HOW DOES THE STOCK MARKET VOLATILITY CHANGE AFTER INCEPTION OF FUTURES TRADING? THE CASE OF THE ISE NATIONAL 30 STOCK INDEX FUTURES MARKET

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ABSTRACT

HOW DOES THE STOCK MARKET VOLATILITY CHANGE AFTER INCEPTION OF FUTURES TRADING? THE CASE OF THE ISE NATIONAL 30 STOCK INDEX FUTURES MARKET

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As the trading volume in TURKDEX, the first and only options and futures exchange in Turkey, increases, it becomes more important to have an understanding of the effect of stock index futures trading on the underlying spot market volatility. In this respect, this thesis analyzes the effect of ISE-National 30 index futures contract trading on the underlying stocks' volatility. In this thesis, spot portfolio volatility is decomposed into two components and this decomposition is applied to a single-factor return-generating model to focus on the relationships among the volatility components rather than on the components in isolation. In order to measure the average volatility and the cross-sectional dispersion of the component securities and the portfolio volatility for each day in the sample period, a simple filtering procedure to recover a series of realized volatilities from a discrete time realization of a continuous time diffusion process is used. Results reveal that inception of futures trading has no significant effect on the volatility of the underlying ISE National 30 index stock market.

Keywords : Stock Market Volatility, Derivatives, Futures, Diffusion Function Estimation

ÖZ

VADELİ İŞLEMLER BAŞLADIKTAN SONRA HİSSE SENEDİ PİYASASININ OYNAKLIĞI NASIL DEĞİŞTİ? İMKB ULUSAL 30 ENDEKSİNE DAYALI VADELİ İŞLEM SÖZLEŞMELERİ ÜZERİNE BİR ÇALIŞMA

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Türkiye'deki ilk ve tek vadeli işlemler ve opsiyon borsası olan VOBAŞ'ta işlem hacmi arttıkça vadeli işlemlerin dayanak spot piyasanın oynaklığına etkilerini anlamak önem kazanmakta olup bu tezde IMKB Ulusal 30 endeksine dayalı vadeli işlem sözleşmelerinin işlem görmesi sonucu dayanak hisse senetlerinden oluşturulan bir portföyün oynaklığındaki değişiklikler araştırılmıştır. Söz konusu etkileşimin araştırılması amacıyla spot portföy oynaklığı bileşenlerine ayrılmış ve vadeli işlem alım satımı sonucu bileşenler arasındaki bağıntılarda bir değişiklik olup olmadığının tespitine yönelik olarak, bu ayrım tek faktörlü bir getiri modeline uygulanmıştır. Spot portföy oynaklığının bileşenlerinin günlük bazda hesaplanmasında ise sürekli diffizyonların kesikli zamandaki gerçekleşmelerine basit bir filtreleme yöntemi uygulanmıştır. Sonuçlar İMKB Ulusal 30 endeksine dayalı vadeli işlem sözleşmelerinin işlem görmesinin İMKB Ulusal 30 endeksinde yer alan hisse senetlerinin oynaklığı üzerinde bir etkisi bulunmadığına işaret etmektedir.

Anahtar Kelimeler: Hisse Senedi Piyasası Oynaklığı, Vadeli İşlem Sözleşmeleri, Türev Araçlar, Diffizyon Fonksiyonu Tahmini

To my mother, grandmother and lovely sister

To my beloved husband

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INTRODUCTION

Origins of organized securities markets in Turkey date back to the second half of the 19th century. The very first securities market was named "Dersaadet Tahvilat Borsası (İstanbul Bond Exchange)" and it was established during the Ottoman Empire period in 1866 (ISE web site). Following the foundation of the Turkish Republic, "Securities and Foreign Exchange Law No. 1447"was passed in 1929 and it provided a basis for an organized Stock Exchange under the name of "Istanbul Securities and Foreign Exchange Bourse." This new stock exchange grew in a short period of time and contributed considerably to the financing of the real sector. Unfortunately, both the 1929 economic crisis and the outbreak of the Second World War ended up hampering the success of the stock exchange. Still, during the post-war period, as a result of the rapidly growing industrial sector, an increasing number of companies offered their shares to the public and faced strong demand from individual investors.

During the first half of 1980s, the Turkish securities markets experienced serious developments in terms of setting up of the legal and institutional infrastructure necessary for sound capital movements within a financial system. The Capital Markets Law (CML) was passed in 1981, and the Decree by Law No.91 establishing the basic principles concerning the foundation and operations of securities exchanges was passed in October 1983. In the following year, the Regulation concerning the foundation and operations of the securities exchanges was published in the Official Gazette. Following the adoption of related regulations in the subsequent period, in December 1985 the Istanbul Stock Exchange (ISE) was officially established and started its operations on January 3, 1986. Currently ISE is comprised of an equities market, a bonds and bills market and an international securities market.

Despite certain macro-economic imbalances, Turkish capital markets have made considerable progress both qualitatively and quantitatively during their relatively short history. This fact reveals itself in the trend of capital market indicators throughout last two decades. The number of corporations with shares traded on the ISE equities market was 80 by the end of 1986, the year in which the ISE was established. As of the end of 2006, the number of corporations registered with the CMB is 604, and 316 of these corporations are listed on the ISE. Despite the dominance of the government as a participant in the Turkish financial markets, the funds that are transferred to the private sector via stock issues have accumulated to as much as US $ 29 billion between 1986 and the year-end 2005 . As of the end of 2006, the number of investors in the ISE has reached 1,068,584 (CMB 2005 year book).

Although the Turkish capital markets have undergone a great progress as an emerging economy, what is evident regarding the economy as a whole is also evident for the capital markets as well: the prices of financial securities are very volatile due to macro-economic imbalances as well as domestic factors such as political stability and international factors such as exchange rates. This fact is supported if the market capitalization and market index level figures are examined. Total market capitalization in accordance with volatility in prices has shown a very uneven pattern over the years. Market capitalization, which was only 938 million US Dollars (0.7 million YTL) by the end of 1986, reached the level of 163.8 billion US Dollars (230 billion YTL) at the end of 2006. When annual changes in market capitalization are analyzed, it is seen that market capitalization had the highest increase in dollar terms during 1999 (236%). Following this increase, market capitalization decreased by 39%, 31% and 27.9% in 2000, 2001 and 2002 respectively and then increased by 100.6% in 2003, 42.1% in 2004, 66% in 2005 and 0.6% in 2006 (CMB 2006 year book). Moreover, in a general pattern of cyclical fluctuations, ISE indices, composed in order to calculate price and return performances of all shares as well as on the basis of relative markets and sectors, both in terms of New Turkish Lira and US dollar were volatile. The following figure presents the percentage return series of ISE National-100 Index closing values for the last 7 years. The return series should be interpreted as the main indicator of volatility in the prices of shares quoted on the National Market.

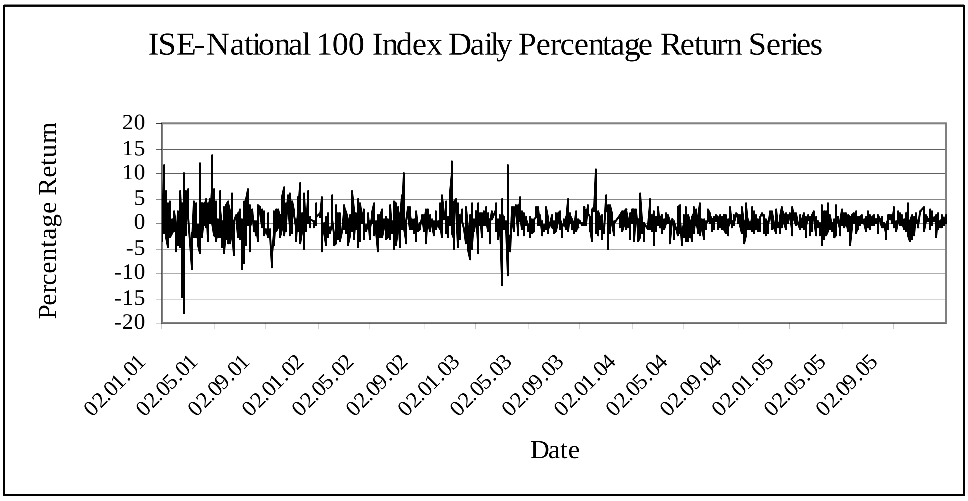

Figure 1: ISE-National 100 Index Daily Percentage Return Series

In the above figure, it is clearly seen that the daily percentage return of ISE-National 100 Index has oscillated in a wide band around 0, implying that the stock market as a whole was volatile. Recall that volatility refers to the standard deviation of the change in the value of a financial instrument within a specific time horizon and is often used to quantify the risk of the instrument over that time period. According to the figure, investors have realized returns that were different from their expected returns so, the riskiness of investing in Turkish capital markets has been more that of most of the other developing markets. This type of an investment environment is only preferable if the investor is a risk taker. However, the theory of investment suggests that different investors may have different choices regarding the level of risk to assume, and even that same investor may prefer to take different risk levels at different times. In order to accommodate these different risk preferences, the financial system has to offer a means by which investors can manage and adjust the level of risk that they take. The derivative markets and derivative instruments are one of the best possible ways of achieving this objective.

In finance, a derivative is a financial instrument the value of which depends on the value of an underlying asset's value. With such an instrument, rather than trading or exchanging the asset itself, market participants enter into an agreement to exchange money, assets or some other value at some future date based on the underlying asset. Examples of underlying assets range from cotton, to shares of common stock, to interest rates. One of the simplest derivative instruments is a futures contract. This is an agreement to buy or sell the underlying asset (or the equivalent cash flows) at a future date. The buying or selling price, the amount of the underlying asset to be exchanged and the date on which the exchange will take place are all determined on the day that the futures contract is created. The exact terms of the derivative (the payments between the counter parties) depend on, but may or may not exactly correspond to, the behavior or performance of the underlying asset. The diverse range of potential underlying assets and payoff alternatives leads to a very large number of different derivative contracts that can be traded in the markets. The main types of derivatives are futures, forwards, options and swaps.

Forward contracts are negotiated between two parties, with no formal regulation or exchange and involve the purchasing (long position) or selling (short position) of a specific quantity of a commodity (e.g., corn or gold), foreign currency, or financial instrument (e.g., bonds or stocks) at a specified price (delivery price), with delivery or settlement at a specified future date (maturity date). The price of the underlying asset for immediate delivery is known as the spot price.

Futures contracts are standardized forward contracts that are traded on an organized exchange and involve the making or taking delivery of a specified quantity of a commodity, a foreign currency, or a financial instrument at a specified price, with delivery or settlement at a specified future date.

A futures contract is entered into through an organized exchange, using banks and brokers. These organized exchanges have clearinghouses, which may be financial institutions or part of the futures exchange. They interpose themselves between the buyer and the seller, guarantee obligations, and make futures liquid with low credit risk. The changes in the value of the underlying asset require a daily mark-to-marking and a cash settlement (i.e. disbursed gains and daily collected losses) for both sides of a futures contract.

Options are the other commonly used derivative instruments and give the holder the right and not the obligation to buy or sell an underlying asset at a specific price on or before a future date. The two main types of option contracts are call options and put options, while some others include stock (or equity) options, foreign currency options, options on futures, caps, floors, collars, and swaptions (options written on swap contracts).

A swap, on the other hand, is a flexible, private, forward-based contract or agreement between two counter parties to exchange streams of cash flows based on an agreed-on (or notional) principal amount over a specified period of time in the future. Swaps are usually entered into through brokers or dealers who take an up-front cash payment or who adjust the interest rates to bare default risk. The two most prevalent swaps are interest rate swaps and foreign currency swaps, while others include equity swaps and commodity swaps.

The derivatives market serves the needs of several groups of users, including those parties who wish to hedge, those who wish to speculate, and arbitrageurs.

- A hedger enters the market to reduce risk. Hedging usually involves taking a position in a derivative financial instrument, which has opposite return characteristics of the asset or position being hedged, and has the purpose of offsetting losses or gains in order to eliminate return volatility.
- A speculator enters the derivatives market in search of profits, and is willing to accept risk. A speculator takes an open position in a derivative product (i.e. there is no offsetting cash flow exposure to offset losses on the position taken in the derivative product).
- An arbitrageur is a speculator who attempts to lock in near-riskless profits that can be earned from price differences by simultaneously buying and selling identical financial instruments at two different prices.

TurkDex, the very first and only options and futures exchange in Turkey, was launched and began its operations on 4 February 2005. The objective of founding the futures and options exchange was to satisfy the hedging, speculation or arbitrage needs of Turkish investors. As of the end of May 2007, only 4 kinds of futures contracts are traded on the TurkDex, although it was launched with the purpose of offering both futures and options. The kinds of futures traded on TurkDex are as follows (Official website of TurkDex):

1) Currency Futures Contracts written either on YTL/EURO rate or YTL/DOLLAR exchange rate
2) Equity Index Futures Contracts written either on ISE National 30 Index or ISE National 100 Index
3) Interest Rate Futures Contracts written either on 91-Day T-Bill interest rates or 365-Day T-Bill interest rates or T-Benchmark
4) Commodity Futures Contracts written on either cotton or wheat or gold.

Among these contracts, the equity index futures contracts is of most relevance for this thesis, since the stock index futures are deemed as one of the most successful financial innovations of the 1980s (Ryoo and Smith, 2004). Today, stock index futures and options trade in developed financial markets all over the world, with new contracts launched nearly every year. In contrast, much of the futures trading in emerging markets is a relatively recent phenomenon. Although Turkey is one of the growing emerging markets, it was not until 4 February 2005 that futures contracts based on the ISE National 30 Index and ISE National 100 Index were introduced. However, since the introduction of the futures contracts on interest rates, indices, commodities and exchange rates, the trading volumes have grown remarkably. The following tables provide the trading volume figures. Table 1 presents the yearly trading volume figures for year 2005 and 2006 at each category of futures contracts, and Table 2 presents the trading volume figures for each of the equity index futures contracts since the inception of TurkDex.

Table 1: The yearly trading volume figures for year 2005 and 2006 for each class of futures written on an different underlying asset category

Category of Underlying Asset	Year 2005 (YTL)	Year 2006 (YTL)	Total (YTL) (04.02.2005-31.12.2006)	Percentage Change (from 2005 to 2006)
Equity Indices	658,743,565	10,608,360,610	11,267,104,175	1510.39
Interest Rates	19,945,793	26,049,053	45,994,846	30.59
Exchange Rates	2,240,018,049	6,747,504,822	8,987,522,871	201.22
Commodities	771,525	4,240,704	5,012,229	449.65
Total (YTL)	2,919,478,931	17,386,155,189	20,305,634,120	495.52

Table 2: The trading volume figures for each of the equity index futures contract

Contract Code	Maturity of the Contract	Underlying Equity Index	Year 2005 (YTL)	Year 2006 (YTL)
101F_IX1001205	December 05*	ISE National 100 Index	72,669,060	
101F_IX1000206	February 06	ISE National 100 Index	19,250,703	
101F_IX1000406	April 06	ISE National 100 Index	3,433,103	
111F_IX0300205	February 05*	ISE National 30 Index	2,250,016	
111F_IX0300405	April 05*	ISE National 30 Index	18,154,085	
111F_IX0300605	June 05*	ISE National 30 Index	40,816,329	
111F_IX0300805	August 05*	ISE National 30 Index	61,593,941	
111F_IX0301005	October 05*	ISE National 30 Index	169,525,652	
111F_IX0301205	December 05*	ISE National 30 Index	230,720,949	
111F_IX0300206	February 06	ISE National 30 Index	29,657,990	
111F_IX0300406	April 06	ISE National 30 Index	10,671,740	
101F_IX1000206	February 06**	ISE National 100 Index		60,063,853
101F_IX1000406	April 06*	ISE National 100 Index		62,295,678
101F_IX1000606	June 06**	ISE National 100 Index		27,361,840
101F_IX1000806	August 06**	ISE National 100 Index		749,085
101F_IX1001006	October 06**	ISE National 100 Index		6,025,955
101F_IX1001206	December 06**	ISE National 100 Index		4,617,085
101F_IX1000207	February 07	ISE National 100 Index		38,300
101F_IX1000407	April 07	ISE National 100 Index		506,295
111F_IX0300206	February 06**	ISE National 30 Index		390,930,918
111F_IX0300406	April 06**	ISE National 30 Index		907,237,368
111F_IX0300606	June 06**	ISE National 30 Index		1,528,584,573
111F_IX0300806	August 06**	ISE National 30 Index		1,802,671,575
111F_IX0301006	October 06**	ISE National 30 Index		2,013,930,633
111F_IX0301206	December 06**	ISE National 30 Index		3,461,245,768
111F_IX0300207	February 07	ISE National 30 Index		277,239,813
111F_IX0300407	April 07	ISE National 30 Index		64,861,875

* Contracts closed between 4 Feb-30 Dec 2005
** Contracts closed between 2 Jan-29 Dec 2006

What is evident from the above tables is that futures written on stock indexes have been the most frequently traded contracts since 2005 and that the trading volume in equity index future contracts is dominated by contracts written on the ISE-National 30 Index. In year 2005, the trading volume in futures contracts with the underlying asset as ISE National 30 Index has been 563,390,702 YTL and this figure has increased to 10,446,702,523 YTL by the end of 2006 (a 1,754% increase). On the other hand, 2005 annual trading volume of ISE-National 30 Index stocks has been 269,970,134,449 YTL and 2006 annual trading volume of ISE-National 30 Index stocks has been 325,157,131,314 YTL, showing that the trading volumes of the futures contracts on ISE-National 30 Index as a percentage of the trading volume in the underlying asset were realized as %0.2 and %3 in 2005 and 2006, respectively. Although the percentage of the trading volume of the equity index futures compared to trading volume of the underlying instrument has been low until the end of 2006, the progress of the trading volume in futures contracts written on ISE-National 30 Index is promising. It looks like as the trading volume in TURKDEX increases, it becomes even more important to have an understanding of the interaction between futures and spot markets, and specifically of the effect of stock index futures trading on the underlying spot market.

Before continuing with the literature survey about the effect of stock index futures trading on the underlying spot market, a detailed description of the ISE-National 30 Index futures contract is given below in order to provide a better understanding of futures trading on TurkDex.

First, let’s focus on the forces that derive an investor to buy or sell the ISE-National 30 Index futures contract. All risk-averse investors like investing their savings in alternatives that provide the highest return for a given level of risk. The stock market is one of these alternatives. Typically, most of the stocks and therefore the stock indexes increase in value when the economy is in an expansionary phase and decline in value when the economy is in a recessionary phase. If an investors, for instance, makes an investment in a portfolio of stocks that also make up the ISE National 30 index, this investor’s holding period return will fluctuate with the phases of the economy. In other words, his returns will increase while the economy is expanding and will decrease while the economy is contracting. This typical volatility in the returns may discourage a lot of risk averse individuals and they may refrain from investing in the stock market because of its high expected risk. However, if the same investors can find a way to decrease or completely eliminate this risk, the stock market may once again become a viable investment alternative, even for the most risk averse investor. The ISE-30 Index Futures contract provides the investors with the opportunity to invest in the stock market based on their expectations regarding the direction of overall economy while hedging themselves against the return volatility by taking an appropriate position in the futures market. If the investor expects that the economy is going to have an expansionary movement, then he can buy a portfolio of stocks that mimic the ISE-30 index and then he can take a short position in the ISE-30 futures contract. With such a combined position, the investor's return volatility will be decreased since the cash market position and the futures market position are constructed to move in opposite directions under all possible scenarios. Therefore, the offsetting positions will generate a return whose volatility is a lot lower than the return that the investor can earn if he took a position only in the stock market and not in the futures market.

