2.1.2}$$

where W_t^1 and W_t^2 are standard Brownian motions having instantaneous correlation ρ. With the help of the Ito Lemma, it can be shown that the variance process follows the Cox-Ingersoll-Ross (CIR) process [7]

$$dv_t = (\delta^2 - 2\beta v_t)dt + 2\delta\sqrt{v_t}dW_t^2,$$

which can be expressed as a standard CIR process as follows:

$$dv_t = \kappa[\theta - v_t]dt + \sigma\sqrt{v_t}dW_t^2. \tag{2.1.3}$$

Then the stochastic volatility model is

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^1, \tag{2.1.4}$$

$$dv_t = \kappa[\theta - v_t]dt + \sigma\sqrt{v_t}dW_t^2, \tag{2.1.5}$$

where

$$d\left\langle W^1, W^2 \right\rangle_t = \rho dt. \tag{2.1.6}$$

For simplicity, it is assumed that the interest rate r is constant and the price at time t of a discount bond that matures at time $t+\tau$ is

$$P(t, t+\tau) = e^{-r\tau}. \tag{2.1.7}$$

The possible no-arbitrage price of a derivative at time t with a final payoff K and maturity date T can be written as

$$U_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}_t\right], \tag{2.1.8}$$

where $\mathbb{Q}$ stands for the risk neutral martingale measure. Since the value of the option at time T is expected to be equal to the final payoff of the derivative, discounted asset prices are martingales under the measure $\mathbb{Q}$:

$$e^{-rt}U_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}U_T | \mathcal{F}_t\right], \tag{2.1.9}$$

$$\tilde{U}_t = \mathbb{E}^{\mathbb{Q}}\left[\tilde{U_T} | \mathcal{F}_t\right]. \tag{2.1.10}$$

Here, $\tilde{U}$ represents the discounted value of the option at time t.

In the model, there are two random sources, so the underlying probability space is represented by $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \mathcal{C}(\mathbb{R}_+; \mathbb{R}^2)$ is the space of all continuous functions from $\mathbb{R}_+$ into $\mathbb{R}^2$. Furthermore, the coordinate process is $(W_t^1, W_t^2) = w_t \in \mathbb{R}^2$; the measure on Ω is such that the two Brownian motions W_t^1 and W_t^2 have the correlation defined by (2.1.6). The filtration $\{\mathcal{F}_t\}$ represents the information on the two correlated Brownian motions W_t^1 and W_t^2. Thus, to transform the measure into the risk neutral martingale measure, the two-dimensional Girsanov theorem is applied.

In the two-dimensional case, the crucial point is on the choice of the market risk premium through which the discounted asset price processes become martingales. For the spot asset, the standard market risk premium satisfying the arbitrage-free condition can be used; however, for the volatility risk premium the specification is not always that simple. Nevertheless, any allowable choice of the price of volatility risk leads to an equivalent martingale measure $\mathbb{Q}$ (see [10]). Here, we note that $\mathbb{Q}$ depends on the choice of the price of volatility risk. As there can be many market volatility risk premiums, there can be many equivalent measures $\mathbb{Q}$, which is an indication of the market incompleteness. Moreover, according to the model, the number of risk sources in the market is greater than the number of the risky assets traded in the market. This also shows that the market model is incomplete (see [1]). However, in [10], it is emphasized that a unique equivalent martingale measure under which derivative contracts are priced is selected by the market. This point of view may be called as "*selecting an approximating complete market*". However, the term can be misperceived because although the discounted asset prices are martingales under this measure, they cannot be replicated by the spot asset and the risk-free bond alone. That is to say, the risk caused by the underlying volatility cannot be hedged with this spot replicating strategy.

In this model, the market price of the volatility risk denoted as $\lambda(S, v, t)$ is given by

$$\lambda(S, v, t)dt \;=\; \gamma Cov[dv, dC_t/C_t], \tag{2.1.11}$$

where C_t is the consumption rate and γ is the relative risk aversion of an investor

[2]. The consumption process that emerges in the CIR model [7] is considered, where consumption growth has constant correlation with the spot asset return of the form

$$dC_t \;=\; \mu_c v_t C_t dt + \sigma_c \sqrt{v_t} C_t dW_t^3. \tag{2.1.12}$$

This makes the market price of volatility risk proportional to v_t; thus, after some arrangements, $\lambda(S, v, t)$ can be expressed as

$$\lambda(S, v, t) = \lambda v, \tag{2.1.13}$$

where λ represents a constant parameter.

By the two-dimensional Ito formula [18], $U(t, S, v)$ satisfies the following partial differential equation:

$$dU_t = \frac{\partial U}{\partial t} dt + \frac{\partial U}{\partial S} dS_t + \frac{\partial U}{\partial v} dv_t + \frac{1}{2}\frac{\partial^2 U}{\partial S^2} d\left\langle S, S\right\rangle_t + \frac{1}{2}\frac{\partial^2 U}{\partial v^2} d\left\langle v, v\right\rangle_t + \frac{\partial^2 U}{\partial S \partial v} d\left\langle S, v\right\rangle_t. \tag{2.1.14}$$

By the Girsanov Theorem [18], the asset price dynamics can be expressed under the equivalent martingale measure as follows:

$$dS_t = rS_t dt + \sqrt{v_t} S_t d\tilde{W}_t^1, \tag{2.1.15}$$

$$dv_t = \left[\kappa[\theta - v_t] - \lambda v_t\right] dt + \sigma \sqrt{v_t} d\tilde{W}_t^2, \tag{2.1.16}$$

where $\tilde{W}^1$ and $\tilde{W}^2$ are Brownian motions under the equivalent risk-neutral martingale measure. The dynamics of the discounted option price is

$$d(e^{-rt}U_t) = e^{-rt}\left(-rUdt + dU_t\right). \tag{2.1.17}$$

Substituting the equations (2.1.15), (2.1.16) and (2.1.14) into (2.1.17) gives

$$\begin{aligned} d(e^{-rt}U_t) = e^{-rt}\left(-rUdt + \frac{\partial U}{\partial t}dt + \frac{\partial U}{\partial S}rS_tdt + \frac{\partial U}{\partial S}\sqrt{v_t}S_td\tilde{W}_t^1\right) + \\ +e^{-rt}\left(\frac{\partial U}{\partial v}\sigma\sqrt{v_t}d\tilde{W}_t^2\frac{1}{2}\frac{\partial^2 U}{\partial S^2}v_tS_t^2dt + \frac{1}{2}\frac{\partial^2 U}{\partial v^2}\sigma^2 v_tdt\right). \end{aligned}$$

As the discounted asset prices are martingales under the equivalent measure $\mathbb{Q}$, we have the following PDE for the option price:

$$\begin{aligned} \frac{1}{2}vS_t^2\frac{\partial^2 U}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + \rho\sigma vS\frac{\partial^2 U}{\partial S\partial v} + rS\frac{\partial U}{\partial S} + \{\kappa[\theta - v]\lambda v\}\frac{\partial U}{\partial v} + \\ +\frac{\partial U}{\partial t} - rU_t \;=\; 0. \end{aligned} \tag{2.1.18}$$

A European call option price with strike price K and maturity T satisfies the PDE (2.1.18) subject to the following boundary conditions:

$$\begin{aligned} U(S,v,T) &= max(0, S_T - K), \\ U(0,v,t) &= 0, \\ \frac{\partial U}{\partial S}(\infty, c, t) &= 1, \\ rS\frac{\partial U}{\partial S}(S,0,t) + \kappa\theta\frac{\partial U}{\partial v}(S,0,t) - rU(S,0,t) + \frac{\partial U}{\partial t}(S,0,t) = 0, \\ U(S,\infty,t) &= S. \end{aligned} \tag{2.1.19}$$

As in the Black-Scholes formula, the form of the solution is expected to be in the following form

$$C(S,v,t) = U(S,v,t) = S_t P_1 - KP(t,T)P_2, \tag{2.1.20}$$

where $P(t,T) = e^{-r(T-t)}$, the first term of the expression stands for the present value of the spot asset, the second term expresses the present value of the strike price payment and P_1 and P_2 are probabilities. Here, the main task is to find these probabilities explicitly. Since (2.1.20) is the solution of (2.1.18), partial differential equations satisfied by these probabilities can be derived by using (2.1.18).

By taking $x = \ln S$, the PDE (2.1.18) is rewritten in terms of probabilities P_1 and P_2 as follows:

$$\frac{\partial U}{\partial S} = P_1 + \frac{\partial P_1}{\partial x} + \frac{1}{S} K e^{-r(T-t)} \frac{\partial P_2}{\partial x}, \tag{2.1.21}$$

$$\frac{\partial U}{\partial v} = S \frac{\partial P_1}{\partial v} + K e^{-r(T-t)} \frac{\partial P_2}{\partial v}, \tag{2.1.22}$$

$$\frac{\partial^2 U}{\partial S \partial v} = \frac{\partial^2 U}{\partial v \partial S} = \frac{\partial P_1}{\partial v} + \frac{\partial^2 P_1}{\partial x \partial v} + \frac{1}{S} K e^{-r(T-t)} \frac{\partial^2 P_2}{\partial x \partial v}, \tag{2.1.23}$$

$$\frac{\partial U}{\partial t} = S \frac{\partial P_1}{\partial t} + rKP_2 + K e^{-r(T-t)} \frac{\partial P_2}{\partial t}, \tag{2.1.24}$$

$$\frac{\partial^2 U}{\partial S^2} = \frac{1}{S} \frac{\partial P_1}{\partial x} + \frac{1}{S} \frac{\partial^2 P_1}{\partial x^2} - \frac{1}{S^2} K e^{-r(T-t)} \frac{\partial P_2}{\partial x} + \frac{1}{S^2} K e^{-r(T-t)} \frac{\partial^2 P_2}{\partial x^2}, \tag{2.1.25}$$

$$\frac{\partial^2 U}{\partial v^2} = S\frac{\partial^2 P_1}{\partial v^2} + Ke^{-r(T-t)}\frac{\partial^2 P_2}{\partial v^2}. \tag{2.1.26}$$

Substituting all the above partial derivatives into the PDE (2.1.18), we obtain the PDE for P_1:

$$\frac{\partial P_1}{\partial t} + \left(r + \frac{1}{2}v\right)\frac{\partial P_1}{\partial x} + (\kappa\theta - \kappa v + \lambda v + \rho\sigma v)\frac{\partial P_1}{\partial v} + \frac{1}{2}v\frac{\partial^2 P_1}{\partial x^2} + \\ + \rho\sigma v\frac{\partial^2 P_1}{\partial v \partial x} + \frac{1}{2}\sigma^2 v\frac{\partial^2 P_1}{\partial v^2} = 0 \tag{2.1.27}$$

subject to the terminal condition

$$P_1(x, v, T, \ln K) \;=\; \mathbb{I}_{\{x \geq \ln K\}}. \tag{2.1.28}$$

The PDE for P_2 turns out to be

$$\left(r - \frac{1}{2}v\right)\frac{\partial P_2}{\partial x} + \frac{\partial P_2}{\partial t} + (\kappa\theta - \kappa v - \lambda v)\frac{\partial P_2}{\partial v} + \frac{1}{2}v\frac{\partial^2 P_2}{\partial x^2} + \rho\sigma v\frac{\partial^2 P_2}{\partial v \partial x} + \\ + \frac{1}{2}\sigma^2 v\frac{\partial^2 P_2}{\partial v^2} = 0 \tag{2.1.29}$$

subject to the terminal condition

$$P_2(x, v, T, \ln K) \;=\; \mathbb{I}_{\{x \geq \ln K\}}. \tag{2.1.30}$$

The partial differential equations can be expressed as follows:

$$\frac{1}{2}v\frac{\partial^2 P_j}{\partial x^2} + \rho\sigma v\frac{\partial^2 P_j}{\partial x \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 P_j}{\partial v^2} + (r + u_j v)\frac{\partial P_j}{\partial x} + (a_j - b_j v)\frac{\partial P_j}{\partial v} + \frac{\partial P_j}{\partial t} = 0 \tag{2.1.31}$$

for $j = 1, 2$, where $u_1 = 1/2$, $u_2 = -1/2$, $a_{1,2} = \kappa\theta$, $b_1 = \kappa + \lambda - \rho\sigma$ and $b_2 = \kappa + \lambda$.

Thus, they can be interpreted as "adjusted" or "risk neutralized" probabilities, and, according to this, the spot asset price and volatility process dynamics can be written as

$$\begin{aligned} dx_t &= [r + u_j v_t]dt + \sqrt{v_t} d\tilde{W}_t^1, \\ dv_t &= (a_j - b_j v_t)dt + \sigma\sqrt{v_t} d\tilde{W}_t^2, \end{aligned} \qquad (2.1.32)$$

for $j = 1, 2$.

Here, P_j is in fact the conditional risk neutral probability that the option expires in-the-money, which can be expressed by

$$P_j(x, v, T; \ln K) = \mathbb{Q}\left[x(T) \geq \ln K | x(t) = x, v(t) = v\right]. \qquad (2.1.33)$$

2.2 Heston's Characteristic Function Method

When $x(t)$ and $v(t)$ follow the risk neutral processes given by (2.1.32), any twice differentiable function $f(x, v, t)$ that is a conditional expectation of some function of x and v at a later date T, $g(x(T), v(T))$ can be expressed as follows:

$$f(x, v, t) := \mathbb{E}^{\mathbb{Q}}\left[g(x(T), v(T)) | x(t) = x, v(t) = v\right] \qquad (2.2.34)$$

subject to the terminal condition

$$f(x, v, T) \;=\; g(x, v). \tag{2.2.35}$$

From this definition, f must be a martingale under the risk neutral probability $\mathbb{Q}$. The Markov property of the processes can be easily verified, thus, for every $s \leq t \leq T$, it holds:

$$E^{\mathbb{Q}}[f(x, v, t)|\mathcal{F}_s] = E^{\mathbb{Q}}[f(x, v, t)|x(s), v(s)].$$

By using the definition of the function f,

$$E^{\mathbb{Q}}[f(x, v, t)|\mathcal{F}_s] = E^{\mathbb{Q}}[E^{\mathbb{Q}}[g(x(T), v(T))|\mathcal{F}_t]|\mathcal{F}_s].$$

By the tower property of conditional expectation, the last term in (2.2) can be written as

$$= E^{\mathbb{Q}}[g(x(T), v(T))|\mathcal{F}_s] = E^{\mathbb{Q}}[g(x(T), v(T))|x(s) = x, v(s) = v] \;=\; f(x, v, s). \tag{2.2.36}$$

By the Ito formula, the partial differential equation that f satisfies can be derived as follows:

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx_t + \frac{\partial f}{\partial v}dv_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}d\left\langle x, x\right\rangle_t + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}d\left\langle v, v\right\rangle_t + \frac{\partial^2 f}{\partial x \partial v}d\left\langle x, v\right\rangle_t, \tag{2.2.37}$$

and, after some substitutions,

$$df = \left(\frac{\partial f}{\partial t} + [r + u_j v]\frac{\partial f}{\partial x} + (a_j - b_j v)\frac{\partial f}{\partial v}dt + \frac{1}{2}v\frac{\partial^2 f}{\partial x^2}\right)dt +$$
$$+ \left(\frac{1}{2}\sigma^2 v\frac{\partial^2 f}{\partial v^2} + \rho\sigma v\frac{1}{2}v\frac{\partial^2 f}{\partial x \partial v}\right)dt + \sqrt{v}\frac{\partial f}{\partial x}d\tilde{W}_t^1 + \sigma\sqrt{v}\frac{\partial f}{\partial v}d\tilde{W}_t^2. \qquad (2.2.38)$$

By using of the martingale property of f, we have the following PDE:

$$\frac{1}{2}v\frac{\partial^2 f}{\partial v^2} + \frac{1}{2}v\sigma^2\frac{\partial^2 f}{\partial v^2} + \rho\sigma v\frac{\partial^2 f}{\partial x \partial v} + [r + u_j v]\frac{\partial f}{\partial x} + (a_j - b_j v)\frac{\partial f}{\partial v} + \frac{\partial f}{\partial t} = 0 \qquad (2.2.39)$$

subject to the terminal condition

$$f(x, v, T) = g(x, v). \qquad (2.2.40)$$

This equation has many uses. With the proper specification of the function $g(x, v)$, the desired solution can be reached directly.

If $g(x, v) := \mathbb{I}_{\{x(T) \geq \ln K\}}$, then the function f can be expressed as

$$f(x, v, t) = \mathbb{E}^{\mathbb{Q}}[\mathbb{I}_{\{x(T) \geq \ln K\}} | x(t) = x, v(t) = v], \qquad (2.2.41)$$

which gives the solution of the conditional probability of the fact that $x(T)$ is greater than $\ln K$ at time $t \leq T$.

For $j = 1, 2$, P_j represents the same probability, that is, the conditional probability of the fact that option expires in-the-money. Moreover, it is observed that P_j satisfies the same partial differential equation (2.2.39) with f and has the terminal condition

$$P_j(x, v, T; \ln K) \;=\; \mathbb{I}_{\{x \geq \ln K\}}. \tag{2.2.42}$$

Thus the conditional probability that the option expires in-the-money at time t can be expressed as follows:

$$\begin{aligned} P_j(x, v, t; \ln K) &= \mathbb{E}^{\mathbb{Q}}[\mathbb{I}_{\{x(T) \geq \ln K\}} | x(t) = x, v(t) = v] & (2.2.43) \\ &= \mathbb{Q}\left(x(T) \geq \ln K | x(t) = x, v(t) = v\right). & (2.2.44) \end{aligned}$$

The probabilities may not be derivable in a closed form; however, to set the function g properly can help to derive the probabilities. By specifying the function g as

$$g(x, v) = e^{i\phi x}, \tag{2.2.45}$$

the function f can be expressed as follows:

$$f(x, v, t) = \mathbb{E}^{\mathbb{Q}}[e^{i\phi x(T)} | x(t) = x, v(t) = v]. \tag{2.2.46}$$

It is clearly seen that the solution of (2.2.39) gives the characteristic function.

Remark 2.2.1. Here, the point why the characteristic function is chosen to obtain the desired probabilities is the fact the characteristic functions alwasy exist [14, 20]. Moreover, the desired probabilities can be derived immediately by using the inversion formula [20].

