SERIES SOLUTION OF THE WAVE EQUATION IN OPTIC FIBER

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ABSTRACT

SERIES SOLUTION OF THE WAVE EQUATION IN OPTIC FIBER

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In this study, the mapped Galerkin method was applied to solve the vector wave equation based on $H-$ field and to obtain the propagation constant in $x-y$ space. The vector wave equation was solved by the transformation of the infinite $x-y$ plane onto a unit square. Two-dimensional Fourier series expansions were used in the solutions. Modal fields and propagation constants of dielectric waveguides were calculated. In the first part of the study, all of the calculations were made in step index fibers. Transverse magnetic fields were obtained in the $u-v$ and $x-y$ space through the solution of the matrix eigenvalue equation. Some graphics were plotted in the light of the results obtained. The results are found to be in accord with the results of other numerical techniques and exact solutions. After that, the propagation constant in $x-y$ space was calculated with ease using the solution of the modal field components. In the second part of the study, the similar calculations were made in graded index fibers.

Keywords: Moment Method, Galerkin Method, Mapped Galerkin Method, Solutions of Vector Wave Equations, Solutions of Eigenvalues.

ÖZ

FİBER OPTİKTE DALGA DENKLEMİNİN SERİ ÇÖZÜMLERİ

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Doktora, Fizik Bölümü
Tez Yöneticisi: Assoc. Prof. Dr. Serhat Çakır

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Bu çalışmada, $x-y$ uzayında mağnetik alanın vektörel dalga denklemi çözümü için haritalanmış Galerkin method uygulandı ve ilerleme sabiti elde edildi. Vektörel dalga denklemi, sonsuz $x-y$ düzleminin birim kare üzerine dönüşümü yapılarak çözüldü. Çözümde iki boyutlu Fourier seri açılımı kullanıldı. Mod alanları ve dielektrik dalga klavuzlarının ilerleme sabitleri hesaplandı. Çalışmanın ilk kısmında, tüm hesaplamalar basamak tipi fiberlerde yapıldı. Matris özdeğer denklemi çözülerek, $u-v$ ve $x-y$ uzayında enine mağnetik alanlar elde edildi. Bu sonuçlar kullanılarak bazı grafikler çizildi. Sonuçlar diğer nümerik tekniklerin sonuçları ve tam çözümler ile uyumlu bulundu. Bundan sonra $x-y$ uzayında ilerleme sabiti, bu alan bileşenlerinden kolayca hesaplandı. Çalışmanın ikinci kısmında, benzer hesaplamalar kırılma indisi yarıçapın fonksiyonu olan fiberlerde yapıldı.

Anahtar Sözcükler: Moment Metot, Galerkin Metot, Haritalanmış Galerkin Metot, Vektörel Dalga Denklemi Çözümleri, Özdeğer Çözümleri.

...To My Mother

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CHAPTER 1

OVERVIEW OF OPTICAL FIBER WAVEGUIDE

1.1 Introduction

Optical waveguides can be classified as integrated optics and optical fibers. Integrated optics is the technology of integrating various optical devices and components of the generation, focusing, splitting, combining, isolation, polarization, coupling, switching, modulation and detection of light, all on a single substrate. Optical waveguides provide the connections between these components. An optical waveguide is a light conduit consisting of a slab strip, or cylinder of dielectric material surrounded by another dielectric material of lower refractive index. The most widely used one of these waveguides is the optical fiber, which is made of two concentric cylinders of low-loss dielectric material such as glass [2]. Optical fibers are used as the transmission systems for various applications because of their certain advantages. The most basic of these advantages are low-loss transmission when coupled with others, small size and weight and large information carrying capacity. Waveguides are usually constructed by combining appropriate dielectrics because of the difficulties of constructing a waveguide from metal

in the visible wave band. The metal will behave as a substance having a complex permittivity [1]. By placing various appropriate forms of optical waveguides on the substrate, optical integrated circuits can be constructed. If we want to change the direction of the light waves, we should use corner-band waveguides, S-shaped waveguides or bent waveguides. The width of waveguides are changed by using the tapered waveguides. Branching waveguides and crossed waveguides are used for splitting, combining and interference. For coupling, couplers such as optical waveguides directional couplers and two-mode couplers are used. Waveguide gratings have a function in the optical integrated circuits as a wavelength filter, mode converter, reflector, resonator, demultiplexes, etc. We can see some waveguide forms in Figure 1.1.

1.2 Basics of Optical Waveguide Analysis

This part briefly discusses electromagnetic theory, the fundamentals of Maxwell equations and the propagation of light. By using wave-and ray-theoretical analysis, the wave equations are obtained.

Light is an electromagnetic wave. The theoretical principles in electromagnetic theory is valid for light. Optical frequencies in a band of electromagnetic spectrum extends from the infrared through the visible to the ultraviolet. The wavelength of light is between 10 nm and 1 mm. The wave optics theory is an approximation of the electromagnetic theory. In this theory, light is explained as

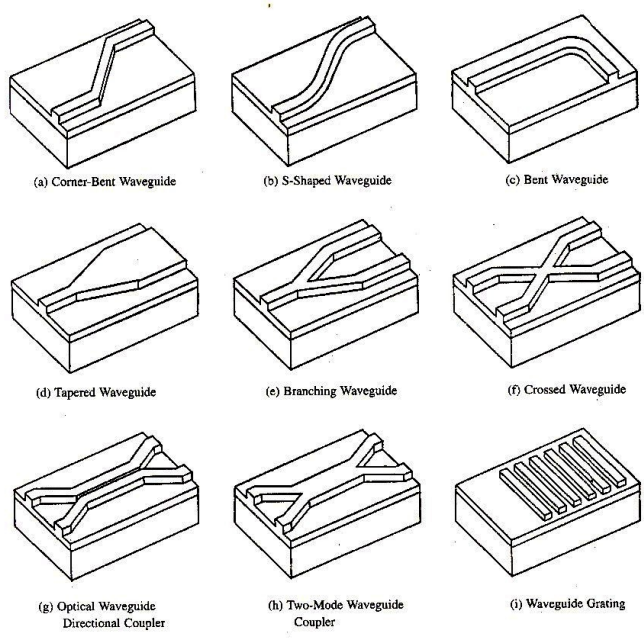

Figure 1.1: The Optical Waveguide Usage [1].

a wave function. This function has a single scalar function of position and time. The ray optics approximation provides a further simplification valid in the limit of short wavelengths [2].

a)Maxwell's Equations

Electromagnetic waves are composed of two fields; electric-and magnetic-field. Its magnetic field can be written by four electromagnetic field vectors which can be indicated as functions of position and time. These vectors verify Maxwell's equations

$$\vec{\nabla} \times \vec{E} = -\partial \vec{B}/\partial t \tag{1.1}$$

$$\vec{\nabla} \times \vec{H} = \partial \vec{D}/\partial t + \vec{J} \tag{1.2}$$

Here $\vec{J}[A/m^2]$ is the electric current density. The following current continuity equation holds for $\vec{J}$ and the charge density $\rho[C/m^3]$

$$\vec{\nabla} \cdot \vec{J} = -\partial\rho/\partial t \tag{1.3}$$

By using Equation 1.1 and Equation 1.3 and vector formulas we get

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{1.4}$$

$$\vec{\nabla} \cdot \vec{D} = \rho \tag{1.5}$$

Equation 1.1, Equation 1.2, Equation 1.4 and Equation 1.5 are called Maxwell's equations. The flow density of the energy carried by an electromagnetic field is expressed by as Poynting vector $\vec{S}[W/m^2]$, which is defined as follows [1]:

$$\vec{S} = \vec{E} \times \vec{H} \tag{1.6}$$

b)Constitutive Relations

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \tag{1.7}$$

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) \tag{1.8}$$

Here $P[C/m^2]$ is polarization and $M[A/m]$ is magnetization. $\varepsilon_0[F/m]$ and $\mu_0[H/m]$ are the permittivity of vacuum and the permeability of free space, respectively and their values are

$$\varepsilon_0 = 8,8541878 \times 10^{-12}[F/m] \tag{1.9}$$

$$\mu_0 = 4\pi \times 10^{-7}[H/m] \tag{1.10}$$

The electric flux density $\vec{D}$ and the magnetic flux density $\vec{B}$ are respectively in a light-transmitting medium. If the medium isotropic, linear and non-dispersive, the polarization and magnetization conform to the following equations

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \tag{1.11}$$

$$\vec{M} = \chi_m \vec{H} \tag{1.12}$$

Here χ_e and χ_m are the electric susceptibility and magnetic susceptibility. By using the last two equations, Equation 1.7 and Equation 1.8 relative permittivity ε_r and relative permeability μ_r can be obtained.

$$\varepsilon_r = 1 + \chi_e \tag{1.13}$$

$$\mu_r = 1 + \chi_m \tag{1.14}$$

permittivity ε and μ are respectively,

$$\varepsilon = \varepsilon_0 \varepsilon_r \tag{1.15}$$

$$\mu = \mu_0 \mu_r \tag{1.16}$$

the constitutive relations

$$\vec{D} = \varepsilon \vec{E} \tag{1.17}$$

$$\vec{B} = \mu \vec{H} \tag{1.18}$$

The refractive index n is as follow

$$n = \sqrt{\varepsilon_r \mu_r} \tag{1.19}$$

If $\mu = \mu_0$

$$n = \sqrt{\varepsilon_r} \tag{1.20}$$

Equation 1.17 and Equation 1.18 are valid in an isotropic medium; that is, the medium constant is a scaler quantity. If the medium constant is dependent on the direction of the electromagnetic field vector, then the medium constant is a tensor quantity that is an anisotropic medium. Constitutive relations for an anisotropic medium are as follows:

$$\vec{D} = [\varepsilon]\vec{E} \tag{1.21}$$

$$\vec{B} = [\mu]\vec{H} \tag{1.22}$$

Here $[\varepsilon]$ and $[\mu]$ are the rank two permittivity and permeability tensors, respectively [1].

1.2.1 Some Definitions for Dielectric Media

- A dielectric medium is said to be *linear* if the vector field (the polarization density) $\vec{P}(r,t)$ is linearly related to the vector field (electric field) $\vec{E}(r,t)$. The principle of superposition then applies [2].

- The medium is said to be *non-dispersive* if its response is instantaneous. $\vec{P}$ at time t is determined by $\vec{E}$ at the same time t and not by prior values of

$\vec{E}$. Non-dispersiveness is clearly an idealization since any physical system has a finite response time.

- The medium is said to be *homogeneous* if the relation between $\vec{P}$ and $\vec{E}$ is independent of the position $\vec{r}$.

- The medium is called *isotropic* if the relation between the vectors $\vec{P}$ and $\vec{E}$ is independent of the direction of the vector $\vec{E}$, so that the medium looks the same from all directions. The vectors $\vec{P}$ and $\vec{E}$ must then be parallel.

- The medium is said to be *spatially non-dispersive* if the relation between $\vec{P}$ and $\vec{E}$ is local. $\vec{P}$ at each position $\vec{r}$ is influenced only by $\vec{E}$ at the same position.

c)Boundary Conditions

If two different media come into contact, we can observe a boundary surface and boundary conditions. For example, if the unit normal vector directed from medium 2 to medium 1 is n, the boundary conditions are given as follows [1]:

1) The tangential component of the electric field must be continuous.

$$\vec{n} \times (\vec{E_1} - \vec{E_2}) = 0 \tag{1.23}$$

2) The tangential component of the magnetic field must be continuous.

$$\vec{n} \times (\vec{H_1} - \vec{H_2}) = 0 \tag{1.24}$$

If there is surface current $\vec{J}_s$ on the boundary surface, it may be discontinuous.

$$\vec{n} \times (\vec{H_1} - \vec{H_2}) = \vec{J}_s \tag{1.25}$$

3) The normal component of the electric flux density must be continuous

$$\vec{n} \cdot (\vec{D_1} - \vec{D_2}) = 0 \tag{1.26}$$

If there exists surface charge ρ_s on the boundary surface, it may be discontinuous.

$$\vec{n} \cdot (\vec{D_1} - \vec{D_2}) = \rho_s \tag{1.27}$$

4) The normal component of the magnetic flux density must be continuous.

$$\vec{n} \cdot (\vec{B_1} - \vec{B_2}) = 0 \tag{1.28}$$

d) Wave Equations

$$\nabla^2 \vec{E} - \varepsilon\mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} + \vec{\nabla}(\frac{\vec{\nabla}\varepsilon}{\varepsilon} \cdot \vec{E}) = 0 \tag{1.29}$$

$$\nabla^2 \vec{H} - \varepsilon\mu_0 \frac{\partial^2 \vec{H}}{\partial t^2} + \frac{\vec{\nabla}\varepsilon}{\varepsilon} \times (\vec{\nabla} \times \vec{H}) = 0 \tag{1.30}$$

If the permittivity is uniform, the following equations are valid

$$\nabla^2 \vec{E} - \varepsilon\mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \tag{1.31}$$

$$\nabla^2 \vec{H} - \varepsilon\mu_0 \frac{\partial^2 \vec{H}}{\partial t^2} = 0 \tag{1.32}$$

We can write fields vectors and flux densities as a function of complex amplitudes ($\tilde{E}, \tilde{H}, \tilde{D}, \tilde{B}$) and with unit angular frequency $w[rad/s]$ [1]

$$\vec{E}(\vec{r}, t) = Re[\tilde{E}(\vec{r}) \exp(jwt)] \tag{1.33}$$

$$\vec{H}(\vec{r},t) = Re[\tilde{H}(\vec{r})\exp(jwt)] \tag{1.34}$$

$$\vec{D}(\vec{r},t) = Re[\tilde{D}(\vec{r})\exp(jwt)] \tag{1.35}$$

$$\vec{B}(\vec{r},t) = Re[\tilde{B}(\vec{r})\exp(jwt)] \tag{1.36}$$

Here $\vec{E}$ and $\vec{H}$ are electric and magnetic field vectors. $\vec{D}$ is electric flux density and $\vec{B}$ is magnetic flux density. We can write Maxwell's equations in another form.

$$\vec{\nabla} \times \vec{E} = -jw\vec{B} = -jw\mu_0\vec{H} \tag{1.37}$$

$$\vec{\nabla} \times \vec{H} = jw\vec{D} = jw\varepsilon\vec{E} \tag{1.38}$$

As a result, we can obtain the following wave equations

$$\nabla^2\vec{E} + k^2\vec{E} + \vec{\nabla}(\frac{\vec{\nabla}\varepsilon}{\varepsilon} \cdot \vec{E}) = 0 \tag{1.39}$$

$$\nabla^2\vec{E} + k^2\vec{E} + \frac{\vec{\nabla}\varepsilon}{\varepsilon} \times (\vec{\nabla} \times \vec{H}) = 0 \tag{1.40}$$

Here $k[rad/m]$ is wave number. If $\varepsilon = \varepsilon_0$, then the wave number is called the free-space wave number k_0

$$k = w\sqrt{\varepsilon\mu_0} \tag{1.41}$$

$$k_0 = w\sqrt{\varepsilon_0\mu_0} \tag{1.42}$$

From the expression Equation 1.15and Equation 1.20

$$k = k_0 n \tag{1.43}$$

For constant permittivity

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0 \tag{1.44}$$

$$\nabla^2 \vec{H} + k^2 \vec{H} = 0 \tag{1.45}$$

The energy flow density at the time-average value is

$$< \vec{S} >= Re[(1/2)\vec{E} \times \vec{H}^*] \tag{1.46}$$

1.3 Optical Communications

Until recently, all communication systems have used electrical cables or have made use of radio-frequency and microwave electromagnetic radiation propagating in free space to transfer information from one point to another [2]. However, the light wave has recently become important for communication systems. The information signal is transmitted by superimposing (modulating) it onto an electromagnetic wave. This wave behaves as a carrier for the information signal. When the modulated information is received at the end of the way the original information signal is obtained by demodulation. In an optical communication system, the carrier is selected from the optical region. The optical spectrum ranges from about $50nm$ (ultraviolet) to about $100\mu m$ (far infrared), the visible region begins at the $400-$ to $700-nm$ band. Until mid-1990s, optical systems specify the band with wavelength instead of frequency as is used in the radio region. However, with the advent of high-speed multiple-wavelength systems, the output

optical sources started to be specified in terms of optical frequency. Optical frequencies ν are related to the wavelengths λ through the fundamental equation $c = \nu\lambda$ [4]. Theoretically, increasing the carrier frequency increases the available transmission bandwidth(frequency extent) of the modulated carrier. Frequencies in the optical range have a wide bandwidth about 10^5 times more than that of a carrier in the radio frequency (RF)range. The transmitting large amounts of information is an advantage in communicating with optical frequencies. In addition, available transmitted power within the transmitted electromagnetic wave increases with carrier frequency. Higher power densities lead to the improvement of the system's performance. Therefore, using higher carrier frequencies increases the capability of the system to achieve higher power densities [3].

1.3.1 Model of System

A typical optical communication can be considered as if a block diagram. A source produces information for the transfer to the definite direction. This information may be of different kinds, such as waveforms in time, digital symbols, etc. Source has its output modulated onto an optical carrier in the second part of this system. This part is an optical transmitter and modulator. In this part, the modulated information with carrier which has frequency in the optical portion of the electromagnetic spectrum is transferred to the optical channel which is the third part of the system. The optical channel may be a free space, a turbulent

atmosphere, a fiber waveguide, etc. The last component of the system is the optical receiver. At the receiver, the light field is optically collected and processed (photodetected).