The underlying asset of this futures contract, the ISE National-30 index, is composed of National Market companies, except for investment trusts. The constituent 30 companies are selected on the basis of pre-determined criteria determined by the ISE administration. The stocks are ranked according to their market values and their daily average trading values. Those stocks which have the highest market values and daily average trading values are included in the ISE National-30, ISE National-50 and ISE National-100 indices (official website of ISE).

The ISE indices are weighted by the market capitalization of the publicly held portion (the stocks kept in custody at Takasbank, except for those kept in non-fungible accounts) of each constituent stock. All of the indices have varying base values and the continuity of the indices is maintained by adjusting these base values (the divisor of the index formula).

The following table provides the contract specifications for the futures contract written on the ISE-National 30 Index.

Table 3: Contract specifications of a future contract on ISE-National 30 Index

Contract Size	Value calculated by dividing the index value by 1.000 and multiplying the quotient by YTL 100 (ISE National-30 Index/1,000)*YTL 100 (Example: 47.325*100=YTL 4,732.5)
Price Quotation	ISE National-30 Index value, divided by 1,000 shall be quoted significant to three decimals.
Daily Price Limit	±%10 of the established Base Price for each contract with a different contract month
Minimum Price Fluctuation (Tick)	0.025 (25 ISE National-30 Index points) Value of one tick corresponds to YTL 2.5
Margins	Initial margin is 600 YTL and maintenance margin is 450 YTL where maintenance level is 75%.
Contract Months	February, April, June, October and December (Contracts with three different expiration months nearest to the current month shall be traded concurrently)
Final Settlement Day	Last business day of each contract month
Last Trading Day	Last business day of each contract month
Settlement Method	Cash Settlement
Final Settlement Price	Arithmetic average of 10 randomly selected, less than 30 seconds apart, ISE National-30 Index values executed at the ISE within the last 15 minutes before the closing of the trading session of the Exchange on the last trading day shall be used as the last settlement price of the futures contract. If the ISE trading session closes before that of the Exchange, calculation method being the same, calculations shall be made based on the ISE National-30 Index values executed during the last 15 minutes before the closing of the ISE trading session
Daily Settlement Price	Daily settlement price is established at the closing of each trading session as follows: 1. Weighted average price of all the transactions performed within the last 10 minutes before the closing of the trading session based on the quantity thereof shall be established as the daily settlement price. 2. If number of transactions performed within the last 10 minutes before the closing of the trading session is less than 10, weighted average of the last 10 transactions before the closing shall be calculated instead.

The rest of this thesis is organized as follows: Chapter Two presents the literature survey on the effects of futures trading on the volatility of the underlying spot market. Chapter Three provides a detailed discussion of the methodology used. Chapter Four presents the data, preliminary statistics and analysis results. Finally, Chapter Five provides conclusions.

CHAPTER 2

LITERATURE SURVEY

The trading of futures on equity indices aims to provide a hedging alternative for the risk taken in the spot market. Therefore, the effect of such trading on the volatility and riskiness of the spot market has critical importance from the point of view of both investors of the stock market, the stock index futures and the regulators such as the CMB. In fact, this is one of the most widely debated issues in the finance literature. There are arguments for and against the introduction of derivative instruments. The main argument against stock index futures trading claims that the existence of a futures market may increase volatility of the stock market which provides the underlying assets for the contracts. This argument is based on the assumption that, because of their high degree of leverage, futures markets are likely to attract risk takers and speculators. The investor is able to take a futures position by only depositing a small percentage of the contract size. For example, to open a position on a ISE-National 30 Index futures, the investor is required to deposit only 600 YTL. The speculative investment strategies of risk takers and speculators are likely to increase the volatility in the market. Another argument against futures trading is that futures markets provide otherwise unattainable trading strategies like index arbitrage and portfolio insurance. Index arbitrage, for example, attempts to detect temporary deviations in futures prices from the theoretical no-arbitrage values found by use of current spot prices. Specifically, arbitragers buy (sell or short sell) the spot stock portfolio and simultaneously sell (buy) the index futures contracts when the futures prices exceed (fall short of) the spot price of the index, net of carrying costs (Chang et al, 1999). Therefore, arbitragers drive spot prices up or down. These investment strategies are also likely to increase the underlying stock market volatility.

There are also arguments that support the introduction of derivative instruments. These arguments claim that futures markets play an important role in price discovery and have a beneficial effect on the underlying spot markets. This viewpoint asserts that speculation in the futures market tends to stabilize cash prices. Futures trading adds more informed traders to the cash market, making it more liquid and, therefore, less volatile. Both of the arguments against and in favor of futures market trading have some theoretical and empirical support. Here are some of the studies that address this issue.

Edwards (1988) is the first to study the effect of S&P500 stock index futures trading on the volatility of the underlying S&P500 index. He searches for the effect between 1972 and 1987 using daily price volatility series of the index by simply comparing the estimated volatility of the S&P500 index before and after the inception of futures trading. Edwards finds a statistically significant decrease in stock market volatility after the introduction of the stock index futures contract.

Lockwood and Linn (1990) study the variance of hourly market returns computed from opening, closing, and intraday hourly values of the Dow Jones Industrial Average (DJIA) for the January 1964-February 1989 period. They perform tests for homoscedasticity (equality of variances between different periods). Contrary to the findings of Edwards (1988), their results indicate that return volatility fell from the opening hour until early afternoon and rose thereafter and was significantly greater for intraday versus overnight periods. Market variance was also shown to change significantly over time: rising after NASDAQ’s start in 1971, rising after trading in stock options began trading in 1973, falling after fixed commissions were eliminated in 1975, rising after trading in stock index futures was introduced in 1982, and falling after margin requirements for stock index futures became larger in 1988.

Bessembinder and Seguin (1992) examine whether greater futures-trading activity is associated with greater equity volatility by using the daily data of S&P500 between January 1978 and September 1989. They decompose each trading activity series into expected and unexpected components and estimate a conditional expected return and a conditional standard deviation for the return data. Bessembinder and

Seguin find evidence that unexpected S&P500 futures trading is positively related with spot market volatility but the relationship between spot market volatility and expected futures volume is negative. This means that active futures markets are associated with decreased rather than increased equity market volatility. These findings are consistent with the theories which predict that active futures markets enhance the liquidity and depth of equity markets.

Darrat and Rahman (1995) focus on the jump volatility and use the FPE/multivariate Granger-causality model to examine whether activities in the futures market and other relevant factors have Granger-caused jump volatility of stock prices. Monthly data on the S&P 500 index spot prices and the S&P 500 index futures trading volume and open interest spanning the period May 1982 through June 1991 are used. The empirical results of this study suggest that futures trading activity, no matter how it is measured, is not a force behind the recent episodes of jump volatility. Darrat and Rahman conclude that S&P 500 index futures volume did not affect the spot market volatility.

Antoniou and Holmes (1995) examine the impact of trading in the FTSE-100 Stock Index Futures on the volatility of the underlying spot market for the case of England. The GARCH family of techniques, suggested first by Bollerslev (1986), is used. Their results suggest that futures trading has led to increased volatility, but that the nature of volatility did not change post-futures. Based on their finding that price changes are integrated pre-futures, but are stationary post-futures, they conclude that the introduction of futures has improved the speed and quality of information flow in the spot market.

Chang et al (1999) study the effects of futures trading on the Nikkei 225 index for the period of September, 1982 to December, 1991. They propose new tests to examine whether stock index futures affect stock market volatility. In their study, spot portfolio volatility is decomposed into two components: the cross-sectional dispersion and the average volatility of returns on the portfolio's constituent securities. They apply the decomposition to a single-factor return-generating model to focus on the relationships among the volatility components rather than (as in traditional tests) on the components in isolation. In order to measure the average volatility and the cross-sectional dispersion of the component securities and the portfolio volatility for each day in the sample period, they use a simple filtering procedure to recover a series of realized volatilities from a discrete time realization of a continuous time diffusion process. This procedure is outlined in papers by Chesney, Elliott, Madan and Yang (Chesney et al., 1993) and Pastorello (1996). The Chang et. al. findings are consistent with the hypotheses that futures trading increases spot portfolio volatility but that there is no volatility ``spillover'' to stocks against which futures are not traded. The increase in volatility attributable to futures trading is small compared with volatility shifts induced by changes in broad economic factors.

Bologna (1999) analyzes the effect of the introduction of stock index futures (a futures contract on the MIB30 stock index) on the volatility of the Italian Stock Exchange for the period of November, 1994 to December, 1997. His study addresses two issues: First, the study analyses whether the reduction of stock market volatility evidenced in the post-futures period is effectively due to the introduction of the futures contract. Second, the study analyzes whether the `futures effect', if confirmed, is immediate or delayed with respect to the moment of the futures trading onset. In his paper, the GARCH family methods are used to show that the introduction of stock index futures per se has led to diminished stock market volatility and no other contingent cause seems to have a systematic reducing effect. Further, the results also suggest that the impact of futures onset on the underlying market volatility is likely to be immediate. These findings are consistent with those theories stating that active and developed futures markets enhance the efficiency of the corresponding spot markets.

Most of these studies examine the impact of the introduction of index futures in one market and thus are unable to make a comparison across markets. Gulen and Mayhew (2000) examine stock market volatility before and after the introduction of index futures trading in twenty-five countries, using various GARCH models augmented with either additive and/or multiplicative dummy. Their statistical model takes care of asynchronous data, conditional heteroscedasticity, asymmetric volatility responses, and the joint dynamics of each country’s index with the world market portfolio. They find that futures trading is related to an increase in conditional volatility in the U.S. and Japan, but in nearly every other country, no significant effect can be found.

Wu Yu (2001) also examines the effect of stock index futures trading on the stock markets of US (S&P 500 Index), UK (FT-SE 100 Index), Japan (Nikkei 225 Index), France (General Share Index), Australia (All Ordinaries Share Index) and Hong Kong (Hang Seng Index) by using the modified Levene statistic and a switching GARCH model for a period of 500 days before and 500 days after the futures trading inception for each index. He finds that stock market volatility increases significantly after the stock index futures are listed on the underlying index with the exception of the London and Hong Kong stock markets.

More recently, Bae et al (2004) examine the effect of the introduction of index futures trading in the Korean markets on spot price volatility and the market efficiency of the underlying KOSPI 200 stocks. They compare this effect relative to the carefully matched non-KOSPI 200 stocks. Employing both an event study approach and a matching-sample approach for the market data during the period of January 1990 to December 1998, they find that the introduction of the KOSPI 200 index futures trading is associated with greater market efficiency but, at the same time, greater spot price volatility in the underlying stock market. They also report that KOSPI 200 stocks experience lower spot price volatility and higher trading efficiency compared to non-KOSPI 200 stocks after the introduction of futures trading. They claim that the trading efficiency gap between the two groups of stocks, however, declines over time and vanishes following the addition of options trading. Overall, their results suggest that while futures trading in Korea increases spot price volatility and market efficiency, there exists volatility spillovers to stocks against which futures are not traded.

Ryoo and Smith (2004) also investigate the impact of stock index futures trading on the Korean stock market by employing GARCH-type methods for the daily and five-minute frequency data for the period between September 1993 and December 1998. They find an increase in volatility and a decrease in the persistence of volatility following the introduction of stock index futures.

As exchange-traded stock index futures and other derivatives continue to play a greater role in financial markets, it is increasingly important to understand the effect of derivatives trading on the underlying spot markets. However, the existent literature on the effects of stock index futures trading on spot market volatility has focused primarily on developed markets, and it is unclear to what extent these results are applicable to emerging markets. Therefore, this thesis aims to produce a contribution to this literature by analyzing the volatility of the stock market after the inception of stock index futures trading for the case of Turkey.

CHAPTER 3

METHODOLOGY

The typical approach adopted in the literature to examine the effect of futures trading on spot market volatility is to compare the spot price volatility prior to the event with that of post-futures. In this thesis, a methodology based on the technique proposed by Chang et.al.(1999) is employed in order to make the carry out a similar before and after comparison on the volatility of the ISE National 30 Index. The model also and examines the sources of the effect of stock index futures trading on the ISE National 30 Index portfolio in detail. This thesis is the first study that analyzes the Turkish financial markets in this context.

Chang et al (1999) propose that the total volatility of a spot portfolio can be decomposed into the components of a cross-sectional dispersion (weighted deviation of each portfolio asset’s return from the portfolio return) and average volatility of returns of the portfolio's constituent securities. The decomposition is applied to a single-factor return-generating model to determine the relationships among the volatility components. In this model, a shift in broad economic factors induces proportional shifts in spot portfolio and average volatility. However, futures-related volatility shifts change the proportionality of this relation in a predictable fashion. The predictions regarding the direction of the shifts in the proportionality of components is discussed in detail in the next section. The model also predicts structural shifts in the relationship between cross-sectional dispersion and spot portfolio (and average) volatility when futures trading begins.

The model is empirically estimated using data from the Turkish financial markets. More specifically, the volatility of the ISE National 30 index stocks and the volatility of non-ISE National 100 index stocks are analyzed and compared against each other during periods that precede and follow the introduction of the ISE National 30 index futures trading on the TURKDEX. The logic behind this comparison is that both set of stocks are susceptible to broad economic disturbances, but only the ISE National 30 index stocks are impacted directly by futures trading. Thus, for the model based on volatility decomposition, shifts in the relationship between the volatility components for the ISE National 30 index stocks but not for the non-ISE National 100 index stocks, are unlikely to be explained by changes in broad economic factors and are more than likely due to the start of futures trading.

3.1. THE DECOMPOSITION

In this section, the relationship driven by Chang et al. between the volatility of a portfolio (*PVOL*), average volatility (*AVOL*) of securities in the portfolio and the expected cross-sectional dispersion (*E(CSD)*) of those securities is discussed. Let's begin with the definition of cross-sectional dispersion at time *t*.

$$CSD_t = \sum_{i=1}^{n} W_{it}\left(r_{it} - r_{pt}\right)^2 \tag{1}$$

In Equation (1), r_{it} (r_{pt}) is the return of security *i* (portfolio *p*) at time *t*, W_{it} is the weight of each stock in the portfolio, and $r_{pt} = \Sigma\, W_{it} * r_{it}$ (a weighted average of individual stock returns within the portfolio). If the returns of all securities in the portfolio move in unison, CSD_t is equal to zero. Conversely, CSD_t is large if the distribution of r_{it} is dispersed. Therefore, CSD_t quantifies the average proximity of individual returns to the realized average portfolio return (Chang et al., 1999). Let's drop the time subscript for convenience and take expectations on both sides of Equation (1):

$$\begin{aligned} E(CSD) &= E\left(\sum_{i=1}^{n} W_i r_i^2\right) + E\left(\sum_{i=1}^{n} W_i r_p^2\right) - 2E\left(\sum_{i=1}^{n} W_i r_p r_i\right) \\ &= E\left(\sum_{i=1}^{n} W_i r_i^2\right) - E\left(r_p^2\right) \end{aligned} \tag{2}$$

where

$$E\left(\sum_{i=1}^{n} W_i r_p^2\right) = E\left(r_p^2\right),$$
$$2E\left(\sum_{i=1}^{n} W_i r_p r_i\right) = 2E\left(r_p \sum_{i=1}^{n} W_i r_i\right) = 2E\left(r_p r_p\right) = 2E\left(r_p^2\right).$$

The first term on the right-hand-side of Equation (2) can be rewritten as follows:

$$\begin{aligned} E\left(\sum_{i=1}^{n} W_i r_i^2\right) &= \sum_{i=1}^{n} W_i E\left(r_i^2\right) = \sum_{i=1}^{n} W_i \left[\sigma^2(r_i) + \left(E(r_i)\right)^2\right] \\ &= \sum_{i=1}^{n} W_i \sigma^2(r_i) + \sum_{i=1}^{n} W_i \left[E(r_i)\right]^2 \end{aligned} \tag{3}$$

In Equation (3), $\sigma^2(r_i)$ is the variance of security i's returns. Let's define $\Sigma_i W_i\sigma^2(r_i)$ as the weighted average volatility (*AVOL*) of the securities of portfolio p and rewrite Equation (3) as follows:

$$E\left(\sum_{i=1}^{n} W_i r_i^2\right) = AVOL + \sum_{i=1}^{n} W_i \left[E(r_i)\right]^2 .$$

The term $E(r_p^2)$ can be rewritten as well:

$$E\left(r_p^2\right) = \sigma^2\left(r_p\right) + \left[E\left(r_p\right)\right]^2 = PVOL + \left[E\left(r_p\right)\right]^2 .$$

Then,

$$E(CSD) = E\left(\sum_{i=1}^{N} W_i r_i^2\right) - E\left(r_p^2\right)$$
$$= AVOL + \sum_{i=1}^{N} W_i \left[E(r_i)\right]^2 - PVOL - \left[E(r_p)\right]^2.$$

If the above equation is solved for PVOL, the final decomposition is obtained as follows:

$$PVOL = AVOL - E(CSD) + \sum_{i=1}^{n} W_i \left[E(r_i)\right]^2 - \left[E(r_p)\right]^2 \qquad (4)$$

In Equation (4), *PVOL* stands for portfolio volatility, *AVOL* stands for average volatilities of the securities in the portfolio, *E(CSD)* stands for expected cross-sectional dispersion, *W* stands for weight of individual securities within the porfolio, *r* stands for return, and *n* stands for number of securities in the portfolio.

The decomposition of PVOL provides a comprehensible framework for understanding the determinants of the volatility of a stock index. As seen from Equation (4), portfolio volatility (*PVOL*) is positively related to the average volatility of securities in the portfolio (*AVOL*) but negatively related to the expected cross-sectional dispersion [*E(CSD)*] of component security returns. *PVOL* is also positively related to a third term, which is the cross-sectional variance of mean returns (*CSVOM*) (Chang et al., 1999). Chang et al. state that in tests not shown in their paper, they found that the cross sectional variance of mean returns (the third term) accounts for less than one percent of the variation in *PVOL*, and, therefore, this term has only a second-order impact on *PVOL* and should be ignored in the analysis that follows. In order to adopt the same methodology as Chang et.al, the third term is shown to have an insignificant effect on the volatility of the ISE National 30 portfolio and is also dropped from the analysis.

