THE MULTIPLE RETAILER INVENTORY ROUTING PROBLEM WITH BACKORDERS

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ABSTRACT

THE MULTIPLE RETAILER INVENTORY ROUTING PROBLEM WITH BACKORDERS

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In this study we consider an inventory routing problem in which a supplier distributes a single product to multiple retailers in a finite planning horizon. Retailers should satisfy the deterministic and dynamic demands of end customers in the planning horizon, but the retailers can backorder the demands of end customers considering the supply chain costs. In each period the supplier decides the retailers to be visited, and the amount of products to be supplied to each retailer by a fleet of vehicles. The decision problems of the supplier are about when, to whom and how much to deliver products, and in which order to visit retailers while minimizing system-wide costs. We propose a mixed integer programming model and a Lagrangian relaxation based solution approach in which both upper and lower bounds are computed. We test our solution approach with test instances taken from the literature and provide our computational results.

Keywords: Inventory Routing Problem, Lagrangian Relaxation, Backordering.

ÖZ

ÇOKLU PERAKENDECİLERDEN OLUŞAN GEÇ TESLİMATLI ENVANTER ROTALAMA PROBLEMİ

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Bu çalışmada tek tedarikçi ve perakendecilerden oluşan bir tedarik zincirinde, tek ürünlü, çok dönemli ve geç teslimatın kabul edilebildiği bir envanter-rotalama problemi işlenmiştir. Perakendecilerin, müşterilerden gelen tahmin yoluyla belirlenmiş talepleri planlama dönemi içinde karşılanmaktadır. Ancak, perakendeciler toplam zincir maliyetlerini gözeterek geç teslimat yapabilmektedir. Tedarikçi, her dönemde kime, ne kadar mal dağıtacağına karar verip, bir araç filosu ile bu dağıtımı yapmaktadır. Problem, tedarikçinin ne zaman, kime, ne kadar mal dağıtacağına ve dağıtım esnasında perakendecileri hangi sırada ziyaret edeceğine, toplam zincir maliyetlerini en azlayarak karar vermesi problemidir. Bu problem için karışık tam sayılı bir model önerilmiş ve model için Lagrange gevşetme yaklaşımına dayalı bir çözüm yöntemi geliştirilmiştir. En iyi çözüm değerleri için bu yolla alt ve üst sınırlar hesaplanmıştır. Çözüm yöntemi literatürden alınan problemlerle test edilmiş, sayısal deney sonuçları verilmiştir.

Anahtar Kelimeler: Envanter-rotalama Problemi, Lagrange Gevşetme Yaklaşımı, Geç Teslimat.

To my tears

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CHAPTER 1

INTRODUCTION

In this thesis, we study an inventory routing problem where there are multiple retailers in a supply chain and their replenishments over a finite planning horizon are planned and realized by a single source (a supplier's depot or a supplier's crossdock facility). In every period, a fleet of (non-) homogenous vehicles departs from the facility and serve a (sub) set of geographically dispersed retailers on a route and comes back to the starting facility. The demands of end customers which are realized at the retailers are dynamic and deterministic in nature. Those retailers which are not replenished in a period satisfy the demands of end customers from inventory or by backlogging. The basic aim of the problem is to minimize the system-wide costs consisting of transportation costs (fixed vehicle dispatching cost, fixed arc usage cost, variable arc usage cost depending on the amount carried on each arc), retailers' inventory holding cost and retailers' backlogging cost while deciding in each period on how many vehicles to dispatch, which retailers to visit, how much to deliver to each retailer to be visited and in which order to visit these retailers. The retailers have capacity limitation on the amount of inventory stocked. It is assumed that during the planning horizon the supplier should satisfy all the demand, while backordering is possible for any period's demand except the last period's demand.

1.1 Motivation

In inventory routing problems, basic decisions are about inventory management and distribution of products to the customers at different levels of the supply chain. There exists a great deal of studies on inventory and distribution management to optimize the related expenditures, but the distribution problem in any period is difficult to solve since it involves a well known NP-hard problem, called Traveling Salesman Problem (TSP). Therefore, the amalgamation of these two management problems leads to a problem difficult to solve.

Since inventory routing problems constitute the subject of many works done in the literature, different solution procedures and algorithms are applied to come up with reasonable outcomes. In most of the studies, minimization of the inventory holding costs and the transportation costs is considered as the major objective of the supply chain. Without routing constraints the distribution problem can be formulated as the NP-hard joint replenishment problem considered in Joneja (1990) in which a joint ordering cost for all parties in addition to individual ordering costs is incurred for orders given in any period. These costs are similar to the fixed vehicle dispatching cost and fixed arc usage costs in the inventory routing problem. However, joint replenishment problem does not consider backorders and sequence dependent fixed arc usage costs. Moreover, none of the finite horizon models with deterministic demand, reviewed in later on, except Chien et al. (1989) and Abdelmaguid and Dessouky (2006), considers backordering as an alternative option for supply chains. Chien et al. (1989) consider a single period problem where the aim is to maximize sales revenue. The single period problem starts with a predetermined inventory quantity and the best possible delivery schedule is tried to be found with respect to transportation and backlogging costs. They apply a Lagrangian Relaxation based solution approach and come up with less than 3% gaps.

However, they did not present the performance of their approach for multi-period problem settings. Abdelmaguid and Dessouky (2006) consider a finite horizon planning problem where variable transportation costs are not included; moreover, their solution approach is different from ours. We propose a Lagrangian Relaxation based solution approach whereas Abdelmaguid and Dessouky (2006) present a heuristic procedure based on backordering decisions and transportation cost estimates, to solve the problem. They come up with upper bounds that deviate about 20% from the upper bounds calculated with CPLEX solver, but they do not calculate lower bounds on the optimal solutions.

1.2 Outline of the study

The chapters of this thesis are organized as follows.

In Chapter 2, we present a review of related literature on inventory routing problems. In Section 2.1, a classification scheme concerning number of suppliers and retailers, length of the planning horizon, vehicle capacity, demand structure, cost structure, inventory policy and performance measures is presented. Then in Section 2.2, we present the literature review, and according to the planning horizon and vehicle routing aspects, another classification scheme is given.

In Chapter 3, we state the characteristics of our inventory routing problem (INVROP) according to the classification scheme presented in Chapter 2. Then in Section 3.1, we mention the differences of the INVROP with Chien et al. (1989) and Abdelmaguid and Dessouky (2006) and state the assumptions of the INVROP. In Section 3.2 a mixed integer formulation (MIP) for the INVROP is presented. Since the INVROP is NP-hard it is almost impossible solving even moderate-sized instances in reasonable times; therefore, complicated (hard to satisfy) constraints are relaxed with the Lagrange multipliers and added to the objective function. In Section 3.3, the Lagrangian based solution approach is presented. In Section 3.4 and Section 3.5, lower bound and upper bound calculation methods using Lagrangian Relaxation are explained in detail. In Section 3.6, a subgradient optimization algorithm for updating Lagrange multipliers is presented.

