SOME PROPERTIES AND CONSERVED QUANTITIES OF THE SHORT PULSE EQUATION

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ABSTRACT

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Short Pulse equation derived by Schafer and Wayne is a nonlinear partial differential equation that describes ultra short laser propagation in a dispersive optical medium such as optical fibers. Some properties of this equation e.g. traveling wave solution and its soliton structure and some of its conserved quantities were investigated. Conserved quantities were obtained by mass conservation law, lax pair method and transformation between Sine-Gordon and short pulse equation. As a result, loop soliton characteristic and six conserved quantities were found.

Keywords: short pulse, nonlinear optics, soliton, conserved quantities, Lax pair

ÖZ

KISA ATMA DENKLEMİNİN BAZI ÖZELLİKLERİ VE KORUNAN NİCELİKLERİ

Erbaş, Kadir Can
Yüksek Lisans, Fizik Bölümü
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Schafer ve Wayne tarafından bulunan kısa atma denklemi dağıtıcı optik ortamlardaki ultra kısa lazer yayılmasını açıklayan lineer olmayan bir diferansiyel denklemdir. Bu denklemin, ilerleyen dalga çözümü yoluyla soliton yapısı gibi özellikleri ve bazı korunan değerleri bulundu. Korunan değerler bulunurken; kütle korunumu denklemi, Lax çiftleri ve Sine-Gordon ile kısa atma denklemi arasındaki dönüşüm kuralları kullanıldı.

Anahtar Kelimeler: kısa atma, lineer olmayan optik, soliton, korunan değerler, Lax çiftleri

To my wife

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I would like to thank Prof. Dr. Ayşe Karasu for her help and patience.

I wish to thank my wife for her support.

I also thank to my family for their great help.

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CHAPTER 1

INTRODUCTION

Fiber optic communication provides transmitting huge amount of data by sending light pulses through an optical fiber. Compared with electrical system based on copper wires, fiber optic communication has some advantages. First, the data transmission capacity is huge. For example, even if hundreds of thousands telephone channels are used with full capacity, a single silica fiber can replace them. Particularly, low bandwidth in communication is a very serious problem of our time. Second fiber optic transmission loss is small therefore there is no need to amplify the signal in a fiber for many tens of kilometers. Third it is very safe and economical. Because of light transferring rather than electrical current, there is no risk of spark and its cost per transported bit is very low [1].

Although it has serious advantages, there are some obstacles in using fiber technology. The transmission capacity of a fiber depends on the fiber length. Dispersion reduces the achievable transmission rate.

One of the important problems of nonlinear pulse propagation is the dispersive pulse broadening effect. It prevents very high data transmission via optical fibers. As a solution to this problem, Hasegawa and Tappert [2] suggested that the dispersive spreading can be balanced by a weak nonlinearity of the index of refraction in the optical fiber. By means of the growth in optical fiber and laser technology, necessary conditions for soliton pulse propagation were reached. Finally, in 1980, the first experimental observation of optical solitons was made.

Recently, Schafer and Wayne [3] proposed an alternative model to approximation of a very short pulse in nonlinear media. Chung, Jones, Schafer and Wayne [3] proved numerically that as the pulse length shortens, the nonlinear Schrödinger equation approximation becomes less and less accurate while the short pulse equation provides a better and better approximation to the solution of Maxwell's equation. Nonlinear Schrödinger Equation (NLSE) assumes that the pulses spectrum is localized around the carrier frequency while short pulse equation assumes the pulse is as short as few cycles of the central frequency [4].

The short pulse equation (SPE) examined in this study was derived by Schafer and Wayne [3] to describe the ultra short pulse propagation in nonlinear media such as optical fibers that is,

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx} \tag{1.1}$$

where subscript denotes partial derivative.

Sakovich and Sakovich studied short pulse equation from the standpoint of its integrability [4]. They showed that the short pulse equation is integrable and possesses a Lax pair. The Lax pair is the most fundamental object in the theory of integrable systems. It is the starting point for using the inverse scattering method on a given integrable equation that is very special because it can be solved analytically.

They also proved that through a chain of transformations the SPE can be related to the sine-Gordon equation. Using this property Sakovich and Sakovich [5] obtained the exact loop and pulse solutions of the SPE from the well known kink and breather solutions of the sine-Gordon equation.

Later Brunelli [6], [7] studied the Hamiltonian structures and obtained the conserved quantities of SPE. Multi soliton solutions and periodic solutions of SPE [8] are found by Parkes very recently.

In this work we studied some properties of SPE and derived the conserved quantities by using mass conservation law, its Lax operators and transformation from sine-Gordon Equation.

CHAPTER 2

NONLINEAR OPTICS

A dielectric medium can be polarized if an electrical field is executed. The polar molecules behave like electric dipoles and possess electrical dipole moment. If there is no external electrical field, these dipole moments directions are random.

The sum of all dipole moment vectors in an infinitesimal volume element is called as polarization vector $\vec{P}$. It gives a quantity of polarization of dielectrics and it is affected by external electric field. Polarization depends on both external field and optical properties of the medium. Therefore it can be seen as the total response of the medium to the external electrical field.

Like the external field, light can also polarize dielectrics. The medium where the light propagates can also be polarized by the electric field component of electromagnetic wave. However, this electric field varies in time. The time lag between electric field E and the response of the medium due to the electrical inertia must be taken into account [9].

Incoherent optics, e.g. laser radiation, assumed that optical characteristics of the medium do not depend on the intensity of light propagating in it. That is, the frequency and polarization of light determines the optical properties such as refractive index. The reason for this assumption is that non-laser radiation does not carry enough electrical field components with respect to inter-atomic and atomic field strength of the medium. After the advent of laser, a light wave becomes an intensive enough to affect atomic fields and the optical properties of the medium [9].

When an optical material is induced by the light with high intensity, nonlinear changes in refractive index of the medium are seen. This phenomenon is known as the optical Kerr effect. This effect leads to the nonlinearity for the pulse propagation in the medium. After the discovery of laser, it was observed and the new type of optics was called as nonlinear optics, while pre-laser optics was called as linear optics [19].

Nonlinear transportation by using soliton pulses is a kind of method which is used for solving the limiting effects of chromatic dispersion. Soliton is a solution of NLSE describing the non-dispersive transmission of an optical pulse in a dispersive medium.

The nonlinear optics was originated in the early 1960's. Discovery of laser offered more amplified light beam with high directionality. It is also more monochromatic, bright and has more degenerate photon. The interaction of laser with nonlinear media leads to some kind of new effects which is studied by nonlinear optics. Classically, in linear medium, polarization vector $\vec{P}$ is considered to be linearly proportional to the electric field E.

$$\vec{P} = \varepsilon_0 \chi . \vec{E} \tag{2.1}$$

χ is the susceptibility of the medium. So the Maxwell equations contained only the first power of electrical field strength $\vec{E}$. It means that no coherent radiation at a new frequency will be generated when there are some electromagnetic waves with different frequencies. However, in some types of media, it was seen that this assumption was violated.

The term nonlinear comes from the dependence of electrical susceptibility to electric field strength of a light wave. The susceptibility χ is a nonlinear function of electric field strength E. Consequently the equation for polarization becomes:

$$\vec{P} = \varepsilon_0 \left[\chi^{(1)} \cdot \vec{E} + \chi^{(2)} : \vec{E}\vec{E} + \chi^{(3)} \vdots \vec{E}\vec{E}\vec{E} .... \right] \tag{2.2}$$

If the electric field strength is sufficiently small, this equation can be approximated as;

$$\vec{P} = \varepsilon_0 \chi . \vec{E} \tag{2.3}$$

as in linear optics.

The susceptibilities (coefficients) depend on the material and they are tensors in general. When it is substituted into Maxwell equations, nonlinear differential equations were found. If the light intensity is low, the linear approximation with a first coefficient gives the correct results approximately. This is a topic of linear optics but the intensity of light is high, like a short pulse laser, nonlinear higher order coefficients must be included as in nonlinear optics. Some observation confirmed the nonlinear optics theories. These are the discovery of frequency mixing effects, optical second and third harmonic generation, sum frequency generation, optical difference frequency generation and optical rectification [10].

If there are no free charges and currents, Maxwell equations in a nonmagnetic dielectric material become:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \qquad \vec{\nabla} \bullet \vec{H} = 0 \qquad \vec{\nabla} \bullet \vec{D} = 0 \tag{2.4}$$

$$\vec{H} = \frac{\vec{B}}{\mu_0} \text{ and } \vec{D} = \varepsilon_0 . E + \vec{P} . \tag{2.5}$$

$\vec{P}$ is the polarization vector of the medium and $\mu_0 \varepsilon_0 = \frac{1}{c^2}$

Therefore the second equation can be written as:

$$\vec{\nabla} \times \vec{B} = \mu_0 . \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \frac{\partial \vec{P}}{\partial t} \tag{2.6}$$

When combined with the first equation after differentiating with respect to time, we obtain;

$$-\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \mu_0 . \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \vec{P}}{\partial t^2} \tag{2.7}$$

$$\nabla^2\vec{E} - \vec{\nabla}(\vec{\nabla}\bullet\vec{E}) = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} + \mu_0\frac{\partial^2\vec{P}}{\partial t^2} \tag{2.8}$$

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \mu_0\frac{\partial^2\vec{P}}{\partial t^2} \tag{2.9}$$

If $\nabla\bullet\vec{D} = 0$, then $\vec{\nabla}\bullet\left(\varepsilon_0\vec{E} + \vec{P}\right) = 0$. Since there are no free charges and currents, $\vec{\nabla}\bullet\vec{E} = -\vec{\nabla}\bullet\vec{P} = 0$. In conclusion, equation (2.9) is obtained. For one dimensional case, if a new parameter is defined as $p = \mu_0.P_x$, the result is:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E = \frac{\partial^2 p}{\partial t^2} \tag{2.10}$$

This final equation is the starting point of Schafer and Wayne for the short pulse equation [3]. When the susceptibility of the material is affected by electric field strength, that is, electric field is as high as to change electrical properties of the medium; susceptibility becomes a nonlinear function of the electrical field. This nonlinearity makes the equation difficult to solve so some approximations are necessary. After imposing nonlinear polarization p in (2.10), they obtained (2.11) with some constants c_1 and c_2.

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c_1^2}\frac{\partial^2 E}{\partial t^2} + \frac{1}{c_2^2}E + \chi^{(3)}\frac{\partial^2}{\partial t^2}\left(E^3\right) \tag{2.11}$$

By defining a new parameter ε and function A_0 and A_1, a multiple scale ansatz was made by Schafer and Wayne in the form of 2.12 and 13 [3].

$$E(x,t) = \varepsilon A_0(\phi, x_1, x_2, ...) + \varepsilon^2 A_1(\phi, x_1, x_2, ...) + ... \tag{2.12}$$

$$\phi = \frac{t - x}{\varepsilon}, \qquad x_n = \varepsilon^n x \tag{2.13}$$

After equating equal powers of ε, equation 2.14 was obtained by the term with the order of ε^1. In (2.14), $\chi^{(3)}$ is the third order susceptibility and c_2=1.59 μm.

$$2\partial_{x1}\partial_\phi A_0 + \frac{1}{c_2^2}A_0 + \chi^{(3)}\partial_\phi^2 A_0^3 = 0 \tag{2.14}$$

Equation (2.14) is the short pulse equation of Schafer and Wayne, and it can be transformed by Robelo transformation with the help of (2.15)-(2.16).

$$A_0 = \frac{1}{\sqrt{\varepsilon\alpha}}u, \qquad \phi = x, \qquad x_1 = \frac{t}{\varepsilon\theta} \tag{2.15}$$

$$\chi^{(3)} = -\varepsilon^2 \frac{\alpha\theta}{3}, \qquad c_2^2 = -\frac{1}{2\varepsilon\theta} \tag{2.16}$$

After this transformation (2.14) becomes:

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx} \tag{2.17}$$

(2.17) is known as the Schafer Wayne short pulse equation which was searched in this study. The subscripts mean partial derivative and u represents the electrical field of the pulse.

When an electromagnetic wave penetrates into a dielectric, its respond depends on frequency ω. Namely the medium's refractive index is a function of the frequency. This property is called as chromatic dispersion. Fiber dispersion is very important for the propagation of short optical pulses because different frequency components in the pulse travel at different speeds $v=c/n(\omega)$. This dispersive pulse broadening effect is a big problem for optical communication with fibers. When short optical pulses ranging from 10 ns to 10 fs propagate through a fiber, both dispersive and nonlinear effects affect the pulse's shape and spectrum [11].