The price of each asset *i* in the portfolio is assumed to have the Markov property. This means that the distribution of $X_{t+\Delta t}$ depends only on the current state X_t and not on the whole history. In other words, given the history, the Markov property suggests that the current state is enough to determine all the distributions of the future, but distribution of current state cannot be calculated over X_t. Therefore, it is not possible to calculate *E(CSD)* or *CSVOM* at time *t.* So, we assume that *CSD* proxies for *E(CSD)* and

$$\sum_{i}^{n} W_i (r_i)^2 - (r_p)^2$$

proxies for

$$\sum_{i}^{n} W_i \left[E(r_i)\right]^2 - \left[E(r_p)\right]^2$$

at time *t.*

Two separate regression models are estimated, one with *PVOL* as the dependent variable and *AVOL* and *CSD* as explanatory variables, and the other with *PVOL* as the dependent variable and *AVOL*, *CSD* and *CSVOM* as explanatory variables. The results are presented in Tables 4 and 5. In both tables, the results in Panel A are for the sample of securities included in the ISE-National 30 index and the results in Panel B are for the sample of securities that were never a part of the ISE-National 100 index throughout the sample period. Not only, R^2 for panel B, in two regressions has taken value of nearly 0.0606 and revealed that explanatory power of these regressions for panel B data is poor, but also do the results for panel A, show that R^2, indicating what proportion of the total variation in response is explained by the model, increased marginally by addition of cross sectional variance of mean returns to the model. However, the increase in R^2 is ignorable since it is very small (from 0.729 to 0.732). More specifically, *CSVOM* is shown to account for less than one percent of the variation in *PVOL*, and, therefore, just like in the study by Chang et.al., this variable is dropped from the model.

Table 4: Regression of components of PVOL excluding CSVOM

Model : $PVOL_t = c_0 + c_1 AVOL_t + c_2 CSD_t + \varepsilon_t$ (Sample period: 07.01.2003-30.05.2006) (t statistics are in parenthesis)			
c_0	c_1	c_2	R^2
Panel A: Results for a portfolio of ISE 30 National Index Stocks			
0.000207 (1.336961)	0.276343 (28.64840)	0.063918 (3.874566)	0.729587
Panel B: Results for a portfolio of Non-ISE 100 National Index Stocks			
0.001235 (1.972134)	-0.130261 (-1.850893)	0.199526 (6.387899)	0.060652

Table 5: Regression of components of *PVOL* including *CSVOM*

.				
Model : $PVOL_t = c_0 + c_1 AVOL_t + c_2 CSD_t + c_3 CSVOM_t + \varepsilon_t$ (Sample period: 07.01.2003-30.05.2006) (t statistics are in parenthesis)				
c_0	c_1	c_2	c_3	R^2
Panel A: Results for a portfolio of ISE 30 National Index Stocks				
0.000179 1.161232	0.276876 28.85457	0.114532 4.703145	-0.021727 -2.812632	0.732935
Panel B: Results for a portfolio of Non-ISE 100 National Index Stocks				
0.001235 1.970089	-0.129946 -1.842958	0.199399 6.373009	-0.001065 -0.095219	0.060666

3.2. ESTIMATING ASSET I'S RETURN VOLATILITY

The next step in methodology construction addresses the question of how to measure the average volatility and the cross sectional dispersion of the component securities and the portfolio volatility for each day in the sample period.

For many years economists, statisticians, and teachers of finance have been involved in developing and testing models of stock price behavior. The most prominent model among these has spun from the theory of random walk (Fama, 1965). Random-walk theorists usually start from the postulate that the major security exchanges can be given as good examples of efficient markets (Fama, 1965). An efficient market is defined as a market where large numbers of rational profitmaximizers actively compete with each other with the objective of predicting the future market values of individual securities. If this competitive environment is truly efficient, then important current information about the securities should be freely available to all market participants (Fama, 1965). In other words, in an efficient market, at any given point in time, the actual price of an asset is the best estimate of its intrinsic, or true economic, value (Fama, 1965). In practice, as a result of market imperfections and uncertainty involved in trading, market participants may not all agree upon the same intrinsic value for an asset. Therefore, discrepancies between actual prices and intrinsic values may be observed. Still, if the markets are efficient, , the number of buy and sell transactions is so large that the actual price of a security is expected to oscillate around the security's intrinsic value (Fama, 1965). This expectation further implies that market efficiency does not require the market price to be equal to the asset's intrinsic value at every point in time. All it requires is that errors in the market price be unbiased, i.e., that prices can be greater than or less than true value, as long as these deviations are random. Hence, the random walk hypothesis claims that successive price changes are identically distributed, independent random variables. Most of the early empirical studies support the random-walk behavior of stock prices: Kendall (1953), Roberts (1959), Alexander (1961), Cootner (1964) and Fama (1965), among many others.

In recent years, it has become increasingly interesting for researchers to use the theory of stochastic processes for describing the uncertainty in financial markets. The random behavior of financial asset prices is a very good candidate for such an endeavor (Kijima, 2003). The following sections of methodology construction make use of the essentials of probability theory, stochastic processes and stochastic differential equations to model the portfolio volatility decomposition.

The most basic probability concept used in financial analysis is the Brownian motion. This concept was first defined to model the random movement of pollens immersed in fluids. It was then applied to the analysis of the behavior of random variables. The use of Brownian motion for modeling in this context assumes that the price of a financial asset i follows a diffusion process which is itself a solution to a stochastic differential equation.

In probability theory, the set of possible outcomes is called the sample space and is generally denoted by Ω. Each outcome ω belonging to the sample space Ω is called an elementary event, whereas a subset A of Ω is called an event. In the terminology of set theory, Ω is the universal set, $\omega \in \Omega$ is an element, and $A \in \Omega$ is a set (Kijima, 2003). In order to make such a probability model more precise, a family of events, F, namely the σ-field generated by Ω, needs to be defined (Kijima,2003). The family F of events satisfies the following properties in order to be classified as the σ-field generated by Ω:

1. $\Omega \in F$,
2. If $A \subset \Omega$ is in F then $A^c \in F$, and
3. If $A_n \subset \Omega, n = 1,2,...,$ are in F then $\bigcup_{n=1}^{\infty} A_n \in F$.

For each event $A \in F$, the probability of event A is denoted by $P(A)$. In modern probability theory, probability represents a set function, defined on F, satisfying the following properties (Kijima,2003):

1. $P(\Omega)=1$,
2. $0 \le P(A) \le 1$, for any event $A \in F$, and
3. For mutually exclusive events $A_n \subset F, n = 1,2,...,$ i.e. $A_i \cap A_j = \phi$ for $i \ne j$,

$$P(A_1 \cup A_2 \cup ...) = \sum_{n=1}^{\infty} P(A_n) .$$

Given a sample space Ω and a σ-field F, if a set of function P defined on F satisfies the aforementioned properties, P is called a probability measure. The triplet (Ω, F, P) is called a probability space (Kijima, 2003).

Given a probability space (Ω, F, P), let X denote a mapping from Ω to an interval I of the real line R. The mapping is called a random variable if for any $a < b, \{\omega : a < X(\omega) \le b\} \in F$ is true (Kijima, 2003). In other words, since the probability P is a set function defined on F, X is said to be a random variable if the probability $P\{\omega : a < X(\omega) \leq b\}$ for any $a < b$ is known, which means, if X is a random variable, then the probability that the realization of X is in the interval $(a,b]$ is known in principle (Kijima, 2003).

A random variable X is continuous, if the set of realizations of X is an interval I of the real line and there exists a non negative function $f(x)$, such that

$$P\{a \leq X \leq b\} = \int_a^b f(x)dx$$

for any $[a,b] \subset I$ and $f(x)$ is the density function of X. Conversely, the density function $f(x)$, $x \in I$ defines the continuous random variable X. Also, if we define

$$P\{X \leq x\} = F(x) = \int_{-\infty}^{x} f(y)dy, \ x \in I$$

where $f(x)=0$ for $x \notin I$, $F(x)$ is called the distribution function and we get

$$f(x) = \frac{d}{dx}F(x), \quad x \in I.$$

Before continuing with stochastic processes, the expectation of random variables needs to be discussed. The expected value of a continuous random variable is defined in terms of the density function (Kijima, 2003). Let $f(x)$ be the density function of random variable X. Then, for any real-valued function $h(x)$, the expectation is defined as the following:

$$E[h(X)] = \int_{-\infty}^{+\infty} h(x)f(x)dx$$

provided that

$$\int_{-\infty}^{+\infty} |h(x)| f(x)dx < \infty .$$

Otherwise, the expectation does not exist. Notice that the mean of the random variable X, if any, is

$$E[X] = \int_{-\infty}^{+\infty} xf(x)dx$$

while the variance of *X* is defined by

$$V[X] = \int_{-\infty}^{+\infty} (x - E[X])^2 f(x)dx.$$

Now consider two continuous random variables, *X* and *Y*, defined from Ω to R^2. The probability that *X* is in the interval $(x_1, x_2]$ and *Y* is in the $(y_1, y_2]$ is denoted by the probability (Kijima, 2003)

$$P\{x_1 < X \le x_2, y_1 < Y \le y_2\}.$$

The conditional expectation of *X* under the event $\{Y = y\}$ is given by

$$E[X|Y = y] = \int_{-\infty}^{+\infty} xdF(x|y), \quad y \in R .$$

provided that

$$\int_{-\infty}^{+\infty} |x| dF(x|y) < \infty ,$$

where

$$F(x|y) = P\{X \le x|Y = y\}$$

denotes the conditional distribution of X under $\{Y = y\}$. Notice that $E[X|Y = y]$ is a function of *Y*. Since, *Y*(*ω*) for some $\omega \in \Omega$, the conditional expectation can be thought of as a composed function of *ω*, whence $E[X|Y(\omega)]$ is a random variable.

A family of random variables $\{X_t; t \ge 0\}$ (or $\{X_t\}$ shortly) with index set $t \ge 0$ representing time is called a continuous stochastic process. A stochastic process is a widely used tool to model a system that varies randomly in time, for instance in modeling the price behavior of securities (Kijima, 2003). On the other hand, for each $\omega \in \Omega$, $X_t(\omega)$ is a realization of X_t, and a function of time *t*. The real valued function is called a sample path or a realization of the process $\{X_t\}$.

Let's fix the date *T* such that $t \in [0, T]$, and $T \in [0, \infty)$ where trading horizon *T* is treated as the terminal date of the economic activity being modeled. The information structure available to the investors is given by an increasing (finite) sequence of sub-σ-field of *F*: it is assumed that F_0 is trivial, that is, it contains only sets of *P*-measure from 0 to 1 (Elliot and Kopp, 2005). It is also assumed that (Ω, F_0) is complete (i.e. any subset of a null set is itself null and F_0 contains all *P*-null sets) and that $F_0 \subset F_1 \subset ... \subset F_T = F$ (Elliot and Koop, 2005). An increasing family of σ-fields is called a filtration $\mathcal{F} = (F_t)_{t \in [0,T]}$. F_t can be thought as containing the information available to investors at time *t*: investors learn without forgetting, but insider trading is not possible (Elliot and Koop, 2005).

A random variable *X* is F_t-measurable (or measurable with respect to F_t) if $\{x_1 < X \le x_2\} \in F_t$ for any $x_1 < x_2$ (Kijima, 2003). If the random variable *X* is F_t-measurable, then it is possible to determine whether or not the event $\{x_1 < X \le x_2\}$ occurs just by examining F_t for any $x_1 < x_2$, which is roughly speaking to know the value of X given the information F_t (Kijima, 2003).

A stochastic process $\{X_t; t \geq 0\}$ is said to be adapted to the filtration $\mathscr{F}$, if each X_t is measurable with respect to F_t (Kijima, 2003).

For two distinct time epochs $t > s$, the random variable X_t - X_s is called an increment of $\{X_t\}$. If $s = t + \Delta t, \Delta t > 0$, the increment is denoted by $\Delta X_t = X_{t+\Delta t} - X_t$ (Kijima, 2003). The time intervals $(s_i, t_i], i = 1,2,\cdots$ are non-overlapping if

$$s_1 < t_1 \leq s_2 < t_2 \leq ... \leq s_i < t_i \leq \cdots.$$

The process $\{X_t; t \geq 0\}$has independent increments, if the increments $X_{t_i} - X_{s_i}$ over non-overlapping intervals $(s_i, t_i]$ are independent (Kijima, 2003).

The definitions of probability spaces, σ-fields, continuous random variables, stochastic processes, filtration and stochastic processes that are adapted to filtrations are all given in order to define the Brownian motion. This process is named after Robert Brown, a Scottish botanist who studied movement of pollens in fluids. He observed that these particles were performing a very random movement and claimed that this was because pollens were alive (Elliot and Koop, 2005). The first approaches to mathematically modeling the Brownian motion were made by L Bachelier and A. Einstein in the first half of 1900's. However, it was N. Wiener who was the first to present a general mathematical treatment of the Brownian motion in 1918 (Beichelt, 2006). The Brownian motion process, simply the Brownian motion, is an essential ingredient in stochastic calculus and plays a vital role in mathematics of finance. It also provides the basis for defining one of the most important classes of Markov processes, namely the diffusion processes.

Let $\{B_t, t \geq 0\}$ be a stochastic process defined on the probability space (Ω, F, P). The process $\{B_t\}$ is a standard Brownian motion if the following are true:

1. It has independent increments,

2. The increment $B_{t+s} + B_t$ is normally distributed with mean 0 and variance

s, independently of time *t*, and

3. Its sample paths are continuous and B_0=0 (Kijima, 2003).

The first property above implies that the increment $B_{t+s} + B_t$ is independent of the history F_t. Moreover, one of the most important properties of a Brownian motion is that its paths are nowhere differentiable (Lamberton and Lapeyre, 2000). In other words, if B_t is a Brownian motion, it can be proved that for almost every $\omega \in \Omega$, there is not any time $t \in R^+$ such that $\mathrm{d}B_t / \mathrm{d}t$ exists. However, by the help of stochastic calculus, it is possible to write $\mathrm{d}B_t / \mathrm{d}t$ in terms of differentials

$$(dB_t)^2 = (B_{t+dt} - B_t)^2 = dt$$

and in terms of integrals

$$\int_0^t (dB_s)^2 = \int_0^t ds = t .$$

We will not the make the proofs of these relations, since such proofs are beyond the scope of this thesis.

For real numbers μ and σ , the process $\{X_t\}$

$$X_t = X_0 + \mu t + \sigma B_t, \qquad t \geq 0$$

is called a Brownian motion with drift μ and diffusion coefficient σ (Kijima, 2003). Recall that for any random variable *X* with mean μ and variance σ^2, the transformation given by

$$Y = \frac{X - \mu}{\sigma}$$

is called the standardization of *X* and if $Y \sim N(0,1)$, then

$$X = \mu + \sigma Y \sim N(\mu, \sigma^2).$$

Since $B_t \sim N(0,t)$, the random variable X_t is distributed by $N(\mu t, \sigma^2 t)$. Therefore, the diffusion coefficient σ can be interpreted as the standard deviation of an increment over the unit of time interval (Kijima, 2003).

Since $B_t \sim N(0,t)$, the random variable X_t is distributed by $N(\mu t, \sigma^2 t)$. Therefore, the diffusion coefficient σ can be interpreted as the standard deviation of an increment over the unit of time interval (Kijima, 2003). This conclusion is crucial to understanding the theory behind construction of an estimator regarding the volatility of an asset’s return. The Brownian motion can be shown to be a special diffusion process and if an estimator for a diffusion function can be found, then it is possible to estimate the volatility of an asset’s return.

In order to start defining diffusion processes, it is necessary to first define the continuous Markov processes since diffusion processes are continuous Markov processes with special characteristics. Furthermore, in order to understand Markov processes, Borel measurable functions should be considered first.

Let G (respectively M) be the family of all open (respectively closed) subsets of R^n. Then σ-field generated by G is equal to the σ-field generated by M. This generated σ-field is called the Borel σ-field of R^n, denoted by B_{R^n} (Körezlioğlu and Hayfavi, 2001). If the function f is defined as $f : \Omega \rightarrow R^n$ whereas (Ω, F) and (R^n, B_{R^n}) are measurable spaces and

$$\forall E \in B_{R^n} : f^{-1}(E) \in \Omega$$

then f is Borel-measurable (Capasso and Bacstein, 2005).

In order to define the Markov process, let (Ω, F, P) be a probability space with filtration $\mathcal{F} = (F_t)_{t \in [0,T]}$. An adapted process $\{X_t\}$ is a continuous-time Markov process with respect to the filtration $\mathcal{F}$ if

$$E\left(f\left(X_t\right)\middle|F_s\right) = E\left(f\left(X_t\right)\middle|X_s\right) \text{ almost surely for all } t \geq s \geq 0.$$

for every bounded real-valued Borel-measurable f defined on R^n (Elliot and Koop, 2005). The Markov property asserts that the distribution of $X_{t+\Delta t}$ depends only on the current state X_t and not on the whole history. In other words, given the history, the Markov property suggests that the current state is enough to determine all the distributions of the future. In the literature of financial engineering, it is common to model a continuous time price process (in this case the diffusion process) in terms of a stochastic differential equation. The stochastic differential equations can be explained as a limit of stochastic difference equations and it is then possible to show how diffusion processes can be represented as stochastic differential equations.

Let $\Delta t > 0$ be sufficiently small, and consider the stochastic difference equation given that $t \geq 0$

$$\Delta X_t = X_{t+\Delta t} - X_t = \mu(X_t,t)\Delta t + \sigma(X_t,t)\Delta B_t, \tag{5}$$

where $\mu(X_t,t)$ and $\sigma(X_t,t)$ are given functions with enough smoothness and where $\Delta B_t \equiv B_{t+\Delta t} - B_t$ is the increment of standard Brownian motion (Kijima, 2003). Suppose that $X_t = x$. Then, the limiting process $\{X_t\}$ as $\Delta t \to 0$, if it exists, is a strong Markov process with continuous sample paths, since the Brownian motion $\{B_t\}$ has similar properties (Kijima, 2003). In addition, the following limits from the above stochastic difference equation can be obtained:

$$\mu(x,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\left[\Delta X_t \middle| X_t = x\right]$$

and

$$\sigma^2(x,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\left[\{\Delta X_t\}^2 \middle| X_t = x\right]$$

A Markov process $\{X_t\}$ in continuous time is called a diffusion process (or diffusion in short) if it has continuous sample paths and the limits regarding $\mu(X_t,t)$ and $\sigma^2(X_t,t)$ exists with $\sigma^2(X_t,t) \neq 0$ (Kijima, 2003).

$$X_t = X_0 + \mu(X_t,t)t + \sigma(X_t,t)B_t, \tag{6}$$

where t is nonnegative. In Equation (6), the function $\mu(X_t,t)$ is called the drift function, whereas $\sigma(X_t,t)$ is called the diffusion function. Moreover, from above equation, it is possible to formally obtain the following stochastic differential equation

$$dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dB_t,$$

under some regularity conditions, where $0 \leq t \leq T$ and T is the given time horizon. This stochastic differential equation is understood to be the differential form of the integral equation

$$X_t - X_0 = \int_0^t \mu(X_u,u)du + \int_0^t \sigma(X_u,u)dB_u \tag{7}$$

where only the first term on the right-hand side is the ordinary (path-by-path) integral, and the second term cannot be calculated as an ordinary integral, in other words in the Riemann-Stieltjes sense, because Brownian motions are nowhere differentiable. The second term can be calculated by help of an Ito integral.