To solve the PDE given by (2.2.39) explicitly, the following solution form is suggested

$$f(x, v, t) \;=\; \exp[C(T-t) + D(T-t)v + i\phi x]. \tag{2.2.47}$$

The partial differential equation that f satisfies is

$$df = \left(\frac{\partial C}{\partial t}f + v\frac{\partial D}{\partial t}f + [r + u_j v]i\phi f + (a_j - b_j v)fD - \frac{1}{2}v\phi^2 f\right)dt + \\ + \frac{1}{2}\sigma^2 v D^2 f dt + \rho\sigma v i\phi D f dt + (...)d\tilde{W}_t^1 + (...)d\tilde{W}_t^2. \tag{2.2.48}$$

Since f is martingale, the PDE that f satisfies is found as

$$\frac{\partial C}{\partial t}f + v\frac{\partial D}{\partial t}f + [r + u_j v]i\phi f + (a_j - b_j v)fD - \frac{1}{2}v\phi^2 f + \\ + \frac{1}{2}\sigma^2 v D^2 f + \rho\sigma v i\phi D f \;=\; 0. \tag{2.2.49}$$

One particular solution for the functions $C(T-t)$ and $D(T-t)$ can be found as follows:

$$v\frac{\partial D}{\partial t}f + u_j v i\phi f + b_j v f D - \frac{1}{2}v\phi^2 f + \frac{1}{2}\sigma^2 v D^2 f + \rho\sigma v i\phi D f \;=\; 0, \tag{2.2.50}$$

$$\frac{\partial C}{\partial t}f + ri\phi f + a_j t D f = \; 0, \tag{2.2.51}$$

After some arrangements, we get two ordinary differential equations:

$$\frac{\partial D}{\partial t} + u_j i\phi - b_j D - \frac{1}{2}\phi^2 + \frac{1}{2}\sigma^2 D^2 + \rho\sigma i\phi D = 0 \qquad (2.2.52)$$

$$\frac{\partial C}{\partial t} + ri\phi + a_j D = 0 \qquad (2.2.53)$$

subject to

$$C(0) = 0, \qquad (2.2.54)$$

$$D(0) = 0. \qquad (2.2.55)$$

Thus, by using the known techniques for ordinary differential equations [15], we solve the ODE as follows:

$$\frac{\partial D}{\partial t} + u_j i\phi - b_j D - \frac{1}{2}\phi^2 + \frac{1}{2}\sigma^2 D^2 + \rho\sigma i\phi D = 0. \qquad (2.2.56)$$

For brevity, we denote the parameters as follows:

$$a = \frac{1}{2}\sigma^2,$$

$$b = \rho\sigma i\phi - b_j,$$

$$c = u_j i\phi - \frac{1}{2}\phi^2. \qquad (2.2.57)$$

Thus, we can use the known technique [15] to solve Riccati differential equations for (2.2.56):

$$c_1 - t = \int \frac{dD}{aD^2 + bD + c} \qquad (2.2.58)$$

where a, b and c are given in (2.2).

Thus,

$$\frac{1}{aD^2 + bD + c} = \frac{1}{a}\left(\frac{A}{D - D_1} + \frac{B}{D - D_2}\right). \tag{2.2.59}$$

where D_1 and D_2 stands for the roots as given by

$$D_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad D_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \tag{2.2.60}$$

and, A and B are found as follows:

$$A = \frac{1}{D_1 - D_2}, \qquad B = \frac{1}{D_2 - D_1}. \tag{2.2.61}$$

Hence, we have

$$c_1 - t \;=\; \frac{1}{a(D_1 - D_2)}\int \frac{1}{D - D_1} dD - \frac{1}{a(D_1 - D_2)}\int \frac{1}{D - D_2} dD, \tag{2.2.62}$$

$$c_1 - t = \frac{1}{a(D_1 - D_2)}\left(\ln\left(\frac{|D - D_1|}{|D - D_2|}\right)\right) + c \tag{2.2.63}$$

for $D(\tau) = D(T - t)$, subject to

$$D(0) = 0, \tag{2.2.64}$$

which implies $T - t = 0$ and so $T = t$, such that by (2.2.63), we get

$$c_1 - c = \frac{\ln(D_1/D_2)}{a(D_1 - D_2)} + T. \tag{2.2.65}$$

Thus we obtain

$$\tau = \frac{1}{a(D_1 - D_2)}\left[\ln\left(\frac{|D - D_1|}{|D - D_2|}\right) - \frac{D_1}{D_2}\right], \tag{2.2.66}$$

$$D = D_1\frac{\left(1 - e^{\tau a(D_1 - D_2)}\right)}{\left(1 - \frac{D_1}{D_2}e^{\tau a(D_1 - D_2)}\right)}. \tag{2.2.67}$$

Let us denote

$$d = \sqrt{b^2 - 4ac} = \sqrt{(\rho\sigma\phi i - b_j)^2 - \sigma^2(2u_j\phi i - \phi^2)}, \tag{2.2.68}$$

then the roots D_1 and D_2 can be expressed as follows:

$$D_1 = \frac{\rho\sigma i\phi - b_j + d}{\sigma^2}, \quad D_2 = \frac{\rho\sigma i\phi - b_j - d}{\sigma^2}. \tag{2.2.69}$$

Then, we obtain

$$a(D_1 - D_2) = \frac{1}{2}\sigma^2(D_1 - D_2) = d. \tag{2.2.70}$$

Let us denote $g := D_1/D_2$, then

$$g = \frac{\rho\sigma i\phi - b_j + d}{\rho\sigma i\phi - b_j - d} \tag{2.2.71}$$

$$D = \frac{\rho\sigma i\phi - b_j + d}{\sigma^2}\left(\frac{1-e^{\tau d}}{1-ge^{\tau d}}\right), \tag{2.2.72}$$

and by substituting D into the equation (2.2.53), the solution of C is found as follows:

$$-dC = ri\phi dt + \frac{a_j(b_j - \rho\sigma\phi + d)}{\sigma^2}\left[\frac{1-e^{d(T-t)}}{1-ge^{d(T-t)}}\right] dt, \tag{2.2.73}$$

$$-C + c_1 = ri\phi t + \frac{a_j(b_j - \rho\sigma\phi + d)}{\sigma^2}\left[\int \frac{1-e^{d(T-t)}}{1-ge^{d(T-t)}} dt\right], \tag{2.2.74}$$

$$\begin{aligned} -C + c_1 &= ri\phi t + \\ &+\frac{a_j(b_j - \rho\sigma\phi + d)}{\sigma^2}\left[\frac{\ln|e^{-d(T-t)} - g|}{d} - \frac{1}{dg}\left(\ln|e^{-d(T-t)} - g| - \ln|e^{-d(T-t)}|\right) + c_2\right]\end{aligned}$$

subject to

$$C(0) = 0. \tag{2.2.75}$$

Then we obtain

$$C = ri\phi\tau - \frac{a_j D_1}{gd}\left[(g-1)\ln|e^{-d(T-t)} - g| + g\ln e^{-d(T-t)} - (g-1)\ln|1-g|\right] \tag{2.2.76}$$

$$= ri\phi\tau + \frac{a_j D_1}{gd}\left[gd\tau - (g-1)\ln\left[\frac{1-gd^{d\tau}}{1-g}\right]\right] \tag{2.2.77}$$

$$= ri\phi\tau + \frac{a_j}{\sigma^2}\left((b_j - \rho\sigma i\phi + d)\tau - \frac{(b_j - \rho\sigma i\phi + d)(g-1)}{gd}\ln\left[\frac{1-ge^{d\tau}}{1-g}\right]\right). \tag{2.2.78}$$

Finally,

$$C = ri\phi\tau + \frac{a_j}{\sigma^2}\left\{(b_j - \rho\sigma i\phi + d)\tau - 2\ln\left[\frac{1-ge^{d\tau}}{1-g}\right]\right\}. \tag{2.2.79}$$

To find the desired probabilities, the inversion formula [14] can be used. We can express the probabilities as follows

$$\begin{aligned} P_j(x, v, t, \ln K) &= \mathbb{Q}\left(x(T) \geq \ln K | x(t) = x, v(t) = v\right) \\ &= 1 - \mathbb{Q}\left(x(T) \leq \ln K | x(t) = x, v(t) = v\right) \end{aligned} \tag{2.2.80}$$

Let us recall the inversion formula

$$F_x(b) - F_x(a) = \frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^{T}\frac{e^{-ita} - e^{-itb}}{it}\Phi_x(u)du, \tag{2.2.81}$$

where $\Phi_x(u)$ represents the characteristic function of the random variable x. Here, we denote the characteristic function as $f_j(x, v, t)$ for $j = 1, 2$ as given by equation (2.2.86). Then, we take $a \to -\infty$ and $b = \ln K$ so that the result yields the probability $\mathbb{Q}\left(x(T) \leq \ln K | x(t) = x, v(t) = v\right)$:

$$F_x(b) - F_x(a) = \frac{1}{2\pi} \lim_{T \to \infty} \int_{-T}^{T} \frac{e^{-i\phi a} - e^{-i\phi b}}{i\phi} f_j(x, v, t; \phi) d\phi \qquad (2.2.82)$$

which can be written as follows (see [20, 11]):

$$P_j(x, v, t, \ln K) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \Re\left[\frac{e^{-i\phi \ln K} f_j(x, v, t; \phi)}{i\phi}\right] d\phi, \qquad (2.2.83)$$

where $\Re$ denotes the real part of a complex number.

The integrals above cannot be eliminated; however, by using approximations, the probabilities can be evaluated. Thus, together with the equation (2.1.20), the equation (2.2.83) gives the solution of the option pricing formula:

$$C(S, v, t) = S_t P_1(x, v, t, \ln K) - K P(t, T) P_2(x, v, t, \ln K) \qquad (2.2.84)$$

$$P_j(x, v, t, \ln K) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \Re\left[\frac{e^{-i\phi \ln K} f_j(x, v, t; \phi)}{i\phi}\right] d\phi, \qquad (2.2.85)$$

where

$$f_j(x, v, t) \;=\; \exp[C(T - t) + D(T - t)v + i\phi x]. \qquad (2.2.86)$$

Here,

$$C = ri\phi\tau + \frac{a_j}{\sigma^2}\left\{(b_j - \rho\sigma i\phi + d)\tau - 2\ln\left[\frac{1-ge^{d\tau}}{1-g}\right]\right\} \tag{2.2.87}$$

and

$$D = \frac{\rho\sigma i\phi - b_j + d}{\sigma^2}\left(\frac{1-e^{\tau d}}{1-ge^{\tau d}}\right), \tag{2.2.88}$$

where

$$g = \frac{\rho\sigma i\phi - b_j + d}{\rho\sigma i\phi - b_j - d} \tag{2.2.89}$$

and

$$d = \sqrt{(\rho\sigma\phi i - b_j)^2 - \sigma^2(2u_j\phi i - \phi^2)}. \tag{2.2.90}$$

CHAPTER 3

BHM'S APPROACH

In this chapter, the new framework proposed by Brody-Huhgston-Macrina(BHM) [4] is closely examined. Firstly, we try to explain the intuition behind the model by following the path BHM used for the explanation of their motivation. Then, we give the assumptions made in the study [4]. After giving the model setting, we closely follow the structure of the paper in a more detailed form. We give the proofs and derivations which are not given in the original paper. For this, we benefit from the two other studies suggested by BHM [3] and Rutkowski-Yu [21], respectively.

According to the BHM approach, asset price dynamics are modeled under the assumption that market participants do not have access to the information about the actual value of the relevant market variables. In other words, it is claimed that market participants acknowledge only partial noisy information about the associated market factors. For example, if an asset is defined by its cash flow structure, then the associated market factor can be the upcoming cash flows. In fact, the associated market factors corresponding to an asset can represent all the market variables which may have an effect on the asset's expected future cash flows. The asset price dynamics are derived based on modeling the structure of the information about those market factors circulating in the market.

In a general setting, as seen in the previous chapters, the economy is generally modeled by a probability space with a filtration generated by a multi-dimensional Brownian motion. Moreover, asset prices are assumed to follow Ito processes which are adapted to this filtration. However, these standard models suffer from tending to show an "ad hoc" nature. Take the Black-Scholes-Merton model, for instance. The underlying price processes are assumed to follow a geometric Brownian motion or as in the Heston model, the variance process is assumed to follow a CIR process. In such standard models, Brownian filtration is certainly sensed to contain all the applicable information, and no inapplicable information. More specifically, in a complete market, the relevant information about the movements of the asset prices is contained in the Brownian filtration. The idea behind this is that there can be a succession of events which can affect price change, and these various effects can be abstractified in the form of this filtration to which asset prices are adapted. The unsatisfactory side of this approach is that it shows that the prices moves as if they were spontaneous. However, in reality, price processes are expected to show more structure. The BHM model suggests an alternative to improve this unsatisfactory side of the standard models and at the same time to avoid their tendency to be of the ad hoc nature.

When assets are traded, prices are formed by the behaviors of investors. The source that affects investor decision concerning future possible transactions can be expressed by two different origins: *i*) investor attitudes toward risk and *ii*) the subjective value of future cash flows. Thus, when a market participant decides to buy or sell an asset, the price at which he is willing to make the transaction is formed according to the information about the possible future cash flows associated with the asset. It can be said that the movements of asset prices contain some information on market participants' expectations about the future value of the asset and so some information about the actual future value of the asset. This elementary observation reveals that the asset price should be seen as the output of the various decisions made concerning possible transactions instead of as an input into such transactions.

To capture the outline described above, the incomplete-information approach is adopted and the so-called market information process ξ_t is defined specifically. The asset price dynamics is derived explicitly by assuming that the market filtration is the one which is generated by this market information process.

3.1 The Model

3.1.1 Basic Definitions and the Assumptions

According to the model, the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ is specified and the market filtration $\{\mathcal{F}_t\}_{0 \leq t < \infty}$ will be stated explicitly. Here, $\mathbb{Q}$ stands for the risk neutral probability measure. All asset price processes and information processes accessible to market participants will be adapted to $\{\mathcal{F}_t\}$.
More specifically,

- The absence of arbitrage and the existence of an established pricing kernel (see, e.g., [6] and references cited therein) is assumed. The existence of a unique risk neutral measure $\mathbb{Q}$ is ensured with these conditions, although the markets considered may be incomplete.

- The short term interest rate r_t is assumed to be deterministic so the default-free discount bond can be expressed as follows:

$$P_{tT} = \exp\left\{-\int_t^T r(u)du\right\}. \tag{3.1.1}$$

- By the absence of arbitrage, the discount bond functions $\{P_{tT}\}_{0\le t\le T<\infty}$ can be written in the form

$$P_{tT} = P_{0T}/P_{0t}, \ \forall t \le T. \tag{3.1.2}$$

- The function $\{P_{0t}\}_{0\le t<\infty}$ is assumed to be strictly decreasing, differentiable and satisfying $0 < P_{0t} \le 1$ and

$$\lim_{t\to\infty} P_{0t} = 0. \tag{3.1.3}$$

- All cash flows occur at pre-determined dates, i.e., its timing is definite, only the amount of the cash flow is random.

3.1.2 Modeling Cash Flows and Asset Prices

Let D_T be a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ having a cumulative distribution function

$$\mathbb{Q}(D_T \le x) = \int_{-\infty}^{x} p(y)dy \quad (x \in \mathbb{R}). \tag{3.1.4}$$

The random variable D_T represents a random cash flow occurring at time T. Thus, it is postulated that D_T takes values only in $[0, \infty)$. Hence, it can be depicted as the single cash flow of an asset paying a single dividend at a predetermined date T.

Let S_t be the value of the cash flow at time t for $0 \leq t < T$ given by

$$S_t = \mathbb{I}_{\{t<T\}} P_{tT} \mathbb{E}^{\mathbb{Q}}[D_T | \mathcal{F}_t]. \tag{3.1.5}$$

The process $\{S_t\}_{0 \leq t < T}$ is in fact the price process of a limited-liability asset paying the single dividend D_T at time T. The convention that when the dividend is paid, the asset price goes "ex-dividend", is adopted; therefore,

$$\lim_{t \to T} S_t = D_T, \qquad S_T = 0. \tag{3.1.6}$$

For a sequence of dividends $D_{T_k}(k = 1, 2, ..., n)$ on the dates T_k, the price is then

$$S_t = \sum_{k=1}^{n} \mathbb{I}_{\{t<T_k\}} P_{tT_k} \mathbb{E}^{\mathbb{Q}}[D_{T_k} | \mathcal{F}_t]. \tag{3.1.7}$$

More generally, when the ex-dividend behavior is taken into account, the price process is described as

$$S_t = \sum_{k=1}^{n} \mathbb{I}_{\{t<T_k\}} P_{tT_k} \mathbb{E}^{\mathbb{Q}}[D_{T_k} | \mathcal{F}_t]. \tag{3.1.8}$$

Furthermore, it is assumed that the discount bond also goes ex-dividend at its maturity date. The price of the bond is given before maturity by the product of the discount factor and the principal. However, at maturity the value of the bond drops to zero. For a coupon bond, when a coupon is paid, the price has a downward jump. Thus, all price processes have the property that they are right-continuous with left limits (cadlaq processes).

The Brownian bridge process (see [18]) $\{\beta_{tT}\}_{0\leq t\leq T}$ is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ such that $\beta_{0T} = 0$ and $\beta_{TT} = 0$ (see [12]). It is known that the mean and the covariance of the Brownian bridge are:

$$\mathbb{E}^{\mathbb{Q}}(\beta_{tT}) = 0 \ \ \forall\ 0 \leq t \leq T, \tag{3.1.9}$$

$$\mathbb{E}^{\mathbb{Q}}(\beta_{sT}\beta_{tT}) = \frac{s(T-t)}{T} \quad \forall\ 0 \leq s \leq t \leq T. \tag{3.1.10}$$

The Brownian bridge process satisfies the following relations [18, 13]:

$$\beta_{tT} = (T-t)\int_0^t \frac{1}{T-s}dW_s, \tag{3.1.11}$$

where $W = \{W_t\}_{0\leq t\leq T}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{Q})$ adapted to its natural filtration $\widetilde{\mathcal{F}}_t := \mathcal{F}_t^W$. Additionally, it can be seen easily that filtrations generated by W_t and β_{tT} coincide:

$$\frac{\beta_{tT}}{T-t} = \int_0^t \frac{1}{T-s}dW_s. \tag{3.1.12}$$

By differentiating (3.1.12), it is seen that

$$\frac{\beta_{tT}}{(T-t)^2}dt + \frac{d\beta_{tT}}{(T-t)} = \frac{1}{T-t}dW_t, \tag{3.1.13}$$

$$\frac{\beta_{tT}}{(T-t)}dt + d\beta_{tT} = dW_t. \tag{3.1.14}$$

Integrating both sides of (3.1.14), the following relations are found:

$$\beta_{tT} \;=\; -\int_0^t \frac{\beta_{sT}}{(T-s)}ds \;+\; W_t, \tag{3.1.15}$$

$$\beta_{uT} \;-\; \beta_{tT} \;=\; -\int_t^u \frac{\beta_{sT}}{(T-s)}ds \;+\; W_u \;-\; W_t. \tag{3.1.16}$$

Furthermore, the Brownian bridge β_{tT} is assumed to be independent of the random variable D_T, and thus the Brownian motion W_t and D_T are independent random variables as well. Then the *enlarged filtration* as given in the study of Rutkowski-Yu [21] is $\mathcal{G} = \widetilde{\mathcal{F}}_t \vee \mathcal{D}_T$ for every $t \in [0,T]$, where $\mathcal{D}_T$ stands for the sigma-algebra generated by the random variable D_T; i.e.,

$$\mathcal{G}_t \;=\; \widetilde{\mathcal{F}}_t \vee \mathcal{D}_T = \;\sigma\{W_u;\; u \;\in [0,t],\; D_T\} = \;\sigma\{\beta_{uT};\; u \;\in [0,t],\; D_T\}.$$

According to this, it is seen that the filtration $\widetilde{\mathcal{F}}_t$ is a sub-filtration of the filtration $\mathcal{G}_t$. The processes β_{tT} and W_t are Brownian bridge and Brownian motion with respect to the filtration $\mathcal{G}_t$, respectively. Lastly, the random variable D_T is $\mathcal{G}_t$-measurable for any $t \;\in [0,T]$.

3.1.3 Modeling the Information Flow

Each market information process is in fact the sum of two terms

- one stands for the partially true information about the value of the associated market variable,
- the other one represents the "noise".