Usually pulse propagation in optical fibers was described by cubic nonlinear Schrödinger equation (NLSE). It can be derived from Maxwell equations if the pulse width is large with respect to oscillations in carrier frequency (central frequency). Schafer and Wayne used the idea that the short pulse is broad in the Fourier domain and studied the propagation of very short pulses in Maxwell equations. Since technology achieved very short pulses in nonlinear media, NLSE does not represent these pulses any more [12].

In 1961 researchers observed the second harmonic generation at an optical frequency in a piezoelectric crystal [10]. After that several frequency mixing effects were observed such as third harmonic generation, sum-frequency generation and difference frequency generation. It was deduced that the linear term in Maxwell equations was invalid and they offered new nonlinear terms by new susceptibilities.

Nonlinear partial differential equations are differential equations with some nonlinear terms. Although linear differential equations can be solved as a superposition of linearly independent solutions, nonlinear differential equations cannot be solved by a superposition.

One of the methods to solve nonlinear partial differential equations is to change the variables for simplicity. The original equation may be transformed into a new equation which is ordinary and linear. Another tactic is the scale analysis in which some negligibly small terms may be ignored and some approximations may be made. In this tactic, general equation is tried to simplify by neglecting small terms as in Navier-Stokes equation which describes the motion of fluids [13].

CHAPTER 3

NONLINEAR DIFFERENTIAL EQUATIONS AND INTEGRABILITY

An integrable model refers to a physical model or set of differential equations whose exact solution may be calculated analytically in terms of elementary or special functions. As an adjective, integrable therefore means solvability of a differential equation. The study of completely integrable non-linear partial differential equation began with the discovery of solitons by Zabusky and Kruskal in the Korteweg-de Vries equation (KdV) in 1965. KDV equation is the first and therefore maybe the most popular nonlinear differential equation which is proven to be completely integrable. Other examples of integrable non-linear models include the KP equation, the non-linear Schrödinger equation, the sine-Gordon equation and the Toda lattice [13].

Many exactly solvable models have soliton solutions which are self-reinforcing waves (wave packets or pulses) that maintain their shape while they travel at constant speed. Before the discovery of solitons, mathematicians believed that nonlinear partial differential equations could not be solved exactly. The soliton solutions are obtained by the inverse scattering transform and Lax representation. Therefore the inverse scattering transform is a very important method for integrable nonlinear differential equations.

“In mathematics, the inverse scattering transform is a procedure for integrating certain nonlinear partial differential equations (PDEs) by first converting them into a system of linear ordinary differential equations (ODEs). The basic idea is not unlike the Fourier transform. It applies to potentials that are rapidly decaying at infinity" [13]. These potentials are seen in time independent linear Schrödinger equation. In this equation potential is represented by u(x).

$$\psi_{1xx} + u(x)\psi_1 = \lambda\psi_1 \tag{3.1}$$

The potential u(x) is the function that satisfies the nonlinear partial differential equation which will be solved. The inverse scattering transform (IST) may be used for exactly solvable models such as the Korteweg-de Vries equation (KDV), the nonlinear Schrödinger equation (NLSE), and the sine-Gordon equation. Solutions typically consist of solitons and some background radiation.

The integrability of a nonlinear partial differential equation depends on finding its Lax pair. Lax pairs are based on an abstract formulation of evolution equations [14]. They associate certain nonlinear evolution equations with linear equations and they are the starting point for using the inverse scattering method in a given integrable equation.

Let u be a function of x and t satisfying the nonlinear evolution equation

$$F(u, u_x, u_{xx}, u_{xt} ....) = 0 \tag{3.2}$$

Lax pairs are pair of linear matrix operators X and T satisfying the equation:

$$\psi_x = X\psi \tag{3.3}$$

$$\psi_t = T\psi \tag{3.4}$$

ψ is a function of vector form (ψ_1 and ψ_2) depending on the function u, its derivatives and the spectral parameter λ. Differentiating equation (3.3) and (3.4) with respect to t and x respectively

$$\psi_{xt} = X_t\psi + X\psi_t \tag{3.5}$$

$$\psi_{tx} = T_x\psi + T\psi_x \tag{3.6}$$

are obtained.

Since x and t derivative operators commute with each other, equations (3.5) and (3.6) are equal to each other.

$$\psi_{xt} = X_t\psi + X\psi_t = T_x\psi + T\psi_x \tag{3.7}$$

Writing (3.3) and (3.4) into (3.7) gives:

$$X_t\psi + XT\psi = T_x\psi + TX\psi \tag{3.8}$$

In terms of operator form:

$$X_t - T_x = TX - XT \tag{3.9}$$

$$X_t - T_x = [T, X] \tag{3.10}$$

This is a commutation relation with Lax operators. X and T operators depend on the function u in the differential equation that will be examined so the commutation relation given in equation (3.10) satisfies the differential equation. If a pair of operators X and T whose compability condition satisfies the equation, the operators are said to be the Lax pair of this differential equation.

After finding the Lax pair, exact solutions of the differential equation can be tried or some conserved quantities can be evaluated. The inverse scattering transform method implies three steps to find solutions of a differential equation. First one is finding the Lax pair. Second one is determining the time evolution of the eigenvalues λ, the norming constants, and the reflection coefficient. Finally, the last one is performing the inverse scattering procedure by solving the Marchenko equation.

In order to find conserved quantities of a differential equation via Lax representation, the x dependence of the eigenfunction ψ is found from equation (3.3). Since ψ is an eigenfunction it must have a constant value at infinity. This leads to the Ricatti equation to be solved by a series ansatz. This procedure is defined in Chapter 5 in detail when the conserved quantities of the short pulse equation are being found.

A conserved quantity of a differential equation corresponds to a functional of the unknown function and its derivatives that does not vary in time. For example, let a differential equation is of the form

$$F(u, u_x, u_t, u_{xt} \ldots) = 0 \tag{3.11}$$

Any functions of u and its derivatives, say H_n, satisfying the equation,

$$\int_{-\infty}^{\infty} H_n (u, u_x, u_t, u_{xt} \ldots) dx = \text{constant} \tag{3.12}$$

H_n are called the conserved quantities of the differential equation (3.11). That is, the value of the integral gives the same value for all time measure of t. There are several ways for finding the conserved quantities of differential equations. The methods we have emphasized in this study are using Lax pair, Hamiltonian structure of a differential equation and mass conservation law.

Soliton is a wave packet or pulse caused by nonlinear and dispersive effects of the medium and maintains its shape while it travels constant speed [13]. It does not represent a periodic wave, but the propagation of a single isolated symmetrical hump of unchanged form. Thus its inventor, John Scott Russel, called at “great wave of translation” [14]

Drazin and Johnson [15] described three properties of solitons: they have a permanent form, they are localized in a region and they can interact with other solitons and emerge from the collision unchanged. Sometimes some phenomena which do not show these properties may be called soliton as in light bullets of nonlinear optics that lose their energy during propagation.

Typically speed of a soliton depends on its amplitude. For example, water waves in shallow water can produce solitons. For water wave, the speed of wave is dependent on the depth of the water. If the water is as deep as its amplitude can be neglected, the speed of water remains constant and does not depend on the amplitude. However, when the water is shallow, it means that the amplitude affects the depth of the water and its speed. In this situation, speed (c) is a function of amplitude (a).

$$c = \sqrt{g(h+a)} \tag{3.13}$$

Therefore, the wave profile z is given as:

$$z = \zeta(x,t) = a\,\mathrm{sech}^2\left(\beta(x-ct)\right) \quad \text{where} \quad \beta^{-2} = 4h^2(h+a)/3a \tag{3.14}$$

In these equations; g is gravitational acceleration, h is the undisturbed depth of water, a is the amplitude of wave and c is the speed of wave. In equation (3.14) it is seen that higher waves travels faster. Both Boussinesq (1871) and Rayleigh (1876) used the John Scott Russell's formula (3.13). they deduced the Russell's formula from the equations of motion for an inviscid incompressible fluid. They also decided the solitary wave profile in equation (3.14) for any a>0 and $a/h \ll 1$. However these scientists did not find any differential equation whose soliton solution is equation (3.14). The differential equation corresponds to their solution was found by Korteveg de Vries in 1895. This famous nonlinear partial differential equation is known as Korteveg de Vries (KDV) equation in the literature is given by

$$\zeta_t = \frac{3}{2}\left(\frac{g}{h}\right)^{1/2}\left(\zeta\zeta_X + \frac{1}{3}\sigma\zeta_{XXX}\right) \quad \text{where} \quad X = \chi + \varepsilon\left(\frac{g}{h}\right)^{1/2} t \tag{3.15}$$

where, χ is a coordinate chosen to be moving with the wave [16].

As an example of the shape of soliton from equation (3.15), the graph of equation (3.16) with $\beta = 5$ and $a = 3$ is shown in the figure (3.1).

$$z = \zeta(x,t) = 3\,\mathrm{sech}^2\left(5w\right) \quad \text{where } w = x - ct \tag{3.16}$$

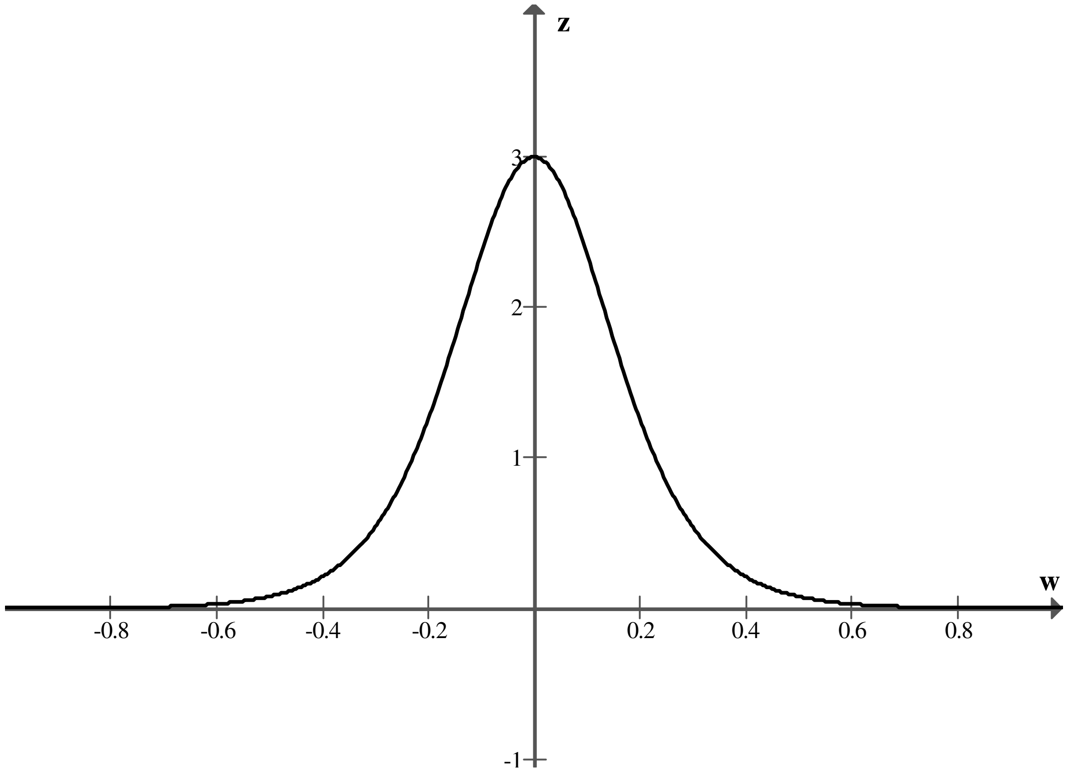

Figure 3.1. Plot of the Function (3.16) with $\beta = 5$ and a=3.

One of the important properties of solitons is their stability. They travel at constant speed and with constant shape. When they collide with each other, they superpose and merge into a single wave packet then, they separate without any change in their original shape. This stability is due to the balance of nonlinearity and dispersion because they have complementary effect on each other. Nonlinearity makes the wave more concentrated and narrower but dispersion spreads it and makes it wider. If these two effects balance each other, soliton becomes stable [17].

In optics, soliton refers to optical field that does not change during its propagation due to the stability caused by balance of nonlinearity and dispersion [13]. Electromagnetic field of light can change the refractive index of the medium. This leads to the change of the speed of light in the medium. Because of this effect, like shallow water waves, soliton phenomenon can be observed in optical waves. Akira Hasegawa [13] suggested that solitons can exist in optical fibers and made important contributions to optical telecommunication by leading soliton based transmission system.