Let $\{X_t; t \geq 0\}$ be adapted to Brownian motion B_t on $[0,T]$, i.e. X_t is function of $B_s, s \leq t$, and

$$\int_0^T E[X_s]^2 ds < \infty.$$

Now, it is possible to define the Ito stochastic integral of $\{X_t\}$. It is denoted by

$$\int_0^t X_s dB_s, \ t \in [0,T].$$

Note that the definition of the Ito stochastic integral is not enough to write the above integral in terms of a Brownian motion. Nonetheless, by the help of Ito's lemma, it is possible to obtain explicit formulae for Ito stochastic integrals (Mikosch, 1998).

Ito's lemma can be considered as the stochastic analogue of the classical chain rule of the differentiation. Let f and g be two differentiable functions. Recall from basic calculus that the classical chain rule in the integral form is as follows:

$$f(g_t) - f(g_0) = \int_0^t f'(g_s) g'_s ds = \int_0^t f'(g_s) dg_s.$$

Now, assume that f is a twice-differentiable function, but replace g_t with a sample path of a Brownian motion B_t. The formula

$$f(B_t) - f(B_s) = \int_s^t f'(B_x) dB(x) + \frac{1}{2}\int_s^t f''(B_x) dx,$$

where $s < t$, becomes a simple form of Ito's lemma or of the Ito formula. For later use, a more general version of the form $f(X_t, t)$ is needed where X_t is given by Equation (7) and is a diffusion process. It is also called an Ito process because the second term on the left-hand-side is defined in terms of the Ito integral.

Let $\{X_t\}$ be an Ito process with representation in Equation (7) and $f(X_t,t)$ be a function whose second order partial derivatives are continuous. Then given $s < t$,

$$
\begin{aligned}
f(X_t,t) - f(X_s,s) &= \int_s^t f_2(X_y,y)dy + \int_s^t f_1(X_y,y)dX_y + \frac{1}{2}\int_s^t f_{11}(X_y,y)d\langle X,X\rangle_y \\
&= \int_s^t f_2(X_y,y)dy + \int_s^t f_1(X_y,y)\mu(X_y,y)dy + \int_s^t f_1(X_y,y)\sigma(X_y,y)dB_y \\
&\quad + \frac{1}{2}\int_s^t f_{11}(X_y,y)(\sigma(X_y,y))^2 dy
\end{aligned}
$$

In papers by Chesney, Elliot, Madan and Yang (CEMY) (Chesney et al.,1993) and Pastorello (1996), a simple filtering procedure to recover a series of realized volatilities from a discrete time realization of a continuous time diffusion process is proposed. CEMY use the measure proposed in their paper to construct point estimates of time-varying asset volatility and covariation with risk factors to test Merton's Intertemporal Capital Asset Pricing Model (Merton, 1973). Chang et al. (1998) use the same measure to construct point estimates of volatility for each asset *i* at time *t*. Following Chang et al. (1998), the same unbiased estimator of asset *i*'s return volatility at time t is used in this study to measure the average volatility and the cross sectional dispersion of the component securities and the portfolio volatility for each day in the sample periods.

Let $\{X_t, t \geq 0\}$ represent the price of asset *i* and be a real-valued process defined as the solution of a stochastic differential equation (SDE)

$$dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dB_t, \tag{8}$$

where $0 \leq t \leq T$, $X_0 = x$, $\{B_t, t \geq 0\}$ is a real Brownian motion defined on a probability space (Ω, F, P), $\mu(X_t,t)$ and $\sigma(X_t,t)$ are drift and diffusion functions, which are deterministic. If X_t is known, $X_{t+\Delta t}$ is given by Ito formula as follows

$$X_{t+\Delta t} = X_t + \int_t^{t+\Delta t} \mu(X_u,u)du + \int_t^{t+\Delta t} \sigma(X_u,u)dB_u, \tag{9}$$

given $0 \le t \le T$. In order to investigate for the values of a continuous time stochastic process defined on [$0,T$] at the discretization times, a discrete time approximation of the process given is needed. Two of the time-discrete approximation schemes, namely Euler and Milstein, are considered to construct an estimator for the volatility of asset *i*'s return.

The Euler or Euler-Maruyama approximation is known to be one of the simplest of approximation methods. For a given discretization

$$0 = \tau_0 < \tau_1 < \cdots < \tau_n < \cdots < \tau_N = T$$

of the time interval [$0,T$], an Euler approximation is a continuous time stochastic process $Y=\{Y_t, 0 \le t \le T\}$ satisfying the iterative scheme

$$Y_{n+1} \approx Y_n + f(\tau_n, Y_n)(\tau_{n+1} - \tau_n) + g(\tau_n, Y_n)(B_{\tau_{n+1}} - B_{\tau_n})$$

for $n = 0,1,2,\cdots,N-1$ with initial value $Y_0 = x$ and $Y_n = Y(\tau_n)$.

We shall also write

$$\Delta_n = \tau_{n+1} - \tau_n$$

for the n^{th} time increment and define the maximum time step as

$$\delta = \max_n \Delta_n \ .$$

Since most of the financial data, especially the time series are observed at points in time such that the distance between two points is equal for all successive pairs of points, an equidistant time discretization is considered in the following manner:

$$\tau_n = 0 + n\delta$$

with $\delta = \Delta_n \equiv \Delta = (T - 0)/N$ for some integer large enough so that $\delta \in (0,1)$.

Before continuing with other approximation scheme, namely the Milstein approximation (Milstein, 1974), it is necessary to mention why other time discrete approximation methods have been developed and how the best approximation scheme can be inferred.

There are two criteria, strong convergence and weak convergence to compare the Euler approximation and other approximations with each other.

A general time discrete approximation Y^δ with maximum step size δ is said to converge strongly to X at time T if

$$\lim_{\delta \downarrow 0} E(\left| X_T - Y^\delta(T) \right|) = 0.$$

Kloeden and Platen (1992) state that although the Euler approximation is the simplest useful time discrete approximation, it is not efficient in numerical sense. Therefore, other time discrete approximation methods should also be considered to compare different time discrete approximations. In order to make this comparison the approximations' rates of strong convergence need to be known.

A time discrete approximation Y^δ converges strongly with order $\gamma > 0$ at time T if there exists a positive constant C, which does not depend on δ, and $\delta_0 > 0$ such that

$$\varepsilon(\delta) = E(\left| X_T - Y^\delta(T) \right|) \le C\delta^\gamma \qquad \text{for each } \delta \in (0, \delta_0).$$

On the other hand, a time discrete approximation Y^δ converges weakly with order $\beta > 0$ to X at time T as δ approaches 0 if for each

$$g \in C_P^{2(\beta+1)}(R^d, R)$$

there exists a positive constant *C*, which is independent from δ, and a finite $\delta_0 > 0$ such that

$$\left| E(g(X_T)) - E(g(Y^{\delta}(T))) \right| \leq C\delta^{\beta} \qquad \text{for each } \delta \in (0, \delta_0)$$

where R^d denotes the d-dimensional Euclidean space and $C^{2(\beta+1)}\left(R^d, R\right)$ denotes the space of $2(\beta+1)$ times continuously differentiable functions defined from R^d to *R*.

Strong and weak convergence criteria lead to the development of different time discrete approximations that are only efficient with respect to one of the two criteria. In particular, time discrete approximations derived with respect to strong convergence criterion using stochastic Taylor expansions are called strong Taylor expansions. Considering the Euler approximation, it is known to converge with weak order $\beta = 1$, in contrast with the strong order $\gamma = 0.5$ and represents the simplest strong Taylor approximation (Kloeden and Platen, 1992).

Now, the scheme proposed by Milstein, which turns out to be an order 1.0 strong Taylor scheme, will be examined. As before, the stochastic integral equation given $0 \leq \tau \leq T$

$$X_{\tau} = X_{\tau-1} + \int_{\tau-1}^{\tau} \mu(X_s, s)\, ds + \int_{\tau-1}^{\tau} \sigma(X_s, s) dB_s. \tag{10}$$

is considered.

The Euler approximation is based on the discretization of the integrals in Equation (10). On the other hand, the Milstein approximation exploits a Taylor-Ito expansion of Equation (10). The Milstein approximation is constructed by applying the Ito's lemma to integrands $\mu(X_t, t)$ and $\sigma(X_t, t)$ in Equation (10) and is given as follows:

$$X_{n+1} \approx X_n + \mu(X_n,\tau_n)\,(\tau_{n+1}-\tau_n) + \sigma(X_n,\tau_n)(B_{\tau_{n+1}} - B_{\tau_n})$$
$$+\ \frac{1}{2}\sigma(X_n,\tau_n)\frac{\partial\sigma(X_n,\tau_n)}{\partial X_n}\left[(\Delta B_n)^2 - \Delta_n\right]$$

for $n = 0,1,2,\cdots,N-1$ with initial value $X_0 = x$ and $X_n = X(\tau_n)$.

It must be noticed that the Milstein approximation scheme is the Euler approximation with an additional correction term containing the squared increments of the Brownian motion.

Suppose that the state of the system is given by a SDE as in Equation (9) and define a stochastic process Y_t as follows:

$$Y_t = \exp X_t = \exp X_0 \exp\left\{\int_0^t \mu(s,X_s)\,ds\ + \int_0^t \sigma(s,X_s)dB_s\right\} \tag{11}$$

given $0 \le t \le T$. Notice that Y_t, given in Equation (11), is defined as $f(t,B_t)$, which is a function of class $C^{1,2}$ (f is one time continuously differentiable with respect to t and two times continuously differentiable with respect to B_t). Then application of Ito formula gives the following:

$$f(t,B_t) = f(0,B_0)\ + \int_0^t \left[\ f_1(s,B_s) + \frac{1}{2}\ f_{22}(s,B_s)\right]ds + \int_0^t f_2(s,B_s)dB_s. \tag{12}$$

Then, given $0 \le t \le T$,

$$Y_t = Y_0 + \int_0^t \left[\mu(X_s,s)Y_s + \frac{1}{2}\sigma^2(X_s,s)Y_s\right]ds + \int_0^t \sigma(X_s,s)Y_s dB_s,$$

$$dY_t = Y_t\left(\mu(X_t,t) + \frac{1}{2}\sigma^2(X_t,t)\right)dt + Y_t\sigma(X_t,t)dB_t.$$

By applying Ito formula in Equation (12) again for some twice differentiable function g that $g'' \neq 0$, then

$$dg(Y_t) = \left[g'(Y_t)Y_t\left(\mu(X_t,t) + \frac{1}{2}\sigma^2(X_t,t) \right) + \frac{1}{2}g''(Y_t)Y_t^2\sigma^2(X_t,t) \right]dt \\ + g'(Y_t)Y_t\sigma(X_t,t)dB_t.$$

If the Milstein approximation scheme is applied to Y_t and $g(Y_t)$ then the following is produced:

$$Y_{t+\Delta t} \approx Y_t + Y_t\left(\mu(X_t,t) + \frac{1}{2}\sigma^2(X_t,t) \right)\Delta t + Y_t\sigma(X_t,t)\Delta B_t \\ + \frac{1}{2}\left\{ Y_t\sigma^2(X_t,t) + Y_t^2\frac{\partial\sigma(X_t,t)}{\partial Y}\sigma(X_t,t) \right\}\left((\Delta B_t)^2 - \Delta t \right) \tag{13}$$

and

$$g(Y_{t+\Delta t}) \approx g(Y_t) + g'(Y_t)Y_t\sigma(X_t,t)\Delta B_t \\ + \left[g'(Y_t)Y_t\left(\mu(X_t,t) + \frac{1}{2}\sigma^2(X_t,t) \right) + \frac{1}{2}g''(Y_t)Y_t^2\sigma^2(X_t,t) \right]\Delta t \\ + \frac{1}{2}\left(\frac{\partial g'(Y_t)}{\partial g(Y_t)}g'(Y_t)Y_t\sigma^2(X_t,t) \right. \\ + (g'(Y_t))^2Y_t\frac{\partial Y_t}{\partial g(Y_t)}\sigma^2(X_t,t) \\ \left. + (g'(Y_t))^2Y_t^2\frac{\partial\sigma(X_t,t)}{\partial g(Y_t)}\sigma(X_t,t) \right)\left((\Delta B_t)^2 - \Delta t \right). \tag{14}$$

Notice that

$$\frac{\partial g'(Y_t)}{\partial g(Y_t)}g'(Y_t) = \frac{\partial g'(Y_t)}{\partial g(Y_t)}\frac{\partial g(Y_t)}{\partial Y_t} = g''(Y_t),$$

$$(g'(Y_t))^2 \frac{\partial Y_t}{\partial g(Y_t)} = \frac{\partial g(Y_t)}{\partial Y_t}\frac{\partial g(Y_t)}{\partial Y_t}\frac{\partial Y_t}{\partial g(Y_t)} = g'(Y_t),$$

$$(g'(Y_t))^2 \frac{\partial \sigma(X_t,t)}{\partial g(Y_t)} = \frac{\partial g(Y_t)}{\partial Y_t}\frac{\partial g(Y_t)}{\partial Y_t}\frac{\partial \sigma(X_t,t)}{\partial g(Y_t)} = g'(Y_t)\frac{\partial \sigma(X_t,t)}{\partial Y_t}.$$

Then $g(Y_{t+\Delta t})$ becomes

$$\begin{aligned} g(Y_{t+\Delta t}) \approx\ & g(Y_t) + g'(Y_t)Y_t\sigma(X_t,t)\Delta B_t \\ & + \left[g'(Y_t)Y_t\left(\mu(X_t,t) + \frac{1}{2}\sigma^2(X_t,t)\right) + \frac{1}{2}g''(Y_t)Y_t^2\sigma^2(X_t,t)\right]\Delta t \\ & + \frac{1}{2}\Big[g''(Y_t)Y^2{}_t\sigma^2(X_t,t) + g'(Y_t)Y_t\,\sigma^2(X_t,t \\ & + g'(Y_t)Y_t^2\frac{\partial \sigma(X_t,t)}{\partial Y_t}\sigma(X_t,t)\Big]\left((\Delta B_t)^2 - \Delta t\right). \end{aligned}$$

From Approximation (13) we have

$$\begin{aligned} \frac{Y_{t+\Delta t} - Y_t}{Y_t} \approx\ & \left(\mu(X_t,t) + \frac{1}{2}\sigma^2(X_t,t)\right)\Delta t + \sigma(X_t,t)\Delta B_t \\ & + \frac{1}{2}\left\{ \sigma^2(X_t,t) + Y_t\ \frac{\partial \sigma(X_t,t)}{\partial Y}\sigma(X_t,t)\right\} ((\Delta B_t)^2\text{-}\Delta t)\ . \end{aligned}$$

From Approximation (14) we have

$$\begin{aligned} \frac{g(Y_{t+\Delta t}) - g(Y_t)}{g'(Y_t)Y_t} \approx\ & \left[\left(\mu(X_t,t) + \frac{1}{2}\sigma^2(X_t,t)\right) + \frac{1}{2}\frac{g''(Y_t)}{g'(Y_t)}Y_t\ \sigma^2(X_t,t)\right]\Delta t \\ & + \sigma(X_t,t)\Delta B_t \\ & + \frac{1}{2}\left(\frac{g''(Y_t)}{g'(Y_t)}Y_t\sigma^2(X_t,t) + \sigma^2(X_t,t) + Y_t\frac{\partial \sigma(X_t,t)}{\partial Y_t}\sigma(X_t,t)\right)\left((\Delta B_t)^2 - \Delta t\right). \end{aligned}$$

Therefore the following approximation can be written:

$$\frac{g(Y_{t+\Delta t})-g(Y_t)}{g'(Y_t)Y_t}-\frac{Y_{t+\Delta t}-Y_t}{Y_t}\approx\frac{1}{2}\frac{g''(Y_t)}{g'(Y_t)}Y_t\ \sigma^2(X_t,t)(\Delta B_t)^2 \tag{15}$$

If Approximation (15) is solved for $\sigma^2(X_t,t)$, the following result is produced:

$$\frac{g'(Y_t)2}{(\Delta B_t)^2 g''(Y_t)Y_t}\left[\frac{g(Y_{t+\Delta t})-g(Y_t)}{g'(Y_t)Y_t}-\frac{Y_{t+\Delta t}-Y_t}{Y_t}\right]\approx\sigma^2(X_t,t). \tag{16}$$

Before taking conditional expectation (Taking an expectation is about estimating the value of σ^2 which is a function of time and price. Since nothing is known about the deterministic continuous function σ^2, its values at discrete times are tried to be found and that is the reason the approximation is produced in the first place. Moreover, parameter estimation of the distribution of a random variable (σ) by common methods such as maximum likelihood method needs information about the distribution and there is no knowledge about this either. After the approximation, taking the expectation generates a point estimate of the unknown function. More information on the estimation of diffusion models can be found in Gourieroux and Jasiak, 2001) of the left hand side of the equation 16 given X_t, recall from probability theory that if the σ-field generated by the random variable Y is contained in F, then $E(Y|F)=Y$. In particular, if Y is a function of X, $\sigma(Y)\subset\sigma(X)$, thus $E(Y|F)=Y$.

Then,

$$E[g'(Y_t)|X_t]=g'(Y_t),$$
$$E[g''(Y_t)|X_t]=g''(Y_t),$$
$$E[Y_t|X_t]=Y_t.$$

Since by definition of Brownian motion $E[(\Delta B_t)^2|X_t]=\Delta t$, an estimate of $\sigma^2(t,X_t)$ is given by

$$V = \frac{g'(Y_t)2}{(\Delta t)g''(Y_t)Y_t}\left[\frac{g(Y_{t+\Delta t}) - g(Y_t)}{g'(Y_t)Y_t} - \frac{Y_{t+\Delta t} - Y_t}{Y_t}\right] \tag{17}$$

CEMY suppose that $g=y^{(1+\alpha)}$ and use the Euler approximation before taking expectations regarding the mean and variance of V_t in order to show that the estimate constructed is the minimum variance unbiased estimator. However, Pastorello (1996) show that these results can be improved without the use of the less precise Euler approximation. Therefore, the approach adopted by Pasterollo is followed in this thesis as well.

