ON THE ALGEBRAIC STRUCTURE OF RELATIVE HAMILTONIAN DIFFEOMORPHISM GROUP

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ABSTRACT

ON THE ALGEBRAIC STRUCTURE OF RELATIVE HAMILTONIAN DIFFEOMORPHISM GROUP

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Let M be smooth symplectic closed manifold and L a closed Lagrangian submanifold of M. It was shown by Ozan that Ham(M,L): the relative Hamiltonian diffeomorphisms on M fixing the Lagrangian submanifold L setwise is a subgroup which is equal to the kernel of the restriction of the flux homomorphism to the universal cover of the identity component of the relative symplectomorphisms.

In this thesis we show that Ham(M,L) is a non-simple perfect group, by adopting a technique due to Thurston, Herman, and Banyaga. This technique requires the diffeomorphism group be transitive where this property fails to exist in our case.

Keywords: Hamiltonian Diffeomorphisms, nontransitive diffeomorphism groups, Lagrangian submanifolds

ÖZ

RÖLATİF HAMİLTON DİFEOMORFİZMALARIN CEBİRSEL YAPISI

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M simplektik bir manifold ve L, M'in kapalı bir Lagrange alt manifoldu olsun. L'i kümece sabit brakan M üzerindeki Hamilton difeomorfizmalarının oluşturduğu Ham(M,L) kümesinin, akı homomorfizmasının, rölatif simplektomorfizmaların birim bileşenine kısıtlanışının çekirdek grubu olduğu Ozan tarafından 2005 yılında gösterilmişti. Bu tezde, Ham(M,L) grubunun basit olmayan mükemmel bir grup olduğu, Thurston, Herman ve Banyaga tarafından geliştirilmiş bir tekniğin uyarlanmasıyla gösterilmiştir. Teknik, grubun transitif olmasını gerektirirken Ham(M,L) grubu transitif değildir.

Anahtar Kelimeler: Hamilton difeomorfizmaları, transitif olmayan gruplar, Lagrange alt manifoldları

To Sultan of my heart, for love and patience...

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CHAPTER 1

INTRODUCTION

From the time that Poincare suggested the notion of a differentiable manifold as the phase space in mechanics, there have been many developments in mathematical physics. The modeling for mechanics, consisting of a symplectic manifold together with a Hamiltonian vector field (or global system of differential equations preserving the symplectic structure) replaced analytical methods by differential topological ones in the study of the phase portrait. This modeling had its own settings and developed the theory of Symplectic Topology.

Just as symplectic manifolds stood for configuration (or phase) spaces of Hamiltonian Mechanical Systems, there are some infinite dimensional Lie groups that take the roles of configuration spaces in fluid dynamics, plasma physics or in quantization. In this respect diffeomorphism groups and their subgroups play an important role in dynamical systems both as phase spaces and as symmetry groups. For example the configuration space of a homogeneous ideal fluid contained in a container M is $\mathrm{Diff}(M)$: the group of self diffeomorphisms on M, and if the fluid is incompressible it is $\mathrm{Diff}_{vol}(M)$: the group of volume preserving diffeomorphism of M to itself. Another set of examples come from the Maxwell-Vlasov equations of plasma physics, which is an infinite dimensional system on a space of symplectomorphisms.

The study of diffeomorphism groups and some certain subgroups may be classified in many different aspects. Most basically one can consider geometric or algebraic properties or their homotopy types.

In this thesis we deal with some algebraic properties of a subgroup of the diffeomorphism group; namely the relative Hamiltonian diffeomorphisms that leave a lagrangian submanifold invariant on a closed symplectic manifold. The main result of this thesis is that this group is perfect. We adopt a technique due to Thurston, Herman and Banyaga that was used to show that some of these groups are simple or at least perfect. We made inevitable modifications to overcome the lack of some properties in the relative group.

1.1 From Perfectness to Simplicity

One of the main problems in the study of diffeomorphism groups is whether a group of diffeomorphisms and/or its certain subgroups is perfect, simple. The simplicity of the commutator subgroup $[\mathrm{Diff}_0^r(M), \mathrm{Diff}_0^r(M)]$ of $\mathrm{Diff}_0^r(M)$ was shown by Epstein in 1970 [6]. Whether the group $\mathrm{Diff}_0^r(M)$ is perfect is a harder question and was shown for $M = T^n,\ r = \infty$ by Herman in 1971 [8]. Then Thurston used this result to generalize it for arbitrary manifolds [22]. He also developed a machinery, now called the "Thurston tricks", to yield the simplicity of $\mathrm{Diff}_0^\infty(M)$. Here we will briefly explain this technique.

Remark 1.1. *Although we consider closed manifolds, i.e. compact without boundary, most of the results are valid on a non-compact manifold if one replaces diffeomorphisms or isotopies by the compactly supported ones. We will make it explicit if even such considerations are not sufficient.*

We first start with some definitions. Let $G \subseteq \mathrm{Diff}^\infty(M)$ be a group of diffeomorphisms.

Definition 1.2. 1. G is said to have the fragmentation property if for any open cover $\mathcal{U} = \{U_i\}_{i \in I}$ and any $g \in G$, there are $g_1,\ g_2, ..., g_s \in G$ with $supp(g_k) \subset U_{i(k)}$ for $k = 1, .., s$ where $i(k) \in I$ and $g = g_1 g_2 ... g_s$.

2. G is said to be strongly n-fold transitive if given a pair $(x_1, ..., x_n)$, $(y_1, ..., y_n)$ of distinct points on M, there exist n diffeomorphisms $g_i \in G$ such that $g_i(x_i) = y_i,\ i = 1, .., n$ and $supp(g_i) \cap supp(g_j) = \emptyset$, $\forall i \neq j$.

3. The subgroup $G_U \subseteq G$ denotes the subgroup consisting of diffeomorphisms which have compact support in the open subset U of M.

Theorem 1.3. *[22](Thurston's trick) Let $G \subseteq \mathrm{Diff}_0^\infty(M)$ be a group of diffeomorphisms which is strongly 2-fold transitive, has the fragmentation property and such that G_U is perfect for each open set $U \subset M$. Then G is simple.*

Remark 1.4. *1. Since the original paper of Thurston is unpublished one can find the proof of this theorem, for instance, in [2].*

2. There is a more direct and shorter proof for the simplicity of $\mathrm{Diff}_0^r(M)$, for $1 \leq r \leq \infty$, due to Epstein and Mather [7, 12]. Since their method involves shrinking and expanding volume of subsets , it can not be generalized to volume-preserving and symplectic diffeomorphisms.

For the rest of this section let G denote the identity component of C^∞ diffeomorphisms on a closed smooth manifold M, i.e. $G = \mathrm{Diff}_0^\infty(M)$. One can easily verify that G is strongly 2-fold transitive and has the fragmentation property. See [2]. As Theorem 1.3 suggests, to show the simplicity of G we need to show perfectness of G_U. We now briefly explain how G_U is shown to be perfect. Recall that the perfectness of a topological group G is equivalent to $H_1(G)$ being trivial, in other words

$$H_1(G) = G/[G, G] = 0 \text{ if and only if } G \text{ is perfect.}$$

Let U be an open subset of a smooth manifold M such that $\bar{U} \subset V$, where V is the domain of a local chart $\phi : V \to \mathbb{R}^n$. The conjugation of elements of G_U by the charts ϕ induces an isomorphism $G_U \approx \mathrm{Diff}_0^\infty(\mathbb{R}^n)$. In other words the local picture of diffeomorphism groups are the same as diffeomorphism group of $\mathbb{R}^n$. Moreover, this local picture is exactly the same picture of the whole group of diffeomorphisms if one considers the abelianizations.

Lemma 1.5. *(Deformation Lemma) Let U be an open subset of a smooth manifold M^n such that $\bar{U} \subset V$, where V is the domain of a local chart $\phi : V \to \mathbb{R}^n$. Then the inclusion $G_U \subset G$ induces an isomorphism: $H_1(G_U) \approx H_1(G)$.*

To conclude that $\mathrm{Diff}_0^\infty(M^n)$ is simple, as the above lemma suggests, we need to find just one manifold for which $\mathrm{Diff}_0^\infty(M^n)$ is perfect. Herman's result that $\mathrm{Diff}_0^\infty(T^n)$ is perfect proves that $\mathrm{Diff}_0^\infty(M^n)$ is simple. We will need a similar result with Herman's and come back to this again in the last chapter.

1.2 Main Results

Let (M, ω) be a connected, closed, symplectic manifold and L a closed Lagrangian submanifold. Let $\mathrm{Symp}(M, L, \omega)$ denote the subgroup of $\mathrm{Symp}(M, \omega)$ consisting of symplectomorphisms leaving the Lagrangian submanifold L invariant and $\mathrm{Symp}_0(M, L, \omega)$ the path component of $\mathrm{Symp}(M, L, \omega)$ containing the identity. The relative symplectomorphism group was introduced by Ozan [16]. In this paper he defined a relative version of the flux homomorphism:

$$\mathrm{Flux} : \widetilde{\mathrm{Symp}}_0(M, L, \omega) \to H^1(M, L, \mathbb{R})$$

and showed that its kernel gives the relative Hamiltonian group $\widetilde{\mathrm{Ham}}(M, L)$, similar to the absolute case.

In this thesis, we will examine some topological and algebraic aspects of these relative groups. The main result will be the perfectness of the group of relative

Hamiltonian diffeomorphism.

Theorem 1.6. *Let M be a connected, closed, symplectic manifold, $L \subset M$ a connected, oriented Lagrangian submanifold such that $M \backslash L$ is connected. Then,* $\mathrm{Ham}(M, L)$ *is a non-simple perfect group.*

In the third chapter we analyze this group in detail. We outline some topological properties of this group first discovered by Ozan.

The relative versions of deformation lemma and the ideas to overcome the lack of transitivity is explained in the fourth chapter. Mainly we prove

Theorem 1.7. *The natural map $\rho : B\overline{\mathrm{Ker} R_{U,U\cap L}} \longrightarrow B\overline{\mathrm{Ham}(M, L)}$ induces an isomorphism*

$$\phi : H_1(B\overline{\mathrm{Ker} R_{U,U\cap L}}; \mathbb{Z}) \longrightarrow H_1(B\overline{\mathrm{Ham}(M, L)}; \mathbb{Z})$$

Here $R_{U,U\cap L}$ denotes the Calabi homomorphism defined for open subsets of closed manifolds, whose details are given in the third chapter. The topological space $B\overline{G}$ of a group G will be reviewed in the appendix. We remark here the relation $H_1(B\overline{G}) \approx \tilde{G}/[\tilde{G}, \tilde{G}] \approx H_1(\tilde{G})$ and address the appendix for the details.

The last chapter involves a proof that $\mathrm{Ham}(T^{2n}, T^n)$ is perfect. We adopt the original proof of Herman's for the absolute case. This is a fundamental step on the way to perfectness of all relative Hamiltonian groups.

CHAPTER 2

PRELIMINARIES

In this chapter we recall some basic properties of classical diffeomorphism groups. We follow mainly Banyaga [2] and Schmid [20].

2.1 The Lie group of C^r-diffeomorphisms

Let M, N be finite dimensional smooth manifolds. Denote by $C^r(M, N)$ the space of all C^r mappings $f : M \to N$, for $1 \leq r \leq \infty$. The topological and smooth structure of the diffeomorphism groups are induced from those on $C^r(M, M)$: C^r automorphisms of the manifold. So we recall these structures on $C^r(M, N)$ first.

Definition 2.1. A C^r diffeomorphism of a smooth manifold M is an invertible element $\phi \in C^r(M, M)$ such that ϕ^{-1} is a C^r map.

The set of all C^r diffeomorphisms is denoted by $\mathrm{Diff}^r(M)$. The composition of mappings in $C^r(M, M)$ gives $\mathrm{Diff}^r(M)$ a group structure and we have the natural inclusions:

$$\mathrm{Diff}^1(M) \supset \mathrm{Diff}^2(M) \supset \cdots \supset \mathrm{Diff}^r(M) \supset \cdots \supset \mathrm{Diff}^\infty(M)$$

where $\mathrm{Diff}^\infty(M)$ is the group of all C^∞ diffeomorphisms. Indeed, $\mathrm{Diff}^r(M)$ is a topological group with compact open topology induced from $C^r(M, M)$.

2.1.1 The compact-open C^r topology

Let $f \in C^r(M,N)$ with $r \leq \infty$. Let (U,ϕ) and(V,ψ) be local charts of M and N respectively such that $f(K) \subset V$, where K is some compact subset of U. For $\epsilon \geq 0$ define $\mathcal{N}^r(f,(U,\phi),(V,\psi),\epsilon,K)$ to be the set of all $g \in C^r(M,N)$ such that $g(K) \subset V$ and if

$$\overline{f} = \psi f \phi^{-1}, \quad \overline{g} = \psi g \phi^{-1}$$

then:

$$\|D^k \overline{f}(x) - D^k \overline{g}(x)\| < \epsilon$$

for all $x \in \phi(K)$ and $0 \leq k \leq r$.

The sets $\mathcal{N}^r(f,(U,\phi),(V,\psi),\epsilon,K)$ form a subbasis for a topology on $C^r(M,N)$, called the compact-open C^r topology. A neighborhood of f in this topology is any finite intersection of the sets of type $\mathcal{N}^r(f,(U,\phi),(V,\psi),\epsilon,K)$. The C^∞ compact-open topology on $C^\infty(M,N)$ is the topology induced by the inclusions $C^\infty(M,N) \subset C^r(M,N)$ for r finite.

The group $\mathrm{Diff}^r(M)$ with this topology is a topological group. It is metrizable and has a countable basis [10]. However, if M is not compact there is no control of what happens at "infinity". We have to restrict to mappings with compact supports to overcome this difficulty. Then, the topology on the group $\mathrm{Diff}^r_c(M)$ of C^r diffeomorphisms with compact supports is given by the direct limit topology induced from $C^r_c(M,M)$ for $r \leq \infty$.

2.1.2 Smooth structure on $\mathrm{Diff}^r(M)$

After the work of Arnold ([1]), showing that, if we assume that the group of diffeomorphisms has the properties of a Lie group we can use these properties to get a better understanding of hydrodynamics, people have tried to show that

$\mathrm{Diff}^r(M)$ is an infinite dimensional Lie group. Especially Omori's studies enlighted the theory in this direction. The details can be found in [2, 20].

Let $f \in C^r(M,N)$ and $\gamma : I \subset \mathbb{R} \to C^r(M,N)$ be a curve with $\gamma(0) = f$. $\dot{\gamma}(0)$ will be a tangent vector to $C^r(M,N)$ at the point f, i.e.

$$\dot{\gamma}(0) = \frac{d\gamma(t)}{dt}|_{t=0} \in T_f C^r(M,N).$$

This should be interpreted as follows: For each $x \in M$, let $\gamma_x : I \subset \mathbb{R} \to N$ be the curve in N given by $\gamma_x(t) = \gamma(t)(x)$. Then $\gamma_x(0) = f(x)$ and $\dot{\gamma}(0) \in T_{f(x)}N$, in other words $\dot{\gamma}(0)$ is a tangent vector to N at the point $f(x)$. Identify $\dot{\gamma}(0) \equiv \dot{\gamma}(0)(x)$; hence $\dot{\gamma}(0)$ can be regarded as a map $\dot{\gamma}(0) : M \to TN$ such that $\dot{\gamma}(0) \in T_{f(x)}N$. This means $\dot{\gamma}(0)$ is a vector field along f. Thus the tangent space of $C^r(M,N)$ at f is

$$T_f C^r(M,N) = \{\xi_f \in C(M,TN) \mid \tau_N \circ \xi_f = f\}.$$

Here $\tau_N : TN \to N$ is the canonical projection. We can identify $T_f C^r(M,N)$ with the space $\Gamma^r(f^*\tau_N)$ of sections of the pull-back bundle $f^*\tau_N$; i.e.
$T_f C^r(M,N) \cong \Gamma^r(f^*\tau_N)$ which is an infinite dimensional vector space.