CHAPTER 4

PROPERTIES OF THE SHORT PULSE EQUATION

4.1 Integrability

Sakovich and Sakovich [4] proved that the Schafer-Wayne short pulse equation (SWSPE) is integrable. After a chain of transformations, SWSPE is related to the sine-Gordon equation. We used this transformation to find some of the conserved quantities of SPE. This transformation will be discussed on the Chapter 5 in detail. Sakovich and Sakovich derived an exact solitary wave solution of the short pulse equation with the help of transformations to the sine-Gordon equation. They try a zero curvature representation (ZCR) in order to find the Lax Pair of short pulse equation.

Finally, they obtained the Lax pair of SPE [4] as:

$$X(u,\lambda)=\begin{pmatrix} \lambda & \lambda u_x \\ \lambda u_x & -\lambda \end{pmatrix} \tag{4.1.1}$$

$$T(u,\lambda)=\begin{pmatrix} \frac{\lambda}{2}u^2+\frac{1}{4\lambda} & \frac{\lambda}{2}u^2u_x-\frac{1}{2}u \\ \frac{\lambda}{2}u^2u_x+\frac{1}{2}u & -\frac{\lambda}{2}u^2-\frac{1}{4\lambda} \end{pmatrix} \tag{4.1.2}$$

Since SPE has a Lax pair, it is integrable. Brunelli has supported the integrability of SPE through its bi-Hamiltonian structure in Ref. [7].

4.2 Bi-Hamiltonian Structure

Brunelli [7] found the conserved Hamiltonians of the SPE and supported its integrability thorough bi-Hamiltonian structure. Victor Thomas and Kofane [18] constructed N soliton solutions of SPE by means of the Wadati-Kono-Ichikawa method. They also showed that collision process of similar solitons behaved differently from collision of dissimilar solitons. They observed that the interaction type depended on the ratio of the eigenvalues.

We shortly discuss the results of Brunelli's study [7] because he deals with conserved quantities of short pulse equation which is our topic. In his study, finding the Lagranian of the short pulse equation is explained in Equation (4.2.1) – Equation (4.2.6).

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx} \tag{4.2.1}$$

$$u_t = \partial^{-1}u + \frac{1}{6}\left(u^3\right)_x \tag{4.2.2}$$

$$u_t = \partial^{-1}u + \frac{1}{2}u^2 u_x \tag{4.2.3}$$

$$\text{Let } u = \phi_x \tag{4.2.4}$$

$$\phi_{xt} = \phi + \frac{1}{2}\phi_x^2\phi_{xx} \tag{4.2.5}$$

$$L = \frac{1}{2}\phi_x\phi_t + \frac{1}{2}\phi^2 - \frac{1}{24}\phi_x^4 \tag{4.2.6}$$

By using the Dirac's theory of constraints, he obtained the Hamiltonians and the Hamiltonian operators for short pulse equation. These Hamiltonians are given below as:

$$H_0 = -\int \sqrt{1+u_x^2}\,dx \tag{4.2.7}$$

$$H_1 = \frac{1}{2}\int u^2\,dx \tag{4.2.8}$$

$$H_2 = \int\left[\frac{1}{24}u^4 - \frac{1}{2}\left(\partial^{-1}u\right)^2\right]dx \tag{4.2.9}$$

$$H_3 = \int\left[\frac{1}{720}u^6 + \frac{1}{2}\left(\partial^{-2}u\right)^2 + \frac{1}{6}\left(\partial^{-2}u^3\right)u - \frac{1}{4}\left(\partial^{-1}u\right)^2 u^2\right]dx \tag{4.2.10}$$

4.3 Loop Soliton Solution

Most partial differential equations have a soliton solution like KDV equation, sine-Gordon equation etc. The solitary solution of short pulse equation can be found by using traveling wave solution. We will introduce the traveling wave solution of short pulse equation shortly and in terms of some special conditions. Short pulse equation shows loop soliton characteristics. Several researches with different perspectives have done to try its solitary solution form. We also tried to a simple special soliton of short pulse equation in this chapter. More detailed solutions were achieved by Ref. [5], [8], [18].

SPE equation that we studied is:

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx} \tag{4.3.1}$$

It can be written as:

$$u_{xt} = u(1+u_x^2) + \frac{u^2}{2}u_{xx} \tag{4.3.2}$$

In order to convert this equation to ordinary differential equation, we can choose a left traveling wave and write:

$$z = x + ct \tag{4.3.1}$$

After this transformation, equation 4.3.2 becomes:

$$cu_{zz} = u(1+u_z^2) + \frac{u^2}{2}u_{zz} \tag{4.3.3}$$

After some calculations, an appropriate form to take integral can be obtained.

$$\frac{u_{zz}}{1+u_z^2}=\frac{-2u}{u^2-2c} \quad (4.3.4)$$

By using the relation, it is obtained that:

$$u_{zz}=\frac{u_{zz}.u_z}{u_z}=u_z\frac{du_z}{du} \quad (4.3.5)$$

$$\frac{u_z du_z}{1+u_z^2}=\frac{-2u}{u^2-2c}du \quad (4.3.6)$$

When integrating both sides of equation (4.3.6), we get

$$\frac{1}{2}\ln(1+u_z^2)=A-\ln(u^2-2c) \quad (4.3.7)$$

with an integration constant A.

Equation (4.3.7) can be written without logarithmic form:

$$\sqrt{1+u_z^2}=\frac{B}{u^2-2c} \quad (4.3.8)$$

$$u_z^2=\frac{B^2}{(u^2-2c)^2}-1 \quad (4.3.9)$$

Then, for the derivative of u with respect to z is found as a function of u.

$$u_z=\pm\sqrt{\frac{B^2}{(u^2-2c)^2}-1}=\pm\sqrt{\frac{B^2-u^4+4cu^2-4c^2}{(u^2-2c)^2}} \quad (4.3.10)$$

The $\pm$ notations imply that there are two solutions for z.

$$\frac{du}{dz}=\pm\frac{\sqrt{B^2-u^4+4cu^2-4c^2}}{u^2-2c} \quad (4.3.11)$$

$$dz=\pm\frac{u^2-2c}{\sqrt{B^2-u^4+4cu^2-4c^2}}du \quad (4.3.12)$$

Now, the integral in equation (2.3.13) will give the solutions of z with respect to u in terms of several choice of B which is integration constant. This integration can be evaluated in terms of elliptic integrals of the first and second kind. (The integrator. http://integrals.wolfram.com/index.jsp)

$$z = \pm \int \frac{u^2 - 2c}{\sqrt{B^2 - u^4 + 4cu^2 - 4c^2}} du \tag{4.3.13}$$

Such integration like (4.3.13) can be evaluated by means of elliptic integrals. However, in this study the exact solution of the short pulse equation is not our main topic. To simplify we accepted the asymptotic condition that u_z goes to zero when u goes to zero. It means that $B^2=4c^2$ in the equation (4.3.9). Therefore we can modify the integral in (4.3.13) as:

$$z = \pm \int \frac{u^2 - 2c}{u\sqrt{4c - u^2}} du \tag{4.3.14}$$

When taking the integral by the integrator (http://integrals.wolfram.com/index.jsp), we get:

$$z = \pm \int \frac{u^2 - 2c}{u\sqrt{4c - u^2}} du = \pm\sqrt{c}\ln\left(\frac{\sqrt{4c - u^2} - 2\sqrt{c}}{cu}\right) \mp \sqrt{4c - u^2} + z_0 \tag{4.3.15}$$

with $z_0=0$

Finally, we obtain two special solutions for z as:

$$z = +\sqrt{c}\ln\left(\frac{\sqrt{4c - u^2} - 2\sqrt{c}}{cu}\right) - \sqrt{4c - u^2} \tag{4.3.16}$$

$$z = -\sqrt{c}\ln\left(\frac{\sqrt{4c - u^2} - 2\sqrt{c}}{cu}\right) + \sqrt{4c - u^2} \tag{4.3.17}$$

These solutions for z were drawn separately by Graph (Version 4.0.1). We selected c=1 in the figure (4.3.1) – (4.3.3).

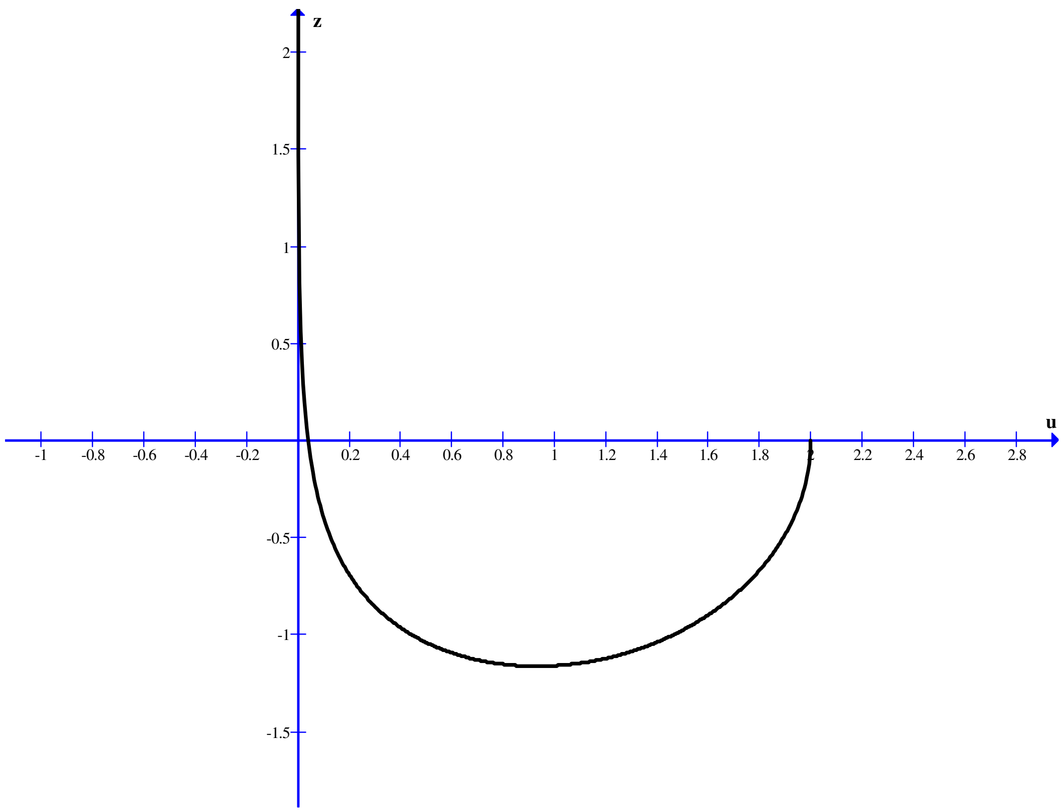

Figure 4.3.1 is graph of the function (4.3.16).

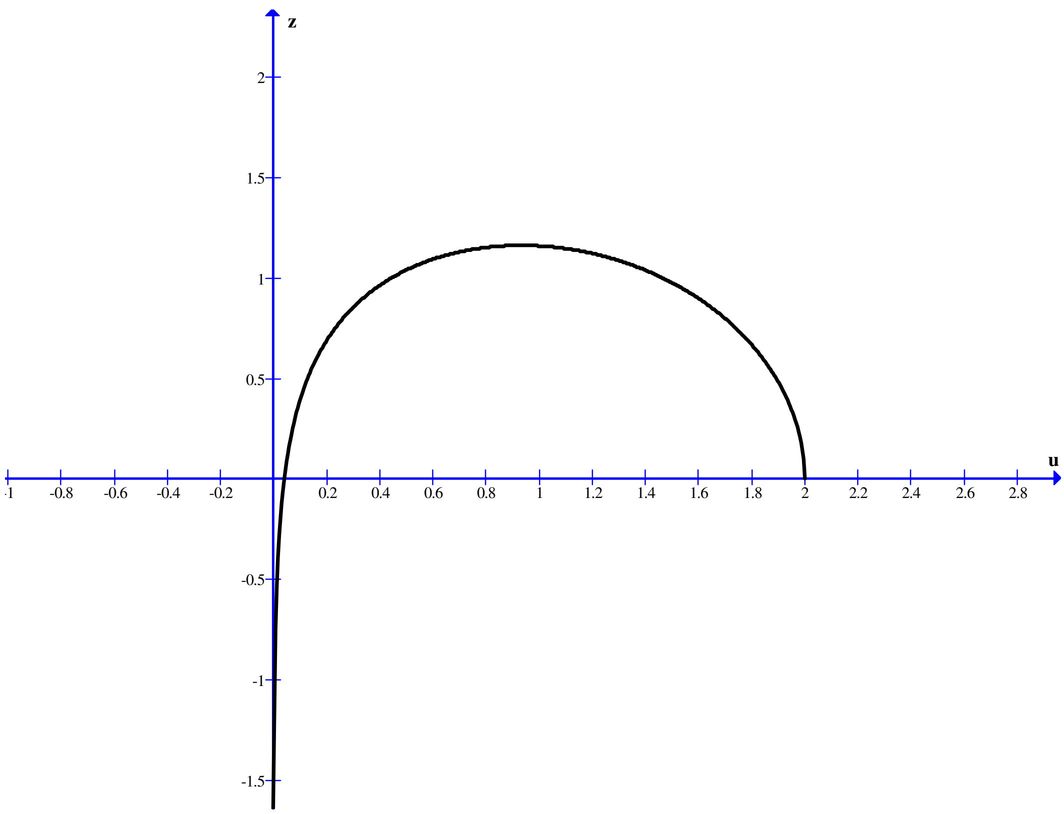

Figure 4.3.2 is graph of the function (4.3.17).