Let's substitute two Milstein approximations of $g(Y_{t+\Delta t})$ and $Y_{t+\Delta t}$ from Approximation (15) in Equation (17), then

$$V_t \cong \frac{g'(Y_t)2}{(\Delta t)g''(Y_t)Y_t}\left[\frac{1}{2}\frac{g''(Y_t)}{g'(Y_t)}Y_t\,\sigma^2(t,X_t)(\Delta B_t)^2\right] = \tilde{V}_t = \frac{\sigma^2(t,X_t)(\Delta B_t)^2}{(\Delta t)} \tag{18}$$

An interesting feature of Equation (18) is that it holds independently of the function g chosen. To strengthen the results, let's suppose that h and z are twice continuously differentiable functions and

$$h : \Re \to \Re \text{ where } h'(x) = \frac{\partial h(x)}{\partial x}, h''(x) = \frac{\partial h(x)}{\partial x}$$
$$z : \Re \to \Re \text{ where } z'(x) = \frac{\partial z(x)}{\partial x}, z''(x) = \frac{\partial z(x)}{\partial x}.$$

By help of Ito's lemma, the following equations can be written

$$\begin{aligned} h(X_{t+\Delta t}) = h(X_t) & \\ & + \int_t^{t+\Delta t} \sigma(X_s, s)h'(X_s)dB_s \\ & + \int_t^{t+\Delta t}\left[\mu(X_s, s)h'(X_s) + \frac{1}{2}\sigma^2(X_s, s)h''(X_s)\right]ds, \end{aligned} \tag{19}$$

$$z(X_{t+\Delta t}) = z(X_t) + \int_t^{t+\Delta t} \sigma(X_s, s) z'(X_s) dB_s + \int_t^{t+\Delta t} \left[\mu(X_s, s) z'(X_s) + \frac{1}{2}\sigma^2(X_s, s) z''(X_s) \right] ds \tag{20}$$

where $0 \le t \le t + \Delta t \le T$.

Now, let's apply Milstein approximation to both of the Equations (19) and (20), then

$$h(X_{t+\Delta_t}) \approx h(X_t) + \left[\mu(X_t, t) h'(X_t) + \frac{1}{2}\sigma^2(X_t, t) h''(X_t) \right] \Delta t + h'(X_t) \Delta B_t + \frac{1}{2} h'(X_t)\, \sigma(X_t, t) \left[h''(X_t) \sigma(X_t, t) + h'(X_t) \frac{\partial \sigma(X_t, t)}{\partial X_t} \right] ((\Delta B_t)^2 - \Delta t)$$

$$z(X_{t+\Delta_t}) \approx z(X_t) + \left[\mu(X_t, t) z'(X_t) + \frac{1}{2}\sigma^2(X_t, t) z''(X_t) \right] \Delta t + z'(X_t) \Delta B_t + \frac{1}{2} z'(X_t) \sigma(X_t, t) \left[z''(X_t) \sigma(X_t, t) + z'(X_t) \frac{\partial \sigma(X_t, t)}{\partial X_t} \right] ((\Delta B_t)^2 - \Delta t)$$

and

$$\frac{h(X_{t+\Delta t}) - h(X_t)}{h'(X_t)} \approx \left[\mu(t, X_t) + \frac{1}{2} \frac{\sigma^2(X_t, t) h''(X_t)}{h'(X_t)} \right] \Delta t + \sigma(X_t, t) \Delta B_t + \frac{1}{2} \sigma(X_t, t) h''(X_t) \sigma(X_t, t) \; ((\Delta B_t)^2 - \Delta t) + \frac{1}{2} \sigma(X_t, t) h'(X_t) \frac{\partial \sigma(X_t, t)}{\partial X_t} ((\Delta B_t)^2 - \Delta t) \tag{21}$$

$$\frac{z(X_{t+\Delta t})-z(X_t)}{z'(X_t)} \approx \left[\mu(X_t,t)+\frac{1}{2}\frac{\sigma^2(X_t,t)z''(X_t)}{z'(X_t)}\right]\Delta t + \sigma(X_t,t)\Delta B_t$$
$$+\frac{1}{2}\sigma(t,X_t)z''(X_t)\sigma(X_t,t)((\Delta B_t)^2-\Delta t) \quad (22)$$
$$+\frac{1}{2}\sigma(t,X_t)z'(X_t)\frac{\partial\sigma(X_t,t)}{\partial X_t}((\Delta B_t)^2-\Delta t)$$

Let's subtract Approximation (22) from Approximation (21), then

$$\frac{h(X_{t+\Delta t})-h(X_t)}{h'(X_t)}-\frac{z(X_{t+\Delta t})-z(X_t)}{z'(X_t)} \approx \left(\mu(X_t,t)+\frac{1}{2}\frac{\sigma^2(X_t,t)h''(X_t)}{h'(X_t)}\right)\Delta t$$
$$-\left(\mu(X_t,t)+\frac{1}{2}\frac{\sigma^2(X_t,t)z''(X_t)}{z'(X_t)}\right)\Delta t$$
$$+\sigma(X_t,t)\Delta B_t-\sigma(X_t,t)\Delta B_t$$
$$+\frac{1}{2}\sigma(X_t,t)h''(X_t)\sigma(t,X_t)((\Delta B_t)^2-\Delta t)$$
$$+\frac{1}{2}\sigma(X_t,t)h'(X_t)\frac{\partial\sigma(X_t,t)}{\partial X_t}((\Delta B_t)^2-\Delta t)$$
$$-\frac{1}{2}\sigma(X_t,t)z''(X_t)\sigma(X_t,t)((\Delta B_t)^2-\Delta t)$$
$$-\frac{1}{2}\sigma(X_t,t)z'(X_t)\frac{\partial\sigma(X_t,t)}{\partial X_t}((\Delta B_t)^2-\Delta t)$$

$$\frac{h(X_{t+\Delta t})-h(X_t)}{h'(X_t)}-\frac{z(X_{t+\Delta t})-z(X_t)}{z'(X_t)} \approx \frac{\Delta t}{2}\sigma^2(X_t,t)\left[\frac{h''(X_t)}{h'(X_t)}-\frac{z''(X_t)}{z'(X_t)}\right]$$
$$+\frac{1}{2}\sigma(X_t,t)h''(X_t)\sigma(t,X_t)((\Delta B_t)^2-\Delta t)$$
$$+\frac{1}{2}\sigma(X_t,t)h'(X_t)\frac{\partial\sigma(X_t,t)}{\partial X_t}((\Delta B_t)^2-\Delta t)$$
$$-\frac{1}{2}\sigma(X_t,t)z''(X_t)\sigma(X_t,t)((\Delta B_t)^2-\Delta t)$$
$$-\frac{1}{2}\sigma(X_t,t)z'(X_t)\frac{\partial\sigma(X_t,t)}{\partial X_t}((\Delta B_t)^2-\Delta t).$$

Recall that if the σ-field generated by the random variable Y is contained in F, then $E(Y|F) = Y$. By this token,

$$
\begin{aligned}
&E\left[h(X_{t+\Delta t})|X_t\right] = h(X_{t+\Delta t}),\\
&E\left[h(X_t)|X_t\right] = h(X_t),\\
&E\left[h'(X_t)|X_t\right] = h'(X_t),\\
&E\left[z(X_{t+\Delta t})|X_t\right] = z(X_{t+\Delta t}),\\
&E\left[z(X_t)|X_t\right] = z(X_t),\\
&E\left[zh(X_t)|X_t\right] = z(X_t).
\end{aligned}
$$

Since the definition of Brownian motion implies $E\left[(\Delta B_t)^2|X_t\right] = \Delta t$, if we take conditional expectation of difference of Approximations (22) and (21), we can conclude that a general V_t given as follows is an estimate of σ^2 (provided that the first denominator is nonzero):

$$
V_t^{hz} = \frac{2}{\Delta t\left[\frac{h''(X_t)}{h'(X_t)} - \frac{z''(X_t)}{z'(X_t)}\right]}\left[\frac{h(X_{t+\Delta t}) - h(X_t)}{h'(X_t)} - \frac{z(X_{t+\Delta t}) - z(X_t)}{z'(X_t)}\right] \qquad (23)
$$

Notice that V_t^{hz} is approximated to the same order by $\tilde{V}_t$, and hence it has (approximately) the same properties (Pastorello, 1996). Pastorello used the general framework provided by Kloeden and Platen (1992) to derive the strong Ito-Taylor approximations of a desired order γ with the aim to find the difference between the two estimators in Equation (23). However, since supplementary terms in a strong Ito-Taylor approximation scheme with an order higher than 1 depend on unknown values of functions μ and σ, this attempt to improve the estimator given in Equation (23) has failed (Pastorello, 1996). Since the derivation of strong Ito-Taylor approximations of a desired order is out of the scope of this thesis, the proof regarding Pastorello's (1996) findings is not provided here. Therefore, let's continue with a pair of h and z such that their choice does not depend on X_t itself.

Let

$$h : \Re \to \Re \text{ and } h(x) = e^{(\beta+\alpha)x}$$
$$z : \Re \to \Re \text{ where } z(x) = e^{(\beta)x}$$

where β and α are the parameters to be defined. If we define Z_t as follows:

$$Z_t = X_{t+\Delta t} - X_t$$

then

$$V_t^{\beta\alpha} = \frac{2}{\Delta t\left[\dfrac{(\beta+\alpha)^2 e^{(\beta+\alpha)X_t}}{(\beta+\alpha)e^{(\beta+\alpha)X_t}} - \dfrac{\beta^2 e^{\beta X_t}}{\beta e^{\beta X_t}}\right]}\left[\frac{e^{(\beta+\alpha)X_{t+\Delta_t}} - e^{(\beta+\alpha)X_t}}{(\beta+\alpha)e^{(\beta+\alpha)X_t}} - \frac{e^{\beta X_{t+\Delta_t}} - e^{\beta X_t}}{\beta e^{\beta X_t}}\right]$$
$$= \frac{2}{\alpha\Delta t}\left[\frac{e^{(\beta+\alpha)Z_t} - 1}{(\beta+\alpha)} - \frac{e^{\beta Z_t} - 1}{\beta}\right]$$

Before taking conditional expectation of the above estimate, let's focus on the expectation of exp(λZ) for an $N(0,1)$ random variable Z

$$E(e^{\lambda Z}) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{\lambda z} e^{-z^2/2} dz = e^{\lambda^2/2}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{-(z-\lambda)/2} dz = e^{\lambda^2/2}, \qquad \lambda \in \mathrm{R}$$

Moreover, the Brownian motion is 0.5 self-similar, that is:

$$T^{1/2}B_{t_1},\ldots,T^{1/2}B_{t_n}) \underline{\underline{d}} (B_{Tt_1},\ldots,B_{Tt_n})$$

for every $T>0$, any choice of $t_i \geq 0, i = 1,\ldots,n$ and $n \geq 1$. Then it follows immediately that

$$E(X_t) = e^{\mu t} E(e^{\sigma B_t}) = e^{\mu t} E(e^{\sigma t^{1/2} B_1}) = e^{(\mu + 0.5\sigma^2)t} \ .$$

Therefore,

$$\begin{aligned} e^{(\beta+\alpha)Z_t} &= \mathrm{e}^{(\beta+\alpha)\mu\Delta t + (\beta+\alpha)\sigma\Delta B_t} \\ \mathrm{E}(e^{(\beta+\alpha)Z_t}) &= \mathrm{e}^{(\beta+\alpha)\mu\Delta t + 0.5(\beta+\alpha)^2\sigma^2\Delta t}. \end{aligned} \tag{24}$$

Then the conditional expectation and variance of the $V_t^{\beta\alpha}$ can be written as follows:

$$E(V_t^{\beta\alpha} | X_t) = \frac{2}{\alpha\Delta t}\left[\frac{e^{(\beta+\alpha)\mu\Delta t + 0.5(\beta+\alpha)^2\sigma^2\Delta t} - 1}{(\beta+\alpha)} - \frac{e^{\beta\mu\Delta t + 0.5\beta^2\sigma^2\Delta t} - 1}{\beta}\right],$$

$$\begin{aligned} Var(V_t^{\beta\alpha} | X_t) &= E((V_t^{\beta\alpha})^2 | X_t) - (E(V_t^{\beta\alpha} | X_t))^2 \\ &= \frac{4}{(\Delta t)^2\alpha^2\beta^2(\beta+\alpha)^2}\Big[\beta^2 e^{2(\beta+\alpha)\mu\Delta t + (\beta+\alpha)^2\sigma^2\Delta t}(e^{(\beta+\alpha)^2\sigma^2\Delta t} - 1) \\ &\quad + (\beta+\alpha)^2 e^{2\beta\mu\Delta t + \beta^2\sigma^2\Delta t}(e^{\beta^2\sigma^2\Delta t} - 1) \\ &\quad - 2\beta(\beta+\alpha)e^{(2\beta+\alpha)\mu\Delta t + 0.5(\beta^{2+}(\beta+\alpha)^2)\sigma^2\Delta t}(e^{\beta(\beta+\alpha)\sigma^2\Delta t} - 1)\Big] \end{aligned}$$

Let

$$
\begin{aligned}
A &= (\beta+\alpha)\mu \\
B &= (\beta+\alpha)^2\sigma^2 \\
C &= \beta\mu \\
D &= \beta^2\sigma^2 \\
E &= (2\beta+\alpha)\mu \\
F &= (\beta^2+(\beta+\alpha)^2)\sigma^2 \\
G &= \beta(\beta+\alpha)\sigma^2 \\
I &= e^{(2A+B)\Delta t} \\
II &= e^{B\Delta t}\text{-1} \\
III &= e^{(2C+D)\Delta tt} \\
IV &= e^{D\Delta t}\text{-1} \\
V &= e^{(C+\frac{D}{2})\Delta t} \\
VI &= e^{G}\text{-1}
\end{aligned}
$$

Therefore, $Var\left[V_t^{\beta\alpha} \middle| X_t\right]$ can be rewritten as follows:

$$
\begin{aligned}
Var(V_t^{\beta\alpha}|X_t) = \frac{4}{(\Delta t)^2\alpha^2\beta^2(\beta+\alpha)^2}\Big[&\beta^2 e^{2A\Delta t+B\Delta t}(e^{B\Delta t}-1) \\
&+(\beta+\alpha)^2 e^{2C\Delta t+D\Delta t}(e^{D\Delta t}-1) \\
&-2\beta(\beta+\alpha)e^{E\Delta t+0.5F\Delta t}(e^{G\Delta t}-1)\Big]
\end{aligned}
\tag{25}
$$

Let's define the expression in square brackets as M, which is a function of Δ*t*. Then, a Taylor expansion of order 3 (around Δ*t*=0) of the expression in squared brackets in Equation (25) is:

$$
M(\Delta t) = M(0) + \frac{\partial M}{\partial \Delta t}\Delta t + \frac{\partial^2 M}{\partial \Delta t^2}\frac{\Delta t^2}{2} + \frac{\partial^3 M}{\partial \Delta t^3}\frac{\Delta t^3}{6} + o(\Delta t) \tag{26}
$$

If Δ*t* is zero then M(Δ*t*), II, IV, VI equal to zero and I, III and V equal to one. Now, the derivation of the remaining expressions in Equation (26) is possible.

$$\frac{\partial M}{\partial \Delta t} = \beta^2 I'II + \beta^2 II'I + (\beta+\alpha)^2 III'IV + (\beta+\alpha)^2 IV'III$$
$$- 2\beta(\beta+\alpha)V'VI - 2\beta(\beta+\alpha)VI'V.$$

$$\frac{\partial^2 M}{\partial \Delta t^2} = \beta^2 I''II + \beta^2 II'I' + \beta^2 II''I + \beta^2 II'I' + (\beta+\alpha)^2 III''IV$$
$$+ (\beta+\alpha)^2 IV'III' + (\beta+\alpha)^2 IV''III + (\beta+\alpha)^2 IV'III' - 2\beta(\beta+\alpha)V'VI'$$
$$- 2\beta(\beta+\alpha)VI''V - 2\beta(\beta+\alpha)V'VI' - 2\beta(\beta+\alpha)V''VI.$$

$$\frac{\partial^3 M}{\partial \Delta t^3} = \beta^2 \left(I'''II + II'I'' + II''I' + II'I'' + II'''I + II''I' + II''I' + II'I''\right)$$
$$+ (\beta+\alpha)^2 \left(III'''IV + IV'III'' + IV'III'' + IV''III' + IV'''III\right)$$
$$+ (\beta+\alpha)^2 (III'IV'' + IV''III' + IV'III'')$$
$$- 2\beta(\beta+\alpha)''\left(V'''VI + VI'V + V''VI' + VI''V' + VI'''V + V'VI'' + V'VI'' + V''VI''\right)$$

If Δt is zero then,

$$\frac{\partial M}{\partial \Delta t}(\Delta t = 0) = \left[\beta^2 B + (\beta+\alpha)^2 D - 2\beta(\beta+\alpha)G\right] = 0$$

$$\frac{\partial^2 M}{\partial \Delta t^2}(\Delta t = 0) = 2\beta^2 II'I' + \beta^2 II''I + 2(\beta+\alpha)^2 IV'III' + (\beta+\alpha)^2 IV''III$$

$$- 4\beta(\beta+\alpha)V'VI' - 2\beta(\beta+\alpha)VI''V$$

$$= 2\beta^2 B(2A+B) + 2(\beta+\alpha)^2 D(2C+D) + (\beta+\alpha)^2 D^2 + \beta^2 B^2$$

$$- 2\beta(\beta+\alpha)(2E+F)G - 2\beta(\beta+\alpha)G^2$$

$$= \left[\beta^2 B(4A+3B) + (\beta+\alpha)^2 D(4C+3D) - 2\beta(\beta+\alpha)(2E+F+G)G\right]$$

$$= \beta^2(\beta+\alpha)^2\sigma^2\left(4(\beta+\alpha)\mu + 3(\beta+\alpha)^2\sigma^2\right) + (\beta+\alpha)^2\beta^2\sigma^2\left(4\beta\mu + 3\beta^2\sigma^2\right)$$

$$- 2\beta^2(\beta+\alpha)^2\sigma^2\left((4\beta+2\alpha)\mu + \beta^2\sigma^2 + (\beta+\alpha)^2\sigma^2 + \beta(\beta+\alpha)\sigma^2\right)$$

$$= \beta^2(\beta+\alpha)^2\sigma^2\left[\left(4(\beta+\alpha)\mu + 3(\beta+\alpha)^2\sigma^2\right)\right.$$

$$+ \left(4\beta\mu + 3\beta^2\sigma^2\right) - 2(4\beta+2\alpha)\mu$$

$$\left. - 4(\beta+\alpha)^2\sigma^2 - 4\beta^2\sigma^2 - 4\beta\alpha\sigma^2 - 4\beta^2\sigma^2 \right]$$

$$= \beta^2(\beta+\alpha)^2\sigma^2\left[8\beta\mu + 4\alpha\mu + (\beta+\alpha)^2\sigma^2 + \beta^2\sigma^2\right.$$

$$\left. - 8\mu\beta - 4\alpha\mu - 2\beta^2\sigma^2 - 2\beta\alpha\sigma^2\right]$$

$$= \beta^2(\beta+\alpha)^2\sigma^4\alpha^2$$

$$\frac{\partial^3 M}{\partial \Delta t^3}(\Delta t = 0) = \beta^2\left(3II'I'' + 3II''I' + II'''\right) + (\beta+\alpha)^2\left(3IV'III'' + 3IV'III'' + II'''\right)$$

