KALUZA-KLEIN MONOPOLE

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EMRE SAKARYA

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submitted by **EMRE SAKARYA** in partial fulfillment of the requirements for the degree of **Master of Science in Physics Department, Middle East Technical University** by,

Prof. Dr. Canan Özgen ______________
Dean, Graduate School of **Natural and Applied Sciences**

Prof. Dr. Sinan Bilikmen ______________
Head of Department, **Physics**

Assoc. Prof. Dr. Bayram Tekin ______________
Supervisor, **Physics Dept., METU**

Examining Committee Members:

Prof. Dr. Ayşe Kalkanlı Karasu ______________
Physics Dept., METU

Assoc. Prof. Dr. Bayram Tekin ______________
Physics Dept., METU

Prof. Dr. Atalay Karasu ______________
Physics Dept., METU

Assoc. Prof. Dr. Bahtiyar Özgür Sarıoğlu ______________
Physics Dept., METU

Assist. Prof. Dr. Ali Ulaş Özgür Kişisel ______________
Mathematics Dept., METU

Date: ______________

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Name, Last Name: Emre Sakarya

Signature :

ABSTRACT

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Sakarya, Emre

M.S., Department of Physics

Supervisor : Assoc. Prof. Dr. Bayram Tekin

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Kaluza-Klein theories generally in $(4+D)$ and more specifically in five dimensions are reviewed. The magnetic monopole solutions found in the Kaluza-Klein theories are generally reviewed and their generalizations to Anti-de Sitter spacetimes are discussed.

Keywords: Kaluza-Klein Theory, Magnetic Monopole, Anti-de Sitter Spacetime

ÖZ

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Kaluza-Klein teorileri genel olarak $(4 + D)$ boyutta ve daha ayrıntılı bir şekilde beş boyutta gözden geçirildi. Bu teorilerde bulunan magnetik monopol çözümleri gözden geçirildi ve bunların Anti-de Sitter uzayzamanlarına genelleştirilmeleri tartışıldı.

Anahtar Kelimeler: Kaluza-Klein Teorisi, Magnetik Monopol, Anti-de Sitter Uzayzamanı

To My Parents...

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CHAPTER 1

INTRODUCTION

1.1 Introduction and Motivation

Kaluza-Klein theories present a scheme for the unification of gravity with other interactions by means of the hypothesis that the gauge symmetries are originally geometrical. To realize this, the existence of extra dimensions is proposed. In the unification programme, the first attempt containing the extra dimensions was made by Nordström [1] in his scalar gravity theory in 1912. This idea gained a permanent place when it was used by Kaluza [2] in 1921 to unify gravity with electromagnetism in five-dimensions. Later, five-dimensional Kaluza's theory was put in a more realistic form by Klein [3] in 1926 who suggested that the extra dimension to be compact. This five dimensional Kaluza-Klein theory was generalized to higher dimensions by de Witt [4], Kerner [5], Trautman [6], Cho [7, 8] and Freund [7] to achieve the unification of gravity with non-Abelian gauge theories.

As in the case of grand unified theories, magnetic monopole solutions occur also in the Kaluza-Klein theories. The first magnetic monopole solution was given independently by Sorkin [9] and Gross and Perry [10] in 1983 to the five-dimensional Kaluza-Klein theory. This solution was generalized to higher dimensions by Lee [11, 12] and Iwazaki [13]. Its non-Abelian extension was made by Perry [14] and Bais and Batenburg [15]. In recent years, also the generalization of the monopole solution to Anti-de Sitter space has gained an increasing importance. Such a generalization to Anti-de Sitter space was first considered by Önemli and Tekin [16], who found that there is no exact analog of monopole solution in five-dimensions.

In the second chapter, we review the general structure of Kaluza-Klein theories and give a detailed description in five-dimensions. In the third chapter, we review the magnetic monopole solutions including both Abelian and non-Abelian types with some of their properties. Lastly in the fourth chapter, we consider the generalization to the Anti-de Sitter spaces.

1.2 Notation and Conventions

In this thesis, we adopt the following conventions: The metric signature is $(-,+,+,\ldots,+)$. The upper case Latin letters "$A,B,C,\ldots$" denote the $(4+D)$-dimensional indices, $\{0,1,2,3,5,6,\ldots,D\}$, the lower case Greek letters denote the usual four dimensions, $\{0,1,2,3\}$, the lower case "i,j,k" run over spatial dimensions in four dimensions, $\{1,2,3\}$, and the lower case Latin indices other than "i,j,k" run over only D-extra dimensions $\{5,6,\ldots,4+D\}$. Besides, all hatted quantities refer to the $(4+D)$-dimensional theory.

Throughout this work, independent of the number of dimensions, the Christoffel symbol is

$$\hat{\Gamma}^K_{LM} = \frac{1}{2}\hat{g}^{KN}(\partial_L \hat{g}_{NM} + \partial_M \hat{g}_{NL} - \partial_N \hat{g}_{LM}), \tag{1.1}$$

the Riemann tensor is

$$\hat{R}^K{}_{LMN} = \partial_M \hat{\Gamma}^K_{LN} - \partial_N \hat{\Gamma}^K_{LM} + \hat{\Gamma}^K_{JM}\hat{\Gamma}^J_{LN} - \hat{\Gamma}^K_{JN}\hat{\Gamma}^J_{LM}, \tag{1.2}$$

the Ricci tensor and the Ricci scalar are

$$\hat{R}_{LM} = \hat{R}^K{}_{LKM} \quad \text{and} \quad \hat{R} = \hat{R}^L{}_L. \tag{1.3}$$

We use units in which $\hbar = c = 1$.

CHAPTER 2

KALUZA-KLEIN THEORIES

2.1 General Framework

Generically, Kaluza-Klein (KK) theories refer to generalizations of pure or a modified version of four-dimensional General Relativity (GR) to a $(4 + D)$-dimensional spacetime by hypothesizing that there exist D extra dimensions, so that they give by a mechanism called spontaneous compactification, ordinary gravity plus gauge theories in their effective low energy sectors [17]. We start with the following $(4+D)$-dimensional line element, by assuming that $(4 + D)$-dimensional spacetime is a Riemann spacetime $V^{(4+D)}$, that is a differentiable manifold with the general metric compatible affine connection and vanishing torsion,

$$d\hat{s}^2 = \hat{g}_{AB} d\hat{z}^A d\hat{z}^B, \tag{2.1}$$

where $\{\hat{z}^A\} = \{x^\mu, y^a\}$ and $\hat{g}_{AB}$ is the metric tensor.

Now, it seems that we have two alternatives for what ways the extra dimensions participate into the description of the $(4+D)$-dimensional spacetime, namely, timelike or spacelike ways. However, if we go into further analysis, we see that the number of timelike dimensions can not be greater than one, because this would lead to casuality problems due to the existence of closed timelike curves [18] arising in the theory. In addition, extra timelike dimensions would yield tachyons in effective four dimensions and the sign of the Maxwell action would be incorrect [19]. Thus, all the extra dimensions have to be spacelike and so, the signature of $(4 + D)$-dimensional metric becomes $(-, +, +, ..., +)$.

The dynamics of the KK theories is determined by the $(4+D)$-dimensional Einstein-Hilbert action with the cosmological constant for generality, plus a term $\hat{I}$ which represents a potential modification of action in terms of matter fields, extra gauge fields or other types of curvature invariants,

$$\hat{S} = \hat{S}_{GR} + \hat{S}' = -\frac{1}{16\pi\hat{G}}\int d^{4+D}\hat{z}\left[\sqrt{|\hat{g}|}(\hat{R} - 2\hat{\Lambda}) + \hat{I}\right], \tag{2.2}$$

where $\hat{g} = \det(\hat{g}_{AB})$, $\hat{R}$ is the Ricci scalar, $\hat{\Lambda}$ is the cosmological constant, and $\hat{G}$ is the $(4 + D)$-dimensional gravitational constant. However, the original purpose is to extract everything from only the spacetime geometry, that is the case $\hat{I} = 0$. If we now assume $\hat{I} = 0$ and so vary only the $\hat{S}_{GR}$ part of (2.2) with respect to the metric, we obtain $(4 + D)$-dimensional ordinary Einstein equations

$$\hat{R}_{AB} - \frac{1}{2}\hat{g}_{AB}\hat{R} + \hat{g}_{AB}\hat{\Lambda} = 0. \tag{2.3}$$

However, if we assume non-zero $\hat{I}$ to represent active matter fields and again vary (2.2), we then obtain

$$\hat{R}_{AB} - \frac{1}{2}\hat{g}_{AB}\hat{R} + \hat{g}_{AB}\hat{\Lambda} = 8\pi\hat{G}\hat{T}_{AB}, \tag{2.4}$$

where $\hat{T}_{AB}$ is energy-momentum tensor obtained from

$$\delta\hat{I} = \frac{1}{2}\int d^{4+D}\hat{z}\sqrt{|\hat{g}|}\hat{T}_{AB}\delta\hat{g}^{AB}. \tag{2.5}$$

2.2 General Structure of the Ground State

The most vital phase of the construction is to pick out a ground state [20] for the theory (2.2), because this ground state determines the general structure of the $(4+D)$-dimensional spacetime and its isometries comprise the local gauge symmetries which we want to describe in a unified way with the ordinary gravity. For the chosen ground state to solve the complete field equations, we may constitute an $\hat{I}$ by assuming that it represents either matter fields or a combination of some curvature invariants. Here, it is important to point out that the original purpose, simplicity and beauty of KK theories is achieved for the cases when $\hat{I}$ is zero, which may be referred to as ‘pure’ KK theories, that is, to describe the gauge fields just from the spacetime geometry by means of the extra dimensions. Nevertheless, in general, there may be a need to introduce $\hat{I}$ with some exceptions such as the supersymmetric version of KK theory where certain matter fields are treated by an underlying supersymmetry and the related geometry of superspace [21]. In this work, we will consider only the non-supersymmetric KK theories. Generally, the ground state of a given field theory is defined as the stable solution of the field equations with the lowest energy. Determining the correct vacuum space of a given theory is not a straightforward procedure which is true especially for gravitational theories, because the energy concept is not well defined and depends on the boundary conditions. We naturally expect the vacuum space $V_0^{(4+D)}$ for the pure $(4+D)$-dimensional gravity

$$\hat{S}_{GR} = -\frac{1}{16\pi\hat{G}}\int d^{4+D}\hat{z}\sqrt{-\hat{g}}(\hat{R}-2\hat{\Lambda}) \tag{2.6}$$

to be maximally symmetric. For instance, when $\hat{\Lambda}=0$, the most trivial choice at first sight seems to be the $(4+D)$-dimensional Minkowski space $V_0^{(4+D)} = M^{(4+D)}$, but then, we have a right to ask for an explanation of why the spacetime appears four-dimensional M^4. Therefore, this can not be a candidate and we see that we must make a choice which is close to M^4 locally. Hence, although a maximally symmetric $V_0^{(4+D)}$ contains more symmetries, we assume the vacuum space to be a direct product of two manifolds, that is $V_0^{(4+D)} = V^4 \times B^D$, where V^4 is a four-dimensional Riemann spacetime with a Lorentzian signature $(-,+,+,+)$ and B^D is a D-dimensional manifold with a Euclidean signature $(+,+,...,+)$. Here, V^4 characterizes the vacuum of the usual four-dimensional spacetime and B^D is assumed to be a compact space, that is, a closed bounded subset of the D-dimensional Euclidean space, which will characterize the internal space. In this way, if we assume the size of the compact space B^D to be very small, $V^4 \times B^D$ can simply appear to be V^4 containing the same four-dimensional symmetry group such as the Poincaré group P_4 for the case $V^4 = M^4$. Consequently, at every point of V^4, there will be a compact space B^D which is so small that we can not observe in daily life. On the other hand, the compactness of B^D will provide us gauge symmetry groups and discrete spectrums, which create the core of the theory.