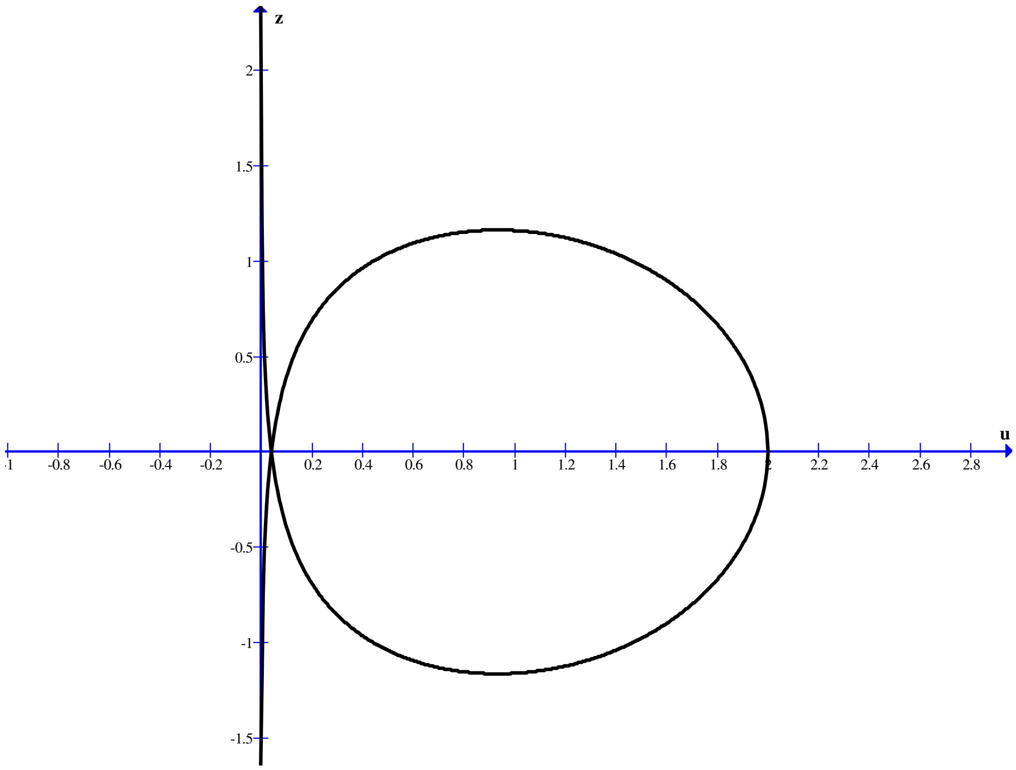

Figure 4.3.3. is the graph of the function (4.3.15)

In figure (4.3.3) two solutions are seen at the same coordinate axes. However this graph belongs to the function z(u). In order to obtain the function u(z) in the differential equation, we should take its inverse function so the coordinate axes in the figure (4.3.3) must be rotated 90 degrees.

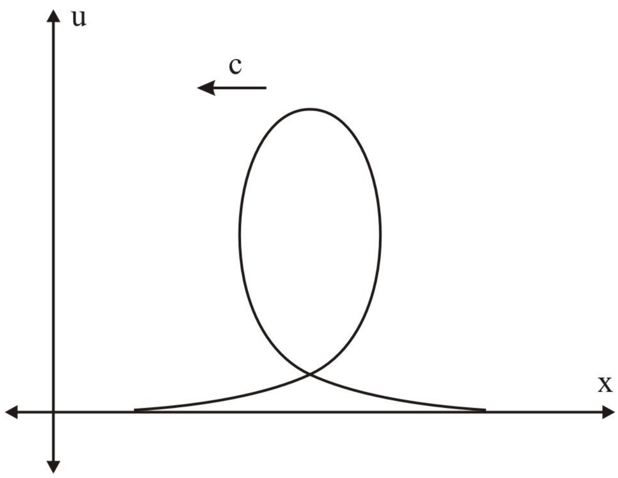

Figure 4.3.4. is the graphical representation of the soliton solution of short pulse equation.

After the rotation, figure (4.3.4) was obtained. This is a loop soliton which intersects with itself. Since we choose z=x+ct , the solitary wave propagates to the left with a speed c.

E.J. Parkes [8] also found periodic and traveling wave solutions of SPE. He concludes that "There is one type of periodic-wave solution that propagates in the positive x-direction. There are several types of periodic-wave solution and one type of solitary-wave solution that propagate in the negative x-direction; there are places on the wave profile of these solutions where the slope goes infinite. Each solution has a corresponding solution that propagates in the same direction and is a mirror image in the x axis."

Sakovich and Sakovich [5] have found short pulse's solitary wave solution via transformation to the sine-Gordon equation. The kink solution of the sine-Gordon equation is known as:

$$z = 4\arctan\left[\exp\left(y+t\right)\right] \tag{4.3.18}$$

According to the transformation, they obtain loop soliton solution of short pulse equation which moves from the right to the left with constant shape and unit speed as:

$$u = \frac{2}{\cosh\left(y+t\right)} \tag{4.3.19}$$

$$x = y - 2\tanh\left(y+t\right) \tag{4.3.20}$$

The two-loop solution and loop-antiloop solution of SPE with different t values and their graphs can be seen in their study [5].

Kuetche and others [18] studied the propagation of loop and hump soliton solutions of the short pulse equation. They found the corresponding N-soliton solutions by means of an inverse scattering method. Performing a more detailed analysis, the properties of the one- and two-soliton solutions have been studied. According to their studies when two-soliton solutions of the previous equation with ‘similar’ or ‘dissimilar’ amplitudes collide, they always shift backwards except when one of them has negative amplitude. With those special soliton solutions, the interaction type depends on the ratio of the two eigenvalues involved. They have also shown that the two basic collision processes with similar and dissimilar amplitudes, respectively, depend strongly upon a critical value of the ratio of the two eigenvalues.

4.4 Transformation to Sine-Gordon Equation

Short pulse equation can be examined by using the similarities with other equations via some change of transformations. Sakovich and Sakovich transformed the short pulse equation to sine-Gordon equation.

$$z_{yt} = \sin z \qquad 4.4.1$$

$$u(x,t) = z_t(y,t) \qquad 4.4.2$$

$$x = w(y,t) \quad w_y = \cos z \quad w_t = -\frac{1}{2} z_t^2 \qquad 4.4.3$$

$$v(x,t) = \left(1 + u_x^2\right)^{-1/2} \qquad 4.4.4$$

$$v(x,t) = w_y(y,t) \qquad 4.4.5$$

$$z(y,t) = \arccos w_y \qquad 4.4.6$$

We used these transformations to get some conserved quantities of the short pulse equation. This procedure will be explained in chapter 5.

CHAPTER 5

CONSERVED QUANTITIES OF THE SHORT PULSE EQUATION

5.1 Conserved Quantities by Mass Conservation Law

Conserved quantities are very important in mathematical physics. They are the quantities which do not vary in time. Many conserved quantities can be found in nonlinear partial differential equations. One of the methods to find conserved quantities is derived from the equation of mass conservation. Let us consider a well known example of this in one dimensional fluid dynamics.

Suppose g(x,t) is the flux of the fluid and f(x,t) is density. Therefore the relation between f and g is:

$$\frac{\partial f}{\partial t}+\frac{\partial g}{\partial x}=0 \qquad (5.1.1)$$

According to mass conservation law, it must be true that g(x,t) has constant values at infinity. That is:

$$\lim_{|x|\to\infty} g(x,t)=\text{constant} \qquad (5.1.2)$$

By using (5.1.1) we can obtain g(x,t) alone and use (5.1.2).

$$g(x,t)\Big|_{-\infty}^{\infty}=-\frac{d}{dt}\left(\int_{-\infty}^{\infty} f(x,t)dx\right)=0 \qquad (5.1.3)$$

From this equation we can conclude that:

$$\int_{-\infty}^{\infty} f(x,t)dx = \text{constant} \quad (5.1.4)$$

It represents the conservation of total mass of the fluid. More generally, if any functions $f(u(x,t), u_x(x,t),\ldots)$ and $g(u(x,t), u_x(x,t),\ldots)$ can be written in the form of (5.1.1), then a conserved quantity of the differential equation of $u(x,t)$ is:

$$c = \int_{-\infty}^{\infty} f[u(x,t), u_x(x,t), u_{xx}(x,t), \ldots..]dx \quad (5.1.5)$$

As an example we can find a conserved quantity for the short pulse equation:

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx} \quad \text{or} \quad u_{xt} = u(1+u_x^2) + \frac{u^2}{2}u_{xx} \quad (5.1.6)$$

Let us assume that f be only a function of u_x for simplicity. That is $f=f(u_x)$.

$$\frac{\partial f(u_x)}{\partial t} = \frac{\partial f}{\partial u_x}\frac{\partial u_x}{\partial t} = \frac{\partial f}{\partial u_x}u_{xt} = \frac{\partial f}{\partial u_x}\left(u(1+u_x^2) + \frac{u^2}{2}u_{xx}\right)$$

$$\frac{\partial f}{\partial u_x}u(1+u_x^2) + \frac{u^2}{2}\frac{\partial f}{\partial u_x}\frac{\partial u_x}{\partial x} = \frac{\partial f}{\partial u_x}u(1+u_x^2) + \frac{u^2}{2}\frac{\partial f}{\partial x}$$

Therefore

$$\frac{\partial f}{\partial u_x}u(1+u_x^2) + \frac{u^2}{2}\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial x} \quad (5.1.7)$$

must be satisfied. In order to satisfy (5.1.7), it must be possible that the left hand-side of the equation is an exact derivative of multiplication of two functions. Therefore; it can be considered as derivative of a multiplication.

$$\frac{\partial f}{\partial u_x}u(1+u_x^2) + \frac{u^2}{2}\frac{\partial f}{\partial x} = \left(\frac{u^2}{2}f\right)_x \quad (5.1.8)$$

If equation (5.1.8) will be satisfied, it must be true that

$$uu_x f = \frac{\partial f}{\partial u_x} u(1+u_x^2) \tag{5.1.9}$$

$$\frac{u_x}{(1+u_x^2)} du_x = \frac{df}{f} \tag{5.1.10}$$

Equation (5.1.10) is consistent with our assumption $f=f(u_x)$. This simple differential equation for f can be integrated with an integration constant c.

$$\frac{1}{2}\ln(1+u_x^2) = c + \ln f \tag{5.1.11}$$

$$f = A\sqrt{1+u_x^2} \tag{5.1.12}$$

Equation (5.1.12) is a general solution for f with a constant A. Therefore a conserved quantity has been found as:

$$\int_{-\infty}^{\infty} \sqrt{1+u_x^2}.dx = \text{const.} \tag{5.1.13}$$

$$\left[\sqrt{1+u_x^2}\right]_t = \left[\frac{u^2}{2}\sqrt{1+u_x^2}\right]_x$$

This is the first conserved quantity of short pulse equation. We also found this quantity by using different methods like Lax pairs and sine-Gordon transformation. Brunelli has also obtained this conserved quantity by using Hamiltonian structure of the short pulse equation.