$$-2\beta(\beta+\alpha)''\left(3V''VI' + 3VI''V' + VI'''\right)$$

$$= \beta^2\left(3B(2A+B)^2 + 3B^2(2A+B) + B^3\right)$$

$$+(\beta+\alpha)^2\left(3D(2C+D)^2 + 3D^2(2C+D) + D^3\right)$$

$$-2\beta(\beta+\alpha)\left(3G\left(E+\frac{F}{2}\right)^2 + 3G^2\left(E+\frac{F}{2}\right) + G^3\right)$$

$$= \beta^2\left(3B(2A+B)^2 + 3B^2(2A+B) + B^3\right)$$

$$+(\beta+\alpha)^2\left(3D(2C+D)^2 + 3D^2(2C+D) + D^3\right)$$

$$-2\beta(\beta+\alpha)\left(3G\left(E+\frac{F}{2}\right)^2 + 3G^2\left(E+\frac{F}{2}\right) + G^3\right)$$

$$= \beta^2 B\left(3(2A+B)^2 + 3B(2A+B) + B^2\right)$$

$$+(\beta+\alpha)^2 D\left(3(2C+D)^2 + 3D(2C+D) + D^2\right)$$

$$-2\beta(\beta+\alpha)G\left(3\left(E+\frac{F}{2}\right)^2 + 3G\left(E+\frac{F}{2}\right) + G^2\right)$$

$$= \beta^2(\beta+\alpha)^2\sigma^2\left[-2\left(3\left(E+\frac{F}{2}\right)^2 + 3G\left(E+\frac{F}{2}\right) + G^2\right)\right]$$

$$+\beta^2(\beta+\alpha)^2\sigma^2\left(3(2C+D)^2 + 3D(2C+D) + D^2\right)$$

$$+\beta^2(\beta+\alpha)^2\sigma^2\left(3(2A+B)^2 + 3B(2A+B) + B^2\right)$$

$$= \beta^2(\beta+\alpha)^2\sigma^2\Big[12A^2 + 18AB + 7B^2 + 12C^2 + 18CD + 7D^2$$

$$-6E^2 - 6EF - \frac{3}{2}F^2 - 6EG - 3FG - 2G^2\Big]$$

where

$$12A^2 + 12C^2 + 18AB + 18CD + 7B^2 + 7D^2 = 12\mu^2\left(2\beta^2 + \alpha^2 + 2\beta\alpha\right)$$

$$+18\mu\sigma^2\left(2\beta^3 + 3\beta^2\alpha + 3\alpha^2\beta + \alpha^3\right)$$

$$+7\sigma^4\left(2\beta^4 + 4\beta^3\alpha + 6\beta^2\alpha^2 + 4\beta\alpha^3 + \alpha^4\right)$$

and

$$6E^2 + 6EF + \frac{3}{2}F^2 + 6EG + 3FG + 2G^2 = 6\mu^2\left(4\beta^2 + \alpha^2 + 4\beta\alpha\right)$$
$$+ 6\mu\sigma^2\left(6\beta^3 + 9\beta^2\alpha + 5\beta\alpha^2 + \alpha^3\right)$$
$$+ 14\sigma^4\beta^4 + 28\sigma^4\beta^3\alpha + 23\sigma^4\beta^2\alpha^2$$
$$+ 9\sigma^4\beta\alpha^3 + \frac{3}{2}\alpha^4\sigma^4$$

Therefore,

$$\frac{\partial^3 M}{\partial \Delta t^3}(\Delta t = 0) = \frac{\Delta t^3}{6}\beta^2\sigma^2(\beta+\alpha)^2\Big[6\mu^2\left(4\beta^2 + 2\alpha^2 + 4\beta\alpha - 4\beta^2 - \alpha^2 - 4\beta\alpha\right)$$
$$+ 6\mu\sigma^2\left(6\beta^3 + 9\beta^2\alpha + 9\alpha^2\beta + 3\alpha^3 - 6\beta^3 - 9\beta^2\alpha - 5\beta\alpha^2 - \alpha^3\right)$$
$$+ \sigma^4\left(14\beta^4 + 28\beta^3\alpha + 42\beta^2\alpha^2 + 28\beta\alpha^3 + 7\alpha^4\right)$$
$$+ \sigma^4\left(-14\beta^4 - 28\beta^3\alpha - 23\beta^2\alpha^2 - 9\beta\alpha^3 - \frac{3}{2}\alpha^4\right)\Big]$$
$$= \frac{\Delta t^3}{6}\beta^2\sigma^2(\beta+\alpha)^2\Big[6\mu^2\left(\alpha^2\right) + 6\mu\sigma^2\left(4\alpha^2\beta + 2\alpha^3\right)$$
$$+ \sigma^4\left(19\beta^2\alpha^2 + 19\beta\alpha^3 + \frac{11}{2}\alpha^4\right)\Big]$$

As a conclusion, Taylor expansion of M of order 3 around Δt=0 is as follows:

$$M(\Delta t) = \frac{\Delta t^2}{2}\beta^2\sigma^4(\beta+\alpha)^2\alpha^2$$
$$+ \frac{\Delta t^3}{6}\beta^2\sigma^2(\beta+\alpha)^2\Big[6\mu^2\left(\alpha^2\right) + 6\mu\sigma^2\left(4\alpha^2\beta + 2\alpha^3\right)$$
$$+ \sigma^4\left(19\beta^2\alpha^2 + 19\beta\alpha^3 + \frac{11}{2}\alpha^4\right)\Big] + o(\Delta t).$$

Then we have

$$Var(V_t^{\beta\alpha}|X_t) = \frac{4}{(\Delta t)^2\alpha^2\beta^2(\beta+\alpha)^2}\left[\frac{\Delta t^3}{6}\beta^2\sigma^2(\beta+\alpha)\left(6\mu^2(\alpha^2)+6\mu\sigma^2\begin{pmatrix}4\alpha^2\beta\\+2\alpha^3\end{pmatrix}\right)\right]$$

$$+\frac{4}{(\Delta t)^2\alpha^2\beta^2(\beta+\alpha)^2}\left[\frac{\Delta t^3}{6}\beta^2\sigma^2(\beta+\alpha)\left(\sigma^4\left(19\beta^2\alpha^2+19\beta\alpha^3+\frac{11}{2}\alpha^4\right)\right)\right]$$

$$+\frac{4}{(\Delta t)^2\alpha^2\beta^2(\beta+\alpha)^2}\frac{\Delta t^2}{2}\beta^2\sigma^4(\beta+\alpha)^2\alpha^2+o(\Delta t)$$

$$=2\sigma^4+\frac{3}{2}\Delta t\left[6\mu^2+\sigma^4\left(19\beta^2+19\beta\alpha+\frac{11}{2}\alpha^2\right)+\mu\sigma^2(24\beta+12\alpha)\right]+o(\Delta t).$$

$$Var(V_t^{\beta\alpha}|X_t) \cong 2\sigma^4+\frac{3}{2}\Delta t\left\{6\mu^2+\sigma^4\left(19\beta^2+19\beta\alpha+\frac{11}{2}\alpha^2\right)+\mu\sigma^2(24\beta+12\alpha)\right\}$$

Minimization of the $Var\left[V_t^{\beta\alpha}|X_t\right]$ requires that derivatives with respect to every variable in the Taylor expansion be equal to 0 (first order conditions). Derivatives of $Var\left[V_t^{\beta\alpha}|X_t\right]$ with respect to β and α are given as follows:

$$\frac{\partial Var(V_t^{\beta\alpha}|X_t)}{\partial\beta}=\frac{2}{3}\left\{(38\beta+19\alpha)\sigma^4+(24\mu)\sigma^2\right\}\Delta t \tag{27}$$

and

$$\frac{\partial Var(V_t^{\beta\alpha}|X_t)}{\partial\alpha}=\frac{2}{3}\left\{(19\beta+11\alpha)\sigma^4+(12\mu)\sigma^2\right\}\Delta t. \tag{28}$$

The solution of Equations (27) and (28) give α=0. However, for α=0, $V_t^{\beta\alpha}$ is not defined. Therefore, let's apply L'Hopital's rule and take the limit of $V_t^{\beta\alpha}$ when α approaches 0.

$$\lim_{\alpha\to 0} V_t^{\beta\alpha}=V_t^{\beta}=\frac{2}{\beta^2\Delta t}\left\{\beta Z_t e^{\beta Z_t}+1-e^{\beta Z_t}\right\}$$

By using the findings in Equation (24), the mean and variance of the V_t^{β} are found as follows

$$
\begin{aligned}
E\left[V_t^{\beta}\middle|X_t\right] &= E\left[\frac{2}{\beta^2\Delta t}\left\{\beta Z_t e^{\beta Z_t} + 1 - e^{\beta Z_t}\right\}\middle|X_t\right] \\
&= \frac{2}{\beta^2\Delta t}\left(1\text{-}E\left[e^{\beta Z_t}\middle|X_t\right] + E\left[\beta Z_t e^{\beta Z_t}\middle|X_t\right]\right) \\
&= \frac{2}{\beta^2\Delta t}\left(1 + e^{\beta\mu\Delta t + 0.5\beta^2\sigma^2\Delta t}\left(\beta\mu\Delta t + \beta^2\sigma^2\Delta t - 1\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
Var\left[V_t^{\beta}\middle|X_t\right] &= E((V_t^{\beta})^2\middle|X_t) - (E(V_t^{\beta}\middle|X_t))^2 \\
&= \frac{4}{\beta^4(\Delta t)^2}\begin{pmatrix} e^{2\beta\mu\Delta t + 2\beta^2\sigma^2\Delta t}\left[\left(\beta\mu\Delta t + 2\beta^2\sigma^2\Delta t - 1\right)^2 + \beta^2\sigma^2\Delta t\right] \\ - e^{2\beta\mu\Delta t + 2\beta^2\sigma^2\Delta t}\left(\beta\mu\Delta t + 2\beta^2\sigma^2\Delta t - 1\right)^2 \end{pmatrix}
\end{aligned}
$$

Let

$$
\begin{aligned}
&A = \beta\mu, \\
&B = \beta^2\sigma^2, \\
&C = \left((A + 2B)\Delta t - 1\right)^2 + B\Delta t, \\
&D = \left((A + B)\Delta t - 1\right)^2, \\
&K = e^{2\Delta t(A+B)}, \\
&L = e^{\Delta t(2A+B)} \\
&R = \begin{pmatrix} e^{2\beta\mu\Delta t + 2\beta^2\sigma^2\Delta t}\left[\left(\beta\mu\Delta t + 2\beta^2\sigma^2\Delta t - 1\right)^2 + \beta^2\sigma^2\Delta t\right] \\ - e^{2\beta\mu\Delta t + 2\beta^2\sigma^2\Delta t}\left(\beta\mu\Delta t + 2\beta^2\sigma^2\Delta t - 1\right)^2 \end{pmatrix} = KC\text{-}LD.
\end{aligned}
$$

The first and second derivatives of C and D with respect to Δt are as follows:

$$
\frac{\partial C}{\partial\Delta t} = -3B - 2A + 2\Delta t(A + 2B)^2, \frac{\partial^2 C}{\partial\Delta t^2} = 2(A + 2B)^2,
$$

$$\frac{\partial D}{\partial \Delta t} = -2B - 2A + 2\Delta t(A+B)^2, \frac{\partial^2 D}{\partial \Delta t^2} = 2(A+B)^2.$$

A Taylor expansion of order 3 (around Δt=0) of the expression R is

$$R(\Delta t) = R(0) + \frac{\partial R}{\partial \Delta t}\Delta t + \frac{\partial^2 R}{\partial \Delta t^2}\frac{\Delta t^2}{2} + \frac{\partial^3 R}{\partial \Delta t^3}\frac{\Delta t^3}{6} + o(\Delta t)$$

If Δt=0, then

$$\begin{aligned}
&R(\Delta t = 0) = 0,\\
&K = 1,\\
&C = 1,\\
&D = 1,\\
&L = 1,\\
&\frac{\partial C}{\partial \Delta t} = -3B - 2A,\\
&\frac{\partial^2 C}{\partial \Delta t^2} = 2(A+2B)^2,\\
&\frac{\partial D}{\partial \Delta t} = -2B - 2A,\\
&\frac{\partial^2 D}{\partial \Delta t^2} = 2(A+B)^2.
\end{aligned}$$

Now, let's take the first, second and third derivatives of R with respect to Δt at Δt=0:

$$\frac{\partial R}{\partial \Delta t} = 2(A+B)KC + KC' - (2A+B)LD - LD'$$

$$\frac{\partial R}{\partial \Delta t}(\Delta t = 0) = 2(A+B) + (-2A - 3B) - (2A+B) + 2(A+B) = 0$$

$$\frac{\partial^2 R}{\partial \Delta t^2} = 2(A+B)K'C + 2(A+B)KC' + K'C' + KC''$$
$$- (2A+B)L'D - (2A+B)LD' - L'D' - LD''$$
$$= 4(A+B)^2 KC + 2(A+B)KC' + K'C' + KC''$$
$$- (2A+B)^2 D - (2A+B)LD' - L'D' - LD''$$

$$\frac{\partial^2 R}{\partial \Delta t^2}(\Delta t = 0) = 4(A+B)^2 + 2(A+B)(-2A-3B) + 2(A+2B)^2$$
$$+ 2(A+B)(-2A-3B) - (2A+B)^2$$
$$+ 2(A+B)(2A+B) + 2(A+B)(2A+B) - 2(A+B)^2$$
$$= 4(A+B)^2 + 4(A+B)(-2A-3B) + 2(A+2B)^2$$
$$- (2A+B)^2 + 4(A+B)(2A+B) - 2(A+B)^2$$
$$= 4A^2 - 4B^2 - 4A^2 - B^2 - 4AB - 2A^2 - 4AB - 2B^2 + 2A^2 + 8AB + 8B^2$$
$$= B^2 .$$

$$\frac{\partial^3 R}{\partial \Delta t^3} = 4(A+B)^2 K'C + 4(A+B)^2 KC' + 2(A+B)K'C'$$
$$+ 2(A+B)KC'' + K'C'' + KC''' - (2A+B)^2 L'D$$
$$- (2A+B)^2 LD' - (2A+B)L'D' - (2A+B)LD''$$
$$- L'D'' - L''D' - L'D'' - LD''' + K''C' + K'C''$$
$$= 8(A+B)^3 C + 4(A+B)^2 K(-2A-3B) - LD'''$$
$$+ 4(A+B)^2 (-2A-3B) + 2(A+B)K2(A+2B)^2$$
$$+ 4(A+B)^2 (-2A-3B) + 2(A+B)2(A+2B)^2$$
$$+ 2(A+B)2(A+2B)^2 - (2A+B)2(A+B)^2$$
$$- (2A+B)^3 D + (2A+B)^2 L2(A+B)$$
$$+ (2A+B)^2 2(A+B) - (2A+B)L2(A+B)^2$$
$$+ (2A+B)^2 2(A+B) - (2A+B)2(A+B)^2 + KC'''$$

$$\frac{\partial^3 R}{\partial \Delta t^3}(\Delta t = 0) = 8(A+B)^3 + 4(A+B)^2(-2A-3B)$$

$$+4(A+B)^2(-2A-3B) - (2A+B)^3$$
$$+2(A+B)2(A+2B)^2 + (2A+B)^2 2(A+B)$$
$$+2(A+B)2(A+2B)^2 - (2A+B)2(A+B)^2$$
$$+(2A+B)^2 2(A+B) - (2A+B)2(A+B)^2$$
$$+(2A+B)^2 2(A+B) - (2A+B)2(A+B)^2$$
$$+2(A+B)2(A+2B)^2 + 4(A+B)^2(-2A-3B)$$
$$= 8(A+B)^3 + 12(A+B)^2(-2A-3B) + 12(A+B)(A+2B)^2$$
$$-(2A+B)^3 + 6(2A+B)^2(A+B) - 6(2A+B)(A+B)^2$$
$$= 2(A+B)^2(4(A+B) - 6(2A+3B) - 3(2A+B))$$
$$+(2A+B)^2(6(A+B) - 2A - B) - 12(A+B)(A+2B)^2$$
$$= 2(A+B)^2(-14A-17B) + (2A+B)^2(4A+5B)$$
$$-12(A+B)(A+2B)^2$$

Therefore the Taylor expansion of R of order 3 around Δt=0 is as follows:

$$R(\Delta t) = \frac{\Delta t^2}{2}\beta^4\sigma^4 + \frac{\Delta t^3}{6}\left[6\mu^2\beta^4\sigma^2 + 24\mu\sigma^4\beta^3 + 18\beta^6\sigma^6\right] + o(\Delta t).$$

Then,

$$Var\left[V_t^{\beta}\middle|X_t\right] = \frac{4}{\beta^4(\Delta t)^2}\left(\frac{\Delta t^2}{2}\beta^4\sigma^4 + \frac{\Delta t^3}{6}\left[6\mu^2\beta^4\sigma^2 + 24\mu\sigma^4\beta^3 + 18\beta^6\sigma^6\right] + o(\Delta t)\right)$$

$$Var\left[V_t^{\beta}\middle|X_t\right] \cong 2\sigma^4 + 4\sigma^2\Delta t\left(\mu^2 + 4\beta\mu\sigma^2 + 3\beta^2\sigma^4\right)$$

Since the estimator that is both unbiased (expectation of the estimator equals the parameter) and has the minimum variance (the variance of the estimator is minimum among the variance of all estimators) is needed, $Var\left[V_t^{\beta\alpha}\middle|X_t\right]$ needs to be minimized. The first order condition with respect to β is as follows:

$$\frac{\partial Var(V_t^{\beta\alpha} | X_t)}{\partial \beta} = 8\sigma^4 \left(2\mu + 3\beta\sigma^2\right)\Delta t = 0.$$

$Var\left[V_t^{\beta\alpha} | X_t\right]$ is minimized when β is equal to $-2\mu/3\sigma^2$. Then, $Var\left[V_t^{\beta\alpha} | X_t\right]$ can be rewritten as:

$$Var\left[V_t^{\beta} | X_t\right] \cong 2\sigma^4 - \frac{4}{3}\sigma^2\mu^2\Delta t.$$

Therefore, following Chang et al. (1998), the following formulas are employed to calculate the volatility of both an asset's and a portfolio's returns. These findings are also used to calculate the *AVOL*.

$$V_{i,t} = \frac{2}{\beta_i^2}\left[1 - e^{\beta_i (X_{i,t+1} - X_{i,t})} + \beta_i (X_{i,t+1} - X_{i,t}) e^{\beta_i (X_{i,t+1} - X_{i,t})}\right]$$

where $\beta_i = -2\mu_i / 3\sigma_i^2$, μ_t denotes the mean of daily returns of asset i over the sample period, σ_t denotes the standard deviation of daily returns of asset i over the sample period and X_t denotes the log of daily closing prices. *AVOL* at time t will be calculated as follows:

$$AVOL_t = \frac{1}{n}\sum_{i=1}^{n} V_{it}$$

PVOL is calculated using the same formula as *AVOL* with only the difference that $\ln(I_t)$ is substituted by X_t, where I_t denotes the closing prices of the ISE-National 30 index stocks (or the Non-ISE-National 100 index stocks)

3.3. THE MODEL

In order to test whether the introduction of stock index futures contracts change the underlying stock market volatility, a typical test that is performed comes in the following format:

$$PVOL_t = a_1 + a_{post} D_{post,t} + \varepsilon_t \qquad (29)$$

In Equation (29), $D_{post,t}$ equals 1 (0) after (before) the futures trading begins and the null hypothesis is that $a_{post} = 0$. Significantly positive (negative) estimates of a_{post} imply that futures trading induces higher (lower) spot market volatility (Chang et al, 1999). A major deficiency of this type of test is the absence of a control variable for the extraneous influences on the stock market volatility. Therefore, decomposition of portfolio volatility into its components is used in forming a single-factor return-generating model, namely the classical linear regression method, to differentiate the volatility impacts caused by futures trading and those caused by changes in broad economic factors. In their paper, Chang et al (1999) propose that the decomposition of spot portfolio volatility into the components of cross- sectional dispersion and average volatility of the portfolio's constituent securities can be used to control for broad market influences. It is assumed that, with the suppression of time subscripts, the return generating process can be represented as follows:

It is assumed that, with the suppression of time subscripts, the return generating process can be represented as

$$R_i = \alpha_i + \beta_i F_i + \varepsilon_i \tag{30}$$

In Equation (30), R_i is the realized return on security i, α_i is the expected return on security i, F is the realization of a zero-mean common factor which represents the broad market factor, β_i is the time-invariant factor loading of security i, and ε_i represents the effect of zero-mean firm-specific information, which is assumed to be independent of the effect of the broad market factors (Chang et al, 1999).