1.4 Advantages of Optical Fiber Communication

Communication using an optical carrier wave guided along a glass fiber has a number of extremely attractive features [4]. It has some unique advantages for certain types of communications [5].

a) Enormous potential bandwidth

Bandwidth is an effective measure of the rate at which information can be sent. The amount of information that can be carried through a system of any kind is limited by the frequency of the carrier wave. Fiber optics handless the higher carrier frequencies, making possible not just a few simultaneous video channel, but hundreds of channels [6]. The optical carrier frequency in the range of 10^{13} to 10^{16} Hz (generally in the near infrared around 10^{14} Hz or 10^5 GHz) yield as far greater potential transmission bandwidth than metallic cable systems (i.e. coaxial cable bandwidth up to around $500MHz$) or even millimeter wave radio systems (i.e. systems currently operating with modulation bandwidth of $700MHz$). At present, the bandwidth available to fiber systems is not fully utilized but modulation at several gigahertz over a hundred kilometers and hundreds of megahertz over three hundred kilometers without repeaters is possible.

Therefore, the information-carrying of optical fiber systems has proved far superior to the best copper cable systems. By comparison the losses in wideband coaxial cable systems restrict the transmission distance to only a few kilometers at bandwidths over one hundered megahertz [4].

The bandwidth characteristics of a fiber differ according to its construction and the method by which it is made. A graded index fiber can easily be given a very broad bandwidth, ranging from 1 to 10 GHz.km. The single-mode fiber also has a very broad bandwidth of more than several tenfold GHz.km. Therefore, it is easy to from a transmission system of 400 Mb/sec. It is also easy to increase the amount of transmitted information by several to ten times much as much by the wavelength division multiplex (WDM) method in which different carrier wavelengths are used for the transmission of information [7]. This wavelength division multiplexed operation, particularly with dense packing of the optical wavelengths (or,essentially, fine frequency spacing), offers the potential for a fiber information-carrying capacity which is many orders of magnitude in excess of that obtained using copper cables or a wideband radio system [4]. Thousands of voice channels can now be multiplexed together and sent over a single fiber strand [8].

b) Small size and weight

Optical fibers have very small diameters which are often no greater than the diameter of a human hair. Hence, even when such fibers are covered with protective coating they are far smaller and much lighter than corresponding copper

cables. This is a tremendous boon towards the alleviation of duct congestion in cities, as well as allowing for an expansion of signal transmission within mobiles such as aircraft, satellites and even ships.

c) Electrical isolation

Optical fibers which are fabricated from glass or sometimes a plastic polymer, are electrical insulators and therefore, unlike their metallic counterparts, they do not exhibit earth loop and interface problems. Furthermore,this property makes optical fiber transmission ideally suited for communication in electrically hazardous environments as the fibers create no acting or spark hazard at abrasions or short circuits [4]. Optical fibers can be used in potentially flammable or explosive environments [7].

d) Immunity to interference and crosstalk.

Optical fibers from a dielectric waveguide and are therefore free from electromagnetic interference (EMI), radio frequency interference (RFI), or switching transients giving electromagnetic pulses (EMP). Hence the operation of an optical fiber communication system is unaffected by transmission through an electrically noisy environment and fiber cable requires no shielding from EMI. The fiber cable is also not susceptible to lightning strikes if used overhead rather than underground [4]. Moreover, leaking of light from optical fibers is extremely minor and almost no external light enters the fiber to propagate. This means that there

is extremely little crosstalk between fibers [7]. Hence, unlike communication using electrical conductors, crosstalk is negligible, even when many fibers are cabled together.

e) Signal security

The light from optical fibers does not radiate significantly and therefore they provide a high degree of signal security. Unlike the situation with copper cables, a transmitted optical signal cannot be obtained from a fiber in a noninvasive manner (i.e without drawing optical power from the fiber). Therefore, in theory, any attempt to acquire a message signal transmitted optically may be detected. This feature is obviously attractive for military, banking and general data transmission (i.e computer network) applications [4].

f) Digital Transmission Method

The primary method of transmitting information via optical fiber is digital- that is, by sequences of optical pulses whose positions, widths, or occurrences can be modulated. The simplest case is binary "on-off" pulse code modulation, in which the presence or absence of pulses within equally spaced time slots(one pulse per slot) corresponds to occurrences of ones or zeros in message, respectively. The information capacity of the fiber link can be measured by how many time slots (or bits) can occur per second [9].

g) Low transmission loss

Loss indicates the amount of data-deterioration that occurs over a given distance. All mediums suffer some loss. But lost in coaxial cable and twisted-pair wires increases with frequency; whereas loss in fiber-optic cable remains stationary over a wide range of frequencies [6]. The development of optical fibers over last twenty years has resulted in the production of optical fiber cables which exhibit very low attenuation or transmission loss in comparison with the best copper conductors. Fibers have been fabricated with losses as low as $0.2dBkm^{-1}$ and this feature has become a major advantage of optical fiber communications. It facilitates the implementation of communication links with extremely wide repeater spacing (long transmission distances without intermediate electronics), thus reducing both system cost and complexity. Together with the already proven modulation bandwidth capability of fiber cable this property provides totally compelling case for the adoption of optical fiber communication in the majority of long-hall telecommunication application.

h) Ruggedness and flexibility

Although protective coatings are essential, optical fibers may be manufactured with very high tensile strengths. Perhaps surprisingly for a glassy substance, the fibers may also be bent to quite small radii or twisted without damage. Furthermore, cable structures have been developed which have proved flexible, compact and extremely rugged. Taking the size and weight advantage into account, these optical fiber cables are generally superior in terms of storage, transportation,

handling and installation to corresponding copper cables,whilst exhibiting and at least comparable strength and durability.

i) System reliability and ease of maintenance

These features primarily stem from the low loss property of optical fiber cables which reduces the requirement for intermediate repeaters or line amplifiers to boost the transmitted signal strength. Hence with fewer repeaters, system reliability is generally enhanced in comparison with conventional electrical conductor systems. Furthermore, the reliability of the optical components is no longer a problem with predicted lifetimes of 20 to 30 years now quite common. Both these factors also tend to reduce maintenance time and cost [4].

j) Free from rust

Glass materials, such as silica, are stable chemically and optical fibers are entirely impervious to corrosion which is not the case with metals. Therefore, optical fibers endure well in adverse environments, such as the ocean floor.

k) Resistance to high temperature

The melting point of silica is about $1900°C$, far above than that of copper or plastics. Therefore, cables made with silica are resistant to high temperature.

l) Principal material in fiber manufacture is glass

Glass, in addition to being abundant and inexpensive, enables low-loss propagation at optical frequencies. However, copper resources are limited [7].

m) Potential low cost

The glass which generally provides the optical fiber transmission medium is made from sand-not a scarce resource. So, in comparison with copper conductors, optical fibers offer the potential for low cost line communication. Although over recent years this potential has largely been realized in the costs of the optical fiber transmission medium which for bulk purchases is now becoming competitive with copper wires (i.e. twisted pairs), it has not yet been achieved in all the other component areas associated with optical fiber communications. For example, the costs of high performance semiconductor lasers and detector photodiodes are still relatively high, as well as some of those concerned with the connection technology (demountable connectors, coupler, etc.). Overall system costs when utilizing optical fiber communication on long-haul links, however, are substantially less than those for equivalent electrical line systems because of the low loss and wideband properties of the optical transmission medium. As incident in (g), the requirement for intermediate repeaters and the associated electronics is reduced, giving a substantial cost advantage. Although this cost benefit gives a net gain for long-haul links it is not always the case in short-haul applications where additional cost incurred, due to the electrical-optical conversion(and vice versa), may be a deciding factor. Nevertheless, there are other possible cost advantages in relation to shipping, handling, installation and maintenance, as well as the features indicated in (c) and (d) which may prove significant in the system choice. The reducing

costs od optical fiber communications has not only provided strong competition with electrical line transmission systems, but also for microwave and millimeter wave radio transmission systems. Although these systems are reasonably wide-band the relatively short span 'line of sight' transmission necessitates expensive aerial towers at intervals no greater than a few tens of kilometers. Hence optical fiber is fast becoming the dominant transmission medium within the major industrialized societies. Many advantages are therefore provided by the use of a lightwave carrier within a transmission medium consisting of an optical fiber [4]. These features allow many possible types of application. They will become widely used in public communication networks. Some applications are:

Connections between microwave receivers and control facilities and the distribution networks of cable-television (CATV) operators.

Transmission of information for control and measurement systems (aircrafts, ships, automobiles, trains, measuring circuits, industrial plants, electrical power system).

Links among computers and high-resolution video terminals used for such purposes as computer-aided design.

High-speed interconnections between computers and peripheral devices, or between computers, or even within segments of single large computers.

Communications and control in hazardous situations such as fuel tanks, since there is no sparking hazard with optical fibers.

Interoffice trunks that carry many simultaneous telephone conservations between local and regional telephone company switching facilities.

Public communication (a) between the local exchange and houses (less than 5-6 km), (b) between exchanges in city (less than 8-9 km), (c) between trunk exchanges (10-30 km).

Underwater communication, international communication (repeater spacing 10-100 km; total length for intercontinental communication is about 10.000 km) [5,10].

1.5 Elements of an Optical Fiber Transmission Link

An optical fiber communication system is similar in basic concept to any type of communication system [4]. The key sections are a transmitter consisting of a light source and its associated drive circuitry, a cable offering mechanical and environmental protection to the optical fibers contained inside, and a receiver consisting of a photodetector plus amplifications and signal-restoring circuitry. Additional component include optical amplifiers, connectors, splices, couplers, and regenerators (for restoring the signal-shape characteristic). The cabled fiber is one of the most important elements in an optical fiber link. In addition to protecting the glass fibers during installation and service, the cable may contain copper wires for powering optical amplifiers or signal regenerators, which are needed periodically for amplifying and reshaping the signal in long-distance links.

Analogous to copper cables, optical fiber cables can be installed either aerially, in ducts, undersea, or buried directly in the ground. As a result of installation and/or manufacturing limitations, individual cable lengths will range from several hundred meters to several kilometers. Practical considerations such as real size and cable weight determine the actual length of a single cable section. Shorter segments tend to be used when the cables pulled ducts. Longer lengths are used in aerial, direct-buried, or undersea applications. Splicing together individual cable sections forms continuous transmission lines for these long-distance links. For undersea installations, the splicing and repeater-installation functions are carried out on board a specially cable-laying ship.

Once the cable is installed, a light source that is dimensionally compatible with the fiber core is used to launch optical power into the fiber. Semiconductor light-emitting diodes (LEDs) and laser diodes are suitable for this purpose, since their light output can be modulated rapidly by simply varying the bias current at the desired transmission rate, thereby producing an optical signal [11]. The term modulation describes the process of varying one of the parameters associated with the carrier wave to enable information to be carried. Variations in the amplitude, irradiance, frequency, phase and polarization can all in principle be used for this purpose. Since, most optical detectors respond to the irradiance of the light falling on them, irradiance modulation is the most popular technique. There are two main modulation methods commonly used to convey information

via irradiance variations, namely *analog* and *digital* [12]. In the system of analog modulation involves the variation of light emitted from the optical source in manner. With digital modulation, however, discrete changes in the light are obtained (i.e. on-off pulses). Although often simpler to implement, analog modulation with an optical fiber communication system is less efficient, requiring a far higher signal to noise ratio at the receiver than digital modulation. Also, the linearity needed for analog modulation is not always provided by semiconductor optical sources, especially at high modulation frequencies. For these reasons, analog optical fiber communication links are generally limited to shorter distances and lower bandwidths than digital links [4]. For high rate systems (usually greater than 1 Gb/s), direct modulation of the source can lead to unacceptable signal distortion. In this case, an external modulator is used to vary the amplitude of a continuous light output from a laser diode source.

After an optical signal is launched into a fiber, it will become progressively attenuated and distorted with increasing distance because of scattering, absorption, and dispersion mechanisms in the glass material. At the receiver, a photodiode will detect the weakened optical signal emerging from the fiber end and converted it to an electric current (referred to as a photocurrent). Silicon photodiodes are used in the 800- to 900-nm region. The prime material in the 1100- to 1600-nm region is an InGaAs alloy.

The design of an optical receiver is inherently more complex than that of the

transmitter, since it has to interpret the content of the weakened and degraded signal received by the photodetector. The principal figure of merit for a receiver is the minimum optical power necessary at the desired data rate to attain either a given error probability for digital system or specified signal- to -noise ratio for an analog system. The ability of a receiver to achieve a certain performance level depends on the photodetector type, the effects of noise in the system, and characteristics of the successive amplification stages in the receiver.

The design and installation of an optical fiber communication system require measurement techniques for verifying the operational characteristics of the constituent components. In addition to optical fiber parameters, system engineers are interested in knowing the characteristics of passive splitters, connectors, and couplers and electro-optic components, such as sources, photodetectors, and optical amplifiers. Furthermore, when a link is being installed and tested, the operational parameters of interest include bit-error rate, timing jitter, and signal- to-noise ratio as indicated by the eye pattern. During actual operation, measurements are needed for maintenance and monitoring functions to determine factors such as fault locations in fibers and the status remotely located optical amplifiers [11].

1.6 Optical Multiple Accessing

Multiple accessing refers to communication operation of having multiple signals propagate over the same link and yet be individually separable at a receiver.

The separability is provided by transmitting the signals with inherent addressing formats superimposed on the signal in some manner. A specific source signal can be recovered by tuning or aligning, the receiver to the addressing format. All other signals, having an undesired address, will be rejected (partially or totally) by the address recovered procedure. The resulting recovered signals containing the proper address will then be separated out, in the presence of detector noise and any interference from the signals that were not totally rejected. This latter interference is referred to as multiple access noise or simply crosstalk [4]. Multiplexing is used in the fiber-optic-systems, taking advantage of the broad bandwidth of the fiber, by combining more than one channel onto a fiber. In doing so, the available optical power is shared among the number of channels multiplexed, either directly as a power division or indirectly as a result of transmission-bandwidth sharing [13].

1.7 Optical Fiber Modes and Configurations

An optical fiber is a cylindrical dielectric waveguide made of low-loss materials such as silica glass. This fiber waveguides confines electromagnetic energy in the form of light to within its surfaces and guides the light in a direction parallel to its axis. Light rays incident on the core cladding boundary at angles grater than the critical angle undergo total internal reflection. Rays of greater inclination to the fiber axis lose part of their power into the cladding at each reflection and

are not guided. The transmission properties of an optical waveguide are directed by its structural characteristics, which have a major effect in determining how an optical signal is affected as it propagates along the fiber and also influences the response of the waveguide to environmental perturbations [11]. As a result of recent technological advances in fabrications, light can be guided through 1 km of glass fiber with a loss as low as $\approx 0.16dB$. Optical fibers are replacing copper coaxial cables as the preferred transmission medium for electromagnetic waves, thereby revolutionizing terrestrial communications. Applications,range form long-distance telephone data communications to computer communications in a local area network [2].

The propagation of light along a waveguide can be described in terms of a set of guided electromagnetics waves called the *modes*of the waveguide. These guided modes are referred to as the *bound* or *trapped* modes of the waveguide. Each of these guided modes consists of a set of simple electromagnetic field configurations which form a standing-wave pattern in the transverse direction (that is, transverse to the waveguide axis). For monochromatic light fields of radian frequency w, a mode travelling in the positive z direction (that is, along the fiber axis) has a time and z dependence given by $e^{j(wt-\beta z)}$.

The factor β is the z component of the wave propagation constant $k = 2\pi/\lambda$ and is the main parameter of interest in describing fiber modes. For guided modes β can only assume certain discrete values, which are determined from

the requirement that the mode field must satisfy Maxwell's equations and the electric and magnetic field boundary conditions at the core-cladding interface. A guided mode travelling in the z direction (along the fiber axis) can be decomposed into a family of superimposed plane waves that collectively form a standing-wave pattern in the direction transverse to the fiber axis. Since with any plane wave we can associate a light ray that is perpendicular to the phase front of the wave, the family of the plane waves corresponding to a particular mode forms a set of rays a *ray congruence.* Each ray of this particular set travels in the fiber at the same angle relative to the fiber axis. We note here that, since only a certain number $M$of discrete guided modes exist in a fiber, the possible angles of the ray congruencies corresponding to these modes are also limited to the same number M [11].

1.7.1 Fiber Types

Although many different configurations of the optical waveguide have been discussed in the literature, the most widely accepted structure is the single solid dielectric cylinder of radius a and index of refraction n_1 shown in Figure 1.2. This cylinder is known as the *core* of the fiber. The core is surrounded by a solid dielectric *cladding* having a refractive index n_2 that is lees than n_1.

The cladding reduces scattering loss resulting from dielectric discontinuities at the core surface, it adds mechanical strength to the fiber and it protects the

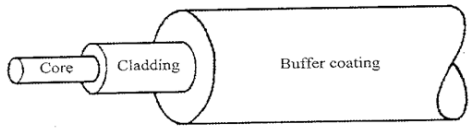

Figure 1.2: An optical fiber is cylindrical dielectric waveguide [11].

core from absorbing surface contaminants with which it could come in contact. In low-and medium-loss fibers the core material is generally glass and is surrounded by either a glass or plastic cladding. Higher-loss plastic core fibers with plastic cladding are also widely in use. In addition, most fibers are encapsulated in an elastic, abrasion-resistant plastic material.