Another conserved quantity has been found by this method. Now let us assume the function f in the equation 5.1.1 depend on just u, that is, f=f(u). Therefore the time derivative of f can be written as:

$$\frac{\partial f(u)}{\partial t} = \frac{\partial f}{\partial u}.\frac{\partial u}{\partial t} \tag{5.1.14}$$

where

$$\frac{\partial u}{\partial t} = u_t = \int u_{xt} dx = \int \left[u + \frac{1}{6}\left(u^3\right)_{xx}\right] dx \tag{5.1.15}$$

$$u_t = \partial^{-1}u + \frac{u^2}{2}u_x \qquad (5.1.16)$$

where ∂^{-1} operator represents the integration with respect to x. Combining equations 5.1.14 and 5.1.16 yields:

$$\frac{\partial f(u)}{\partial t} = \frac{\partial f}{\partial u}.\left(\partial^{-1}u + \frac{u^2}{2}u_x\right) \qquad (5.1.17)$$

We define another parameter w as:

$$\partial^{-1}u = w \quad \Leftrightarrow \quad u = w_x \qquad (5.1.18)$$

so equation 5.1.17 becomes:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial w_x}\left(w + \frac{w_x^2}{2}w_{xx}\right) \qquad (5.1.19)$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial w_x}w + \frac{w_x^2}{2}\frac{\partial f}{\partial x} \qquad (5.1.20)$$

According to equation 5.1.1, the time derivative of f must be equal to x derivative of another function.

$$\frac{\partial f}{\partial w_x}w + \frac{w_x^2}{2}\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial x} \qquad (5.1.21)$$

This equation can be satisfied if equation 5.1.22 and 5.1.23 are assumed to be valid.

$$\frac{\partial f}{\partial w_x} = w_x \qquad (5.1.22)$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial w_x}\frac{\partial w_x}{\partial x} = w_x.w_{xx} \qquad (5.1.23)$$

Then equation 5.1.21 becomes:

$$w_x.w + \frac{w_x^3}{2}w_{xx} = \left(\frac{w^2}{2} + \frac{w_x^4}{8}\right)_x \qquad (5.1.24)$$

Consequently, equation 5.1.1 was satisfied if and only if equation 5.1.22 is true. The function f can be found by 5.1.22

$$\frac{\partial f}{\partial w_x} = w_x \quad \Leftrightarrow \quad f = \frac{1}{2} w_x^2 \tag{5.1.25}$$

$$f(u) = u^2 \tag{5.1.26}$$

$$\left[u^2\right]_t = \left[\left(\partial^{-1}u\right)^2 + \frac{u^2}{4}\right]_x$$

One can try mass conservation law 5.1.1 for 5.1.26. If it replaces in 5.1.1, it is seen that $f=u^2$ is a density function for short pulse equation. As a consequence the quantity:

$$\int_{-\infty}^{\infty} u^2 dx \tag{5.1.27}$$

is a conserved quantity for the short pulse equation. This conserved quantity can also result by other techniques. u^2 is proportional to the intensity which is the average power per unit area transported by an electromagnetic wave. The integration in (5.1.27) means that total power transported by the wave remains constant.

5.2 Conserved Quantities of the Short Pulse Equation by Use of Lax Pair

Conserved quantities can also be calculated by the help of Lax pairs. The Lax pairs of the short pulse equation found by Sakovich and Sakovich [4] are given below:

$$X(u,\lambda) = \begin{pmatrix} \lambda & \lambda u_x \\ \lambda u_x & -\lambda \end{pmatrix} \tag{5.2.1}$$

$$T(u,\lambda) = \begin{pmatrix} \frac{\lambda}{2}u^2 + \frac{1}{4\lambda} & \frac{\lambda}{2}u^2 u_x - \frac{1}{2}u \\ \frac{\lambda}{2}u^2 u_x + \frac{1}{2}u & -\frac{\lambda}{2}u^2 - \frac{1}{4\lambda} \end{pmatrix} \tag{5.2.2}$$

Lax equations with these linear operators are:

$$\psi_x = X\psi \quad \text{and} \quad \psi_t = T\psi \tag{5.2.3}$$

$$X\psi = \psi_x \;\Rightarrow\; \begin{pmatrix} \lambda & \lambda u_x \\ \lambda u_x & -\lambda \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_{1x} \\ \psi_{2x} \end{pmatrix} \tag{5.2.4}$$

$$\lambda\psi_1 + \lambda u_x \psi_2 = \psi_{1x} \tag{5.2.5}$$

$$\lambda u_x \psi_1 - \lambda\psi_2 = \psi_{2x} \tag{5.2.6}$$

The linear equations can be solved for Ψ_1

$$\psi_2 = \frac{\psi_{1x} - \lambda\psi_1}{\lambda u_x} \quad , \quad \psi_{2x} = \frac{\psi_{1xx}}{\lambda u_x} - \frac{\psi_{1x}}{\lambda u_x^2} u_{xx} - \frac{\psi_{1x}}{u_x} + \frac{\psi_1}{u_x^2} u_{xx} \tag{5.2.7}$$

$$\lambda u_x \psi_{1x} - \lambda \frac{\psi_{1x} - \lambda\psi_1}{\lambda u_x} = \frac{\psi_{1xx}}{\lambda u_x} - \frac{\psi_{1x}}{\lambda u_x^2} u_{xx} - \frac{\psi_{1x}}{u_x} + \frac{\psi_1}{u_x^2} u_{xx} \tag{5.2.8}$$

$$\psi_{xx} - \frac{u_{xx}}{u_x}\psi_x + (\lambda \frac{u_{xx}}{u_x} - \lambda^2 - \lambda^2 u_x^2)\psi = 0 \tag{5.2.9}$$

5.2.9 is a linear differential equation for ψ. To solve this equation, we can try a solution as:

$$\psi = e^{\lambda x + \theta(x,\lambda)} \tag{5.2.10}$$

Since ψ is an eigenfunction, its behavior at infinity should not be exponential growth. Therefore

$$\lim_{x\to\infty} \theta(x,\lambda) = \text{constant.} \tag{5.2.11}$$

If we substitute ψ into the equation, we get;

$$\psi = e^{\lambda x + \theta(x,\lambda)} \Rightarrow \theta_{xx} + \theta_x^2 + 2\lambda\theta_x - \frac{u_{xx}}{u_x}\theta_x - \lambda^2 u_x^2 = 0 \tag{5.2.12}$$

This is the Ricatti equation that may be solved by series ansatz.

$$\text{Let } \theta_x = \sum_{n=0}^{\infty} \frac{c_n}{\lambda^{n-1}} = (\lambda c_0 + c_1 + \frac{c_2}{\lambda} + \frac{c_3}{\lambda^2} + ...) \tag{5.2.13}$$

(where c_n are functions of u, u_x, u_{xx} etc.)

The functions c_n can be find by equating equal powers o λ.

$$\text{for } \lambda^2: \quad 2c_0 + c_0^2 - u_x^2 = 0 \tag{5.2.14}$$

$$\text{for } \lambda: \quad c_{0,x} + 2c_1 + 2c_0 c_1 - \frac{u_{xx}}{u_x} c_0 = 0 \tag{5.2.15}$$

$$\text{for } \lambda^0: \quad c_{1x} + c_1^2 + 2c_0 c_2 + 2c_2 - \frac{u_{xx}}{u_x} c_1 = 0 \tag{5.2.16}$$

$$c_0 = -1 \mp \sqrt{1+u_x^2} \tag{5.2.17}$$

$$c_1 = \frac{\frac{u_{xx}}{u_x} c_0 - c_{0,x}}{2(1+c_0)} \tag{5.2.18}$$

$$c_2 = \frac{\frac{u_{xx}}{u_x} c_1 - c_{1x} - c_1^2}{2(1+c_0)} \tag{5.2.19}$$

$$\vdots$$

$$\vdots$$

This algorithm can be formulized as:

$$c_{n+1} = \frac{\frac{u_{xx}}{u_x} c_n - c_{n,x} - \sum_{k=1}^{n} c_k c_{n-k+1}}{2(c_0+1)} \quad \text{if } n \geq 1 \tag{5.2.20}$$

In order to find c_1, we must calculate c_{0x}.

$$c_{0x} = \mp \frac{u_x u_{xx}}{\sqrt{1+u_x^2}} \tag{5.2.21}$$

If 5.2.17 and 5.2.21 are written in 5.2.18, c_1 is obtained as:

$$c_1 = \frac{u_{xx}}{2u_x}\left(\frac{1}{1+u_x^2} \pm \frac{1}{\sqrt{1+u_x^2}}\right) \quad (5.2.22)$$

Since this term is an exact derivative, it cannot be a conserved quantity. Now let us calculate c_2 for both c_0 values in 5.2.17 according to 5.2.19.

$$\text{for} \quad c_0 = -1 + \sqrt{1+u_x^2} \quad (5.2.23)$$

$$c_1 = \frac{u_{xx}}{2u_x}\left(\frac{1}{1+u_x^2} - \frac{1}{\sqrt{1+u_x^2}}\right) \quad (5.2.24)$$

$$c_{1x} = \left(\frac{u_{xxx}}{2u_x} - \frac{u_{xx}^2}{2u_x^2}\right)\left(\frac{1}{1+u_x^2} - \frac{1}{\sqrt{1+u_x^2}}\right) + \frac{u_{xx}^2}{2}\left(\frac{1}{\left(1+u_x^2\right)^{3/2}} - \frac{2}{\left(1+u_x^2\right)^2}\right) \quad (5.2.25)$$

$$c_1^2 = \frac{u_{xx}^2}{4u_x^2}\left(\frac{1}{\left(1+u_x^2\right)^2} + \frac{1}{\left(1+u_x^2\right)} - \frac{2}{\left(1+u_x^2\right)^{3/2}}\right) \quad (5.2.26)$$

When equations (5.2.23)-(5.2.26) are written in (5.2.19), c_2 is obtained as in (5.2.27).

$$c_2 = \frac{u_{xx}^2}{8u_x^2}\left(3v^{-3/2} - 4v^{-1} - v^{-5/2} + 2v^{-2}\right) + \frac{u_{xxx}}{4u_x}\left(-v^{-3/2} + v^{-1}\right) + \frac{u_{xx}^2}{4}\left(2v^{-5/2} - v^{-2}\right) \quad (5.2.27)$$

Let us try another value of c_2 for a different value of c_0.

$$\text{for} \quad c_0 = 1 - \sqrt{1+u_x^2} \quad (5.2.28)$$

$$c_1 = \frac{u_{xx}}{2u_x}\left(\frac{1}{1+u_x^2} + \frac{1}{\sqrt{1+u_x^2}}\right) \quad (5.2.29)$$

$$c_{1x} = \left(\frac{u_{xxx}}{2u_x} - \frac{u_{xx}^2}{2u_x^2}\right)\left(\frac{1}{1+u_x^2} + \frac{1}{\sqrt{1+u_x^2}}\right) - \frac{u_{xx}^2}{2}\left(\frac{1}{\left(1+u_x^2\right)^{3/2}} - \frac{2}{\left(1+u_x^2\right)^2}\right) \quad (5.2.30)$$

$$c_1^2 = \frac{u_{xx}^2}{4u_x^2}\left(\frac{1}{\left(1+u_x^2\right)^2}+\frac{1}{\left(1+u_x^2\right)}+\frac{2}{\left(1+u_x^2\right)^{3/2}}\right) \tag{5.2.31}$$

If (5.2.29)-(5.2.31) and (5.2.19) are combined for c_2, it gives

$$c_2 = \frac{u_{xx}^2}{8u_x^2}\left(-3v^{-3/2}-4v^{-1}+v^{-5/2}+2v^{-2}\right)+\frac{u_{xxx}}{4u_x}\left(v^{-3/2}+v^{-1}\right)-\frac{u_{xx}^2}{4}\left(2v^{-5/2}+v^{-2}\right) \tag{5.2.32}$$

Simply c_2 is:

$$c_2 = \left(\frac{u_{xx}}{4u_x}\left(\pm v^{-3/2}+v^{-1}\right)\right)_x \pm \frac{u_{xx}^2}{8\left(1+u_x^2\right)^{5/2}}$$

When c_3 is calculated, it will be an exact derivative. Therefore, c_4 should be calculated. Because c_4 is too long for practical purpose, we ignore the next conserved quantities. Consequently we have found three conserved quantities by the help of Lax Pair. They are seen in (5.2.33)-(5.2.35).