If W_i represents the weight of security i in a portfolio consisting of N securities, *AVOL*, *PVOL* and *E(CSD)* can be expressed, respectively, in terms of Equation (30) in the following manner:

$$\begin{aligned} AVOL &= \left[\sum_{i=1}^{N} W_i b_i^2\right]\sigma_F^2 + \sum_{i=1}^{N} W_i \sigma_{\varepsilon i}^2, \\ PVOL &= \left[\sum_{i=1}^{N} W_i^2 b_i^2 + \sum_{j=1}^{N}\ \sum_{i=1}^{N} 2W_i W_j b_i b_j\right]\sigma_F^2 + \sum_{i=1}^{N} W_i^{\,2}\sigma_{\varepsilon i}^2, \\ E(CSD) &= \sum_{i=1}^{N} W_i(\alpha_i - \alpha_p)^2 + \left[\sum_{i=1}^{N} W_i (b_i - b_p)^2\right]\sigma_F^2 + \sum_{i=1}^{N} W_i \sigma_{\varepsilon i}^2 \end{aligned} \tag{31}$$

In Equation (31), b_p is the weighted average of factor loadings of all securities in the portfolio. The last terms in Equation (31) show the contributions of firm-specific information to the respective volatility measures (Chang et al, 1999). They are identical for *AVOL* and *E(CSD)*, but smaller for *PVOL* (Chang et al, 1999). The first terms for *AVOL* and *PVOL* and the second term for *E(CSD)* in Equation (31) represent the contributions of the broad market factor to the volatility measures (Chang et al, 1999). With the assumption that the loadings (b_i's) and weights (W_i's) in Equation (31) are time-invariant (an assumption required by the classical linear regression model), the contributions of the broad market factor are the products of σ_F^2 and the three different constants (Chang et al, 1999). Partials of each expression with respect to σ_F^2 yield:

$$\frac{\partial AVOL}{\partial \sigma_F^2} = \left[\sum_{i=1}^{N} W_i b_i^2\right] > 0,$$
$$\frac{\partial PVOL}{\partial \sigma_F^2} = \left[\sum_{i=1}^{N} W_i^2 b_i^2 + \sum_{j=1}^{N} \sum_{i=1}^{N} 2W_i W_j b_i b_j\right] = b_p^2 > 0,$$
$$\frac{\partial E(CSD)}{\partial \sigma_F^2} = \left[\sum_{i=1}^{N} W_i (b_i - b_p)^2\right] = \left[\sum_{i=1}^{N} W_i b_i^2\right] - b_p^2 > 0.$$

First, ceteris paribus, increases in the volatility of the broad market factor increases each of the volatility measures (Chang et al, 1999). Thus, conclusions drawn from models based on Equation (29) that do not control for changes in broad market influences can be seriously defective (Chang et al, 1999). Second, the increments for the volatility measures are the respective bracketed terms in Equation (31) suggesting that, ceteris paribus, if σ_F^2 increases, the increments in the volatility measures are proportional to the measures' systematic components (Chang et al, 1999). If, for example, the bracketed terms for *AVOL* and *PVOL* equal 2 and 1 respectively, then the exposure to systematic influences of *AVOL* are twice as large as the exposure of *PVOL* before and after a volatility shift in the broad economic factor (Chang et al, 1999). Therefore, extraneous shifts in broad market factors can be controlled for by including *AVOL* as a regressor in Equation (29) (Chang et al, 1999).

The ex-post realization of the volatility measures is open to disturbances in both systematic and unsystematic risk, and probably also to random interventions by arbitragers (Chang et al, 1999). Thus, the effects of arbitrage-motivated program trading (program trading is the execution on a stock market of a large number of simultaneous buy or sell orders and is generally triggered by a computer program that detects an arbitrage opportunity or suggests some other reason for quickly establishing a large portfolio of stock) on the structural relations between the volatility measures are contrasted with the effects induced by changes in broad economic factors (Chang et al, 1999). As program trading is typically independent of information specific to individual securities, prices of all securities in a basket tend to move simultaneously in the same way (Chang et al, 1999). The arbitrage factor A can be assumed to affect the return-generating process for securities in the basket as follows:

$$R_i = \alpha_i + \beta_i F_i + A + \varepsilon_i$$

In order to capture the nature of arbitrage-motivated trading, Chang et al. (1999) also assume that all securities in the basket have the same loading relative to the arbitrage force. Since index arbitrage appears randomly, factor A's presence in the return-generating process is also random. Thus, the uncertainty of the arbitrage factor is an additional source of return volatility for securities in the basket (Chang et al., 1999). This implies the following:

$$AVOL = \left[\sum_{i=1}^{N} W_i b_i^2\right]\sigma_F^2 + \sigma_A^2 + \sum_{i=1}^{N} W_i \sigma_{\varepsilon i}^2,$$

$$PVOL = \left[\sum_{i=1}^{N} W_i^2 b_i^2 + \sum_{j=1}^{N}\ \sum_{i=1}^{N} 2W_i W_j b_i b_j\right]\sigma_F^2 + \sigma_A^2 + \sum_{i=1}^{N} W_i^2 \sigma_{\varepsilon i}^2,$$

$$E(CSD) = \sum_{i=1}^{N} W_i(\alpha_i - \alpha_p)^2 + \left[\sum_{i=1}^{N} W_i (b_i - b_p)^2\right]\sigma_F^2 + (0)\sigma_A^2 + \sum_{i=1}^{N} W_i \sigma_{\varepsilon i}^2$$

The effects of an increase in σ_A^2 on *AVOL*, *PVOL* and *E(CSD)* differ substantially from those due to an increase σ_F^2 in two prominent ways. First, since uncertainty caused by arbitrage effects cannot be diversified, an increase in σ_A^2 is not diversifiable (Chang et al., 1999). That is,

$$\frac{\partial AVOL}{\partial \sigma_A^2} = \frac{\partial PVOL}{\partial \sigma_A^2} = 1 > 0 \quad \text{whereas} \quad \frac{\partial AVOL}{\partial \sigma_F^2} > \frac{\partial PVOL}{\partial \sigma_F^2} > 0 \qquad (32)$$

Second, though an increase in σ_F^2 has a positive effect on *E(CSD)*, an increase in σ_A^2 has no effect (Chang et al., 1999). Hence,

$$\frac{\partial E(CSD)}{\partial \sigma_A^2} = 0.$$

These comparative statics leads us to the fact that for a single-factor model changes in *PVOL* and *E(CSD)* due either to arbitrage or to broad market factors equals the change in *AVOL* (Chang et al., 1999). This result stems directly from the decomposition of *PVOL*. At the same time, diversifiable broad economic disturbances trigger larger (but proportional) shifts in *AVOL* than in *PVOL*; whereas, the nondiversifiable effects of program trading trigger identical shifts in both volatility measures (Chang et al., 1999). When the identical effects of program trading are summed with the proportional effects of broad economic disturbances, *PVOL* should increase proportionately more than *AVOL* because *PVOL* is smaller than *AVOL* due to diversification (Chang et al., 1999). Remember the previous example in which the exposures of *AVOL* and *PVOL* to unit changes in σ_F^2 were 2 and 1, respectively. Equation (34) indicates that the exposures of *AVOL* and *PVOL* to a unit change in σ_A^2 are both 1. Therefore, the exposures of *AVOL* and *PVOL* to total volatility shifts (induced by broad market factors and arbitrage activities) become 3 and 2, respectively (Chang et al., 1999). The ratio of total exposure (1.50) is smaller compared to the ratio of the exposure to broad economic influences (2.00) because the exposure from arbitrage is not diversifiable, pointing to a structural shift in the relationship between *AVOL* and *PVOL* when futures trading starts (Chang et al., 1999). Similar arguments also hold for a structural shift in the relationship between *E(CSD)* and *AVOL* (or *PVOL*) (Chang et al., 1999). Based on these arguments, Chang et al. (1999) predict that futures trading increases the spot portfolio volatility. In their study, Chang et al. (1999) empirically test for the validity of such a prediction and claim that their findings support this prediction. It should also be remembered that the literature presents conflicting results about the effect of futures trading on the volatility of the underlying stock market. Therefore, in this study, the comparative statics above lead to the following four testable implications (Chang et al., 1999):

H1: Volatility shifts in broad economic factors induce proportional shifts in *PVOL* and *AVOL*, but *PVOL* increases relative to *AVOL* when futures trading begins.

$$PVOL_t = c_0 + c_1 AVOL_t + c_{post} D_{post,t} + \varepsilon_t , \tag{33}$$

In Equation (33), $D_{post,t}$ is the dummy variable that equals 1 for the post-futures period and 0 for the pre-futures period. With c_1 constant, the structural shift between *AVOL* and *PVOL* should show itself in the intercept and will be captured by the coefficient on the dummy variable (Chang et al., 1999). Therefore, a positive estimate of c_{post} would be consistent with the hypothesis of an arbitrage-induced increase in volatility related to futures trading (Chang et al., 1999).

H2: Volatility shifts in broad economic factors induce proportional shifts in *E(CSD)* and *AVOL*, but *E(CSD)* decreases relative to *AVOL* when futures trading begins.

$$CSD_t = c_0 + c_1 AVOL_t + c_{post} D_{post,t} + \varepsilon_t , \tag{34}$$

In Equation (34), *CSD* proxies for the unobservable *E(CSD)*. A significantly negative estimate of c_{post} would be consistent with H2. The model predicts a negative shift in *CSD* because with c_1 constant, the *AVOL* correction overstates the volatility effect of a shift in broad economic factors on *CSD* (since *CSD* is influenced only by broad economic factors, whereas *AVOL* is influenced both by broad factors and by futures-related arbitrage activity) (Chang et al., 1999).

H3: Volatility shifts in broad economic factors induce proportional shifts in *E(CSD)* and *PVOL*, but *E(CSD)* decreases relative to *PVOL* when futures trading begins.

$$CSD_t = c_0 + c_1 PVOL_t + c_{post} D_{post,t} + \varepsilon_t, \quad (35)$$

Since, as a result of diversification *PVOL* is influenced less by idiosyncratic factors compared to *AVOL*, it should control better for broad market influences than *AVOL* in tests of arbitrage-induced shifts in the structural relations between *E(CSD)* and these other volatility components. Thus, aforementioned shifts in the intercept should be more obvious for Equation (35) than for Equation (34) (Chang et al., 1999).

H4: Following the start of futures trading, changes in E(CSD) are smaller relative to changes in PVOL.

$$CSD_t = c_0 + c_1 PVOL_t + c_{post}\left(D_{post,t} * PVOL_t\right) + \varepsilon_t, \quad (36)$$

H1 through H3 test for shifts in the intercept with the slope coefficient held constant. If the intercept is held constant, however, H4 predicts that the slope will change if futures trading affects stock market volatility (Chang et al., 1999). A significantly negative estimate of c_{post} would be consistent with H4 (Chang et al., 1999).

The above hypotheses are tested by using data on both the ISE-National 30 and non-ISE-National 100 stocks. Both groups are susceptible to general economic disturbances; however, only the ISE National 30 stocks are impacted directly from futures trading. Thus, changes in the relationships among the volatility components for ISE National 30 stocks but not for the non-ISE National 100 stocks are unlikely to be explained by changes in broad factors and further, these changes can be attributed to the influence of futures trading.

CHAPTER 4

RESULTS AND ANALYSIS

4.1. DATA

This study analyzes whether futures trading affects the volatility of the underlying stock market by examining the volatility of the ISE-30 National Index while controlling for broad economic factors. The methodology adopted calls for the creation of two stock portfolios. The first portfolio contains those stocks that have been included in the ISE-30 National Index for the entire sample period (from January 2, 2003 to May 30, 2006). There are 18 stocks that satisfy this condition. The volatility of the underlying stock market is proxied by the volatility of this portfolio. The second portfolio contains those stocks that were never included in either the ISE-30 or the ISE-100 National Indices during the entire sample period. There are 140 stocks that satisfy this condition. This portfolio of stocks serves as the "control portfolio." By measuring the volatility changes for this particular portfolio, it is possible to account for the impact of broad economic factors on stock market volatility. Since these stocks are not included in either of the national indices, they are not expected to be influenced by futures trading as there are no futures contracts being traded on these particular stocks.

The closing prices of the 158 stocks that make up the two portfolios are obtained from the database of the ISE. The sample period is from 2nd January 2003 to 30th May 2006. The data set includes a total of 636 observations. 369 of these observations are from the first sub-period prior to the introduction of stock index futures trading and the remaining 267 observations are from the second sub-period subsequent to the introduction of stock index futures trading.

Two equally-weighted portfolios are constructed for each sub-period. If weights are not constant over time, the classical linear regression used for estimating the volatility decomposition would produce meaningless results. Moreover, the weight of each stock is assumed to be equal in order to capture the nature of arbitrage-motivated trading.

Continuously-compounded daily percentage returns are estimated as the log price relative for each portfolio. That is, for a portfolio with daily closing price of P_t (equally-weighted daily closing prices of firms in the portfolio), the return R_t is defined as $log(Pt/P_{t-1})$.

Table 6 provides basic descriptive statistics for the log-return series of ISE National 30 Index and the Non-ISE National 100 Index portfolios. The Jarque-Bera tests reveal that the skewness and kurtosis figures for each of the return series (including sub-periods) are different than those from a normal distribution. This test evaluates the hypothesis that X has a normal distribution with unspecified mean and variance against the alternative that X does not have a normal distribution. The test is based on the sample skewness and kurtosis of X. For a true normal distribution, the sample skewness should be near 0 and the sample kurtosis should be near 3. The Jarque-Bera test determines whether the sample skewness and kurtosis are unusually different from their expected values as measured by a chi-square statistic. Since the Jarque-Bera probability for each of the return series is nearly zero, the null hypotheses that the series fit a normal distribution are rejected.

Also in Table 6, the variance figures provide an initial indication of the volatility the ISE National 30 Index portfolio. The pre-futures ISE National 30 Index portfolio volatility is greater than that of post-futures. Based on an F-Variance ratio test, this reduction in variance is statistically significant at the %1 level (Table 7 provides the test results). This result is an initial indication that the introduction of index futures did not destabilize the spot market. However, inferences cannot be drawn from these figures alone, as these variance calculations do not take into account the market-wide movements and the time-varying nature of volatility.

Table 6: Summary statistics of return series of ISE National 30 Index stocks portfolio and of Non-ISE National 100 Index stocks portfolio for before and after the futures periods

	ISE National 30 Index Portfolio Return	Non-ISE National 100 Index Portfolio Return	ISE National 30Index Portfolio Return Before Futures Trading	ISE National 30 Index Portfolio Return After Futures Trading	Non-ISE National 100 Index Portfolio Return Before Futures Trading	Non-ISE National 100 Index Portfolio Return After Futures Trading
Mean	-0.0000250	0.0029670	-0.0000707	0.0000381	0.0040010	0.0015370
Median	0.0014010	0.0021860	0.0016550	0.0006920	0.0023220	0.0018050
Maximum	0.1049700	0.4681430	0.1049700	0.0472120	0.4681430	0.0923290
Minimum	-0.3478100	-0.0895430	-0.3478100	-0.0816220	-0.0749160	-0.0895430
Std. Dev.	0.0266630	0.0288450	0.0316410	0.0176630	0.0325360	0.0227620
Skewness	-3.8386850	6.7154510	-3.8965830	-0.5253510	8.1099790	-0.3237290
Kurtosis	49.5991700	109.0819000	42.8665400	4.4624940	114.3487000	5.7634340
Jarque-Bera	59106.26000	302994.80000	25369.90000	36.07687	194672.40000	89.62044
Probability	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
Observations	636	636	369	267	369	267

Table 7: Test for equality of variances between ISE National 30 Index stocks portfolio return series before futures trading and ISE National 30 Index stocks portfolio return series after futures trading

Method	df	Value	Probability
F-test	(266, 368)	3.208954	0.0000
Siegel-Tukey		3.194275	0.0014
Bartlett	1	93.90893	0.0000
Levene	(1, 634)	12.74495	0.0004
Brown-Forsythe	(1, 634)	12.26443	0.0005

Category Statistics

Variable	Count	Std. Dev.	Mean Abs. Mean Diff.	Mean Abs. Median Diff.	Mean Tukey-Siegel Rank
IMKB30_PORT_RET_AFTER	267	0.017663	0.013501	0.013498	345.8614
IMKB30_PORT_RET_BEFORE	369	0.031641	0.019359	0.019272	298.7019
All	636	0.026663	0.016900	0.016848	318.5000

Bartlett weighted standard deviation: 0.026683

4.2. RESULTS AND ANALYSIS

Table 8 presents mean daily estimates and standard deviations (in parenthesis) of *AVOL* and *PVOL* for the total sample and for the two subperiods. The ISE National 30 Index portfolio includes 18 stocks with no missing data over the sample period. The Non- ISE National 100 Index portfolio consists of 140 stocks, also with no missing data over the sample period. Mean *PVOL* estimates are higher for pre-futures period compared to the estimates from the post-futures period. This result supports the conclusions driven from the figures in Tables 6 and 7. At first glance, the introduction of futures trading does not seem to have a destabilizing effect on the volatility of the underlying asset-the ISE National 30 Index. However, as stated earlier, this result needs to be analyzed in further detail by taking into account the effect of market-wide factors and the passage of time on the volatility of the stock market.