Remark 2.2. *If $r = \infty$ then the space $\Gamma^\infty(f^*\tau_N)$ of C^∞ sections with the uniform C^∞-topology is a Frechet space; (a metrizable topological vector space), defined by the sequence of seminorms $(|\ \ |_p)_{p \in \mathbb{N}}$*

$$|\xi|_p = \max_{\substack{0 \le i \le p \\ 0 \le j \le p}} \sup_{x \in U_i} \|D^j \xi_i(x)\| < \infty.$$

Here ξ_i is the local representative of $\xi \in \Gamma^\infty(f^*\tau_N)$ in a chart U_i of M. If $0 \le r < \infty$ then $\Gamma^r(f^*\tau_N)$ is a Banach space with norm $\|\xi\| = max_{0 \le p < r}|\xi|_p$, with $|\xi|_p$ as above. So to do analysis on $\mathrm{Diff}^\infty(M)$ one has to refine things. But to a topologist this fact is not too much annoying. Structures in which a kind of inverse function theorem works, namely Nash-Moser type implicit function *theorem, still do exist. Omori presents* $\mathrm{Diff}^\infty(M)$ *as an ILH (inverse limit of a Hilbert space) manifold.*

To define the local charts $\Phi_f : \mathcal{V}(f) \subset C^r(M,N) \longrightarrow W \subset T_f C^r(M,N)$ around a neighborhood $\mathcal{V}(f)$ of $f \in C^r(M,N)$, we first start with choosing a Riemannian metric g on N in order to get an exponential mapping $exp_x : U_x \subset T_x N \to N$ on some neighborhood U_x of zero in $T_x N$. For each $v_x \in T_x N$, there is a unique geodesic α_x through x whose tangent vector at x is v_x, i.e. $\alpha_x(0) = x$ and $\dot{\alpha_x}(0) = v_x$. Then define

$$exp_x(v_x) := \alpha_x(1), \quad v_x \in T_x N$$

In general exp_x is a local diffeomorphism from a neighborhood of $0 \in T_x N$ onto a neighborhood of $x \in N$; i.e. there is an open ball $D_x^{\lambda(x)} \subset U_x$ centered at 0 with radius $\lambda(x)$ onto an open neighborhood $\mathcal{N}_x$ of x in N. There exists a $\delta(x) \geq 0$ such that $\mathcal{N}_x \subset B(x, \delta(x))$, where $B(x, \delta(x))$ is the d-ball in N centered at x with radius $\delta(x)$. Here d is the metric on N induced by the Riemannian metric g. If N is compact, there exists a uniform δ and a uniform λ such that for any $x \in N$ $exp_x(D_x^\lambda) \subset B(x, \lambda)$. Moreover exp_x can be extended to a map $exp : TN \to N$ such that the map

$$Exp := (\tau_N, exp) : TN \to N \times N, \quad Exp(v_x) = (x, exp_x(v_x))$$

is a diffeomorphism from a neighborhood $\mathcal{O}(0)$ of the zero section in TN onto a neighborhood $\mathcal{U}(\Delta)$ of diagonal $\Delta \subset N \times N$.

Let $\mathcal{V}(f) = \{g \in C^r(M,N) | \sup_{x \in N} d(f(x), g(x)) \leq \delta\}$. This defines a C^0 neighborhood of f. Note that if f is the identity map $id : M \to M$ then

$$\mathcal{V}(id_M) = \{f \in C^r(M,M) \mid graph(f) \subset \mathcal{U}(\Delta)\}$$

For any $g \in \mathcal{V}(f)$, define $\Phi_f(g) \in T_f C^r(M,N)$ by

$$\Phi_f(g) = exp_{f(x)}^{-1}(g(x)),$$

which is a bijection of $\mathcal{V}(f)$ with an open neighborhood $\mathcal{W}$ of $0 \in T_f C^r(M, N)$, for all $x \in M$. Its inverse is given by

$$\begin{aligned} \overline{exp} : \mathcal{W} \quad \subset T_f C^r(M,N) \quad &\longrightarrow \quad C^r(M,N) \\ \rho \quad &\longmapsto \quad exp \circ \rho \end{aligned}$$

which is a homeomorphism, hence is a local chart. One can show that the transition map between two overlapping charts is "smooth" [20].

Example 2.3.

1. For the case $N = M$ and $f = id_M$, $T_f C^r(M, N)$ is just the set of C^r vector fields on M. Hence $\mathrm{Diff}^r(M)$ is a smooth manifold modeled on $\chi^r(M)$ of C^r vector fields on M which is a Banach space. If $r = \infty$, $\mathrm{Diff}^\infty(M)$ is still a manifold on $\chi(M)$ of C^∞ vector fields on M, however the latter is a Frechet space as we mentioned in Remark 2.2.

2. Let $R_\alpha \in \mathrm{Diff}_0^\infty(T^n)$ denote the rotation by $\alpha \in T^n$. If $\pi : \mathbb{R}^n \to T^n$ denotes the covering map and $\tilde{\beta} \in \mathbb{R}^n$ is a lift of $\beta \in T^n$, then $R_\alpha(v) = \pi(\tilde{\alpha} + \tilde{v})$. Let $\lambda \in T^n$ be close enough to α so that $R_\lambda \in \mathcal{V}(R_\alpha)$. If $\Phi_{R_\alpha} : \mathcal{V}(R_\alpha) \to T_{R_\alpha} C^\infty(T^n, T^n)$ denotes the chart near R_α, then $\Phi_{R_\alpha}(R_\lambda) : T^n \to \mathbb{R}^n$ is the map given by $\Phi_{R_\alpha}(R_\lambda)(x) = \tilde{x} + \tilde{\lambda} - \tilde{\alpha}$.

Proposition 2.4. $\mathrm{Diff}^r(M)$ *and* $\mathrm{Diff}_c^r(M)$ *are locally contractible. Hence they are locally connected by smooth arcs.*

Although $\mathrm{Diff}^r(M)$ is an infinite dimensional Lie group with Lie algebra $\chi^r(M)$, the nice relations between a finite dimensional Lie group G and its Lie algebra $\boldsymbol{g}$ may fail to exist for $\mathrm{Diff}^r(M)$. For instance, the exponential mapping is neither one-to-one nor onto near the identity. (See [2, pp.8-9] for examples). It is still important to know the structure of the Lie algebra, because we can use this knowledge to construct interesting 1-parameter groups inside the Lie group and deduce information about the structure of the entire Lie group.

It is well-known that any C^∞ vector field X on M with compact support generates a flow $\phi_t \in \mathrm{Diff}_c^\infty(M)$. We get the family of diffeomorphisms ϕ_t as the trajectories of the time-dependent differential equation:

$$\frac{d}{dt}\phi_t(x) = X(\phi_t(x)), \quad \phi_0(x) = x.$$

The diffeomorphism ϕ_1 is called the time one map of the flow. The correspondence $X \mapsto \phi_1$ is a well defined map $Exp : \chi_c(M) \to \mathrm{Diff}_c^\infty(M)$ called the exponential map of the Lie group $\mathrm{Diff}_c^\infty(M)$. When a smooth manifold M is equipped with some interesting geometric structure, there exists a distinguished class of vector fields which generate a local 1-parameter group of diffeomorphisms preserving the structure. If M is oriented, for instance, $\mathrm{Diff}_+(M) \subset \mathrm{Diff}_c^\infty(M)$ is the subgroup of orientation preserving diffeomorphisms on M. For a fixed volume form ω on M $\mathrm{Diff}_\omega(M)_c$ is the group of volume preserving diffeomorphisms with compact support, i.e. $\mathrm{Diff}_\omega(M)_c = \{\phi \in \mathrm{Diff}_c^\infty(M) \| \ \phi^*\omega = \omega\}$. The group of diffeomorphisms that preserve the symplectic structure on a symplectic manifold M is the next important set of examples.

2.2 The Group of Symplectomorphisms

Let (M^{2n}, ω) be a symplectic manifold, i.e. ω is a closed 2-form such that ω^n ($\neq 0$) is a volume form on M. The group of symplectomorphisms

$$\mathrm{Symp}(M, \omega) = \{\phi \in \mathrm{Diff}^\infty(M) \mid \phi^*\omega = \omega\}$$

is of fundamental importance for the study of Symplectic Topology (in addition to its role in plasma physics). For example the symplectic rigidity theorem, being the basis of symplectic topology, states that "$\mathrm{Symp}(M, \omega)$ is C^0-closed in $\mathrm{Diff}(M)$". Or, for instance, one can consider the Arnold conjecture which estimates bounds for the fixed points of Hamiltonian symplectomorphisms.

2.2.1 Symplectic Manifolds

In this section we first recall some fundamentals of Symplectic Topology, basic definitions and examples. For more details one can see McDuff and Salamon's book [14] or da Silva's survey [5]. The followings are the classical examples of symplectic manifolds.

Example 2.5.
1. Let $M = \mathbb{R}^{2n}$ with the coordinates $(x_1, ..., x_n, y_1, ..., y_n)$. Then the 2-form defined by $\omega_{st} = \sum_{i=1}^{n} dx_i \wedge dy_i$ is called the standard symplectic form on $\mathbb{R}^{2n}$.

2. The 2-sphere S^2, with the standard symplectic form on S^2 is induced by the standard inner (dot) and exterior (vector) products: $\omega_p(u, v) :=< p, u \times v >$, for $u, v \in T_pS^2 = \{p\}^{\perp}$. This is the standard area form on S^2 with total area 4π. In terms of cylindrical polar coordinates $0 \leq \theta < 2\pi$ and $-1 \leq z \leq 1$ away from the poles, it is written $\omega = d\theta \wedge dz$. Since $[\omega] \in H^2(M, \mathbb{R})$ is a non-zero class for a symplectic manifold (M, ω), S^n is symplectic only for $n = 2$.

3. **(Cotangent Bundles)** Let $(U, x_1, .., x_n)$ be a coordinate chart of a smooth manifold M such that $(T^*U, x_1, .., x_n, \xi_1, .., \xi_n)$ becomes a coordinate chart for T^*M: the cotangent bundle of M. Then, if $\alpha = \sum_{i=1}^{n} \xi_i dx_i$ is the Liouville 1-form, the canonical symplectic form on the cotangent bundle is given by $\omega_{can} = -d\alpha = \sum_{i=1}^{n} dx_i \wedge d\xi_i$.

4. Let (M, ω) be a symplectic manifold. Then the product manifold $M \times M$ is also symplectic with the symplectic form $(-\omega) \oplus (\omega)$.

Symplectic manifolds have special submanifolds that arise naturally both in physics and geometry.

Definition 2.6. $L^n \subset M^{2n}$ is a Lagrangian submanifold of a symplectic manifold

(M, ω), if $\omega|_{TL} = 0$.

Example 2.7. 1. $L = \{(x_1, .., x_n, 0, .., 0) \mid x_i \in \mathbb{R}\} \subset \mathbb{R}^{2n}$ is a Lagrangian submanifold.

2. Any 1-dimensional submanifold of a symplectic surface is Lagrangian.

3. The zero section $M_0 = \{(x, \xi) \in T^*M \mid \xi = 0 \text{ in } T_x^*M\}$ diffeomorphic to M is a Lagrangian submanifold of the cotangent bundle of any smooth manifold M. Hence any smooth manifold is a Lagrangian submanifold!

4. The diagonal $\Delta = \{(p, p) \mid p \in M\} \subset (M \times M, (-\omega) \oplus (\omega))$ diffeomorphic to M is a Lagrangian submanifold. Indeed, this is a particular case of the following, which is due to Weinstein.

Theorem 2.8. *[23] Let (M, ω) be a symplectic manifold and $\psi : M \to M$ be a diffeomorphism. Then ψ is a symplectomorphism if and only if its graph*

$$\text{graph}(\psi) = \{(p, \psi(p)) | \; p \in M\} \subset M \times M$$

is a Lagrangian submanifold of $(M \times M, (-\omega) \oplus (\omega))$.

Example 2.7.4 is the case $\psi = id_M$ of the above theorem. The following result, due to Weinstein, classifies Lagrangian embeddings up to local symplectomorphism.

Theorem 2.9. ***(Weinstein Tubular Neighborhood Theorem)*** *Let (M, ω) be a symplectic manifold, L a compact Lagrangian submanifold, ω_{can} the canonical symplectic form on T^*L, $i_0 : L \hookrightarrow T^*L$ the Lagrangian embedding as the zero section, and $i : L \hookrightarrow M$ the Lagrangian embedding given by the inclusion. Then there are neighborhoods U_0 of L in T^*L, U of L in M, and a diffeomorphism $\varphi : U_0 \to U$ such that $\varphi^*\omega = \omega_{can}$ and $\varphi \circ i_0 = i$.*

2.2.2 Symplectic and Hamiltonian Diffeomorphisms

$\mathrm{Symp}(M, \omega)$ is by definition equipped with C^∞-topology and as first observed by Weinstein in [23] it is locally path connected. Let $\mathrm{Symp}_0(M, \omega)$ denote the path component of $id_M \in \mathrm{Symp}(M, \omega)$. For any $\psi \in \mathrm{Symp}_0(M, \omega)$, let $\psi_t \in \mathrm{Symp}(M, \omega)$ for all $t \in [0, 1]$, such that $\psi_0 = id_M$ and $\psi_1 = \psi$. There exists a unique family of vector fields (corresponding to ψ_t)

$$X_t : M \longrightarrow TM \quad \text{such that} \quad \frac{d}{dt}\psi_t = X_t \circ \psi_t. \tag{2.1}$$

The vector fields X_t are called symplectic since they satisfy $\mathcal{L}_{X_t}\omega = 0$, where $\mathcal{L}_{X_t}\omega$ denotes the Lie derivative of the form ω along the vector field X_t. By Cartan's formula

$$\mathcal{L}_{X_t}\omega = i_{X_t}(d\omega) + d(i_{X_t}\omega).$$

Hence X_t is a symplectic vector field if and only if $i_{X_t}\omega$ is closed for all t. If moreover $i_{X_t}\omega$ is exact, that is to say $i_{X_t}\omega = dH_t$, $H_t : M \to \mathbb{R}$ a family of smooth functions, then X_t are called Hamiltonian vector fields. In this case the corresponding diffeomorphism ψ is called a Hamiltonian diffeomorphism and H_1 is a Hamiltonian for ψ. The Hamiltonian diffeomorphisms form a group as a subgroup in the identity component of the group of symplectomorphisms, $\mathrm{Ham}(M, \omega) \subseteq \mathrm{Symp}_0(M, \omega)$. Thus we have a sequence of groups and inclusions:

$$\mathrm{Ham}(M) \hookrightarrow \mathrm{Symp}_0(M) \hookrightarrow \mathrm{Symp}(M) \hookrightarrow \mathrm{Diff}^\infty(M)$$

The most important elementary theorems in Symplectic Topology, the Darboux's theorem and the Moser's theorem, tell us the first observations about the groups $\mathrm{Ham}(M)$ and $\mathrm{Symp}(M)$.

Theorem 2.10. ***(Darboux's Theorem)*** *Every symplectic form is locally diffeomorphic to the standard symplectic form $\omega_{st} = \sum_{i=1}^{n} dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$.*

Theorem 2.11. ***(Moser's Theorem)*** *Any path $\omega_t,\ t \in [0,1]$ of cohomologous symplectic forms on a closed manifold M is induced by an isotopy $\Phi_t : M \to M$ of the underlying manifold, i.e. $\Phi_t^*(\omega_t) = \omega_0$ $\Phi_0 = id$.*

Since there are no local invariants of symplectic structures, by Darboux's theorem, these groups are infinite dimensional. Due to Moser's theorem $\mathrm{Symp}(M,\omega)$ and $\mathrm{Ham}(M,\omega)$ depend only on the diffeomorphism class of the form ω. In fact when ω_t varies along a path of cohomologous forms the topological or algebraic properties of these groups do not change .

2.2.3 Ham(M) and Flux Homomorphism

Since Hamiltonian diffeomorphisms can not be described as diffeomorphisms preserving some certain structure, there are some complications that one encounters while studying this group. For instance, it was not known until very recently that the Flux conjecture is true:

Flux Conjecture: On a closed, symplectic manifold the limit of a C^∞ convergent sequence of Hamiltonian diffeomorphisms in $\mathrm{Symp}(M,\omega)$ is a Hamiltonian diffeomorphism.

The proof of this theorem is due to Ono [15]. There is a useful characterization of Hamiltonian diffeomorphisms as the kernel of a group homomorphism and the name of the above conjecture will now be appearent.

Definition 2.12. Let ψ_t, $t \in [0,1]$, be a loop of symplectomorphisms on a smooth symplectic manifold M. Then the flux homomorphism $\widetilde{\mathrm{Flux}} : \widetilde{\mathrm{Symp}_0}(M,\omega) \to H^1(M,\mathbb{R})$ is given by

$$\widetilde{\mathrm{Flux}}(\{\psi_t\}) = \int_0^1 [i_{X_t}\omega]dt \ \in H^1(M,\mathbb{R}),$$

where X_t is defined by $\frac{d}{dt}\psi_t = X_t \circ \psi_t$.