2.2.1 Maximally symmetric spacetime in the ground state

Since our assumed vacuum space is a direct product space $V_0^{(4+D)} = V^4 \times B^D$, the general form of its metric is as follows:

$$\hat{g}^0_{AB}(x,y) = \begin{pmatrix} \hat{g}^0_{\mu\nu}(x) & 0 \\ 0 & \hat{g}^0_{ab}(y) \end{pmatrix}, \tag{2.7}$$

where $g^0_{\mu\nu}(x)$ is the metric of V^4 and $g^0_{ab}(y)$ is the metric of B^D. Now then, we again physically assume the four-dimensional vacuum space V^4 to be a maximally symmetric spacetime [17], that is a space of constant curvature

$$R_{\mu\nu\lambda\rho} = \frac{\lambda}{3}\left(g^0_{\mu\lambda}g^0_{\nu\rho} - g^0_{\mu\rho}g^0_{\nu\lambda}\right), \tag{2.8}$$

where λ is the four-dimensional cosmological constant. Therefore, V^4 is an Einstein space

$$R_{\mu\lambda} = \lambda g^0_{\mu\lambda}. \tag{2.9}$$

Here it is important to note also that we could write a more general form of the vacuum metric [22] $\hat{g}^0_{AB}$ with maintaining the maximal symmetry property of V^4. This is the warped product metric in the following form,

$$\hat{g}^0_{MN}(x,y) = \begin{pmatrix} f(y)g^0_{\mu\nu}(x) & 0 \\ 0 & g^0_{ab}(y) \end{pmatrix}. \tag{2.10}$$

However, in our consideration we will take the warp-factor as $f(y) = 1$.

Now, using (2.9) in the pure gravitational field equations (2.3), that is when $\hat{I} = 0$, we find that B^D is an Einstein space too,

$$R^D_{ab} = \rho g^0_{ab}, \tag{2.11}$$

where $\rho = \lambda$.

At this point, we have three different cases [22] according to the sign of λ. Firstly, when $\lambda > 0$, V^4 becomes a de Sitter spacetime with $SO(1,4)$ symmetry group but does not admit a positive energy theorem which is needed for stability and supersymmetry, so we immediately drop this option. Secondly, when $\lambda = 0$, V^4 becomes a Minkowski spacetime M^4 with the usual Poincaré group and admits a positive energy theorem and supersymmetry. Thirdly, when $\lambda < 0$, V^4 becomes an anti-de Sitter spacetime with $SO(2,3)$ group and again admits a positive energy theorem and supersymmetry. On the other hand, by a theorem in [23] saying that compact Einstein spaces with Euclidean signature and $\rho = \lambda < 0$ have no continuous symmetries, we eliminate the third one, that is, the anti de Sitter option, because we know that the gauge symmetries are continuous symmetries. Therefore, eventually we are left with the second case, which is $V^4 = M^4$ and Ricci tensors of both spaces become zero, $R_{\mu\lambda} = 0$ and $R_{ab} = 0$. This is equivalent to saying that when $\hat{I} = 0$, both spaces must be flat, $M^4 \times B^D$.

2.2.2 Structure of the internal space

Now, we should determine the structure of the internal compact manifold B^D, in a way that its isometry group contains Abelian or non-Abelian gauge group of interactions we want. In the case of flat internal compact space, $R_{ab} = 0$, we must restrict our consideration to spaces like a D-torus

$$B^D = \underbrace{S^1 \times S^1 \times ... \times S^1}_{D-times}, \tag{2.12}$$

or a $K3$-surface which is a compact complex surface [22]. The solution of D-torus yields just Abelian gauge symmetries and the solution for $K3$-surface yields no symmetries by any means.

The very simple way of obtaining non-Abelian gauge symmetries is to regard B^D as a coset space, but these type of spaces are Einstein spaces with $\rho \neq 0$. Therefore, we must choose between two cases: one of which V^4 is not a flat spacetime or one of which is $V^4 = M^4$ with $\hat{I} \neq 0$.

2.2.3 Criteria for the ground state

We have some fundamental criteria for a ground state to obey [24]. First of all, we know that it should be the lowest energy state. However, in gravity usually the comparison of energies will be impossible or pointless, because the definition of gravitational energy differs with the boundary conditions. To be able to make a comparison between any two solutions, their asymptotical structure should be the same. For instance, we cannot make a comparison between $M^4 \times B^D$ whose energy is zero and M^{4+D} whose energy is also zero, because their asymptotical structure is totally different.

Secondly, the solution must be stable both classically and semiclassically. By classical stability, we mean the stability of ground state under small perturbations. Given an arbitrary ground state $V^4 \times B^D$, we expand a general metric around this ground state and if there are no imaginary frequencies, that is, no exponentially growing modes, this ground state is classically stable, because an imaginary frequency w_k will imply the existence of exponential factors, $exp(\pm |w_k| t)$ which increase arbitrarily with time. By semiclassical stability, we mean the behavior of the ground state against a process of semiclassical barrier penetration. If there are two states which correspond to the local minima of an energy functional and are separated by a finite potential barrier, the system can tunnel to the more stable state. This tunneling process is associated with an instanton solution of the corresponding Euclidean field theory. Semiclassical instability can be investigated as follows. First, we find a local minimum of the corresponding Euclidean field theory which is the ground state that we want to analyze, then we search for a bounce solution that asymptotically approaches this local minimum. At the end, we look at the behavior of this bounce solution against small perturbations. If it is unstable, its contribution to the ground state energy will be imaginary which shows the instability of the ground state. At the same time, we can analyze the semiclassical stability by means of the positive energy theorem. For the case of $M^4 \times B^D$ whose energy is zero, if the positive energy theorem holds, in the sense that all solutions with the same asymptotic behavior have positive energy, then $M^4 \times B^D$ is semiclassically stable.

2.3 Five-Dimensional Kaluza-Klein Theory

Now, we start with the simplest KK theory, which is the five-dimensional theory originally constructed by Kaluza and Klein. Initially, the five-dimensional line element is $d\hat{s}^2 = \hat{g}_{MN} d\hat{z}^M d\hat{z}^N$, where $\{\hat{z}^M\} = \{x^\mu, y\}$, and the general metric tensor has the form

$$\hat{g}_{MN} = \begin{pmatrix} \hat{g}_{\mu\nu} & \hat{g}_{\mu 5} \\ \hat{g}_{5\nu} & \hat{g}_{55} \end{pmatrix}. \tag{2.13}$$

The theory is determined by the five-dimensional Einstein-Hilbert action without a cosmological constant or any matter fields, that is $\hat{I} = 0$,

$$\hat{S}_{GR} = -\frac{1}{16\pi\hat{G}} \int d^5\hat{z} \sqrt{-\hat{g}} \hat{R}, \tag{2.14}$$

and the equations of motion in vacuum

$$\hat{R}_{AB} - \frac{1}{2}\hat{g}_{AB}\hat{R} = 0 \Rightarrow \hat{R}_{AB} = 0. \tag{2.15}$$

The action (2.14) is invariant under five-dimensional general coordinate transformations

$$\hat{z}^A \to \hat{z}'^A = \hat{z}'^A(\hat{z}^B). \tag{2.16}$$

Accordingly, the metric transforms as

$$\hat{g}'_{AB}(\hat{z}') = \frac{\partial \hat{z}^C}{\partial \hat{z}'^A} \frac{\partial \hat{z}^D}{\partial \hat{z}'^B} \hat{g}_{CD}(\hat{z}). \tag{2.17}$$

2.3.1 Ground state of the five-dimensional theory

In five dimensions, the form of the ground state is $V_0^5 = V^4 \times B^1$. Since in one dimension $R_{\mu\nu\lambda\rho} = 0$, any one-dimensional space B^1 is flat. In this case, according to the discussions made before, the four-dimensional space should also be flat, $V^4 = M^4$. Now, we choose our compact space as a circle, $B^1 = S^1$ of fixed radius L, so that the vacuum has topology of $M^4 \times S^1$ rather than maximally symmetric M^5 and the extra dimension is periodic with period $2\pi L$

$$y = y + 2\pi L. \tag{2.18}$$

As a result, if we assume the radius of S^1 to be very small, the vacuum $M^4 \times S^1$ can simply appear to be M^4 containing the usual Poincaré group.

We can regard the ground state $M^4 \times S^1$ as a kind of spontaneous symmetry breaking [14]. The symmetry group of M^5, namely, the five-dimensional Poincare group P_5 is spontaneously broken to the symmetry group of $M^4 \times S^1$, $P_4 \times U(1)$. The emergence of the ground state in this way is called spontaneous compactification. Here, the point is that we have a compact internal space, with which we can account for the gauge transformation as a rotation in this space.

2.3.2 The harmonic expansion

The periodicity of the fifth dimension allows us to expand any dynamical variable $\hat{\gamma}(x^\mu, y)$ in the five-dimensional space including the metric tensor in terms of the complete set of the harmonics on S^1

$$\hat{\gamma}(x^\mu, y) = \sum_{n=-\infty}^{\infty} \hat{\gamma}^{(n)}(x^\mu) Y_n(y), \tag{2.19}$$

where $Y_n(y)$ are the eigenfunctions of the operator ∂_y^2

$$Y_n(y) = e^{iny/L}. \tag{2.20}$$

This is a Fourier expansion of $\hat{\gamma}(x^\mu, y)$ with the following orthogonality condition

$$\int_{y_0}^{y_0+L} dy Y_n^*(y) Y_m(y) = 2\pi L \delta_{nm}. \tag{2.21}$$

In this way, we can determine the spectrum of excitations for any dynamical variable first by expanding around its ground state.

2.3.3 Expansion around the ground state

By introducing local coordinates, the metric of the vacuum space $M^4 \times S^1$ can be written as

$$\hat{\eta}_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.22}$$

where $\eta_{\mu\nu}$ is the metric of M^4. As seen, the metric has the same form with M^5, because the spaces $M^4 \times S^1$ and M^5 are locally isomorphic [17].