The price of an asset is given by the conditional expectation of the future cash flows under the risk neutral measure. Differently, the market filtration in the conditional expectation is taken as the filtration generated by the market information processes $\{\xi_t\}_{0\leq t\leq T}$ which is modeled as follows:

$$\xi_t = \sigma t D_T + \beta_{tT}. \tag{3.1.17}$$

Here σ stands for the emerging rate of the true information, D_T is the random variable representing the associated market variable, and the Brownian bridge process β_{tT} models the noisy information such as rumors, speculations and general disinformation about the relevant market variables in the market.
For the sake of simplicity, here we only give the case of one single cash flow occurring at time T. The process $\{\xi_t\}$ as seen in equation (3.1.17) is the sum of two terms. The term $\sigma t D_T$ stands for the "true information" about the approaching cash flow. The process $\{\beta_{tT}\}_{0\leq t\leq T}$ is a standard Brownian bridge over the time interval $[0, T]$, so it takes zero values at time 0 and T. It holds $\beta_{0T} = 0$ and $\beta_{TT} = 0$ and it is in fact a Gaussian process having zero mean, $t(T-t)/T$ variance and $s(T-t)/T$ is the covariance between β_{sT} and β_{tT} for $s \leq t$. Thus, the information contained in the bridge process actually represents the pure noise.

The market filtration is assumed to be equal to the filtration generated by $\{\xi_t\}$, i.e., $\{\mathcal{F}_t\} = \{\mathcal{F}_t^\xi\}$, where $\{\mathcal{F}_t^\xi\} = \sigma\{\xi_u;\ u \in [0,t]\}$ for $t \in [0, T]$. Hence, the dividend D_T is $\mathcal{F}_T$-measurable, but not $\mathcal{F}_t$-measurable. The Brownian bridge is not adapted to the market filtration $\{\mathcal{F}_t\}$. Thus it is not accessible to market participants which reflects the fact that market participants cannot perceive the true information without the noise in the market until the dividend is paid. This models the fact that market perceptions play an important role in determining asset prices.

3.1.4 Markov Property of the Information Process

According to the following lemma, to compute the conditional expectation in the equation (3.1.5), it will be enough to take the conditional expectation with respect to the sigma-sub algebra generated by ξ_t since the process $\{\xi_t\}$ satisfies the Markov property:

$$\mathbb{Q}(\xi_t \leq x | \mathcal{F}_s^{\xi}) = \mathbb{Q}(\xi_t \leq x | \xi_s) \tag{3.1.18}$$

for all $x \in \mathbb{R}$ and all s, t such that $0 \leq s \leq t \leq T$.

Lemma 3.1.1. *The process ξ satisfies the Markov property with respect to its natural filtration $\mathcal{F}^{\xi}$.*

Proof. Here we follow the exact path suggested in the study of [21, 3]. It suffices to show that

$$\mathbb{Q}(\xi_t \leq x | \xi_s, \xi_{s_1}, \xi_{s_2}, ..., \xi_{s_k}) = \mathbb{Q}(\xi_t \leq x | \xi_s)$$

for any times $T \geq t > s > s_1 > s_2 > ... > s_k > 0$. It holds

$$\xi_s/s - \xi_{s_1}/s_1 = \beta_{sT}/s - \beta_{s_1T}/s_1, \quad \forall T > t > s > s_1 > 0.$$

Note that for any $t > s > s_1$, β_{tT} and $\frac{\beta_{sT}}{s} - \frac{\beta_{s_1T}}{s_1}$ are Gaussian random variables having zero covariance which can be easily verified as follows:

$$Cov(\beta_{tT}, \frac{\beta_{sT}}{s} - \frac{\beta_{s_1T}}{s_1}) = \mathbb{E}(\beta_{tT}\left(\frac{\beta_{sT}}{s} - \frac{\beta_{s_1T}}{s_1}\right))$$

$$= \frac{1}{s}\mathbb{E}(\beta_{tT}\beta_{sT}) - \frac{1}{s_1}\mathbb{E}(\beta_{tT}\beta_{s_1T})$$

$$= \frac{1}{s}Cov(\beta_{tT}, \beta_{sT}) - \frac{1}{s_1}Cov(\beta_{tT}, \beta_{s_1T})$$

$$= \frac{1}{s}\left(s \wedge t - \frac{st}{T}\right) - \frac{1}{s_1}\left(s_1 \wedge t - \frac{s_1t}{T}\right).$$

Since $t > s > s_1$,

$$= \frac{1}{s}\left(s - \frac{st}{T}\right) - \frac{1}{s_1}\left(s_1 - \frac{s_1t}{T}\right) = 0.$$

More generally, $\frac{\beta_{sT}}{s} - \frac{\beta_{s_1T}}{s_1}$, $\frac{\beta_{s_2T}}{s_2} - \frac{\beta_{s_3T}}{s_3}$, for any $s > s_1 > s_2 > s_3$ are all independent Gaussian random variables.

Furthermore, since ξ_t and ξ_s are independent of $\frac{\beta_{sT}}{s} - \frac{\beta_{s_1T}}{s_1}, \frac{\beta_{s_1T}}{s_1} - \frac{\beta_{s_2T}}{s_2}, ..., \frac{\beta_{s_{k-1}T}}{s_{k-1}} - \frac{\beta_{s_kT}}{s_k}$, the result is as follows:

$$\mathbb{Q}(\xi_t \leq x|\xi_s, \xi_{s_1}, \xi_{s_2}, ..., \xi_{s_k}) = \mathbb{Q}(\xi_t \leq x|\xi_s, \frac{\xi_s}{s} - \frac{\xi_{s_1}}{s_1}, \frac{\xi_{s_1}}{s_1} - \frac{\xi_{s_2}}{s_2}, ..., \frac{\xi_{s_{k-1}}}{s_{k-1}} - \frac{\xi_{s_k}}{s_k})$$

$$= \mathbb{Q}(\xi_t \leq x|\xi_s, \frac{\beta_{sT}}{s} - \frac{\beta_{s_1T}}{s_1}, \frac{\beta_{s_1T}}{s_1} - \frac{\beta_{s_2T}}{s_2}, ..., \frac{\beta_{s_{k-1}T}}{s_{k-1}} - \frac{\beta_{s_kT}}{s_k})$$

$$= \frac{\mathbb{Q}(\xi_t \leq x, \xi_s \leq y_s, \frac{\beta_{sT}}{s} - \frac{\beta_{s_1T}}{s_1} \leq y_{s_1}, \frac{\beta_{s_1T}}{s_1} - \frac{\beta_{s_2T}}{s_2} \leq y_{s_2}, ..., \frac{\beta_{s_{k-1}T}}{s_{k-1}} - \frac{\beta_{s_kT}}{s_k} \leq y_{s_k})}{\mathbb{Q}(\xi_s \leq y_s, \frac{\beta_{sT}}{s} - \frac{\beta_{s_1T}}{s_1} \leq y_{s_1}, \frac{\beta_{s_1T}}{s_1} - \frac{\beta_{s_2T}}{s_2} \leq y_{s_2}, ..., \frac{\beta_{s_{k-1}T}}{s_{k-1}} - \frac{\beta_{s_kT}}{s_k} \leq y_{s_k})}$$

$$= \frac{\mathbb{Q}(\xi_t \leq x, \xi_s \leq y_s)\mathbb{Q}(\frac{\beta_{sT}}{s} - \frac{\beta_{s_1T}}{s_1} \leq y_{s_1})...\mathbb{Q}(\frac{\beta_{s_{k-1}T}}{s_{k-1}} - \frac{\beta_{s_kT}}{s_k} \leq y_{s_k})}{\mathbb{Q}(\xi_s \leq y_s)\mathbb{Q}(\frac{\beta_{sT}}{s} - \frac{\beta_{s_1T}}{s_1} \leq y_{s_1})...\mathbb{Q}(\frac{\beta_{s_{k-1}T}}{s_{k-1}} - \frac{\beta_{s_kT}}{s_k} \leq y_{s_k})}$$

$$= \frac{\mathbb{Q}(\xi_t \leq x, \xi_s \leq y_s)}{\mathbb{Q}(\xi_s \leq y_s)} = \mathbb{Q}(\xi_t \leq x|\xi_s).$$

This completes the proof.

3.1.5 The Derivation of the Conditional Density

By using the Markov property and the fact that D_T is $\mathcal{F}_T$-measurable, the asset price process is expressed as follows:

$$S_t = \mathbb{I}_{\{t<T\}} P_{tT} \mathbb{E}^{\mathbb{Q}}[D_T|\mathcal{F}_t] = \mathbb{I}_{\{t<T\}} P_{tT} \mathbb{E}^{\mathbb{Q}}[D_T|\xi_t]. \tag{3.1.19}$$

As D_T stands for the dividend payoff of the risky asset, it can have a continuous distribution and then the conditional expectation in (4.1.6) can be written in the following form

$$\mathbb{E}^{\mathbb{Q}}[D_T|\xi_t] = \int_0^\infty x\pi_t(x)dx, \tag{3.1.20}$$

where $\pi_t(x)$ represents the conditional probability density function for the random variable D_T:

$$\pi_t(x) = \frac{d}{dx}\mathbb{Q}(D_T \leq x|\xi_t). \tag{3.1.21}$$

By the Bayes formula, the conditional probability density can be expressed in the form

$$\pi_t(x) = \frac{\rho(\xi_t|D_T = x)p(x)}{\rho(\xi_t)}, \tag{3.1.22}$$

where $p(x)$ represents the priori probability density function for D_T which will be assumed as an initial condition, and $\rho(\xi_t)$ and $\rho(\xi_t|D_T = x)$ denotes the probability density function and the conditional probability density function for the random variable ξ_t given $D_T = x$, respectively.

As the probability density function of the random variable ξ_t can be written as follows:

$$\rho(\xi_t) = \int_{-\infty}^{\infty} \rho(\xi_t|D_T = x)p(x)dx = \int_0^{\infty} \rho(\xi_t|D_T = x)p(x)dx, \tag{3.1.23}$$

the conditional probability density function $\pi_t(x)$:

$$\pi_t(x) = \frac{\rho(\xi_t|D_T = x)p(x)}{\int_0^{\infty} \rho(\xi_t|D_T = x)p(x)dx}. \tag{3.1.24}$$

Since β_{tT} is a Gaussian random variable for every $0 \leq t \leq T$, the conditional probability density function for the random variable ξ_t given $D_T = x$ is a Gaussian density with mean $\sigma t x$ and variance $t(T-t)/T$:

$$\rho(\xi_t | D_T = x) = \frac{1}{\sqrt{2\pi}\sqrt{\frac{t(T-t)}{T}}} \exp\left(-\frac{1}{2}\frac{(\xi_t - \sigma t x)^2}{\frac{t(T-t)}{T}}\right)$$

$$= \sqrt{\frac{T}{2\pi t(T-t)}} \exp\left(-\frac{(\xi_t - \sigma t x)^2 T}{2t(T-t)}\right).$$

Substituting the expression into the Bayes formula gives

$$\pi_t(x) = \frac{p(x)\sqrt{\frac{T}{2\pi t(T-t)}} \exp\left(-\frac{(\xi_t - \sigma t x)^2 T}{2t(T-t)}\right)}{\int_0^\infty p(x)\sqrt{\frac{T}{2\pi t(T-t)}} \exp\left(-\frac{(\xi_t - \sigma t x)^2 T}{2t(T-t)}\right) dx} \tag{3.1.25}$$

$$= \frac{p(x) \exp\left(-\frac{\xi_t^2}{2t(T-t)}\right) \exp\left(\frac{2\sigma x \xi_t t T - \sigma^2 x^2 t^2 T}{2t(T-t)}\right)}{\exp\left(-\frac{\xi_t^2}{2t(T-t)}\right) \int_0^\infty p(x) \exp\left(\frac{2\sigma x \xi_t t T - \sigma^2 x^2 t^2 T}{2t(T-t)}\right) dx}$$

$$= \frac{p(x) \exp\left(\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right)}{\int_0^\infty p(x) \exp\left(\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right) dx}. \tag{3.1.26}$$

Proposition 3.1.2. *The information-based price process $\{S_t\}_{0 \le t \le T}$ of a limited-*

liability asset that pays a single dividend D_T at time T with a priori distribution

$$\mathbb{Q}(D_T \leq y) = \int_0^y p(x)dx$$

is given by

$$S_t = \mathbb{I}_{\{t<T\}} P_{tT} \frac{\int_0^\infty xp(x)\exp\left(\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t)\right)dx}{\int_0^\infty p(x)\exp\left(\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t)\right)dx}, \tag{3.1.27}$$

where $\xi_t = \sigma t D_T + \beta tT$ is the market information process.

Proof.

We know that the price process satisfies (3.1.19):

$$\begin{aligned} S_t &= \mathbb{I}_{\{t<T\}} P_{tT} \mathbb{E}^{\mathbb{Q}}[D_T|\xi_t] \\ &= \mathbb{I}_{\{t<T\}} P_{tT} \int_0^\infty x\pi_t(x)dx \\ &= \mathbb{I}_{\{t<T\}} P_{tT} \int_0^\infty x \left(\frac{p(x)\exp\left(\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t)\right)}{\int_0^\infty p(x)\exp\left(\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t)\right)dx} \right) dx \end{aligned}$$

$$= \mathbb{I}_{\{t<T\}} P_{tT} \frac{\int_0^\infty xp(x)\exp\left(\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t)\right)dx}{\int_0^\infty p(x)\exp\left(\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t)\right)dx}.$$

3.2 Asset Price Dynamics in the Case of a Single Cash Flow

The stochastic differential equation of which the price process $\{S_t\}$ is the solution is derived so that it can be possible to analyze the properties of the price process and to compare it with other models. In order to obtain the dynamics of the price process $\{S_t\}$, the conditional expectation of D_T with respect to the market information ξ_t is denoted as follows:

$$D_{tT} \;=\; D(\xi_t, t) \;=\; \mathbb{E}^{\mathbb{Q}}[D_T|\xi_t], \tag{3.2.28}$$

where

$$D(\xi_t, t) = \frac{\int_0^\infty xp(x)\exp\left(\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t)\right)dx}{\int_0^\infty p(x)\exp\left(\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t)\right)dx}. \tag{3.2.29}$$

Lemma 3.2.1. *Let D_{tT} be the process given by the equation (3.2.28). Then the dynamics of D_{tT} under $\mathbb{Q}$ are*

$$dD_{tT} = \frac{\sigma T}{T-t}V_t\left[\frac{1}{T-t}\left(\xi_t - \sigma T D_{tT}\right)dt + d\xi_t\right], \tag{3.2.30}$$

where V_t stands for the conditional variance of the dividend:

$$V_t = \int_0^\infty x^2\pi_t(x)dx - \left(\int_0^\infty x\pi_t(x)dx\right)^2. \tag{3.2.31}$$

Then the spot asset price dynamics is:

$$dS_t = r_tS_tdt + \Gamma_{tT}dD_{tT}, \tag{3.2.32}$$

where

$$\Gamma_{tT} = P_{tT}\frac{\sigma T}{T-t}V_t.$$

Proof. To find the differential equation for the conditional expectation $\{D_{tT}\}$, the Ito rules are used as follows:

$$dD(\xi,t) = \frac{\partial D(\xi,t)}{\partial t}dt + \frac{\partial D(\xi,t)}{\partial \xi}d\xi + \frac{1}{2}\frac{\partial^2 D(\xi,t)}{\partial \xi^2}d\left\langle \xi,\xi\right\rangle_t$$

$$\frac{\partial D(\xi,t)}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\int_0^\infty xp(x)\exp(\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t))dx}{\int_0^\infty p(x)\exp(\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t))dx}\right)$$

and by denoting

$$A(\xi,x,t) = \frac{T}{T-t}\left(\sigma x\xi - \frac{1}{2}\sigma^2x^2t\right),$$

we get

$$\frac{\partial D(\xi,t)}{\partial t} = \frac{\frac{\partial}{\partial t}\left(\int_0^\infty xp(x)exp\left\{A(\xi,x,t)\right\}dx\right)\left(\int_0^\infty p(x)\exp\left\{A(\xi,x,t)\right\}dx\right)}{\left(\int_0^\infty p(x)\exp(A(\xi,x,t))dx\right)^2}$$
$$-\frac{\left(\int_0^\infty xp(x)exp\left\{A(\xi,x,t)\right\}dx\right)\frac{\partial}{\partial t}\left(\int_0^\infty p(x)\exp\left\{A(\xi,x,t)\right\}dx\right)}{\left(\int_0^\infty p(x)\exp(A(\xi,x,t))dx\right)^2}$$

$$= \frac{\left(\int_0^\infty xp(x)\frac{\partial A(\xi,x,t)}{\partial t}exp\left\{A(\xi,x,t)\right\}dx\right)\left(\int_0^\infty p(x)exp\left\{A(\xi,x,t)\right\}dx\right)}{\left(\int_0^\infty p(x)exp\left\{A(\xi,x,t)\right\}dx\right)^2} -$$
$$-\frac{\left(\int_0^\infty xp(x)exp\left\{A(\xi,x,t)\right\}dx\right)\left(\int_0^\infty p(x)\frac{\partial A(\xi,x,t)}{\partial t}exp\left\{A(\xi,x,t)\right\}dx\right)}{\left(\int_0^\infty p(x)exp\left\{A(\xi,x,t)\right\}dx\right)^2}$$

$$= \frac{\int_0^\infty xp(x)\frac{\partial A(\xi,x,t)}{\partial t}exp\left\{A(\xi,x,t)\right\}dx - D_{tT}\int_0^\infty p(x)\frac{\partial A(\xi,x,t)}{\partial t}exp\left\{A(\xi,x,t)\right\}dx}{\int_0^\infty p(x)exp\left\{A(\xi,x,t)\right\}dx}.$$

Because of

$$\frac{\partial A(\xi,x,t)}{\partial t} = \frac{\sigma xT}{T-t}\left(\frac{1}{T-t}\left(\xi-\sigma xT\right)\right), \tag{3.2.33}$$

we conclude

$$\frac{\partial D(\xi,t)}{\partial t} = \frac{\sigma T\xi}{(T-t)^2}\int_0^\infty x^2\pi_t(x)dx - D_{tT}\frac{\sigma T\xi}{(T-t)^2}\int_0^\infty x^2\pi_t(x)dx +$$
$$+D_{tT}\frac{\sigma^2T^2}{2(T-t)^2}\int_0^\infty x^2\pi_t(x)dx - \frac{\sigma^2T^2}{2(T-t)^2}\int_0^\infty x^3\pi_t(x)dx$$

$$= \frac{\sigma T\xi}{(T-t)^2}\left(\int_0^\infty x^2\pi_t(x)dx - \left(\int_0^\infty x\pi_t(x)dx\right)^2\right) +$$
$$+\frac{\sigma^2T^2}{2(T-t)^2}D_{tT}\left(\int_0^\infty x^2\pi_t(x)dx - \int_0^\infty x^3\pi_t(x)dx\right)$$

$$= \frac{\sigma T\xi}{(T-t)^2}V_t + \frac{\sigma^2T^2}{2(T-t)^2}\left(D_{tT}V_t + D_{tT}^3 - \int_0^\infty x^3\pi_t(x)dx\right).$$