Recall that the universal cover of a space is just the set of equivalence classes of paths in that space with fixed ends. The notation $\{\psi_t\}$ denotes the equivalence class of homotopic isotopies that have fixed ends $\psi_0 = id,\ \psi_1 = \psi$. Further, if we let

$$\Gamma = \widetilde{\mathrm{Flux}}(\pi_1(\mathrm{Symp}_0(M,\omega))) \subset H^1(M,\mathbb{R})$$

then, $\widetilde{\mathrm{Flux}}$ induces a well defined homomorphism on $\mathrm{Symp}_0(M,\omega)$, also called the Flux:

$$\mathrm{Flux} : \mathrm{Symp}_0(M,\omega) \to H^1(M,\mathbb{R})/\Gamma.$$

The group Γ is called the Flux group, and the flux conjecture is equivalent to Γ being discrete [18]. The following theorem due to Banyaga exhibits the relation between Ham(M) and the Flux homomorphism.

Theorem 2.13. *Let $\psi \in \mathrm{Symp}_0(M)$. Then ψ is a Hamiltonian symplectomorphism if and only if there exists a symplectic isotopy $\psi_t \in \mathrm{Symp}_0(M),\ t \in [0,1]$ such that $\psi_0 = id,\ \psi_1 = \psi,$ Flux$(\{\psi_t\}) = 0$. Moreover, if Flux$(\{\psi_t\}) = 0$ then $\{\psi_t\}$ is isotopic with fixed end points to a Hamiltonian isotopy.*

2.2.4 Algebraic aspects of $\mathrm{Symp}(M,\omega)$ and $\mathrm{Ham}(M,\omega)$

Theorem 2.13 gives first informations about the algebraic structure of $\mathrm{Symp}_0(M)$ and $\mathrm{Ham}(M)$. $\mathrm{Symp}_0(M)$ can no longer be simple since $\mathrm{Ham}(M)$ is a normal subgroup. Indeed $\mathrm{Symp}_0(M)$ is not even perfect unless M is simply connected. We have the following commutative diagram.

$$\begin{array}{ccccc}
\pi_1(\mathrm{Ham}(M)) & \to & \pi_1(\mathrm{Symp}_0(M)) & \xrightarrow{\mathrm{Flux}} & \Gamma \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{\mathrm{Ham}}(M) & \longrightarrow & \widetilde{\mathrm{Symp}}_0(M) & \xrightarrow{\widetilde{\mathrm{Flux}}} & H^1(M,\mathbb{R}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Ham}(M) & \longrightarrow & \mathrm{Symp}_0(M) & \xrightarrow{\mathrm{Flux}} & H^1(M,\mathbb{R})/\Gamma
\end{array}$$

In [3] Banyaga showed that $\mathrm{Ham}(M)$ is simple. Therefore any other "natural" homomorphism from $\mathrm{Symp}_0(M)$ to an arbitrary group G must factor through the flux homomorphism. The proof relies on symplectic versions of Thurston's arguments that we introduced in the first chapter.

CHAPTER 3

THE RELATIVE HAMILTONIANS

Let (M, ω) be a connected, closed symplectic manifold, L a Lagrangian submanifold of M. We choose L to be connected so that the relative flux homomorphism is onto:

$$\text{Flux}_{rel} : \widetilde{\text{Symp}}_0(M, L, \omega) \to H^1(M, L; \mathbb{R}).$$

Here $\widetilde{\text{Symp}}_0(M, L, \omega)$ is the universal cover of the identity component of the group of symplectomorphisms of M that leave L setwise invariant. The Flux_{rel} is defined as

$$\text{Flux}_{rel}(\{\ \psi\}) = \int_0^1 [i_{X_t}\omega]dt$$

where $\{\psi_t\} \in \widetilde{\text{Symp}}_0(M, L, \omega)$ and X_t is the vector field given by

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t.$$

Note that since ψ_t leaves L invariant, for any $p \in L$, $X_t(p) \in T_pL$.

Remark 3.1. *We use* Flux *for all versions of the flux homomorphisms, e.g. absolute, relative. The homomorphism should be understood from the context. If the symplectic form is once mentioned we generally drop* ω *in* $\text{Symp}_0(M, L, \omega)$ *etc. and write* $\text{Symp}_0(M, L)$.

Notation: Let M be a manifold, $L \subset M$ a submanifold. If f is meant to be a map of M that leave L setwise invariant then we write $f : (M, L) \to (M, L)$.

Let $\mathrm{Ham}(M, L) \subset \mathrm{Symp}_0(M, L)$ be the subgroup consisting of symplectomorphisms ψ such that there is a Hamiltonian isotopy $\psi_t : (M, L) \to (M, L)$, $t \in [0, 1]$ such that $\psi_0 = id$, $\psi_1 = \psi$; i.e. ψ_t is a Hamiltonian isotopy of M such that $\psi_t(L) = L$ for any $t \in [0, 1]$. So if X_t is the vector field associated to ψ_t we have $i_{X_t}\omega = dH_t$ for $H_t : M \to \mathbb{R}$. Since L is Lagrangian ($w|_L = 0$), H_t is locally constant on L. We have the following characterization.

Theorem 3.2. *[16] $\psi \in \mathrm{Symp}_0(M, L)$ is a Hamiltonian symplectomorphism iff there exists a symplectic isotopy $\psi_t : [0, 1] \to \mathrm{Symp}_0(M, L)$ such that $\psi_0 = id$, $\psi_1 = \psi$ and $\mathrm{Flux}(\{\psi_t\}) = 0$. Moreover, if $\mathrm{Flux}(\{\psi_t\}) = 0$ then $\{\psi_t\}$ is isotopic with fixed end points to a Hamiltonian isotopy through points in $\mathrm{Symp}_0(M, L)$.*

3.1 Relative Calabi Homomorphism

Let (M, ω) be a noncompact symplectic manifold. If $h_c(M)$ is the subalgebra of hamiltonian vector fields with compact support then for each $X \in h_c(M)$ there is a unique function f_X with compact support such that

$$i_X\omega = df_X.$$

Proposition 3.3. *Let $X \in h_c(M)$, then $X \longmapsto \int_M f_X \cdot \omega^n$ is a surjective homomorphism of Lie algebras $r : h_c(M) \to \mathbb{R}$.*

The natural place of this infinitesimal version of the Calabi homomorphism is the universal cover of the compactly supported Hamiltonian diffeomorphism.

Definition 3.4. Let (M, ω) be a non-compact symplectic manifold, $\widetilde{\mathrm{Ham}_c}(M)$ be the universal cover of the compactly supported Hamiltonian diffeomorphisms on M. Then the Calabi homomorphism $\tilde{R} : \widetilde{\mathrm{Ham}_c}(M) \to \mathbb{R}$ is defined by

$$\{\phi_t\} \longmapsto \int_0^1 \int_M H_t \omega^n dt,$$

where H_t is given by $i_{X_t}\omega = dH_t$ and $\frac{d}{dt}\phi_t = X_t \circ \phi_t$.

Recall that an element $\{\phi_t\}$ of $\widetilde{\mathrm{Ham}}_c(M)$ is an equivalence class of homotopic Hamiltonian isotopies with fixed ends.

A local version of the Calabi homomorphism for compact manifolds may be defined by the above formulation i.e.

$$\tilde{R}_U : \widetilde{\mathrm{Ham}}_U(M)_c \to \mathbb{R}.$$

Here $\widetilde{\mathrm{Ham}}_U(M)_c$ denotes the universal cover of the compactly supported Hamiltonian diffeomorphisms, whose supports are contained in a contractible open subset U of M.

The relative Calabi diffeomorphism may be defined by the same formula. Let (M^{2n}, ω) be a noncompact symplectic manifold and $L^n \subset M^{2n}$ be a Lagrangian submanifold i.e. $\omega|_L = 0$. If $\mathrm{Ham}_c(M, L)$ is the group of Hamiltonian diffeomorphisms of M that leave L invariant, then

$$\tilde{R}_{rel} : \widetilde{\mathrm{Ham}}_c(M, L) \to \mathbb{R}$$

$$\{\Phi_t\} \longmapsto \int_0^1 \int_M H_t \omega^n dt$$

is the relative Calabi homomorphism. That this homomorphism is a well-defined surjective homomorphism can be proved almost the same as the absolute case (see, for example Banyaga [2, p.103]).

Similarly, the relative Calabi homomorphism can be defined for compact manifolds. Namely, if $\widetilde{\mathrm{Ham}}_{U,U\cap L}(M, \omega)_c$ denotes the universal cover of Hamiltonian diffeomorphisms supported in U that leave the Lagrangian submanifold L invariant then

$$\tilde{R}_{U,U\cap L} : \widetilde{\mathrm{Ham}}_{U,U\cap L}(M, \omega) \to \mathbb{R}$$

$$\{\Phi_t\} \longmapsto \int_0^1 \int_M H_t(\omega)^n dt$$

is again a surjective homomorphism. (Here H_t is given by $i_{\dot{\Phi}_t}\omega = dH_t$.)

Remark 3.5. *If $\tilde{R}_* : \tilde{G}_* \to \mathbb{R}$ denotes versions of the Calabi homomorphisms in the universal cover setting, then we can write the induced homomorphisms for the underlying groups. Namely, if Λ denotes the image of $\pi_1(G_*)$ under $\tilde{R}_*$, then*

$$R_* : G_* \to \mathbb{R}/\Lambda$$

is a well-defined homomorphism.

3.2 Relative Weinstein Charts

In order to show perfectness of in $\mathrm{Ham}(M, L)$, we first establish that it has the fragmentation property. This needs some technical preperation. We first recall the relative versions of Weinstein forms and charts.

Let $\psi \in \mathrm{Symp}_0(M, L)$ be sufficiently C^1-close to the identity. Then there corresponds a closed 1-form $\sigma = C(\psi) \in \Omega^1(M)$ defined by $\Psi(graph(\psi)) = graph(\sigma)$. Here $\Psi : \mathcal{N}(\Delta) \to \mathcal{N}(M_0)$ is the symplectomorphism between the tubular neighborhoods of the Lagrangian submanifolds diagonal ($\Delta \subset (M \times M, (-\omega) \oplus \omega)$) and the zero section ($M_0 \subset (T^*M, \omega_{can})$) of the tangent bundle with $\Psi^*(\omega_{can}) = (-\omega) \oplus \omega$. Note that since $\psi \in \mathrm{Symp}_0(M, L)$ the corresponding 1-form vanish on TL, i.e. $\sigma|_{TqL} = 0$ for any $q \in L$.

As a consequence we have the following due to Ozan:

Lemma 3.6. *[16] If $\psi \in \mathrm{Symp}_0(M, L, \omega)$ is sufficiently C^1-close to the identity and $\sigma = C(\psi_t) \in \Omega'(M)$ then $\psi \in \mathrm{Ham}(M, L)$ iff $[\sigma] \in \Gamma(M, L)$.*

$\Gamma(M, L)$ is the relative flux group defined as the image of the fundamental group of $\mathrm{Symp}_0(M, L, \omega)$ under the flux homomorphism.

$$\Gamma(M, L) = \widetilde{\mathrm{Flux}}(\pi_1(\mathrm{Symp}_0(M, L, \omega))) \subseteq H^1(M, L, \mathbb{R}).$$

Definition 3.7. The correspondence

$$C : \mathrm{Symp}_0(M, L, \omega) \to Z^1(M, L)$$

$$h \longmapsto C(h)$$

is called a Weinstein chart of a neighborhood of $id_M \in \mathrm{Symp}_0(M, L, \omega)$ into a neighborhood of zero in the set of closed 1-forms that vanish on TL. The form $C(h)$ is called a Weinstein form.

With these definitions in mind we have the following.

Corollary 3.8. *Let (M, ω) be a symplectic manifold, L a Lagrangian submanifold. Any $h \in \mathrm{Ham}(M, L)$ can be written as a finite product of $h_i \in \mathrm{Ham}(M, L)$ close enough to id_M to be in the domain of the Weinstein chart and such that their Weinstein forms are exact.*

Proof. As the above lemma suggests, every smooth path $\psi_t \in \mathrm{Ham}(M, L)$ is generated by Hamiltonian vector fields. Let h_t be any isotopy in $\mathrm{Ham}(M, L)$ to the identity such that $\frac{d}{dt}h_t = X_t(h_t)$ where $i_{X_t}\omega = df_t$, $h_0 = id_M$, $h_1 = h$ and $f_t : M \to \mathbb{R}$ are Hamiltonians for all $t \in [0, 1]$. Let N be an integer large enough so that

$$\Phi_t^i = \left[h_{\left(\frac{N-i}{N}\right)t} \right]^{-1} h_{\left(\frac{N-i+1}{N}\right)t}$$

is in the domain of the Weinstein chart. If we let $h_i = \Phi_1^i$ then we have $h = h_N h_{N-1} ... h_1$. The mapping $t \longmapsto [C(\Phi_t^i)]$ is a continuous map of the interval $[0, 1]$ into the countable group $\Gamma(M, L)$ [16]. Hence it must be constant. Thus $[C(\Phi_t^i)] = 0$. □

3.3 The Fragmentation Lemma

We are ready to state the relative symplectic fragmentation lemma.

Lemma 3.9. *Let $\mathcal{U} = (U_j)_{j \in I}$ be an open cover of a compact, connected, symplectic manifold (M, ω) and h be an element of $\mathrm{Ham}(M, L)$ for a Lagrangian submanifold L of M. Then h can be written*

$$h = h_1 h_2 ... h_N,$$

where each $h_i \in \mathrm{Ham}_c(M, L)$, $i = 1, .., N$ *is supported in* $U_{j(i)}$ *for some* $j(i) \in I$. *Moreover, if* M *is compact , we may choose each* h_i *such that* $R_{U_i, U_i \cap L}(h_i) = 0$, *where we made the identification* $U_{j(i)} := U_i$.

Proof. We use the notation of Corollary 3.8. By Corollary 3.8 any $h \in \mathrm{Ham}(M, L)$ can be written as $h = h_1...h_N$ where each $h_i \in \mathrm{Ham}(M, L)$ is close to id_M to be in the domain V of the Weinstein chart

$$C : V \subset \mathrm{Symp}_0(M, L) \to C(V) \subset Z^1(M, L)$$

and such that $C(h_i)$ is exact.

Start with an open cover $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$ of M and a partition of unity $\{\lambda_i\}$ subordinate to it. Let K be a compact subset of M containing the support of h. Let $\mathcal{U}_k = \{U_0, ..., U_N\}$ be the finite subcover for K such that $U_i \cap U_{i+1} \neq \emptyset$. Then consider the functions

$$\mu_0 = 0 \quad , \quad \mu_j = \sum_{i \leq j} \lambda_i$$

for $j = 1, 2, ..., N$. Note that for any $x \in K$ $\mu_N(x) = 1$ and $\mu_i(x) = \mu_{i-1}(x)$ for $x \notin U_i$.

Let $\tilde{\mu}_i$ be defined as in the Equation B.1. Since this operator is bounded there is an open neighborhood $V_0 \subset V$ of $id \in \mathrm{Symp}_c(M, L)$ with

$$\tilde{\mu}_i(C(h)) \in C(V) = W \text{ for all } i = 1, ..., N \text{ and } h \in V_0$$

We will fragment such $h \in V_0$. Consider

$$\psi_i = C^{-1}(\mu_i(\tilde{C}(h))) \in \mathrm{Ham}(M, L).$$

Note that $\psi_{i-1}(x) = \psi_i(x)$ for $x \notin U_i$ since $\mu_{i-1}(x) = \mu_i(x)$ in that case. Therefore $(\psi_{i-1}^{-1}\psi_i)(x) = x$ if $x \notin U_i$. Hence, $h_i = (\psi_{i-1})^{-1}(\psi_i)$ is supported in U_i. On K we have $\mu_N = 1$, $\mu_0 = 0$, $\psi_N = h$, and $\psi_0 = id$. Therefore

$$h = \psi_N = (\psi_0^{-1}\psi_1)(\psi_1^{-1}\psi_2)...(\psi_{N-1}^{-1}\psi_N) = h_1 h_2 ... h_N.$$

For the second statement define the isotopies $h_t^i = \psi_{i-1}(t)\psi_i(t)$ where $\psi_i(t) = C^{-1}(t\mu_i(\tilde{C}(h)))$. A classical result due to Calabi states that the Lie algebra of locally supported Hamiltonian diffeomorphisms is perfect [4]. Since for each t, $\dot{h_t^i}$ is a Hamiltonian vector field parallel to L, we can write $\dot{h_t^i}$ as a sum of commutators. In other words we have

$$\dot{h_t^i} = \sum_j [X_t^{ji}, Y_t^{ji}],$$

where X_t^{ji} and $Y_t^{ji}$are again Hamiltonian vector fields (not necessarily parallel to L). By the cut-off lemma below X_t^{ji} and Y_t^{ji} can be chosen to vanish outside of an open set whose closure contain U_i. If u_t^i is the unique function supported in U_i with $i_{\dot{h_t^i}}\omega = du_t^i$ then $du_t^i = \sum_j \omega(X_t^{ji}, Y_t^{ji})$ since both functions above have the same differential and both have compact supports. Therefore

$$\int_{U_i} u_t^i \omega^n = \int_M u_t^i \omega^n = \sum_j \int \omega(X_t^{ji}, Y_t^{ji})\omega^n = 0$$

Therefore

$$\int_0^1 \int_{U_i} u_t^i \omega^n = R_{U_i, U_K \cap L}(h_i) = 0.$$

□

The cut-off lemma we used in the proof of the fragmentation lemma is as follows.