We can now expand a general five-dimensional metric $\hat{g}_{MN}$, which is again a solution of $\hat{G}_{MN} = 0$, around its ground state value $\hat{\eta}_{MN}$ to the first order as a small fluctuation

$$\hat{g}_{MN}(x^\mu, y) = \hat{\eta}_{MN} + \hat{h}_{MN}(x^\mu, y). \tag{2.23}$$

Now, assuming $|\hat{h}_{MN}| \ll 1$ and keeping $\hat{h}_{MN}$ to the first order, the first order gravitational field equations are found as

$$\hat{G}_{MN} = \partial_N \partial_M \hat{h} + \hat{\Box} \hat{h}_{MN} - \partial_N \partial_P \hat{h}^N_{\ M} - \partial_P \partial_M \hat{h}^P_{\ N} - \hat{\eta}_{MN} \left(\hat{\Box} \hat{h} - \partial_A \partial_B \hat{h}^{AB} \right) = 0, \tag{2.24}$$

where $\hat{h} \equiv \hat{h}^M_{\ M}$ and the d'Alembertian operator $\hat{\Box} \equiv \hat{\eta}^{MN} \partial_M \partial_N$. Einstein tensor $\hat{G}_{MN}$ can be simplified by writing $\hat{h}_{MN}$ in terms of its trace reverse, which is defined as

$$\hat{\bar{h}}_{MN} = \hat{h}_{MN} - \frac{1}{2} \hat{\eta}_{MN} \hat{h}, \tag{2.25}$$

and $\hat{h}_{MN}$ can be written as,

$$\hat{h}_{MN} = \hat{\bar{h}}_{MN} - \frac{1}{3} \hat{\eta}_{MN} \hat{\bar{h}}, \tag{2.26}$$

so that $\hat{h} = -\frac{2}{3} \hat{\bar{h}}$. Substituting these expressions in (2.24), $\hat{G}_{MN}$ becomes

$$\hat{G}_{MN} = \hat{\Box} \hat{\bar{h}}_{MN} + \hat{\eta}_{MN} \partial_A \partial_B \hat{\bar{h}}^{AB} - \partial_N \partial_P \hat{\bar{h}}^P_{\ M} - \partial_M \partial_P \hat{\bar{h}}^P_{\ N} = 0. \tag{2.27}$$

Under infinitesimal general coordinate transformations $\hat{z}'^M = \hat{z}^M + \hat{\xi}^N(\hat{z})$, while $\hat{\eta}_{MN}$ stays the same, $\hat{h}_{MN}$ transforms as follows:

$$\hat{h}'_{MN} = \hat{h}_{MN} - \partial_M \hat{\xi}_N - \partial_N \hat{\xi}_M. \tag{2.28}$$

This transformation can be regarded as a gauge transformation of a symmetric tensor field in the flat background, because $\hat{h}_{MN}$ and $\hat{h}'_{MN}$ both solve the field equations

(2.24). We can now further simplify field equations (2.27) by making a gauge transformation of $\hat{\bar{h}}_{MN}$ with an appropriate gauge condition [25]. The transformed $\hat{\bar{h}}'^{MN}$ is equal to

$$\hat{\bar{h}}'^{MN} = \hat{\bar{h}}^{MN} - \partial^M \hat{\xi}^N - \partial^N \hat{\xi}^M + \hat{\eta}^{MN} \partial_P \hat{\xi}^P, \tag{2.29}$$

and the gauge condition we impose is the Lorenz gauge condition, $\partial_M \hat{\bar{h}}'^{MN} = 0$, leading to

$$\partial_M \hat{\bar{h}}'^{MN} = \partial_M \hat{\bar{h}}^{MN} - \hat{\Box}\hat{\xi}^N = 0 \Rightarrow \hat{\Box}\hat{\xi}^N = \partial_M \hat{\bar{h}}^{MN}. \tag{2.30}$$

In this gauge condition, by dropping bars and primes, field equations (2.27) take the form

$$\hat{G}_{MN} = \hat{\Box}\hat{h}_{MN}(x^\mu, y) = 0. \tag{2.31}$$

2.3.4 The effective low energy sector

The effective low energy sector describes the theory at energies low compared to the inverse size of the internal dimension of the KK theory. We see that (2.31) is the five-dimensional Klein-Gordon equation, and $\hat{h}_{MN}$ is a massless field in five dimensions. By Fourier expanding $\hat{h}_{MN}$ using (2.19), we obtain an infinite number of Klein-Gordon equations, that is, an infinite number of fields in four dimensions. For each mode n in four dimensions, we have

$$(\Box - \frac{n^2}{L^2})\hat{h}^{(n)}_{MN}(x^\mu) = 0. \tag{2.32}$$

All modes are massive with $m_n^2 = n^2/L^2$ except for the zero mode $h^{(0)}_{MN}(x^\mu)$. If L is of the order of Planck length, all the massive modes will have very high energies, which is greater than $\hbar c/L \approx 10^{19}$GeV, therefore it will be very difficult to excite them. For low energy effective theory where $E \ll \hbar c/L$, we will consider only the $n = 0$ mode of the general metric

$$\hat{g}_{MN}(x^\mu, y) = \sum_{n=-\infty}^{+\infty} \hat{g}^{(n)}_{MN}(x^\mu) e^{iny/L}, \tag{2.33}$$

which results in dropping the dependence on y, $\hat{g}_{MN} \approx \hat{g}^{(0)}_{MN}(x^\mu)$.

At the very beginning, first Kaluza [2] introduced this assumption

$$\partial_y \hat{g}_{AB} = 0, \tag{2.34}$$

called *cylinder condition*, that is, the variations of all state-quantities, including the components of the five-dimensional metric $\hat{g}_{AB}$, with respect to the fifth coordinate is so small that we can ignore them. Also, the corresponding isometry will result in a local U(1) gauge symmetry.

2.3.5 Residual coordinate transformations and the general metric

The cylinder condition puts limits on the general coordinate transformations (2.16). By taking out the metric $\hat{g}_{MN}$, from the transformation (2.17) and using (2.34), we get

$$\frac{\partial}{\partial y}\left[\left(\frac{\partial \hat{z}'^M}{\partial \hat{z}^R}\right)\left(\frac{\partial \hat{z}'^N}{\partial \hat{z}^L}\right)\hat{g}'_{MN}(\hat{z}')\right] = 0. \tag{2.35}$$

Now, we see that the partial derivatives $\partial x'^\mu/\partial x^\lambda, \partial x'^\mu/\partial y, \partial y'/\partial x^\lambda$ and $\partial y'/\partial y$ must not depend on y. We also require that the first four coordinates belong to the usual four-dimensional spacetime, that is to say, $\partial x'^\mu/\partial y$ and $\partial x^\mu/\partial y'$ vanish. In this way, the most general coordinate transformation we can construct is

$$x^\mu \to x'^\mu = x'^\mu(x^\nu), \tag{2.36}$$

$$y \to y' = ay + f(x^\nu), \tag{2.37}$$

and the inverse transformation is

$$x'^\mu \to x^\mu = x^\mu(x'^\nu), \tag{2.38}$$

$$y' \to y = \frac{y' - f(x^\mu(x'^\nu))}{a}, \tag{2.39}$$

where a is a constant for all coordinates and $a \neq 0$. Under the transformation (2.36), the metric components transform as a tensor

$$\hat{g}'_{\mu\nu} = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu}\hat{g}_{\lambda\rho}, \tag{2.40}$$

$$\hat{g}'_{\mu 5} = \frac{\partial x^\lambda}{\partial x'^\mu}\hat{g}_{\lambda 5} \quad \text{and} \quad \hat{g}'_{55} = \hat{g}_{55}, \tag{2.41}$$

and under the transformation (2.37), they transform as

$$\hat{g}'_{\mu\nu} = \hat{g}_{\mu\nu} - \hat{g}_{\nu 5}(\partial_\mu f) - \hat{g}_{\mu 5}(\partial_\nu f) + \hat{g}_{55}(\partial_\mu f)(\partial_\nu f), \tag{2.42}$$

$$\hat{g}'_{\mu 5} = a(\hat{g}_{\mu 5} - \hat{g}_{55}(\partial_\mu f)), \tag{2.43}$$

$$\hat{g}'_{55} = a^2\hat{g}_{55}. \tag{2.44}$$

Moreover, we know that if a four-dimensional spacetime is stationary, that is $\partial_0 g_{\mu\nu} = 0$, the line element can be written in terms of temporal $d\lambda$ and spatial dl separations [26]

$$ds^2 = g_{00} d\lambda^2 + dl^2, \tag{2.45}$$

where

$$d\lambda = dx^0 + \frac{g_{k0}}{g_{00}} dx^k \quad \text{and} \quad dl^2 = \left(g_{kl} - \frac{g_{k0} g_{l0}}{g_{00}} \right) dx^k dx^l. \tag{2.46}$$

The $d\lambda$ and dl are invariant under $x^0 \to x'^0 = x^0 + f(x^k)$ and under $x^k \to x'^k = x'^k(x^r)$. By analogy, in five-dimensional spacetime we can separate the line element into two parts as a result of our assumption (2.34), $d\hat{s}^2 = d\hat{l}^2 + g_{55} d\hat{\lambda}^2$, where $d\hat{\lambda}$ and $d\hat{l}$ are

$$d\hat{\lambda} = dy + \frac{\hat{g}_{\mu 5}}{\hat{g}_{55}} dx^\mu, \tag{2.47}$$

$$d\hat{l}^2 = \left(\hat{g}_{\mu\nu} - \frac{\hat{g}_{\mu 5} \hat{g}_{5\nu}}{\hat{g}_{55}} \right) dx^\mu dx^\nu. \tag{2.48}$$

These quantities are invariant under (2.36) and (2.37).

Now in the light of these, we will introduce a parametrization of the five-dimensional metric of the effective theory. Firstly, we can take $\hat{g}_{55}$ most generally as

$$\hat{g}_{55} = \Psi(x^\mu), \tag{2.49}$$

where Ψ is a dimensionless non-zero scalar field. This field will be later called the dilaton field. Secondly, we see that the transformation of the quantities $\hat{g}_{\mu 5}$ in (2.43) resembles a gauge transformation if we assume that $\hat{g}_{\mu 5}$ contains $\hat{g}_{55}$ in itself. Therefore, we define $\hat{g}_{\mu 5}$ as

$$\hat{g}_{\mu 5} = \kappa \hat{g}_{55} A_\mu(x^\mu) \Rightarrow \hat{g}_{\mu 5} = \kappa \Psi A_\mu(x^\mu), \tag{2.50}$$

where A_μ is the electromagnetic four-vector potential and κ is a proportionality constant, which we will take to be $\kappa = 1$ without loss of generality. Now then, we see that the transformation (2.43) is equivalent to

$$A'_\mu = A_\mu - \partial_\mu f, \tag{2.51}$$

where $A'_\mu = (\hat{g}'_{\mu 5}/a\Psi)$. Here, we see the most vital point of our construction since gauge transformation has now become equivalent to the movement in the fifth dimension. Finally, the invariance of (2.48) with respect to the translation in y and the fact that the term in parentheses of (2.48) is a tensor under (2.36) suggest that this is the unique alternative to identify with the usual four-dimensional line element $d\hat{l}^2 = ds^2$. So we take the four-dimensional metric tensor as

$$g_{\mu\nu} = \hat{g}_{\mu\nu} - \frac{\hat{g}_{\mu 5}\hat{g}_{5\nu}}{\hat{g}_{55}} \Rightarrow g_{\mu\nu} = \hat{g}_{\mu\nu} - \Psi A_\mu A_\nu. \tag{2.52}$$

In this way, the most general five-dimensional metric and the line element take the form

$$\hat{g}_{MN} = \begin{pmatrix} g_{\mu\nu} + \Psi A_\mu A_\nu & \Psi A_\mu \\ \Psi A_\nu & \Psi \end{pmatrix}, \tag{2.53}$$

$$d\hat{s}^2 = g_{\mu\nu}dx^\mu dx^\nu + \Psi(A_\mu dx^\mu + dy)^2. \tag{2.54}$$

From the identity $\hat{g}_{MN}\hat{g}^{NL} = \delta^L_M$, the inverse metric is

$$\hat{g}^{MN} = \begin{pmatrix} g^{\mu\nu} & -A^\mu \\ -A^\nu & \Psi^{-1} + A_\mu A^\mu \end{pmatrix}. \tag{2.55}$$

2.3.6 The dimensionally reduced action

We can now embed the metric (2.53) into the action (2.14) by writing everything in terms of the four-dimensional quantities. With a row reduction on (2.53), we find the determinant as

$$\hat{g} = \det(\hat{g}_{MN}) = \Psi\det(g_{\mu\nu}) = \Psi g, \tag{2.56}$$

and we calculate the following Christoffel symbols

$$\begin{aligned}
\hat{\Gamma}^\mu_{\nu\lambda} &= \Gamma^\mu_{\nu\lambda} + \frac{1}{2}\left[\Psi A_\lambda F_\nu{}^\mu + \Psi A_\nu F_\lambda{}^\mu - A_\lambda A_\nu \partial^\mu\Psi\right], \\
\hat{\Gamma}^\mu_{5\lambda} &= \frac{1}{2}\left[\Psi F_\lambda{}^\mu - A_\lambda\partial^\mu\Psi\right], \quad \hat{\Gamma}^\mu_{55} = -\frac{1}{2}\partial^\mu\Psi, \\
\hat{\Gamma}^5_{\nu\lambda} &= \frac{1}{2}(\nabla_\lambda A_\nu + \nabla_\nu A_\lambda) + \frac{1}{2}A^\rho(A_\nu A_\lambda\partial_\rho\Psi - A_\nu F_{\lambda\rho}\Psi - A_\lambda F_{\nu\rho}\Psi) \\
&\quad + \frac{1}{2}\Psi^{-1}(A_\nu\partial_\lambda\Psi + A_\lambda\partial_\nu\Psi), \\
\hat{\Gamma}^5_{5\lambda} &= \frac{1}{2}\left[A_\lambda A^\rho\partial_\rho\Psi + A^\rho F_{\rho\lambda}\Psi + \Psi^{-1}\partial_\lambda\Psi\right] \quad \text{and} \quad \hat{\Gamma}^5_{55} = \frac{1}{2}A^\rho\partial_\rho\Psi,
\end{aligned} \tag{2.57}$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$.