Similarly,

$$\frac{\partial D(\xi,t)}{\partial \xi} = \frac{\partial}{\partial \xi}\left(\frac{\int_0^\infty xp(x)\exp(A(\xi,x,t))dx}{\int_0^\infty p(x)\exp(A(\xi,x,t))dx}\right)$$

$$= \frac{\int_0^\infty xp(x)\frac{\partial A(\xi,x,t)}{\partial \xi}exp\left\{A(\xi,x,t)\right\}dx - D_{tT}\int_0^\infty p(x)\frac{\partial A(\xi,x,t)}{\partial \xi}exp\left\{A(\xi,x,t)\right\}dx}{\int_0^\infty p(x)exp\left\{A(\xi,x,t)\right\}dx},$$

where

$$\frac{\partial A(\xi,x,t)}{\partial \xi} = \frac{T\sigma x}{T-t}$$

$$= \frac{\sigma T}{T-t}\left(\frac{\int_0^\infty x^2 p(x) exp\left\{A(\xi,x,t)\right\}dx - D_{tT}\int_0^\infty xp(x)exp\left\{A(\xi,x,t)\right\}dx}{\int_0^\infty p(x)exp\left\{A(\xi,x,t)\right\}dx}\right)$$

$$= \frac{\sigma T}{T-t}V_t. \tag{3.2.34}$$

Furthermore,

$$\frac{\partial^2 D(\xi,t)}{\partial \xi^2} = \frac{\partial^2}{\partial \xi^2}\left(\frac{\int_0^\infty xp(x)\exp(\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t))dx}{\int_0^\infty p(x)\exp(\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t))dx}\right)$$

$$= \frac{\partial}{\partial \xi}\left(\frac{\int_0^\infty xp(x)\frac{\partial A(\xi,x,t)}{\partial \xi}e^{A(\xi,x,t)}dx}{\int_0^\infty p(x)e^{A(\xi,x,t)}dx}\right) - \frac{\partial}{\partial \xi}\left(D_{tT}\frac{\int_0^\infty xp(x)e^{A(\xi,x,t)}dx}{\int_0^\infty p(x)e^{A(\xi,x,t)}dx}\right)$$

$$= \left(\frac{\sigma T}{T-t}\right)^2\frac{\int_0^\infty x^3p(x)exp(A(\xi,x,t))dx}{\int_0^\infty p(x)exp(A(\xi,x,t))dx} -$$
$$-\left(\frac{\sigma T}{T-t}\right)^2\left(\frac{\int_0^\infty x^2p(x)exp(A(\xi,x,t))dx}{\int_0^\infty p(x)exp(A(\xi,x,t))dx}\right)\left(\frac{\int_0^\infty xp(x)exp(A(\xi,x,t))dx}{\int_0^\infty p(x)exp(A(\xi,x,t))dx}\right) +$$
$$+\left(\frac{\sigma T}{T-t}\right)^2 V_tD_{tT} + \left(\frac{\sigma T}{T-t}\right)^2 D_{tT}\frac{\int_0^\infty x^2p(x)exp(A(\xi,x,t))dx}{\int_0^\infty p(x)exp(A(\xi,x,t))dx} -$$
$$-\left(\frac{\sigma T}{T-t}\right)^2\left(\frac{\int_0^\infty xp(x)exp(A(\xi,x,t))dx}{\int_0^\infty p(x)exp(A(\xi,x,t))dx}\right)^2$$

$$= \left(\frac{\sigma T}{T-t}\right)^2\int_0^\infty x^3\pi_t(x)dx - \left(\frac{\sigma T}{T-t}\right)^2 D_{tT}\int_0^\infty x^2\pi_t(x)dx +$$
$$+\left(\frac{\sigma T}{T-t}\right)^2 V_tD_{tT} + \left(\frac{\sigma T}{T-t}\right)^2 D_{tT}\left(\int_0^\infty x^2\pi_t(x)dx - D_{tT}^2\right)$$

$$= \left(\frac{\sigma T}{T-t}\right)^2 \left(\int_0^\infty x^3 \pi_t(x)dx - D_{tT}V_t - D_{tT}^3 - 2V_t D_{tT}\right). \qquad (3.2.35)$$

Substituting the expressions of $\partial D(\xi,t)/\partial t$, $\partial D(\xi,t)/\partial \xi$ and $\partial^2 D(\xi,t)/\partial \xi^2$ given above, the dynamic equation of $D(\xi,t)$ is obtained as follows:

$$dD(\xi,t) = \frac{\sigma T}{(T-t)^2}\xi_t V_t dt + \frac{(\sigma T)^2}{2(T-t)^2}\left(D_{tT}V_t + D_{tT}^3 - \int_0^\infty x^3\pi_t(x)dx\right)dt +$$
$$+\frac{\sigma T}{T-t}V_t d\xi + \frac{1}{2}\frac{(\sigma T)^2}{(T-t)^2}\left(\int_0^\infty x^3\pi_t(x)dx - D_{tT}V_t - D_{tT}^3 - 2V_t D_{tT}\right)dt$$

$$= \frac{\sigma T}{T-t}V_t\left(\frac{1}{T-t}(\xi_t - \sigma T D_{tT})dt + d\xi_t\right),$$

$$dD_{tT} = \frac{\sigma T}{T-t}V_t\left[\frac{1}{T-t}\left(\xi_t - \sigma T D_{tT}\right)dt + d\xi_t\right],$$

where V_t stands for the conditional variance of the dividend

$$V_t = \int_0^\infty x^2\pi_t(x)dx - \left(\int_0^\infty x\pi_t(x)dx\right)^2. \qquad (3.2.36)$$

Then, we obtain the asset price dynamics as given by

$$dS_t = r_t S_t dt + \frac{\sigma T}{T-t}V_t\left[\frac{1}{T-t}\left(\xi_t - \sigma T D_{tT}\right)dt + d\xi_t\right]. \qquad (3.2.37)$$

This ends the proof of the lemma.

3.2.1 Dynamics of the Information Process

It is known that the Brownian bridge satisfies the following dynamics (see [13])

$$d\beta_{tT} = -\frac{\beta_{tT}}{(T-t)}dt + dW_t, \tag{3.2.38}$$

$$\beta_{0T} = 0,$$

and the information process

$$\begin{aligned}
\xi_t &= \sigma t D_T + \beta_{tT}, \\
d\xi_t &= \sigma D_T dt + d\beta_{tT} \\
&= \sigma D_T dt - \frac{\beta_{tT}}{(T-t)} + dW_t \\
&= \sigma D_T dt - \frac{\xi_t - \sigma t D_T}{(T-t)}dt + dW_t \\
&= \frac{\sigma D_T (T-t) - \xi_t + \sigma t D_T}{T-t}dt + dW_t \\
&= \frac{1}{T-t}(\sigma D_T T - \xi_t)dt + dW_t.
\end{aligned}$$

Thus, the information process ξ_t is a continuous semimartingale, and its quadratic variation is $\langle \xi, \xi \rangle_t = t$ for every $t \in [0, T]$.

Information-driven Brownian Motion

A new process $\{W_t\}_{0 \le t < T}$ is defined as follows

$$W_t = \xi_t - \int_0^t \frac{1}{T-s} \left(\sigma T D_{sT} - \xi_s \right) ds \tag{3.2.39}$$

After some rearrangement of terms, it is found that D_{tT} satisfies the following stochastic differential equation

$$dD_{tT} = \frac{\sigma T}{T-t} V_t dW_t.$$

The asset price process $\{S_t\}_{0 \leq t < T}$ satisfies the following stochastic differential equation

$$dS_t = r_t S_t dt + \Gamma_{tT} dW_t,$$

where r_t represents the short rate that is given by $r_t = -d \ln P_{0t}/dt$, and the absolute price volatility Γ_{tT} is as expressed in this way:

$$\Gamma_{tT} = P_{tT} \frac{\sigma T}{T-t} V_t.$$

When the expected dividend process $\{D_{tT}\}$ is examined, it is seen that it is an $\{F_t\}$-martingale which can be easily seen as follows: for $t \leq s < T$

$$D_{tT} = \mathbb{E}^{\mathbb{Q}} \left[D_T | F_t \right]$$

and

$$\mathbb{E}^{\mathbb{Q}} \left[D_{sT} | F_t \right] = \mathbb{E}^{\mathbb{Q}} \left[\mathbb{E}^{\mathbb{Q}} \left[D_T | F_s \right] | F_t \right] = \mathbb{E}^{\mathbb{Q}} \left[D_T | F_t \right] = D_{tT}.$$

Thus, $\{W_t\}$ must be an $\{F_t\}$-martingale.

Proposition 3.2.2. *[4] The process $\{W_t\}$ defined by equation (3.2.39) is a stan-*

dard $\{F_t\}$-Brownian motion under the risk neutral measure $\mathbb{Q}$.

Proof. For the proof, we follow the original proof of the BHM [4]. Therefore, it is sufficient to show the following two axioms

(i) $\{W_t\}$ is an $\{F_t\}$-martingale.

(ii) $(dW_t)^2 = dt$.

The conditional expectation can be expressed as follows: for $t \leq u < T$

$$\mathbb{E}^Q\left[W_u|F_t^\xi\right] = \mathbb{E}^Q\left[W_u|\xi_t\right] = \mathbb{E}^Q\left[\xi_u|\xi_t\right] + \mathbb{E}^Q\left[\int_0^u \frac{1}{T-s}\xi_s ds|\xi_t\right]$$

$$-\mathbb{E}^Q\left[\int_0^u \frac{\sigma T}{T-s}D_{sT}ds|\xi_t\right]$$

$$= \mathbb{E}^Q\left[\xi_u|\xi_t\right] + \int_0^t \frac{1}{T-s}\xi_s ds - \int_0^t \frac{\sigma T}{T-s}D_{sT}ds + \mathbb{E}^Q\left[\int_t^u \frac{1}{T-s}\xi_s ds|\xi_t\right]$$

$$-\mathbb{E}^Q\left[\int_t^u \frac{\sigma T}{T-s}D_{sT}ds|\xi_t\right]$$

$$= \mathbb{E}^Q\left[\xi_u|\xi_t\right] + \int_0^t \frac{1}{T-s}\xi_s ds - \int_0^t \frac{\sigma T}{T-s}D_{sT}ds + \int_t^u \frac{1}{T-s}\mathbb{E}^Q\left[\xi_s|\xi_t\right]ds$$

$$-\int_t^u \frac{\sigma T}{T-s}D_{tT}ds$$

for $t \leq s$

$$\mathbb{E}^Q\left[\beta_{sT}|F_t^\xi\right] = \mathbb{E}^Q\left[\mathbb{E}^Q\left[\beta_{sT}|\sigma(\beta_{kT}; k \leq t, D_T)\right]|F_t^\xi\right].$$

As ξ_t is a Markov process,

$$\mathbb{E}^Q\left[\beta_{sT}|\xi_t\right] = \mathbb{E}^Q\left[\mathbb{E}^Q\left[\beta_{sT}|\sigma(\beta_{tT}, D_T)\right]|\xi_t\right]$$

$$= \mathbb{E}^Q \left[\mathbb{E}^Q \left[\beta_{sT} | D_T, \beta_{tT} \right] | \xi_t \right] = \mathbb{E}_t^Q \left[\mathbb{E}^Q \left[\beta_{sT} | \beta_{tT} \right] \right].$$

By making use of the fact that the random variable $\beta_{sT}/(T-s) - \beta_{tT}/(T-t)$ is independent of β_{tT}, the inner conditional expectation $\mathbb{E}^Q \left[\beta_{sT} | \beta_{tT} \right]$ is found as follows:

$$\mathbb{E}^Q \left[\frac{\beta_{sT}}{T-s} - \frac{\beta_{tT}}{T-t} | \beta_{tT} \right] = \mathbb{E}^{\mathbb{Q}} \left[\frac{\beta_{sT}}{T-s} - \frac{\beta_{tT}}{T-t} \right] = 0,$$

$$\mathbb{E}^Q \left[\beta_{sT} | \beta_{tT} \right] = \frac{T-s}{T-t} \beta_{tT},$$

$$E_t^Q \left[\beta_{sT} \right] = E_t^Q \left[\frac{T-s}{T-t} \beta_{tT} \right] = \frac{T-s}{T-t} E_t^Q \left[\beta_{tT} \right],$$

$$E_t^Q \left[\xi_s \right] = E_t^Q \left[\sigma s D_T + \beta_{sT} \right] = \sigma s D_{tT} + \frac{T-s}{T-t} E_t^Q \left[\beta_{tT} \right].$$

From this, it follows:

$$E_t^Q \left[W_u \right] = E_t^Q \left[\xi_u \right] + \int_0^t \frac{(\xi_s - \sigma T D_{sT})}{T-s} ds - \int_t^u \sigma D_{tT} ds + \int_t^u \frac{\mathbb{E}_t^Q \left[\beta_{tT} \right]}{T-t} ds.$$

Let us recall that the definition of the process $\{W_t\}$

$$W_t - \xi_t = \int_0^t \frac{1}{T-s} \xi_s ds - \int_0^t \frac{\sigma T}{T-s} D_{sT} ds$$

After some arrangements,

$$E_t^Q \left[W_u \right] = E_t^Q \left[\xi_u \right] + W_t - \xi_t - \sigma D_{tT}(u-t) + \frac{1}{T-t} \mathbb{E}_t^Q \left[\beta_{tT} \right] (u-t)$$

$$= \sigma u D_{tT} + \frac{T-u}{T-t} \mathbb{E}_t^{\mathbb{Q}} \left[\beta_{tT} \right] + W_t - \xi_t + \frac{u-t}{T-t} \mathbb{E}^{\mathbb{Q}} t \left[\beta_{tT} \right] - \sigma D_{tT}(u-t)$$

$$= W_t - \xi_t + \mathbb{E}^{\mathbb{Q}} t \left[\beta_{tT}\right] + \sigma t D_{tT}.$$

Finally, using the fact that $\xi_t = \mathbb{E}_t^{\mathbb{Q}}\left[\xi_t\right]$, it is found that the process $\{W_t\}$ satisfies the martingale property:

$$\mathbb{E}_t^{Q}\left[W_u\right] = W_t. \tag{3.2.40}$$

Furthermore, the second axiom can be seen easily:

$$(dW_t)^2 = \left(d\xi_t - \frac{1}{T-t}\left(\sigma T D_{tT} - \xi_t\right) dt\right)^2 = (d\xi_t)^2 = dt.$$

Thus, it is concluded that the process $\{W_t\}$ is an $\{F_t\}$-Brownian motion.

3.2.2 The Derivation of the Dynamics of the Conditional Density Process

A slightly different way of finding the dynamics of the price process $\{S_t\}$ can be as follows.

Firstly, the dynamics of the conditional probability process $\pi_t(x)$ is found by pursuing the same path as above:

$$\pi_t(x) = \frac{p(x)\exp\left(\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right)}{\int_0^\infty p(x)\exp\left(\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right) dx}. \tag{3.2.41}$$

An application of the Ito rules to (3.2.41) gives the dynamics of the conditional density process as follows:

$$d\pi_t(x) = \frac{\partial \pi_t(x)}{\partial t}dt + \frac{\partial \pi_t(x)}{\partial \xi}d\xi + \frac{1}{2}\frac{\partial^2 \pi_t(x)}{\partial \xi^2}d\left\langle \xi, \xi \right\rangle_t .$$

By using the same notation used above, we have

$$A(\xi, x, t) = \frac{T}{T-t}\left(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t\right).$$

Thus,

$$\frac{\partial \pi_t(x)}{\partial t} = \frac{\partial}{\partial t}\left(\frac{p(x)exp\left(A(\xi,x,t)\right)}{\int_0^\infty p(x)exp\left(A(\xi,x,t)\right)dx}\right)$$

$$= \frac{\sigma T \xi_t}{(T-t)^2}\pi_t(x)\left(x - \int_0^\infty x\pi_t(x)dx\right) - \frac{(\sigma T)^2}{2(T-t)^2}\pi_t(x)\left(x^2 - \int_0^\infty x^2\pi_t(x)dx\right)$$

$$= \frac{\sigma T \xi_t}{(T-t)^2}\pi_t(x)\left(x - D_{tT}\right) - \frac{(\sigma T)^2}{2(T-t)^2}\pi_t(x)\left(x^2 - V_t - D_{tT}^2\right),$$

$$\frac{\partial \pi_t(x)}{\partial \xi} = \frac{\partial}{\partial \xi}\left(\frac{p(x)exp\left(A(\xi,x,t)\right)}{\int_0^\infty p(x)exp\left(A(\xi,x,t)\right)dx}\right)$$

$$= \frac{\sigma T}{T-t}x\pi_t(x) - \frac{\sigma T}{T-t}\pi_t(x)\int_0^\infty x\pi_t(x)dx = \frac{\sigma T}{T-t}\pi_t(x)(x - D_{tT}),$$

$$\frac{\partial^2 \pi_t(x)}{\partial \xi^2} = \frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \xi}\left(\frac{p(x)exp\left(A(\xi,x,t)\right)}{\int_0^\infty p(x)exp\left(A(\xi,x,t)\right)dx}\right)\right)$$

$$= \frac{(\sigma T)^2}{(T-t)^2}x^2\pi_t(x) - \frac{(\sigma T)^2}{(T-t)^2}x\pi_t(x)D_{tT} + \frac{(\sigma T)^2}{(T-t)^2}\pi_t(x)(x - D_{tT})D_{tT} +$$

$$+ \frac{(\sigma T)^2}{(T-t)^2}\pi_t(x)V_t$$

$$= \frac{\sigma T}{T-t}\pi_t(x)(x - D_{tT}) + \left(\frac{\sigma T}{T-t}\right)^2\pi_t(x)\left(xD_{tT} - D_{tT}^2 + V_t\right)$$

$$= \left(\frac{\sigma T}{T-t}\right)^2 \pi_t(x)\left(x^2 - 2xD_{tT} + D_{tT}^2 - V_t\right).$$

Substituting the expressions of $\partial\pi_t(x)/\partial t$, $\partial\pi_t(x)/\partial\xi$ and $\partial^2\pi_t(x)/\partial\xi^2$ calculated above, the dynamic equation for the conditional probability process $\pi_t(x)$ is found as follows:

$$d\pi_t(x) = \frac{\sigma T\xi_t}{(T-t)^2}\pi_t(x)\left(x - D_{tT}\right) - \frac{(\sigma T)^2}{2(T-t)^2}\pi_t(x)\left(x^2 - V_t - D_{tT}^2\right)dt +$$

$$+\frac{\sigma T}{T-t}\pi_t(x)(x - D_{tT})d\xi_t + \frac{1}{2}\left(\frac{\sigma T}{T-t}\right)^2 \pi_t(x)\left(x^2 - 2xD_{tT} + D_{tT}^2 - V_t\right)dt$$

$$= \frac{\sigma T}{T-t}\pi_t(x)\left((x - D_{tT})\left(d\xi_t - \frac{1}{T-t}(\sigma T D_{tT} - \xi_t)dt\right)\right).$$

Finally, the dynamics of the conditional probability process is

$$d\pi_t(x) = \frac{\sigma T}{T-t}\pi_t(x)(x - D_{tT})dW_t. \qquad (3.2.42)$$

As the asset price process is given by

$$S_t = \mathbb{I}_{\{t<T\}} P_{tT} \int_0^\infty x\pi_t(x)dx.$$

Therefore, the same result for the dynamics of the asset price process can easily be verified as follows:

$$dS_t = r_t S_t dt + P_{tT}\int_0^\infty x\, d\pi_t(x)dx$$

$$= r_t S_t dt + P_{tT} \int_0^\infty x \left(\frac{\sigma T}{T-t}(x - D_{tT})\pi_t(x) dW_t \right) dx$$

$$= r_t S_t dt + P_{tT} \frac{\sigma T}{T-t} \left(\int_0^\infty x^2 \pi_t(x) dx - D_{tT} \int_0^\infty x \pi_t(x) dx \right) dW_t,$$

Hence,

$$dS_t = r_t S_t dt + P_{tT} \frac{\sigma T}{T-t} V_t dW_t.$$

3.3 Time-Dependent Information Emerging Rate

In this section, a generalization of the model to the case where the emerging rate of the true information depends on time is considered as given in the original study [4]. When the parameter σ in the definition of the market information process is taken as being time-dependent, the expected dividend process and the asset price dynamics are examined. Without deviating from the path they follow, we present the proofs and the derivations, which are not given detailed in the study [4].