This materials adds further strength to the fiber and mechanically isolates or buffers the fibers from small geometrical irregularities, distortions or roughnesses of adjacent surfaces. These perturbations could otherwise cause scattering losses induced by random microscopic bends that can arise when the fibers are incorporated into cables or supported by other structures. Variations in the material composition of the core give rise to the two commonly used fiber types. The refractive index of the core is uniform throughout and undergoes an abrupt change at the cladding boundary. This is called a *step-index* fiber. In The core refractive index is made to vary as a function of the radial distance from the center of the fiber. This type is a *graded-index fiber* [11]. Both the step-and the graded-index fibers can be further divided into single-mode and multimode

classes. As the name implies, a single-mode fiber sustains only one mode of propagation, whereas multimode fibers contain many hundreds of modes. A few typical size of single-and multimode fibers are given in Figure 1.3 to provide an idea of the dimensional scale. Multimode fibers offer several advantages compared single-mode fibers. The larger core radii of multimode fibers make it easier to launch optical power into the fiber and facilitate the connecting together of similar fibers. Another advantage is that light can be launched into a multimode fiber using a light-emitting-diode (LED) source, whereas single mode fibers must generally be excited with laser diodes. Although LED's have less optical output power than laser diodes,they are easier to make, are less expensive, require less complex circuitry and have longer lifetimes than laser diodes, thus making them more desirable in many applications [11]. A disadvantage of multimode fibers is that they suffer from intermodal dispersion. Briefly, intermodal dispersion can be described as follows. When an optical pulse is launched into a fiber, optical power in the pulse is distributed over all (or most) of modes of the fiber. Each of the modes that can be propagate in a multimode fiber travels at a slightly different velocity. This means that the modes in a given optical pulse arrive at the fiber end at slightly different times, thus causing the pulse to spread out in time as it travels along the fiber. This effect, which is known as *intermodal dispersion*, can be reduced by using a graded-index profile in the fiber core. This

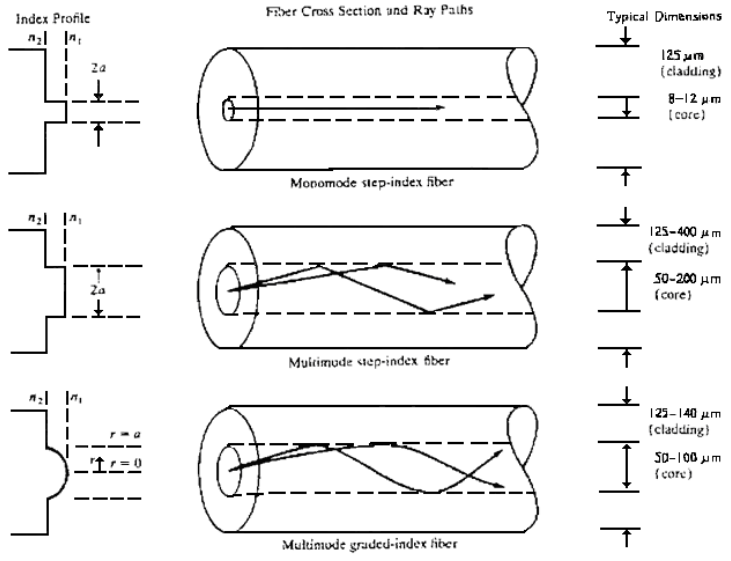

Figure 1.3: Comparison of single-and multi-mode step-and graded-index optical fibers [11].

allows graded-index fibers to have much larger band widths (data rate transmission capabilities) than step-index fibers. Even higher bandwidths are possible in single-mode fibers, where intermodal dispersion effects are not present [11].

1.7.2 Step-Index Fibers

A step-index fiber is a cylindrical dielectric waveguide specified by its core and cladding refractive indices, n_1 and n_2 and the radii a and b. Example of standard core cladding diameters $2a/2b$ are $8/125, 50/125, 62.5/125, 85/125, 100/140$ (units of μm). The refractive indices differ only slightly, so that the fractional refractive change is small ($\Delta << 1$) [2].

$$\Delta = \frac{n_1 - n_2}{n_1}$$

Almost all fibers currently used in optical communication systems are made of fused silica glass (SiO_2) of high chemical purity. Slight changes in the refractive index are made by the addition of low concentrations of doping materials (titanium, germanium, or boron example). The refractive index n_1 is the range from 0.001 and 0.02

1.7.3 Graded Index Fibers

The index of the core in the graded index fiber has a maximum value at the center and decreases toward the edges. The advantage of using this type of fiber is that the modal dispersion can be greatly reduced. The axial ray travels the shortest distance, but it moves the slowest since it sees the largest average index of refraction. The off axis meridional rays move in sinusoidal paths like those shown in Figure 1.4. The higher order modes have a larger amplitude and, therefore, a longer path length but they spend more time in the lower index material. As a result, they travel farther but faster. Hence, it is possible for them to arrive at their destination at about the same time as the axial ray. Physically, the rays follow a curved path. For propagation below the axis the ray is continually bent upward because the vibrating electric field sees a different index of refraction above the direction of propagation than it does below. For a ray above the axis,

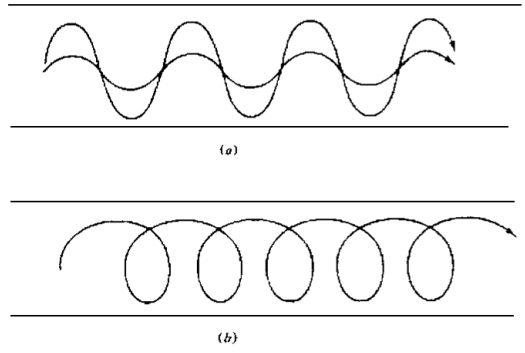

Figure 1.4: The (a) sinusoidal path of a meridional ray and (b) helical path of a skew ray in a graded index fiber [14].

the ray sees a smaller index of refraction above the direction of propagation than it does below. Thus, it slows down when it is vibrating below the direction of propagation; this braking effect bends the ray down.

The input angle of incidence determines the amplitude of the ray path and the mode order. The highest order mode that will propagate is the mode with an amplitude equal to the fiber radius. If the amplitude is greater than a, then the ray will not be totally internally reflected. Clearly, the larger the fiber radius the larger than allowed amplitude, the larger the number of modes that can propagate[14].

Index grating is an ingenious method for reducing the pulse spreading caused by the differences in the group velocities of modes of a multimode fiber. The core of a graded-index fiber has a varying refractive index, highest in the center and decreasing gradually to its lowest value at the cladding. The phase velocity

of of light is therefore minimum at the center and increasing gradually with the radial distance. Rays of the most axial mode travel the shortest distance at the smallest phase velocity. Rays of the most oblique mode zigzag at a greater angle and travel a longer distance, mostly in a medium where the phase velocity is high. Thus, the disparities in distances are compensated by opposite disparities in the phase velocities. As a consequence, the differences in the group velocities and the travel times are expected to be reduced.

The core refractive index is a function $n(r)$ of the radial position r and the cladding refractive index is a constant n_2. The highest value of $n(r)$ is $n(0) = n_1$ and the lowest value occurs at the core radius $r = a, n(a) = n_2$ as illustrated in Figure 1.5 [2].

Rays in meridional planes follow oscillatory planar trajectories, whereas skewed rays follow helical trajectories with turning points forming cylindrical caustic surfaces, as illustrated in Figure 1.6. Guided rays are confined within the core and

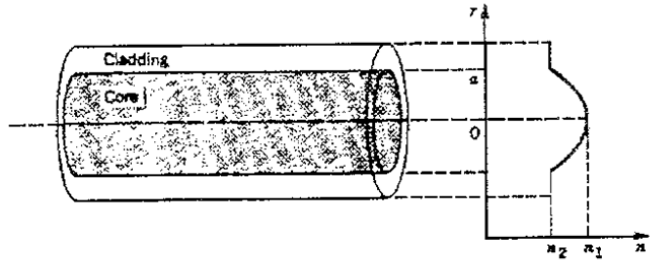

Figure 1.5: Geometry and refractive-index profile of a graded-index fiber [4].

do not reach the cladding.

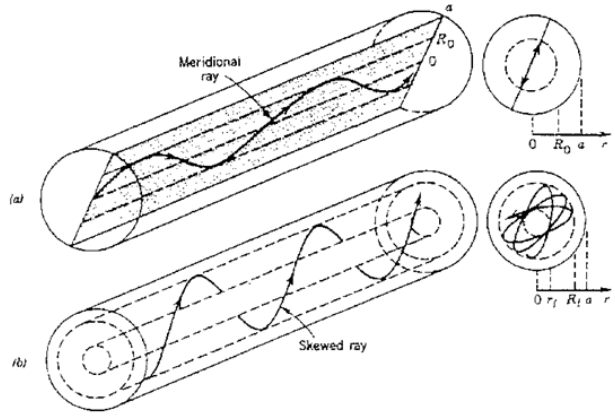

Figure 1.6: Guided rays in the core of a graded-index fiber.(a) A meridional ray confined to meridional plane inside a cylinder of radius R_0.(b) A skewed ray follow a helical trajectory confined within two cylindrical shells of radii r_1 and R_1 [2].

The modes of the graded-index fiber may be determined by writing the Helmholtz equation with $n = n(r)$, solving for the spatial distribution of the field components, and using Maxwell's equations and the boundary conditions to obtain the characteristics equation as was done in the step-index case. This procedure is in general difficult [4].

CHAPTER 2

BASIC THEORY

2.1 Mathematical Approaches to the Slab Optical Waveguide

Dielectric slabs are the simplest optical waveguides. They consist of three planes. Because of their simple geometry, guided and radiation modes of slab waveguides can be described by simple mathematical expressions. To understand the waveguiding properties of more complex dielectric waveguides the knowledge of slab waveguides may be useful [15]. Slab optical guides can be of two types: the symmetric guide and the asymmetric guide. If the refractive indices are similar at the top and bottom layers, the slab is symmetric, as indicated, whereas the refractive indices are different in an asymmetric slab in Figure 2.1 [16]. The refractive index of core region is n_1 and the refractive index of substrate is n_2. The refractive index of the top medium is indicated as n_3. The core material of the symmetric guide is completely embedded inside the substrate (cladding) material. The asymmetric guide consists of a film layer as the guiding core layer, with air or some other covering material as the top cladding layer, and substrate as the bottom cladding layer [16].

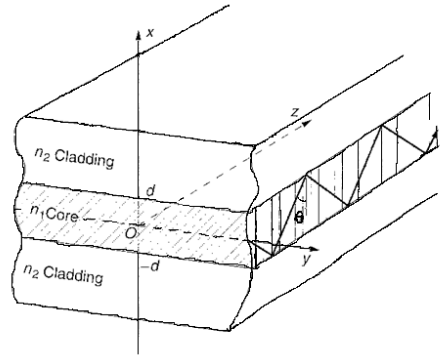

Figure 2.1: Geometry of the slab optical guide [16].

For mode guidance, refractive indices should bring about condition in which n_1 is larger than n_2 and n_3. The modes of symmetric slabs waveguides are simpler than those of asymmetric slabs. The lowest-order mode of a symmetric slab waveguide can propagate arbitrarily at low frequencies because there is not a cutoff frequency of this mode. However, at low operating frequencies all modes of asymmetric slabs have cutoff [15]. In this chapter, some methods of analysis will be explained. These are given below:

- The wave optic approach
- The coefficient approach
- The transmission matrix method
- The modified ray model method

2.2 Wave Optics Approach

In this approach, we need Maxwell's equations to analyze the optical waveguide. When the dielectric material is linear, isotropic and does not have currents and free charges, we can write Maxwell's equations in the following form

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2.1}$$

$$\vec{\nabla} \times \vec{H} = -\frac{\partial \vec{D}}{\partial t} \tag{2.2}$$

$$\vec{\nabla} \cdot \vec{D} = 0 \tag{2.3}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.4}$$

where $\vec{D} = \varepsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$. The parameter ε is the permittivity (or dielectric constant) and μ is the permeability of the medium [11].

Maxwell's equations give the relationships between the electric and magnetic fields. By using these equations standard wave equations can be obtained. Taking the curl of Equation 2.1 and making use of Equation 2.2 the equation below can be written

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\mu \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H}) = -\varepsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2} \tag{2.5}$$

Using the vector identity

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} \tag{2.6}$$

and using Equation 2.3, Equation 2.5 becomes

$$\nabla^2 \vec{E} = \varepsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2} \tag{2.7}$$

In the same way we can obtain Equation 2.2 as standard wave equation

$$\nabla^2 \vec{H} = \varepsilon\mu \frac{\partial^2 \vec{H}}{\partial t^2} \tag{2.8}$$

We assume that wave propagate in the z-direction and is sinusoidally time varying. The propagation constant β is in the z direction. The electric and magnetic field components of the wave are expressed as

$$\vec{E} = \vec{E_0}(x, y)e^{j(\beta z - wt)} \tag{2.9}$$

$$\vec{H} = \vec{H_0}(x, y)e^{j(\beta z - wt)} \tag{2.10}$$

Two assumptions are used for simplify analysis

Assumption 1. No component varies in the y direction:

$$\frac{\partial}{\partial y} = 0 \tag{2.11}$$

Assumption 2. There is no magnetic field component in the z-direction. (TM modes)

$$\vec{H_z} = 0 \tag{2.12}$$

According to the first assumption, a wave does not vary in the y direction. This means that the field extends indefinitely in the y direction (in reality there is no such a wave). Note also that an infinite dimension does not quarantee

Assumption 1. The infinity of the dimension of the layer is certainly a necessary condition for $\partial/\partial y = 0$, but not a sufficient condition [16].

The second assumption leads to a way of dividing the solutions. The solutions are separated into two waves:

- TE (transverse electric) mode; If the solution has only transverse field and no longitudinal electric field, that is, $E_z = 0$, the modes are called TE modes.

- TM (transverse magnetic) mode; If the solution has only transverse and no longitudinal magnetic field, that is, $H_z = 0$, the modes are called TM modes.

Standard wave equation for magnetic field can be written as a function of refractive index $n_{1,2}$ and k. By using Equation 2.8 and Equation 2.10 the following equation can be obtained:

$$\begin{aligned}
\nabla^2\vec{H} &= \varepsilon\mu\frac{\partial}{\partial t}(\frac{\partial}{\partial t}\vec{H} = \vec{H}_0(x,y)e^{j(\beta z - wt)}) \\
\nabla^2\vec{H} &= \varepsilon\mu\frac{\partial}{\partial t}(jw\vec{H}) \\
\nabla^2\vec{H} &= \varepsilon\mu(-w^2\vec{H}) \\
\nabla^2\vec{H} + \varepsilon\mu w^2\vec{H} &= 0 \\
\nabla^2\vec{H} + (n_{1,2}k)^2\vec{H} &= 0 \qquad (2.13)
\end{aligned}$$

Here $k = w\sqrt{\varepsilon\mu}$ and the medium with refractive index n.

Equation 2.13 with the Assumption 1, the y-component of Equation 2.10 is found

as follow

$$\begin{aligned} \nabla^2 \vec{H}_y + (n_{1,2}k)^2 \vec{H}_y &= 0 \\ \frac{\partial^2 \vec{H}_y}{\partial x^2} + \frac{\partial^2 \vec{H}_y}{\partial z^2} + (n_{1,2}k)^2 \vec{H}_y &= 0 \\ \frac{\partial^2 \vec{H}_y}{\partial x^2} - \beta^2 \vec{H}_y + (n_{1,2}k)^2 \vec{H}_y &= 0 \\ \frac{\partial^2 \vec{H}_y}{\partial x^2} + (n_{1,2}^2 k^2 - \beta^2) \vec{H}_y &= 0 \end{aligned} \tag{2.14}$$

Here $n_{1,2}$ is the core and cladding layers refractive values.

Equation 2.14 has two kinds of solutions: trigonometric and exponential solutions. If $(n_{1,2}^2 k^2 - \beta^2)$ has positive value, trigonometric solutions are valid. If it has negative value, exponential solutions are valid. We need to decide that which solution is required in our physical conditions. For example, in the core region wave is oscillatory. Therefore, we should use trigonometric solution. However, in the cladding region, solution should have decaying nature (exponential solution) [16].

$$n_1^2 k^2 - \beta^2 = K^2 \tag{2.15}$$

$$n_2^2 k^2 - \beta^2 = -\gamma^2 \tag{2.16}$$

First condition is always positive (in the core region) and β should be smaller than $n_1 k$, the second condition is always negative (in the cladding region) and β should be larger than the $n_2 k$. The range of β is

$$n_1 k > \beta > n_2 k \tag{2.17}$$

According to the above conditions Equation 2.14 has a solution of core region as follows

$$H_y = AcosKx + BsinKx \tag{2.18}$$

and the solution inside the cladding region is

$$H_y = Ce^{-\gamma x} + De^{\gamma x} \tag{2.19}$$

In these equations, A, B, C, and D are the constants. We may use the boundary conditions to determine the constants. In the cladding layers, solution should not be infinite if x approaches ∞. Therefore, in the upper cladding region D is zero and lower cladding region C is zero.

$$H_y = Ce^{-\gamma x}, x > d \tag{2.20}$$

$$H_y = De^{\gamma x}, x < -d \tag{2.21}$$

The solution in the core region may be written in the two parts:

$$H_y = AcosKx \tag{2.22}$$

$$H_y = BsinKx \tag{2.23}$$

According to trigonometric relations $cosKx$ is an even function and $sinKx$ is an odd function. Therefore, the first equation is called even-mode solution and the second is called odd-mode solution. As a result we can say that if the amplitude distribution of the incident wave is symmetric with respect to the x at

slab optical guide, B should be zero (no odd-mode solution). However, incident wave amplitude distribution is asymmetric, A should be zero (no even-mode solution) [16].