Meanwhile, from the following identity arising from the symmetries of the Riemann tensor

$$\partial_K(\hat{\Gamma}_{LMN} + \hat{\Gamma}_{MNL} + \hat{\Gamma}_{NLM}) = \partial_N\hat{\Gamma}_{KLM} + \partial_L\hat{\Gamma}_{KMN} + \partial_M\hat{\Gamma}_{KNL}, \tag{2.58}$$

where we fix the index $K = 5$ and let others run over all values, we see that the electromagnetic Bianchi identity and exchangeability of partial derivatives on Ψ occur

$$F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} + F_{\mu\nu,\lambda} = 0, \tag{2.59}$$

$$\Psi_{,\mu\nu} - \Psi_{,\nu\mu} = 0. \tag{2.60}$$

Next, we calculate Ricci scalar as

$$\hat{R} = R - \frac{1}{4}\Psi F_{\mu\nu}F^{\mu\nu} - \frac{2}{\Psi^{1/2}}\Box\Psi^{1/2}. \tag{2.61}$$

Substituting $\hat{R}$ into the action (2.14), we obtain the effective theory action as

$$\hat{S} = -\frac{1}{16\pi\hat{G}}\int d^4x dy(\sqrt{-g}\Psi^{1/2})(R - \frac{1}{4}\Psi F_{\mu\nu}F^{\mu\nu} - \frac{2}{\Psi^{1/2}}\Box\Psi^{1/2}). \tag{2.62}$$

Here we see that the compact structure of the fifth dimension guarantees the integral over the fifth dimension not to diverge, therefore integrating over y we get a constant $2\pi L$

$$\hat{S} = -\frac{1}{16\pi G}\int d^4x(\sqrt{-g}\Psi^{1/2})(R - \frac{1}{4}\Psi F_{\mu\nu}F^{\mu\nu} - \frac{2}{\Psi^{1/2}}\Box\Psi^{1/2}), \tag{2.63}$$

where $G = \hat{G}/2\pi L$. When we write the integral as

$$\hat{S} = -\frac{1}{16\pi G}\left[\int d^4x(\sqrt{-g}\Psi^{1/2})(R - \frac{1}{4}\Psi F_{\mu\nu}F^{\mu\nu}) - 2\int d^4x\sqrt{-g}\Box\Psi^{1/2}\right], \tag{2.64}$$

we see that the second integral in the parentheses is a surface term therefore we can drop it, because it does not affect the equations of motion, so in this way we end up with the following integral

$$\hat{S} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}(\Psi^{1/2}R - \frac{1}{4}\Psi^{3/2}F_{\mu\nu}F^{\mu\nu}). \tag{2.65}$$

Now, we see that this action contains the GR and Maxwell's theory coupled to a scalar field. To make this clear, we have to put the action into a more familiar form in which the coefficient of $\sqrt{-g}R$ is constant. Also in this form, the variation is simpler and it yields the typical Einstein's equations. This can be realized with a conformal transformation under which both $\hat{g}_{MN}$ and Ψ are rescaled. We apply a standard result [27] stating that if two n-dimensional metrics, with $n > 2$, are related through a conformal transformation such as

$$\hat{g}'_{MN} = \Omega^2\hat{g}_{MN} \Leftrightarrow \hat{g}'^{MN} = \Omega^{-2}\hat{g}^{MN}, \tag{2.66}$$

where Ω is a non-zero differentiable function, then their Ricci scalars have the following relationship

$$R' = \Omega^{-2}R - 2(n-1)\Omega^{-3}\Omega_{;MN}g^{MN} - (n-1)(n-4)\Omega^{-4}\Omega_{;M}\Omega_{;N}g^{MN}. \tag{2.67}$$

With $n = 5$, we transform $\hat{g}_{MN} \rightarrow \hat{g}'_{MN} = \Omega^2\hat{g}_{MN}$, then the term in the action transforms as $\sqrt{-g}\Psi^{1/2}R \rightarrow \sqrt{-g'}R' = \sqrt{-g}\Psi^{1/2}\Omega^3 R$. Thus, if we take $\Omega = \Psi^{-1/6}$, we can put the action into the desired form. Inserting these into (2.67) yields the transformed metric and the Ricci scalar

$$\hat{g}'_{MN} = \Psi^{-1/3}\begin{pmatrix} g_{\mu\nu} + \Psi A_\mu A_\nu & \Psi A_\mu \\ \Psi A_\nu & \Psi \end{pmatrix}, \tag{2.68}$$

$$\hat{R}' = \Psi^{1/3}(\hat{R} - \frac{5}{3}\Psi^{-2}g^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi + \frac{4}{3}\Psi^{-1}g^{\mu\nu}\partial_{\mu\nu}\Psi), \tag{2.69}$$

where $\hat{R}$ is defined in (2.61). Using (2.68) and (2.69) in (2.14) and dropping the total divergence, we find the low energy action

$$\hat{S} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}(R - \frac{1}{4}\Psi F_{\mu\nu}F^{\mu\nu} + \frac{1}{6}\frac{\partial_\mu\Psi\partial^\mu\Psi}{\Psi^2}). \tag{2.70}$$

We can regard the low energy sector of the KK theory as describing a massless scalar coupled to gravity and electromagnetism. Starting with such a higher dimensional theory, the process of obtaining a unified description for the effective four-dimensional spacetime by considering small fluctuations around ground state is called *dimensional reduction*.

2.3.7 The general dynamics

We now consider the motion of a classical point test particle of mass m in five-dimensional spacetime having the action

$$\hat{S}_m = m\int d\tau(\hat{g}_{MN}\frac{dz^M}{d\tau}\frac{dz^N}{d\tau})^{1/2}, \tag{2.71}$$

where τ is the proper time. The motion is given by a five-dimensional geodesic equation

$$\frac{d^2z^M}{d\tau^2} + \hat{\Gamma}^M_{NL}\frac{dz^N}{d\tau}\frac{dz^L}{d\tau} = 0. \tag{2.72}$$

Using the effective theory metric (2.53), the geodesic equation becomes

$$\ddot{y} + \hat{\Gamma}^5_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda + 2\hat{\Gamma}^5_{5\lambda}\dot{y}\dot{x}^\lambda + \hat{\Gamma}^5_{55}\dot{y}^2 = 0, \tag{2.73}$$

$$\ddot{x}^\mu + \hat{\Gamma}^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda + 2\hat{\Gamma}^\mu_{5\lambda}\dot{y}\dot{x}^\lambda + \hat{\Gamma}^\mu_{55}\dot{y}^2 = 0, \tag{2.74}$$

where dot refers to the derivative $\frac{d}{d\tau}$. For the Killing vector $K = K^M\partial_M = \partial_y$, we find the corresponding conserved quantity

$$K_M\frac{dz^M}{d\tau} = C \Rightarrow \Psi A_\mu\dot{x}^\mu + \Psi\dot{y} = C, \tag{2.75}$$

where C is a constant. We know this is the first integral of (2.73). If we use (2.75) in (2.74), the four-dimensional geodesic equation (2.74) becomes

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = \frac{q}{m}F^\mu{}_\lambda\frac{dx^\lambda}{d\tau} + \frac{q^2}{2m^2}\frac{\partial^\mu\Psi}{\Psi^2}, \tag{2.76}$$

describing the motion of a particle with mass m and charge q in the usual curved spacetime, provided that $C = q/m$. This determines the dimensions of C as $(A.s/kg)$. If $q\sqrt{16\pi G}$ is identified with the charge of the particle, then on the right of (2.76) we see the usual Lorentz force plus an interaction with the scalar field Ψ.

For a particle of mass m, the five-dimensional Lagrangian L contained in (2.71) is

$$\hat{L} = m\hat{g}_{MN}\frac{dz^M}{d\tau}\frac{dz^N}{d\tau}, \tag{2.77}$$

by which we can calculate the conjugate five-momentum from the usual definition

$$\hat{p}_M = \frac{\partial\hat{L}}{\partial\dot{z}^M} \Rightarrow \hat{p}_M = m\hat{g}_{MN}\dot{z}^N. \tag{2.78}$$

Then the five-momentum becomes

$$\hat{p}_\mu = mg_{\mu\lambda}\dot{x}^\lambda + A_\mu\hat{p}_5, \tag{2.79}$$

$$\hat{p}_5 = m(\Psi A_\lambda\dot{x}^\lambda + \Psi\dot{y}) \Rightarrow \hat{p}_5 = mC = q. \tag{2.80}$$

This means that the electric charge is essentially the fifth component of the five-momentum of a massive particle. In addition, (2.79) agrees with the generalized momenta of a particle of charge q interacting with an electromagnetic field, so we can identify $\hat{p}_\mu$ with π_μ.

We can now continue the analysis of the fifth dimension by considering a complex scalar field $\hat{\varphi}(x^\mu, y)$ in the five-dimensional space [10], with the following action

$$\hat{S} = \int d^5\hat{z}\sqrt{-\hat{g}}[(\partial_M\hat{\varphi})(\partial_N\hat{\varphi}^\dagger)\hat{g}^{MN}], \tag{2.81}$$

whose equations of motion are

$$\hat{\Box}\hat{\varphi} = 0 \ \text{ and } \ \hat{\Box}\hat{\varphi}^\dagger = 0. \tag{2.82}$$

By using the inverse metric (2.55) and the Fourier expansion of the field $\hat{\varphi}(x^\mu, y)$, we obtain the following expression for the term in the action

$$(\partial_M \hat{\varphi})(\partial_N \hat{\varphi}^\dagger)\hat{g}^{MN} = \left| \left(\partial_\mu + i\frac{A_\mu n}{L} \right) \hat{\varphi} \right|^2 + \Psi^{-1}\frac{n^2}{L^2} \left| \hat{\varphi} \right|^2. \tag{2.83}$$

If we compare (2.83) with the standard form of the minimal coupling $\partial_\mu \to \partial_\mu - iq_n A_\mu$, we may identify the electric charge q_n and the mass m_n of the n^{th} Fourier mode $\hat{\varphi}^{(n)}(x^\mu)e^{iny/L}$ as

$$q_n = \frac{n}{L} \quad \text{and} \quad m_n^2 = \frac{n^2}{L^2}. \tag{2.84}$$

Here, we notice that we can obtain a formula relating the radius of the extra dimension with the elementary charge. After we scale the gauge field $A_\mu \to (16\pi G)^{-1/2} A_\mu$ to get the proper normalization, we find the elementary charge, for $n = 1$, as

$$e = \frac{\sqrt{16\pi G}}{L}. \tag{2.85}$$

If we could deduce the radius L from any other argument, we would have an explanation for the quantization and the calculation of the numerical value of the elementary charge e.