According to this case, the market information process is defined as follows

$$\xi_t \;=\; D_T \int_0^t \sigma_s ds \;+\; \beta_{tT}, \tag{3.3.43}$$

where the function $\{\sigma_s\}_{0 \le s \le T}$ is taken to be a nonnegative deterministic function of time. Furthermore, the following condition is assumed:

$$0 \;<\; \int_0^T \sigma_s^2 ds \;<\; \infty. \tag{3.3.44}$$

The risky asset price process is given by

$$S_t = \mathbb{I}_{\{t<T\}} P_{tT} \mathbb{E}^{\mathbb{Q}}[D_T|\mathcal{F}_t], \tag{3.3.45}$$

where the market filtration is assumed to be generated by the information process $\{\xi_t\}$ defined by equation (3.3.43). Here, a change of measure technique is used to find out the conditional expectation given in the equation (3.3.45). The conditional density process $\{\pi_t(x)\}$ is defined as

$$\pi_t(x) = \frac{d}{dx}\mathbb{Q}(D_T \leq x|\mathcal{F}_t). \tag{3.3.46}$$

Proposition 3.3.1. *[4] Let the information process $\{\xi_t\}$ be defined by equation (3.3.43). Then, the conditional probability density process $\{\pi_t(x)\}$ for the random variable D_T is given by*

$$\pi_t(x) = \frac{p(x)e^{x\left(\frac{1}{T-t}\xi_t\int_0^t\sigma_s ds+\int_0^t\sigma_s d\xi_s\right)-\frac{1}{2}x^2\left(\frac{1}{T-t}\left(\int_0^t\sigma_s ds\right)^2+\int_0^t\sigma_s^2 ds\right)}}{\int_0^\infty p(x)e^{x\left(\frac{1}{T-t}\xi_t\int_0^t\sigma_s ds+\int_0^t\sigma_s d\xi_s\right)-\frac{1}{2}x^2\left(\frac{1}{T-t}\left(\int_0^t\sigma_s ds\right)^2+\int_0^t\sigma_s^2 ds\right)}dx} \tag{3.3.47}$$

and the conditional expectation looks as follows:

$$D_tT = \frac{\int_0^\infty xp(x)e^{x\left(\frac{1}{T-t}\xi_t\int_0^t\sigma_s ds+\int_0^t\sigma_s d\xi_s\right)-\frac{1}{2}x^2\left(\frac{1}{T-t}\left(\int_0^t\sigma_s ds\right)^2+\int_0^t\sigma_s^2 ds\right)}}{\int_0^\infty p(x)e^{x\left(\frac{1}{T-t}\xi_t\int_0^t\sigma_s ds+\int_0^t\sigma_s d\xi_s\right)-\frac{1}{2}x^2\left(\frac{1}{T-t}\left(\int_0^t\sigma_s ds\right)^2+\int_0^t\sigma_s^2 ds\right)}dx}. \tag{3.3.48}$$

Thus, the asset price process $\{S_t\}$ is given by

$$S_t = \mathbb{I}_{\{t<T\}} P_{tT}\ D_{tT}. \tag{3.3.49}$$

Proof. The probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with a filtration $\{\mathcal{G}_t\}_{0\leq t<\infty}$ and W_t is a standard $\{\mathcal{G}_t\}$-Brownian motion. Moreover, β_{tT} and D_T are assumed to be independent; D_T is assumed to be $\mathcal{G}_0$-measurable and bounded. Then the Brownian bridge process $\{\beta_{tT}\}$ can be expressed as a stochastic integral:

$$\beta_{tT} \;=\; (T-t)\int_0^t \frac{1}{T-s}dW_s. \tag{3.3.50}$$

Here, β_{tT} is adapted to $\{\mathcal{G}_t\}$. The deterministic nonnegative process $\{\nu_t\}_{0\le t\le T}$ is defined as follows:

$$\nu_t \;=\; \sigma_t \;+\; \frac{1}{T-t}\int_0^t \sigma_s ds, \tag{3.3.51}$$

satisfying the following relation

$$\int_0^t \frac{1}{T-s}\nu_s ds \;=\; \frac{1}{T-t}\int_0^t \sigma_s ds, \tag{3.3.52}$$

which can easily be verified by differentiation. The process $\{\Lambda_t\}_{0\le t<T}$ is defined by the relation

$$\frac{1}{\Lambda_t} \;:=\; \exp\left(-D_T\int_0^t \nu_s dW_s - \frac{1}{2}D_T^2\int_0^t \nu_s^2 ds\right), \tag{3.3.53}$$

which can be used as a change of measure density from $\mathbb{Q}$ to $\mathbb{B}_T$ on $\mathcal{G}_U$ for a fixed time horizon $U\in(0,T)$:

$$d\mathbb{B}_T \;=\; \Lambda_U^{-1}\, d\mathbb{Q}. \tag{3.3.54}$$

Then, the process $\{W_t^*\}_{0\le t<U}$ defined by

$$W_t^* \;= D_T\int_0^t \nu_s ds + W_t, \tag{3.3.55}$$

that is a $\mathbb{B}_T$-Brownian motion. Note that D_T has the same probability law with respect to $\mathbb{B}_T$ and $\mathbb{Q}$. Moreover, the process $\{\xi_t\}$ defined by (3.3.43) is a $\mathbb{B}_T$-Brownian bridge:

$$\begin{aligned}
\xi_t &= D_T \int_0^t \sigma_s ds + \beta_{tT} \\
&= D_T \int_0^t \sigma_s ds + (T-t)\int_0^t \frac{1}{T-s} dW_s \\
&= D_T \int_0^t \sigma_s ds + (T-t)\int_0^t \frac{1}{T-s}(dW_s^* - D_T \nu_s ds) \\
&= D_T \left(\int_0^t \sigma_s ds - (T-t)\int_0^t \frac{1}{T-s}\nu_s ds \right) + (T-t)\int_0^t \frac{1}{T-s} dW_s^* .
\end{aligned}$$

By using the relation given by (3.3.52), we obtain

$$\xi_t = (T-t)\int_0^t \frac{1}{T-s} dW_s^*, \tag{3.3.56}$$

which is in fact the stochastic integral representation of a Brownian bridge process.

Hence, the conditional probability density process can be derived by using a variation of the Kallianpur-Striebel formula (see [9, 5]) for the conditional expectation

$$\mathbb{E}^{\mathbb{Q}}[f(D_T)|\mathcal{F}_t^{\xi}] = \frac{\mathbb{E}^{\mathbb{B}_T}[f(D_T)\Lambda_t|\mathcal{F}_t^{\xi}]}{\mathbb{E}^{\mathbb{B}_T}[\Lambda_t|\mathcal{F}_t^{\xi}]}. \tag{3.3.57}$$

The process $\{\Lambda_t\}$ can be expressed in terms of $\{\xi_t\}$

$$\Lambda_t = \exp\left(D_T \int_0^t \nu_s dW_s + \frac{1}{2} D_T^2 \int_0^t \nu_s^2 ds \right). \tag{3.3.58}$$

By substituting (3.3.55), we get

$$\Lambda_t = \exp\left(D_T \int_0^t \nu_s (dW_s^* - D_T \nu_s ds) + \frac{1}{2} D_T^2 \int_0^t \nu_s^2 ds \right)$$

$$= \exp\left(D_T \int_0^t \nu_s dW_s^* - \frac{1}{2} D_T^2 \int_0^t \nu_s^2 ds \right). \tag{3.3.59}$$

Differentiating the equation (3.3.56) gives

$$d\xi_t = -\frac{\xi_t}{T-t} dt + dW_s^*. \tag{3.3.60}$$

Substituting (3.3.60) into (3.3.59) yields

$$\Lambda_t = \exp\left(D_T \int_0^t \nu_s d\xi_s + D_T \int_0^t \nu_s \frac{\xi_s}{T-s} ds - \frac{1}{2} D_T^2 \int_0^t \nu_s^2 ds \right),$$

$$\begin{aligned} d\left(\int_0^t \nu_s d\xi_s + \int_0^t \nu_s \frac{\xi_s}{T-s} ds \right) &= \nu_t \left(d\xi_t + \frac{1}{T-t} \xi_t dt \right) \\ &= \left(\sigma_t + \frac{1}{T-t} \int_0^t \sigma_s ds \right) \left(d\xi_t + \frac{1}{T-t} \xi_t dt \right) \\ &= d\left(\frac{1}{T-t} \xi_t \int_0^t \sigma_s ds + \int_0^t \sigma_s d\xi_s \right). \end{aligned} \tag{3.3.61}$$

Integrating both sides of (3.3.61) gives

$$\int_0^t \nu_s d\xi_s + \int_0^t \nu_s \frac{\xi_s}{T-s} ds = \frac{1}{T-t} \xi_t \int_0^t \sigma_s ds + \int_0^t \sigma_s d\xi_s. \tag{3.3.62}$$

Similarly,

$$\nu_t^2 dt = \left(\sigma_t^2 + \frac{2}{T-t} \sigma_t \int_0^t \sigma_s ds + \frac{1}{(T-t)^2} \left(\int_0^t \sigma_s ds \right)^2 \right) dt \tag{3.3.63}$$

$$= d\left(\frac{1}{T-t}\left(\int_0^t \sigma_s ds\right)^2 + \int_0^t \sigma_s^2 ds\right). \tag{3.3.64}$$

Thus, $\{\Lambda_t\}$ can be expressed as follows:

$$\Lambda_t = exp\left[D_T(\tfrac{\xi_t}{T-t}\int_0^t \sigma_s ds + \int_0^t \sigma_s d\xi_s) - \tfrac{1}{2}D_T^2(\tfrac{1}{(T-t)}(\int_0^t \sigma_s ds)^2 + \int_0^t \sigma_s^2 ds)\right], \tag{3.3.65}$$

$$\mathbb{Q}\left(D_T \leq x|\mathcal{F}_t^\xi\right) = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{\{D_T\leq x\}}|\mathcal{F}_t^\xi\right] = \frac{\mathbb{E}^{\mathbb{B}_T}\left[\mathbb{I}_{\{D_T\leq x\}}\Lambda_t|\mathcal{F}_t^\xi\right]}{\mathbb{E}^{\mathbb{B}_T}\left[\Lambda_t|\mathcal{F}_t^\xi\right]}, \tag{3.3.66}$$

$$\mathbb{Q}\left(D_T \leq x|\mathcal{F}_t^\xi\right) = \frac{\int_0^x p(y)e^{y(\frac{\xi_t}{T-t}\int_0^t \sigma_s ds + \int_0^t \sigma_s d\xi_s) - \frac{1}{2}y^2(\frac{1}{(T-t)}(\int_0^t \sigma_s ds)^2 + \int_0^t \sigma_s^2 ds)}}{\int_0^\infty p(y)e^{y(\frac{\xi_t}{T-t}\int_0^t \sigma_s ds + \int_0^t \sigma_s d\xi_s) - \frac{1}{2}y^2(\frac{1}{(T-t)}(\int_0^t \sigma_s ds)^2 + \int_0^t \sigma_s^2 ds)}}. \tag{3.3.67}$$

An alternative expression for the conditional expectation process $\{\pi_t(x)\}$, written in terms of $\{W_t^*\}$, is given by

$$\pi_t(x) = \frac{p(x)\exp\left(x\int_0^t \nu_s dW_s^* \ - \frac{1}{2}x^2\int_0^t \nu_s^2 ds\right)}{\int_0^t p(x)\exp\left(x\int_0^t \nu_s dW_s^* \ - \frac{1}{2}x^2\int_0^t \nu_s^2 ds\right)dx}. \tag{3.3.68}$$

Similarly, the conditional expectation of the random variable D_T is represented as

$$D_{tT} = \frac{\int_0^t xp(x)\exp\left(x\int_0^t \nu_s dW_s^* \ - \frac{1}{2}x^2\int_0^t \nu_s^2 ds\right)dx}{\int_0^t p(x)\exp\left(x\int_0^t \nu_s dW_s^* \ - \frac{1}{2}x^2\int_0^t \nu_s^2 ds\right)dx} \tag{3.3.69}$$

$$= \frac{\int_0^\infty xp(x)e^{x(\frac{\xi_t}{T-t}\int_0^t \sigma_s ds + \int_0^t \sigma_s d\xi_s) - \frac{1}{2}x^2(\frac{1}{(T-t)}(\int_0^t \sigma_s ds)^2 + \int_0^t \sigma_s^2 ds)}}{\int_0^\infty p(x)e^{x(\frac{\xi_t}{T-t}\int_0^t \sigma_s ds + \int_0^t \sigma_s d\xi_s) - \frac{1}{2}x^2(\frac{1}{(T-t)}(\int_0^t \sigma_s ds)^2 + \int_0^t \sigma_s^2 ds)}}. \tag{3.3.70}$$

Lemma 3.3.2. *Let D_{tT} be the process given by the equation (3.3.74). Then the dynamics of D_{tT} under $\mathbb{Q}$ are*

$$dD_{tT} \;=\; \nu_t V_t \left(\frac{1}{T-t}\xi_t \;-\; \nu_t D_{tT} \right) dt \;+\; \nu_t V_t d\xi_t \tag{3.3.71}$$

where $\{V_t\}$ represents the conditional variance of the random variable D_T:

$$V_t = \int_0^\infty x^2 \pi_t(x) dx - \left(\int_0^\infty x \pi_t(x) dx \right)^2. \tag{3.3.72}$$

Then the spot asset price dynamics are:

$$dS_t = r_t S_t dt + \Gamma_{tT} dD_{tT}, \tag{3.3.73}$$

where

$$\Gamma_{tT} = P_{tT} \nu_t V_t.$$

Proof. By using the Ito formula, the dynamics of the expected dividend process is found as follows:

$$D_{tT} = D(\xi_t, t) = \frac{\int_0^\infty xp(x)e^{x(\frac{\xi_t}{T-t}\int_0^t \sigma_s ds + \int_0^t \sigma_s d\xi_s) - \frac{1}{2}x^2(\frac{1}{(T-t)}(\int_0^t \sigma_s ds)^2 + \int_0^t \sigma_s^2 ds)}}{\int_0^\infty p(x)e^{x(\frac{\xi_t}{T-t}\int_0^t \sigma_s ds + \int_0^t \sigma_s d\xi_s) - \frac{1}{2}x^2(\frac{1}{(T-t)}(\int_0^t \sigma_s ds)^2 + \int_0^t \sigma_s^2 ds)}}. \tag{3.3.74}$$

We denote

$$A(\xi_t, t) = \frac{\xi_t}{T-t}\int_0^t \sigma_s ds + \int_0^t \sigma_s d\xi_s,$$

$$B(t) = \frac{1}{(T-t)}(\int_0^t \sigma_s ds)^2 + \int_0^t \sigma_s^2 ds$$

and

$$D(\xi_t, t) = \frac{\Delta_1}{\Delta_2}.$$

Thus, we can express D_{tT} in terms of $A(\xi_t, t)$ and $B(t)$:

$$D(\xi_t, t) = \frac{\int_0^\infty xp(x)e^{xA(\xi_t,t)-\frac{1}{2}x^2B(t)}}{\int_0^\infty p(x)e^{xA(\xi_t,t)-\frac{1}{2}x^2B(t)}}.$$

The Ito formula:

$$dD(\xi, t) = \frac{\partial D(\xi, t)}{\partial t}dt + \frac{\partial D(\xi, t)}{\partial \xi}d\xi + \frac{1}{2}\frac{\partial^2 D(\xi, t)}{\partial \xi^2}d\left\langle \xi, \xi \right\rangle_t. \quad (3.3.75)$$

The first term of the formula:

$$\frac{\partial D(\xi, t)}{\partial t} = \frac{1}{\Delta_2^2}\left(\frac{\partial \Delta_1}{\partial t}\Delta_2 - \frac{\partial \Delta_2}{\partial t}\Delta_1\right)$$

$$= \frac{1}{\Delta_2}\left(\frac{\partial \Delta_1}{\partial t} - \frac{\partial \Delta_2}{\partial t}D(\xi_t, t)\right),$$

$$\frac{\partial \Delta_1}{\partial t} = \int_0^\infty xp(x)\left(x\frac{\partial A(\xi_t, t)}{\partial t} - \frac{1}{2}x^2\frac{\partial B(t)}{\partial t}\right)\exp\left\{xA(\xi_t, t) - \frac{1}{2}x^2B(t)\right\}dx,$$

where

$$\frac{\partial A(\xi_t, t)}{\partial t} = \frac{\xi_t}{T-t}\left(\frac{1}{T-t}\int_0^t \sigma_s ds + \sigma_t\right) = \frac{\xi_t}{T-t}\nu_t,$$

$$\frac{\partial B(t)}{\partial t} = \left(\frac{1}{T-t} \int_0^t \sigma_s ds + \sigma_t \right)^2 = \nu_t^2.$$

Thus,

$$\frac{\partial \Delta_1}{\partial t} = \frac{\xi_t}{T-t} \nu_t \int_0^\infty x^2 p(x) e^{xA(\xi_t,t) - \frac{1}{2}x^2 B(t)} dx - \frac{1}{2}\nu_t^2 \int_0^\infty x^3 p(x) e^{xA(\xi_t,t) - \frac{1}{2}x^2 B(t)} dx,$$

$$\frac{\partial \Delta_2}{\partial t} = \frac{\xi_t}{T-t} \nu_t \int_0^\infty x p(x) x e^{xA(\xi_t,t) - \frac{1}{2}x^2 B(t)} dx - \frac{1}{2}\nu_t^2 \int_0^\infty x^2 p(x) e^{xA(\xi_t,t) - \frac{1}{2}x^2 B(t)} dx,$$

$$\frac{\partial D(\xi,t)}{\partial t} = \frac{\xi_t}{T-t} \nu_t V_t + \frac{1}{2}\nu_t^2 \left(D_{tT} \int_0^\infty x^2 \pi_t(x) dx - \int_0^\infty x^3 \pi_t(x) dx \right),$$

where V_t is as defined in (3.3.72).