In Chapter 4, we present the computational results of the proposed approach. In Section 4.1, we present our computational experiment settings. In Section 4.2, we give the details of basic test instances, which are taken from the literature. In Section 4.3, the performance measures used for the tests are introduced. The results of basic test instances are presented in Section 4.4. Then, in Section 4.5, we present the results obtained when best parameter settings, determined in Section 4.4, are applied to larger settings. Lastly, in Section 4.6, for benchmarking purposes, we present the results obtained when the proposed Lagrangian Relaxation based solution algorithm is applied to the problems of Abdelmaguid and Dessouky (2006).

In Chapter 5, a generalized version of the INVROP is presented as single supplier multiple retailer inventory routing problem with backorders (SSMRIRB), in which we let supplier keep inventories. In Section 5.1, the assumptions of the SSMRIRB are stated. In Section 5.2, mixed integer programming formulation for the SSMRIB is given. In Section 5.3, a Lagrangian based solution approach is presented. The SSMRIRB is decomposed into three subproblems as supplier subproblem (SSP), retailer subproblem (RSP) and distribution subproblem (DSP). Solution methods that can be used in solving these three subproblems are explained in detail. Then how to use these three problems in the lower bound and upper bound computations are defined.

In Chapter 6, we conclude the study, present our contributions, and discuss possible directions for future works.

CHAPTER 2

LITERATURE REVIEW ON INVENTORY ROUTING PROBLEM

In inventory routing problems, decisions that enclose the routing of vehicles and the inventory policies of suppliers and retailers are combined together, in order to decrease system-wide costs. Importance given to each aspect can be different. For example, there are cases in which only direct shipments are considered since the importance is given to the inventory side, or that an inventory policy is adopted according to the least cost vehicle tours.

In this chapter, we present a classification scheme for classifying previous studies on inventory routing problem in the literature. Then the classification of previous work -ordered with respect to publication year- is presented in detail.

2.1 Classification scheme

In order to classify the related literature, a similar system with Baita, Ukovich, Pesenti, and Favaretto (1998) and Pinar (2005) is used. It consists of ten elements. The elements of the classification scheme are defined below.

2.1.1 Start point-End point (E):

The first parameter denotes the number of suppliers (or depot) and the second parameter denotes the number of retailers.

- E(1,1): One-to-one.
- E(1,M): One-to-many.
- E(M,M): Many-to-many.

2.1.2 Planning horizon (P):

Shows the number of periods the model is designed for.

- P(1): Designed for single period.
- P(T): Designed for a finite number (T) of periods.
- P(∞): Designed for infinite horizon.

2.1.3 Vehicle (V):

It denotes the capacities of the vehicles and the number of vehicles available.

- C: Homogeneous fleet of vehicles (each vehicle has the same capacity).
- C_V: Heterogeneous fleet of vehicles (each vehicle *v* has different capacity).
- 1: Single vehicle.
- M: Multiple vehicles.
- NC: There is no constraint on number of vehicles.
- DV: Number of vehicles is a decision variable in the model.

2.1.4 Demand structure:

- Dynamic: Demands may change over the planning horizon.
- Stationary: Demands do not change and are constant over the entire horizon.
- Deterministic: Demands are assumed to be known a priori.

- Stochastic: Demands are not known a priori.

2.1.5 Inventory (I):

Whether the supplier(s) and the retailers hold inventory or not is depicted. The first parameter is defined for the supplier(s) and the second is defined for retailers.

- Y: Holding inventory is allowed.
- N: Holding inventory is not allowed.

2.1.6 Backordering (B):

Whether backordering is allowed or not is depicted. The first parameter is defined for the supplier(s) and the second is defined for retailers.

- Y: Backordering is allowed.
- N: Backordering is not allowed.

2.1.7 Ordering (O):

Whether fixed ordering (setup for production) cost is applied or not. The first parameter is defined for the supplier(s) and the second is defined for retailers.

- Y: Fixed ordering (setup) cost is applied.
- N: Fixed ordering (setup) cost is not applied.

2.1.8 Inventory policy:

This component is set in order to specify the inventory control policy of the problem. If there is no specific policy defined and the model output specifies when to replenish and to whom to replenish, then it is written “endogenous” in that section.

2.1.9 Transportation cost:

- Fixed: Fixed dispatching or usage cost of vehicles is applied.
- Distance: Transportation cost is applied based on the distance traveled.
- Amount: Transportation cost is applied based on the amount of products carried.

2.1.10 Performance measures:

How the effectiveness or the powerful aspects of a solution approach are measured in the study; i.e. the gap between the lower and upper bounds, comparisons of model solutions with benchmarked results, or reasonable cost reductions (decrements in total cost or transportation cost) etc.

2.2 Literature review

In the study of Federgruen and Zipkin (1984) summarized in Table 2.1, an integrated problem of allocating given supply among several locations and their routing is considered. Distinctive feature of the study is that demand is stochastic.

A mathematical formulation of the problem and an algorithm that can be adapted to deterministic-demand case are presented. First, the inventory allocation problem is solved with relaxing vehicle capacity constraints. Second, the routing problem is solved by generating cuts. Finally, 3-opt heuristic is used in order to improve the results. Improvement stage has two phases, in the first phase only the switches between adjacent routes are considered whereas in the second stage all possible switches are considered.

According to the results, the algorithm yields 6-7% savings in operating costs and 20% reduction in the number of vehicles required.

Table 2.1 Federgruen and Zipkin (1984) in Operations Research

Component	***Characteristic***
Start Point-End Point	E(1, M)
Planning Horizon	P(1)
Vehicle(s)	V(C_V, M)
Demand Structure	Stochastic
Inventory	I(N, Y)
Backordering	B(N, Y)
Ordering	O(N, N)
Inventory Policy	Endogenous
Transportation Cost	Distance
Performance Measure(s)	% Cost reduction (relative to the benchmarked results)

In the study of Burns, Hall, Blumenfeld (1985) summarized in Table 2.2, direct shipping and peddling strategies are compared.

In direct shipping trucks visit only one customer and in peddling trucks visit more than one customer.

An economic order quantity (EOQ)-like solution method is applied such that the closed form of the solutions is derived as in the case of EOQ.

It is found that sending EOQ for direct shipping strategy and full truck load for peddling strategy are economical.

Table 2.2 Burns, Hall, Blumenfeld and Daganzo (1985) in Operations Research

Component	***Characteristic***
Start Point-End Point	E(1, M)
Planning Horizon	P(∞)
Vehicle(s)	V(C, NC)
Demand Structure	Stationary, Deterministic
Inventory	I(N, Y)
Backordering	B(N, N)
Ordering	O(N, Y)
Inventory Policy	EOQ
Transportation Cost	Fixed + Distance
Performance Measure(s)	Effects of parameters on total cost, inventory cost, distribution cost

In the work of Blumenfeld, Burns, Diltz and Daganzo (1985) summarized in Table 2.3, transportation, inventory holding and production setup costs are considered in a deterministic environment. The cost tradeoffs between inventory holding and transportation costs, and setup and inventory costs are examined for three different network structures. In direct shipment setting vehicles go from suppliers to retailers directly. In "via consolidation terminal setting", vehicles must visit a cross-docking terminal.