2.4 The Non-Abelian Generalization

We now again look at the case of a general KK theory in $(4 + D)$ dimensions for the non-Abelian extension. We expect the symmetries of the internal compact space B^D to yield the non-Abelian gauge symmetries in the massless low energy sector for the effective observer in four dimensions.

2.4.1 The general metric

We now investigate the general structure of the space $V^{(4+D)}$ [17]. Inspired by the general structure of the vacuum space $V_0^{(4+D)} = V^4 \times B^D$, the general $V^{(4+D)}$ is assumed to have a fiber bundle structure, that is, a topological space which looks locally like a direct product of two spaces, $V^4 \times V^D$, where V^4 is the base manifold and V^D is the fiber. For every point x^μ of V^4, we have a D-dimensional hypersurface which is a Riemann space V^D. By using this property of $V^{(4+D)}$, we can determine the general form of the metric in $V^{(4+D)}$. We first set up both coordinate bases and orthonormal bases in $V^{(4+D)}$, $\hat{e}_A = \{\hat{e}_\alpha, \hat{e}_a\}$ and $\hat{e}_{\bar{B}} = \left\{\hat{e}_{\bar{\beta}}, \hat{e}_{\bar{b}}\right\}$ respectively. Here, $\hat{e}_\alpha$

and $\hat{e}_{\bar{\beta}}$ belong to V^4 and others belong to V^D. We can express any coordinate basis vectors in terms of the orthonormal basis vectors, $\hat{e}_A = (p_A^{\ \beta}\hat{e}_{\bar{\beta}} + p_A^{\ b}\hat{e}_{\bar{b}})$, where p's are components. Next, we demand that a displacement in the spacetime V^4 should be orthogonal to the internal space V^D. The underlying physical foundation is that a translation in the internal space V^D should leave the metric of the physical four-dimensional spacetime invariant. This is equivalent to saying that the expansion $\hat{e}_a = (p_a^{\ \beta}\hat{e}_{\bar{\beta}} + p_a^{\ b}\hat{e}_{\bar{b}})$ in V^D must contain only its orthonormal partner $\hat{e}_{\bar{b}}$ in V^D, i.e. $p_a^{\ \beta} = 0$. Thus, a general p in matrix notation is as follows

$$p_A^{\ B}(x,y) = \begin{pmatrix} p_\alpha^{\ \beta} & 0 \\ p_\alpha^{\ b} & p_a^{\ b} \end{pmatrix}, \tag{2.86}$$

and from $p_A^{\ B} q^A_{\ C} = \delta^B_C$, its inverse $q^A_{\ B}(x,y)$ is

$$q^A_{\ B}(x,y) = \begin{pmatrix} q^\alpha_{\ \beta} & q^a_{\ \beta} \\ 0 & q^a_{\ b} \end{pmatrix}. \tag{2.87}$$

The metric of $V^{(4+D)}$ can be written in terms of the components p

$$\hat{g}_{AD} = p_A^{\ B} p_D^{\ C} \eta_{BC} \tag{2.88}$$

and then the main blocks in $\hat{g}_{AB}$ are

$$\hat{g}_{\mu\nu} = p_\mu^{\ \alpha} p_\nu^{\ \beta} \eta_{\alpha\beta} + p_\mu^{\ r} p_\nu^{\ s} \eta_{rs}, \qquad \hat{g}_{\mu a} = p_\mu^{\ c} p_a^{\ d} \eta_{cd},$$
$$\hat{g}_{a\nu} = p_a^{\ c} p_\nu^{\ d} \eta_{cd} \ \text{ and } \ \hat{g}_{ab} = p_a^{\ c} p_b^{\ d} \eta_{cd}. \tag{2.89}$$

Now, metrics of V^4 and V^D are

$$g_{\mu\nu} = p_\mu^{\ \alpha} p_\nu^{\ \beta} \eta_{\alpha\beta} \ \text{ and } \ g_{ab} = p_a^{\ c} p_b^{\ d} \eta_{cd} \tag{2.90}$$

respectively. Thus, by using (2.90) and $B^a_\alpha := q^a_{\ b} p_\alpha^{\ b}$ in (2.89), the general form of the $(4+D)$-metric becomes

$$\hat{g}_{AB} = \begin{pmatrix} g_{\mu\nu} + g_{ab} B^a_\mu B^b_\nu & B^b_\mu g_{ba} \\ g_{ab} B^b_\nu & g_{ab} \end{pmatrix}, \tag{2.91}$$

and its inverse is given as

$$\hat{g}^{AB} = \begin{pmatrix} g^{\mu\nu} & -B^a_b g^{\mu b} \\ -B^a_b g^{b\nu} & g^{ab} + g^{cd} B^a_c B^b_d \end{pmatrix}. \tag{2.92}$$

Although this derivation is not entirely rigorous, the form of this metric is unique and the full proof can be found in [8]. As it is seen, the form of the metric covers the one we found for the five-dimensional KK theory.

2.4.2 Residual coordinate transformations

The fiber structure of the $4+D$-dimensional space puts also a limitation on the most general coordinate transformations, $z' = z'(z)$. If we look at the transformation of $p_\alpha{}^\beta$ under $z' = z'(z)$

$$p'^{\ \beta}_{\alpha} = \frac{\partial x^\sigma}{\partial x'^\alpha} p_\sigma{}^\beta + \frac{\partial y^a}{\partial x'^\alpha} p_a{}^\beta, \tag{2.93}$$

we see that to guarantee the condition $p_a{}^\beta = 0$ of the previous section, we need to allow only the coordinate transformations where y does not depend on x'

$$z' = z'(z) \longrightarrow x' = x'(x) \ \text{ and } \ y' = y'(x,y). \tag{2.94}$$

The components $g_{\mu\nu}$ and g_{ab} of the general metric (2.91) transform as a tensor in V^{4+D} under the reduced coordinate transformations (2.94)

$$g'_{\mu\nu}(x') = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu} g_{\lambda\rho}(x) \ \text{ and } \ g'_{ab}(y') = \frac{\partial y^c}{\partial y'^a}\frac{\partial y^d}{\partial y'^b} g_{cd}(y). \tag{2.95}$$

The B^b_ν transform as follows under (2.94)

$$B'^a_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\left[\frac{\partial y'^a}{\partial y^b} B^b_\nu - \frac{\partial y'^a}{\partial x^\nu}\right]. \tag{2.96}$$

2.4.3 Isometries of the compact internal space

Infinitesimal general coordinate transformations in B^D is written as

$$y^a \to y'^a = y^a + \varepsilon^i K^a_i, \tag{2.97}$$

where $i = 1, ..., n$. If this transformation is an isometry, that is, preserving the form of the metric g_{ab}, the vectors K^a_i are Killing vectors and they satisfy the Killing equation $\nabla_a K_{ib} + \nabla_b K_{ia} = 0$.

We assume the compact space B^D has a set of n linearly independent global Killing vectors K^a_i, $i = 1, ..., n$, where n is $D \leq n \leq \frac{1}{2}D(D+1)$ [14]. Thus, B^D must have a positive curvature scalar [23]. The Killing vectors have the following commutation relations via the Lie bracket

$$[K_i, K_j]^a = f_{ij}{}^k K^a_k, \tag{2.98}$$

where $f_{ij}{}^k K^a_k = K^b_i \partial_b K^a_j - K^b_j \partial_b K^a_i$. (2.98) forms the Lie algebra of the isometry group G in B^D, which will be identified as a gauge group in the effective low energy sector. By choosing B^D appropriately, we can get almost any gauge group. This can be done either by choosing B^D as the group manifold of G or by choosing B^D as a coset space G/H so that its isometry group will be G.

2.4.4 Non-Abelian gauge transformations

Now, we consider the following infinitesimal coordinate transformations in B^D

$$x^\mu \to x'^\mu = x^\mu \quad \text{and} \quad y^a \to y'^a = y^a + \varepsilon^i(x) K_i^a(y). \tag{2.99}$$

Let the functions B_μ^a in the general metric (2.91) be in the form $B_\mu^a = K_i^a(y) A_\mu^i(x)$. Then, by using (2.96), we can find the infinitesimal transformation of $A_\mu^i(x)$ under the coordinate transformation (2.99) as

$$\delta A_\mu^i = -\partial_\mu \varepsilon^i(x) + f_{jk}{}^i A_\mu^j \varepsilon^k(x). \tag{2.100}$$

This is exactly an infinitesimal transformation of the non-Abelian gauge field with respect to the gauge transformation of $\delta y^a = \varepsilon^i(x) K_i^a(y)$.

2.4.5 Spontaneous compactification

KK theory can be regarded as a theory with spontaneous symmetry breaking where the maximal symmetry of $M^{(4+D)}$, the group of general coordinate transformations in (4+D) dimensions, that is (4+D)-dimensional Poincare group $P_{(4+D)}$ is spontaneously broken to the product of the four-dimensional coordinate transformation group of V^4 and the group of coordinate transformations in compact space B^D.

CHAPTER 3

KALUZA-KLEIN MONOPOLE

All grand unified theories in which an original semi-simple symmetry group G' is spontaneously broken by a Higgs mechanism to a subgroup H including a local $U(1)$ necessarily contain soliton-like solutions [28, 29]. If this $U(1)$ group is identified with the gauge group of electromagnetic theory, these solutions turn out to be magnetic monopoles. The existence and classification of these monopoles can be realized topologically by the first homotopy group of H, $\pi^1(H)$, or equivalently by the second homotopy group of (G'/H), $\pi^2(G'/H)$, because H is a subgroup of G'. Similarly in KK theories, we know there is a kind of symmetry breaking in which $(4+D)$-dimensional Poincaré gopup P^{4+D} has been spontaneously broken to $P^4 \times G$ by means of the spontaneous compactification. Therefore, the existence of magnetic monopoles in the KK theories is correspondingly related to the topology of gauge groups G which is unified with gravity. If we assume the spacetime is locally $M \times \Sigma$ where M is asymptotically flat and Σ is a D-dimensional compact manifold, we can determine the expected monopoles by examining the topological structure of spacelike infinity [14]. At spacelike infinity, the spacetime can be regarded as a G-valued bundle over S^2 where G is the isometry group of Σ and S^2 is the boundary of three-dimensional space. Non-trivial bundles are classified by $\pi^1(G)$ whose elements are associated with the charge of the expected monopoles. For $G = Sp(n)$, $SU(n)$, E_6, E_7 or E_8, $\pi^1(G) = 0$, the non-empty $\pi^1(G)$ occurs for the groups $G = U(n)$ and $G = SO(n)$. These are $\pi^1(U(n)) = \mathbb{Z}$, the set of integers, and $\pi^1(SO(n)) = \mathbb{Z}_2$, the cyclic group of order two, corresponding to monopoles of Abelian and non-Abelian gauge theories, respectively.