Lemma 3.10. *Let $\varphi_t \in \mathrm{Ham}(M, L)$ be an isotopy of a smooth symplectic manifold (M, ω) leaving a Lagrangian submanifold L invariant. Let $F \subset M$ be a closed subset and $U, V \subset M$ open subsets such that $U \subset \overline{U} \subset V$ with $\cup_{t \in [0,1]}\varphi_t(F) \subset U$. Then there is an isotopy $\overline{\varphi_t} \in \mathrm{Symp}(M, L)$ supported in V and coincides with φ_t on U.*

Proof. We choose a smooth function $\lambda_t(x) = \lambda(x, t)$ which equals to 1 on $U \times [0, 1]$, 0 outside of $V \times [0, 1]$. Let f_t denote the family of Hamiltonians corresponding to φ_t, i.e. $i_{\dot{\varphi}_t}\omega = df_t$. Define $\overline{X}(x, t) = X_{(\lambda_t \cdot f_t)} + \partial/\partial t$ on $M \times [0, 1]$

where $X_{(\lambda_t \cdot f_t)}$ is the Hamiltonian vector field given by $i_{X_{(\lambda_t \cdot f_t)}}\omega = d(\lambda_t \cdot f_t)$.The desired isotopy is the integral curves of the vector field $\overline{X}(x,t)$. $\square$

CHAPTER 4

THE DEFORMATION LEMMA

In this chapter, the main step in the proof of perfectness of $\mathrm{Ham}(M, L)$ will be shown. Namely the Deformation Lemma will be proved. Roughly speaking, Deformation Lemma says that the local picture of the Hamiltonian diffeomorphisms is the same as the global one as far as the first homologies are concerned. Since locally supported diffeomorphisms are the same for all manifolds we can conclude that $\mathrm{Ham}(M, L)$ is perfect for all M, if it is perfect for just one manifold. We remark here that in the absolute case of the differential category, i.e. for $\mathrm{Diff}^r_0(M)$ Deformation Lemma is true for all levels of homology. See Mather for proofs, [11] and [13].

The proof of deformation lemma needs some technical preparation.

Throughout this chapter we work with a closed, connected, symplectic manifold M; a closed, connected, oriented Lagrangian submanifold $L \subset M$ such that $M \backslash L$ is still connected. Note that, if M has dimension at least 4, then $M \backslash L$ is connected for any Lagrangian submanifold L.

4.1 Transitivity properties

Since any element of $\mathrm{Diff}^r(M, L)$ leaves L invariant, the groups $\mathrm{Diff}^r(M, L)$, $\mathrm{Symp}(M, L)$, or $\mathrm{Ham}(M, L)$ can not be transitive. Nevertheless, we have the following.

Lemma 4.1. *Let M be a connected, closed, symplectic manifold, $L \subset M$ a connected, oriented Lagrangian submanifold such that $M \backslash L$ is connected. For each $x \in M \backslash L$ and $y \in M \backslash L$ there exists an isotopy $\phi_t \in \mathrm{Ham}(M, L)$ such that $\phi_t(x) = y$. Similarly for each $x \in L$ and $y \in L$ such ϕ_t exists.*

Proof. First we assume $x \in L$ and $y \in L$ are arbitrarily close inside the domain U of a Darboux chart $\varphi : U \to \mathbb{R}^{2n}$. Let V be the vector $V = \varphi(y) - \varphi(x) \in \mathbb{R}^{2n}$ and $h : M \to \mathbb{R}$ be the smooth function defined by $dh = i_V\omega$. Let U_1 be an open set such that $U \subset \overline{U} \subset U_1$. Choose a smooth function $\lambda : \mathbb{R}^{2n} \to \mathbb{R}$ such that $\lambda|_U = ||V||$ and $\lambda|_{\mathbb{R}^{2n}\backslash U_1} = 0$. Consider the map $f = \lambda h$, which coincides with h on U and 0 on $\mathbb{R}^{2n}\backslash U_1$. Then the desired isotopy ϕ_t is the image of the isotopy that can be found by integrating the smooth family of the vector fields X_t where $X_0 = 0$ and $X_1 = X_f$ with $df = i_{X_f}\omega$ under φ.

For the other case, i.e. for $x, y \notin L$ choose U such that $U \cap L = \emptyset$, the above arguments apply equally well. If the points x and y are apart, choose a continuous path $c : [0, 1] \to M$ such that $c(0) = x$ and $c(1) = y$. Subdivide $[0, 1]$ into subintervals $[s_k, s_{k+1}],\ k = 1, ..., N$ so that each consecutive points $c(s_j)$ and $c(s_{j+1})$ are within the domain of a Darboux chart. Hence there is a hamiltonian diffeomorphism h_j isotopic to identity with support in a small neighborhood of $c(s_j)$ and $c(s_{j+1})$ such that $h_j(c(s_j)) = c(s_{j+1})$. The diffeomorphism $h = h_N h_{N-1} ... h_1$ maps x to y. □

4.1.1 A Special Open Cover

With the above result in mind, the following crucial lemma will be a key in the proof of the deformation lemma. Let $\mathrm{Symp}_{U,U\cap L}$ denote the group of symplectic diffeomorphisms that are supported on an open set $U \subset M$ of a symplectic manifold M, leaving a Lagrangian submanifold L invariant.

Lemma 4.2. *Let (M, ω) be a closed, connected, symplectic manifold, L an oriented, connected Lagrangian submanifold such that $M \backslash L$ is connected, $U \subset M$ an embedded symplectic ball with $L \cap U \neq \emptyset$. Then M has a finite open cover $\mathcal{V} = \{V_i\}$ by balls such that if $V_i \cap V_j \neq \emptyset$ then $V_i \cap V_j$ is diffeomorphic to a ball. Furthermore, for each i and j there are symplectic isotopies $\phi_t^i \in \mathrm{Ham}(M, L)$ and $H_t^{i,j} \in \mathrm{Symp}_{U, U \cap L}$ such that $\phi_1^i(V_i) \subset U$, $H_1^{i,j}(\phi_1^i(V_i \cap V_j)) = \phi_1^j(V_i \cap V_j)$.*

Note that if $V_i \cap L = \emptyset$ then $\phi_t^i(V_i) \cap L = \emptyset$ and if $V_i \cap L \neq \emptyset$ then $\phi_t^i(V_i) \cap L \neq \emptyset$. In order to prove this lemma we need some results about isotopies of relative symplectic embeddings.

Definition 4.3. An embedding $f : M \to M'$ of two symplectic manifolds $(M, \omega), (M', \omega')$ is a symplectic embedding if $f^*\omega' = \omega$. Two such embeddings $f,\ f'$ are isotopic if there exists a smooth family $f_t : M \to M'$ of symplectic embeddings such that $f = f_0,\ f' = f_1$.

Let $x_1, y_1, ..., x_n, y_n$ be coordinates on $\mathbb{R}^{2n}$ and ω_{st} denote the standard symplectic structure $\omega_{st} = \sum dx_i \wedge dy_i$. Then $L = \mathbb{R}^n$ given by $y_1 = y_2 = ... = y_n = 0$ is a Lagrangian submanifold of $(\mathbb{R}^{2n}, \omega_{st})$.

Definition 4.4. Let $Sp(2n)$ denote the group of symplectic matrices. It is well known that this group is connected. Now let $Sp(2n, n) \subset Sp(2n)$ be the subgroup of symplectic transformations $(\mathbb{R}^{2n}, L) \to (\mathbb{R}^{2n}, L)$, where L is as above. A typical element X of $Sp(2n, n)$ is of the form:

$$X = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}.$$

Remark 4.5. *The group $Sp(2n, n)$ has two connected components. See Appendix for a proof of this result. An element $X \in Sp(2n, n)$ as above is in the identity component if and only if $det(A) > 0$.*

It is a well known result that any symplectic embeddings of the unit ball into $\mathbb{R}^n$ are isotopic. See [2] for instance. The relative version of this result is the following.

Lemma 4.6. *Let B be the unit ball in $\mathbb{R}^{2n}$ equipped with the standard symplectic structure ω_{st}. Then any two symplectic embedding of the pairs $(B, B \cap L)$ into $(\mathbb{R}^{2n}, L)$ are isotopic through isotopies in* $\mathrm{Symp}(\mathbb{R}^{2n}, L)$ *if and only if the orientations on $B \cap L$ induced by the embeddings and the orientations obtained from the orientations of L are the same.*

Proof. Let $f : (B, B \cap L) \to (\mathbb{R}^{2n}, L)$ be a symplectic embedding. Assume first that $f(B) \cap L \neq \emptyset$. It suffices to show that f is isotopic to the natural embedding $i : B \hookrightarrow \mathbb{R}^{2n}$ if and only if it preserves the orientation on L. Note that, by setting $\overline{f} = T \circ f$, where T is the translation such that $T(f(0)) = 0$, we can assume that f is isotopic to a symplectic embedding $\overline{f}$ which fixes the origin. For each $t \in [0, 1]$, let $R_t(x) = tx,\ x \in \mathbb{R}^{2n}$. If ω is the restriction of $\omega_{st} \in \Omega^2(\mathbb{R}^{2n})$ into $B \cap L \neq \emptyset$, then $R_t^* \omega = t^2 \omega$. Therefore, $R_t^{-1} \circ \overline{f} \circ R_t : B \to \mathbb{R}^{2n}$ becomes a symplectic embedding for all $t \in (0, 1]$. We have

$$\overline{f}'(0) = \lim_{t \to 0} \frac{\overline{f}(tx)}{t} = \lim_{t \to 0} (R_t^{-1} \circ \overline{f} \circ R_t)(x)$$

Thus,

$$H_t = \begin{cases} R_t^{-1} \overline{f} R_t & 0 < t \leq 1 \\ \overline{f}'(0) & t = 0 \end{cases}$$

is a continuous family of symplectic embeddings with $H_1 = \overline{f}$. Note that for any $x \in L$ and any $t \in (0, 1]$ we have $H_t(x) \in L$. Hence $H_0 \in Sp(2n, n)$. Since f preserves the orientation on L, H_0 is in the identity component of $Sp(2n, n)$. Therefore there is a smooth path $G_t \in Sp(2n, n)$ from id to H_0. So composing these isotopies and smoothing via change of parameters (see [2] proof of Corollary 1.2.2, page 5), if necessary, gives the desired isotopy.

If $B \cap L = \emptyset$ then the result follows immediately from the absolute case. $\square$

Next, we show that the support of the isotopies between two symplectic embeddings of the ball have some precision.

Lemma 4.7. *Let $V \subset B_{r/8}$ be an open convex subset of $\mathbb{R}^{2n}$, where $B_{r/8}$ is the ball centered at 0 with radius $r/8$. There is an $\epsilon > 0$, such that if a symplectic embedding $h : (V, V \cap L) \to (B_{r/8}, B_{r/8} \cap L)$ satisfies*

$$1 - \epsilon \leq \frac{||h'(x)(y)||}{||y||} \leq 1 + \epsilon \qquad \text{for all } x, y \in V \tag{4.1}$$

then there exists a symplectic isotopy H_t of $\mathbb{R}^{2n}$ with support in B_r and such that $H_{1|V} = h$.

Proof. We will prove the proposition for the case $h(V) \cap L \neq \emptyset$ and make use of the previous isotopy introduced for this case. If $h(V) \cap L = \emptyset$, then the proof of the absolute version of this proposition works equally well. See [2], pp120-121. Assume $h(0) = 0$. Mean value theorem and the condition (4.1) imply that

$$||\frac{h(tx)}{t}|| \leq ||x||(1 + \epsilon) \text{ for all } x \in V \ t \in [0, 1].$$

Thus $\frac{h(tx)}{t} \in B_{r/4}$, for all $x \in V$, $t \in (0, 1]$ and as a result $h'(0)(V) \in B_{r/4}$. Due to inequality (4.1), $h'(0)$ is close to $U(2n, n) \subset Sp(2n, n)$: the maximal compact subgroup of $Sp(2n, n)$. Let $p : T(U(2n, n)) \to U(2n, n)$ be a C^∞ tubular neighborhood around $U(2n, n)$ in $Sp(2n, n)$. Identifying $T(U(2n, n))$ with a neighborhood of $U(2n, n)$ in $Sp(2n, n)$, we may think $h'(0) \in T(U(2n, n))$. We can get an isotopy $g_t \in Sp(2n, n)$ from $h'(0)$ to the identity, by composing the paths a_t, b_t, where a_t is the shortest ray joining $h'(0)$ to $p(h'(0))$ in $T(U(2n, n))$ and b_t joins $p(h'(0))$ to the identity in $U(2n, n)$. This lets $g_t(V) \subset B_{r/2}$. Let now G_t be the isotopy from h to the identity obtained by composing the isotopy from h to $h'(0)$ of previous lemma and the path g_t above. Let $\tilde{G}_t$ be the smoothing of this isotopy via change of parameters. Clearly $\tilde{G}_t(V) \subset B_{r/2}$.

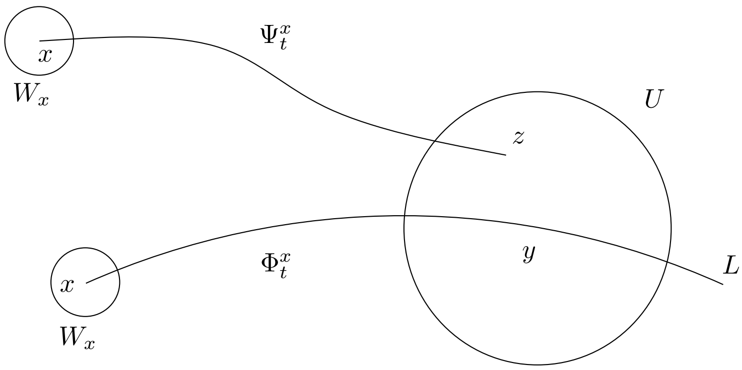

Figure 4.1: Isotopies carrying W_x

Now let H_t be the symplectic isotopy with compact support such that $\dot{H}_t = u \cdot f_t$ where u is a smooth function equal to 1 on $B_{r/2}$ and 0 outside of B_r and f_t is a hamiltonian of the isotopy $\tilde{G}_t$. Clearly $supp(H_t) \subset B_r$ and $H_{1|_V} = h$. $\square$

Proof. **(of Lemma 4.2)** We start with a choice of a hermitian metric g compatible with ω. Without loss of generality assume y, the center of U, is on L (take a smaller ball inside U centered at a point on L if necessary). For each $x \in L$ there exists a symplectic isotopy $\Phi_t^x \in \mathrm{Ham}(M, L)$ such that $\Phi_1^x(x) = y$ by Lemma 4.1. Since $M \backslash L$ is connected, for any $x \in M \backslash L$ there is an isotopy $\Psi_t^x \in \mathrm{Ham}(M, L)$ to a generic point $z \in U \backslash L$ with $\Psi_1^x(x) = z$, whose support is away from L. Since these isotopies were constructed by successive compositions of translations in local charts we may assume that the differentials $dx(\Phi_1^x)$ and $dx(\Psi_1^x)$ send the hermitian metric of T_xM to that of T_yM and T_zM. Let $x \in M$ and $W_x$be a geodesic ball centered at x with radius δ_x. We choose δ_x small enough so that $\Phi_1^x(W_x) \subset U$ and $\Psi_1^x(W_x) \subset U$. Clearly $W_x \cap W_{x'}$ is diffeomorphic to an open ball. For $x \in M \backslash L$ choose δ_x smaller, if necessary, so that $W_x \cap L = \emptyset$.