The second term of (3.3.75) is

$$\frac{\partial D(\xi,t)}{\partial \xi} = \frac{1}{\Delta_2^2} \left(\frac{\partial \Delta_1}{\partial \xi} \Delta_2 - \frac{\partial \Delta_2}{\partial \xi} D(\xi,t) \right),$$

where

$$\frac{\partial \Delta_1}{\partial \xi} = \nu_t \int_0^\infty x^2 p(x) exp \left\{ xA(\xi,t) - \frac{1}{2}x^2 B(t) \right\} dx,$$

$$\frac{\partial \Delta_2}{\partial \xi} = \nu_t \int_0^\infty x p(x) exp \left\{ xA(\xi,t) - \frac{1}{2}x^2 B(t) \right\} dx.$$

Thus,

$$\frac{\partial D(\xi,t)}{\partial t} = \nu_t \int_0^\infty x^2 \pi_t(x) dx - D_{tT}^2 \nu_t = \nu_t V_t.$$

The third term of (3.3.75) is

$$\frac{\partial^2 D(\xi,t)}{\partial \xi^2} = \frac{1}{\Delta_2}\frac{\partial^2 \Delta_1}{\partial \xi^2} - \frac{1}{\Delta_2^2}\frac{\partial \Delta_1}{\partial \xi}\frac{\partial \Delta_2}{\partial \xi} - \frac{D(\xi,t)}{\Delta_2}\frac{\partial^2 \Delta_2}{\partial \xi^2} + \frac{D(\xi,t)}{\Delta_2^2}\left(\frac{\partial \Delta_2}{\partial \xi}\right)^2 -$$

$$-\frac{1}{\Delta_2}\frac{\partial \Delta_2}{\partial \xi}\frac{\partial D(\xi,t)}{\partial \xi},$$

$$\frac{\partial^2 \Delta_1}{\partial \xi^2} = \nu_t^2 \int_0^\infty x^3 p(x) exp\left\{xA(\xi_t,t) - \frac{1}{2}x^2 B(t)\right\} dx,$$

$$\frac{\partial^2 \Delta_2}{\partial \xi^2} = \nu_t^2 \int_0^\infty x^2 p(x) exp\left\{xA(\xi_t,t) - \frac{1}{2}x^2 B(t)\right\} dx,$$

$$\frac{\partial^2 D(\xi,t)}{\partial \xi^2} = \nu_t^2 \int_0^\infty x^3 \pi_t(x)dx - \nu_t^2 D_{tT}\int_0^\infty x^2 \pi_t(x)dx - \nu_t^2 D_{tT}\int_0^\infty x^2 \pi_t(x)dx +$$

$$+\nu_t^2 D_{tT}\left(\int_0^\infty x\pi_t(x)dx\right)^2 - \nu_t^2 D_{tT} V_t$$

$$= \nu_t^2 \int_0^\infty x^3 \pi_t(x)dx - \nu_t D_{tT}^2 \int_0^\infty x^2 \pi_t(x)dx - 2\nu_t^2 D_{tT} V_t.$$

Substituting the expressions of $\partial D(\xi,t)/\partial t$, $\partial D(\xi,t)/\partial \xi$ and $\partial^2 D(\xi,t)/\partial \xi^2$ given above into (3.3.75):

$$dD_{tT} = \frac{\xi_t}{T-t}\nu_t V_t dt + \frac{1}{2}\nu_t^2\left(D_{tT}\int_0^\infty x^2\pi_t(x)dx - \int_0^\infty x^3 \pi_t(x)dx\right)dt + \nu_t V_t d\xi_t +$$

$$+\frac{1}{2}\nu_t^2\int_0^\infty x^3\pi_t(x)dxdt - \frac{1}{2}\nu_t^2 D_{tT}\int_0^\infty x^2\pi_t(x)dxdt - \nu_t^2 D_{tT} V_t dt$$

$$= \frac{\xi_t}{T-t}\nu_t V_t dt + \nu_t V_t d\xi_t + \nu_t^2 D_{tT} V_t dt$$

$$= \nu_t V_t \left(\frac{1}{T-t} \xi_t - \nu_t D_{tT} \right) dt + \nu_t V_t d\xi_t.$$

This completes the proof.

A new process $\{W_t\}$ is defined so as to be an $\{\mathcal{F}_t\}$-Brownian motion:

$$W_t = \xi_t + \int_0^t \frac{1}{T-s} \xi_s ds - \int_0^t \nu_s D_{sT} ds. \tag{3.3.76}$$

Proposition 3.3.3. *The process $\{W_t\}$ defined by equation (3.3.71) is a standard $\{\mathcal{F}_t\}$-Brownian motion under the risk neutral measure $\mathbb{Q}$.*

Proof. For the proof, a close analogy with the proof of the proposition (3.2.2) as given in [4] is following.
Firstly, it will be shown that $\{W_t\}$ is an $\{\mathcal{F}_t^{\xi}\}$-martingale and, then, $(dW_t)^2 = dt$ will be verified. For $u \geq t$,

$$\mathbb{E}^{\mathbb{Q}}[W_u|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[\xi_u|\mathcal{F}_t] + \mathbb{E}^{\mathbb{Q}} \left[\int_0^u \frac{1}{T-s} \xi_s ds | \mathcal{F}_t \right] - \mathbb{E}^{\mathbb{Q}} \left[\int_0^u \nu_s D_{sT} ds | \mathcal{F}_t \right]. \tag{3.3.77}$$

Since the two terms $\int_0^t \frac{1}{T-s} \xi_s ds$ and $\int_0^t \nu_s D_{sT} ds$ are $\{\mathcal{F}_t\}$-measurable, the expression given by equation (3.3.77) can be written as follows:

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[W_u|\mathcal{F}_t] &= \mathbb{E}^{\mathbb{Q}}[\xi_u|\mathcal{F}_t] + \int_0^t \frac{1}{T-s} \xi_s ds - \int_0^t \nu_s D_{sT} ds + \\ &+ \mathbb{E}^{\mathbb{Q}} \left[\int_t^u \frac{1}{T-s} \xi_s ds | \mathcal{F}_t \right] - \mathbb{E}^{\mathbb{Q}} \left[\int_t^u \nu_s D_{sT} ds | \mathcal{F}_t \right]. \end{aligned} \tag{3.3.78}$$

By the martingale property of D_{tT},

$$\mathbb{E}^{\mathbb{Q}}[D_{sT}|\mathcal{F}_t] = D_{tT}, \tag{3.3.79}$$

for $t \leq s$, and of the conditional expectation $\mathbb{E}^{\mathbb{Q}}[\xi_s|\mathcal{F}_t]$, we get

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[\beta_{sT}|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}[\beta_{sT}|\mathcal{G}_t]|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}[\beta_{sT}|\sigma\{\beta_{vT}; v \leq, D_T\}]|\mathcal{F}_t] = \\ = \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}[\beta_{sT}|\beta_{tT}, D_T]|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}[\beta_{sT}|\beta_{tT}]|\mathcal{F}_t], \end{aligned} \tag{3.3.80}$$

for $t \leq s$. It is already known from (3.2.1) that

$$\mathbb{E}^{\mathbb{Q}}[\beta_{sT}|\beta_{tT}] = \frac{T-s}{T-t}\beta_{tT}.$$

Then, it follows

$$\mathbb{E}^{\mathbb{Q}}[\beta_{sT}|\mathcal{F}_t] = \frac{T-s}{T-t}\mathbb{E}^{\mathbb{Q}}[\beta_{tT}|\mathcal{F}_t]. \tag{3.3.81}$$

Thus, the conditional expectation $\mathbb{E}^{\mathbb{Q}}[\xi_s|\mathcal{F}_t]$ can be expressed as follows:

$$\mathbb{E}^{\mathbb{Q}}[\xi_s|\mathcal{F}_t] = D_{tT}\int_0^s \sigma_v dv + \frac{T-s}{T-t}\mathbb{E}^{\mathbb{Q}}[\beta_{tT}|\mathcal{F}_t] \tag{3.3.82}$$

and, by the definition of the process $\{W_t\}$ given by (3.3.76),

$$\int_0^t \frac{1}{T-s}\xi_s ds - \int_0^t \nu_s D_{sT} ds = W_t - \xi_t. \tag{3.3.83}$$

Finally, substituting (3.3.82), (3.3.83) into (3.3.78) yields

$$\mathbb{E}^{\mathbb{Q}}[Wu|\mathcal{F}_t] = D_{tT}\int_t^u \sigma_v dv + \frac{T-u}{T-t}\mathbb{E}^{\mathbb{Q}}[\beta_{tT}|\mathcal{F}_t] + \mathbb{E}^{\mathbb{Q}}[\beta_{tT}|\mathcal{F}_t]\int_t^u \frac{1}{T-t}ds$$
$$-D_{tT}\int_t^u \nu_s ds + D_{tT}\int_t^u \frac{1}{T-s}\left(\int_0^s \sigma_v dv\right) ds + W_t - \xi_t$$

$$= W_t - \xi_t + \mathbb{E}^{\mathbb{Q}}[\beta_{tT}|\mathcal{F}_t] - D_{tT}\int_t^u \nu_s ds + D_{tT}\int_0^u \sigma_v dv + D_{tT}\int_t^u \nu_s - \sigma_s ds$$

$$= W_t - \xi_t + \mathbb{E}^{\mathbb{Q}}[\beta_{tT}|\mathcal{F}_t] + D_{tT}\int_0^t \sigma_v dv = W_t.$$

Moreover, it is obvious from the definition of the process $\{W_t\}$ that the quadratic variation is $(dW_t)^2 = dt$. Thus, it is concluded that $\{W_t\}$ is an $\{\mathcal{F}_t\}$-Brownian motion.

Hence, the dynamics of the expected dividend process D_{tT} can be written in terms of Brownian motion

$$dD_{tT} = \nu_t V_t dW_t, \tag{3.3.84}$$

and the dynamics of the asset price dynamics is given by

$$dS_t = r_t S_t dt + \Gamma_{tT} dW_t, \tag{3.3.85}$$

where the asset price volatility process $\{\Gamma_{tT}\}$ is given by

$$\Gamma_{tT} = \nu_t P_{tT} V_t. \tag{3.3.86}$$

Here, V_t stands for the conditional variance of the random variable D_T and it is given by

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[(D_T - \mathbb{E}^{\mathbb{Q}}[D_T|\mathcal{F}_t])^2|\mathcal{F}_t\right]. \tag{3.3.87}$$

The dynamics of $\{V_t\}$ is obtained [4] as follows:

$$dV_t = -\nu_t^2 V_t^2 dt + \nu_t \kappa_t dW_t, \tag{3.3.88}$$

where κ_t denotes the third conditional moment of D_T given by

$$\kappa_t = \mathbb{E}^{\mathbb{Q}}\left[(D_T - \mathbb{E}^{\mathbb{Q}}[D_T|\mathcal{F}_t])^3|\mathcal{F}_t\right]. \tag{3.3.89}$$

CHAPTER 4

OPTION VALUATION UNDER INCOMPLETE INFORMATION

In this chapter, the pricing of a European-style call option having an underlying asset price dynamics presented in the previous chapter is closely examined.

4.1 Valuation Formula

A call option on such an asset, with strike price K and maturity date t, is considered for the derivative valuation problem. The underlying asset pays a single dividend D_T at time $T > t$. The risk neutral value of the option under incomplete information is given as follows: For $s = 0$ it holds

$$C_0 = P_{0t}\mathbb{E}^{\mathbb{Q}}\left[(S_t - K)^+\right], \tag{4.1.1}$$

and for any time $s \leq t < T$ we have

$$C_s = P_{st}\mathbb{E}^{\mathbb{Q}}\left[(S_t - K)^+ | \mathcal{F}_t^{\xi}\right]. \tag{4.1.2}$$

Firstly, the derivation of the valuation formula of C_0 is given.

It is known that the asset price process as given in Proposition (3.1.2) have the following form:

$$S_t = \mathbb{I}_{\{t<T\}} P_{tT} \int_0^\infty x\pi_t(x)dx. \tag{4.1.3}$$

Substituting the equation (4.1.3) into the option valuation formula given by equation (4.1.1), it is found that

$$C_0 = P_{0t}\mathbb{E}^{\mathbb{Q}}\left[(P_{tT}\int_0^\infty x\pi_t(x)dx - K)^+\right] \tag{4.1.4}$$

$$= P_{0t}\mathbb{E}^{\mathbb{Q}}\left[(\int_0^\infty (P_{tT}x - K)\pi_t(x)dx)^+\right], \tag{4.1.5}$$

where $\pi_t(x)$ is as given by equation (3.1.26):

$$\pi_t(x) = \frac{p(x)\exp\left(\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t)\right)}{\int_0^\infty p(x)\exp\left(\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t)\right)dx}. \tag{4.1.6}$$

For convenience, a density process is defined as expressed in this way:

$$p_t(x) := p(x)\exp\left(\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t)\right); \tag{4.1.7}$$

thus, the conditional density process can be written as

$$\pi_t(x) = \frac{p_t(x)}{\int_0^\infty p_t(x)dx}. \tag{4.1.8}$$

Substituting the equation (4.1.8) into the option valuation formula given in (4.1.5) yields

$$C_0 = P_{0t}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\int_0^\infty p_t(x)dx}(\int_0^\infty (P_{tT}x - K)p_t(x)dx)^+\right] \tag{4.1.9}$$

$$= P_{0t}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\Lambda_t}(\int_0^\infty (P_{tT}x - K)p_t(x)dx)^+\right], \tag{4.1.10}$$

where Λ_t is denoted as

$$\Lambda_t = \int_0^\infty p_t(x)dx = \int_0^\infty p(x)\exp\left(\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t)\right)dx. \tag{4.1.11}$$

Here, the process Λ_t is used to introduce an equivalent probability measure on $(\Omega, \mathcal{F}_t^\xi)$ for every $t \ < \ T$.

Lemma 4.1.1. *[21] The dynamics of the process Λ_t given by equation (4.1.11) is given by*

$$d\Lambda_t = \Lambda_t\left[\frac{T}{T-t}\sigma\xi_tD_{tT}dt + \frac{T}{T-t}\sigma\xi_tD_{tT}d\xi_t\right] \tag{4.1.12}$$

for any $t \in [0, T)$

Proof. We follow the path given in the original proof.

By using Ito's rule, we get

$$d\Lambda_t = \frac{\partial\Lambda_t}{\partial t}dt + \frac{\partial\Lambda_t}{\partial\xi}d\xi + \frac{1}{2}\frac{\partial^2\Lambda_t}{\partial\xi^2}d\left\langle\xi,\xi\right\rangle_t, \tag{4.1.13}$$

$$d\Lambda_t = \frac{T\sigma}{(T-t)^2}\xi_t \left(\int_0^\infty xp_t(x)dx\right)dt + \frac{T\sigma^2}{2(T-t)^2}\left(\int_0^\infty x^2p_t(x)dx\right)dt$$
$$+\frac{T\sigma}{(T-t)}\left(\int_0^\infty xp_t(x)dx\right)d\xi_t + \frac{(T\sigma)^2}{2(T-t)^2}\left(\int_0^\infty x^2p_t(x)dx\right)dt \qquad (4.1.14)$$

$$= \frac{T\sigma}{(T-t)^2}\xi_t \left(\int_0^\infty xp_t(x)dx\right)dt + \frac{T\sigma}{(T-t)}\left(\int_0^\infty xp_t(x)dx\right)d\xi_t \qquad (4.1.15)$$

$$= \Lambda_t \left[\frac{T\sigma\xi_t}{(T-t)^2}\left(\int_0^\infty x\frac{p_t(x)}{\int_0^\infty p_t(x)dx}dx\right)dt + \frac{T\sigma}{(T-t)}\left(\int_0^\infty x\frac{p_t(x)}{\int_0^\infty p_t(x)dx}dx\right)d\xi_t\right] \qquad (4.1.16)$$

$$= \Lambda_t \left[\frac{T\sigma}{(T-t)^2}\xi_t D_{tT}dt + \frac{T\sigma}{T-t}D_{tT}d\xi_t\right]. \qquad (4.1.17)$$

Lemma 4.1.2. *[21] The dynamics of the process $\Psi_t := \Lambda_t^{-1} = \frac{1}{\Lambda_t}$ is given by*

$$d\Psi_t = \Psi_t\left[-\frac{T\sigma}{T-t}D_{tT}\right]\left[d\xi_t + \frac{\xi_t}{T-t}dt - \frac{T\sigma}{T-t}D_{tT}dt\right] \qquad (4.1.18)$$

which is in fact

$$d\Psi_t = \Psi_t\left[-\frac{T\sigma}{T-t}D_{tT}\right]dW_t. \qquad (4.1.19)$$

Proof. By the same path given in [21], the Ito application and to (4.1.17) gives

$$d\Psi_t = d\left(\frac{1}{\Lambda_t}\right) = -\frac{1}{\Lambda_t^2}d\Lambda_t + \frac{1}{\Lambda_t^3}d\left\langle \Lambda, \Lambda\right\rangle_t \qquad (4.1.20)$$

$$= -\frac{1}{\Lambda_t}\left[\frac{T\sigma}{(T-t)^2}\xi_t D_{tT}dt + \frac{T\sigma}{T-t}D_{tT}d\xi_t\right] + \frac{1}{\Lambda_t}\left(\frac{T\sigma}{T-t}\right)^2 D_{tT}^2 d\left\langle \xi, \xi \right\rangle_t \quad (4.1.21)$$

$$= -\Psi_t\left[\frac{T\sigma}{(T-t)^2}\xi_t D_{tT}dt + \frac{T\sigma}{T-t}D_{tT}d\xi_t\right] + \Psi_t\left(\frac{T\sigma}{T-t}\right)^2 D_{tT}^2 dt \qquad (4.1.22)$$

$$= \Psi_t\left[-\frac{T\sigma}{(T-t)}D_{tT}\right]\left[\frac{\xi_t}{T-t}dt + d\xi_t - \frac{T\sigma}{T-t}D_{tT}dt\right]. \qquad (4.1.23)$$

Recall that

$$dW_t = d\xi_t + \frac{\xi_t}{T-t}dt - \frac{T\sigma}{T-t}D_{tT}dt, \qquad (4.1.24)$$

where W_t is a standard Brownian motion defined as the information driven Brownian motion by equation (3.2.39). Thus,

$$d\Psi_t = \Psi_t\left[-\frac{T\sigma}{(T-t)}D_{tT}\right]dW_t. \qquad (4.1.25)$$

Corollary 4.1.3. *The process Ψ_t, $t \in [0, T)$, is a Radon-Nikodym density process with respect to $\mathbb{Q}$, and so it is a strictly positive $\mathbb{Q}$-martingale with $\Psi_0 = 1$*

$$\frac{d\mathbb{B}_T}{d\mathbb{Q}}|\mathcal{F}_t = \Psi_t = \frac{1}{\Lambda_t} \qquad (4.1.26)$$

for $t \in [0, T)$.

Proof. It is clearly seen from the definition of $\Psi_t = \frac{1}{\Lambda_t}$ that it is a strictly positive process and $\Psi_0 = \frac{1}{\Lambda_0} = 1$.

Furthermore, Ψ_t is a $\mathbb{Q}$-martingale which can be verified easily by making use of

(4.1.25):

$$d\Psi_t = \Psi_t \left[-\frac{T\sigma}{(T-t)} D_{tT} \right] dW_t,$$

where W_t is a $\mathbb{Q}$-Brownian motion and for any $t \in [0,u]$, for any $0 \le u < T$, and D_{tT} is a bounded process. Thus, the *R-N* density process Ψ is

$$\frac{d\mathbb{B}_T}{d\mathbb{Q}} | \mathcal{F}_t = \Psi_t = \frac{1}{\Lambda_t} \tag{4.1.27}$$

for $t \in [0,T)$.