3.1 Abelian Kaluza-Klein Monopoles

3.1.1 The Sorkin-Gross-Perry monopole

We start now with the simplest one, Sorkin-Gross-Perry (SGP) monopole, which is the first monopole found in the original five dimensional KK theory independently by Sorkin [9] and by Gross and Perry [10] in 1983. We are searching for non-singular, static and topologically stable solutions of the field equations $\hat{R}_{MN} = 0$ of the five-dimensional KK theory, which may describe the spatially localized lumps of matter. The solution we look for will attain meaning in the original parametrization of the general KK metric

$$\hat{g}_{MN} = \begin{pmatrix} g_{\mu\nu} + \Psi A_\mu A_\nu & \Psi A_\mu \\ \Psi A_\nu & \Psi \end{pmatrix}. \tag{3.1}$$

For topological reasons, we naturally demand our solutions to approach the vacuum solution at least locally, $\hat{g}_{MN} \to \hat{\eta}_{MN}$, at spatial infinity, $r \to \infty$. In our situation, this corresponds to saying that our solutions will be asymptotically locally flat because of our periodic fifth dimension in vacuum space structure, $M^4 \times S^1$. According to our parametrization of the effective theory metric (3.1), the vacuum solution (2.22) is characterized by

$$g_{\mu\nu} = \eta_{\mu\nu}, \quad A_\mu = 0 \;\text{ and }\; \Psi = 1. \tag{3.2}$$

A static metric is defined as a metric that has a timelike Killing vector field K which is orthogonal to spacelike hypersurfaces. In other words, for a static spacetime, the metric components can be made to satisfy

$$\partial_t \hat{g}_{MN} = 0 \;\text{ and }\; \hat{g}_{0N} = \delta_{0N}, \tag{3.3}$$

by selecting an appropriate coordinate system, that is, the metric components are all independent from the time coordinate t and there are no cross terms involving dt. In this way, the field equations can be reduced to

$$\hat{R}_{ij} = \hat{R}_{5j} = \hat{R}_{55} = 0 \tag{3.4}$$

by taking $\hat{g}_{00} = -1$. Thus, $\hat{g}_{MN}$ may split as $\hat{g}_{MN} = -dt^2 + \hat{g}_{ij} dz^i dz^j$ where $i, j = (1, 2, 3, 5)$ and, $\hat{R}_{00}$ and $\hat{R}_{0N}$ are automatically satisfied. This also corresponds to choosing the $A_0 = 0$ gauge. Now obviously, (3.4) implies that we are searching for a solution to the four-dimensional Euclidean gravity with signature $(+, +, +, +)$. Then the only possible interpretation of y in four dimensions is to take it as a periodic Euclidean time. We know that we can generate lots of solutions to these equations using the Schwarzschild, Kerr or Taub-NUT metrics by Wick rotating the time coordinate $t \to i\tau$ to obtain the Euclideanized form. For instance, the simplest solution turns out to be the Euclidean Schwarzschild metric

$$d\hat{s}^2 = -dt^2 + ds^2_{SCH} = -dt^2 + (1-\frac{2m}{r})^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 + (1-\frac{2m}{r})d\tau^2, \tag{3.5}$$

where (r, θ, ϕ) denote the usual spherical coordinates. This metric has an Euclidean character in the range $r \geq 2m$, therefore the $r = 0$ spacetime singularity is out of the Euclidean range and the apparent coordinate singularity at $r = 2m$ can be removed by taking τ as an angular variable with period $8\pi m$, $0 \leq \tau \leq 8\pi m$.

To identify our solution as a monopole, we expect the solution leads to a monopole field in the effective four-dimensional spacetime in the form of $A_\phi = 4m(1 - \cos\theta)$ having the Dirac-string singularity. Now, the Taub-NUT metric ds^2_{TN} provides this form with the coordinates (τ, r, θ, ϕ)

$$ds^2_{TN} = \Psi(d\tau + 4m(1-\cos\theta)d\phi)^2 + \frac{1}{\Psi}(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2), \tag{3.6}$$

where $\Psi = \frac{r}{r+4m}$. This metric depends on a parameter m which can be positive or negative. For $m > 0$ case, ds^2_{TN} becomes a non-singular solution of the Euclidean equation for the self-dual Riemann tensor ($\tilde{R}_{\alpha\beta\mu\nu} = \epsilon_{\alpha\beta}{}^{\gamma\delta}R_{\gamma\delta\mu\nu}$) and it is interpreted as a gravitational instanton [30]. Later, we will see that the parameter m is proportional to the mass of the monopole.

Now, by replacing the τ coordinate with our periodic coordinate y and adding the $-dt^2$ in (3.6), we can write our solution explicitly as

$$\begin{aligned} d\hat{s}^2 = -dt^2 + ds^2_{TN} = -dt^2 &+ \Psi(dy + 4m(1-\cos\theta)d\phi)^2 \\ &+ \frac{1}{\Psi}(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2), \end{aligned} \tag{3.7}$$

where (r, θ, ϕ) are the spherical polar coordinates of the spatial part of the usual four-dimensional spacetime. If we match our parametrization with this solution, all components of A_μ except for A_ϕ vanish

$$A_\phi = 4m(1-\cos\theta), \tag{3.8}$$

leading to

$$\mathbf{E} = 0 \quad \text{and} \quad \mathbf{B} = \frac{4m\mathbf{r}}{r^3}, \tag{3.9}$$

which is that of a magnetic monopole.

A quick check of the solution (3.7) reveals that there may be two singularities, at $\theta = \pi$ and at $r = 0$. Firstly, the metric has a singularity as $\theta \to \pi$, because the loop formed by ϕ about $\theta = \pi$ axis does not shrink to zero, namely, the term $(1 - \cos\theta)$ does not go to zero. This corresponds to the Dirac-string singularity of the monopole. To avoid this, we can make a change of coordinates $y \to y' = y + 8m\phi$ around the $\theta = \pi$ axis, then the related part transforms as

$$dy + 4m(1 - \cos\theta)d\phi \to dy' - 4m(1 + \cos\theta)d\phi, \tag{3.10}$$

and the metric becomes regular here. However, with this coordinate set, we again have the same singularity when $\theta \to 0$. Therefore, we must use two sets of coordinate systems to cover the whole space. For the northern hemisphere, we use the set (t, r, θ, ϕ, y) and for the southern hemisphere we use (t, r, θ, ϕ, y'). This way, we see that the Dirac-string singularity becomes a coordinate singularity in the SGP monopole. As for the second singularity, when $r \to 0$ the singular part of the metric behaves like

$$\frac{r}{4m}dy^2 + \frac{4m}{r}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2). \tag{3.11}$$

If we let $r = r'^2$, then (3.11) becomes

$$16m\left[dr'^2 + \left(\frac{r'}{8m}\right)^2 dy^2\right] + 4mr'^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.12}$$

Here, by considering the metric of a two-dimensional plane in polar coordinates $ds^2 = dr^2 + r^2 d\varphi^2$, we see that we can eliminate the singularity if we associate $y/8m$ with φ. To eliminate the singularity we assume that y is an angular variable with period $16\pi m$. Therefore, we must identify $16\pi m$ with $2\pi L$ since our solution approaches to the vacuum $M^4 \times S^1$ for large r, which leads to

$$m = \frac{L}{8}. \tag{3.13}$$

Now, the circle characterized by y shrinks to a point at $r = 0$. Moreover, the other singularity seen in front of the dr^2 in (3.11) disappears too. Hence, we see that the solution (3.7) is regular everywhere.

3.1.1.1 Charge of the SGP monopole

To determine the magnetic charge of the monopole, we first scale the magnetic field properly $\mathbf{B} \to (16\pi G)^{-1/2}\mathbf{B}$

$$\mathbf{B} = \frac{1}{\sqrt{16\pi G}} \left[\frac{(16\pi G)^{-1/2} 4m\mathbf{r}}{r^3} \right]. \tag{3.14}$$

Then, the magnetic charge $g = 4m/\sqrt{16\pi G}$ and from $m = L/8$, it becomes $g = L/(2\sqrt{16\pi G})$. The charge g is fixed by the fifth dimension similar to the case of the electric charge. Here, we see that if the quantization of the electric charge is dictated by topology, it is possible to obtain a magnetic monopole. Lastly, by writing L in terms of the electric charge from the discussion of the previous chapter (2.85), we see that our monopole obeys the Dirac quantization rule:

$$g = \frac{1}{2e}. \tag{3.15}$$

3.1.1.2 Mass of the SGP monopole

For an asymptotically flat spacetime for which one has at least $g_{AB} \to \eta_{AB} + O(1/r)$ as $r \to \infty$, we can calculate a conserved momentum belonging only to the gravitational field by using the conserved energy-momentum pseudo-tensor of the gravitational field [26], which is given as

$$t^{AB} = \frac{1}{16\pi\hat{G}(-\hat{g})} \partial_C \partial_D [(-\hat{g})(\hat{g}^{AB}\hat{g}^{CD} - \hat{g}^{AC}\hat{g}^{BD})]. \tag{3.16}$$

The corresponding conservation is provided by

$$\frac{\partial}{\partial x^B}(-\hat{g})t^{AB} = 0, \tag{3.17}$$

and the corresponding conserved quantities are given as

$$P^A = \int (-\hat{g})t^{A0} dV. \tag{3.18}$$

In our five-dimensional SGP metric, there is nothing other than the gravitational field, so we can calculate the mass P^0 of the monopole by using (3.18). At spatial infinity, t^{00} is found as

$$t^{00} \simeq -\nabla^2(\frac{1}{\Psi}). \tag{3.19}$$

Therefore, the mass of the monopole is

$$M = P^0 = \frac{-2\pi R}{16\pi\hat{G}} \int d^3x \nabla^2(\frac{1}{\Psi}) = \frac{m}{G}. \tag{3.20}$$

Since, we know m in terms of G and e, we can determine the mass. Remember that we already knew the fifth component of momentum would correspond to the conserved charge. Now this can be verified by calculating the component P^5. At spatial infinity, we find t^{50} approximately as

$$t^{50} \simeq \frac{1}{16\pi\hat{G}} \frac{1}{\sqrt{|\hat{g}|}} \partial_i[\sqrt{|\hat{g}|}(\partial^0 A^i - \partial^i A^0)] = \frac{\nabla.\mathbf{E}}{16\pi\hat{G}}. \tag{3.21}$$

3.1.1.3 Multi-monopole generalization

We can easily generalize the monopole solution to the static multi-monopole case. Basically, our aim is to construct a metric that will contain a four-potential of an arrangement of N monopoles sitting at $\mathbf{r} = \mathbf{r}_n$ where $n = 1, 2, ..., N$. Firstly, Ψ is calculated from

$$\frac{1}{\Psi} = 1 + \sum_{n=1}^{N} \frac{4m}{|\mathbf{r} - \mathbf{r}_n|}. \tag{3.22}$$

Next, the four-potential and the magnetic field is given as

$$\partial_i \left(\frac{1}{\Psi} \right) = -\epsilon_{ijk} \partial_j A_k = B_i. \tag{3.23}$$

Finally, the corresponding solution takes the form

$$d\hat{s}^2 = -dt^2 + \Psi(A_i dx^i + dy)^2 + \frac{1}{\Psi}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2). \tag{3.24}$$

Similarly, as long as y is periodic with $16\pi m$, this metric is regular everywhere. Besides, since the monopoles do not interact with each other, the total mass of this configuration is simply NM, where each one of the monopoles has the same mass M. Here, we notice that we cannot construct this solution with monopoles having different masses, because at each point $\mathbf{r} = \mathbf{r}_n$ we would have a singularity which could be eliminated with different periods imposed on y coordinate, to avoid all of the singularities each monopole should have the same mass. We can also form the analogous antimonopole solutions simply by changing the sign of $\mathbf{B}$.