$$\int_{-\infty}^{\infty}\sqrt{1+u_x^2}\,dx \tag{5.2.33}$$

$$\int_{-\infty}^{\infty}\left[\frac{u_{xx}^2}{8u_x^2}\left(-3v^{-3/2}-4v^{-1}+v^{-5/2}+2v^{-2}\right)+\frac{u_{xxx}}{4u_x}\left(v^{-3/2}+v^{-1}\right)-\frac{u_{xx}^2}{4}\left(2v^{-5/2}+v^{-2}\right)\right]dx \tag{5.2.34}$$

$$\int_{-\infty}^{\infty}\left[\frac{u_{xx}^2}{8u_x^2}\left(3v^{-3/2}-4v^{-1}-v^{-5/2}+2v^{-2}\right)+\frac{u_{xxx}}{4u_x}\left(-v^{-3/2}+v^{-1}\right)+\frac{u_{xx}^2}{4}\left(2v^{-5/2}-v^{-2}\right)\right]dx \tag{5.2.35}$$

(5.2.33) has been obtained by using different methods such as mass conservation, transformation and Hamilton in this study. After some calculations, (5.2.34) and (5.2.35) can be written as:

$$\int_{-\infty}^{\infty} \frac{u_{xx}^2}{\left(1+u_x^2\right)^{5/2}} dx \qquad (5.2.36)$$

and the mass conservation law is

$$\left[\frac{u_{xx}^2}{\left(1+u_x^2\right)^{5/2}}\right]_t = \left[\frac{u^2 u_{xx}^2}{2\left(1+u_x^2\right)^{5/2}} - \frac{2}{\left(1+u_x^2\right)^{1/2}}\right]_x \qquad (5.2.37)$$

5.3 Conserved Quantities of the Sine-Gordon Equation by Use of Lax Pair

Other conserved quantities for the short pulse equation were derived from sine-Gordon equation via transformations found by Sakovich. These transformations are explained in section 2.4. Sine-Gordon and short pulse equations are

$$Z_{yt} = \sin z \quad \text{(SGE)} \qquad (5.3.1)$$

$$u_{xt} = u + 1/6(u^3)_{xx} \quad \text{(SPE)} \qquad (5.3.2)$$

The transformation rules:

$$\begin{aligned} &\upsilon(x,t) = \left(u_x^2+1\right)^{-1/2}, \\ &x = \omega(y,t), \qquad \upsilon(x,t) = \omega_y(y,t), \\ &z(y,t) = \arccos \omega_y \end{aligned} \qquad (5.3.3)$$

$$\begin{aligned} &u(x,t) = z_t(y,t), \qquad x = \omega(y,t): \\ &\omega_t = -\frac{1}{2} z_t^2 \end{aligned} \qquad (5.3.4)$$

The conserved quantities of sine-Gordon equation in (5.3.1) can be found by Lax representation. The Lax operators of SGE are seen in (5.3.5)-(5.3.6).

$$X(z,\lambda) = \begin{pmatrix} -i\lambda & -\frac{1}{2}z_y \\ \frac{1}{2}z_y & i\lambda \end{pmatrix} \quad (5.3.5)$$

$$T(z,\lambda) = \begin{pmatrix} \frac{i}{4\lambda}\cos z & \frac{i}{4\lambda}\sin z \\ \frac{i}{4\lambda}\sin z & -\frac{i}{4\lambda}\cos z \end{pmatrix} \quad (5.3.6)$$

Lax equations with these linear operators are:

$$\psi_y = X\psi \quad \text{and} \quad \psi_t = T\psi \quad (5.3.7)$$

According to (5.3.7) the matrix equation for ψ_1 and ψ_2 are:

$$X\psi = \psi_y \;\Rightarrow\; \begin{pmatrix} -i\lambda & -\frac{1}{2}z_y \\ \frac{1}{2}z_y & i\lambda \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_{1y} \\ \psi_{2y} \end{pmatrix} \quad (5.3.8)$$

$$-i\lambda\psi_1 - \frac{1}{2}z_y\psi_2 = \psi_{1y} \quad (5.3.9)$$

$$\frac{1}{2}z_y\psi_1 + i\lambda\psi_2 = \psi_{2y} \quad (5.3.10)$$

In order to combine (5.3.9) and (5.3.10), ψ_2 and ψ_{2x} should be obtained as:

$$\psi_2 = -\frac{2}{z_y}\left(\psi_{1y} + i\lambda\psi_1\right) \quad (5.3.11)$$

$$\psi_{2y} = \frac{2z_{yy}}{z_y^2}\left(\psi_{1y} + i\lambda\psi_1\right) - \frac{2}{z_y}\left(\psi_{1yy} + i\lambda\psi_{1y}\right) \quad (5.3.12)$$

If (5.3.11) and (5.312) are written in (5.3.10), an equation depending on only ψ_1 is obtained as (5.3.13).

$$\frac{1}{2}z_y\psi_1 - i\lambda\frac{2}{z_y}\left(\psi_{1y} + i\lambda\psi_1\right) = \frac{2z_{yy}}{z_y^2}\left(\psi_{1y} + i\lambda\psi_1\right) - \frac{2}{z_y}\left(\psi_{1yy} + i\lambda\psi_{1y}\right) \qquad (5.3.13)$$

(5.3.14) is a linear differential equation for ψ_1 depending on y, t and λ.

$$\psi_{1yy} - \frac{z_{yy}}{z_y}\psi_{1y} - \left[i\lambda\frac{z_{yy}}{z_y} - \frac{1}{4}z_y^2 - \lambda^2\right]\psi_1 = 0 \qquad (5.3.14)$$

This eigenfunction equation can be solved by trying a solution of the form in (5.3.15).

$$\psi_1 = e^{-i\lambda y + \theta(y,\lambda)} \qquad (5.3.15)$$

If (5.3.15) is replaced in (5.3.14) a differential equation with $\theta(x,t,\lambda)$ is obtained in (5.3.16). Meanwhile it is necessary to recall that θ goes to a constant value when y goes to infinity.

$$\lim_{y\to\infty}\theta(y,t,\lambda) = \text{constant}$$

$$\theta_{yy} - \left(\frac{z_{yy}}{z_y} + 2i\lambda\right)\theta_y + \theta_y^2 + \frac{1}{4}z_y^2 = 0 \qquad (5.3.16)$$

To solve this equation a series solution for θ_y like (5.3.17) can be tried. In the equation, c_n's represent functions of y and t.

$$\theta_y = \sum_{n=0}^{\infty}\frac{c_n(y,t)}{\left(2i\lambda\right)^{n+1}} \qquad (5.3.17)$$

$$\theta_{yy} = \sum_{n=0}^{\infty} \frac{c_{n,y}(y,t)}{(2i\lambda)^{n+1}} \quad (5.3.18)$$

$$\theta_y^2 = \sum_{k=0}^{\infty} \frac{c_k}{(2i\lambda)^{k+1}} \sum_{n=0}^{\infty} \frac{c_n}{(2i\lambda)^{n+1}} \quad (5.3.19)$$

When (5.3.17)-(5.3.19) are placed into (5.3.16) equation (5.3.20) is obtained.

$$\sum_{n=0}^{\infty} \frac{c_{n,y}(y,t)}{(2i\lambda)^{n+1}} - \left(\frac{z_{yy}}{z_y} + 2i\lambda\right) \sum_{n=0}^{\infty} \frac{c_n(y,t)}{(2i\lambda)^{n+1}} + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{c_k c_{n-1-k}}{(2i\lambda)^{n+1}} + \frac{1}{4} z_y^2 = 0 \quad (5.3.20)$$

The solutions of c_n's for different powers of λ give the values of c_n's. They will be the conserved quantities of SGE because of the asymptotic behavior of θ. This asymptotic behavior is

$$\lim_{y \to \infty} \theta(y,t,\lambda) = \text{constant} \quad (5.3.21)$$

so θ can be defined as:

$$\theta(y,t,\lambda) = \int_{-\infty}^{y} \theta_y(y,t,\lambda)\, dy \quad (5.3.22)$$

$$\lim_{y \to \infty} \theta(y,t,\lambda) = \int_{-\infty}^{\infty} \theta_y(y,t,\lambda)\, dy = \text{constant} \quad (5.3.23)$$

$$\int_{-\infty}^{\infty} c_n(y,t,\lambda)\, dy = \text{constant} \quad (5.3.24)$$

Calculating c_n for different values of λ gives:

$$\text{for } \lambda^0: \quad -c_0 + \frac{1}{4} z_y^2 = 0 \quad (5.3.25)$$

$$\text{for } \lambda^{-1}: \quad c_{0y} - \frac{z_{yy}}{z_y} c_0 - c_1 = 0 \tag{5.3.26}$$

$$\text{for } \lambda^{-2}: \quad c_{1y} - \frac{z_{yy}}{z_y} c_1 + c_0^2 - c_2 = 0 \tag{5.3.27}$$

$$\vdots$$

$$\text{for } \lambda^{-n}\ (n \geq 2): \quad c_{(n-1)y} - \frac{z_{yy}}{z_y} c_{n-1} - c_n + \sum_{k=0}^{n-2} c_k c_{n-k-2} = 0 \tag{5.3.28}$$

$$c_0 = \frac{1}{4} z_y^2 \tag{5.3.29}$$

$$c_1 = \frac{1}{4} z_y z_{yy} \tag{5.3.30}$$

$$c_2 = \frac{1}{4} z_y z_{yyy} + \frac{1}{16} z_y^4 \tag{5.3.31}$$

$$\vdots$$

$$c_n = c_{(n-1)y} - \frac{z_{yy}}{z_y} c_{n-1} + \sum_{k=0}^{n-2} c_k c_{n-k-2} \tag{5.3.32}$$

$$c_3 = \frac{1}{4} z_y z_{4y} + \frac{5}{16} z_y^3 z_{yy} \tag{5.3.33}$$

$$c_4 = \frac{1}{4} z_y z_{5y} + \frac{7}{16} z_y^3 z_{yyy} + \frac{11}{16} z_y^2 z_{yy}^2 + \frac{1}{32} z_y^6 \tag{5.3.34}$$

Equation (5.3.29)-(5.3.34) are conserved integrals of the SGE but the integration of c_1 and c_3 give exact derivatives. Therefore, just functions with even index produce a conserved quantity.

For a conserved quantity f,

$$\frac{\partial f}{\partial t} + \frac{\partial g}{\partial y} = 0 \tag{5.3.35}$$

$$\int_{-\infty}^{\infty} f(y,t)\, dy = \text{constant} \tag{5.3.36}$$

For this reason c_0 must satisfy (5.3.35) with an appropriate function g. Of course, if c_0 is a conserved quantity, there must be a function g according to (5.3.35).

$$\frac{\partial c_0}{\partial t} = \left(\frac{1}{4} z_y^2\right)_t = \frac{1}{2} z_y z_{yt} \tag{5.3.37}$$

By the help of (5.3.1), z_{yt} can be written in (5.3.37) and (5.3.38) is obtained.

$$\left(\frac{1}{4} z_y^2\right)_t = \frac{1}{2} z_y z_{yt} = \frac{1}{2} z_y \sin z \tag{5.3.38}$$

$$\frac{1}{2} z_y \sin z = \left(-\frac{1}{2}\cos z\right)_y \tag{5.3.39}$$

$$\left(\frac{1}{4} z_y^2\right)_t = \left(\frac{1}{2}\cos z\right)_y \tag{5.3.40}$$

According to equation (5.3.40), equation (5.3.41) is a conserved quantity of sine-Gordon equation.

$$\int_{-\infty}^{\infty} \frac{1}{4} z_y^2 \, dy = \text{constant} \tag{5.3.41}$$

Because sine-Gordon equation in (5.3.1) is symmetric for y and t, independent variables (y and t) in (5.3.40) can be interchanged with each other. That is (5.3.42) is also valid.

$$\left(\frac{1}{2} z_t^2\right)_y = \left(1 - \cos z\right)_t \tag{5.3.42}$$

It means that (5.3.43) is also a conserved quantity.

$$\int_{-\infty}^{\infty} (1 - \cos z).dy = \text{constant} \tag{5.3.43}$$

We can use this symmetry in all conserved quantities of SGE.