Table 8: Summary statistics of portfolio volatilities, average volatilities for ISE National 30 Index stocks portfolio and Non-ISE National 100 Index stocks portfolio

Sample Period	Portfolio of ISE 30 National Index Stocks	Portfolio of Non-ISE 100 National Index Stocks
A. Total period		
PVOL	0.001354(0.007365)	0.001595(0.014622)
AVOL	0.003860(0.020827)	0.003958(0.008351)
B.Pre-futures period		
PVOL	0.001876(0.009599)	0.002119(0.019122)
AVOL	0.004519(0.024907)	0.004550(0.008442)
C.Post-futures period		
PVOL	0.000634(0.001118)	0.000872(0.002050)
AVOL	0.002955(0.013310)	0.003162(0.008186)

The next step in the analysis is to estimate and decompose volatilities based on the Chang et.al.'s methodology that is described in Chapter 3. Recall that Chang et. al.'s (1998) a priori expectation is that, under normal conditions, futures trading should destabilize the underlying spot market. However, this study has no such a priori expectation, based on the conflicting results from the literature regarding the issue. This thesis decomposes the spot portfolio volatility into two components: cross-sectional dispersion and average volatility of returns. In this model, on the one hand, a shift in broad economic factors causes proportional shifts in spot portfolio and average volatility. On the other hand, futures-related volatility shifts change the proportionality of this relationship. The model also predicts structural shifts in the relationship between cross-sectional dispersion and spot portfolio (and average) volatility when futures trading begins. The main argument of the decomposition is that if there volatility shifts that are observed for the ISE National 30 Index portfolio but not for the Non- ISE National 100 Index portfolio, then these shifts cannot be explained solely by the influence of market-wide factors but, instead, the introduction of futures trading must be the source of the shifts in volatility.

A formal analysis of structural shifts in the relationship between *PVOL* and *AVOL* is presented in Table 9. In Table 9, the null hypothesis H1 that volatility shifts in broad economic factors induce proportional shifts in *PVOL* and *AVOL*, but *PVOL* increases relative to *AVOL* when futures trading begins, is tested. With c_1, estimated as a single value for each portfolio for the whole sample period, constant the structural shift between *AVOL* and *PVOL* should show itself in the intercept and be captured by the coefficient on the dummy variable. Therefore, a significant and positive estimate of c_{post} for the ISE National 30 Index portfolio and an insignificant estimate of c_{post} for the Non- ISE National 100 Index portfolio would be consistent with the hypothesis of an arbitrage-induced increase in volatility related to futures trading. In Table 9, it is observed that c_{post} takes a value of -0.000773 for the ISE National 30 Index portfolio and a value of -0.001260 for the Non- ISE National 100 Index portfolio. The values in parenthesis are t-statistics and neither of the c_{post} estimates are statistically significant. The insignificant c_{post} estimates imply that the introduction of futures trading did not have a destabilizing effect on the underlying stock market.

Table 9: Test of H1 that the volatility shifts in broad economic factors induce proportional shifts in PVOL and AVOL, but PVOL increases relative to AVOL when futures trading begins

Model A:

$$PVOL_t = c_0 + c_1 AVOL_t + c_{post} D_{post} + \varepsilon_t$$

(Sample period: 07.01.2003-30.05.2006)
(t statistics are in parenthesis)

c_0	c_1	c_{post}	R^2
Panel A: Results for a portfolio of ISE 30 National Index Stocks			
0.000521	0.300027	-0.000773	0.725849
(2.5521)	(40.7096)	(-2.4884)	
Panel A: Results for a portfolio of Non-ISE 100 National Index Stocks			
0.002161	-0.009175	-0.001260	0,001804
(2.6162)	(-0.1314)	(-1.0677)	

The second null hypothesis H2 states that volatility shifts in broad economic factors induce proportional shifts in *E(CSD)* (*CSD* proxies for *E(CSD)*, because *E(CSD)* is unobservable) and *AVOL*, but *E(CSD)* decreases relative to *AVOL* when futures trading begins. According to earlier arguments, while futures-related basket trading strategies may increase spot portfolio volatility, they may have little effect on the cross-sectional dispersion of constituent security returns. Thus, if the estimate of c_{post} is negative and significant for the ISE National 30 Index portfolio but not for the Non- ISE National 100 Index portfolio, then this result would imply that the introduction of futures trading increases the volatility of the underlying stock market. In Table 9, it is seen that the estimates of c_{post} are negative and insignificant for both portfolios. This finding once again implies that the introduction of futures trading did not have a destabilizing effect on the underlying stock market; in fact, futures trading seems to have no effect of the volatility of the ISE National 30 Index portfolio.

Table 10: Test of H2 that volatility shifts in broad economic factors induce proportional shifts in E(CSD) and AVOL, but E(CSD) decreases relative to AVOL when futures trading begins

Model B: $CSD_t = c_0 + c_1 AVOL_t + c_{post} D_{post} + \varepsilon_t$ (Sample period: 07.01.2003-30.05.2006) (t statistics are in parenthesis)			
c_0	c_1	c_{post}	R^2
Panel A: Results for a portfolio of ISE 30 National Index Stocks			
-0.000011	0.380761	-0.000488	0.425392
(-0.0230)	(21.5812)	(-0.6566)	
Panel A: Results for a portfolio of Non-ISE 100 National Index Stocks			
0.002321	0.632468	-0.001037	0.080791
(2.2755)	(7.3356)	(-0.7118)	

Table 11 reports the results of testing for the third null hypothesis, H3, which states that shifts in broad economic factors induce proportional shifts in *E(CSD)* and *PVOL*, but *E(CSD)* decreases relative to *PVOL* when futures trading begins. Just like H2, if the estimate of c_{post} is negative and significant for the ISE National 30 Index portfolio but not for the Non- ISE National 100 Index portfolio, then this result implies that the introduction of futures trading increases the volatility of the underlying stock market. The c_{post} estimates for both portfolios is statistically insignificant. This result is consistent with the findings in Tables 9 and 10 and implies that the inception of futures trading on the TURKDEX had no effect on the portfolio volatility or the cross-sectional dispersion of the underlying stock market.

Table 11: Test of H3 that shifts in broad economic factors induce proportional shifts in E(CSD) and PVOL, but E(CSD) decreases relative to PVOL when futures trading begins

Model C: $CSD_t = c_0 + c_1 PVOL_t + c_{post} D_{post} + \varepsilon_t$ (Sample period: 07.01.2003-30.05.2006) (t statistics are in parenthesis)			
c_0	c_1	c_{post}	R^2
Panel A: Results for a portfolio of ISE 30 National Index Stocks			
-0.000201	1.018222	0.000181	0.378475
(-0.3944)	(19.5674)	(0.2332)	
Panel A: Results for a portfolio of Non-ISE 100 National Index Stocks			
0.004555	0.301138	-0.001538	0.057189
(4.7459)	(6.0533)	(-1.0448)	

Tests of H4, which states that relative to changes in *PVOL*, changes in *E(CSD)* will be smaller after futures trading begins, are reported in Table 12. Recall that the first three hypotheses are related to shifts in the intercept of the volatility model while the slope coefficient is held constant. If the intercept is held constant instead, the fourth null hypothesis, H4, predicts that the slope changes if futures trading affects stock market volatility. A significant and negative estimate of c_{post} would be consistent with H4. Although they are similar, regressions (35) and (36) test slightly different versions of the stability of the relationship between *CSD* and *PVOL*. Regression (35) holds the covariance constant and tests for shifts in the mean of *CSD* as trading regimes change. Regression (36) holds the mean constant and tests for shifts in the covariance. Other things equal, the model predicts that the covariance between *CSD* and *PVOL* will decline if futures-related trading increases the volatility of *PVOL* but not *CSD*. The c_{post} estimates for both portfolios are insignificant. This finding does not support the claim that futures trading on the TURKDEX increased spot portfolio volatility.

Table 12: Test of H3 that relative to changes in PVOL, changes in E(CSD) will be smaller after futures trading begins than before

Model D:

$$CSD_t = c_0 + c_1 PVOL_t + c_{post}(D_{post} * PVOL_t) + \varepsilon_t$$

(Sample period: 07.01.2003-30.05.2006)
(t statistics are in parenthesis)

c_0	c_1	c_{post}	R^2
Panel A: Results for a portfolio of ISE 30 National Index Stocks			
0.000044	1.021190	-0.652918	0.380200
(0.1093)	(19.6884)	(-1.3466)	
Panel A: Results for a portfolio of Non-ISE 100 National Index Stocks			
0.004065	0.306471	-0.450317	0.056667
(5.3907)	(6.1477)	(-0.8611)	

One noteworthy observation about the results in the above tables is the difference in the R^2 values between the two portfolios. Recall that R^2 indicates what proportion of the total variation in PVOL is explained by the model (Myers et al, 2002). R^2 changes between zero and one with values closer to one implying a good fit of the model, whereas values close to zero point to a poor fit. Within the context of the models estimated in this study, it is plausible to expect that macroeconomic disturbances explain a greater portion of the portfolio variance compared to futures trading. Therefore, in the case of either portfolio, an explanatory variable which is affected by macroeconomic disturbances, namely *AVOL*, can reasonably be expected to generate an above- average explanatory power for the model (should reveal itself as an R^2 close to one). In Table 9, the R^2 for the ISE National 30 Index portfolio is 0.725, pointing to a successful fit, whereas the $R^2$is only 0.001 for the Non-ISE National 100 Index portfolio. Similar differences are observed for the other three hypothesis tests in Tables 10, 11, and 12, respectively. The difference in R^2 values implies that the estimated models have a much higher explanatory power when the portfolio consists of ISE-30 stocks but not when the portfolio consists of Non-ISE-100 stocks This difference in explanatory model can be explained by the fact that the Non-ISE-100 stocks happen to be those stocks that are not included in any of the national indices during the entire sample period due to their small market capitalization and low trading volume. As such, these stocks are claimed to be susceptible to manipulation which could explain the failure of the models in explaining the volatility changes in this group of stocks. If these stocks are indeed subject to manipulation, this would mean that the return-generating process for these stocks is not a result of the interaction between the market forces of supply and demand, which are themselves shaped by broad economic factors. Hence, a model that decomposes stock return volatility based on broad economic factors cannot be expected to work well in explaining the volatility changes for such a group of stocks.

Generally speaking, the test results imply that the inception of futures trading on the TURKDEX had no effect on the volatility of the underlying stock market, ISE. The models seem to have a much higher explanatory model for the portfolio of ISE-30 stocks compared to the portfolio of Non-ISE-100 stocks. These results are likely to be caused by the extremely small volume of futures trading compared to the volume of trading in the stock market. Also, possible manipulative influences, especially on the Non-ISE-100 stocks, may give way to the failure of models in explaining the changes in the volatility of this group of stocks. It seems like unless the trading volume and the depth of the futures markets reach high levels compared to the underlying stock market, finding a link between the volatility of the stock market and trading of the futures is going to prove to be an almost impossible task.

CHAPTER 5

CONCLUSION

Although Turkish capital markets, as an emerging economy, have undergone a great progress, what is evident regarding the economy as a whole is also evident for the capital markets: the prices of the securities are extremely volatile due to macro-economic imbalances as well as domestic factors such as political instability and international factors like fluctuating exchange rates. Recall that volatility refers to the standard deviation of the change in the value of a financial instrument within a specific time horizon and is often used to quantify the risk of the instrument over that time period. The Turkish capital markets are categorized as emerging financial markets and expose domestic and foreign investors to a great deal of risk due to high volatility. In such an environment, financial instruments, such as futures contracts, that may offer hedging opportunities attract even more attention.

In order to satisfy the hedging as well as speculative needs of investors, TurkDex, the very first and only options and futures exchange in Turkey, has been launched and began its operations on 4 February 2005. As of the end of May 2007, only four futures contracts are traded on the TurkDex, although it was launched with the purpose of offering both futures and options. The futures contracts traded on TurkDex are as follows (Official website of TurkDex):

1) Currency Futures Contracts written either on YTK/EURO rate or YTL/DOLAR rate

2) Equity Index Futures Contracts written either on ISE National 30 Index or ISE National 100 Index

3) Interest Rate Futures Contracts written either on 91 Days T-Bill interest rates or 365 Days T-Bill interest rates or T-Benchmark

4) Commodity Futures Contract written on either cotton or wheat or gold.

Among these contracts, the contracts written on the ISE- National 30 Index are the focus point of this thesis. Stock index futures are deemed to be one of the most successful financial innovations of the 1980s and parallel to this opinion, the most actively traded contracts on the TurkDex have been the contracts written on the ISE-30 and ISE-100 National Indices. As a matter of fact, the trading volume in equity index future contracts is dominated by contracts written on the ISE-National 30 Index.

In the year 2005, the trading volume of the ISE-30 index futures was 563,390,702 YTL and this volume increased to 10,446,702,523 YTL by the end of 2006. During the same period, the 2005 trading volume of the ISE-National 30 Index stocks was 269,970,134,449 YTL and this volume increased to 325,157,131,314 YTL in 2006. These statistics show that the futures trading volume was 0.2% and 3% of the stock trading volume in years 2005 and 2006 respectively. Although these percentages are rather low, the progress of the trading volume in futures contracts written on the ISE-National 30 Index is promising (it increased by 20-fold from year 2005 to year 2006). Therefore, as the trading volume in TURKDEX increases, it becomes more important to understand the interaction between future and spot markets, and especially the potential impact of stock index futures trading on the underlying spot market. Since the trading of futures on equity indices aims to provide a hedging outlet for the risk taken in the spot market, the effect of such trading on the volatility and riskiness of the spot market is very important for both the investors of stock index futures and the regulators of financial markets. It is usually the case that if futures trading increases the volatility in the spot market, thereby increasing the level of risk faced by the investors in the spot market, the regulatory body responds by passing new regulations to help investors to hedge themselves against the risk in spot market. However, the new regulation itself may end up causing the risk assumed in the spot market since now that it is possible to form hedged positions, investors are more willing to take on risk in the spot market. This is a contradiction in and of itself and therefore, it is worthwhile to investigate the possible influence of futures trading on the volatility of the underlying stock market. The main argument against stock index futures trading claims that futures market may increase stock market volatility. This argument is based on the assumption that, because of their high degree of leverage (i.e. the investor is able to open a position by only depositing a small percentage of the contract size at the beginning), futures markets are likely to attract risk takers and speculator traders. The speculative investing strategies are likely to increase the underlying asset volatility. Another point against futures trading is that futures markets provide otherwise unattainable trading strategies, such as index arbitrage and portfolio insurance with arbitragers driving the spot prices up or down continuously through the positions they take and reverse. There are also arguments claiming that futures markets play an important role in price discovery and have a beneficial effect on the underlying cash markets. This viewpoint holds that speculation in the futures market tends to stabilize cash prices. Futures trading adds more informed traders to the cash market, making it more liquid and, therefore, less volatile. Both of the arguments against and in favor of futures market trading have some theoretical and empirical support and therefore the results in the literature are conflicting. The existent literature on the effects of stock index futures trading on spot market volatility focuses primarily on developed markets, and it is unclear to what extent these results are applicable to emerging markets. Therefore, in this thesis a marginal contribution to the literature is attempted by analyzing the volatility of the stock market before and after the introduction of stock index futures in Turkish capital markets.

The study analyzes whether futures trading causes the volatility of the stocks included in ISE 30 National Index to increase while controlling for the effect of broad economic factors on such volatility. Two portfolios are formed to examine the possible sources of change in volatility. The first portfolio is made up of stocks that were included in the ISE 30 National Index for the whole sample period. The second portfolio consists of stocks that were not included in any of the stock indices during the entire sample period and therefore were not the underlying security of any derivative instrument traded. There are a total of 158 stocks that satisfy the above criteria. Among these stocks, 18 were included in the ISE 30 National Index throughout the sample period, whereas the remaining 140 were not included in any index during the same period. Both sets of stocks are susceptible to broad economic disturbances, but only the ISE 30 National Index stocks are impacted directly by futures trading. Thus, shifts in the relationship between the volatility components for the ISE National 30 Index portfolio but not for the Non-ISE National 100 Index portfolio are unlikely to be explained by changes in broad economic factors alone. The closing prices of the 158 stocks are obtained from the database of ISE over the period from 2nd January 2003 to 30th May 2006. The data set includes a total of 636 observations, of which 369 observations belong to the sub-period prior to the introduction of stock index futures and the remaining 267 observations belong to the second sub-period subsequent to the introduction of stock futures. Two equally-weighted portfolios are constructed for each sub-period and continuously-compounded percentage returns are estimated as the log price relative for each portfolio.

This thesis adopts the methodology proposed by Chang et al (1999). Chang et.al. propose new tests to examine whether stock index futures affect stock market volatility by decomposing spot portfolio volatility (*PVOL*) is into three components: average volatility of mean returns (*AVOL*), expected cross sectional dispersion (*E(CSD)*) and cross-sectional variance of mean returns (*CSVOM*). After the decomposition, a formula is constructed for measuring the average volatility and the cross-sectional dispersion of the component securities and the portfolio volatility for each day in the sample period by using a simple filtering procedure. This filter recovers a series of realized volatilities from a discrete time realization of a continuous-time diffusion process outlined in papers by Chesney, Elliott, Madan and Yang (Chesney et al., 1993) and Pastorello (1996). During this stage of volatility decomposition, it was observed that the cross-sectional variance of mean returns accounts for less than one percent of the variation in *PVOL* and therefore this term has only a second-order impact on *PVOL* . Therefore, this component was ignored in the analysis. After excluding *CSVOM* from the analysis, the decomposition is applied to a single-factor return-generating model to focus on the relationships among the volatility components rather than on the components in isolation which was the traditional method of analyses. The following hypotheses are tested:

H1: Volatility shifts in broad economic factors induce proportional shifts in

PVOL and AVOL, but PVOL increases relative to AVOL when futures trading begins.

H2: Volatility shifts in broad economic factors induce proportional shifts in E(CSD) and AVOL, but E(CSD) decreases relative to AVOL when futures trading begins.

H3: Volatility shifts in broad economic factors induce proportional shifts in E(CSD) and PVOL, but E(CSD) decreases relative to PVOL when futures trading begins.

H4: Relative to changes in PVOL, changes in E(CSD) will be smaller after futures trading begins than before.

Generally speaking, the test results imply that the inception of futures trading on the TURKDEX had no effect on the volatility of the underlying stock market. The models seem to have a much higher explanatory model for the portfolio of ISE National 30 Index stocks compared to the portfolio of Non-ISE National 100 Index stocks. These results are likely to be caused by the extremely small volume of futures trading compared to the volume of trading in the stock market. Also, possible manipulative influences, especially on the Non-ISE National 100 Index portfolio, may give way to the failure of models in explaining the changes in the volatility of this group of stocks. It seems like unless the trading volume and the depth of the futures markets reach high levels compared to the underlying stock market, finding a link between the volatility of the stock market and trading of the futures is going to prove to be an almost impossible task.

For further research, it would be illuminating to study the same question by the use of different methodologies such as GARCH-type models. It may also be interesting to repeat the same tests once enough time passes and stock index futures trading in TurkDex reaches a high level of trading volume. Moreover, lengthening the sample period may increase the statistical significance of the results.

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