3.1.1.4 Gravitational mass of the monopole

Although, it was calculated the inertial mass of the monopole solution, the gravitational mass of the monopole turns out to be zero [10]. This is evident from the form of the solution metric, because the Newtonian force applied on a slowly moving test particle, which is proportional to $\frac{1}{2}\nabla\hat{g}_{00}$ vanishes. Therefore, all kinds of solutions have static geodesics which correspond to test particles sitting at rest with respect to the monopole. More explicitly, these geodesics are in the form of $x^0(\tau) = \tau$, $\mathbf{x}(\tau) = \mathbf{x}_0$ and $y(\tau) = y_0$, where the parameter used is the proper time τ. However, this is not because the principle of equivalence is violated where we have preserved the general five-dimensional covariance, this occurs because of the violation of Birkhoff's theorem in five dimensions.

Birkhoff's theorem states that any solution of the field equations for a localized lump in four dimensions reduce to the Schwarzschild metric at spatial infinity

$$ds^2_{SCH} \simeq -(1-\frac{2M}{r})dt^2 + (1-\frac{2M}{r})^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{3.25}$$

so that Schwarzschild metric is the unique spherically symmetric solution. From (3.25), we see that we have an inertial mass of M and a gravitational potential of $\frac{1}{2}(g_{00}+1) = M/r$. However, we see that this theorem is not true in KK theory. For instance, we have two such independent spherically symmetric solutions, one is the Schwarzschild solution (3.5) with $g_{0A} = \delta_{0A}$ and the other one is again Schwarzschild metric but this time with $g_{5A} = \delta_{5A}$. Only for this last type of solutions, we have an equality between the inertial mass and the gravitational mass. But, our monopole solutions are all the first type.

3.1.2 Generalization to $(4 + D)$ dimensions

The SGP monopole solution can be generalized to $(4+D)$ dimensions for given KK theories whose vacuum structure is $M^4 \times T$, in which T has an Abelian isometry group. Naturally, we again consider static metrics for these solutions. Such a generalization was first made by Lee in 1984 [11, 12]. We start with the following metric in $(4 + D)$ dimensions

$$d\hat{s}^2 = g_{\mu\nu}(x^\mu)dx^\mu \otimes dx^\nu + \Phi_{ab}(x^\mu)\theta^a \otimes \theta^b, \tag{3.26}$$

where θ^a is

$$\theta^a = dy^a + A_\mu{}^a(x^\mu)dx^\mu, \qquad a = 1, ..., D. \tag{3.27}$$

Here, we note that if for example Φ_{ab} is taken to be diagonal, our compact manifold becomes the D-torus, $T = S^1 \times S^1 \times ... \times S^1$. Now, we impose the staticity and spherical symmetry conditions on the four-dimensional metric to obtain the following metric ansatz

$$g_{\mu\nu}(x)dx^\mu \otimes dx^\nu = -e^{2\Psi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\Omega. \tag{3.28}$$

To construct the monopole field, the following non-vanishing components of electromagnetic field strength tensor should read

$$F^a_{\ \theta\phi} = g^a \sin\theta \quad \text{and} \quad F^{atr} = p^a(r). \tag{3.29}$$

Finally, we make the following ansatz for Φ,

$$\Phi_{ab}(x^\mu) = (exp(2\chi(r)))_{ab}. \tag{3.30}$$

Under these, it was shown that the field equations can be completely integrated [12]. In five dimensions, the complete solutions, which correspond to all spherically symmetric monopoles, were obtained [31]. These solutions in general have gravitational masses and are singular at the origin except in the SGP limit. Also in six dimensions, the solutions were obtained by reducing the field equations to the Toda lattice problem [11].

3.2 Non-Abelian Kaluza-Klein Monopoles

We now consider the monopole solutions in non-Abelian KK theories. By assuming our gauge group to be $SO(n)$, which makes our compact manifold S^n, we can construct monopole solutions to these non-Abelian theories. Such a construction was first made by Perry in 1984 [14]. We assume the compact manifold Σ to be S^2, which has constant curvature and can be represented by a line element

$$ds^2_\Sigma = d\psi^2 + \sin^2\psi d\chi^2. \tag{3.31}$$

Σ has a set of three Killing vectors,

$$\begin{aligned} K^1 &= -\sin\chi\frac{\partial}{\partial\psi} - \cot\psi\cos\chi\frac{\partial}{\partial\chi}, \\ K^2 = \cos\chi\frac{\partial}{\partial\psi} - \cot\psi\sin\chi\frac{\partial}{\partial\chi} \quad &\text{and} \quad K^3 = \frac{\partial}{\partial\chi}. \end{aligned} \tag{3.32}$$

For simplicity, we choose a gauge such that $A^1 = A^2 = 0$. Next, we take the following spherically symmetric metric in six-dimensions

$$\begin{aligned} d\hat{s}^2 = &-h^2(r)dt^2 + g^2(r)dr^2 + k^2(r)(d\theta^2 + \sin^2\theta d\phi^2) \\ &+p^2(r)[d\psi^2 + \sin^2\psi(d\chi + \frac{A^3}{p(r)})^2]. \end{aligned} \tag{3.33}$$

If we choose $A^3 = p(r)[q(r)\cos\theta d\phi - s(r)dt]$, then this represents both a magnetic q and electric s charged particle. Unless we define $q = \frac{n}{2}$ where n is an integer, the string singularities along $\psi = 0$ or $\psi = \pi$ occur. When $n = 1$ mod 2, the space is topologically non-trivial, because it is homeomorphic to the twisted S^2 bundle over S^2 at spatial infinity. Therefore, we can naturally expect $\mathbb{Z}_2$ number of topological excitations.

In such a theory, the compactification can be realized most easily by introducing an auxiliary antisymmetric tensor field derived from an antisymmetric potential [19]. The tensor field can be four-indexed or two-indexed because of duality. However, there are no monopole solutions with two-indexed case [14]. Therefore, it is assumed that we have the following four-indexed tensor field

$$\hat{F}_{ABCD} = \nabla_{[A}\hat{A}_{BCD]}, \tag{3.34}$$

where $\hat{A}_{BCD}$ is a three-indexed antisymmetric potential. The action of the theory is

$$\hat{S} = -\frac{1}{16\pi\hat{G}}\int d^6\hat{z}\sqrt{|\hat{g}|}\left[(\hat{R} - 2\hat{\Lambda}) + \frac{1}{8}\hat{F}_{ABCD}\hat{F}^{ABCD}\right]. \tag{3.35}$$

In [14], it was shown that with the chosen metric ansatz (3.33), there are solutions to the corresponding field equations

$$\hat{R}_{AB} - \frac{1}{2}\hat{g}_{AB}\hat{R} + \hat{g}_{AB}\hat{\Lambda} = 8\pi\hat{G}(\hat{F}_{ACDE}F_B{}^{CDE} - \frac{1}{8}\hat{g}_{AB}\hat{F}_{CDEF}\hat{F}^{CDEF}). \tag{3.36}$$

CHAPTER 4

KALUZA-KLEIN MONOPOLE IN ADS SPACETIME

It is important to generalize the KK monopoles to spaces of non-zero constant curvature namely, de Sitter (dS) and anti-de Sitter (AdS) spacetimes, for several reasons such as the completeness of topic, the current theoretical interest in these spaces thanks to the AdS/Conformal Field Theory (CFT) duality and the recent experimental data pointing out to a cosmological constant of dS type. For AdS spaces, there are open hypersurfaces using which we can define magnetic flux, so that we expect monopole solutions. In contrast, for dS spaces, since there are closed hypersurfaces with a boundary, we cannot define the magnetic flux and do not expect the existence of monopole solutions. In spite of this, we can expect neutral solutions such as magnetic dipoles [16].

4.1 SGP Monopole in the AdS Background

Now, we start with a non-singular solution, the original SGP monopole in five dimensions. We mostly follow the discussion made in a paper written by Önemli and Tekin [16]. We want to find a solution which will characterize the SGP monopole completely in a five-dimensional AdS spacetime. Therefore, we put a bunch of criteria to be satisfied by the solution. First of all, we expect our solution to be asymptotically locally AdS rather than asymptotically AdS, similar to the case of the SGP monopole, because we again need to compactify the extra spatial dimension. Secondly, since we know that as the cosmological constant $\hat{\Lambda} \to 0$, we should recover the Minkowski spacetime in a continuous way, to claim that our solution is a true AdS analog, we correspondingly expect our solution to reduce to the original SGP monopole in the limit where again $\hat{\Lambda} \to 0$. Thirdly, the solution should be static. Finally, as the SGP monopole reduces to vacuum when its charge goes to zero, in the same zero charge limit, our solution should also reduce smoothly to the AdS vacuum, that is to a background metric with $\hat{\Lambda} < 0$.

4.1.1 The no-go result in the five-dimensional AdS spacetime

For such a solution, one possibility inspired from the case of the SGP solution is to use a four-dimensional gravitational instanton type space, such as the AdS-Taub-NUT. However, in our case we can not form the solution from a direct product of time coordinate with a gravitational instanton so that we could easily add time coordinate to the instanton metric. Instead, the solution should be of the form of a warped product metric

$$\hat{g}_{AB}(\hat{z}) = \begin{pmatrix} k(t) & 0 \\ 0 & h(t) g_{ab}(r, \theta, \phi, y) \end{pmatrix}, \tag{4.1}$$

where $a, b = \{1, 2, 3, 4\}$. However, this introduces an explicit time dependence in the metric like a cosmological solution, hence we skip this option.

Another way which we will follow is to first search for a suitable background metric of a five-dimensional spacetime in which a static SGP monopole can be defined. In this search, the topology of the background will be essential in the sense that one of the spatial dimensions should be compactified on a circle with an arbitrary radius. In addition, this background has to be an Einstein space with a negative cosmological constant $\hat{\Lambda} = -2L^2$, satisfying the field equations:

$$\hat{R}_{AB} = -2L^2 \hat{g}_{AB}. \tag{4.2}$$

In five dimensions, the standard solution of (4.2) is the maximally symmetric AdS_5 spacetime, which can be written in static form as

$$d\hat{s}^2 = -\cosh^2(Lr/\sqrt{2})dt^2 + dr^2 + (2/L^2)\sinh^2(Lr/\sqrt{2})d\Omega_3, \tag{4.3}$$

where the metric on S^3 can be taken as $d\Omega_3 = d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2)$. However, it is impossible to interpret this metric as a suitable background, since we cannot compactify any one of the ψ, θ or ϕ coordinates on an arbitrary circle, they are all fixed by the condition of being Einstein space. In addition, since the magnetic field of our solution should spread radially in three non-compact spatial dimensions, we expect for the effective four-dimensional part of the solution to contain an explicit spatial part of the AdS_3 form. As seen, there is no way to extract AdS_3 from (4.3). We therefore assume the topology of the background to be a product space $\mathrm{AdS}_2 \times \mathrm{AdS}_3$ in the form

$$d\hat{s}^2 = (-1/2)e^{-2\sqrt{2}Ly}dt^2 + dy^2 + dr^2 + (1/L^2)\sinh^2(Lr)d\Omega_2, \tag{4.4}$$

whose AdS_2 and AdS_3 parts are

$$ds^2_{AdS_2} = (-1/2)e^{-2\sqrt{2}Ly}dt^2 + dy^2 \quad \text{and} \quad ds^2_{AdS_3} = dr^2 + (1/L^2)\sinh^2(Lr)d\Omega_2,$$

where $d\Omega_2 = d\theta^2 + \sin^2\theta d\phi^2$ and y is the compact extra dimension. Here, we see that the spatial part of spacetime is characterized only by the AdS_3 part.