Choose a coordinate system $\alpha : U \to \mathbb{R}^{2n}$ such that $d_y\alpha : T_yM \to \mathbb{R}^{2n}$ and $d_z\alpha : T_zM \to \mathbb{R}^{2n}$ give a hermitian frame (We can choose z close onough to y if necessary). Then for any $x \in M$, $d_x(\alpha \circ \Phi_1^x) : T_xM \to \mathbb{R}^{2n}$ (or $d_x(\alpha \circ \Psi_1^x)$) becomes a hermitian frame. Note that, if x' is close to x, then $d_{x'}(\alpha \circ \Phi_1^x) : T_{x'}M \to \mathbb{R}^{2n}$ (or $d_{x'}(\alpha \circ \Psi_1^x)$) is still close to a hermitian map. Therefore, if W_x is sufficiently small and $p \in W_x \cap W_{x'}$, the mapping

$$f_{xx'}(p) = d_{(\alpha \circ \Phi_1^x(p))}(\alpha \cdot \Phi_1^{x'} \cdot (\Phi_1^x)^{-1} \cdot \alpha^{-1})$$

$$(\text{or } f_{xx'}(p) = d_{(\alpha \circ \Psi_1^x(p))}(\alpha \cdot \Psi_1^{x'} \cdot (\Psi_1^x)^{-1} \cdot \alpha^{-1})$$

is close to a hermitian map. To see this, note that for all $x \in M$, there is a positive $\delta_x' \leq \delta_x$ such that if $r(W_x) \leq \delta_x'$ for all $u \in W_x \cap W_{x'}$

$$1 - \epsilon \leq \frac{||f_{xx'}(p)(u)||}{||u||} \leq 1 + \epsilon.$$

The existence of such ϵ was shown in Lemma 4.7. We will construct the special open cover out of such geodesic ball of radius less than δ_x'.
Clearly, $\{W_x\}_{x \in L}$ is an open cover for the compact submanifold L. Then this has a finite subcover $\{W_i\}_{i=1}^N$ for L. Choose a tubular neighborhood $\mathcal{V}$ for L so that $\mathcal{V} \subset \cup_{i=1}^N W_i$. Since $M \backslash \mathcal{V}$ is also compact, we can let $\{W_i\}_{i=N+1}^K$ to be the finite subcover for $M \backslash \mathcal{V}$ of $\{W_x\}_{x \in M \backslash L}$. Then the special open cover for M is $\{W_i\}_{i=1}^K$. Let x_i denote the center of W_i and let ϕ_t^i be the symplectic isotopies corresponding to the W_i such that $\phi_1^i(W_i) \subset U$ and let $f_{ij}(p)$ be the change of coordinates

$$f_{ij}(p) = d_{(\alpha \phi_1^i(p))}(\alpha \cdot \phi_1^j \cdot (\phi_1^i)^{-1} \alpha^{-1}).$$

We have three cases to consider depending on whether x_i or x_j belongs to L. For the first case, let $x_i, x_j \in M \backslash L$. See Figure 4.2.

Note that in this case, both $\phi_1^i(W_i \cap W_j)$ and $\phi_1^j(W_i \cap W_j)$ are diffeomorphic to balls embedded inside a ball centered at $z \in M$ away from L. Recall that by Lemma 4.6 any two symplectic embeddings of the relative unit balls in $\mathbb{R}^{2n}$ are isotopic. Hence there is a symplectic isotopy $\tilde{H}_t^{ij} \in \text{Symp}(\mathbb{R}^{2n}, \mathbb{R}^n)$, such that $\tilde{H}_1^{ij}(\alpha \phi_1^i(W_i \cap W_j)) = \alpha \phi_1^j(W_i \cap W_j)$. In other words $\tilde{H}_1^{ij}$ equals $\alpha \phi_1^j (\phi_1^i)^{-1} \alpha^{-1}$ on

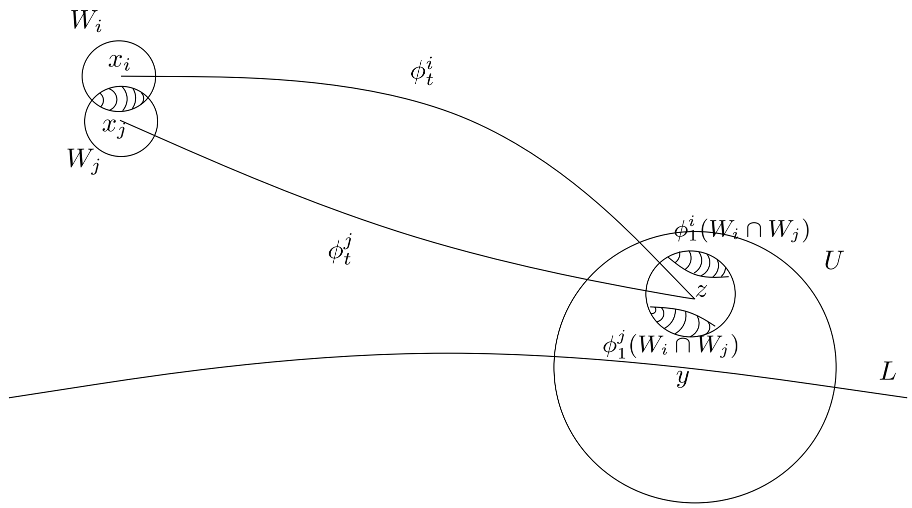

Figure 4.2: Isotopies carrying W_i and W_j when $x_i, x_j \in M \backslash L$

$\alpha\phi_1^i(W_i \cap W_j)$. The condition on $f_{ij}(p) = d_{(\alpha\phi_1^i(p))}(\alpha \cdot \phi_1^j \cdot (\phi_1^i)^{-1}\alpha^{-1})$ with Lemma 4.7, implies that $\tilde{H}_t^{ij}$ can be assumed to be supported in $\alpha(U)$. Therefore setting

$$H_t^{ij} = \alpha^{-1}\tilde{H}_t^{ij}\alpha \in \mathrm{Symp}_{U,U\cap L}$$

gives the desired isotopy.

Next , consider $x_i \in M \backslash L$ and $x_j \in L$ as in Figure 4.3.

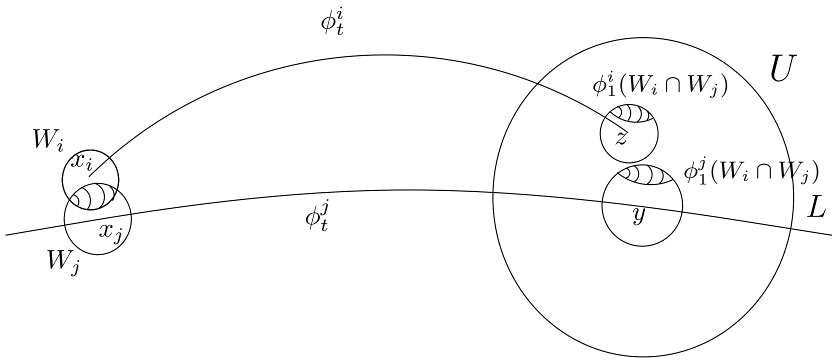

Figure 4.3: Isotopies carrying W_i and W_j when $x_i \in M \backslash L$ and $x_j \in L$

In this case, note that we have chosen W_i to be away from L hence $W_i \cap W_j$ is also away from L. Since $\phi_t^j \in \mathrm{Ham}(M, L)$ for all t, $\phi_1^j(W_i \cap W_j)$ does not intersect L and hence $\phi_1^i(W_i \cap W_j)$ and $\phi_1^j(W_i \cap W_j)$ can be contained in a ball $V \subset U$. Now the isotopy of the previous case works equally well in this case.

The isotopies in the above two cases are the isotopies of the absolute version. In other words, any isotopy of the symplectic embeddings of the unit balls $\phi_1^i(W_i \cap W_j)$ and $\phi_1^j(W_i \cap W_j)$, one can construct the desired one, as long as it is supported in a set not intersecting L.

Finally, consider the case in which $x_i, x_j \in L$. See the Figure 4.4 below. In this case both $\phi_1^i(W_i \cap W_j)$ and $\phi_1^j(W_i \cap W_j)$ are in a ball centered at y, the center of U.

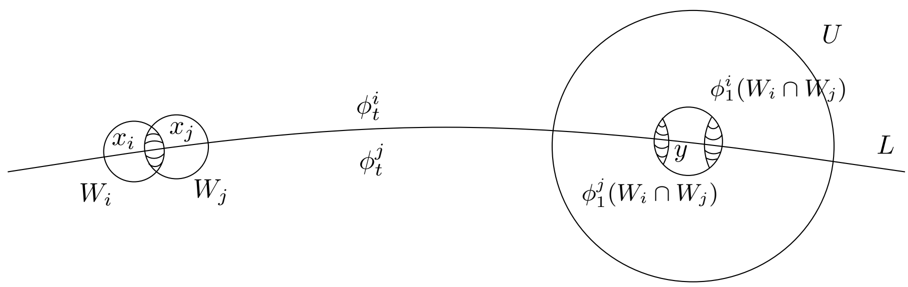

Figure 4.4: Isotopies carrying W_i and W_j when $x_i, x_j \in L$

The isotopy will be constructed out of the isotopy of Lemma 4.6, namely $\tilde{H}_t^{ij} \in \mathrm{Symp}(\mathbb{R}^{2n}, \mathbb{R}^n)$. As in the previous cases, we let

$$H_t^{ij} = \alpha^{-1} \tilde{H}_t^{ij} \alpha \in \mathrm{Symp}_{U, U \cap L}.$$

This proves Lemma 4.2. $\square$

4.2 Proof of Deformation Lemma

Let (M,ω) be a connected closed symplectic manifold, $L \subset M$ a closed Lagrangian submanifold. Define a subcomplex $B\overline{\mathrm{Ham}(M,L)}$ of $B\overline{\mathrm{Symp}(M,L)}$ as usual: an n-simplex c of $B\overline{\mathrm{Ham}(M,L)}$ is a smooth map $c : \Delta^n \to \mathrm{Symp}(M,L)$ such that for any path $\gamma : [0,1] \to \Delta^n$, if $\tilde{c} = c \circ \gamma$, then $\dot{\tilde{c}}_t$ is a family of Hamiltonian vector fields parallel to L. This implies that the class $[\tilde{c}_t]$ is an element of $\mathrm{Ham}(M,L)$. Let U be a fixed embedded symplectic ball in M with $U \cap L \neq \emptyset$, and $\mathrm{Symp}_{U,U\cap L}$ be the subgroup of $\mathrm{Symp}(M,L)$ consisting of elements with compact supports in U and leaving $U \cap L$ invariant. Similarly we define the subcomplex $B\overline{\mathrm{Ker}R_{U,U\cap L}}$ of $B\overline{\mathrm{Symp}_{U,U\cap L}}$: An n-simplex c of $B\overline{\mathrm{Ker}R_{U,U\cap L}}$ is a smooth map $c : \Delta^n \to \mathrm{Symp}_{U,U\cap L}$ such that $c\circ\gamma(t) \in \mathrm{Ker}R_{U,U\cap L}$ for any smooth map $\gamma : [0,1] \to \Delta^n$.

Definition 4.8. Support of an n-simplex $c : \Delta^n \to \mathrm{Diff}(M)$, denoted by *supp*$(c)$, is the set $\{x \in M | c(\sigma)(x) \neq x \text{ for some } \sigma \in \Delta^n\}$

We are ready to state the relative symplectic deformation lemma.

Remark 4.9. *In what follows we work with an open cover $\mathcal{U} = \{U_i\}$ such that if $U_1, U_2 \in \mathcal{U}$ are not disjoint then $U_1 \cup U_2 \subset V_i$, where V_i is an element of the special open cover $\mathcal{V}$ of Lemma 4.2. The existence of such U_i can be seen as follows. Since $\mathcal{V}$ is an open cover for a compact manifold the Lebesgue number δ of $\mathcal{V}$ is well defined. Let $\mathcal{U} = \{U_i\}$ be an open cover of M such that each U_i is a ball of radius $\delta/2$. Now if $x \in U_1 \cap U_2$, we have $U_1 \cup U_2 \subset B(x,\delta) \subset V_i$.*

Theorem 4.10. *The natural map $\rho : B\overline{\mathrm{Ker}R_{U,U\cap L}} \longrightarrow B\overline{\mathrm{Ham}(M,L)}$ induces an isomorphism*

$$\rho_* : H_1(B\overline{\mathrm{Ker}R_{U,U\cap L}};\mathbb{Z}) \longrightarrow H_1(B\overline{\mathrm{Ham}(M,L)};\mathbb{Z})$$

Proof. To show the surjectivity of ρ_* we let $\alpha \in \mathrm{H}_1(\mathrm{B}\overline{\mathrm{Ham}(M,L)},\mathbb{Z})$ be represented by an isotopy Φ_t in $\mathrm{Ham}(M,L)$. Let $\mathcal{V} = \{V_i\}$ be the open cover in

Lemma 4.2 so that there exist a diffeomorphism $h_i \in \mathrm{Ham}(M, L)$ with $h_i(V_i) \subset U$.

By the relative symplectic fragmentation lemma we can write $\Phi_t = \Phi_t^1 \dots \Phi_t^N$ where $\Phi_t^i \in \mathrm{Ham}(M, L)$ is supported in V_i and $R_{V_i, V_i \cap L}(\Phi_t^i) = 0$ for all $t \in [0, 1]$. Note that $R_{V_i, V_i \cap L}(h_i \Phi_t^i h_i^{-1}) = 0$.

Let $\beta_i \in \mathrm{H}_1(\mathrm{B}\overline{\mathrm{Ker} \mathrm{R}_{U, U \cap L}}, \mathbb{Z})$ be the class of the isotopy $h_i \Phi_t^i h_i^{-1}$. Then $\alpha = \rho_*(\beta_1 + \beta_2 + \ldots + \beta_N)$ since Φ_t^i and $h_i \Phi_t^i h_i^{-1}$ are homologous in $\mathrm{B}\overline{\mathrm{Ham}(M, L)}$ by Proposition C.2.

For the injectivity we first divide Δ^2 into $\frac{m(m+1)}{2}$ little squares and triangles as in Figure 4.5 below. Let

$$\mu_{i_1} = \sum_{j \le i_1} \lambda_j, \qquad 0 \le i_1 \le m, \quad \mu_0 = 0 \tag{4.2}$$

$$\mu_{i_2} = \sum_{j \le i_2} \lambda_j, \qquad 0 \le i_2 \le m \tag{4.3}$$

Here λ_i is a partition of unity subordinate to the cover of $\mathcal{U}$ of the Remark 4.9. Define a mapping $f : \Delta^2 \times M \to \Delta^2 \times M$ as follows: for $0 \le i_1, i_2 \le m$ and $x \in M$

$$f\Big((\frac{i_1}{m}, \frac{i_2}{m}), x \Big) = \Big(\big(\mu_{i_1}(x), \mu_{i_1}(x) \big), x \Big)$$

If one defines

$$\mu_s = \big(-ms + (1 + i_1) \big) \mu_{i_1} + (ms - i_1) \mu_{i_1 + 1} \tag{4.4}$$

$$\mu_t = \big(-mt + (1 + i_2) \big) \mu_{i_2} + (mt - i_2) \mu_{i_2 + 1} \tag{4.5}$$

we see that

$$\tilde{f}(s, t) = \Big(\big(\mu_s(x), \mu_t(x) \big), x \Big)$$

extends f linearly to all of $\Delta^2 \times M$ for $(s, t) \in \Delta^2$, where

$$\frac{i_1}{m} \le s \le \frac{i_1 + 1}{m}, \quad \frac{i_2}{m} \le t \le \frac{i_2 + 1}{m}$$

(We use f again for the extension $\tilde{f}$ if it is understood from the context.) Let z be a 1-cocycle of $B\overline{\mathrm{Ker}\mathrm{R}_{U,U\cap L}}$ such that $\rho_*([z]) = 0 \in H_1(B\overline{\mathrm{Ham}(M,L)}, \mathbb{Z})$. This implies that z bounds a 2-chain on $B\overline{\mathrm{Ham}(M,L)}$. For the injectivity we must show that z bounds a 2-chain in $B\overline{\mathrm{Ker}R_{U,U\cap L}}$.