By using the boundary conditions the values of the constants in the core region solutions can be determined. For example; the continuity of the tangential $\vec{H}$ field is used at $x = d$.

$$AcosKd = Ce^{-\gamma d} \tag{2.24}$$

Calculating the constant C and by using the Equation 2.20, an equation for the even TM modes can be found

$$H_y = A(cosKd)e^{-\gamma(x-d)} \tag{2.25}$$

In the same way, for the even TM modes for three regions the following equations are obtained:

$$H_y = A(cosKd)e^{-\gamma(x-d)} \tag{2.26}$$

$$H_y = AcosKx \tag{2.27}$$

$$H_y = A(cosKd)e^{\gamma(x+d)} \tag{2.28}$$

and for the odd-TM modes

$$H_y = B(sinKd)e^{-\gamma(x-d)} \tag{2.29}$$

$$H_y = BsinKx \tag{2.30}$$

$$H_y = -B(sinKd)e^{\gamma(x+d)} \tag{2.31}$$

Using Maxwell's equations, E_x and E_z as a function of H_y can be written

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \tag{2.32}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2.33}$$

E_x and E_z are

$$E_x = \frac{\beta}{\omega \varepsilon_r \varepsilon_0} H_y \tag{2.34}$$

$$E_z = \frac{j}{\omega \varepsilon_r \varepsilon_0} \frac{\partial H_y}{\partial x} \tag{2.35}$$

In the core region ε_r is n_1^2 and in the cladding region ε_r is n_2^2 [16].

The solutions of even and odd TM modes contain parameters of K and γ. More boundary conditions can be used to determine these values.

For even TM modes:

$$n^2 AK sinKd = \gamma C e^{-} \gamma d \tag{2.36}$$

the above equation was used for the continuity of the tangential E field (E_z), at $x = d$

$$\begin{aligned} E_{z1} &= E_{z2} \\ \frac{j}{\omega \varepsilon_{r1} \varepsilon_0} \frac{\partial H_y}{\partial x} &= \frac{j}{\omega \varepsilon_{r2} \varepsilon_0} \frac{\partial H_y}{\partial x} \\ \frac{1}{\varepsilon_{r1}} (AK sinKd) &= \frac{1}{\varepsilon_{r2}} \gamma C e^{-\gamma d} \\ \frac{\varepsilon_{r2}}{\varepsilon_{r1}} AK sinKd &= \gamma C e^{-\gamma d} \\ n^2 AK sinKd &= \gamma C e^{-\gamma d} \end{aligned} \tag{2.37}$$

where Equation 2.20, Equation 2.22 and Equation 2.35 were used. By dividing Equation 2.37 by Equation 2.24

$$n^2 KdtanKd = \gamma d \tag{2.38}$$

The last equation is called characteristic equation for the even TM modes. We need this equation to solve the K and γ. Eliminating the β from the equations Equation 2.15 and Equation 2.16 we find that equation

$$n_1^2 k^2 - n_2^2 k^2 = K^2 + \gamma^2 \tag{2.39}$$

multiplying both side of equation by d

$$\begin{aligned} (kd)^2(n_1^2 - n_2^2) &= (Kd)^2 + (\gamma d)^2 \\ V^2 &= (Kd)^2 + (\gamma d)^2 \end{aligned} \tag{2.40}$$

V is called the normalized thickness of the guide. It is an important parameter of the characteristics of the guide. Equation 2.38 and Equation 2.38 and Equation 2.40 have graphical solutions as shown in Figure 2.2. The first equation has a $tanKd$ curve, the second one has a circle with radius V as solid lines. In this figure intersection points are mode of propagation (a solution) of an even TM modes. Similarly the characteristic equation for the odd TM modes can be found. The continuity of H_y at $x = d$ and the continuity of E_z give

$$BsinKd = Ce^{-\gamma d} \tag{2.41}$$

$$n^2 BKcosKd = -C\gamma e^{-\gamma d} \tag{2.42}$$

The characteristic equation for the odd TM mode (in the figure as dashed lines)

$$-n^2 Kdcot Kd = \gamma d \tag{2.43}$$

The intersections between the solid lines and the circle are the even (order)TM modes and those between the dashed lines and the circle are the odd (order) TM modes. Even numbered subscripts are used for even modes and odd numbers for odd modes. The subscripts are called the *order of the mode* or *mode number* [16].

The disappearance of a particular mode is called the *cutoff* of that mode. As the cutoff condition is approached, the value of γ approaches zero, and the effective depth of the evanescent wave in the cladding layer increases. When γ reaches zero, the evanescent wave is present throughout the cladding layer and

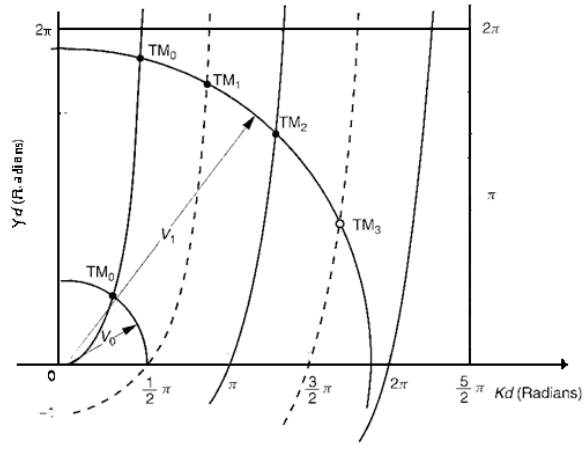

Figure 2.2: Graphical solutions for the even TM modes in solid lines and odd TM modes in dashed lines. $V = kd\sqrt{n_1^2 - n_2^2}$ [16].

the light energy cannot be sustained inside the core. As long as V is greater than $\pi/2$, more than one mode can be excited simultaneously. Among the excited modes, the higher order modes are more susceptible to the conditions outside the guide because γ is smaller and the effective depth of the evanescent wave is deeper. When V is less than $\pi/2$, there exists only one mode and no other modes can be excited. This mode is called the *dominant mode* ($TM_0 mode$).

Multi-mode guides support more than one mode (a slab optical guide support only one mode). In multi-mode guides there are different propagation constants. Therefore, each mode arrives at the receiving point at a different phase and signal is distorted. This distortion is called *mode dispersion.*

Since the number of modes increases for every $\pi/2$-radian increase in V, the highest mode number N is the largest integer that still satisfies

$$\frac{2\pi}{\lambda} d\sqrt{n_1^2 - n_2^2} > N\frac{\pi}{2} \tag{2.44}$$

As a result it can be said that the total number of TM modes with the zero order mode is $N + 1$.

2.2.1 Effective Index of Refraction

The parameter β in $\exp(j\beta z)$ is the z component of the propagation vector and can be determined by the boundary conditions on the electromagnetic fields at the core-cladding interface.

We have already known in free space, the propagation constant is

$$k = \frac{2\pi}{\lambda} \tag{2.45}$$

where λ is the wavelength vacuum. If the free space is now filled with a medium with refractive index n_0, the wavelength will be contracted to λ/n_0 and the rate of advance phase per distance will become larger: namely the propagation constant β_0 in such a medium is

$$\beta_0 = \frac{2\pi}{\lambda/n_0} = n_0 k \tag{2.46}$$

The refractive index of the filling medium is given by

$$n_0 = \frac{\beta_0}{k} \tag{2.47}$$

Let us think of the previous case became in an optical guide. For example; for the z-direction of the propagation constant and x-direction of the K. Index of refraction N can be written as follow

$$N = \frac{\beta}{k} \tag{2.48}$$

where N is called the *effective index of refraction.* The effective index of refraction can be found from Figure 2.3.

$$\beta = n_1 k sin\theta \tag{2.49}$$

$$N = n_1 sin\theta \tag{2.50}$$

2.2.2 TE Modes

In this section TE mode solutions by using similar methods in TM mode solutions can be obtained.

TE modes have only transverse and no longitudinal electric field.

$$E_z = 0 \tag{2.51}$$

There is no variation of the field occurs in the y-direction

$$\frac{\partial}{\partial y} = 0 \tag{2.52}$$

Let us take a solution for y component of electric field

$$E_y = E_y(x, y)e^{j(\beta z - wt)} \tag{2.53}$$

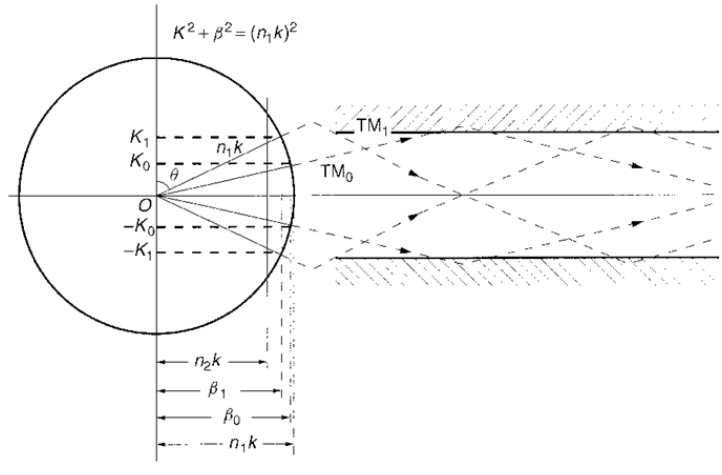

Figure 2.3: $K - \beta$ circle and component waves [16].

and Equation 2.53 by inserting into the wave equation Equation 2.13

$$\begin{aligned} \nabla^2\vec{E} + (n_{1,2}k)^2\vec{E} &= 0 \\ \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} + (n_{1,2}k)^2\vec{E} &= 0 \\ \frac{\partial^2 E_y}{\partial x^2} + E_y(n_{1,2}^2k^2 - \beta^2) &= 0 \end{aligned} \tag{2.54}$$

In core and cladding regions are valid the following conditions:

$$(n_1k)^2 - \beta^2 = K^2 \tag{2.55}$$

$$(n_2k)^2 - \beta^2 = -\gamma^2 \tag{2.56}$$

These conditions give us the solution of Equation 2.54. In the core even and odd TE modes:

$$E_y = AcosKx \tag{2.57}$$

$$E_y = BsinKx \tag{2.58}$$

In the cladding region:

$$E_y = Ce^{-\gamma x} \tag{2.59}$$

$$E_y = De^{\gamma x} \tag{2.60}$$

The first one is valid at the upper layer and the second one at the lower layer. The constants of these equations A, B, C and D are determined with help of boundary condition. Maxwell's equation $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ is used to find H_z:

$$H_z = \frac{1}{jw\mu_{1,2}}\frac{\partial E_y}{\partial x} \tag{2.61}$$

The magnetic permeability of the core and cladding layers is $\mu_{1,2}$. The z-component of magnetic field in the core region [16] :

$$H_z = \frac{K}{jw\mu_1}(-AsinKx) \tag{2.62}$$

$$H_z = \frac{K}{jw\mu_1}(BcosKx) \tag{2.63}$$

and in the cladding region [6]:

$$H_z = \frac{\gamma}{jw\mu_2}(-Ce^{-\gamma x}) \tag{2.64}$$

$$H_z = \frac{\gamma}{jw\mu_2}De^{\gamma x} \tag{2.65}$$

From the boundary conditions the characteristic equations can be obtained. For the even TE modes (the continuity of E_y at $x = d$):

$$AcosKd = Ce^{-\gamma d} \tag{2.66}$$

By using the continuity of H_z

$$-\frac{K}{jw\mu_1}AsinKd = -\frac{\gamma}{jw\mu_2}Ce^{-\gamma d} \tag{2.67}$$

Similar to the even TM modes, the characteristic equation for the even TE modes can be found

$$(\frac{\mu_2}{\mu_1})KdtanKd = \gamma d \tag{2.68}$$

the characteristic equation for the odd TE modes are

$$-(\frac{\mu_2}{\mu_1})KdcotKd = \gamma d \tag{2.69}$$

Here, when the magnetic material is not used, then the factor is $\mu_2/\mu_1 = 1$ [16].

2.3 Coefficient Matrix Method

In the first section the characteristic equations were obtained by the method of wave optics. In this section the coefficient matrix method will be explained. The simplest method is used for this method. Figure 2.1 shows us this geometry. In the core and cladding regions the solution of H_y was discussed in the previous section. The constants A, B, C and D were derived using the boundary conditions. As a result the general solution for H_y in the three layers of a symmetric guide can be summarized.

$$H_{y1} = AcosKx + BsinKx \tag{2.70}$$

$$H_{\gamma 2} = Ce^{-\gamma x} \tag{2.71}$$

$$H_{\gamma 3} = De^{\gamma x} \tag{2.72}$$

The above solutions are respectively in core, in upper layer and lower layer. Let us continue the tangential $\vec{H}$ field is the with continuity at $x = d$ and $x = -d$,

$$AcosKd + BsinKd = Ce^{-\gamma d} \tag{2.73}$$

$$AcosKd - BsinKd = De^{-\gamma d} \tag{2.74}$$

In the last two equations there are four unknown constants.

The tangential components of the $\vec{E}$ field in the three regions are summarized with the help of the Equation 2.35.

$$E_{z1} = \frac{jK}{w\varepsilon_0\varepsilon_{r1}}(-AsinKx + BcosKx) \tag{2.75}$$

$$E_{z2} = \frac{-jC\gamma}{w\varepsilon_o\varepsilon_{r2}}e^{-\gamma x} \tag{2.76}$$

$$E_{z3} = \frac{jD\gamma}{w\varepsilon_0\varepsilon_{r2}}e^{\gamma x} \tag{2.77}$$

These solutions are respectively in core, in upper and lower regions. The continuity of the tangential components of the $\vec{E}$ field at $x = d$ gives:

$$\frac{K}{\varepsilon_{r1}}(-AsinKd + BcosKd) = \frac{-C}{\varepsilon_{r2}}\gamma e^{-\gamma d} \tag{2.78}$$

and at the $x = -d$ is

$$\frac{K}{\varepsilon_{r1}}(AsinKd + BcosKd) = \frac{D}{\varepsilon_{r2}}\gamma e^{-\gamma d} \tag{2.79}$$

Equations 2.73 to 2.79 are written in the matrix form to obtain solution [16]

$$\begin{bmatrix} cosKd & sinKd & -e^{-\gamma d} & 0 \\ cosKd & -sinKd & 0 & -e^{-\gamma d} \\ -K'sinKd & K'cosKd & \gamma' e^{-\gamma d} & 0 \\ K'sinKd & K'cosKd & 0 & -\gamma' de^{-\gamma d} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0$$

where

$$K' = K\frac{1}{\varepsilon_{r1}} = \frac{K}{n_1^2} \tag{2.80}$$

$$\gamma' = \gamma\frac{1}{\varepsilon_{r2}} = \frac{\gamma}{n_2^2} \tag{2.81}$$

If there is a non-zero solution for A, B, C, and D, then the determinant has to be zero. After some algebra the characteristic equation can be found. Firstly, the second row is multiplied by the γ' and then the fourth row is subtracted from the

second row. At the end of this calculation new determinant is reached.

$$\Delta = -\frac{1}{2}e^{-2\gamma d}sin2Kd \begin{bmatrix} 1 & 1 & -1 \\ \gamma' - K'tanKd & -\gamma' - K'cotKd & 0 \\ -K'tanKd & K'cotKd & \gamma' \end{bmatrix}$$

Secondly, the first row is multiplied by the γ' and the third row is added to the first row. As a result the characteristic equation becomes

$$\Delta = e^{-2\gamma d}sin2Kd(\gamma' - K'tanKd)(\gamma' + K'cotKd) = 0 \tag{2.82}$$

The determinant is zero when

$$2Kd = n\pi \tag{2.83}$$

or

$$\gamma d = (\frac{n_2}{n_1})^2 Kdtan Kd \tag{2.84}$$

$$\gamma d = -(\frac{n_2}{n_1})^2 Kdcot Kd \tag{2.85}$$

the results are the same as the characteristic equations in the previous section. The values of Kd in the Equation 2.83 are on the cutoff points of the TM modes [16].

2.4 Transmission Matrix Method (General Guides)

In the coefficient matrix method multilayer guides are not useful because they reason a large size of matrix. However, this method is used for multilayers. The geometry of a multilayer is described by the Figure 2.4. This geometry consists

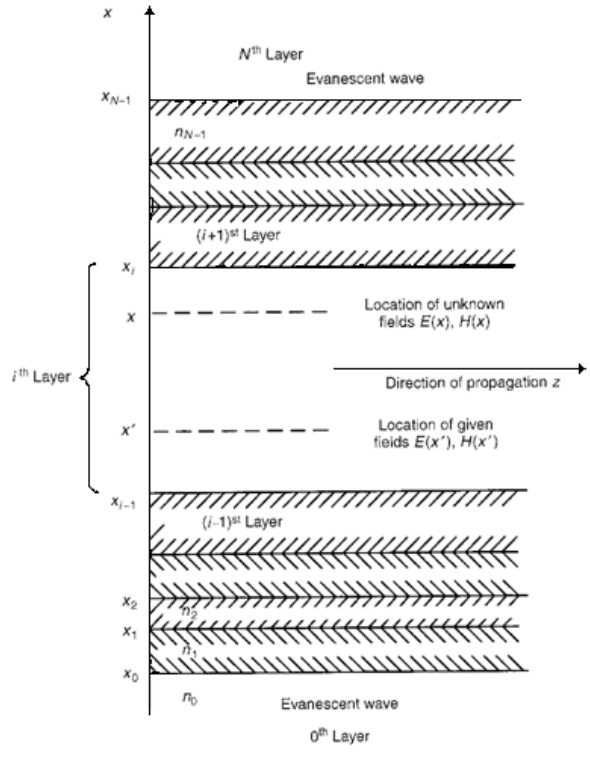

Figure 2.4: Geometry of a layered medium [16].

of N layers which are established as the one on the top of the other. These layers are parallel to the direction of propagation. The guide is treated layer by layer. The field is expressed by a 2×2 matrix in each layer [16].