According to the authors, for the case in which production and transportation scheduling are independent, the total costs can be minimized by determining optimal shipment sizes using EOQ methods for each link separately. For the case in which production and transportation scheduling are synchronized, the shipment sizes on different links that are interdependent must be optimized simultaneously with production scheduling decisions.

Table 2.3 Blumenfeld, Burns, Diltz and Daganzo (1985) in Transportation Research

Component	***Characteristic***
Start Point-End Point	E(M, M)
Planning Horizon	P(∞)
Vehicle(s)	V(C, NC)
Demand Structure	Stationary, Deterministic
Inventory	I(Y, Y)
Backordering	B(N, N)
Ordering	O(Y, N)
Inventory Policy	EOQ
Transportation Cost	Fixed
Performance Measure(s)	Minimization of total costs

The problem considered in Benjamin (1989) summarized in Table 2.4, is a combination of lot sizing problem and transportation problem. It essentially does not deal with routing aspect. Direct shipment -proportional to the amount shipped- is used. One-to-many environment is decomposed in to one-to-one problem for each retailer and EOQ-like solution approach is used for solving the inventory problem of retailers. Also the problem is modified to m-suppliers, n-retailers case in which a simultaneous solution procedure GINO, which is a generalized reduced gradient algorithm, is applied. Linear programming relaxation solution is used as lower bound on the optimal solution value.

Moreover, a heuristic algorithm, which is based on sequentially solving separate sets of variables as opposed to simultaneously solving all, i.e. using GINO, is presented. It is observed that GINO yields improvements between 0.02% and 80% over sequential solutions. When GINO and heuristic are compared, the solution value of heuristic is 0.2% better than the solution value of GINO.

Table 2.4 Benjamin (1989) in Transportation Science

Component	***Characteristic***
Start Point-End Point	E(1, M)
Planning Horizon	P(∞)
Vehicle(s)	V(NC, NC)
Demand Structure	Stationary, Deterministic
Inventory	I(Y, Y)
Backordering	B(N, N)
Ordering	O(Y, Y)
Inventory Policy	EOQ
Transportation Cost	Distance
Performance Measure(s)	Total cost (production, distribution, inventory holding)

The problem in Chien, Balakrishnan and Wong (1989) summarized in Table 2.5, is a single period revenue maximization problem. The costs considered are transportation costs and backordering cost. A fixed amount of product (given as a problem parameter) is distributed to a set of customers that are geographically dispersed in order to maximize profit, which is equal to the difference between sales revenue and total cost. Our study is similar to Chien et al. (1989) in the sense of structure and variable definitions; all the similarities and distinctions will be presented in the next chapter.

Table 2.5 Chien, Balakrishnan and Wong (1989) in Transportation Science

Component	***Characteristic***
Start Point-End Point	E(1, M)
Planning Horizon	P(1)
Vehicle(s)	V(C_V, M)
Demand Structure	Stationary, Deterministic
Inventory	I(N, N)
Backordering	B(N, Y)
Ordering	O(N, N)
Inventory Policy	Endogenous
Transportation Cost	Fixed (vehicle specific) + Distance (vehicle specific)
Performance Measure(s)	% Gap between UB and LB, CPU time

A mixed integer formulation of the problem and its Lagrangian relaxation based solution algorithm are provided. The problem is decomposed into two subproblems; inventory allocation subproblem and customer assignment/vehicle utilization subproblem. Former one is solved using a greedy heuristic, and the latter one is also solved with a similar heuristic after the subproblem is further decomposed into continuous knapsack problems. For each retailer, the heuristic finds the best alternative customer to go in order to maximize profit by assigning the maximum amount (that is, the minimum between truck capacity and demand to the least cost customer). The solutions obtained are used as upper bounds on the objective of mixed integer formulation. In order to get lower bounds (feasible solutions), an add-drop heuristic is applied after obtaining upper bound solutions. Flow variables directed from depot determine the number of vehicles used. The customers that have positive flow variables are designated as visiting customers and assigned to the same vehicle. Tour costs are calculated according to the previous assignments. Then a feasibility check is done according to vehicle capacity. If capacity of a vehicle is exceeded the excess amount is deducted from the customer with least profit. If a vehicle has excess capacity, the customer with the highest profit is assigned to that vehicle if any unassigned customers exist.

The authors come up with results that are close to the optimal solutions with only 3% gap.

In the study of Gallego and Simchi-Levi (1990) summarized in Table 2.6, a lower bound on the long-run average cost (ordering, holding and transportation costs) over all inventory-routing strategies is given. Upper bound is found by using direct shipments with fully loaded truck loads.

In the study, effectiveness, which is defined as the "100% times the ratio of the infimum of the long-run average cost over all strategies to the long-run average cost of the strategy in question," is used as performance measures. It is stated that if the economic lot size over all retailers is more than 71% of truck capacity, direct shipping is at least 94% effective.

Anily and Federgruen (1990) summarized in Table 2.7, is dealing with fixed-partitioning policies, which help to partition demand points into a set of regions. Anily and Federgruen (1990) tries to find upper bounds on the minimal long-run average costs among all strategies in the class of replenishment strategies and heuristic solutions for the setting in consideration.

Table 2.6 Gallego and Simchi-Levi (1990) in Management Science

Component	***Characteristic***
Start Point-End Point	E(1, M)
Planning Horizon	P(∞)
Vehicle(s)	V(C, M)
Demand Structure	Stationary, Deterministic
Inventory	I(N, Y)
Backordering	B(N, N)
Ordering	O(N, Y)
Inventory Policy	Endogenous
Transportation Cost	Distance
Performance Measure(s)	Effectiveness

Table 2.7 Anily and Federguen (1990) in Management Science

Component	***Characteristic***
Start Point-End Point	E(1, M)
Planning Horizon	P(∞)
Vehicle(s)	V(C, NC)
Demand Structure	Stationary, Deterministic
Inventory	I(N, Y)
Backordering	B(N, N)
Ordering	O(N, N)
Inventory Policy	EOQ
Transportation Cost	Fixed (per route) + Distance (unit magnitude)
Performance Measure(s)	Gap between UB and LB, CPU Time

Rather than considering all distribution strategies, a subset of strategies in which collection of regions (set of retailers) is specified to cover all retailers is considered. Depending on that, if a retailer belongs to more than one region, then each fractional portion is also assigned to each of these regions. If a retailer in a given region is supplied, all the other retailers assigned to that region are also supplied.