The conserved coefficient c_2 (5.3.31) can be simplified by use of derivation rule for multiplication. The multiplication $z_y z_{yyy}$ can be written as:

$$z_y z_{yyy} = \left(z_y z_{yy}\right)_y - z_{yy}^2 \tag{5.3.44}$$

Since $\left(z_y z_{yy}\right)_x$ is an exact derivative, it can be omitted from conserved quantity. For this reason, the corresponding conserved quantity becomes:

$$\int_{-\infty}^{\infty} \left(\frac{1}{16} z_y^4 - \frac{1}{4} z_{yy}^2\right).dy = \text{constant} \tag{5.3.45}$$

That is, (5.3.35) is valid for c_2 with an appropriate function g. Taking derivative of c_2 with respect to t, derivative of g with respect to y will appear. T derivative of c_2 is:

$$\left(\frac{1}{4} z_y^4 - z_{yy}^2\right)_t = z_y^3 z_{yt} - 2 z_{yy} z_{yyt} \tag{5.3.46}$$

It is known that $z_{yt} = \sin z$ and $z_{yyt} = z_y \cos z$. So t derivative becomes:

$$\left(\frac{1}{4} z_y^4 - z_{yy}^2\right)_t = z_y^3 \sin z - 2 z_{yy} z_y \cos z = -\left(z_y^2 \cos z\right)_y \tag{5.3.47}$$

$$\left(\frac{1}{4} z_y^4 - z_{yy}^2\right)_t + \left(z_y^2 \cos z\right)_y = 0 \tag{5.3.48}$$

We can use y and t symmetry for SGE in (5.3.48) so the independent variables are replaced with each other as (5.3.49).

$$\left(\frac{1}{4}z_t^4 - z_{tt}^2\right)_y + \left(z_t^2 \cos z\right)_t = 0 \tag{5.3.49}$$

Because of the equation (5.3.49), equation (5.3.50) is a conserved quantity for SGE.

$$\int_{-\infty}^{\infty} z_t^2 \cos z \, dy = \text{constant} \tag{5.3.50}$$

As for c_4, we can use the rule for derivative for multiplication again in order to simplify it. The first term in (5.3.51) is modified as seen in (5.3.52)-(5.3.54) so it can be changed from the conserved quantity.

$$c_4 = \frac{1}{4}z_y z_{5y} + \frac{7}{16}z_y^3 z_{yyy} + \frac{11}{16}z_y^2 z_{yy}^2 + \frac{7}{16}z_y^6 \tag{5.3.51}$$

$$z_y z_{5y} = \left(z_y z_{4y}\right)_y - z_{yy} z_{4y} \tag{5.3.52}$$

$$z_{yy} z_{4y} = \left(z_{yy} z_{3y}\right)_y - z_{3y} z_{3y} \tag{5.3.53}$$

$$z_y z_{5y} = \left(z_y z_{4y} - z_{yy} z_{3y}\right)_y + z_{3y}^2 \tag{5.3.54}$$

The second term can be modified as:

$$z_y^3 z_{yyy} = \left(z_y^3 z_{yy}\right)_y - 3z_y^2 z_{yy}^2 \tag{5.3.55}$$

With this modification, by omitting exact derivatives and multiplying by 32, the conserved quantity c_4 is written as:

$$\int_{-\infty}^{\infty} \left(8z_{3y}^2 - 20z_y^2 z_{yy}^2 + z_y^6\right) dy = \text{constant} \tag{5.3.56}$$

If (5.3.56) is true, (5.3.57) can be written for this conserved quantity. This equation satisfies the sine-Gordon equation.

$$\left(8z_{3y}^2-20z_y^2z_{yy}^2+z_y^6\right)_t+\left[\left(\frac{2}{3}z_y^4-8z_{yy}^2\right)\cos z\right]_y+\frac{16}{3}\left(z_y^3z_{yy}\right)_{yt}=0 \qquad (5.3.57)$$

$$\left(8z_{3y}^2+z_y^6-20z_y^2z_{yy}^2\right)_t=\left(8z_{yy}^2\cos z-6z_y^4\cos z-16z_{yy}z_y^2\sin z\right)_y \qquad (5.3.58)$$

$$\left(8z_{3t}^2+z_t^6-20z_t^2z_{tt}^2\right)_y=\left(8z_{tt}^2\cos z-6z_t^4\cos z-16z_{tt}z_t^2\sin z\right)_t \qquad (5.3.59)$$

From (5.3.57), the symmetric form (5.3.59) can be deduced.

$$\left(8z_{3t}^2-20z_t^2z_{tt}^2+z_t^6\right)_y+\left[\left(\frac{2}{3}z_t^4-8z_{tt}^2\right)\cos z\right]_t+\frac{16}{3}\left(z_t^3z_{tt}\right)_{ty}=0 \qquad (5.3.60)$$

According to (5.3.60), another conserved quantity is written as:

$$\int_{-\infty}^{\infty}\left(z_t^4-12z_{tt}^2\right)\cos z\,dy=\text{constant} \qquad (5.3.61)$$

As a result, the conserved quantities of SGE are listed as controlled in the book of *Drazin & Johnson "Solitons: an introduction. P. 119".*

$$\int_{-\infty}^{\infty}z_y^2\,dy=\text{constant} \qquad (5.3.62)$$

$$\int_{-\infty}^{\infty}(1-\cos z).dy=\text{constant} \qquad (5.3.63)$$

$$\int_{-\infty}^{\infty} (z_y^4 - 4z_{yy}^2).dy = \text{constant} \tag{5.3.64}$$

$$\int_{-\infty}^{\infty} z_t^2 \cos z \, dy = \text{constant} \tag{5.3.65}$$

$$\int_{-\infty}^{\infty} \left(8z_{3y}^2 - 20z_y^2 z_{yy}^2 + z_y^6\right) dy = \text{constant} \tag{5.3.66}$$

$$\int_{-\infty}^{\infty} \left(z_t^4 - 12z_{tt}^2\right) \cos z \, dy = \text{constant} \tag{5.3.67}$$

5.4 Finding Conserved Quantities of SPE by Transforming from SGE

Transformation rules in chapter 4 can be modified as:

$$z_{yt} = \sin z \quad \text{(SGE)} \tag{5.4.1}$$

$$u_{xt} = u + 1/6(u^3)_{xx} \quad \text{(SPE)} \tag{5.4.2}$$

$$\frac{\partial x}{\partial y} = \cos z = \frac{1}{\sqrt{1+u_x^2}} \tag{5.4.3}$$

$$\frac{\partial x}{\partial t} = -\frac{1}{2} z_t^2 = -\frac{1}{2} u^2 \text{ and } \frac{\partial x}{\partial y} = \cos z \tag{5.4.4}$$

Applying the transformation (3.3.3) and (3.3.4) to the conserved quantities in (5.3.62)-(5.3.67), we can obtain the other conserved quantities of the short pulse equation.

Let us start with (5.3.62). z_y represents the derivative of z with respect to y so we transform it a function of u in terms of x variable. In (5.4.3) we can differentiate both sides with respect to y and we get (5.4.5).

$$-z_y \sin z = -\frac{u_x u_{xy}}{\left(1+u_x^2\right)^{3/2}} \quad (5.4.5)$$

$$u_{xy} = \frac{\partial u_x}{\partial y} = \frac{\partial u_x}{\partial x}\frac{\partial x}{\partial y} = u_{xx} \cos z \quad (5.4.6)$$

With the help of (5.4.6) we can write:

$$-z_y \sin z = -\frac{u_x}{\left(1+u_x^2\right)^{3/2}} u_{xx} \cos z \quad (5.4.7)$$

By trigonometric identities, (5.4.3) can be described by (5.4.8).

$$\tan z = u_x \quad (5.4.8)$$

Then z_y is defined in terms of u and x as:

$$z_y = \frac{u_{xx}}{\left(1+u_x^2\right)^{3/2}} \quad (5.4.9)$$

From (5.4.3), dy can be written as:

$$dy = \sqrt{1+u_x^2}\, dx \quad (5.4.10)$$

Finally, the conserved quantity of SGE in (5.3.62) is transformed for short pulse equation as seen in equation (5.4.11).

$$\int_{-\infty}^{\infty} \frac{1}{2} z_y^2 .dy \Rightarrow \int_{-\infty}^{\infty} \frac{u_{xx}^2}{2(1+u_x^2)^{5/2}} .dx \quad (5.4.11)$$

$$\int_{-\infty}^{\infty} \frac{u_{xx}^2}{(1+u_x^2)^{5/2}} .dx = \text{cons}\tan t \quad (5.4.12)$$

Since (5.4.12) is a conserved quantity for the short pulse equation, it can be verified by mass conservation law in (3.1.1). That is, the time derivative of the function in (5.4.12) must be and exact derivative of any function with respect to x. We tried to verify and found the result as:

$$\left[\frac{u_{xx}^2}{(1+u_x^2)^{5/2}}\right]_t = \left[\frac{u^2}{2}u_{xx}^2.\left(1+u_x^2\right)^{-5/2} - 2\left(1+u_x^2\right)^{-1/2}\right]_x \qquad (5.4.13)$$

The second conserved quantity of SGE is (5.3.63). If cos z and dy in (5.4.3) and (5.4.10) are written in (5.3.63), it is obtained that

$$\int_{-\infty}^{\infty}(1-\cos z).dy \Rightarrow \int_{-\infty}^{\infty}(\sqrt{1+u_x^2}-1).dx \qquad (5.4.14)$$

Therefore another conserved quantity of SPE is:

$$\int_{-\infty}^{\infty}\sqrt{1+u_x^2}.dx \qquad (5.4.15)$$

This quantity was found by mass conservation law in section 3.1 in equation (3.1.13).

Equation (5.3.64) is another conserved quantity of SGE. We have found z_y in (5.4.9). For z_{yy}, we differentiate it with respect to y again.

$$z_{yy} = \frac{\partial z_y}{\partial x}\frac{\partial x}{\partial y} = \left[\frac{u_{xx}}{(1+u_x^2)^{3/2}}\right]_x.\left(1+u_x^2\right)^{-1/2} \qquad (5.4.16)$$

$$z_{yy} = \left[\frac{u_{xxx}}{(1+u_x^2)^{3/2}} - \frac{3u_x u_{xx}^2}{(1+u_x^2)^{5/2}}\right].\left(1+u_x^2\right)^{-1/2} \qquad (5.4.17)$$

$$z_{yy} = \frac{u_{xxx}}{(1+u_x^2)^2} - \frac{3u_x u_{xx}^2}{(1+u_x^2)^3} \qquad (5.4.18)$$

If (5.4.9) and (5.4.18) are replaced (5.3.45), we find a new conserved quantity for SPE as:

$$\int_{-\infty}^{\infty}(z_y^4-4z_{yy}^2).dy \Rightarrow \int_{-\infty}^{\infty}\left(\frac{24.u_x.u_{xx}^2u_{xxx}}{(1+u_x^2)^{9/2}}-\frac{4u_{xxx}^2}{(1+u_x^2)^{7/2}}+\frac{u_{xx}^4(1-36u_x^2)}{(1+u_x^2)^{11/2}}\right).dx \qquad (5.4.19)$$

$$\frac{24u_xu_{xx}^2u_{xxx}}{(1+u_x^2)^{9/2}}=\frac{8u_x}{(1+u_x^2)^{9/2}}\left(u_{xx}^3\right)_x=\left(\frac{8u_x}{(1+u_x^2)^{9/2}}u_{xx}^3\right)_x-8u_{xx}^3\left(\frac{u_x}{(1+u_x^2)^{9/2}}\right)_x$$

$$\left(\frac{u_x}{(1+u_x^2)^{9/2}}\right)_x=\frac{u_{xx}}{(1+u_x^2)^{9/2}}-\frac{9u_x^2u_{xx}}{(1+u_x^2)^{11/2}} \qquad (5.4.20)$$

$$\frac{24u_xu_{xx}^2u_{xxx}}{(1+u_x^2)^{9/2}}=\left(\frac{8u_x}{(1+u_x^2)^{9/2}}u_{xx}^3\right)_x-8u_{xx}^3\left(\frac{u_{xx}}{(1+u_x^2)^{9/2}}-\frac{9u_x^2u_{xx}}{(1+u_x^2)^{11/2}}\right)$$

$$\int_{-\infty}^{\infty}\left[-8u_{xx}^3\left(\frac{u_{xx}}{(1+u_x^2)^{9/2}}-\frac{9u_x^2u_{xx}}{(1+u_x^2)^{11/2}}\right)-\frac{4u_{xxx}^2}{(1+u_x^2)^{7/2}}+\frac{u_{xx}^4(1-36u_x^2)}{(1+u_x^2)^{11/2}}\right]dx \qquad (5.4.21)$$

$$\int_{-\infty}^{\infty}\left(\frac{4u_{xxx}^2}{(1+u_x^2)^{7/2}}+\frac{7u_{xx}^4(1-4u_x^2)}{(1+u_x^2)^{11/2}}\right)dx=\text{constant} \qquad (5.4.22)$$