Thus, for every $u \in [0,t]$, Ψ_u, for $0 \le t < T$ is the *Radon-Nikodym* density with $\Psi_0 = 1$ and it follows that $\mathbb{E}^{\mathbb{Q}}[\Psi_t] = 1$ where t is given as the option maturity date. The R-N density can be expressed as follows:

$$\frac{1}{\Lambda_t} = \Psi_t = \exp\left[\int_0^t \left(-\frac{T\sigma}{(T-u)} D_{uT} \right) dW_u - \frac{1}{2} \int_0^t \left(-\frac{T^2\sigma^2}{(T-u)^2} D_{uT}^2 \right) du \right]. \tag{4.1.28}$$

Hence, the process W_t^* can be defined so as to be a Brownian motion under the equivalent measure $\mathbb{B}_T$, which will be called "Bridge measure". Hence, the $\mathbb{B}_T$-Brownian motion can be expressed as follows:

$$W_u^* = W_u + \int_0^u \left(\frac{T\sigma}{(T-s)} D_{sT} \right) ds \tag{4.1.29}$$

for every $u \in [0,t]$. Then, W^* is a standard Brownian motion on the space $(\Omega, \mathcal{F}_t, \mathbb{B}_T)$. For the measure change back form $\mathbb{B}_T$ to $\mathbb{Q}$ on $(\Omega, \mathcal{F}_t)$, the suitable density Λ_t with respect to $\mathbb{B}_T$ is given by

$$\Lambda_t = \frac{1}{\Psi_t} = \exp\left[\int_0^t \left(\frac{T\sigma}{(T-u)} D_{uT} \right) dW_u + \frac{1}{2} \int_0^t \left(\frac{T^2\sigma^2}{(T-u)^2} D_{uT}^2 \right) du \right] \tag{4.1.30}$$

$$= \exp\left[\int_0^t \left(\frac{T\sigma}{(T-u)}D_{uT}\right)dW_u^* - \frac{1}{2}\int_0^t \left(\frac{T^2\sigma^2}{(T-u)^2}D_{uT}^2\right)du\right]. \qquad (4.1.31)$$

Proposition 4.1.4. *For any fixed $t \in [0,T)$, the information process ξ_u, $u \in [0,t]$, follows a Brownian bridge process on the equivalent probability space $(\Omega, \mathcal{F}_t, \mathbb{B}_T)$.*

Proof. Firstly, the Brownian motion driven by information process may be recalled:

$$dW_t = d\xi_t + \frac{\xi_t}{T-t}dt - \frac{T\sigma}{T-t}D_{tT}dt$$

and

$$W_u^* = W_u + \int_0^u \left(\frac{T\sigma}{(T-s)}D_{sT}\right)ds. \qquad (4.1.32)$$

Thus, the following result is obvious

$$dW_u^* = d\xi_u + \frac{\xi_u}{T-u}dt - \frac{T\sigma}{T-u}D_{uT}du + \left(\frac{T\sigma}{(T-u)}D_{uT}\right)du \qquad (4.1.33)$$

$$= d\xi_u + \frac{\xi_u}{T-u}dt, \qquad (4.1.34)$$

which gives the standard Brownian bridge definition

$$\xi_u = -\int_0^u \frac{\xi_s}{T-s}ds + W_u^* \qquad (4.1.35)$$

for $u \in [0,t]$ on the probability space $(\Omega, \mathcal{F}_t, \mathbb{B}_T)$.

Hence, the option pricing formula given in equation (4.1.10) can be expressed as

$$C_0 = P_{0t}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\Lambda_t}\left(\int_0^\infty (P_{tT}x - K)p_t(x)dx\right)^+\right]$$

$$= P_{0t}\mathbb{E}^{\mathbb{B}_T}\left[\left(\int_0^\infty (P_{tT}x - K)p_t(x)dx\right)^+\right]$$

$$= P_{0t}\mathbb{E}^{\mathbb{B}_T}\left[\left(\int_0^\infty (P_{tT}x - K)p(x)\exp\left(\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t)\right)dx\right)^+\right]. \quad (4.1.36)$$

To give the option valuation formula more explicitly, a constant ξ^* is defined as a critical value satisfying the following condition:

$$\int_0^\infty (P_{tT}x - K)p(x)\exp\left(\frac{T}{T-t}(\sigma x\xi^* - \frac{1}{2}\sigma^2x^2t)\right)dx = 0. \quad (4.1.37)$$

Then, the option price formula given by equation (4.1.36) can be written as

$$C_0 = P_{0t}\int_{\xi^*}^\infty\int_0^\infty (P_{tT}x - K)p(x)\exp\left(\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t)\right)\ dx\ d\mathbb{B}_T^\xi. \quad (4.1.38)$$

By the Fubini Theorem,

$$C_0 = P_{0t}\int_0^\infty (P_{tT}x - K)p(x)\int_{\xi^*}^\infty \exp\left(\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t)\right)\ d\mathbb{B}_T^\xi\ dx. \quad (4.1.39)$$

Here, it is already known that ξ_t is a standard Brownian bridge under $\mathbb{B}_T$, then it is a Gaussian random variable with zero mean and $t(T-t)/T$ variance under this probability measure:

$$\xi_t \sim \; N\left(0, \frac{t(T-t)}{T}\right).$$

Then, for any $t \in [0, T)$, the inner integral term can be reformulated as follows:

$$\int_{\xi^*}^{\infty} \exp\left(\frac{T}{T-t}(\sigma xy - \frac{1}{2}\sigma^2 x^2 t)\right) \frac{\sqrt{T}}{\sqrt{2\pi t(T-t)}} \exp\left(-\frac{y^2 T}{2t(T-t)}\right) dy$$

$$= \int_{\xi^*}^{\infty} \frac{\sqrt{T}}{\sqrt{2\pi t(T-t)}} \exp\left(-\frac{1}{2}\left(\frac{y^2 T}{t(T-t)} - \frac{2T}{T-t}\sigma xy + \frac{T}{T-t}\sigma^2 x^2 t\right)\right) dy$$

$$= \int_{\xi^*}^{\infty} \frac{\sqrt{T}}{\sqrt{2\pi t(T-t)}} \exp\left(-\frac{1}{2}\left(y\sqrt{\frac{T}{t(T-t)}} - \sigma x\sqrt{\frac{tT}{T-t}}\right)^2\right) dy$$

$$= \int_{\xi^*}^{\infty} \frac{\sqrt{T}}{\sqrt{2\pi t(T-t)}} \exp\left(-\frac{1}{2}\left(\frac{(y-\sigma xt)}{\sqrt{\frac{t(T-t)}{T}}}\right)^2\right) dy.$$

In fact, this integral gives the probability that Y is greater than ξ^* for a fixed time t:

$$Pr(Y \geq \xi^*) = \int_{\xi^*}^{\infty} \frac{\sqrt{T}}{\sqrt{2\pi t(T-t)}} \exp\left(-\frac{1}{2}\left(\frac{(y-\sigma xt)}{\sqrt{\frac{t(T-t)}{T}}}\right)^2\right) dy \qquad (4.1.40)$$

with the random variable Y distributed as follows:

$$Y \sim N\left(\sigma x t, \sqrt{\frac{t(T-t)}{T}}\right). \tag{4.1.41}$$

Thus, the standard normal distribution can be used to express this probability:

$$Pr(Y \geq \xi^*) \; = Pr\left(Z \geq \xi^* \sqrt{\frac{T}{t(T-t)}} - \sigma x \sqrt{\frac{tT}{(T-t)}}\right), \tag{4.1.42}$$

where Z is a standard normal random variable.

Since Z is a standard normal distribution having symmetric probability density function, the probability given by (4.1.42) can be written as follows:

$$Pr\left(Z \geq z^* - \sigma x \sqrt{\tau}\right) \;=\; Pr\left(Z \leq -z^* + \sigma x \sqrt{\tau}\right), \tag{4.1.43}$$

where

$$z^* \;=\; \xi^* \sqrt{\frac{T}{t(T-t)}}, \tag{4.1.44}$$

$$\tau \;=\; \frac{tT}{(T-t)}. \tag{4.1.45}$$

This gives the following result for the option valuation formula:

$$C_0 \;=\; P_{0t} \int_0^\infty (P_{tT} x - K) p(x) N(-z^* + \sigma x \sqrt{\tau}) dx, \tag{4.1.46}$$

where $N(x)$ denotes the standard normal distribution function. Hence, the valuation formula for the option price at $t = 0$ takes the following form:

$$C_0 \;=\; P_{0t}\int_0^\infty P_{tT}xp(x)N(-z^* + \sigma x\sqrt{\tau})dx - P_{0t}K\int_0^\infty p(x)N(-z^* + \sigma x\sqrt{\tau})dx. \tag{4.1.47}$$

The price of a European call option at time $s \in [0, t)$ with maturity date t can be expressed in this way:

$$C_s = P_{st}\mathbb{E}^{\mathbb{Q}}[(S_t - K)^+|\mathcal{F}_s]. \tag{4.1.48}$$

After a change of the measure and by the help of the Radon-Nikodym density given by equation (4.1.28), the option price can be represented by

$$C_s = \frac{P_{st}}{\Lambda_s}\,\mathbb{E}^{\mathbb{B}_T}[\Lambda_t(S_t - K)^+|\mathcal{F}_s] \tag{4.1.49}$$

$$= \frac{P_{st}}{\Lambda_s}\,\mathbb{E}^{\mathbb{B}_T}\left[\left(\int_0^\infty (P_{tT}x - K)p_t(x)dx\right)^+|\mathcal{F}_s\right]. \tag{4.1.50}$$

Here, $p_t(x)$ given by equation (4.1.6) is a function of ξ_t. Then, the calculation can be simplified by the fact that ξ_t is a $\mathbb{B}_T$-Brownian bridge. Furthermore, a $\mathbb{B}_T$-Gaussian process Z_{st} which will be independent of $\{\xi_u\}_{0\le u\le s}$ is defined as follows:

$$Z_{st} := \frac{\xi_t}{T-t} - \frac{\xi_s}{T-s}. \tag{4.1.51}$$

Their independence property can be verified easily by examining its covariance structure since they are all $\mathbb{B}_T$-Gaussian processes. The covariance between the variables Z_{st} and ξ_u for any $u \in [0, s]$ and for $t \geq s$ can be computed in this way:

$$Cov(Z_{st}, \xi_u) = Cov(\frac{\xi_t}{T-t} - \frac{\xi_s}{T-s}, \xi_u)$$

$$= \frac{1}{T-t}\mathbb{E}^{\mathbb{B}_T}\left[\xi_t \xi_u\right] - \frac{1}{T-s}\mathbb{E}^{\mathbb{B}_T}\left[\xi_s \xi_u\right]$$

$$= \frac{1}{T-t}\left(u \wedge t - \frac{ut}{T}\right) - \frac{1}{T-s}\left(u \wedge s - \frac{us}{T}\right) \; = \; 0.$$

Thus, the fact that the process ξ_t can be written as a function of Z_{st} and ξ_s can make the conditional expectation a more standard one. According to this, $p_t(x)$ can be expressed as follows:

$$p_t(x) = p(x)\exp\left(\frac{T}{T-s}\sigma x \xi_s + T\sigma x Z_{st} - \frac{T}{2(T-t)}\sigma^2 x^2 t\right), \qquad (4.1.52)$$

where Z_{st} is a $\mathbb{B}_T$-Gaussian process with a distribution

$$Z_{st} \sim N\left(0, \frac{t-s}{(T-s)(T-t)}\right). \qquad (4.1.53)$$

By substituting the equation (4.1.52), the option price can be expressed as follows:

$$C_s = \frac{P_{st}}{\Lambda_s}\mathbb{E}^{\mathbb{B}_T}\left[\left(\int_0^\infty (P_{tT}x - K)p(x)e^{\frac{T\sigma x \xi_s}{T-s} + T\sigma x Z_{st} - \frac{T\sigma^2 x^2 t}{2(T-t)}}dx\right)^+ |\mathcal{F}_s\right]. \qquad (4.1.54)$$

Proposition 4.1.5. *The price C_s admits the following representation under the measure $\mathbb{B}_T$:*

$$C_s = \frac{P_{st}}{\Lambda_s}\left[\mathbb{E}^{\mathbb{B}_T}(P_{tT}\zeta_t \mathbb{I}_A|\mathcal{F}_s) - \mathbb{E}^{\mathbb{B}_T}(K\Lambda_t \mathbb{I}_A|\mathcal{F}_s)\right], \qquad (4.1.55)$$

where $A = \{P_{tT}\zeta_t > K\Lambda_t\}$ *and*

$$\zeta_t = \int_0^\infty xp(x)\exp\left(\frac{T}{T-s}\sigma x\xi_s + T\sigma x Z_{st} - \frac{T}{2(T-t)}\sigma^2 x^2 t\right)dx, \qquad (4.1.56)$$

$$\Lambda_t = \int_0^\infty p(x)\exp\left(\frac{T}{T-s}\sigma x\xi_s + T\sigma x Z_{st} - \frac{T}{2(T-t)}\sigma^2 x^2 t\right)dx. \qquad (4.1.57)$$

Proof. We follow closely the original proof of Rutkowski-Yu [21]

$$C_s = \frac{P_{st}}{\Lambda_s}\mathbb{E}^{\mathbb{B}_T}\left[(P_{tT}\zeta_t - K\Lambda_t)^+ |\mathcal{F}_s\right]. \qquad (4.1.58)$$

Set A is defined as $A := \{P_{tT}\zeta_t > K\Lambda_t\}$, and thus C_s can be represented by using the indicator function $\mathbb{I}_A$:

$$C_s = \frac{P_{st}}{\Lambda_s}\mathbb{E}^{\mathbb{B}_T}\left[(P_{tT}\zeta_t - K\Lambda_t)\,\mathbb{I}_A|\mathcal{F}_s\right] \qquad (4.1.59)$$

$$= \frac{P_{st}}{\Lambda_s}\left[\mathbb{E}^{\mathbb{B}_T}\left[P_{tT}\zeta_t\mathbb{I}_A|\mathcal{F}_s\right] - \mathbb{E}^{\mathbb{B}_T}\left[K\Lambda_t\mathbb{I}_A|\mathcal{F}_s\right]\right]. \qquad (4.1.60)$$

Proposition 4.1.6. *The price of a call option on a single dividend paying risky stock with a maturity date* t *and strike price* K *has the following form:*

$$C_u = S_u M_1(u,t,T) - KP_{ut}M_2(u,t,T), \qquad (4.1.61)$$

where $A = \{P_{tT}\zeta_t > K\Lambda_t\}$ *and*

$$M_1(u,t,T) \;=\; \mathbb{E}^{\mathbb{B}_T}\left(\frac{\zeta_t}{\zeta_u}\,\mathbb{I}_A|\mathcal{F}_u\right),\quad M_2(u,t,T) \;=\; \mathbb{E}^{\mathbb{B}_T}\left(\frac{\Lambda_t}{\Lambda_u}\,\mathbb{I}_A|\mathcal{F}_u\right). \tag{4.1.62}$$

Proof. Here, we again follow closely the proof of Rutkowski-Yu [21]. By recalling the underlying stock price representation for any $u \in [0,t)$

$$S_u = P_{uT}D_{uT}, \tag{4.1.63}$$

where D_{uT} represents the expected dividend process which can be expressed as follows:

$$D_{uT} \;=\; \frac{\zeta_u}{\Lambda_u}, \tag{4.1.64}$$

and so the single dividend paying risky asset price is:

$$S_u = P_{uT}\,\frac{\zeta_u}{\Lambda_u}. \tag{4.1.65}$$

From the previous Proposition, the option price

$$C_u = \frac{P_{ut}}{\Lambda_u}\mathbb{E}^{\mathbb{B}_T}\left[P_{tT}\zeta_t\mathbb{I}_A|\mathcal{F}_u\right] - \frac{P_{ut}}{\Lambda_u}\mathbb{E}^{\mathbb{B}_T}\left[K\Lambda_t\mathbb{I}_A|\mathcal{F}_u\right]. \tag{4.1.66}$$

Since the interest rates are assumed to be deterministic:

$$C_u = \frac{P_{uT}}{\Lambda_u}\mathbb{E}^{\mathbb{B}_T}\left[\zeta_t\mathbb{I}_A|\mathcal{F}_u\right] - KP_{ut}\mathbb{E}^{\mathbb{B}_T}\left[\frac{\Lambda_t}{\Lambda_u}\mathbb{I}_A|\mathcal{F}_u\right]. \tag{4.1.67}$$

Substituting the stock price representation into the equation (4.2.116), we get

$$C_u = S_u \mathbb{E}^{\mathbb{B}_T}\left[\frac{\zeta_t}{\zeta_u}\mathbb{I}_A|\mathcal{F}_u\right] - KP_{ut}\mathbb{E}^{\mathbb{B}_T}\left[\frac{\Lambda_t}{\Lambda_u}\mathbb{I}_A|\mathcal{F}_u\right] \quad (4.1.68)$$

$$= S_u M_1(u,t,T) - KP_{ut}M_2(u,t,T). \quad (4.1.69)$$

This completes the proof.

Hence, it is seen that the price of a European call option can be expressed in a form similar to the Black-Scholes-Merton model.