In the experimentation part, a set of randomly generated test instances is used. Eight different settings are tested and these include several variants of the original problem such as, uncapacitated and capacitated cases. According to the results, the gap between upper and lower bounds ranges from 1% to 19% for the original model and from 0.1% to 42% for the scenarios considered.

Table 2.8 Chandra (1993) in Journal of Operational Research Society

Component	***Characteristic***
Start Point-End Point	E(1, M)
Planning Horizon	P(T)
Vehicle(s)	V(C, NC)
Demand Structure	Dynamic, Deterministic
Inventory	I(Y, Y)
Backordering	B(N, N)
Ordering	O(Y, N)
Inventory Policy	Endogenous
Transportation Cost	Fixed + Distance
Performance Measure(s)	% Reduction of inventory holding, ordering and transportation costs

In the study of Chandra (1993) summarized in Table 2.8, a coordination of customer and warehouse replenishment decisions is investigated. Coordinated decisions involve replenishment quantities of both retailers and supplier and the distribution routes. A mixed-integer formulation of the problem is presented. It is decomposed into two subproblems: multi-product, multi-period warehouse ordering problem and distribution planning problem.

The solution algorithm starts with solving two subproblems sequentially. First the ordering problem is solved and then the distribution problem is solved. Distribution problem is solved until no further improvement is obtained by using insertion, nearest neighbor, and swap heuristics. In the decoupled approach, it is observed that replenishment amounts are not affected by the solutions obtained from distribution subproblem; however, in the consolidation process supply quantities are adapted according to the results obtained from distribution subproblem.

In the experiments on randomly generated problem instances, it is observed that, on average, consolidation process yields better results than decoupled approach ranging from 3% to 11% improvement over the decoupled approach.

Table 2.9 Anily and Federgruen (1993) in Operations Research

Component	***Characteristic***
Start Point-End Point	E(1, M)
Planning Horizon	P(∞)
Vehicle(s)	V(C, NC)
Demand Structure	Stationary, Deterministic
Inventory	I(Y, Y)
Backordering	B(N, N)
Ordering	O(Y, N)
Inventory Policy	Endogenous
Transportation Cost	Fixed + Distance
Performance Measure(s)	%Gap between UB and LB

In the study of Anily and Federgruen (1993) summarized in Table 2.9, a variation of their previous work given in Table 2.6 is examined. Depot is allowed to keep inventory; therefore, central stock keeping is possible.

A similar solution strategy with their previous work is used, such that lower bounds are computed by using external partitioning algorithm. Upper bounds are computed by using modified circular regional partitioning algorithm. When the regions are partitioned, the problem turns into EOQ.

The gap between upper and lower bounds ranges between 6% and 12%. For the set of partitioning strategies in which regions cover all retailers, the gap between the proposed strategy (after applying external partitioning algorithm, a modified circular regional partitioning algorithm is used and finally a rounding procedure is applied) and the lower bound is less than 6% for problems with large number of retailers.

Table 2.10 Anily (1994) in EJOR

Component	***Characteristic***
Start Point-End Point	E(1, M)
Planning Horizon	P(∞)
Vehicle(s)	V(C, NC)
Demand Structure	Stationary, Deterministic
Inventory	I(N, Y)
Backordering	B(N, N)
Ordering	O(Y, N)
Inventory Policy	Endogenous
Transportation Cost	Fixed (per tour) + Distance
Performance Measure(s)	%Gap between UB and LB

In the work of Anily (1994) summarized in Table 2.10, the same problem in Anily and Federgruen (1990) is studied and generalizes the results obtained for the case in which holding costs are retailer specific. Partitioning of retailers into regions is done by taking retailer specific holding cost into account.

The experiments show that the gap between upper and lower bounds is always less than 10%. Moreover, the solutions found by the heuristic defined, converge to a lower bound when the number of retailers is increased to infinity.

In the study of Chandra and Fisher (1994) summarized in Table 2.11, production, inventory and distribution decisions are considered together. Making production and distribution decisions separately and making coordinated decisions are compared.

Table 2.11 Chandra and Fisher (1994) in EJOR

Component	***Characteristic***
Start Point-End Point	E(1, M)
Planning Horizon	P(T)
Vehicle(s)	V(C, NC)
Demand Structure	Dynamic, Deterministic
Inventory	I(Y, Y)
Backordering	B(N, N)
Ordering	O(Y, N)
Inventory Policy	Endogenous
Transportation Cost	Fixed (vehicle specific) + Variable (route specific)
Performance Measure(s)	% Reduction of inventory holding, ordering and transportation costs

In the decoupled approach, first a production schedule is determined in order to minimize the costs of production and inventory holding. Then a distribution problem is solved with given supply amounts. In the coordinated approach, it is allowed to change production schedule depending on the distribution schedule. The cost reduction obtained by coordinating production and distribution decisions ranges from 3% to 20%.

In the study of Viswanathan and Mathur (1997) summarized in Table 2.12, designed for distribution of multiple products. A new replenishment policy, called stationary nested joint replenishment policy, is defined. The authors use "stationary policy term" if replenishing items are equally spaced points in time; and "nested policy term" when replenishment times of an item are the multiples of the replenishment times of items that have smaller replenishment intervals. In order to use multiple intervals, power-of-two policies, in which the replenishment intervals are the power-of-two multiples of the base planning period, are adopted. The objective is to come out with replenishment intervals and quantities for each item and vehicle routes in order to minimize inventory holding and transportation costs.

Heuristic algorithms are developed for both uncapacitated and capacitated problem settings. At first, the marginal setup cost of adding an item to the existing set of items is calculated. A modified version of standard EOQ formula, where marginal costs are treated as setup costs, is used to find approximated replenishment intervals. In the last step, the item with the lowest replenishment interval is added to the set of items to be replenished.

The results of the heuristic algorithm are compared with Anily and Federgruen (1990)'s heuristic. It is observed that in most cases the heuristic gives better results. However, as the problem size gets larger, the Anily and Federgruen heuristic improves significantly.

Table 2.12 Viswanathan and Mathur (1997) in Management Science

Component	***Characteristic***
Start Point-End Point	E(1, M)
Planning Horizon	P(∞)
Vehicle(s)	V(C, NC)
Demand Structure	Stationary, Deterministic
Inventory	I(N, Y)
Backordering	B(N, N)
Ordering	O(N, Y)
Inventory Policy	Endogenous
Transportation Cost	Fixed (vehicle usage)+ Distance + Fixed (customer specific)
Performance Measure(s)	Average cost and CPU time

In the study of Chan, Federgruen and Simchi-Levi (1998) summarized in Table 2.13, fixed partition policies in which retailers are partitioned into a number of regions that are supplied separately are considered. Zero inventory ordering policies in which retailers are supplied only if their inventory level reaches to zero are also considered. Lower bounds on the cost of any feasible solution are also presented.

Moreover, an alternative mathematical programming based heuristic is presented where a partition of regions is generated and then each region is assigned to a vehicle. Vehicles visit all retailers in regions at equidistant epochs for identifying close-to-optimal fixed partitioning policies.
In computational experimentations on a set of randomly generated problem instances, the gap between heuristic solution and lower bound is found to be less than 19%.