The proof of (5.4.22) is given by the mass conservation law:

$$\left(\frac{4u_{xxx}^2}{(1+u_x^2)^{7/2}}+\frac{7u_{xx}^4(1-4u_x^2)}{(1+u_x^2)^{11/2}}\right)_t= \left[\frac{2u^2u_{xxx}^2+4u_{xx}^2(1+6u_x^2)+8uu_{xx}^3}{(1+u_x^2)^{7/2}}+\frac{7u^2u_{xx}^4(1-4u_x^2)}{2(1+u_x^2)^{11/2}}\right]_x \qquad (5.4.23)$$

Equation (5.3.65) can also be transformed for SPE by the help of (5.4.3) and (5.4.4). It is very easy to see (5.4.24) is a conserved quantity of SPE.

$$\int_{-\infty}^{\infty}(z_t^2.\cos z).dy \Rightarrow \int_{-\infty}^{\infty}u^2.dx \qquad (5.4.24)$$

$$\int_{-\infty}^{\infty} u^2 .dx = \text{constant} \quad (5.4.25)$$

(5.4.25) has also been obtained by mass conservation law in section 3.1. As for the conserved quantity in (5.3.67), we must find z_{tt} in terms of u, x and t. From (5.4.4) it can be deduced that z_t=u so $z_{tt} = \partial^{-1} u$.

$$\begin{aligned} &z_t(y,t) = u(x,t) \\ &z_{tt} = u_t + u_x \frac{\partial x}{\partial t} = u_t - \frac{1}{2} u^2 u_x \\ &z_{tt} = \partial^{-1} u \end{aligned} \quad (5.4.26)$$

$$\int_{-\infty}^{\infty} (\frac{1}{9} z_t^4 - \frac{4}{3} z_{tt}^2).\cos z \, dy \Rightarrow \int_{-\infty}^{\infty} \left[\frac{u^4}{9} - \frac{4}{3} \left(\partial^{-1} u \right)^2 \right] .dx \quad (5.4.27)$$

$$\int_{-\infty}^{\infty} \left[u^4 - 12 \left(\partial^{-1} u \right)^2 \right] .dx = \text{constant} \quad (5.4.28)$$

(5.4.28) is a conserved quantity of SPE. It is also proven by the mass conservation law as follows.

$$\left[u^4 - 12 \left(\partial^{-1} u \right)^2 \right]_t = \left[\left(\partial^{-2} u \right)^2 + u^6 \right]_x \quad (5.4.29)$$

Finally we derived the last conserved quantity from the transformation by transforming (5.3.66).

$$z_{yyy} = \frac{\partial z_{yy}}{\partial y} = \frac{\partial z_{yy}}{\partial x} \frac{\partial x}{\partial y} = \frac{\partial z_{yy}}{\partial x} \frac{1}{\sqrt{1+u_x^2}} \quad (5.4.30)$$

$$z_{yyy} = \frac{u_{xxxx}}{(1+u_x^2)^{5/2}} - \frac{3u_{xx}^3 + 10 u_x u_{xx} u_{xxx}}{(1+u_x^2)^{7/2}} + \frac{18 u_x^2 u_{xx}^3}{(1+u_x^2)^{9/2}} \quad (5.4.31)$$

z_y, z_{yy} and z_{yyy} are given in (5.4.9), (5.4.18) and (5.4.31) respectively. If they are replaced in (5.3.66), the last conserved quantity is obtained as:

$$\int_{-\infty}^{\infty}\left[\begin{array}{l}8u_{4x}^2 v^{-9/2}+u_{xx}^6\left(1800v^{-13/2}-4500v^{-15/2}+2773v^{-17/2}\right)\\ +u_{xx}^2u_{3x}^2\left(800v^{-11/2}-820v^{-13/2}\right)-160u_x u_{xx}u_{3x}u_{4x}v^{-11/2}\\ u_{xx}^3u_{4x}\left(240v^{-11/2}-288v^{-13/2}\right)+u_x u_{xx}^4u_{3x}\left(3000v^{-15/2}-2400v^{-13/2}\right)\end{array}\right]dx \tag{5.4.32}$$

With some calculations, (5.4.32) can be expressed as:

$$\int_{-\infty}^{\infty}\left[\begin{array}{l}8u_{4x}^2 v^{-9/2}+80u_x u_{3x}^3 v^{-11/2}+12u_{xx}^2u_{3x}^2\left(77v^{-13/2}-60v^{-11/2}\right)\\ +\frac{11}{5}u_{xx}^6\left(1080v^{-13/2}-3276v^{-15/2}+2275v^{-17/2}\right)\end{array}\right]dx=\text{const.} \tag{5.4.33}$$

$$\text{where } v=1+u_x^2$$

The last conserved quantity can be written as derivative form as:

$$\left\{\begin{array}{l}8u_{4x}^2 v^{-9/2}+80u_x u_{3x}^3 v^{-11/2}+12u_{xx}^2u_{3x}^2\left(77v^{-13/2}-60v^{-11/2}\right)\\ +\frac{11}{5}u_{xx}^6\left(1080v^{-13/2}-3276v^{-15/2}+2275v^{-17/2}\right)\end{array}\right\}_t =$$
$$\left\{\begin{array}{l}42u_{xx}^4\left[-40v^{-9/2}+96v^{-11/2}-55v^{-13/2}\right]+40u^2u_x u_{3x}^3v^{-11/2}-864uu_{xx}^5\left(v^{-11/2}+v^{-13/2}\right)\\ \frac{11}{10}u^2u_{xx}^6\left(1080v^{-13/2}-3276v^{-15/2}+2275v^{-17/2}\right)+30fu_x u_{xx}^2u_{3x}v^{-9/2}+4u^2u_{4x}^2v^{-9/2}\\ +u_{3x}^2\left[80uu_{xx}v^{-9/2}+462u^2u_{xx}^2v^{-13/2}+80v^{-7/2}-72v^{-9/2}\right]\end{array}\right\}_x \tag{5.4.34}$$

$$\text{where } v=1+u_x^2$$

CHAPTER 6

CONCLUSION

In this study, we investigated some properties and tried to find some conserved quantities of the Schafer-Wayne short pulse equation. In order to find the solitary properties of the short pulse equation, a traveling wave solution was tried by defining a new parameter z. The parameter z was taken as z=x+ct due to the assumption that the soliton was a left going wave with speed c. After the change of variables, an ordinary nonlinear differential equation was obtained in (4.3.6). Integrating this equation, u_z was obtained as a function of u as in (4.3.11). After integration, z was found as a function of u for a left going soliton with speed c (4.3.15).

Some researchers e.g. Sakovich [5], Victor [18], Parkes [8] have searched its solitary solution by numerical methods. In this study, the shape of its soliton was shown by graphical and analytical methods. When the graph of equation (4.3.15) was drawn, the general soliton shape was obtained as Figure (4.3.4) which is a loop soliton.

The conserved quantities of short pulse equation were investigated by three ways: mass conservation law, from Lax equations and transformation from sine-Gordon equation. By mass conservation law a conserved quantity f, which is a function of u and its derivatives, was calculated directly from the relation (5.1.1).

In (5.1.1) g is also a function of u and its derivatives. Shortly it can be said that if the time derivative of the function $f(u, u_x, u_{xx}, ..)$ is equal to x derivative of another function $g(u, u_x, u_{xx}, ..)$, f is a conserved quantity of the differential eaquation.

It was assumed that f is a function of u_x, then f was calculated by inspection. From this assumption, first conserved quantity in (5.1.27) was found.

In a similar way, f was considered as a function of only u and second conserved quantity was calculated by the help of (5.1.27) as:

As a second method, Lax equation of the short pulse equation was used in order to find the conserved quantities. In this method, Lax equations from Lax pairs were written eigenfunction equation was solved by a series ansatz. These solutions for the series ansatz for different powers of eigenvalues gave the conserved quantities of the short pulse equation. The conserved quantities in (5.2.33)-(5.2.36) was found from Lax pair.

Another method for conserved quantities used in this study is transformation from sine-Gordon equation to short pulse equation which was found by Sakovich and Sakovich [4]. With this method, conserved quantities of sine-Gordon equation were found first by using Lax equations like short pulse equation. After finding three conserved quantities from Lax pair. Three more conserved quantities were found by the symmetrical property of independent variables of the sine-Gordon equation. These six conserved quantities found by sine-Gordon transformation are listed in the appendix.

In conclusion, it is deduced that short pulse equation has a loop soliton solution and six conserved quantities were found. These conserved quantities were confirmed by mass conservation law and some of them are the same with the conserved quantities found by Brunelli who used Hamiltonian methods [7].

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APPENDIX

LIST OF CONSERVED QUANTITIES

$$\int_{-\infty}^{\infty} \sqrt{1+u_x^2}\, dx = \text{const.}$$

$$\int_{-\infty}^{\infty} u^2\, dx = \text{const.}$$

$$\int_{-\infty}^{\infty} \left[u^4 - 12\left(\partial^{-1}u\right)^2 \right] dx = \text{const.}$$

$$\int_{-\infty}^{\infty} \frac{u_{xx}^2}{(1+u_x^2)^{5/2}}\, dx = \text{const.}$$

$$\int_{-\infty}^{\infty} \left[\frac{4u_{xxx}^2}{(1+u_x^2)^{7/2}} + \frac{7u_{xx}^4(1-4u_x^2)}{(1+u_x^2)^{11/2}} \right] dx = \text{const.}$$

$$\int_{-\infty}^{\infty} \left[\begin{array}{l} 8u_{4x}^2 v^{-9/2} + 80u_x u_{3x}^3 v^{-11/2} + 12u_{xx}^2 u_{3x}^2 \left(77v^{-13/2} - 60v^{-11/2}\right) \\ + \frac{11}{5} u_{xx}^6 \left(1080v^{-13/2} - 3276v^{-15/2} + 2275v^{-17/2}\right) \end{array} \right] dx = \text{const.}$$

$$v = 1 + u_x^2$$

CONSERVED QUANTITIES IN THE DERIVATIVE FORM

$$\left[\sqrt{1+u_x^2}\right]_t = \left[\frac{u^2}{2}\sqrt{1+u_x^2}\right]_x$$

$$\left[u^2\right]_t = \left[\left(\partial^{-1}u\right)^2 + \frac{u^2}{4}\right]_x$$

$$\left[\frac{u_{xx}^2}{(1+u_x^2)^{5/2}}\right]_t = \left[\frac{u^2}{2}u_{xx}^2.\left(1+u_x^2\right)^{-5/2} - 2\left(1+u_x^2\right)^{-1/2}\right]_x$$

$$\left[u^4 - 12\left(\partial^{-1}u\right)^2\right]_t = \left[\left(\partial^{-2}u\right)^2 + u^6\right]_x$$

$$\left(\frac{4u_{xxx}^2}{(1+u_x^2)^{7/2}} + \frac{7u_{xx}^4(1-4u_x^2)}{(1+u_x^2)^{11/2}}\right)_t = \left[\frac{2u^2u_{xxx}^2 + 4u_{xx}^2(1+6u_x^2) + 8uu_{xx}^3}{(1+u_x^2)^{7/2}} + \frac{7u^2u_{xx}^4(1-4u_x^2)}{2(1+u_x^2)^{11/2}}\right]_x$$

$$\left\{\begin{aligned}&8u_{4x}^2v^{-9/2} + 80u_xu_{3x}^3v^{-11/2} + 12u_{xx}^2u_{3x}^2\left(77v^{-13/2} - 60v^{-11/2}\right)\\ &+\frac{11}{5}u_{xx}^6\left(1080v^{-13/2} - 3276v^{-15/2} + 2275v^{-17/2}\right)\end{aligned}\right\}_t =$$

$$\left\{\begin{aligned}&42u_{xx}^4\left[-40v^{-9/2} + 96v^{-11/2} - 55v^{-13/2}\right] + 40u^2u_xu_{3x}^3v^{-11/2} - 864uu_{xx}^5\left(v^{-11/2} + v^{-13/2}\right)\\ &\frac{11}{10}u^2u_{xx}^6\left(1080v^{-13/2} - 3276v^{-15/2} + 2275v^{-17/2}\right) + 30fu_xu_{xx}^2u_{3x}v^{-9/2} + 4u^2u_{4x}^2v^{-9/2}\\ &+u_{3x}^2\left[80uu_{xx}v^{-9/2} + 462u^2u_{xx}^2v^{-13/2} + 80v^{-7/2} - 72v^{-9/2}\right]\end{aligned}\right\}_x$$

$$v = 1 + u_x^2$$