4.2 Option Pricing with Time-Dependent Information Flux Rate

The dynamics of a single dividend paying risky asset price when the market information process has a time dependent information flux rate are as found in equation (3.3.85):

$$dS_t = r_t S_t dt + \Gamma_{tT} dW_t, \quad (4.2.70)$$

where the asset price volatility process $\{\Gamma_{tT}\}$ is given by

$$\Gamma_{tT} = \nu_t P_{tT} V_t. \quad (4.2.71)$$

Here V_t stands for the conditional variance of the random variable D_T. The market information process

$$\xi_t \;=\; D_T \int_0^t \sigma_s ds \;+\; \beta_{tT} \tag{4.2.72}$$

and the conditional probability density is found as follows:

$$\pi_t(x) = \frac{p(x)e^{x\left(\frac{1}{T-t}\xi_t \int_0^t \sigma_s ds + \int_0^t \sigma_s d\xi_s\right) - \frac{1}{2}x^2\left(\frac{1}{T-t}\left(\int_0^t \sigma_s ds\right)^2 + \int_0^t \sigma_s^2 ds\right)}}{\int_0^\infty p(x)e^{x\left(\frac{1}{T-t}\xi_t \int_0^t \sigma_s ds + \int_0^t \sigma_s d\xi_s\right) - \frac{1}{2}x^2\left(\frac{1}{T-t}\left(\int_0^t \sigma_s ds\right)^2 + \int_0^t \sigma_s^2 ds\right)} dx} \tag{4.2.73}$$

or, alternatively, in terms of the $\mathbb{B}_T$-Brownian motion:

$$\pi_t(x) = \frac{p(x)\exp\left(x\int_0^t \nu_s dW_s^* \; - \frac{1}{2}x^2 \int_0^t \nu_s^2 ds\right)}{\int_0^t p(x)\exp\left(x\int_0^\infty \nu_s dW_s^* \; - \frac{1}{2}x^2 \int_0^t \nu_s^2 ds\right) dx}. \tag{4.2.74}$$

A European style call option on such an asset, with strike price K and maturity date t, is considered. The underlying asset pays a single dividend D_T at time $T > t$. The risk neutral value of the option conditioned on this market information process is given as follows: For time 0 it holds

$$C_0 = P_{0t}\mathbb{E}^{\mathbb{Q}}\left[(S_t - K)^+\right], \tag{4.2.75}$$

and for any time $s \leq t < T$ we have

$$C_s = P_{st}\mathbb{E}^{\mathbb{Q}}\left[(S_t - K)^+ | \mathcal{F}_t^\xi\right]. \tag{4.2.76}$$

Firstly, the derivation of the valuation formula of C_0 is presented. As in the equation (4.1.5), the option price at $t = 0$ can be written as:

$$C_0 = P_{0t}\mathbb{E}^{\mathbb{Q}}\left[(\int_0^\infty (P_{tT}x - K)\pi_t(x)dx)^+ \right], \tag{4.2.77}$$

where the conditional density process is given by the equation (4.2.73). By the same analogy used above, the change of measure density process is defined by

$$\Lambda_t = \int_0^\infty p(x)e^{x\left(\frac{1}{T-t}\xi_t \int_0^t \sigma_s ds + \int_0^t \sigma_s d\xi_s\right) - \frac{1}{2}x^2\left(\frac{1}{T-t}\left(\int_0^t \sigma_s ds\right)^2 + \int_0^t \sigma_s^2 ds\right)} dx. \tag{4.2.78}$$

Lemma 4.2.1. *The dynamics of the process Λ_t given by equation (4.2.78) are*

$$d\Lambda_t = \Lambda_t \left(\frac{\xi_t \nu_t}{T-t} D_{tT}dt + \nu_t D_{tT} d\xi_t \right). \tag{4.2.79}$$

Proof. As given in [21], by applying Ito's formula to (4.2.78):

$$d\Lambda_t = \frac{\partial \Lambda_t}{\partial t}dt + \frac{\partial \Lambda_t}{\partial \xi}d\xi_t + \frac{1}{2}\frac{\partial^2 \Lambda_t}{\partial \xi^2} d\left\langle \xi, \xi \right\rangle_t$$

$$\frac{\partial \Lambda_t}{\partial t} = \frac{\xi_t \nu_t}{T-t}\int_0^\infty x p_t(x)dx - \frac{1}{2}\nu_t^2 \int_0^\infty x^2 p_t(x)dx$$

$$\frac{\partial \Lambda_t}{\partial \xi} = \nu_t \int_0^\infty x p_t(x)dx$$

$$\frac{\partial^2 \Lambda_t}{\partial \xi^2} = \nu_t^2 \int_0^\infty x^2 p_t(x)dx$$

$$\begin{aligned} d\Lambda_t &= \Lambda_t \left(\frac{\xi_t \nu_t}{T-t}\frac{1}{\Lambda_t}\int_0^\infty x p_t(x)dxdt + \nu_t \int_0^\infty x p_t(x)dxd\xi_t \right) \\ &= \Lambda_t \frac{\xi_t \nu_t}{T-t}\int_0^\infty x\pi_t(x)dxdt + \Lambda_t \nu_t \int_0^\infty x\pi_t(x)dxd\xi_t \end{aligned}$$

$$= \Lambda_t \frac{\xi_t \nu_t}{T-t} D_{tT} dt + \Lambda_t \nu_t D_{tT} d\xi_t$$

$$= \Lambda_t \left(\frac{\xi_t \nu_t}{T-t} D_{tT} dt + \nu_t D_{tT} d\xi_t \right).$$

This completes the proof.

Lemma 4.2.2. *The dynamics of the process* $\Phi_t := \Lambda_t^{-1} = 1/\Lambda_t$ *are*

$$d\Phi_t = \Phi_t \left[-\nu_t D_{tT} \right] \left[d\xi_t + \frac{\xi_t}{T-t} dt - \nu_t D_{tT} dt \right], \tag{4.2.80}$$

which is in fact

$$d\Phi_t = \Phi_t \left[-\nu_t D_{tT} \right] dW_t. \tag{4.2.81}$$

Proof. As we know the dynamics of the process Λ_t, an application of the Ito formula gives the result as follows:

$$d\Phi_t = d\left(\frac{1}{\Lambda_t} \right) = -\frac{1}{\Lambda_t} d\Lambda_t - \frac{1}{2} \frac{2}{\Lambda_t^3} \Lambda_t^2 \nu_t^2 D_{tT}^2 dt$$

$$= \frac{-1}{\Lambda_t} \left(\frac{\xi_t \nu_t}{T-t} D_{tT} dt + \nu_t D_{tT} d\xi_t \right) - \frac{1}{\Lambda_t} \nu_t^2 D_{tT} dt$$

$$= -\Phi_t \frac{\xi_t \nu_t}{T-t} D_{tT} dt - \Phi_t \nu_t D_{tT} d\xi_t - \Phi_t \nu_t^2 D_{tT}^2 dt$$

$$= -\Phi_t D_{tT} \nu_t \left(\frac{\xi_t}{T-t} dt + d\xi_t + \nu_t D_{tT} dt \right).$$

We know from the Proposition (3.3.3) that $\{\mathcal{F}_t\}$-Brownian motion $\{W_t\}$ can be expressed as follows:

$$W_t = \xi_t + \int_0^t \frac{1}{T-s} \xi_s ds - \int_0^t \nu_s D_{sT} ds.$$

Thus,

$$d\Phi_t = \Phi_t \left[-\nu_t D_{tT}\right] dW_t.$$

Corollary 4.2.3. *The process Φ_t, $t \in [0, T)$, is a Radon-Nikodym density process with respect to $\mathbb{Q}$, and so it is a strictly positive $\mathbb{Q}$-martingale with $\Phi_0 = 1$.*

Proof. It is clearly seen from the definition of $\Phi_t = 1/\Lambda_t$ that it is a strictly positive process and $\Phi_0 = 1/\Lambda_0 = 1$.

Furthermore, Φ_t is a $\mathbb{Q}$-martingale which is obvious by equation (4.2.81)

$$d\Phi_t = \Phi_t \left[-\frac{T\sigma}{(T-t)} D_{tT}\right] dW_t,$$

where W_t is a $\mathbb{Q}$-Brownian motion and for any $t \in [0, u]$ for any $0 \leq u < T$, D_{tT} is a bounded process:

$$\frac{d\mathbb{B}_T}{d\mathbb{Q}}|\mathcal{F}_t = \Phi_t = \frac{1}{\Lambda_t} \tag{4.2.82}$$

for $t \in [0, T)$.

For every $u \in [0, t]$, Φ_u is the $Radon - Nikodym$ density for $0 \leq t < T$ with $\Phi_0 = 1$, and it follows that $\mathbb{E}^{\mathbb{Q}}[\Phi_t] = 1$, where t is given as the option maturity date. Thus, the $R - N$ density can be expressed as follows:

$$\frac{1}{\Lambda_t} = \Phi_t = \exp\left[\int_0^t \left(-\nu_u D_{uT}\right) dW_u - \frac{1}{2}\int_0^t \left(\nu_u^2 D_{uT}^2\right) du\right]. \tag{4.2.83}$$

Hence, the process W_t^* can be defined so as to be a Brownian motion under the equivalent measure $\mathbb{B}_T$:

$$W_u^* = W_u + \int_0^u (\nu_s D_{sT})\, ds \tag{4.2.84}$$

for every $u \in [0, t]$.

Proposition 4.2.4. *For any fixed $t \in [0, T)$, the information process ξ_u given by (3.3.43), $u \in [0, t]$, follows a Brownian bridge on the equivalent probability space $(\Omega, \mathcal{F}_t, \mathbb{B}_T)$.*

Proof. Firstly, the Brownian motion driven by information process is recalled

$$dW_t = d\xi_t + \frac{\xi_t}{T-t}dt - \frac{T\sigma}{T-t}D_{tT}dt$$

and

$$W_u^* = W_u + \int_0^u (\nu_s D_{sT})\, ds. \tag{4.2.85}$$

Thus, the result is obvious

$$dW_u^* = d\xi_u + \frac{\xi_u}{T-u}dt - \nu_u D_{uT}du + (\nu_u D_{uT})\, du \tag{4.2.86}$$

$$= d\xi_u + \frac{\xi_u}{T-u}dt, \tag{4.2.87}$$

which gives the standard Brownian bridge definition

$$\xi_u = -\int_0^u \frac{\xi_s}{T-s}ds + W_u^* \tag{4.2.88}$$

for $u \in [0, t]$, on the probability space $(\Omega, \mathcal{F}_t, \mathbb{B}_T)$, for $t < T$.

Therefore, the option pricing formula when the market information process has a time-dependent information flux rate can be written as

$$C_0 = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\Lambda_t}\left(\int_0^\infty (P_{tT}x - K)p(x)\exp\left(x\int_0^t \nu_s dW_s^* - \frac{1}{2}x^2\int_0^t \nu_s^2 ds\right) dx\right)^+\right]. \tag{4.2.89}$$

Here, one term is distributed as follows:

$$\int_0^t \nu_s dW_s^* \;\sim\; N\left(0, \int_0^t \nu_s^2 ds\right), \tag{4.2.90}$$

under the Bridge measure $\mathbb{B}_T$. Thus,

$$w_t^{-1}\int_0^t \nu_s dW_s^* \;\sim\; N(0,1), \tag{4.2.91}$$

where

$$w_t^2 = \int_0^t \nu_s^2 ds. \tag{4.2.92}$$

Hence, the option price

$$C_0 = P_{0t}\mathbb{E}^{\mathbb{B}_T}\left[\left(\int_0^\infty (P_{tT}x - K)p(x)\exp\{xw_tY - \frac{1}{2}x^2 w_t^2\}dx\right)^+\right], \tag{4.2.93}$$

$$C_0 = P_{0t}\frac{1}{\sqrt{2\pi}}\int_{y=-\infty}^\infty e^{-\frac{y^2}{2}}\left(\int_{x=0}^\infty (P_{tT}x - K)p(x)\exp\{xw_ty - \frac{1}{2}x^2 w_t^2\}dx\right)^+ dy. \tag{4.2.94}$$

It is observed that there exists a critical value y^* for y, such that

$$\int_0^\infty (P_{tT}x - K)p(x)\exp\{xw_ty^* - \frac{1}{2}x^2w_t^2\}dx = 0. \tag{4.2.95}$$

As a result, the option price can be expressed as

$$C_0 = P_{0t}\frac{1}{\sqrt{2\pi}}\int_{y=y^*}^\infty e^{-\frac{1}{2}y^2}\left(\int_{x=0}^\infty (P_{tT}x - K)p(x)\exp\{xw_ty - \frac{1}{2}x^2w_t^2\}dx\right)dy \tag{4.2.96}$$

$$= P_{0t}\int_{x=0}^\infty (P_{tT}x - K)p(x)\int_{y=y^*}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\exp\{xw_ty - \frac{1}{2}x^2w_t^2\}dydx$$

$$= P_{0t}\int_{x=0}^\infty (P_{tT}x - K)p(x)\int_{y=y^*}^\infty \frac{1}{\sqrt{2\pi}}\exp\{-\frac{1}{2}(y - w_tx)^2\}dydx$$

$$= P_{0t}\int_{x=0}^\infty (P_{tT}x - K)p(x)N(w_tx - y^*)dx$$

$$= P_{0T}\int_0^\infty xp(x)N(w_tx - y^*)dx - P_{0t}K\int_0^\infty p(x)N(w_tx - y^*)dx. \tag{4.2.97}$$

Hence, we conclude that for the option price process, $s \leq t < T$, a very similar form to the one offered in the case of a constant information flux rate can be used in that case, as well.

The price of a European call option at time $s \in [0, t)$ with maturity date t can be expressed as follows:

$$C_s = P_{st}\mathbb{E}^{\mathbb{Q}}[(S_t - K)^+|\mathcal{F}_s]. \tag{4.2.98}$$

After a change of the measure by the help of the Radon-Nikodym density given by equation (4.1.28), the option price takes the following form:

$$C_s = \frac{P_{st}}{\Lambda_s}\, \mathbb{E}^{\mathbb{B}_T}[\Lambda_t(S_t - K)^+|\mathcal{F}_s] \tag{4.2.99}$$

$$= \frac{P_{st}}{\Lambda_s}\, \mathbb{E}^{\mathbb{B}_T}\left[\left(\int_0^\infty (P_{tT}x - K)p_t(x)dx\right)^+ |\mathcal{F}_s\right], \tag{4.2.100}$$

where

$$p_t(x) = e^{x\left(\frac{1}{T-t}\xi_t \int_0^t \sigma_s ds + \int_0^t \sigma_s d\xi_s\right) - \frac{1}{2}x^2\left(\frac{1}{T-t}\left(\int_0^t \sigma_s ds\right)^2 + \int_0^t \sigma_s^2 ds\right)}. \tag{4.2.101}$$

Proposition 4.2.5. *When the market information process has a time dependent information flux rate, the price C_s admits the following representation under $\mathbb{B}_T$:*

$$C_s = \frac{P_{st}}{\Lambda_s}\left[\mathbb{E}^{\mathbb{B}_T}(P_{tT}\zeta_t\mathbb{I}_A|\mathcal{F}_s) - \mathbb{E}^{\mathbb{B}_T}(K\Lambda_t\mathbb{I}_A|\mathcal{F}_s)\right], \tag{4.2.102}$$

where $A = \{P_{tT}\zeta_t > K\Lambda_t\}$,

$$\zeta_t = \int_0^\infty xp(x)e^{x\left(\frac{1}{T-t}\xi_t \int_0^t \sigma_s ds + \int_0^t \sigma_s d\xi_s\right) - \frac{1}{2}x^2\left(\frac{1}{T-t}\left(\int_0^t \sigma_s ds\right)^2 + \int_0^t \sigma_s^2 ds\right)} dx \tag{4.2.103}$$

and

$$\Lambda_t = \int_0^\infty p(x) e^{x\left(\frac{1}{T-t}\xi_t \int_0^t \sigma_s ds + \int_0^t \sigma_s d\xi_s\right) - \frac{1}{2}x^2\left(\frac{1}{T-t}\left(\int_0^t \sigma_s ds\right)^2 + \int_0^t \sigma_s^2 ds\right)} dx, \qquad (4.2.104)$$

or alternatively

$$\Lambda_t = \int_0^t p(x) \exp\left(x \int_0^\infty \nu_s dW_s^* \; - \frac{1}{2}x^2 \int_0^t \nu_s^2 ds\right) dx \qquad (4.2.105)$$

and

$$\zeta_t = \int_0^t xp(x) \exp\left(x \int_0^\infty \nu_s dW_s^* \; - \frac{1}{2}x^2 \int_0^t \nu_s^2 ds\right) dx. \qquad (4.2.106)$$

Proof. The proof is very straightforward along the same way as for Proposition 4.1.5. Indeed,

$$C_s = \frac{P_{st}}{\Lambda_s} \mathbb{E}^{\mathbb{B}_T} \left[\left(P_{tT}\zeta_t - K\Lambda_t \right)^+ | \mathcal{F}_s \right]; \qquad (4.2.107)$$

here, the set A is defined as $A \;:=\; \{P_{tT}\zeta_t > K\Lambda_t\}$, and so C_s can be represented by using the indicator function $\mathbb{I}_A$

$$C_s = \frac{P_{st}}{\Lambda_s} \mathbb{E}^{\mathbb{B}_T} \left[\left(P_{tT}\zeta_t - K\Lambda_t \right) \mathbb{I}_A | \mathcal{F}_s \right] \qquad (4.2.108)$$

$$= \frac{P_{st}}{\Lambda_s} \left[\mathbb{E}^{\mathbb{B}_T} \left[P_{tT}\zeta_t \mathbb{I}_A | \mathcal{F}_s \right] - \mathbb{E}^{\mathbb{B}_T} \left[K\Lambda_t \mathbb{I}_A | \mathcal{F}_s \right] \right]. \qquad (4.2.109)$$

Proposition 4.2.6. *The price of a call option on a single dividend paying risky* *stock with a maturity date t and strike price K has the following form:*

$$C_u = S_u M_1(u,t,T) - KP_{ut} M_2(u,t,T), \tag{4.2.110}$$

where $A = \{P_{tT}\zeta_t > K\Lambda_t\}$ *and*

$$M_1(u,t,T) \;=\; \mathbb{E}^{\mathbb{B}_T}\left(\frac{\zeta_t}{\zeta_u}\, \mathbb{I}_A | \mathcal{F}_u \right), \quad M_2(u,t,T) \;=\; \mathbb{E}^{\mathbb{B}_T}\left(\frac{\Lambda_t}{\Lambda_u}\, \mathbb{I}_A | \mathcal{F}_u \right). \tag{4.2.111}$$

Proof. The proof of the above Proposition is given by applying the same procedure as in Proposition 4.1.6. The underlying stock price representation for any $u \in [0,t)$:

$$S_u = P_{uT} D_{uT}, \tag{4.2.112}$$

where D_{uT} represents the expected dividend process which can be expressed as follows

$$D_{uT} \;=\; \frac{\zeta_u}{\Lambda_u}. \tag{4.2.113}$$

So, the single dividend paying risky asset price equals

$$S_u = P_{uT}\, \frac{\zeta_u}{\Lambda_u}. \tag{4.2.114}$$

From the previous Proposition, the option price can be represented as

$$C_u = \frac{P_{ut}}{\Lambda_u} \mathbb{E}^{\mathbb{B}_T}\left[P_{tT}\zeta_t \mathbb{I}_A | \mathcal{F}_u \right] - \frac{P_{ut}}{\Lambda_u} \mathbb{E}^{\mathbb{B}_T}\left[K\Lambda_t \mathbb{I}_A | \mathcal{F}_u \right]. \tag{4.2.115}$$

Since the interest rates are assumed to be deterministic, we can express the option price in this way:

$$C_u = \frac{P_{uT}}{\Lambda_u}\mathbb{E}^{\mathbb{B}_T}\left[\zeta_t \mathbb{I}_A|\mathcal{F}_u\right] - KP_{ut}\mathbb{E}^{\mathbb{B}_T}\left[\frac{\Lambda_t}{\Lambda_u}\mathbb{I}_A|\mathcal{F}_u\right]. \tag{4.2.116}$$

Now, substituting the stock price representation into the equation (4.2.116) yields

$$C_u = S_u\mathbb{E}^{\mathbb{B}_T}\left[\frac{\zeta_t}{\zeta_u}\mathbb{I}_A|\mathcal{F}_u\right] - KP_{ut}\mathbb{E}^{\mathbb{B}_T}\left[\frac{\Lambda_t}{\Lambda_u}\mathbb{I}_A|\mathcal{F}_u\right] \tag{4.2.117}$$

$$= S_u M_1(u,t,T) - KP_{ut}M_2(u,t,T). \tag{4.2.118}$$

This completes the proof.

CHAPTER 5

CONCLUSION

In this study, we have presented two strong option pricing models which have two totally different intuitions and approaches to the derivative pricing problem. They differ from the Black-Scholes-Merton model in that, one of them treats the case of a stochastic volatility, and the other one is based on a noise process represented by a Brownian bridge. In the stochastic volatility case, suggested by Heston [11], the option pricing formula is obtained via the characteristic function method. The other model proposed by Brody-Hughston-Macrina [4] gives the option pricing formula and the spot asset price dynamics by modeling the structure of the information accessible in the market.

We have proved the results for the risky asset and option price processes under the assumption of deterministic interest rate, these results were stated in the papers [11, 4]. Furthermore, by adopting the same analogy used for defaultable bond option prices introduced by the work of Rutkowski-Yu [21], we have presented the option price process which was not given in the original paper [4].

In this thesis, the author looked for a bridging some gaps existing in original works [11, 4], and for understandable arrangements which could serve the interested reader for his futher use of it and his studies.

Because of the time constraint of this study, we have left the calibration of the models for the Turkish market for future works. In fact, the information-based asset pricing model suggested by Brody-Hughston-Macrina [4, 3] has not been calibrated yet. Its validity in the market is still unknown. Moreover, since it is an observed and experienced fact that asset prices have jumps, the information-based approach can be extended to capture this empirical property of the data to further research.

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