Table 2.13 Chan, Federgruen and Simchi-Levi (1998) in Operations Research

Component	***Characteristic***
Start Point-End Point	E(1, M)
Planning Horizon	P(∞)
Vehicle(s)	V(C, NC)
Demand Structure	Stationary, Deterministic
Inventory	I(N, Y)
Backordering	B(N, N)
Ordering	O(N, N)
Inventory Policy	Endogenous
Transportation Cost	Fixed + Distance
Performance Measure(s)	% Gap between Heuristic result and LB

In the study of Fumero and Vercellis (1999) summarized in Table 2.14, production and distribution decisions are incorporated. Lagrangian relaxation is used to break constraints in order to obtain easy-to-solve subproblems. Four subproblems are obtained by the relaxation, production, inventory, distribution, and routing. Solutions of these four subproblems give lower bound to the objective of the original problem. Upper bound is the feasible solution with the minimum cost value, generated by a heuristic. Two approaches that are synchronized (i.e. Lagrangian relaxation solution procedure) and decoupled are tested in the study. In the latter approach, production decisions are carried out independently, while in the former approach, production plan affects other decisions and is affected by them.

Randomly generated test instances are used in experimentations. On the average, the gap between upper and lower bounds is 5.5%. It should be noted that they measure the variation from the upper bound unlike other problems measuring the variation from the lower bound. Average improvement gained by relaxation as compared to the continuous Linear programming relaxation is 15%.

Table 2.14 Fumero and Vercellis (1999) in Transportation Science

Component	*Characteristic*
Start Point-End Point	E(1, M)
Planning Horizon	P(T)
Vehicle(s)	V(C, M)
Demand Structure	Dynamic, Deterministic
Inventory	I(Y, Y)
Backordering	B(N, N)
Ordering	O(Y, N)
Inventory Policy	Endogenous
Transportation Cost	Fixed + Distance + Amount
Performance Measure(s)	% Gap between UB and LB, % Gap between V_R^* and V_C^*

* V_R is the Lagrangian lower bound and V_C is the optimal value of linear programming relaxation

In the work of Kim and Kim (2000) summarized in Table 2.15, a multi-period inventory management and distribution planning problem is considered. Distinctive feature of the problem is that vehicles can make several trips in a time period. However, the study is not dealing with the routing aspect; rather direct deliveries are considered for distribution planning.

Mixed integer formulation of the problem is presented. Main problem is decomposed into two subproblems by Lagrangian relaxation: one is making schedules of vehicles and the other one is determination of delivery quantities and inventory levels at retailers. Vehicle scheduling problem can further be decomposed into many single period, single vehicle scheduling problems. Each of these problems has the knapsack problem characteristic and is solved by dynamic programming algorithm. The second subproblem which is a production planning problem with LP structure can be solved easily. For establishing feasible solutions, a two phase heuristic is used. In the first phase the second subproblem is solved and then the first subproblem is solved. If there are retailers whose demands are not satisfied, the number of trips is increased. In the second phase, in order to reduce total costs, the number of trips is adjusted while maintaining the feasibility of solutions.

Table 2.15 Kim and Kim (2000) in Journal of Operational Research Society

Component	***Characteristic***
Start Point-End Point	E(1, M)
Planning Horizon	P(T)
Vehicle(s)	V(C_V, M)
Demand Structure	Dynamic, Deterministic
Inventory	I(N, Y)
Backordering	B(N, N)
Ordering	O(N, N)
Inventory Policy	Endogenous
Transportation Cost	Distance + Amount
Performance Measure(s)	% Gap between UB and LB, CPU Time

In the study 120, randomly generated test instances are generated. The overall average percentage gap between upper and lower bounds is 1.04% and as the number of retailers increases the gap decreases. Maximum CPU time is 648.71

minutes for the largest test instance considering 50 vehicles and 140 retailers. In order to the compare the solutions gathered from the proposed heuristic and best feasible solutions values found by CPLEX, 20 small sized test instances are generated and average percentage error is 0.26%.

In the work of Cachon (2001) summarized in Table 2.16, three inventory control policies are considered for managing a retailers shelf space while considering transportation costs. In the system, multiple products of single retailer are examined. Demand of the retailer of each product is stochastic; therefore, should be estimated in advance. Retailer pays per unit of self space required, holding cost for the inventory kept and shortage cost for the unsatisfied demand of end customer. The objective is to minimize the total expected costs (transportation costs, shelf space costs, inventory holding costs, shortage costs) per unit time.

Three inventory control policies are minimum quantity continuous review policy (Q, S); full service periodic review policy (S, T); and minimum quantity periodic review policy (Q, S|T). In the minimum quantity policy inventory is reviewed continuously and a truck is dispatched when Q units of products have been ordered. In the full service periodic review policy, the inventory status of the retailer is reviewed in every T units of time and enough trucks are dispatched in order to replenish all the shelves of the retailer. In this policy, self space is minimized but truck utilization is decreased since a truck may be dispatched for one unit of product. In the minimum quantity periodic review policy, in every T units of time the retailer reviews its inventory status and a truck is dispatched if at least Q units are ordered. In this setting, T is an exogenous parameter, and if the retailer does not have the ability to determine that parameter, this policy (controlling the Q variable) is applicable. This policy may cause lost sales of some products due to Q parameter; therefore, the retailer should determine the portion of demand of each product to satisfy.

Table 2.16 Cachon (2001) in Manufacturing and Service Operations Management

Component	*Characteristic*
Start Point-End Point	E(1, 1)
Planning Horizon	P(∞)
Vehicle(s)	V(C, NC)
Demand Structure	Stochastic
Inventory	I(N, Y)
Backordering	B(N, Y)
Ordering	O(N, N)
Inventory Policy	(Q, S), (S, T), (Q, S|T)
Transportation Cost	Fixed
Performance Measure(s)	Ratios of costs of three inventory policies with respect to optimal values

It is stated that minimum quantity continuous review policy provides a cost that is not much greater than the lower bound if there is a long lead time or if the ration of shortage penalty cost to the self space cost is small where the lower bound is the optimal policy under demand allocation.

Two EOQ-like heuristic methods are used to estimate Q and S variables which are order quantity and self space amount, respectively. In both heuristics the stochastic variables in the cost functions are replaced with their means.

In order to test the findings, 972 randomly generated scenarios are used. For each scenario optimal (Q, S) policy is evaluated. Q-heuristic and S-heuristic results are compared and it is observed that Q-heuristic provides good performance with respect to S-heuristic. On the average Q-heuristic gives 3.7% higher results than the optimal whereas S-heuristic gives 15.7% higher results.

Among the feasible policies considered, continuous review policy gives the best results. But the quality of results of periodic review policies increases when T and transportation costs are low.