For the moment, we can make a prediction about the required form of the magnetic field in AdS_3. This form is evident from the term in front of the metric on two sphere S^2 in (4.4). By comparing this with the vacuum of the SGP monopole, we conclude that the form of the field is

$$\mathbf{B}_{AdS} = \frac{4mL^2\hat{\mathbf{r}}}{\sinh^2(Lr)}. \tag{4.5}$$

As $L \to 0$, this $\mathbf{B}_{AdS}$ reduces smoothly to the field of the flat SGP monopole case. $\mathbf{B}_{AdS}$ actually represents an Abelian hyperbolic monopole [32, 33]. We see that by the conservation of magnetic flux, the total flux $\Phi = \int \mathbf{B}_{AdS}.d\mathbf{s}$ should be the same both in flat case and AdS case. The total magnetic flux of hyperbolic monopole at a great distance is found as $\Phi = 16\pi m$, which is the same as that of the SGP monopole. Hence, the argument above becomes concrete. In addition, the basic form of the four-potential is preserved with $\mathbf{B}_{AdS}$.

Now, we write the most general static ansatz metric on our assumed background (4.4) in the following form:

$$\begin{aligned} d\hat{s}^2 = (-1/2)a^2(r,y)e^{-2\sqrt{2}Ly}dt^2 &+ b^2(r,y)\left[dy + 4m(1-\cos\theta)d\phi\right]^2 \\ &+ v^2(r,y)\left[dr^2 + (1/L^2)\sinh^2(Lr)d\Omega_2\right]. \end{aligned} \tag{4.6}$$

We expect that there exists a solution in this form which can solve the field equations (4.2) with our constraints. Applying these constraints, in the limit $m \to 0$ to return to the background metric (4.4), we should have

$$a^2(r,y) = b^2(r,y) = v^2(r,y) \to 1, \tag{4.7}$$

and in the limit $L \to 0$ to reach the SGP monopole, we should also have

$$a^2(r,y) \to 2 \quad \text{and} \quad b^{-2}(r,y) = v^2(r,y) \to 1 + 4m/r. \tag{4.8}$$

To identify the three unknown functions a, b and v, we will utilize the following three vanishing components of the Ricci tensor

$$\hat{R}_{\theta y} = \frac{8m^2L^2b^2(1-\cos\theta)}{\sinh^2(Lr)v^2\sin\theta}\left[\frac{\partial_y v}{v} - \frac{\partial_y a}{a} - 3\frac{\partial_y b}{b} + \sqrt{2}L\right] = 0, \tag{4.9}$$

$$\begin{aligned}\hat{R}_{\theta\phi} &= \frac{2m(1-\cos\theta)^2}{\sin\theta}\left[\frac{\partial_y a}{a} + \frac{\partial_y v}{v} + \frac{\partial_y b}{b} - \sqrt{2}L\right] \\ &+ \left[\frac{2m(1-\cos\theta)^2}{\sin\theta}\frac{16m^2L^2b^2}{\sinh^2(Lr)v^2}\left(\frac{\partial_y v}{v} - \frac{\partial_y a}{a} - 3\frac{\partial_y b}{b} + \sqrt{2}L\right)\right] = 0,\end{aligned} \tag{4.10}$$

$$\hat{R}_{ry} = \frac{\partial_y v}{v}\left(\frac{\partial_r(abv)}{abv} + \frac{\partial_r(bv)}{bv}\right) - 2\frac{\partial_r\partial_y v}{v} + \sqrt{2}L\frac{\partial_r(a/b)}{a/b} + \frac{\partial_r(b/\partial_y a)}{ab/\partial_y a} = 0. \tag{4.11}$$

We see that (4.9) reduces to

$$\left[\frac{\partial_y v}{v} - \frac{\partial_y a}{a} - 3\frac{\partial_y b}{b} + \sqrt{2}L\right] = 0, \tag{4.12}$$

and if we put this equation in (4.10), then we get

$$\left[\frac{\partial_y a}{a} + \frac{\partial_y v}{v} + \frac{\partial_y b}{b} - \sqrt{2}L\right] = 0. \tag{4.13}$$

Now, from (4.12) and (4.13) we find b in terms of a

$$b(r,y) = \left(\frac{e^{\sqrt{2}Ly}g(r)}{a(r,y)}\right)^{1/2}, \tag{4.14}$$

where $g(r)$ is an arbitrary integral constant. Then by using (4.14) in (4.12), we find v in terms of a as

$$v(r,y) = \left(\frac{e^{\sqrt{2}Ly}k(r)}{a(r,y)g(r)}\right)^{1/2}, \tag{4.15}$$

where $k(r)$ is again an integral constant. Finally, after using $b(r,y)$ and $v(r,y)$ in (4.11), it becomes

$$\frac{3}{2}\frac{\partial_r a(r,y)}{a(r,y)^2}\left[\sqrt{2}La(r,y) - \partial_y a(r,y)\right] = 0. \tag{4.16}$$

This generates two possibilities for $a(r,y)$, which are

$$a(r,y) = e^{\sqrt{2}Ly}s(r), \tag{4.17}$$

and

$$a(r,y) = a(y). \tag{4.18}$$

Thus, we obtain two different sets of solutions for the metric ansatz according to the value of a. The first general solution, obtained by using (4.17) is

$$d\hat{s}^2 = (-1/2)s^2(r)dt^2 + \frac{g(r)}{s(r)}\left[dy + 4m(1-\cos\theta)d\phi\right]^2 + \frac{k(r)}{s(r)g(r)}\left[dr^2 + (1/L^2)\sinh^2(Lr)d\Omega_2\right]. \tag{4.19}$$

However, it is seen that there is no way to restore the background metric in the limit $m \to 0$. One of the most apparent reasons is that the term in front of dt^2 does not have a y dependence. Next, the other general solution, obtained by using (4.18) is

$$d\hat{s}^2 = (-1/2)a^2(y)e^{-2\sqrt{2}Ly}dt^2 + \frac{e^{\sqrt{2}Ly}g(r)}{a(y)}\left[dy + 4m(1-\cos\theta)d\phi\right]^2 + \frac{e^{\sqrt{2}Ly}k(r)}{a(y)g(r)}\left[dr^2 + (1/L^2)\sinh^2(Lr)d\Omega_2\right]. \tag{4.20}$$

However, this solution also does not allow us to recover the background metric. This is seen in the limit $m \to 0$ (4.7), where we should have

$$\lim_{m\to 0} a^2(y) = 1, \tag{4.21}$$

and for $b(r,y)$ and $v(r,y)$,

$$\lim_{m\to 0} \frac{e^{\sqrt{2}Ly}g(r)}{a(y)} = 1 \quad \text{and} \quad \lim_{m\to 0} \frac{e^{\sqrt{2}Ly}k(r)}{a(y)g(r)} = 1. \tag{4.22}$$

It is obvious that all of these limits can not be satisfied simultaneously. Since (4.22), $a(y) \to (1 \text{ or } -1)$, there is no way to realize the desired limits (4.22) without any y dependence at least in the function $g(r)$.

Consequently, it turns out that in five-dimensions there is no monopole solution which smoothly reduces to the SGP monopole in the form of the chosen ansatz (4.6). The topology of the chosen background does not allow for a monopole-like solution.

Basically, it seems impossible to construct a background which is close to AdS_4 by using AdS_5, since in AdS spacetime we cannot change the topology as easily as in the Minkowski case, where we could change it by compactifying one of the spatial dimensions of M^5.

4.2 Monopole in the Six-Dimensional AdS Spacetime

Now, we turn to the same problem in six dimensions by using two extra dimensions y and z this time. We take our background vacuum metric again as a direct product space of $\mathrm{AdS}_2\times\mathrm{AdS}_4$, in the following form

$$d\hat{s}^2 = -e^{-2Lz}dt^2 + dz^2 + \frac{3}{2}\left[dr^2 + (1/L^2)\sinh^2(Lr)d\Omega_2 + \cosh^2(Lr)dy^2\right], \quad (4.23)$$

whose AdS_2 and AdS_4 parts are

$$ds^2_{AdS_2} = e^{-2Lz}dt^2 + dz^2 \quad \text{and} \quad ds^2_{AdS_4} = \frac{3}{2}\left[dr^2 + \frac{1}{L^2}\sinh^2(Lr)d\Omega_2 + \cosh^2(Lr)dy^2\right],$$

where again $d\Omega_2 = d\theta^2 + \sin^2\theta d\phi^2$. Both parts have one extra dimension which can be compactified on an arbitrary circle. This time the spatial AdS_3 which supports the monopole is embedded in the AdS_4 part, and therefore we again assume the same form of the magnetic field (4.5) for the hyperbolic monopole.

We write the following static metric ansatz

$$\begin{aligned} d\hat{s}^2 = &-e^{-2Lz}dt^2 + dz^2 + \frac{3}{2}H^2(r)[\frac{1}{V(r)}\left[dr^2 + (1/L^2)\sinh^2(Lr)d\Omega_2\right] \\ &+ \cosh^2(Lr)V(r)\left[dy + 4m(1-\cos\theta)d\phi\right]^2]. \end{aligned} \quad (4.24)$$

The four-dimensional part (r, θ, ϕ, y) above completely resembles the spatial part of the flat case monopole. This four-dimensional part of the metric was studied before by Pedersen [34], who found that the staticity and spherical symmetry leaves just two functions, $H(r)$ and $V(r)$. The $m \to 0$ and $L \to 0$ limits of these functions should read

$$H(r) = V(r) \to 1 \quad \text{and} \quad H(r) \to 1 \quad (4.25)$$

respectively. In the magnetic field, since r has been replaced by $\sinh(Lr)/L$, from the value of $V(r)$ in the flat case, we can take the ansatz for $V(r)$ as

$$\frac{1}{V(r)} = 1 + \frac{4mL}{\sinh(Lr)}. \quad (4.26)$$

From the field equations, $H(r)$ is then found as follows

$$H(r) = \frac{1}{1 - 4mL\sinh(Lr)}. \quad (4.27)$$

Therefore, in six dimensions we find the monopole with all the desired properties as

$$\begin{aligned} d\hat{s}^2 = &-e^{-2Lz}dt^2 + dz^2 + \frac{3}{2[1-4mL\sinh(Lr)]^2}[(1 + \frac{4mL}{\sinh(Lr)})[dr^2 \\ &+(1/L^2)\sinh^2(Lr)d\Omega_2] + \cosh^2(Lr)(1 + \frac{4mL}{\sinh(Lr)})^{-1}\left[dy + 4m(1-\cos\theta)d\phi\right]^2]. \end{aligned} \quad (4.28)$$

This solution, which is asymptotically locally AdS, describes a monopole sitting at rest at $r = 0$. We see that there is a singularity at $r = 0$, but it is absent if we make y to be periodic with period $16\pi m$. Thus, for regularity of the metric we should compactify one of the coordinates. There is also a coordinate singularity and we cover the whole space again with the two coordinate sets. The curvature invariants indicate that the metric is non-singular.

CHAPTER 5

CONCLUSION

In this thesis, we have made a general review of Kaluza-Klein theories and its monopole solutions. These cover both Abelian and non-Abelian versions. We have especially given emphasis on the five-dimensional Kaluza-Klein theory and its regular, static and spherically symmetric soliton solution, namely, Sorkin-Gross-Perry monopole [9, 10]. We have then discussed the monopole type solutions in the anti-de Sitter spacetime, especially again the counterpart of Sorkin-Gross-Perry monopole [16]. We have seen that there is no monopole type static solution which can describe the Sorkin-Gross-Perry monopole in the five-dimensional anti-de Sitter background. We can however construct the monopole solutions in the anti-de Sitter spacetime by relaxing some of the conditions we have put on the solution such as staticity or the number of dimensions.

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