Suppose z bounds the 2-chain $c = \sum_{j=1} c_j,\ c_j : \Delta^2 \to \mathrm{Ham}(M,L)$. We can assume that each c_j maps Δ^2 inside a small neighborhood of id_M. We can do this by taking m large enough so that each subdivision is sufficiently small. In that case by Corollary 3.8 the Weinstein form of each $c_j(\sigma)$ is exact (i.e. $[C(c_j(\sigma))] = 0\ \forall \sigma \in \Delta^2$). Since $C(c_j(\sigma))$ are exact 1-forms we can choose a smooth family of functions $u_j(\sigma)$ such that

$$C(c_j(\sigma)) = du_j(\sigma) \text{ for all } \sigma \in \Delta^2$$

as Palamodov's theorem suggests (see Equation B.1 and remarks thereof). Consider the 1-forms $C(c_j(\sigma))$ as forms on $\Delta^2 \times M$ and define a 2-chain $\tilde{c}(\sigma) = C^{-1}(f^*C(c_j(\sigma)))$. Here f is the map defined above. So if $\sigma = (s,t)$ and $C(c_j(s,t)) = du_j(s,t)$ then $f^*C(c_j(s,t)) = du(\mu_s(x), \mu_t(x))$. Note that $dc_j = d\tilde{c_j}$.

Let's denote the inclusion of $K_{i_1i_2} \times M$ or of $L_{i_k} \times M$ into $\Delta^2 \times M$ by j, where $K_{i_1i_2}$ is the little square and L_{i_k}s are the little triangles in Figure 4.5. Define $c^j_{i_1i_2}$ to be the 2-chain

$$c^j_{i_1i_2}(\sigma) = C^{-1}(j^*f^*C(c(\sigma)))(C^{-1}(j^*f^*C(c(\frac{i_1}{m}, \frac{i_2}{m}))))^{-1}$$

Similarly, define 2-chain $c^j_{i_k}$. This gives $\tilde{c_j} = \sum c^j_{i_1i_2} + c^j_{i_k}$. Therefore

$$z = d(\sum_j \sum_{i_1,i_2} c^j_{i_1i_2} + c^j_{i_k}).$$

The definition of f forces $supp(c^j_{i_1i_2}) \subset U_{i_1+1} \cup U_{i_2+1}$ and if $U_{i_1+1} \cap U_{i_2+1} = \emptyset$ then $dc^j_{i_1i_2} = 0$. If we denote the subset of the 2-simplices $c^j_{i_1i_2}$ such that $U_{i_1+1} \cap U_{i_2+1} \neq \emptyset$ to which we add the simplices $c^j_{i_k}$ by $\{\overline{c}^j_{i_1i_2}\}$ then we see that

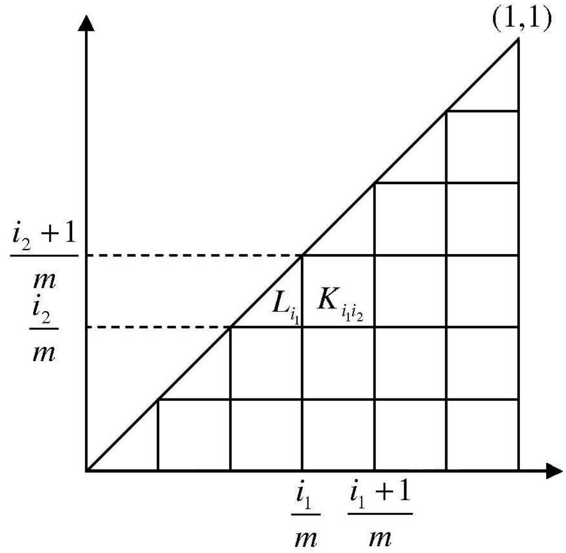

Figure 4.5: Subdivision of Δ^2

$z = dc = \sum \overline{c}^{j}_{i_1 i_2}$. Therefore z bounds a 2-chain consisting of sum of 2-simplices which have supports in the union of two intersecting open sets of the cover $\mathcal{V}$ of Lemma 4.2. According to this lemma $\mathcal{U}$ has an open set containing $V_k \cup V_j$ if $V_k \cap V_j \neq \varnothing$. Hence z bounds a 2 chain $c' = \sum c'_i$ where $supp(c'_i) \subset V_{l(i)} = V_i$. For $K = 0, 1, 2$ let ∂_K be the face operator so that

$$z = dc' = \sum_{K,j} (-1)^K \tau_j^K$$

where $\tau_j^K = \partial_K c'_j$. Let $h_j^K \in \mathrm{Ham}(M, L)$ be such that $h_j^K(supp \tau_j^K) \subset U$. Existence of such diffeomorphism are shown in Lemma 4.2. We may take $h_j^K = id_M$ if $supp(\tau_j^K) \subset U$. (U is the open set fixed since the beginning of the proof). Let $\overline{\tau_j}^K = h_j^K \tau_j^K (h_j^K)^{-1}$, then $supp(\overline{\tau_j}^K) \subset U$ and $z = \sum_{j,K} (-1)^K \overline{\tau_j}^K$. Since $C_1(B\overline{G})$ is the free abelian group over 1−simplices the chain $h_j^K + \tau_j^K - h_j^K$ is the same as τ_j^K

Let $g_j \in \mathrm{Ham}(M, L)$ such that $g_j(V_j) \in U$, whose existence is shown in

Lemma 4.2. We need to show that

$$\sum_{K=0}^{2}(-1)^K\overline{\tau_j}^K \quad and \quad d(g_jc_j'g_j^{-1})$$

are homologous in $B\overline{\mathrm{Ker}R_{U,U\cap L}}$ to finish injectivity. Let $\widetilde{h_j}^K$ and $\widetilde{g_j}$ be restrictions of h_j^K and g_j to V_j. By Lemma 4.2 a diffeomorphism $H_j^K \in G_{U,U\cap L}$ such that

$$H_j^K \circ \widetilde{h_j}^K(V_j) = \widetilde{g_j}(V_j)$$

exists. This yields

$$\widetilde{h_j}^K\tau_j^K(\widetilde{h_j}^K)^{-1} = h_j^K\tau_j^K({h_j}^K)^{-1}$$

and

$$g_j\tau_j^K g_j^{-1} = \widetilde{g_j}\tau_j^K\widetilde{g_j}^{-1} = H_j^K(\widetilde{h_j}^K\tau_j^K(\widetilde{h_j}^K)^{-1})(H_j^K)^{-1} = H_j^K(\widetilde{\tau_j}^K)(H_j^K)^{-1}$$

According to Proposition C.2, $H_j^K(\widetilde{\tau_j}^K)(H_j^K)^{-1}$ and $\widetilde{\tau_j}^K$ are homologous in $B\overline{\mathrm{Ker}R_{U,U\cap L}}$. Thus

$$d(g_jc_j'g_j^{-1}) = \sum_{K=0}^{2}(-1)^K g_j\tau_j^K g_j^{-1}$$

is homologous to $\sum_{K=0}^{2}(-1)^K\widetilde{\tau_j}^K$.

$\square$

CHAPTER 5

THE RELATIVE HERMAN THEOREM

In order to prove the perfectness of the group of relative symplectic hamiltonians, we need to show that this is true at least for a manifold. As in the absolute versions of both the smooth and the symplectic category the n-torus is the candidate. The aim of this chapter is to prove that $\mathrm{Ham}(T^{2n}, T^n)$ is perfect. We will adopt the proof for the absolute case due to Herman-Sergeraert and Banyaga. Perfectness of $\mathrm{Ham}(T^{2n}, T^n)$ needs the smooth version for the torus.

5.1 Relative Herman Theorem

Definition 5.1. A point $\gamma = (\gamma_1, ..., \gamma_n) \in \mathbb{R}^n$ is said to satisfy the diophantine condition if there exist $c, d > 0$ such that for any $(k_0, k_1, ..., k_n) \in \mathbb{Z} \times (\mathbb{Z}^n - \{0\})$ we have

$$\left| k_o + \sum_{i=1}^{n} k_i \gamma_i \right| \geq \frac{c}{(\sum_{i=1}^{n} |k_i|)^d}.$$

We say that $\alpha \in T^n$ satisfies a diophantine condition if some lift $\tilde{\alpha} \in \mathbb{R}^n$ satisfies a diophantine condition. The proof of the following theorem will be discussed in the appendix. For the absolute version see [9]. Throughout this chapter, by an element $\alpha \in (T^{2n}, T^n)$ we mean that α is of the form $\alpha = (\alpha_1, 0, \alpha_3, o, ..., \alpha_{2n-1}, 0)$, where $\alpha_i \in S^1_i$ of factors of $T^{2n} = S^1_1 \times .... \times S^1_{2n}$.

Theorem 5.2. *Let* $\alpha \in (T^{2n}, T^n)$ *satisfy a diophantine condition. There is a neighborhood* U *of the identity in* $\mathrm{Diff}_0^\infty(T^{2n}, T^n)$ *and a smooth map* $s : U \to \mathrm{Diff}_0^\infty(T^{2n}, T^n) \times (T^{2n}, T^n)$ *such that if*

$$s(\varphi) = (\psi, \lambda), \quad \varphi \in U$$

then

$$\varphi = R_\lambda \psi R_\alpha \psi^{-1}.$$

Therefore

$$\varphi = R_{\lambda+\alpha} R_\alpha^{-1} \psi R_\alpha \psi^{-1} = R_{\lambda+\alpha}[R_\alpha^{-1}, \psi].$$

This yields the Relative Herman theorem in the smooth category below.

Theorem 5.3. $\mathrm{Diff}_0^\infty(T^{2n}, T^n)$ *is perfect*

Proof. By the above theorem any small diffeomorphism φ of the torus is the composition of a rotation and a commutator. Hence it is enough to show that any rotation is a product of commutators. Note that a rotation $R_\lambda \in \mathrm{Diff}_0^\infty(T^{2n}, T^n)$ means that $R_\lambda \in \mathrm{Diff}_0^\infty(T^{2n})$ is a rotation with $\lambda = (x_1, 0, x_3, 0, ..., x_{2n-1}, 0) \in T^{2n}$. So the proof of the absolute Herman theorem (see [2] for instance) works perfectly well in the relative case. We include the proof of Herman for the sake of completeness.

The natural embedding of S^1 into $T^{2n} = S^1 \times ... \times S^1$ allows us to write any rotation R_λ, $\lambda \in T^{2n}$ as $R_\lambda = R_{\lambda_1} \circ ... \circ R_{\lambda_{2n}}$, where $\lambda_i \in S^1$. So it is enough to show that rotations of circles are product of commutators.

If H is the group of biholomorpic transformations of the disk $D = \{Z \in \mathbb{C} |\, \|Z\| < 1\}$, the Schwars lemma says any $g \in H$ can be written as:

$$z \mapsto g(z) = \frac{\alpha(z-a)}{1 - \bar{a}z}, \quad z \in \Delta, \quad \alpha \in \partial D = S^1, \text{ and } \quad a \in \Delta.$$

Such g extends uniquely into a diffeomorhism of S^1. Therefore we get an injective homomorphism $H \hookrightarrow \mathrm{Diff}_0^\infty(T^1)$. Note that $H \approx PSL(2, \mathbb{R})$ and hence

H is perfect. For $a = 0$, g is just a rotation. Hence the group of rotations of S^1 injects into H and therefore any R_λ, $\lambda \in S^1$ is a product of commutators in $H \hookrightarrow \mathrm{Diff}_0^\infty(T^{2n}, T^n)$. $\square$

Remark 5.4. *Rybicki showed that the above theorem is true for all (M, N), where M is a smooth manifold and $N \subset M$ a submanifold in [19].*

5.2 Relative Flux of Torus (T^{2n}, T^n)

Recall that the m-torus T^m is the quotient of $\mathbb{R}^m$ by $\mathbb{Z}^m$. Denote by $p : \mathbb{R}^m \to T^m$ the canonical projection. For each $x \in T^m$ denote its lift in $\mathbb{R}^m$ by $\tilde{x}$ i.e. $p(\tilde{x}) = x$. Then $R_x(\theta) = p(\tilde{x} + \tilde{\theta})$ is the rotation by x, which is a symplectomorphim.

If $x = (x_1, 0, x_3, 0, ..., x_{2n-1}, 0) \in (T^{2n}, T^n)$, then $R_x \in \mathrm{Symp}(T^{2n}, T^n)$. Hence (T^{2n}, T^n) is a subgroup of $\mathrm{Symp}(T^{2n}, T^n)$. In fact the natural map

$$j : (\mathbb{R}^{2n}, \mathbb{R}^n) \to \widetilde{\mathrm{Symp}}(T^{2n}, T^n)$$

given by $x \mapsto j(x) = R_{p(tx)}$ $t \in [0, 1]$, covers the injection
$(T^{2n}, T^n) \to \mathrm{Symp}(T^{2n}, T^n)$.

As was shown by Ozan the relative flux maps $\widetilde{\mathrm{Symp}}(T^{2n}, T^n)$ surjectively to $H^1(T^{2n}, T^n, \mathbb{R}) \cong \mathbb{R}^n$ and $\Gamma(T^{2n}, T^n) \subset H^1(T^{2n}, T^n, \mathbb{Z}) \cong \mathbb{Z}^n$ is a subgroup. To see that $\Gamma(T^{2n}, T^n) = H^1(T^{2n}, T^n, \mathbb{Z}) \cong \mathbb{Z}^n$, let $x \in (\mathbb{R}^{2n}, \mathbb{R}^n)$. Consider a basis $\{c_1, ..., c_n\}$ of $H_1(T^{2n}, T^n, \mathbb{Z})$, where each c_i is represented by the loops in (T^{2n}, T^n) which rotate each odd factor i.e. the projection on T^{2n} of the following curves in $\mathbb{R}^{2n}$

$$c_i(t) = (0, ..., 0, t, 0, ..., 0) \quad i = 1, 3, ..., 2n - 1,$$

(t at the ith factor).

Then relative flux is given by

$$\int_0^1 \int_0^1 (c_i \circ j(tx))^* w \; dsdt$$

where $(c_i \circ (j(tx)))(s,t) = p(t_{x_1}, ..., t_{x_{i-1}}, t_{x_i} + s, t_{x_{i+1}}, ..., t_{x_{2n}})$. Thus $\widetilde{Flux}(j(x)) = (0, x_1, 0, x_3, ..., 0, x_{2n-1}) \in (\mathbb{R}^{2n}, \mathbb{R}^n) \cong \mathbb{R}^n$.

Proposition 5.5. *The restriction of relative flux to $(T^{2n}, T^n) \cong T^n \subset Sym(T^{2n}, T^n)$ is the identity isomorphism $J : T^n \to T^n$.*

$$
\begin{array}{ccccc}
(\mathbb{R}^{2n}, \mathbb{R}^n) & \hookrightarrow \widetilde{\mathrm{Symp}}(T^{2n}, T^n) \to & & H^1(T^{2n}, T^n; \mathbb{Z}) \\
\downarrow & \downarrow & & \downarrow \\
(T^{2n}, T^n) & \hookrightarrow \mathrm{Symp}(T^{2n}, T^n) \to & & H^1(T^{2n}, T^n; \mathbb{Z})/\Gamma(T^{2n}, T^n) \\
\| & & & \| \\
T^n & \longrightarrow & & \mathbb{R}^n/\mathbb{Z}^n = T^n
\end{array}
$$

The absolute version of the following theorem is due to Banyaga [2]. Now we are ready to prove the main theorem of this section.

Theorem 5.6. $H_1(B\overline{\mathrm{Ham}(T^{2n}, T^n, \mathbb{Z})}) = 0$.

Proof. We first show that

$$\widetilde{\mathrm{Ham}}(T^{2n}, T^n) = ker(Flux_{rel}) = [\widetilde{\mathrm{Symp}}(T^{2n}, T^n), \widetilde{\mathrm{Symp}}(T^{2n}, T^n)].$$

To do this we must establish $\widetilde{\mathrm{Ham}}(T^{2n}, T^n) \subset [\widetilde{\mathrm{Symp}}(T^{2n}, T^n), \widetilde{\mathrm{Symp}}(T^{2n}, T^n)]$. Let $\alpha \in (T^{2n}, T^n)$ satisfying a diophantine condition. Then by Theorem 5.2 there is a neighborhood V of the rotation R_α in $\mathrm{Diff}_0^\infty(T^{2n}, T^n)$, being the domain of a smooth map $s : V \to \mathrm{Diff}_0^\infty(T^{2n}, T^n) \times (T^{2n}, T^n)$, such that if $\Phi \in V$ and $s(\Phi) = (\psi, \beta)$, then $\Phi = R_\beta \psi R_\alpha \psi^{-1}$. If $\{\Phi_t\} \in \widetilde{\mathrm{Ham}}(T^{2n}, T^n)$ such that Φ_t is an isotopy in $\mathrm{Ham}(T^{2n}, T^n)$, small enough to be in V, then there are smooth families $\psi_t \in \mathrm{Diff}_0^\infty(T^{2n}, T^n)$ and $\beta_t \in (T^{2n}, T^n)$ satisfying

$$\Phi_t = R_{\beta_t}(\phi_t) R_\alpha \psi_t^{-1}.$$

We have $w = \Phi_t^* w = (\psi_t^{-1})^*(R_\alpha^*[(\psi_t)^* w])$, hence $\psi_t^* w = R_\alpha^*(\psi_t)^* w$.