In the work of Kleywegt, Nori and Savelsbergh (2002) summarized in Table 2.17, the supplier is ought to make decisions regarding which customers to serve, how much to deliver to each customer to be served, how to combine the customers into vehicle routes and to assign vehicles to the routes in order to maximize expected discounted value (revenues minus costs) over an infinite horizon. Retailers are responsible from inventory holding cost and shortage penalty for unsatisfied demand. A distinctive feature of the study is that unsatisfied demand is treated as lost sales and could not be satisfied in future periods. The problem is formulated as a discrete time Markov decision process where the states are the current inventory levels of the retailers and the action space consists of all possible decisions satisfying vehicle capacity constraints and the storage capacities of the retailers.

In order to solve the Markov decision process three computational tasks should be done: estimation of the optimum value of the value function, estimation of the expected value necessary for the estimation of the value function, and the maximization problem defined in the value function. Since the problem is NP-hard, a special case of this problem, inventory routing problem with direct deliveries is examined. The routes consist of single customer and must satisfy the workload, time window and capacity constraints. For this special case, in order to estimate value function the problem is decomposed into customer subproblems. These subproblems are solved optimally and then combined by using a knapsack formulation to find a good approximation. Four different algorithms for approximation are presented.

Table 2.17 Kleywegt, Nori and Savelsbergh (2002) in Transportation Science

Component	***Characteristic***
Start Point-End Point	E(1, M)
Planning Horizon	P(∞)
Vehicle(s)	V(C, M)
Demand Structure	Stochastic
Inventory	I(N, Y)
Backordering	B(N, Y)
Ordering	O(N, N)
Inventory Policy	Endogenous
Transportation Cost	Distance
Performance Measure(s)	Comparison of the optimal values with approximation policies

10 benchmarked test instances are used to compare results obtained and parametric value approximation yields the best results.

In the work of Bertazzi, Paletta and Speranza (2002) summarized in Table 2.18, an order-up-to-level inventory policy is examined. According to the minimum and maximum inventory levels that are predetermined, the retailers are supplied with a single vehicle. The problem is defined for multiple products; however, a single product case is solved in the computations.

The order-up-to-level inventory policy is such that each retailer is supplied before the retailer reaches its minimum inventory level with an amount filling its inventory level up to its maximum.

Table 2.18 Bertazzi, Paletta and Speranza (2002) in Transportation Science

Component	***Characteristic***
Start Point-End Point	E(1, M)
Planning Horizon	P(T)
Vehicle(s)	V(C, 1)
Demand Structure	Dynamic, Deterministic
Inventory	I(Y, Y)
Backordering	B(N, N)
Ordering	O(N, N)
Inventory Policy	Order up-to-level
Transportation Cost	Distance
Performance Measure(s)	Total cost, number of visits, delivery quantity

A two-step heuristic method is suggested for the solution. In the first step, retailers are listed according to nondecreasing order of average number of time units needed to consume the maximum inventory. Then an iterative procedure is applied. In each iteration, a retailer is inserted in the solution, and a network representing the incremental cost due to the insertion of the specified retailer is created. And the shortest path of the network is found at the end of first step. In the second step, the solution obtained in the first step is improved if possible.

Results of the algorithm is compared with every and latest heuristics (every-heuristic tends to supply each retailer in every time period, and latest heuristic tends to supply the retailers that will be in stock-out position in the next period if not supplied in the current period). On average, every-heuristic yields 14% and latest-heuristic yields 5% error with respect to the heuristic solution presented.

Moreover, several results under different objectives such that transportation cost, inventory cost at retailers, transportation cost plus inventory cost at supplier, etc. are investigated.

In the work of Bertazzi and Speranza (2002) summarized in Table 2.19, minimization of transportation and inventory holding costs of multiple products for the single link problem is examined. In the problem it is tried to determine when to make shipments, how much of each product to ship and how much starting inventory is needed for both the supplier and the retailer at time zero.

Table 2.19 Bertazzi and Speranza (2002) in Transportation Science

Component	***Characteristic***
Start Point-End Point	E(1, 1)
Planning Horizon	P(∞)
Vehicle(s)	V(C, NC)
Demand Structure	Stationary, Deterministic
Inventory	I(Y, Y)
Backordering	B(N, N)
Ordering	O(N, N)
Inventory Policy	Endogenous
Transportation Cost	Fixed
Performance Measure(s)	Total cost

Three cases of the problem are defined, the continuous case, the discrete case with given frequencies and the case with discrete shipping times. In the continuous case all products are shipped at a unique frequency and a single vehicle is used to ship all products. In order to determine the unique frequency, a nonlinear constrained optimization model, which has closed form solution, should be solved.

In the discrete case with given frequencies, it is assumed that shipments can be made only with given frequencies so that time between these shipments is integer. Moreover, it is assumed that for each frequency the quantity of each product shipped at every shipment is constant. Although resulting problem is NP-hard due to the integrality constraints, an exact algorithm of Speranza and Ukovich (1996) which is able to solve up to 10,000 products and 15 frequencies is used.

In the case with discrete shipping times, the set of shipping times is integer and finite. The quantity of each product to ship, the number of vehicles to use and the initial inventory levels at time zero should be calculated. This problem is also NP-hard.

16 randomly generated test instances are generated to see the effect of discretization of the shipping times and the cost difference between time based strategies and frequency based strategies. According to the results, discretization of the shipping times can have an influence on total cost with an average increase of 20%. On average, time based strategies generate 1.2% lower total costs than frequency based shipping strategies.

In the work of Tang, Yung and Ip (2004) summarized in Table 2.20, the problem of integrating decisions of production lot sizing, ordering and transportation is considered. The related costs are setup costs of suppliers, inventory holding costs of suppliers, holding costs of retailers, ordering costs of retailers, and transportation costs. The problem is separated into two layers, where in the first layer combined decisions of assigning production and lot size to suppliers are made, in the second layer combined decisions of transportation and order quantity with multiple products are made. More specifically, in the first layer the amount of each type of products to be produced and the lot size for each supplier to meet the total demand from the destinations at the minimum total production costs are computed. In this layer a two step assignment heuristic is used. In the first step of the heuristic, the individual production lot size for each type of product and each supplier is determined. In the second step of the heuristic, solutions of the first step is combined using assignment problem.

Table 2.20 Tang, Yung and Ip (2004) in Journal of Manufacturing Systems

Component	***Characteristic***
Start Point-End Point	E(M, M)
Planning Horizon	P(∞)
Vehicle(s)	V(NC, NC)
Demand Structure	Dynamic, Deterministic
Inventory	I(Y, Y)
Backordering	B(N, N)
Ordering	O(Y, Y)
Inventory Policy	Endogenous
Transportation Cost	Distance
Performance Measure(s)	Total cost, CPU Time

In the second layer, the solutions of the first layer are used. The amounts of units shipped annually and the order quantity per time between the suppliers and the destinations at the minimum total cost of transportation, inventory holding and ordering within the capacities are computed. Upper bound for that problem can be obtained by solving a transportation problem which is constructed by the modification of some constraints of the combined transportation and order quantity problem with the transportation simplex method. Transportation heuristic used in the second layer starts with the solutions obtained from the upper bound. Thens in an iterative manner, flow variables of order quantity and shipping quantity are calculated. When the order quantities are calculated, remaining problem is an LP and easy to solve.