The general solutions of the TM wave from Equation 2.18 and Equation 2.35 as the fields in the ith layer are written.

$$H_y(x) = AcosK_ix + BsinK_ix \tag{2.86}$$

$$E_z(x) = -AZ_isinK_ix + BZ_icosK_ix \tag{2.87}$$

where the factor Z_i is

$$Z_i = \frac{jK_i}{w\varepsilon_0\varepsilon_{ri}} \tag{2.88}$$

$$Z_i = \frac{jK_i}{w\varepsilon_0 n_i^2} \tag{2.89}$$

$$K_i = \sqrt{(n_i k)^2 - \beta^2} \tag{2.90}$$

The given values of $H(x')$ and $E(x')$ at $x = x'$:

$$H_y(x') = AcosK_i x' + BsinK_i x' \tag{2.91}$$

$$E_z(x') = -AZ_i sinK_i x' + BZ_i cosK_i x' \tag{2.92}$$

The values A and B are obtained by solving the simultaneous Equation 2.90 and Equation 2.91. The solutions for A and B are put back into the Equation 2.86 and Equation 2.87 to reach

$$\begin{bmatrix} H_y(x) \\ E_z(x) \end{bmatrix} = \begin{bmatrix} cosK_i(x-x') & \frac{1}{Z_i}sinK_i(x-x') \\ -Z_i sinK_i(x-x') & cosK_i(x-x') \end{bmatrix} \begin{bmatrix} H_y(x') \\ E_z(x') \end{bmatrix}$$

This equation is the transmission matrix that relates the fields at $x = x'$ with those at $x = x$. The x' and x can be set at any point as long as they belong to the same layer [16].

In Figure 2.5, a three layer slab guide is shown as an example. The core-cladding lower boundary was placed at $x = 0$ and the upper boundary at $x = 2d$. This geometry needs the following transmission matrix [16]:

$$\begin{bmatrix} H_y(2d) \\ E_z(2d) \end{bmatrix} = \begin{bmatrix} cos2Kd & \frac{1}{Z_1}sin2Kd \\ -Z_1 sin2Kd & cos2Kd \end{bmatrix} \begin{bmatrix} H_y(0) \\ E_z(0) \end{bmatrix}$$

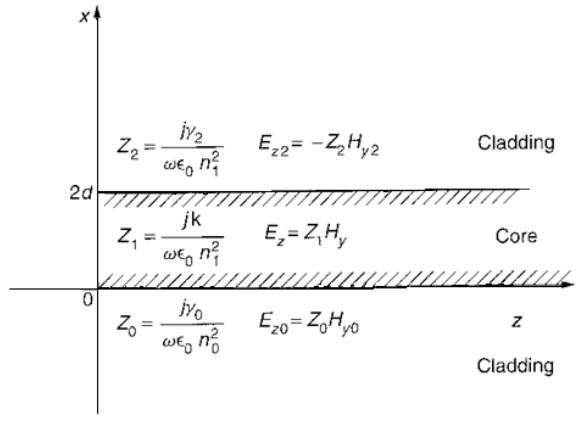

Figure 2.5: Geometry of the three layer slab guide [16].

The geometry of the layers alone can never determine the field inside the medium. It is only after the fields on the boundary are specified that the field inside the medium is determined. The specification of the field can be the field H_y or E_z itself, as in the case of a given incident field to the boundary; or the field specification can be an unbounded travelling wave, ratio of reflected to incident waves (standing wave), or an evanescent wave. These are called boundary fields. This section deals with guides whose boundary fields are evanescent waves [16].

The unbounded field in the bottom layer is the evanescent wave $H_{y0}(x)$, which is expressed by the bottom equation of Equation 2.20. This equation is inserted into Equation 2.35 to obtain

$$E_{z0}(x) = \frac{j}{w\varepsilon_r\varepsilon 0}\frac{\partial}{\partial x}De^{\gamma_0 x} = Z_0 H_{y0}(x) \tag{2.93}$$

where

$$Z_0 = \frac{j\gamma_0}{w\varepsilon_0 n_0^2} \tag{2.94}$$

The unbounded field $H_{y2}(x)$ in the top layer is also an evanescent wave and is expressed by the top equation of Equation 2.20 into Equation 2.35 gives [16]

$$E_{z2}(x) = \frac{j}{w\varepsilon_0\varepsilon_2}\frac{\partial}{\partial x}Ce^{-\gamma_2 x} = -Z_2 H_{y2}(x) \tag{2.95}$$

where

$$Z_2 = \frac{j\gamma_2}{w\varepsilon_0 n_2^2} \tag{2.96}$$

For simplicity,

$$\begin{aligned} H_{y0}(0) &= H_y(0) \\ H_{y2}(2d) &= H_y(2d) \\ E_{z0}(0) &= E_z(0) \\ E_{z2}(2d) &= E_z(2d) \end{aligned} \tag{2.97}$$

The $E_z(2d)$ and $E_z(0)$ in matrix form are rewritten in terms of $H_y(2d)$ and $H_y(0)$ applying the boundary conditions to Equation 2.93 and Equation 2.95 and rearranged as [16]

$$\begin{bmatrix} cos2Kd + \frac{Z_0}{Z_1}sin2Kd & -1 \\ Z_0 cos2Kd - Z_1 sin2Kd & Z_2 \end{bmatrix} \begin{bmatrix} H_y(0) \\ H_y(2d) \end{bmatrix} = 0$$

In order that $H_y(0)$ and $H_y(2d)$ have nonzero solutions, the determinant has to

be zero. The determinant is

$$(Z_0 + Z_2)cos2Kd - (Z_1 - \frac{Z_0 Z_2}{Z_1})sin2Kd = 0 \tag{2.98}$$

or

$$\frac{Z_0/Z_1 + Z_2/Z_1}{1 - Z_0 Z_2/Z_1^2} = tan2Kd \tag{2.99}$$

If the guide is symmetric and $Z_0 = Z_2$

$$\frac{2(Z_0/Z_1)}{1 - (Z_0/Z_1)^2} = tan2Kd \tag{2.100}$$

we have already know from the trigonometric relations

$$tan2Kd = \frac{2tanKd}{1 - tan^2 Kd} \tag{2.101}$$

is inserted into Equation 2.100. After some calculations

$$(\frac{\gamma}{n^2 K} - tanKd)(\frac{\gamma}{n^2 K}tanKd + 1) = 0 \tag{2.102}$$

where

$$\frac{\gamma}{n^2 K} = \frac{Z_0}{Z_1} \tag{2.103}$$

The Equation 2.102 contains two characteristic equations; the first factor for the even TM modes and the second for the odd TM modes.

The field is connected to one layer to the next layer by the continuity relation. So $(N-1)$ layer can be reached. In the ith layer the matrix takes the form as the following equation

$$\begin{bmatrix} H_y(x_i) \\ E_z(x_i) \end{bmatrix} = \begin{bmatrix} T_i(x_i - x_{i-1}) \end{bmatrix} \begin{bmatrix} H_y(x_{i-1}) \\ E_z(x_{i-1}) \end{bmatrix} \tag{2.104}$$

here in the two outermost layers mode has to be an evanescent wave. $T_i(x_i - x_{i-1})$ is the transmission matrix of the ith layer,

$$T_i(x_i - x_{i-1}) = \begin{bmatrix} cosK_i(x_i - x_{i-1} & \frac{1}{Z_i} sinK_i(x_i - x_{i-1}) \\ -Z_i sinK_i(x_i - x_{i-1}) & cosK_i(x_i - x_{i-1}) \end{bmatrix} \tag{2.105}$$

and $T_i(x_i - x_{i-1})$ was derived from Equation 2.86 and Equation 2.87

$$H_y(x) = Ccosh\gamma ix + Dsinh\gamma ix \tag{2.106}$$

$$E_z(x) = Z_i Csinh\gamma ix + Z_i Dcosh\gamma ix \tag{2.107}$$

where

$$\gamma_i = \sqrt{\beta^2 - (n_i k)^2} \tag{2.108}$$

$$Z_i = \frac{j\gamma_i}{w\varepsilon_0 \varepsilon_{ri}} \tag{2.109}$$

The values of C and D are chosen in such away that H and E become $H(x')$ and $E(x')$ at $x = x'$. The obtained matrix is

$$T_i^e(x_i - x_{i-1}) = \begin{bmatrix} cosh\gamma_i(x_i - x_{i-1}) & \frac{1}{Z_i} sinh\gamma_i(x_i - x_{i-1}) \\ Z_i sinh\gamma_i(x_i - x_{i-1}) & cosh\gamma_i(x_i - x_{i-1}) \end{bmatrix} \tag{2.110}$$

Repeated use of the Equation 2.104 to connect the field starting from the first layer up the (N-1)st layer gives $[T_{N-1}][T_{N-2}].....[T_1]$:

$$\begin{bmatrix} H_y(x_{N-1}) \\ E_z(x_{N-1}) \end{bmatrix} = [T_{N-1}][T_{N-2}...[T_N] \begin{bmatrix} H_y(x_0) \\ E_z(x_o) \end{bmatrix} \tag{2.111}$$

The product of the T matrices is rewritten as [6]:

$$\begin{bmatrix} H_y(x_{N-1}) \\ E_z(x_{N-1}) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} H_y(x_0) \\ E_z(x_o) \end{bmatrix} \tag{2.112}$$

Now, the field at the top boundary of the zeroth layer is:

$$E_z(x_0) = Z_0 H_y(x_0) \tag{2.113}$$

where

$$Z_0 = \frac{j\gamma_0}{w\varepsilon_0 n_0^2} \tag{2.114}$$

and

$$\gamma_0 = \sqrt{\beta^2 - (n_0 k)^2} \tag{2.115}$$

The field at the lower boundary of the free-space layer, which is the top boundary of the $N-1$ layer has a similar relationship as Equation 2.95 and

$$E_z(x_{N-1}) = -Z_N H_y(x_{N-1}) \tag{2.116}$$

where

$$Z_N = \frac{j\gamma_N}{w\varepsilon_0 n_N^2} \tag{2.117}$$

and

$$\gamma_N = \sqrt{\beta^2 - (n_N k)^2} \tag{2.118}$$

The Nth layer is free space. The phase match condition in the z direction is automatically satisfied across every layer boundary. The expressions for $E_z(x_{N-1})$

and $E_z(x_0)$ are rewritten in terms of $H_y(x_{N-1})$ and $H_y(x_0)$ from Equation 2.113 and Equation 2.116 and are incorporated into Equation 2.112, [16]:

$$\begin{bmatrix} A + Z_0 B & -1 \\ C + Z_0 D & Z_N \end{bmatrix} \begin{bmatrix} H_y(x_0) \\ H_y(x_{N-1}) \end{bmatrix} = 0 \tag{2.119}$$

In order that the solutions for $H_y(x_0)$ and $H(x_{N-1})$ are nonzero, the determinant of Equation 2.119 has to be zero [16]. Thus, the characteristic equation is finally

$$Z_N A + Z_0 Z_N B + C + Z_0 D = 0 \tag{2.120}$$

where the values of Z_0 and Z_N are available from Equation 2.114 and Equation 2.117.

2.5 Transmission Matrix Method (Symmetric Guide)

The calculations become a very simpler form because of the symmetric distribution of the indices of refraction. A symmetric geometry is shown in Figure 2.6. There the symmetry exists respect to $x = 0$ and the indices of refraction from the bottom to the top layer are arranged as $n_0, n_1, n_2, ...n_{n-1}, n_{n-1}, ...n_2, n_1, n_0$. The zeroth layer take into account as unbounded. The transmission matrix then becomes [16]

$$[T_1][T_2]....[T_{n-1}][T_{n-1}]...[T_2][T_1] \tag{2.121}$$

Equation 2.121 is divided into two layer groups:

$$T_+ = [T_1][T_2]...[T_{n-1}] \tag{2.122}$$

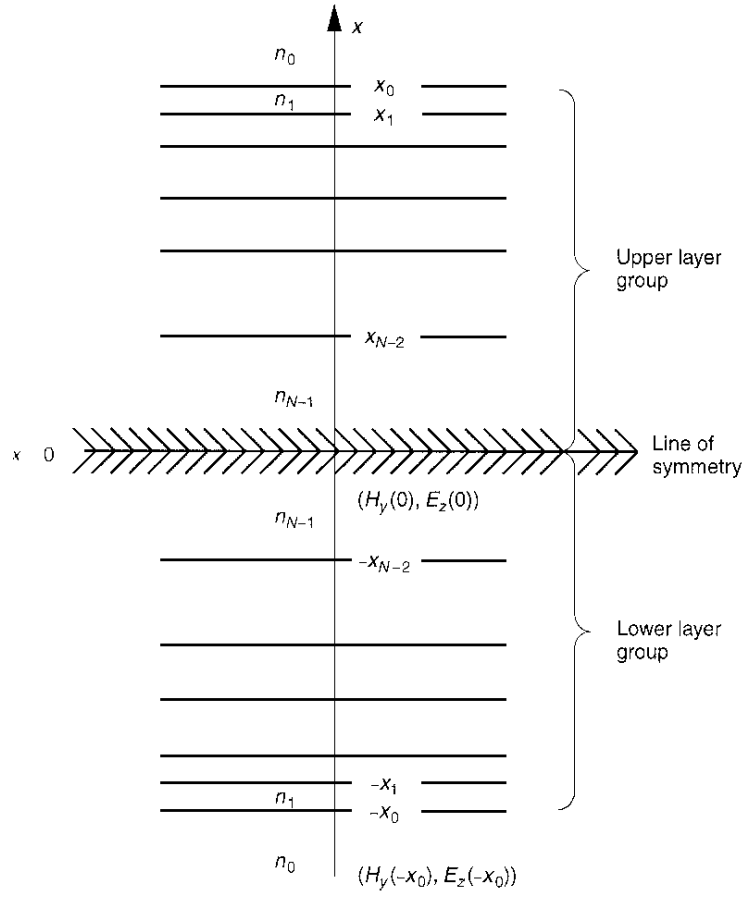

Figure 2.6: Geometry of layers with symmetry with respect to $x = 0$ [16].

$$T_{-} = [T_{n-1}]...[T_2][T_1] \tag{2.123}$$

$$[T_1][T_2] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2.124}$$

$$[T_2][T_1] = \begin{bmatrix} d & b \\ c & a \end{bmatrix} \tag{2.125}$$

In both formulas the two diagonal elements in $[T_i]$ are identical. Let us suppose that

$$T_{-} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{2.126}$$

T_{+} consist of the same matrices but in reverse order in multiplication.

$$T_{+} = \begin{bmatrix} D & B \\ C & A \end{bmatrix} \tag{2.127}$$

the characteristic matrix[16]

$$[T] = [T_{+}][T_{-}] \tag{2.128}$$

and as a result can be obtained

$$[T] = \begin{bmatrix} AD + BC & 2BD \\ 2AC & AD + BC \end{bmatrix} \tag{2.129}$$

The $[T]$ can be obtained by the first calculating A, B, C, and D of T_{-} and then using them in Equation 2.129. The characteristic matrix obtained involves only about one-half of the number of matrix multiplications.

The calculation of the modes in the symmetric structure can be simplified further by calculating the even and odd modes separately. Even and odd refers to H_y for the TM modes [16].

A special feature of the even modes is that not only is the field distribution in the upper layer group symmetric with that in the lower layer group with respect

to $x = 0$, but also the derivative dH_y/dx has to vanish at $x = 0$, for a smooth connection.

$$\frac{dH_y}{dx} = 0 \tag{2.130}$$

this means that

$$E_z(0) = \frac{j}{w\varepsilon_0\varepsilon_r}\frac{dH_y}{dx} = 0 \tag{2.131}$$

At the lowest boundary the electric field as a function of y component of magnetic field

$$E(-x_0) = Z_0 H_y(-x_0) \tag{2.132}$$

The field in the lower layer group is, from Equation 2.112 and Equation 2.131

$$\begin{bmatrix} H_y(0) \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} H_y(-x_0) \\ Z_0 H_y(-x_0) \end{bmatrix} \tag{2.133}$$

$$\begin{bmatrix} A + Z_0 B & -1 \\ C + Z_0 D & 0 \end{bmatrix} \begin{bmatrix} H_y(-x_0) \\ H_y(0) \end{bmatrix} = 0 \tag{2.134}$$

For $H_y(-x_0)$ and $H_y(0)$ to exist, the determinant has to vanish and the characteristic equation for the even modes is given by [16]

$$C + Z_0 D = 0 \tag{2.135}$$

The characteristic equation for the odd TM modes is derived. A special feature of the odd modes is that not only is the field in the upper layer group the negative

of the corresponding field in the lower layer group, but also the field itself has to vanish at $x = 0$ and $H_y = 0$ so as to connect at $x = 0$. [16]

$$\begin{bmatrix} 0 \\ E_z(0) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} H_y(-x_0) \\ Z_0 H_y(-x_0) \end{bmatrix} \tag{2.136}$$

and after calculations the last equation

$$\begin{bmatrix} A + Z_0 B & 0 \\ C + Z_0 D & -1 \end{bmatrix} \begin{bmatrix} H(-x_0) \\ E_z(0) \end{bmatrix} = 0 \tag{2.137}$$

For $H(-x_0)$ and $E_z(0)$ to exist, the determinant has to vanish and the characteristic equation for the odd modes is given by

$$A + Z_0 B = 0 \tag{2.138}$$

2.6 Modified Ray Model Method

The mode is a quantization of the propagation constant. At the angles other than the quantized angles, no light propagate in the guide. Figure 2.7 shows the path of the component rays inside the optical guide [16].