The diophantine condition on α implies that R_α is an irrational rotation and hence has a dense orbit. Since $\psi_t^* w$ is invariant by this rotation with dense orbit it must be a constant form, i.e.

$$w_t = \psi_t^* w = \sum_{i \le j} a_{ij}^t dx_i \wedge dx_j$$

where a_{ij}^t's are constant. Since w_t and w have the same periods, $a_{ij}^t = \delta_{ij}$. Hence $(\psi_t^{-1})^* w = w$ and thus we see that $\psi_t \in \mathrm{Symp}(T^{2n}, T^n)$. Since $\Phi_t \in \mathrm{Ham}(T^{2n}, T^n)$ we have

$$\begin{aligned} 0 = Flux([\Phi_t]) &= Flux(R_{\beta_t}) - Flux([\psi_t]) + Flux(R_\alpha) + Flux([\psi_t]) \\ &= Flux(R_{\beta_t}) + Flux(R_\alpha) = j(\beta_t + \alpha) = \beta_t + \alpha. \end{aligned}$$

Therefore $\beta_t = -\alpha$ for all t hence $\Phi_t = R_\alpha^{-1}\psi_t R_\alpha \psi_t^{-1}$. This shows that $\widetilde{\mathrm{Ham}}(T^{2n}, T^n) = [\widetilde{\mathrm{Symp}}(T^{2n}, T^n), \widetilde{\mathrm{Symp}}(T^{2n}, T^n)]$ and fixing the parameter t to be 1 gives $\mathrm{Ham}(T^{2n}, T^n) = [\mathrm{Symp}(T^{2n}, T^n), \mathrm{Symp}(T^{2n}, T^n)]$.

Consider the setting

$$\begin{aligned} b_t &= JFlux(\psi_t) \\ u_t &= -\alpha + b_t \\ \bar{\psi}_t &= R_{b_t}(\psi_t)^{-1} \\ \hat{\psi}_t &= (R_{b_t})^{-1}\psi_t \end{aligned}$$

With this we have

$$\begin{aligned} \phi_t &= R_{u_t}\hat{\psi}_t R_{u_t}^{-1}\bar{\psi}_t \\ \hat{\psi}_t &= R_{b_t}(\bar{\psi}_t)^{-1}R_{b_t}^{-1} \end{aligned}$$

Clearly $\hat{\psi}_t$ is Hamiltonian. Therefore by fragmentation lemma there are relative symplectic isotopies $\psi_t^j,\ j = 1, ..., N$ supported in the ball U_j of any open cover

$\mathcal{U} = \{U_i\}$ such that $\hat{\psi}_t = \psi_t^1 \psi_t^2 ... \psi_t^N$. Then

$$\phi_t = (\prod_{i=1}^{N} R_{u_t}^{-1} \psi_t^i R_{u_t})(\prod_{i=1}^{N} R_{b_t} (\psi_t^{N+1-i})^{-1} R_{b_t}^{-1})$$

There exists balls B_j and B_j' such that

$$U_j \cup R_{u_t}^{-1}(U_j) \subset B_j, \; U_j \cup R_{b_t}(U_j) \subset B_j'$$

since R_{u_t} and R_{b_t} are close to the identity. By Lemma 3.10 there are relative symplectic isotopies f_t^i and g_t^i supported in D_j and D_j' respectively and equal to R_{u_t} on B_j and to R_{b_t} on B_j' respectively. Recall that we require $\bar{B}_j \subset D_J$ nd $\bar{B}_j' \subset D_j'$. Since supported in balls f_t^i and g_t^i are Hamiltonian indeed. This gives

$$R_{u_t}^{-1} \psi_t^i R_{u_t} = (f_t^i)^{-1} \psi_t^i f_t^i$$

since their supports are contained in $R_{u_t}^{-1}(U_i) = (f_t^i)^{-1}(U_i)$, and the above diffeomorphisms coincide. Therefore,

$$\phi_t = (\prod_{i=1}^{N} (f_t^i)^{-1} \psi_t^i f_t^i)(\prod_{i=1}^{N} (g_t^{N+1-i})^{-1} (\psi_t^{N+1-i})^{-1} (g_t^{N+1-i})$$

Note that all the isotopies in above equation are Hamiltonian. Changing the order of the terms in the final expression of ϕ_t results in id_M. This means that the image of ϕ_t in $H_1(\widetilde{\mathrm{Ham}}(M, L))$ by the canonical mapping
$\widetilde{\mathrm{Ham}}(T^{2n}, T^n) \to [\widetilde{\mathrm{Ham}}(T^{2n}, T^n), \widetilde{\mathrm{Ham}}(T^{2n}, T^n)]$ is trivial.
Thus $H_1(\widetilde{\mathrm{Ham}}(T^{2n}, T^n)) = 0$.
Since $H_1(\widetilde{\mathrm{Ham}}(T^{2n}, T^n)) = H_1(B\overline{\mathrm{Ham}(T^{2n}, T^n)}, \mathbb{Z})$ the proof is complete. $\square$

5.3 Proof of the Main Theorem

$\mathrm{Ham}(M,L)$ is not simple because of the following. Consider the sequence of groups and homomorphisms:

$$0 \longrightarrow \mathrm{Ker}\varphi \longrightarrow \mathrm{Ham}(M,L) \xrightarrow{\varphi} \mathrm{Diff}^{\infty}(L) \longrightarrow 0,$$

where φ is just restriction to L. Therefore $\mathrm{Ker}\varphi$ consists of Hamiltonian diffeomorphisms of M that are identity when restricted to L. Clearly, $\mathrm{Ker}\varphi$ is a closed subgroup.

For the perfectness, we need to show that $H_1(\mathrm{Ham}(M,L)) = 0$, where M is a connected, closed, symplectic manifold, $L \subset M$ a connected, oriented Lagrangian submanifold such that $M \backslash L$ is connected. This is equivalent to $H_1(\widetilde{\mathrm{Ham}}(M,L)) = 0$, where $\widetilde{\mathrm{Ham}}(M,L)$ is the universal cover of $\mathrm{Ham}(M,L)$. As noted in Appendix B, we have $H_1(B\overline{\mathrm{Ham}}(M,L),\mathbb{Z}) = H_1(\widetilde{\mathrm{Ham}}(M,L))$. By the Deformation Lemma 4.10 $H_1(B\overline{\mathrm{Ham}}(M,L),\mathbb{Z})$ is the same for any (M,L) satisfying above properties. Hence, if $H_1(\mathrm{Ham}(M,L)) = 0$ for just one pair (M,L), then it is true for all (M,L). Now the result follows from the Theorem 5.2.

Corollary 5.7. *The commutator subgroup* $[\mathrm{Symp}_0(M,L), \mathrm{Symp}_0(M,L)]$ *is perfect and equals to* $\mathrm{Ham}(M,L)$.

Proof. $[\mathrm{Symp}_0(M,L), \mathrm{Symp}_0(M,L)] \subset \mathrm{Ham}(M,L)$, since $\mathrm{KerFlux}_{rel} = \mathrm{Ham}(M,L)$. Since $\mathrm{Ham}(M,L)$ is perfect we have

$$\mathrm{Ham}(M,L) = [\mathrm{Ham}(M,L), \mathrm{Ham}(M,L)] \subset [\mathrm{Symp}_0(M,L), \mathrm{Symp}_0(M,L)].$$

□

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APPENDIX A

THE GROUP Sp(2n,n)

The aim of this chapter is to show that the group $Sp(2n,n)$ of has two components.

Let $w = dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n$ denote the standard symplectic bilinear form on $\mathbb{R}^{2n}$. Let P be a typical element of $Sp(2n,n)$.

$$P = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}.$$

Let $(Px)_i$ denote the i$^t h$ component of the image of $x = (x_1, .., x_n, y_1, .., y_n) \in \mathbb{R}^{2n}$. Then $(Px)_1 = a_{11}x_1 + ... + a_{n1}x_n$,...,$(Px)_n = a_{1n}x_1 + ... + a_{nn}x_n$ (or $Px_i = Ax_i = a_{1i}x_1 + ... + a_{ni}x_n$, for all $i \in \{1, 2, ..., n\}$ and
$(Py)_1 = b_{11}x_1 + ... + b_{n1}x_n + c_{11}y_1 + ... + c_{n1}y_n$,..., $(Py)_n = b_{1n}x_1 + ... + b_{nn}x_n + c_{1n}y_1 + ... + c_{nn}y_n$

(or, $(Py)_i = b_{1i}x_1 + ... + b_{ni}x_n + c_{1i}y_1 + ... + c_{ni}y_n$, for all $i \in \{1, 2, ..., n\}$). We calculate $d(Px)_k \wedge d(Py)_k$ to check the conditions on P to be symplectic.

We have, $d(Px)_k \wedge d(Py)_k = \sum_{i,j=1}^{n}(a_{ik}b_{jk})dx_i \wedge dx_j + \sum_{i,j=1}^{n}(a_{ik}c_{jk})dx_i \wedge dy_j$

$$\sum_{k=1}^{n} d(Px)_k \wedge d(Py)_k = \sum_{i,j,k=1}^{n} (a_{ik}b_{jk})dx_i \wedge dx_j + \sum_{i,j,k=1}^{n} (a_{ik}c_{jk})dx_i \wedge dy_j$$

Hence we must have, for fixed $i < j$, $\sum_{k=1}^{n}(a_{ik}b_{jk} - a_{jk}b_{ik})dx_i \wedge dx_j = 0$

for fixed $i < j$, $\sum_{k=1}^{n} a_{ik}c_{jk} = 0$

for $i = j$, $\sum_{k=1}^{n} a_{ik}c_{ik} = 1$.

In other words, $\sum_{k=1}^{n} a_{ik}c_{jk} = \begin{cases} 0\ , & i < j \\ 1\ , & i = j \end{cases}$.

$\sum_{k=1}^{n} a_{ik}c_{jk} = 0$ and $\sum_{k=1}^{n} a_{ik}c_{ik} = 1$ for all $i < j$ and all i respectively implies that $AC^T = I$ and hence $C = (A^T)^{-1}$. Also, $\sum_{k=1}^{n}(a_{ik}b_{jk} - a_{jk}b_{ik}) = 0$ implies that $\sum_{k=1}^{n}(a_{ik}b^T_{kj} - a_{jk}b^T_{ki}) = 0$. $A \in GL(n, \mathbb{R})^+ (i.e. \det A > 0)$.

$(AB^T)_{ij} = (AB^T)_{ji}$ for all i, j, hence AB^T is symmetric.

Thus,

$$P = \begin{bmatrix} A & B \\ 0 & (A^T)^{-1} \end{bmatrix}$$

is in $Sp(2n, n)$ with $AB^T = BA^T$ (If A is in $O(n)$ then $B = AB^T A$.)

Since $A \in GL(n, \mathbb{R})^+$, let $\gamma(t) \in GL(n, \mathbb{R})^+, \gamma(0) = I, \gamma(1) = A$. Let $D = AB^T$ and note that since D is symmetric, tD is also symmetric for all t.

Let $\beta = tD^T(\gamma(t)^{-1})^T$, which gives $\beta(0) = 0, \beta(1) = BA^T(A^{-1})^T = B$. Hence

$$P(t) = \begin{bmatrix} \gamma(t) & \beta(t) \\ 0 & (\gamma(t)^T)^{-1} \end{bmatrix}$$

is in $Sp(2n, n)$ for all t, and

$$P(0) = I_{2n} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

$$P(1) = P = \begin{bmatrix} A & B \\ 0 & (A^T)^{-1} \end{bmatrix}.$$

Thus we have found a path from the identity matrix I to an arbitrary matrix $P \in Sp(2n, n)$ provided that its first $n \times n$ block matrix A has positive determinant. This is equivalent that while P leaves a Lagrangian subspaces $L \subset \mathbb{R}^{2n}$ invariant, it does not change its orientation either. Therefore we have proved:

Proposition A.1. *$Sp(2n, n)$ has two components:*

$$\left\{ \begin{bmatrix} A & B \\ 0 & (A^T)^{-1} \end{bmatrix} \mid \det A > 0, AB^T = BA^T \right\}$$

and

$$\left\{ \begin{bmatrix} A & B \\ 0 & (A^T)^{-1} \end{bmatrix} \mid \det A < 0, AB^T = BA^T \right\}.$$

APPENDIX B

PALAMADOV OPERATOR

The following arguments belong to Banyaga [2]. We include these into the thesis for the sake of completeness.

Let $f : M \to \mathbb{R}$ be a smooth function that is locally constant on the Lagrangian submanifold L of the symplectic manifold M. Denote the set of such functions as $C^\infty_L(M)$. Such f induces a continuous linear operator $\tilde{f} : B^1(M, L) \to B^1(M, L)$ as follows: A classical result due to Palamadov [17] asserts that there is a continuous linear map $\sigma_p : B^p(M) \to \wedge^{p-1}(M)$ such that $\omega = d(\sigma_p(\omega))$ for all $\omega \in B^p(M)$, where $p = 0, 1, ..., \dim M$. In particular the case $p = 1$ gives

$$\sigma : B^1(M) \to C^\infty(M).$$

If we denote the set of exact 1-forms that evaluates zero on TL by $B^1(M, L)$ then the above map induces

$$\sigma_{rel} : B^1(M, L) \to C^\infty_L(M).$$

Then define the linear functional $\tilde{f}$ as

$$\tilde{f}(\xi) = d(f\sigma_{rel}(\xi)) \tag{B.1}$$

Note that this operator is bounded.

APPENDIX C

THE SIMPLICIAL SET $B\overline{G}$

The Deformation Lemma is proved on a topological group $B\overline{G}$ constructed out of a discrete group G. We include this section whose original is due to Banyaga [2], to make the thesis a complete, readable manuscript.

Let G be a connected group. Define $S(G)$, the singular complex of G as $S(G) = \{G_n\}$ where G_n is the set of continuous mappings $f : \Delta^n \to G$. Here Δ^n is the standard n-simplex in $\mathbb{R}^n$. Then G acts on the right on $S(G)$ by $(c, g) \mapsto c \cdot g^{-1}$ where $c \in G_n$, $(cg^{-1})(x) = c(x) \cdot g^{-1}$ $x \in \Delta^n$. The quotient space $B\overline{G} = S(G)/G$ is a simplicial set whose n-simplices $(B\overline{G})_n$ can be identified with continuous mappings $c : \Delta^n \to G$ with $c(v_0) = e$, where e is the neuter element of G. We have the usual face and the degeneracy operations :

$$\partial_i : (B\overline{G})_n \to (B\overline{G})_{n-1} \qquad s_i : (B\overline{G})_n \to (B\overline{G})_{n+1} \quad 0 \le i \le n$$

satisfying

$$\begin{array}{ll} \partial_i \partial_j = \partial_{j-1} \partial_i & i \le j \\ s_i s_j = s_{j+1} s_i & i \le j \\ \partial_i s_j = s_{j-1} \partial_i & i \le j \\ \partial_j s_j = id = \partial_{j+1} s_j & i \ge j+1 \end{array}$$

$B\overline{G}$ is, moreover, a Kan complex: for any $n+1$ n-simplices, $x_0, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{n+1} \in (B\overline{G})_n$ such that $\partial_i x_j = \partial_{j-1} x_i,\ i \le j,\ i \ne k,\ j \ne k$ there exists an $(n+1)$-simplex x such that $\partial_i x = x_i,\ i \ne k$. Two simplices

$x,\ x' \in (B\overline{G})_n$ are said to be homotopic, $x \sim x'$, if $\partial_i x = \partial x'$ for $0 \le i \le n$ and there exists an (n+1)-simplex y, called a homotopy between x and x' such that $\partial_n y = x, \partial_{n+1} y = x'$ and $\partial_i y = s_{n-1}\partial_i x = s_{n-1}\partial_i x'$ for $0 \le i \le n$. This is an equivalence relation. If one denotes the unique element of $(B\overline{G})_0$ with $\emptyset$ then $B\overline{G}_n(\emptyset)$ to be the set of n-simplices of $B\overline{G}$) which are homotopic to $s_{n-1}s_{n-2}...s_1 s_0(\emptyset)$. Denote this element again by $\emptyset$. The quotient group $B\overline{G}_n(\emptyset)/\sim$ is called the n^{th} homotopy group of $B\overline{G}$ and denoted by $\pi_n(B\overline{G}, \emptyset)$. Since $B\overline{G}_1(\emptyset) = B\overline{G}_1$ ($(B\overline{G})_0$ having a unique element), we have $\pi_1(B\overline{G}, \emptyset) = (B\overline{G})_1/\sim$. Recall that $\sigma_1, \sigma_2 : \Delta^1 = [0,1] \to G$ are homotopic if and only if there is a continuous map; $H : \Delta^2 \to G$ with $H(0) = e,\ \partial_1 H = \sigma_1,\ \partial_2 H = \sigma_2,\ \partial_0 = \emptyset$. The last equation means that H is a homotopy between the paths $\sigma_1(t)$ and $\sigma_2(t)$ with fixed extremities $\sigma_1(0) = \sigma_2(0) = e$ and $\sigma_1(1) = \sigma_2(1)$. Indeed, for t in the face $[1,2]$ of Δ^2, $(\partial_0 H)(t) = H(t)H(1^{-1}) = e$. In particular $H(2) = H(1)$.