The overall procedure can be summarized as follows; the combined assignment of production and lot size problem is solved with an assignment heuristic. Then, annual production amounts that are obtained from the first problem are used in the solution of combined transportation and order quantity problem with a transportation heuristic. Finally, solutions of two problems are used to calculate the objective function value.

Two-layer-decomposition method is compared with the nonlinear programming Quasi Newton Method for eight randomly generated settings. In all settings, proposed method gives better results in both total cost and CPU time. It saves 2% to 9% cost over than Quasi Newton Method.

In the study of Bertazzi, Paletta and Speranza (2005) summarized in Table 2.21, a variant of order-up-to-level policy, called fill-fill-dump policy, in which order-up-to-level quantity is shipped to all but the last retailer on each delivery route and the quantity supplied to the last retailer is the minimum of order-up-to-level quantity and the remaining vehicle capacity. Production setup costs defined in this paper can be treated as ordering costs in inventory routing problem.

Two decomposition procedures for the model are stated. The first one consists of separating the production problem from the distribution problem, while the second one consists of the same setting by moving the variable production costs from the production subproblem to the distribution subproblem. Two heuristic algorithms are presented where the order of problems solved in the procedure differs only in the two heuristics: either production subproblem or distribution subproblem is solved firstly.

Table 2.21 Bertazzi, Paletta and Speranza (2005) in Journal of Heuristics

Component	***Characteristic***
Start Point-End Point	E(1, M)
Planning Horizon	P(T)
Vehicle(s)	V(C, NC)
Demand Structure	Dynamic, Deterministic
Inventory	I(Y, Y)
Backordering	B(N, N)
Ordering	O(Y, N)
Inventory Policy	Order up-to-level
Transportation Cost	Fixed + Distance
Performance Measure(s)	Total cost, number of vehicles, number of visits

According to the results, fill-fill-dump policy obtains better results with respect to order-up-to-level policy. On 73% of the test instances, fill-fill-dump policy generates the best solution values.

In the study of Pinar and Sural (2006) summarized in Table 2.22, the problem introduced in Bertazzi, Paletta and Speranza (2002) is considered where the available amount of product at the supplier is constant. They propose a Lagrangian relaxation based solution procedure. It is the first study to develop a mixed integer programming formulation for the problem in Bertazzi, Paletta and Speranza (2002).

The upper bounds obtained are better than those of “every” heuristic. However, the upper bounds of Bertazzi et al. are slightly better than the upper bounds of Pinar and Sural (2006) with an average of 4%.

Table 2.22 Pinar and Sural (2006) in Proceedings of the Material Handling Research Colloquium

Component	***Characteristic***
Start Point-End Point	E(1, M)
Planning Horizon	P(T)
Vehicle(s)	V(C, 1)
Demand Structure	Dynamic, Deterministic
Inventory	I(Y, Y)
Backordering	B(N, N)
Ordering	O(N, N)
Inventory Policy	Order up-to-level
Transportation Cost	Distance
Performance Measure(s)	% Gap, CPU time, Total cost

In the study of Abdelmaguid and Dessouky (2006) summarized in Table 2.23, backordering is considered as distinctive feature of the model. In each period deliveries are made only if any retailer’s inventory level reaches to zero. If a retailer carries inventory to the next period, it is not served.

In the algorithm, transportation cost of each retailer is calculated such that a retailer’s transportation cost is the reduction in cost if that retailer is removed from the delivery tour. Inventory and backorder decision subproblems are solved given these transportation costs. Then how much to deliver to each customer is determined by solving a vehicle routing problem.

Table 2.23 Abdelmaguid and Dessouky (2006) in International Journal of Production Research

Component	***Characteristic***
Start Point-End Point	E(1, M)
Planning Horizon	P(T)
Vehicle(s)	V(C_V, M)
Demand Structure	Dynamic, Deterministic
Inventory	I(N, Y)
Backordering	B(N, Y)
Ordering	O(N, Y)
Inventory Policy	Endogenous
Transportation Cost	Fixed + Distance
Performance Measure(s)	Total cost and CPU time

The solution values computed with the proposed heuristic algorithm deviates at most 20% from the upper bounds calculated by trying to solve the original mixed integer programming model with CPLEX solver.

In the work of Lei, Liu, Ruszczynski and Park (2006) summarized in Table 2.24, integrated problem of production, inventory and transportation is examined. The objective of the problem is the determination of the operation schedules to coordinate production, inventory holding and transportation so that the customer demand, transportation travel times, vehicle capacity constraints, plant production and storage constraints are all satisfied while the remaining operational cost over the planning horizon is minimized.

In the problem, since backordering is not allowed, the suppliers are able to use outsourcing when the capacities of its vehicles are insufficient. Moreover, vehicles can make multiple trips in each period.

Table 2.24 Lei, Liu, Ruszczynski and Park (2006) in IIE Transactions

Component	***Characteristic***
Start Point-End Point	E(M, M)
Planning Horizon	P(T)
Vehicle(s)	V(C_V, M)
Demand Structure	Dynamic, Deterministic
Inventory	I(Y, Y)
Backordering	B(N, N)
Ordering	O(N, N)
Inventory Policy	Endogenous
Transportation Cost	Amount + Time
Performance Measure(s)	Total cost and CPU time

The mixed integer formulation of the problem is presented. Authors solve this model with a two-phase approach. In phase one, a restricted version of the main problem is solved in the sense that only direct deliveries are allowed. Since solution to that problem is always feasible to the main problem, a set of solution values for quantities to be produced, kept as inventory and transported per time period are obtained.

In the second phase, a heuristic transporter routing algorithm, called load consolidation is used. The algorithm removes all less-than-truck-load assignments of phase one, and consolidates those assignments subject to transporter capacities and time window constraints.

Load consolidation algorithm is compared with the results obtained by solving the first problem with CPLEX and the second problem with load consolidation algorithm. For small problem settings, the average deviation is 1.98%, for larger settings load consolidation algorithm yields better results.

The solution values of the load consolidation algorithm are also compared with the solutions obtained by solving the whole model with CPLEX. In 34 of the 48 cases, load consolidation yields the same or better results in one minute, whereas CPLEX is run for 2 hours.