The bell-shaped curve signifies the transverse distribution of the light. The field at $x = x_1$ for the downward component wave is considered. As illustrated in the figure the field at any given point along $x = x_1$ is the sum of numerous contributing fields due to the transverse spread of the beam [16]. If the beam were a plane wave, the transverse spread would be wider and the interference would

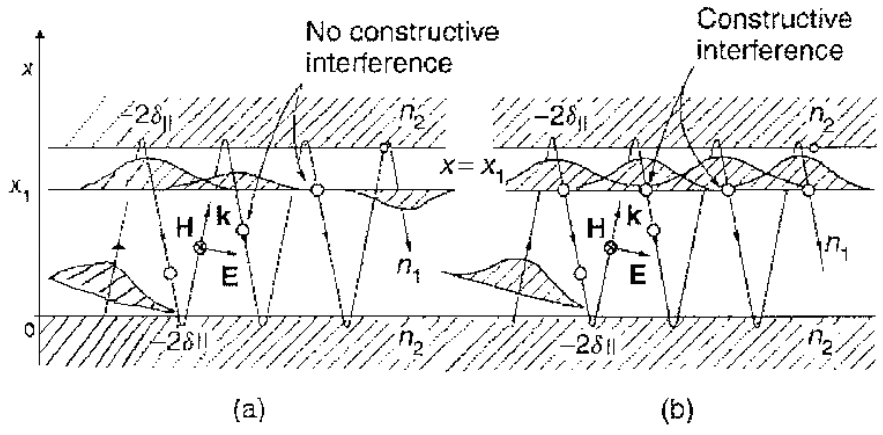

Figure 2.7: Explanation of the quantization in the x direction .(a) Random phase at $x = x_1$. (b) In phase at $x = x_1$. $4Kd - 4\delta = 2\mu\pi$ [16].

be even bigger. If the phases of the contributing fields are random, the resultant travelling wave field becomes null. If, however, the phase shift associated with one round trip is an integral multiple of 2π radians, all the contributing fields interfere constructively, and the field builds up. The phase shift associated with one round trip includes the phase shifts due to the total internal reflections at the top and bottom boundaries as well as the phase shift due to the round-trip distance inside the guide [16].

The same is true with the upward component wave. The field at any given point is the sum of many contributing fields, which must interfere constructively for the field to build up and the same condition applies that the round-trip phase shift be an integral multiple of 2π.

The upward and downward component waves propagating in opposite directions again interfere and generate a standing wave pattern, which is the mode pattern of the guide.

The amount of the phase shift associated with the reflection from the boundary depends on the direction of the polarization of the component plane wave. If the TM wave is used, the direction of the $\vec{H}$ field is perpendicular to the plane of propagation and hence the direction of the polarization of the $\vec{E}$ field is parallel to the plane of propagation. The total round-trip phase shift for a $2d - thick$ guide for constructive interaction is

$$4Kd - 4\delta_{\parallel} = 2\mu\pi \tag{2.139}$$

where μ is an integer and the phase shift due to the total internal reflections at each boundary is $-2\delta_{\parallel}$. Equation 2.139 is sometimes called the *dispersion equation* [16].

$$tan(Kd - \frac{\pi}{2}\mu) = \frac{\gamma}{n^2 K} \tag{2.140}$$

where

$$n = \frac{n_2}{n_1} \tag{2.141}$$

The integer μ is the mode number and separating it into even and odd modes gives

$$tanKd = \frac{\gamma}{n^2 K} \tag{2.142}$$

$$-cotKd = \frac{\gamma}{n^2 K} \tag{2.143}$$

CHAPTER 3

SOLUTIONS OF THE VECTOR WAVE EQUATIONS BASED ON H- FIELD FOR GRADED INDEX OPTICAL WAVEGUIDES BY USING MAPPED GALERKIN METHOD

3.1 Basics of Galerkin Method

In semiconductor or silica based optoelectronic integrated circuits, or photonic integrated circuits and planer lightwave circuits, used the optical rib wave guides and rectangular waveguides. Therefore, guided mode solutions are very important to photonic device design because it is essential to analyze the mode characteristics for designing various devices. Many different ways were used to obtain modal analysis of optical waveguides. If the waveguide structure was manufactured with materials whose dielectric constants are very nearly identical in a large field intensity regions, we need a scalar solution. The simple scalar wave equations can be used to find the mode solution. However, if the wave guide structure consist of a large refractive index differences in regions of high field intensity, then we need a vector solution. In this case one should be used vector solutions of Maxwell's equations.

There are some useful methods to find the vector solution, such as based on variational principles, the finite element method [17], [18]. Another popular approach involves solutions of difference equations that are obtained by replacing the differentiation operators in the partial differential equations by difference operators [19].

The above-mentioned methods have a large matrix equations, especially for the two-dimensional problems. This matrix solutions required special numerical methods. Therefore an alternative method (Galerkin's Method) which has smaller matrix equations was started has been put into practice. The Galerkin's method, expressing the unknown field as series expansions in terms of a complete set of orthogonal basis functions, results in small matrices [19]. Firstly, Henry and Verbeek proposed the Galerkin's method for solving the scalar wave equation of optical waveguides [20]. In this waveguides, the refractive index profiles was arbitrary. This method was used for solving the vector wave equation by Marcuse [19]. He derived a general vector formulation for optical waveguides based on this method with trigonometric basis functions. In this formulation the waveguide has a rectangular bounding box whose dimensions are large enough to ensure that the calculated fields are zero at this artificial boundary. The determining how far the rectangular bounding box has not certain rules. This case effects the accuracy of the calculated results. Although the approach of selecting Hermite-Gaussian functions as basis functions rather than trigonometric functions can

eliminate the artificial boundary, this is valid only for optical waveguide with homogeneous cladding. The use of trigonometric basis functions with tangent-type variable transformation can implement the elimination of the boundary for inhomogeneous-cladding waveguides such as optical rib waveguides [20]. By using this transformation, the transverse plane is mapped onto a unit square. At the unit square boundary, the fields are equal to zero. This method is called mapped Galerkin method. It can be applied in homogeneous and inhomogeneous cladding waveguides. The mapped Galerkin method for solving the scalar wave equation for optical waveguides was first proposed by Hewlett and Ladouceur [21]. For the guasi-vector wave equation Lo and Li used the same method [22].

3.2 Mathematical Formulation

The magnetic or electric field vectors can first be eliminated by using Maxwell's equations. Then the vector wave equations can be written for the electric or magnetic field alone. There are two components of the electric or magnetic field (E_x and E_y or H_x and H_y) for the transverse components of the vector wave equations. These components are obtained from the other field components. Therefore, the solution of the guided mode problem for vector waves can be reduced to solving two simultaneous equations for either E_x and E_y or H_x and H_y.

3.2.1 Vector Wave Equation for H-field

The using magnetic wave equation may appear useful, because for the dielectric structure all components of magnetic field are continuously dielectric interfaces [23]. For monochromatic fields with angular frequency w and time dependence as $exp(-jwt)$ the following source-free Maxwell's equations

$$\vec{\nabla} \times \vec{E} = -jw\mu_0\vec{H} \tag{3.1}$$

$$\vec{\nabla} \times \vec{H} = jw\varepsilon_0 n^2\vec{E} \tag{3.2}$$

where $\vec{E}$ and $\vec{H}$ are the electric and magnetic fields vectors, ε_0 and μ_0 are the electric permittivity and the magnetic permeability in vacuum, respectively and $n = n(x, y, z)$ is the refractive index of the guiding medium. Taking the curl of (2) and substituting $\vec{\nabla} \times \vec{E}$ from (1) we can write following equations:

$$\begin{aligned}
\vec{\nabla} \times \vec{\nabla} \times \vec{H} &= jw\varepsilon_0 n^2\vec{E} \\
&= jw\varepsilon_0 n^2(-jw\mu_0\vec{H}) \\
\vec{\nabla} \times (1/n^2\vec{\nabla} \times \vec{H}) &= w^2 n^2\varepsilon_0\mu_0\vec{H} \\
&= k_0^2\vec{H} \\
\vec{\nabla} \times (1/n^2\vec{\nabla}\vec{H}) - k_0^2\vec{H} &= 0
\end{aligned}$$

where $k_0 = w\sqrt{\varepsilon_0\mu_0}$ is the wave number in vacuum. Using vector identity

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{H}) - \nabla^2\vec{H} \tag{3.3}$$

and we have already known from Maxwell's equation

$$\vec{\nabla} \cdot \vec{H} = 0 \tag{3.4}$$

$$\begin{aligned} \vec{\nabla}(\frac{1}{n^2}\vec{\nabla} \times \vec{H}) &= \vec{\nabla}(\frac{1}{n^2} \times \vec{\nabla} \times \vec{H} + \frac{1}{n^2}\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) \\ &= \vec{\nabla}(\frac{1}{n^2}) \times (\vec{\nabla} \times \vec{H}) + \vec{\nabla}(\vec{\nabla} \cdot \vec{H}) - \nabla^2 \vec{H} \end{aligned}$$

$$\begin{aligned} \vec{\nabla}(\frac{1}{n^2}) \times \vec{\nabla} \times \vec{H} - \nabla^2 \vec{H} - k_0^2 \vec{H} &= 0 \\ \nabla^2 \vec{H} - n^2 \vec{\nabla}(1/n^2) \times \vec{\nabla} \times \vec{H} + k_0^2 n^2 \vec{H} &= 0 \end{aligned} \tag{3.5}$$

where $\vec{H}$ is a magnetic field function. $\vec{H}(x, y, z) = \vec{h}(x, y)exp(-j\beta z)$ (n is not vary along the z-direction and we can use separation method).

$n = n(x, y)$ is the refractive index and $\beta = k_0 n_{eff}$ is the propagation constant. n_{eff} is the effective index.

$$\vec{h}(x, y) = h_x(x, y)\hat{x} + h_y(x, y)\hat{y} + h_z(x, y)\hat{z} \tag{3.6}$$

here $\hat{x}, \hat{y}$, and $\hat{z}$ are unit vectors in $x-, y-$, and $z-$ directions, ∇^2 can be write as follow.

$$\begin{aligned} \nabla^2 \vec{H} &= \frac{\partial^2 \vec{H}}{\partial x^2} + \frac{\partial^2 \vec{H}}{\partial y^2} + \frac{\partial^2 \vec{H}}{\partial z^2} \\ &= \frac{\partial^2 \vec{H}}{\partial x^2} + \frac{\partial^2 \vec{H}}{\partial y^2} + \vec{h}(x, y)e^{-j\beta z}(-\beta^2) \\ \nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \beta^2 \end{aligned} \tag{3.7}$$

We can reduce Equation 3.5 to two coupled equation (h_x and h_y).

$$\underbrace{\nabla^2\vec{H}}_{a} - \underbrace{n^2\vec{\nabla}(\frac{1}{n^2})}_{b} \times \underbrace{(\vec{\nabla \times H})}_{c} + \underbrace{k_o^2 n^2 \vec{H}}_{d} = 0 \tag{3.8}$$

$$\nabla^2\vec{H} = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \beta^2)(h_x\hat{x} + h_y\hat{y} + h_z\hat{z})e^{-j\beta z} \tag{3.9}$$

$\vec{H}$ has not $z-$ component, because the propagation direction is $z-$ direction.

$$\begin{aligned}
\nabla^2\vec{H} &= (\frac{\partial^2 h_x}{\partial x^2}\hat{x} + \frac{\partial^2 h_x}{\partial y^2}\hat{x} - \beta^2 h_x\hat{x} + \frac{\partial^2 h_y}{\partial x^2}\hat{y} + \frac{\partial^2 h_y}{\partial y^2}\hat{y} - \beta^2 h_y\hat{y})e^{j\beta z} \\
a &= \nabla^2\vec{H} = [\frac{\partial^2 h_x}{\partial x^2}\hat{x} + \frac{\partial^2 h_x}{\partial y^2}\hat{x} + \frac{\partial^2 h_y}{\partial x^2}\hat{y} + \frac{\partial^2 h_y}{\partial y^2}\hat{y}]e^{-j\beta z} - \\
&\qquad [\beta^2 h_x\hat{x} + \beta^2 h_y\hat{y}]e^{-j\beta z}
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
b &= -n^2\vec{\nabla}(\frac{1}{n^2}) \\
&= -n^2(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y})(\frac{1}{n^2}) \\
&= -n^2\frac{\partial}{\partial x}(\frac{1}{n^2})\hat{x} - n^2\frac{\partial}{\partial y}(\frac{1}{n^2})\hat{y} \\
&= -\frac{\frac{\partial}{\partial x}(n^{-2})}{n^{-2}}\hat{x} - \frac{\frac{\partial}{\partial y}(n^{-2})}{n^{-2}}\hat{y} \\
b &= -n^2\vec{\nabla}(\frac{1}{n^2}) \\
&= -\frac{\partial ln(n^{-2})}{\partial x}\hat{x} - \frac{\partial ln(n^{-2})}{\partial y}\hat{y}
\end{aligned} \tag{3.11}$$

$$c = \nabla \times \vec{H} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ h_x e^{-j\beta z} & h_y e^{-j\beta z} & e^{-j\beta z} \end{bmatrix}$$

$$= \hat{x}[\frac{\partial}{\partial y}(e^{-j\beta z}) - \frac{\partial}{\partial z}(h_y e^{-j\beta z})] - \hat{y}[\frac{\partial}{\partial x}(e^{-j\beta z}) - \frac{\partial}{\partial z}(h_x e^{-j\beta z})]$$
$$+\hat{z}[\frac{\partial}{\partial x}(h_y e^{-j\beta z}) - \frac{\partial}{\partial y}(h_x e^{-j\beta z})]$$

$$\vec{\nabla} \times \vec{H} = \hat{x}[h_y e^{-j\beta z} j\beta] - \hat{y}[h_x e^{-j\beta z} j\beta] + \hat{z}[\frac{\partial h_y}{\partial x} e^{-j\beta z} - \frac{\partial h_x}{\partial y} e^{-j\beta z}] \tag{3.12}$$

we will continue multiplying the second and thirty terms in Equation 3.8.

$$b \times c = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{-\partial ln(n^{-2})}{\partial x} & \frac{-\partial ln(n^{-2})}{\partial y} & 0 \\ (j\beta h_y e^{-j\beta z}) & (-j\beta h_x e^{-j\beta z}) & (\frac{\partial h_y}{\partial x} e^{-j\beta z} - \frac{\partial h_x}{\partial y} e^{-j\beta z}) \end{bmatrix}$$

$$\begin{aligned} &= \hat{x}[\frac{\partial ln(n^{-2})}{\partial y}\frac{\partial h_y}{\partial x} e^{-j\beta z} + \frac{\partial ln(n^{-2})}{\partial y}\frac{\partial h_x}{\partial y} e^{-j\beta z} - 0] \\ &-\hat{y}[\frac{\partial ln(n^{-2})}{\partial x}\frac{\partial h_y}{\partial x} e^{-j\beta z} + \frac{\partial ln(n^{-2})}{\partial x}\frac{\partial h_x}{\partial y} e^{-j\beta z} - 0] + \\ &\hat{z}[-\frac{\partial ln(n^{-2})}{\partial x} h_x(-j\beta) e^{-j\beta z} + \frac{\partial ln(n^{-2})}{\partial x} h_y(j\beta) e^{-j\beta z}] \end{aligned} \tag{3.13}$$

As a result, we can write $a + (\vec{b} \times \vec{c}) + d = 0$

$$[\frac{\partial^2 h_x}{\partial x^2}\hat{x} + \frac{\partial^2 h_x}{\partial y^2}\hat{x} + \frac{\partial^2 h_y}{\partial x^2} + \frac{\partial^2 h_y}{\partial y^2}] + [2\frac{\partial lnn}{\partial y}\frac{\partial h_y}{\partial x} - 2\frac{\partial lnn}{\partial y}\frac{\partial h_x}{\partial y}]\hat{x} -$$

$$[2\frac{\partial lnn}{\partial x}\frac{\partial h_y}{\partial x} - 2\frac{\partial lnn}{\partial x}\frac{\partial h_x}{\partial y}]\hat{y} - \underbrace{[-2\frac{\partial lnn}{\partial x} j\beta h_x - 2\frac{\partial lnn}{\partial y} j\beta h_y]}_{imaginary}\hat{z} +$$

$$[k_0^2 n^2 h_x \hat{x} + k_0^2 n^2 h_y \hat{y}] + [-\beta^2 h_x \hat{x} - \beta^2 h_y \hat{y}] = 0 \tag{3.14}$$

In this result, there is a real solution only $\hat{x}$ and $\hat{y}$ directions. Therefore, we can omit $z-$ direction (imaginary part of solution). As a result, we have obtained

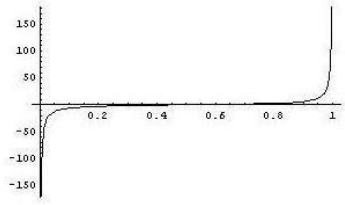

Figure 3.1: The transformation $x - y$ space onto a unit square in u, v space. $\alpha_x = \alpha_y = 2$.