Proposition C.1. *For any path connected topological group G, $\pi_1(B\overline{G}) = \tilde{G}$: the universal covering of G.*

The homology of $B\overline{G}$ is defined in a standard way: Let $C_n(B\overline{G})$ be the free abelian group generated by n-simplices. Define a differential

$$d = \sum_{i=0}^{n} (-1)^i \partial_i :\ C_n(K) \longrightarrow C_{n-1}(K)$$

Then $C(B\overline{G}) = (\oplus C_n(B\overline{G}), d)$ is a chain complex whose homology is $H_*(B\overline{G}, \mathbb{Z})$. $H_*(B\overline{G}, K)$ for any abelian group K is defined as the homology of $C(B\overline{G}) \otimes K$. As usual we have $H_1(B\overline{G}, \mathbb{Z}) = \pi_1(B\overline{G}, \emptyset)/[\pi_1(B\overline{G}, \emptyset), \pi_1(B\overline{G}, \emptyset)] = H_1(\tilde{G})$ is the abelianization of $\tilde{G}$. This means that a path $h : [0,1] \to G$ with $h(0) = e$ determines the zero element in $H_1(B\overline{G}, \mathbb{Z})$ if and only if $h(t)$ is homotopic relatively to ends to a path $g(t)$ of the form $g(t) = [u_1(t), v_1(t)]...[u_m(t), v_m(t)]$, where u_i, v_i are continuous paths in G starting at e. The following remark will be used in the proof of the deformation lemma.

Proposition C.2. *If $v : [0,1] \to G$ is a 1-simplex of $B\overline{G}$, where G is a path connected topological group, and $g \in G$ then the 1-simplices $I_g(v) : t \mapsto g \cdot v(t) \cdot g^{-1}$ and $v : t \mapsto v(t)$ are homologous*

Proof. Equivalently we show $t \mapsto g \cdot v(t) \cdot g^{-1} \cdot v(t)^{-1} = [g, v(t)]$ is homologous to zero. Since G is path connected we consider the 1-simplex in G given by a path from g to e, i.e. $tv(t)]$ and $t \mapsto [g_t, v(t)]$ are homologous. Define a homotopy $H_{(s,t)},\ (s,t) \in [0,1] \times [0,1]$ between them with fixed extremities. Setting $H_{(s,t)} = [g_{s+t-st}, v(t)]$ yields $H_{(0,t)} = [g_t, v(t)],\ H_{(1,t)} = [g, v(t)],\ H_{(s,0)} = [g_s, v(0)] = [g_s, e] = e,\ H_{(s,1)} = [g, v(1)]$. □

Remark C.3. *Let G be a topological group and G_δ be the underlying discrete group (i.e. with discrete topology). Then $i : G_\delta \to G$ identity map is continuous. Since any continuous map can be turned into a fibration denote by $\bar{G}$ the homotopy fiber of this map. Then $B\bar{G}$ is nothing than the classifying space $B(\bar{G})$ of $\bar{G}$. See [2] for a discussion.*

APPENDIX D

HERMAN-SERGERAERT THEOREM

In this section we include the proof of the relative Herman Sergeraert theorem, which is exactly the same proof for the absolute case of Herman and Sergeraert. We include this proof for the sake of completeness, following Banyaga's book [2]. The proof relies on Nash-Moser-Sergeraert implicit function theorem. The details can be found in Sergeraert's thesis [21]. The category in which the proof works is called the "$\mathcal{L}$ category".

Definition D.1. An object $\mathcal{L}$ is a quadriple (E, B, η, ρ) where

(i) E is a Frechet space and B is an open set of E.

(ii) $\eta = (|, |_i)_{i\in\mathbb{N}}$ is an increasing family of semi-norms defining the topology of E.

(iii) $\rho = ((S_t)_{t\in(0,\infty)})$ is an increasing family of smoothing operators $s_t : E \to E$ such that

$$|S_t x|_{i+k} \leq t^k |x|_i$$

$$|x - S_t x|_i \leq C_{ik} |t|^{-k} |x|_{i+k}.$$

An object (E, B, η, ρ) simply denoted (E, η, ρ) is called an $\mathcal{L}$-Frechet space. If η and ρ are understood, we say simply that E is an $\mathcal{L}$-Frechet space.

Definition D.2. Let (E, B, η, ρ) be an $\mathcal{L}$-object and $F_1, ..., F_q, G$ $\mathcal{L}$-Frechet spaces. A mapping $f : B \times F_1 \times ... \times F_q \to G$ is called a $C^r(0 \leq r \leq \infty)$

$q - \mathcal{L}$-morphism if

(i) f is linear in the last q variables.

(ii) $\forall k$, $0 \leq k \leq r+1$, $\exists d_k > 0$ (independent of i) such that $\forall i \in \mathbb{N}$, the map

$$f : (B \times F_1 \times ... \times f_q, |\cdot|_{i+d_k}) \to (G, |\cdot|_i)$$

is C^k.

(iii) if $d^k f$ denotes the kth-derivative of f with respect to the first variable, then

$$d^k f : B \times F_1 \times ... \times F_q \times E^k \to G$$

satisfies

$$\begin{aligned} |(d^k f)(x; y_1, ..., y_q, ..., \tilde{x}_1, ..., \tilde{x}_k)|_i &\leq C_{i,k}(1 + |x|_{i+d_k})|y_1|_0...|y_q|_0|\tilde{x}_1|_0...|\tilde{x}_k|_0 \\ &+ \sum_{i=1}^{q} |y_1|_0...|y_{i-1}|_0|y_i|_{l+d_k}|y_{i+1}|_0...|y_q|_0|\tilde{x}_i|_0...|\tilde{x}_k|_0 \\ &+ \sum_{i=1}^{k} |y_i|_0...|y_q|_0|\tilde{x}_i|_0...|\tilde{x}_{i-1}|_0|\tilde{x}_i|_{l+d_k}...|\tilde{x}_{i+1}|_0|\tilde{x}_k|_0 \end{aligned}$$

where

$$x \in B, y_i \in F_i, 1 \leq l \leq q, \tilde{x}_i \in E_i, 1 \leq l \leq k.$$

An $\mathcal{L} - \mathcal{O}$-morphism is simply called an $\mathcal{L}$-morphism. If in the definition above, d_k depends on i, we say that f is a weak-$\mathcal{L}$-morphism.

Theorem D.3. *[8] Let (E, B, η, ρ) be an $\mathcal{L}$-object and F an $\mathcal{L}$-Frechet space. Let $f : B \to F$ be a $C^r(2 \leq r \leq \infty)$ $\mathcal{L}$-morphism. Let $x_0 \in B$, $y_0 = f(x_0)$. Assume there exists $C^p(0 \leq p \leq r-1)$ $1 - \mathcal{L}$-morphism. $L : B \times F \to E$ such that if $x \in B$, $y \in F$, $df(x, L(x,y)) = y$. Then there exists an $\mathcal{L}$-object $(F, C, \tilde{\eta}, \tilde{\rho})$ and a C^p weak-$\mathcal{L}$-morphism $s : C \to B$ such that $f \circ s = id_C$.*

Remark D.4. *Throughout this section we will use the identification $T^n \approx (T^{2n}, T^n)$ i.e. an element $\alpha \in T^n$ must be understood as an element of the* *form*

$$\alpha = (\alpha_1, 0, \alpha_2, 0, ..., \alpha_{2n-1,0}) \in (T^{2n}, T^n) \approx T^n.$$

Proof. **(of Theorem 5.2)** In our situation we must show that the map

$$\Phi_\alpha : \mathrm{Diff}_0^\infty(T^{2n}, T^n) \times T^n \to \mathrm{Diff}_0^\infty(T^{2n}, T^n)$$

$$(\psi, \lambda) \longmapsto R_\lambda \psi R_\alpha \psi^{-1}$$

is a C^∞ $\mathcal{L}$-morphism between the C^∞, $\mathcal{L}$-groups $\mathrm{Diff}_0^\infty(T^{2n}, T^n) \times T^n$ and $\mathrm{Diff}_0^\infty(T^{2n})$ and that its differential at $(id_{T^{2n}}, 0)$ has an inverse in the $\mathcal{L}$-category. Recall that $\alpha \in (T^{2n}, T^n)$ satisfies a diophantine condition.

Writing ϕ_α and its differential in local coordinates near the identity in $\mathrm{Diff}_0^\infty(T^{2n}, T^n)$ we get

$$\tilde{\Phi}_\alpha : X^\infty(T^{2n}, T^n) \times T^n \to T_{R_\alpha} C^\infty((T^{2n}, T^n), (T^{2n}, T^n))$$

$$(\xi, \lambda) \longmapsto (1+\xi)^{-1} \circ R_\alpha \circ (1+\xi) + \lambda - \alpha$$

$(1+\xi)$, here denotes the diffeomorphim $x \longmapsto x + \xi(x)$ of $\mathbb{R}^{2n}$ for ξ C^1-small. Denote by $1+\mu$ its inverse $(1+\xi)^{-1}$. One can verify that Φ_α is a C^∞ $\mathcal{L}$-morphism.

To show that its differential is invertible near id we first write its differential. If $d_x f : T_x M \to T_{f(x)} N$ denotes the differential of $f : M \to N$ then for $(\xi, \lambda) \in X^\infty(T^{2n}, T^n) \times T^n$ one gets

$$d_{(\xi,\lambda)} \tilde{\Phi}_\alpha : X^\infty(T^{2n}, T^n) \times \mathbb{R}^n \to T(T_{R_\alpha} C^\infty((T^{2n}, T^n), (T^{2n}, T^n)))$$

$$\approx T_{(R_\alpha)}(C^\infty((T^{2n}, T^n), (T^{2n}, T^n)))$$

and for $x \in (T^{2n}, T^n)$, $\hat{\xi} \in X^\infty(T^{2n}, T^n)$, $\hat{\lambda} \in \mathbb{R}^n$ we have

$$((d_(\xi, \lambda)\Phi_\alpha), \hat{\xi}, \hat{\lambda})(x) = (d_{[(R_\alpha \circ (1+\xi))(x)]}(1+\mu))(\hat{\xi}(x))$$

$$-(d_{[(R_\alpha \circ (1+\xi))(x)]}(1+\mu))(\hat{\xi}(x))(\xi((1+\mu) \circ R_\alpha \circ (1+\xi))(x)) + \hat{\lambda}.$$

In particular $(d\Phi_\alpha)(0,0)(\hat{\xi},\hat{\lambda}) = \hat{\xi} - (\hat{\xi}\circ R_\alpha) + \hat{\lambda}$. To find an $\mathcal{L}$-section of $d\hat{\Phi}_\alpha$ in a neighborhood of $(0,0)$ we have to solve for $\hat{\xi},\hat{\lambda}$

$$d\hat{\Phi}_\alpha(\xi,\lambda)(\hat{\xi},\hat{\lambda}) = \eta$$

for given ξ and η. To simplify this equation we multiply on the right by $1+\mu$ and on the left by $(d(1+\mu))\circ(1+\mu)\circ R_\alpha$. Setting

$$\begin{aligned}\tilde{\xi} &= \hat{\xi}\circ(1+\mu)\\ \tilde{\eta} &= d(1+\mu)\circ(1+\mu)\circ R_\alpha\circ\eta\circ(1+\mu)\\ \chi(\xi) &= d(1+\mu)\circ(1+\mu)\circ R_\alpha\end{aligned}$$

Then we have to solve

$$\tilde{\xi} - \tilde{\xi}\circ R_\alpha = \tilde{\eta} - \chi(\xi)\cdot\hat{\lambda}$$

or $\tilde{\xi}(x) - \tilde{\xi}(x+\alpha) = \tilde{\eta}(x) - \chi(\xi)(x)\cdot\hat{\lambda}$
for all $x = (x_1,...,x_n)\in T^n$.

Consider the Haar measure dx on T^n. The equality
$\int_{T^n}\tilde{\xi}(x)dx = \int_{T^n}\tilde{\xi}(x+\alpha)dx$ gives

$$\int_{T^n}\tilde{\eta}(x)dx = (\int_{T^n}\chi(\tilde{\xi})(x)dx)\cdot\hat{\lambda}.$$

Since ξ is C^1-close to zero, the matrix $A = \int_{T^n} x(\tilde{\xi})dx$ is close to the identity, so it is invertible. Thus we can get $\hat{\lambda} = \frac{1}{A}\int_{T^n}\tilde{\eta}(x)dx$.

We will use the Fourier expansion of $\tilde{\eta}(x) - \chi(\xi)(x)d\hat{\lambda} = \sum_{k\in\mathbb{Z}^n-\{0\}} b_k e^{2i\pi<k,x>}$ to compute the Fourier expansion of $\tilde{\xi}(x) = \sum_{k\in\mathbb{Z}^n-\{0\}} a_k e^{2i\pi<k,x>}$, where $a_k, b_k\in\mathbb{C}^n$, $a_{-k} = \bar{a}_k$; $b_{-k} = \bar{b}_k$ and $x = (x_1,...,x_n)\in\mathbb{R}^n$, $k = (k_1,...,k_n)\in\mathbb{Z}^n$, $< x,k >= \sum_{i=1}^n k_i x_i$. The choice of $\hat{\lambda}$ forces b_0 to be zero. Plugging these into the equation to solve we get $a_0 = 0$ and $a_k = \frac{b_k}{1-e^{2i\pi<k,\alpha>}}$, for $k\neq 0$. This with the diophantine condition on α imply

$$|a_k| \le C|b_k||k|^d,$$

where C is a constant depending only on α. Let $L^2(T^n, dx, \mathbb{R}^n)$ denote the space of mappings $f : T^n \to \mathbb{R}^n$, which are square integrable with respect to the Haar measure dx. Let $a_k(f)$ denote the k^{th} Fourier coefficient of $f \in L^2(T^n, dx, \mathbb{R}^n)$. For C^r maps from $T^n \to \mathbb{R}^n$ let

$$H^r(T^n, \mathbb{R}^n) = \{f \in L^2(T^n, dx, \mathbb{R}^n); \sum_{k \in \mathbb{Z}^n} (1 + |k|^2)^n |a_k(f)|^2 < \infty\}.$$

Let $C_0^r(T^n, \mathbb{R}^n)$ resp $H_0^r(T^n, \mathbb{R}^n)$ denote the subset consisting of elements with $a_0(f) = 0$. The inequality $|a_k| \leq C|b_k||k|^d$ implies that the map

$$L : \sum_{k \in \mathbb{Z}^n - \{0\}} b_k(f) e^{2i\pi <k,x>} \longmapsto \sum_{k \in \mathbb{Z}^n - \{0\}} (\frac{b_k(f)}{1 - e^{2i\pi <k,x>}}) e^{2i\pi <k,x>}$$

maps $H_0^r(T^n, \mathbb{R}^n)$ into $H_0^{r-d}(T^n, \mathbb{R}^n)$. Hence, by Sobolov embedding theorem L maps $C_0^r(T^n, \mathbb{R}^n)$ into $C_0^{r-s}(T^n, \mathbb{R}^n)$ where $s = d + \lfloor n/2 \rfloor + 1$. Hence, we have solved for a_k and got $\tilde{\xi}$.

This shows that the linear mapping $L : C_0^r(T^n, \mathbb{R}^n) \to C_0^{r-s}(T^n, \mathbb{R}^n)$ is a 1-$\mathcal{L}$-morphism. Now, Theorem D.3 implies that there exists a neighborhood U of the rotation R_α in $\mathrm{Diff}_0^\infty(T^{2n}, T^n)$ and a smooth map $s : U \to \mathrm{Diff}_0^\infty(T^{2n}, T^n) \times (T^{2n}, T^n)$ such that $\phi_\alpha \circ s = id|_U$. $\square$

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