the following equations for h_x and h_y (two-coupled equation) [23].

$$\frac{\partial^2 h_y}{\partial x^2} + \frac{\partial^2 h_y}{\partial y^2} - 2\frac{\partial lnn}{\partial x}[\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}] + k_0^2 n^2 h_y - \beta^2 h_y = 0 \tag{3.15}$$

$$\frac{\partial^2 h_x}{\partial x^2} + \frac{\partial^2 h_x}{\partial y^2} - 2\frac{\partial lnn}{\partial y}[\frac{\partial h_x}{\partial y} - \frac{\partial h_y}{\partial x}] + k_0^2 n^2 h_x - \beta^2 h_x = 0 \tag{3.16}$$

3.2.2 Mapped Galerkin Method In Step Index Fibers

By using the following transformation functions we can map x-y plane onto a unit square in u-v space in Figure 3.1. Thus, we can eliminate the need of enclosing waveguide within a rectangle [20].

$$\begin{aligned} x &= \alpha_x tan[\pi(u - \frac{1}{2})] \\ y &= \alpha_y tan[\pi(v - \frac{1}{2})] \end{aligned} \tag{3.17}$$

α_x and α_y are scaling parameters in the $x-$ and $y-$ directions. Therefore,

the boundary conditions are automatically satisfied for all guided modes. From transformation equation we can obtain transformed coupled equation $x = x(u)$ and $y = y(v)$

$$\begin{aligned}
\frac{\partial h}{\partial x} &= \frac{\partial h}{\partial u}\frac{\partial u}{\partial x} \\
\frac{\partial^2 h}{\partial x^2} &= \frac{\partial}{\partial x}(\frac{\partial h}{\partial u}\frac{\partial u}{\partial x}) \\
&= [\frac{\partial}{\partial x}(\frac{\partial h}{\partial u})]\frac{\partial u}{\partial x} + \frac{\partial h}{\partial u}\frac{\partial^2 u}{\partial x^2} \\
&= [(\frac{\partial}{\partial u}\frac{\partial u}{\partial x})(\frac{\partial h}{\partial u})]\frac{\partial u}{\partial x} + \frac{\partial h}{\partial u}\frac{\partial^2 u}{\partial x^2} \\
&= \frac{\partial u}{\partial x}\frac{\partial^2 h}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial h}{\partial u}\frac{\partial^2 u}{\partial x^2} \\
\frac{\partial^2 h}{\partial x^2} &= (\frac{du}{dx})^2\frac{\partial^2 h}{\partial u^2} + \frac{\partial h}{\partial u}\frac{d^2 u}{dx^2}
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
\frac{\partial h}{\partial y} &= \frac{\partial h}{\partial v}\frac{\partial v}{\partial y} \\
\frac{\partial^2 h}{\partial y^2} &= \frac{\partial}{\partial y}(\frac{\partial h}{\partial v}\frac{\partial v}{\partial y}) \\
&= [\frac{\partial}{\partial y}(\frac{\partial h}{\partial v})]\frac{\partial v}{\partial y} + \frac{\partial h}{\partial v}\frac{\partial^2 v}{\partial y^2} \\
&= [(\frac{\partial}{\partial v}\frac{\partial v}{\partial y})(\frac{\partial h}{\partial v})]\frac{\partial v}{\partial y} + \frac{\partial h}{\partial v}\frac{\partial^2 v}{\partial y^2} \\
&= \frac{\partial v}{\partial y}\frac{\partial^2 h}{\partial v^2}\frac{\partial v}{\partial y} + \frac{\partial h}{\partial v}\frac{\partial^2 v}{\partial y^2} \\
\frac{\partial^2 h}{\partial y^2} &= (\frac{dv}{dy})^2\frac{\partial^2 h}{\partial v^2} + \frac{\partial h}{\partial v}\frac{d^2 v}{dy^2}
\end{aligned} \tag{3.19}$$

The transformed coupled equations [23];

$$\begin{aligned}
&(\frac{du}{dx})^2\frac{\partial^2 h_y}{\partial u^2} + \frac{d^2 u}{dx^2}\frac{\partial h_y}{\partial u} + (\frac{dv}{dy})^2\frac{\partial^2 h_y}{\partial v^2} + \frac{d^2 v}{dy^2}\frac{\partial h_y}{\partial v} - \\
&2\frac{du}{dx}\frac{\partial lnn}{\partial u}[\frac{du}{dx}\frac{\partial h_y}{\partial u} - \frac{dv}{dy}\frac{\partial h_x}{\partial v}] + k_0^2 n^2 h_y - \beta^2 h_y = 0
\end{aligned} \tag{3.20}$$

$$(\frac{du}{dx})^2\frac{\partial^2 h_y}{\partial u^2}+\frac{d^2u}{dx^2}\frac{\partial h_y}{\partial u}+(\frac{dv}{dy})^2\frac{\partial^2 h_y}{\partial v^2}+\frac{d^2v}{dy^2}\frac{\partial h_y}{\partial v}-$$

$$2(\frac{du}{dx})^2\frac{\partial lnn}{\partial u}\frac{\partial h_y}{\partial u}+2\frac{du}{dx}\frac{dv}{dy}\frac{\partial lnn}{\partial u}\frac{\partial h_x}{\partial v}+k_0^2n^2h_y-\beta^2h_y=0 \qquad (3.21)$$

In the same way we can write other coupled equation [23];

$$(\frac{du}{dx})^2\frac{\partial^2 h_x}{\partial u^2}+\frac{d^2u}{dx^2}\frac{\partial h_x}{\partial u}+(\frac{dv}{dy})^2\frac{\partial^2 h_x}{\partial v^2}+\frac{d^2v}{dy^2}\frac{\partial h_x}{\partial v}-$$

$$2(\frac{dv}{dx})^2\frac{\partial lnn}{\partial v}\frac{\partial h_x}{\partial v}+2\frac{du}{dx}\frac{dv}{dy}\frac{\partial lnn}{\partial v}\frac{\partial h_y}{\partial u}+k_0^2n^2h_x-\beta^2h_x=0 \qquad (3.22)$$

where $h_y=h_y(u,v), h_x=h_x(u,v)$ and $n=n(u,v)$. The transverse magnetic field components (similar to the solutions described [22]) are expanded as a complete of orthogonal basis functions [23].

$$\begin{aligned} h_y(u,v) &= \sum_{k=1}^{N_x\times N_y} c_k^y\phi_k(u,v) \\ &= \sum_{p=1}^{N_x}\sum_{q=1}^{N_y} c_{pq}^y\varphi_p(u)\varphi_q(v) \end{aligned} \qquad (3.23)$$

$$\begin{aligned} h_x(u,v) &= \sum_{k=1}^{N_x\times N_y} c_k^x\phi_k(u,v) \\ &= \sum_{p=1}^{N_x}\sum_{q=1}^{N_y} c_{pq}^x\varphi_p(u)\varphi_q(v) \end{aligned} \qquad (3.24)$$

Substituting Equation 3.27 into Equation 3.25, multiplying by $\phi_k^{'}(u,v)$ and integrating over the unit square, we have the following system of coupled linear equations [23];

$$\sum_{k=1}^{N_x\times N_y} c_k^y[\int_0^1\int_0^1(\frac{du}{dx})^2\frac{\partial^2}{\partial u^2}\phi_k\phi_{k'}dudv+\int_0^1\int_0^1\frac{d^2u}{dx^2}\frac{\partial}{\partial u}\phi_k\phi_{k'}dudv+$$

$$\int_0^1\int_0^1 (\frac{dv}{dy})^2 \frac{\partial^2}{\partial v^2}\phi_k\phi_{k'}dudv + \int_0^1\int_0^1 \frac{d^2v}{dy^2}\frac{\partial}{\partial v}\phi_k\phi_{k'}dudv +$$

$$\int_0^1\int_0^1 k_0^2 n^2\phi_k\phi_{k'}dudv - 2\int_0^1\int_0^1 (\frac{du}{dx})^2 \frac{\partial lnn}{\partial u}\frac{\partial}{\partial u}\phi_k\phi_{k'}dudv] +$$

$$\sum_{k=1}^{N_x\times N_y} c_k^x 2\int_0^1\int_0^1 \frac{du}{dx}\frac{dv}{dy}\frac{\partial lnn}{\partial u}\frac{\partial}{\partial v}\phi_k\phi_{k'}dudv \;=\; \beta^2 \sum_{k=1}^{N_x\times N_y} c_k^y \int_0^1\int_0^1 \phi_k\phi_{k'}dudv$$

We can write for the first for the four terms similar to the solutions described in [23];

$$I = \sum_{i=1}^{4} I_i \tag{3.25}$$

and we can write fifth term as $P_{k,k'}$ [23]

$$P_{k,k'} = \int_0^1\int_0^1 k_0^2\varphi_p(u)\varphi_q(v)n^2(u,v)\varphi_{p'}(u)\varphi_{q'}dudv \tag{3.26}$$

$$I_5 = -2\int_0^1\int_0^1 (\frac{du}{dx})^2\frac{\partial lnn}{\partial u}\phi_{k'}dudv \tag{3.27}$$

$$I_6 = 2\int_0^1\int_0^1 \frac{du}{dx}\frac{dv}{dy}\frac{\partial lnn}{\partial u}\frac{\partial\phi_k}{\partial v}dudv \tag{3.28}$$

$$A_{k',k}^{yy} = \sum_{i=1}^{4} I_i + P_{k',k} + I_5 \tag{3.29}$$

$$A_{k',k}^{yx} = I_6 \tag{3.30}$$

By using the last two equations [23]

$$\sum_{k=1}^{N_x\times N_y} [A_{k',k}^{yy}c_k^y + A_{k',k}^{yx}c_k^x] = \beta^2 c_{k'}^y \tag{3.31}$$

Similar as above

$$\sum_{k=1}^{N_x \times N_y} [A^{xx}_{k',k} c^x_k + A^{xy}_{k',k} c^y_k] = \beta^2 c^x_{k'} \tag{3.32}$$

$$A^{xx}_{k',k} = \sum_{i=1}^{4} I_i + P_{k',k} + I_8 \tag{3.33}$$

$$A^{xy}_{k',k} = I_7 \tag{3.34}$$

$$I_7 = -2 \int_0^1 \int_0^1 (\frac{dv}{dy})^2 \frac{\partial lnn}{\partial v} \frac{\partial \phi_k}{\partial v} \phi_{k'} du dv \tag{3.35}$$

$$I_8 = 2 \int_0^1 \int_0^1 \frac{du}{dx} \frac{dv}{dy} \frac{\partial lnn}{\partial v} \frac{\partial \phi_k}{\partial u} \phi_{k'} du dv \tag{3.36}$$

For example;

$$I_5 = -2 \int_0^1 \int_0^1 (\frac{du}{dx})^2 \frac{\partial lnn}{\partial u} \frac{\phi_k}{\partial u} \phi_{k'} du dv$$

$$x = \alpha_x tan[\pi(u - \frac{1}{2})]$$

$$\frac{x}{\alpha_x} = tan[\pi(u - \frac{1}{2})]$$

$$\pi(u - \frac{1}{2}) = arctan(\frac{x}{\alpha_x})$$

$$\pi u = arctan(\frac{x}{\alpha_x}) + \frac{\pi}{2}$$

$$u = \frac{1}{\pi} arctan(\frac{x}{\alpha_x}) + \frac{1}{2} \tag{3.37}$$

$$\frac{du}{dx} = \frac{1}{\pi \alpha_x} (\frac{1}{1 + \frac{x^2}{\alpha_x^2}}) \tag{3.38}$$

$$\frac{d^2 u}{dx^2} = -\frac{2x}{\pi} \frac{1}{\alpha_x^3} (\frac{1}{1 + \frac{x^2}{\alpha_x^2}})^2 \tag{3.39}$$

Similarly we can obtain another component transformation equation as in the following

$$y = \alpha_y tan[\pi(v - \frac{1}{2})]$$

$$v = \frac{1}{\pi} arctan(\frac{y}{\alpha_x}) + \frac{1}{2} \tag{3.40}$$

$$\frac{dv}{dy} = \frac{1}{\pi\alpha_y}(\frac{1}{1+\frac{y^2}{\alpha_y^2}}) \tag{3.41}$$

$$\frac{d^2v}{dy^2} = -\frac{2y}{\pi}\frac{1}{\alpha_y^3}(\frac{1}{1+\frac{y^2}{\alpha_y^2}})^2 \tag{3.42}$$

$$\begin{aligned}\frac{\partial\phi_k}{\partial u} &= \frac{\partial}{\partial u}(\varphi_p(u)\varphi_q(v)) \\ &= \frac{\partial}{\partial u}(\sqrt{2}sin(p\pi u)\sqrt{2}sin(q\pi v)) \\ &= 2[cos(p\pi u)(p\pi)sin(q\pi v) + cos(q\pi v)(q\pi)sin(p\pi u)]\end{aligned} \tag{3.43}$$

$$\begin{aligned}\phi_{k'} &= \varphi_{p'}(u)\varphi_{q'}(v) \\ \phi_{k'} &= \sqrt{2}sin(p'\pi u)\sqrt{2}sin(q'\pi v) \\ &= 2sin(p'\pi u)sin(q'\pi v)\end{aligned} \tag{3.44}$$

by using these equations we can write

$$\begin{aligned}I_5 &= -2\int_0^1\int_0^1 \frac{1}{\alpha_{x^2}\pi^2} sin(\pi u)^4 \frac{\partial lnn}{\partial u} 2[cos(p\pi u)(p\pi)sin(q\pi v) + \\ &= cos(q\pi v)(q\pi)sin(p\pi u)]2sin(p'\pi u)sin(q'\pi v)dudv\end{aligned} \tag{3.45}$$

In the same way we can obtain other I_i .

At the end of the mathematical formulation with mapped Galerkin method Equation 3.31 and Equation 3.32 can be obtained in a compact form matrix eigenvalue equation [23]

$$\begin{bmatrix} A^{yy} & A^{yx} \\ A^{xy} & A^{xx} \end{bmatrix} \begin{bmatrix} C^y \\ C^x \end{bmatrix} = \beta^2 \begin{bmatrix} C^y \\ C^x \end{bmatrix} \tag{3.46}$$

Table 3.1: Eigenvalues for matrix eigenvalues in step index fibers. $N_x \times N_y = 16$.

Number	β^2-values
1)	150.94
2)	150.08
3)	147.48
4)	146.62
5)	143.38
6)	144.24
7)	149.56
8)	150.43
9)	142.57
10)	144.12
11)	145.81
12)	147.35
13)	150.81
14)	150.3
15)	148.75
16)	149.27

Here C^y and C^x are the Fourier expansion coefficients.

$$\begin{aligned} C^y &= [c_1^y, c_2^y, c_3^y, ..., c_{N_x \times N_y}^y]^T \\ C^x &= [c_1^x, c_2^x, c_3^x, ..., c_{N_x \times N_y}^x]^T \end{aligned} \tag{3.47}$$

A MATLAB program was constituted to solve this matrix eigenvalue equation. It is presented in Appendix. By using this program we obtained the eigenvalues in Table 3.1.

In the resulting matrix, Fourier expansion coefficients are the eigenvectors. By using the MATLAB program we computed eigenvectors. We have already known the guided modes of an optical waveguide corresponding to the real and positive eigenvalues and their eigenvectors. $h_y(u, v)$ and $h_x(u, v)$ were expanded

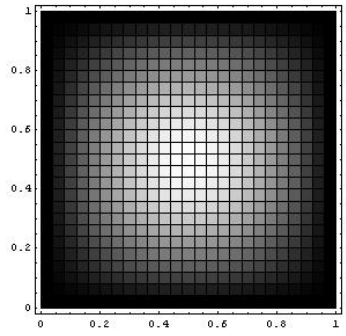

Figure 3.2: Field patterns of the transverse $h_y(u,v)$ component of the HE_{11} mode in circular step-index fiber. The scaling parameter $\alpha_u = \alpha_v = 4$.

as a complete of orthogonal basis functions in Equation 3.23 and Equation 3.24. Now, we want to obtain these modal fields in $u-v$ plane. For this reason we will use eigenvectors (results of the eigenvalue matrix). Here we obtained $h_y(u,v)$ as an example the fundamental mode HE_{11} mode for step-index fibers.

$$\begin{aligned} h_y(u,v) &= c_{11}^{y}\varphi_1(u)\varphi_1(v) \\ &= 1.5709 * sin(\pi u)sin(\pi v) \end{aligned} \tag{3.48}$$

If we want to see the solution as a plot in $u-v$ space in Figure 3.2.

Figure 3.3 was obtained by Xiao, Zhang and Sun [23]. They studied with step index fibers. In their study, the number of basis functions is $N_x = N_y = 25$. In our study, the number of basis functions is $N_x = N_y = 4$. The Figure 3.2 is similar the Figure 3.3. If the number of basis functions increases, the similarity between two graphics is better than small number of basis functions. However,

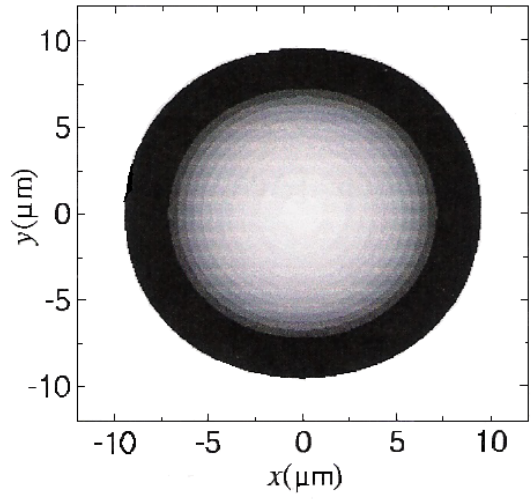

Figure 3.3: Field patterns of the transverse H-field of the HE_{11} mode in the step index fiber. h_y component [23].

the larger number of basis functions requires the larger computational time.

Now, we want to calculate this modal field for $x-y$ plane. We should use again transformation equations. From Equation 3.17 we have written equations

$$\begin{aligned} u &= \frac{1}{\pi}\arctan(\frac{x}{\alpha_x}) + 1/2 \\ v &= \frac{1}{\pi}\arctan(\frac{y}{\alpha_y}) + 1/2 \end{aligned} \tag{3.49}$$

By using these transformations, we can plot the Figure 3.4 and we can calculate $h_y(x,y)$.

$$\begin{aligned} h_y(x,y) &= 1.5709\sin\{\pi[\frac{1}{\pi}\arctan(\frac{x}{\alpha_x}) + 1/2]\}\sin\{\pi[\frac{1}{\pi}\arctan(\frac{y}{\alpha_y}) + 1/2]\} \\ &= 1.5709\sin[\arctan(\frac{x}{\alpha_x}) + \frac{\pi}{2}]\sin[\arctan(\frac{y}{\alpha_y}) + \frac{\pi}{2}] \end{aligned}$$

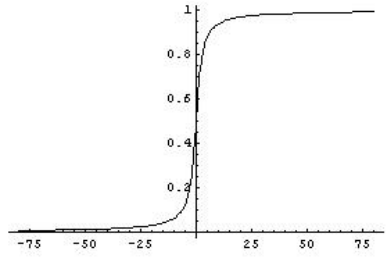

Figure 3.4: The transformation $u - v$ space onto a unit square in x, y space. $\alpha_u = \alpha_v = 2$.

$$= \quad 1.5709(\frac{1}{\sqrt{1+\frac{x^2}{\alpha_x^2}}}\frac{1}{\sqrt{1+\frac{y^2}{\alpha_y^2}}}) \tag{3.50}$$

The last equation is a solution for Equation 3.15. Similarly we can obtain $h_x(u, v)$ and $h_x(x, y)$. Examples were given in the following graphics for the HE_{11} mode in the step-index circular Figure 3.5 and rectangular Figure 3.6 fibers

Figure 3.7 and Figure 3.6 are similar. In Figure 3.7, the number of basis functions were used as $N_x = N_y = 25$ [23]. In Figure 3.6, the number of basis functions were used as $N_x = N_y = 3$. In the core region, both of them have a rectangular profile.

We can use Equation 3.50 to solve β in Equation 3.15 and Equation 3.16 for three layer rectangular step-index fiber. Here n is constant and we can neglect

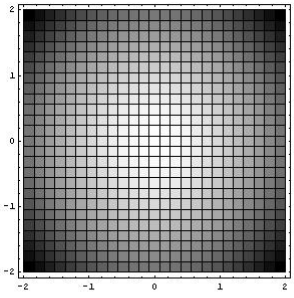

Figure 3.5: Field patterns of the transverse $h_y(x, y)$ component of the HE_{11} mode in circular step-index fiber. The scaling parameter $\alpha_x = \alpha_y = 4$.

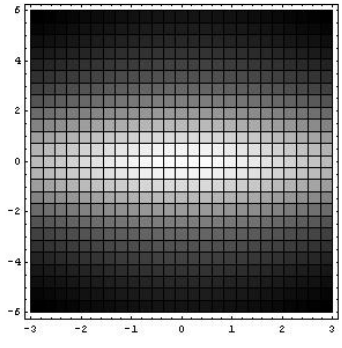

Figure 3.6: Field patterns of the transverse h_y component of the HE_{11} mode in rectangular step-index fiber. The scaling parameter $\alpha_x = 4$ and $\alpha_y = 2$ and $y/x = 2$.

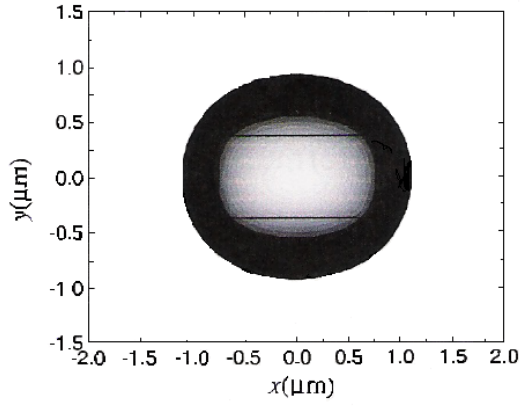

Figure 3.7: Field patterns of the transverse H- field for the fundamental mode of a buried rectangular waveguide. h_y component [23].

the third term. We have already mentioned in Chapter 2, the rectangular form step-index fiber. We will use the assumptions in Equation 2.11 and Equation 2.12. As a result, the Equation 3.15 and Equation 3.16 are written as follows.

$$\frac{\partial^2 h_y}{\partial x^2} + (k^2 n^2 - \beta^2) h_y = 0 \tag{3.51}$$

$$\frac{\partial^2 h_x}{\partial x^2} + (k^2 n^2 - \beta^2) h_x = 0 \tag{3.52}$$

In $u - v$ plane β values were calculated from matrix eigenvalue equations. Now we calculate β values the in $x - y$ plane substituting Equation 3.50 into Equation 3.51. It was solved by using MATHEMATICA. $n_1 = 1.45$ and $\lambda = 1.15\mu m$. β has two values:

$$\beta = -2.5217\pi$$

$$\beta = 2.5217\pi \tag{3.53}$$

By using Maxwell's equations and $h_y(x,y)$ we will obtain $e_x(x,y)$ and $e_z(x,y)$. (In Chapter 2 has already obtained these equations). Electric field components;

$$\begin{aligned} e_x &= \frac{\beta}{w\varepsilon_r\varepsilon_0} h_y \\ e_x &= \frac{12.44}{\omega\varepsilon_r\varepsilon_0}\left(\frac{1}{\sqrt{1+\frac{x^2}{\alpha_x^2}}}\right)\left(\frac{1}{\sqrt{1+\frac{y^2}{\alpha_y^2}}}\right) \end{aligned} \tag{3.54}$$

Here $\varepsilon_r = n_1^2$ is depend on medium, $n_1 = 1.45$ and $\beta = 2.5217\pi$.

$$\begin{aligned} e_z &= \frac{j}{w\varepsilon_r\varepsilon_0}\frac{\partial h_y}{\partial x} \\ e_z &= \frac{-1.5709j}{\omega\varepsilon_r\varepsilon_0}\left(\frac{x}{(\sqrt{1+\frac{x^2}{\alpha_x^2}})^{3/2}}\right)\left(\frac{1}{\sqrt{1+\frac{y^2}{\alpha_y^2}}}\right) \end{aligned} \tag{3.55}$$

The other field components of the TM modes are as follows;

$$E_y = 0 \tag{3.56}$$

$$H_x = 0 \tag{3.57}$$

$$H_z = 0 \tag{3.58}$$

3.3 The Mapped Galerkin's Method for Graded Index Fibers

The last section gives solutions to a step index fibers. We constituted a MATLAB program to solve wave equation in $u - v$ space. In this program there are some elements which contain refractive index, its logarithmic relations, and their derivations. These elements are zero value for step-index fibers. Now, we seek

Table 3.2: Eigenvalues for matrix eigenvalues in graded index fibers. $N_x \times N_y = 9$.

Number	β^2-values
1)	19.6737+26.5655j
2)	19.6737-26.5655j
3)	22.6911+15.5882j
4)	22.6911-15.5882j
5)	25.1549+6.8621j
6)	25.1549-6.8621j
7)	28.1707
8)	25.3320
9)	26.9108

the solutions by the same method for refractive index with functions of the $x-y$ or $u-v$ space. Firstly, we give a definition of function for the refractive index in $u-v$ space.

$$n(u,v) = n_0 e^{-sin(u\frac{\pi}{2}) \times sin(v\frac{\pi}{2})} \tag{3.59}$$

for calculation was taken $n_0 = 1.45$ and $\lambda = 1.15\mu m$. The resulting matrix equation is the form of Equation 3.46. The eigenvalues were calculated by using the MATLAB program. The results of calculations were given in Table 3.2. The guided modes of an optical waveguide correspond to the real, positive eigenvalues and their related eigenvectors. By using eigenvectors of the first order modes we can obtain a solution $h_y(u-v)$ in $u-v$ space.

$$\begin{aligned} h_y(u,v) &= c_{11}^y \varphi_1(u)\varphi_1(v) \\ &= 19.6737 * sin(\pi u) sin(\pi v) \end{aligned} \tag{3.60}$$

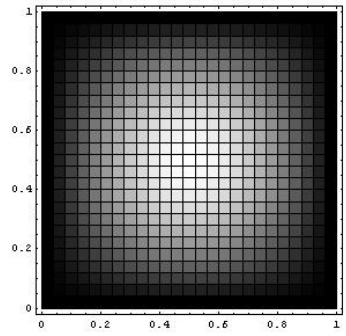

Figure 3.8: Field patterns of the transverse $h_y(u-v)$ component of the HE_{11} mode in circular graded-index fiber. The scaling parameter $\alpha_u = \alpha_v = 4$.

and by using this equation we can plot field in $u-v$ space in Figure 3.8 by using transformation equations we can calculate solution vector in $x-y$ space. And by using the transformation equations we have written $h_y(u-v)$ as a function of $h_y(x-y)$.

$$\begin{aligned} h_y(x,y) &= 19.6737 \sin\{\pi[\frac{1}{\pi}\arctan(\frac{x}{\alpha_x}) + 1/2]\} \sin\{\pi[\frac{1}{\pi}\arctan(\frac{y}{\alpha_y}) + 1/2]\} \\ &= 19.6737 \sin[\arctan(\frac{x}{\alpha_x}) + \frac{\pi}{2}] \sin[\arctan(\frac{y}{\alpha_y}) + \frac{\pi}{2}] \\ &= 19.6737(\frac{1}{\sqrt{1+\frac{x^2}{\alpha_x^2}}} \frac{1}{\sqrt{1+\frac{y^2}{\alpha_y^2}}}) \end{aligned} \tag{3.61}$$

we can plot in $x-y$ space for $h_y(x,y)$ in Figure 3.9.

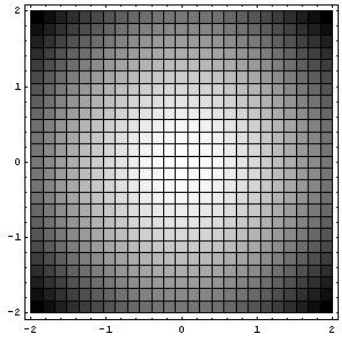

Figure 3.9: Field patterns of the transverse $h_y(x-y)$ component of the HE_{11} mode in circular graded-index fiber. The scaling parameter $\alpha_x = \alpha_y = 4$.

CHAPTER 4

CONCLUSIONS AND REMARKS

In this study, firstly we start with the source-free Maxwell's equations to obtain the vector wave equation based on $H-$ field.

In the next step, this equation was reduced to two coupled equations for $h_x(x, y)$ and $h_y(x, y)$. Then the rectangle waveguide structures were mapped onto a unit square in $u - v$ plane(Mapped Galerkin's Method). For this reason, the transformation functions were used.

For the result equations the solutions are expanded as a complete of orthogonal basis functions. After some algebra, we have system of coupled linear equations. These linear equations contain very long and complex elements. The system was written as matrix elements.

For the matrix elements, a compact form was constituted. This matrix was an eigenvalue equation form. It contained β^2 (eigenvalues) and $C^{y,x}$ (Fourier expansion coefficient).

A MATLAB computer program was constituted to obtain eigenvalues and their related eigenvectors. During the operation of this program we saw that

the coupling terms were zero. Therefore, this equation was reduced to the quasi-vector matrix eigenvalue equation. In this program, the number of basis functions $N_x \times N_y$ should be larger but the computational time was too much long. Therefore, we have studied smaller set.

By using the eigenvalue matrix results (eigenvectors) in $u - v$ space we have calculated $h_y(u, v)$ and $h_x(u, v)$. Then, we mapped the modal fields back to $x - y$ space by using the transformation equations. With the help of the $h_y(x, y)$ we have calculated an eigenvalue for coupled equation $h_y(x, y)$. After that, we have written other fields components.

By using these solution equations, we have obtained some density plots. These plots are fit step-index fibers [23]. Plots give better results for the larger the number of basis functions.

This method has been applied for step-index fibers. The result matrix size is smaller then the other methods. The solution permits us to calculate propagation constants in $x - y$ space without any numerical technique and other matrix equation solution. This method is very useful for quasi-and full- vector analysis of the optical waveguide [23].

In the last section, this method have applied to solution of graded index fibers. After the definition of refractive index in $u - v$ space, a MATLAB computer program was constituted. We have obtained to solutions of modal fields in $u - v$ and $x - y$ spaces. We can say that, the Mapped Galerkin Method can be applied

graded-index fibers with an appropriate function of refractive index in the interval of $[0, 1]$ in $u - v$ space.

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APPENDIX

```
clear
syms u v
n=1.5; % reflection index
alphax =2;%waveguide
alphay =1;%      scale
x=alphax*tan(pi *(u-0.5));
y=alphay*tan(pi *(v-0.5));
Nx = 9;% resolution
Ny = 9;%       scale
klimit=Nx.*Ny; kklimit=Nx.*Ny; counter = 0;
for k = 1:1:klimit
    for kk = 1:1:kklimit
counter = counter+1 p = floor((k-1)/Ny)+1;
q = mod((k-1),Ny)+1;
pp= floor((kk-1)/Ny)+1;
qq = mod((kk-1),Ny)+1; Phi_p= 2^0.5*
sin(pi*p*u); Phi_q= 2^0.5* sin(pi*q*v);
Phi_k= Phi_p*Phi_q;
Phi_pp= 2^0.5* sin(pi*pp*u);
Phi_qq= 2^0.5* sin(pi*qq*v); Phi_kk=
Phi_pp*Phi_qq;
I1_uv= (1/(1+x.^2/alphax.^2)).^2/(alphax*pi).^2
* diff( diff( Phi_k,u) ,u) * Phi_kk;
I1_v = int (I1_uv, u ,0,1);
I1 = int (I1_v, v ,0,1);
I2_uv=-2*x*(1/(1+x.^2/alphax.^2)).^2/(alphax.^3*pi)
*diff( Phi_k,u) * Phi_kk;
I2_v=int (I2_uv, u ,0,1);
I2=int (I2_v, v ,0,1);
I3_uv= (1/(1+y.^2/alphay.^2)).^2/(alphay*pi).^2
* diff( diff( Phi_k,v) ,v) * Phi_kk;
I3_v = int (I3_uv, u ,0,1);
I3=int (I3_v, v ,0,1);
I4_uv=-2*y*(1/(1+y.^2/alphay.^2)).^2
/(alphay.^3*pi) *diff(Phi_k,v)
* Phi_kk; I4_v = int (I4_uv, u ,0,1);
I4=int (I4_v, v ,0,1);
I5_uv=2*(1/(1+x.^2/alphax.^2)).^2
./(alphax*pi).^2
```

```
*diff(Phi_k.*diff(log(n),u),u)*Phi_kk;
I5_v=int (I5_uv, u ,0,1);
I5 = int (I5_v, v ,0,1);
I6_uv=-4*x*(1/(1+x.^2/alphax.^2)).^2/(alphax.^3*pi)
*Phi_k*diff(log(n),u)* Phi_kk;
I6_v=int (I6_uv, u ,0,1); I6 = int
(I6_v, v ,0,1); I7_uv=2*(1/(1+x.^2/alphax.^2))
/alphax*pi
*(1/(1+y.^2/alphay.^2))/alphay*pi
*diff(Phi_k*diff(log(n),v),u)*Phi_kk;
I7_v=int (I7_uv, u ,0,1);
I7 = int (I7_v, v ,0,1);
I8_uv=2*1/(1+x.^2/alphax.^2)/alphax*pi
*(1/(1+y.^2/alphay.^2))/alphay*pi
*diff(Phi_k*diff(log(n),u),v)*Phi_kk;
I8_v=int (I8_uv, u ,0,1);
I8 = int (I8_v, v ,0,1);
I9_uv=2*((1/(1+y.^2/alphay.^2))/alphay*pi).^2
*diff(Phi_k*diff(log(n),v),u)*Phi_kk;
I9_v=int (I9_uv, u ,0,1); I9
= int (I9_v, v ,0,1);
I10_uv=-4*y*(1/(1+y.^2/alphay.^2)).^2/alphay.^3
*pi*Phi_k*diff(log(n),v)*Phi_kk;
I10_v=int (I10_uv, u ,0,1); I10=
int (I10_v, v ,0,1);

P_uv=Phi_k*Phi_kk*n.^2;
P_v=int (P_uv, u ,0,1);
P= int (P_v,
v,0,1);
IT=I1+I2+I3+I4;
Ayy(k,kk)=IT+P+I5;
Ayx(k,kk)=I6;
Axy(k,kk)=I7;
Axx(k,kk)=IT+P+I8;
    end
end betax = double((eig(Axx)).^0.5);
betay=double((eig(Ayy)).^0.5);
```

VITA

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