In the work of Yung, Tang, Ip and Wang (2006) summarized in Table 2.25, multi-product case of Tang, Yung and Ip (2004) is examined. As in the single product case, multi-product problem is decomposed into two layers; however, in the multi-product decomposition Lagrange multipliers are used. In the first layer annual production amounts of suppliers, transportation flows and production lot sizes are determined. This layer is decomposed into two subproblems, where in the first one allocating production capacity among product types for each supplier and assigning transportation flows between the suppliers and retailers are determined, in the second one given a certain assigned production for each type of product lot sizes are determined. The assignment heuristic used in this layer starts with an initial feasible solution by solving an upper bound linear program. Then, closed form formulations are used to find local optimal solutions. Until the termination condition is satisfied, in an iterative manner, local optimal solutions are computed. The optimal solution is the minimum of all local optimal solutions.

In the second layer annual transportation quantity of each product and quantity per order for individual supplier retailer pair are determined. In this layer revision of the heuristic defined in Benjamin (1989) is used. Like the assignment heuristic, this heuristic starts with an initial solution and continues iteratively.

Table 2.25 Yung, Tang, Ip and Wang (2006) in Transportation Science

Component	***Characteristic***
Start Point-End Point	E(M, M)
Planning Horizon	P(∞)
Vehicle(s)	V(NC, NC)
Demand Structure	Stationary, Deterministic
Inventory	I(Y, Y)
Backordering	B(N, N)
Ordering	O(Y, Y)
Inventory Policy	Endogenous
Transportation Cost	Amount
Performance Measure(s)	Total cost and CPU time

11 randomly generated test instances of the same size are used to compare the Lagrangian relaxation with heuristics results with the results obtained by Fmincon, a traditional nonlinear programming technique and the algorithm used in Tang, Yung and Ip (2004). In all cases, proposed algorithm yields the same or better results. Moreover, 7 randomly generated test instances of different sizes are used to test the quality of the results in different settings. The proposed algorithm saves 1.5% to 8% cost and requires less CPU time.

The study of Solyali and Sural (2007) summarized in Table 2.26, considers a variant of Bertazzi, Paletta and Speranza (2002) and differs with cost structure from Fumero and Vercellis (1999). In Fumero and Vercellis (1999), transportation costs are proportional to the amount shipped and distance traveled whereas transportation costs only depend on distance traveled in Solyali and Sural (2007).

On average, a Lagrangian based solution approach in Solyali and Sural (2007), yields better results than "every" and "latest" heuristics given in Bertazzi, Paletta, and Speranza (2002).

Table 2.26 Solyali and Sural (2007) in Technical Report of Department of Industrial Engineering, METU

Component	***Characteristic***
Start Point-End Point	E(1, M)
Planning Horizon	P(T)
Vehicle(s)	V(C, M)
Demand Structure	Dynamic, Deterministic
Inventory	I(Y, Y)
Backordering	B(N, N)
Ordering	O(Y, N)
Inventory Policy	Order up-to-level
Transportation Cost	Fixed + Distance
Performance Measure(s)	% Gap and CPU time

In the study of Savalsbergh and Song (2008) summarized in Table 2.27, more realistic assumptions than the prior works such that limited product availabilities at facilities and prohibition of out-and-back tours are applied. They present MIP formulation of the problem. For solving the problem, they tried to reduce the problem size by using connectivity lists and adding valid inequalities to the formulation. In order to do that, they determine the delivery and non-delivery periods for each customer, and transportation availability of each location. They solve CVRPs by two separation heuristics. One is integer connected components separation heuristic and the other is connected components separation heuristic. In prior heuristic, they detect delivery cover inequalities that are violated and adding these inequalities to the problem. The latter heuristic, which is used only when the prior heuristic fails to detect violated inequalities, seeks the violation for each supernode where supernodes in a period are defined as the nodes included in the tour of the respective period.

They tested their algorithm on three data sets. More specifically they seek the effect of delivery cover inequalities. They show that it takes less time (on average 111 seconds) when cover inequalities are used than the default setting (on average 4823 seconds).

On average, the %IP gap is 4.06% and the %LP gap is 17.66% of the algorithm where %IP gap is the gap between the IP solution calculated by CPLEX and the heuristic solution; and %LP gap is the gap between the LP relaxation result and the heuristic solution.

Table 2.27 Savalsbergh and Song (2008) in Computers & Operations Research

Component	***Characteristic***
Start Point-End Point	E(M, M)
Planning Horizon	P(T)
Vehicle(s)	V(C, M)
Demand Structure	Dynamic, Deterministic
Inventory	I(N, Y)
Backordering	B(N, N)
Ordering	O(N, N)
Inventory Policy	Endogenous
Transportation Cost	Distance
Performance Measure(s)	CPU time, %IP Gap and %LP Gap

The studies related with inventory routing concept in the literature that are listed in this chapter can be classified into three groups according to planning horizon. The groups are exhibited in Table 2.28.

Table 2.28 Classification of reviewed studies according to planning horizon and route cost estimation

Planning Horizon	Table number of articles
P(1)	1 (R), 5 (R)
P(T)	8 (R), 11 (R), 14 (R), 15, 18 (R), 21 (R), 22 (R), 23 (R), 24 (R), 26 (R), 27 (R)
P(∞)	2, 3, 4, 6, 7, 9, 10, 12, 13,16, 17, 19, 20, 25

In the table, (R) denotes that in the related article routing aspect is specifically considered, in the other articles only estimates of delivery routes are made or direct deliveries that cover single customer in each route are used.

It is observed that when the planning horizon is infinite, routing problems are naturally relaxed by estimating routing costs or using direct deliveries; however, the finite horizon models consider routing problem in detail.

CHAPTER 3

THE MULTIPLE RETAILER INVENTORY ROUTING PROBLEM WITH BACKORDERS

In this chapter, we first describe the multiple retailer inventory routing problem with backorders, called INVROP, and then present its classification scheme. Next we list our assumptions related with the INVROP. Then, we formulate the INVROP as a mixed integer programming model, compare our model M(INVROP) (model of inventory routing problem) with previous work in the literature, namely, Chien et al. (1989) and Abdelmaguid and Dessouky (2006), and then state the assumptions we made in this mathematical formulation. Since INVROP is NP-hard, we use Lagrangian relaxation for solving the problem. The suggested relaxation on the mixed integer formulation of the problem is discussed at the end of the chapter.

The INVROP integrates inventory and routing decisions. In each period, the supplier decides whether to dispatch vehicles for distribution so as to serve a set of geographically dispersed retailers or not. Since the supplier is dealing with only dispatching, it can be considered as a crossdock unit in the problem. The supplier is assumed to be able to satisfy the demand in the system, but the system may let retailers backlog their external demands. The main control mechanism is to decide whether to satisfy the end customer demand from the current distribution, or from the inventory at the retailers, or by backlogging so that service is given in some future period. The inventory and distribution decisions are considered together and given to minimize system wide costs. The costs consist of retailer specific holding cost and backlogging cost, vehicle specific dispatching cost, distance and amount based transportation cost.

In this setting, each vehicle distributes specified amounts to the retailers, which are listed to be served for the period in consideration. The lists of customers to be served are prepared by the supplier. Any retailer that is not in the list (i.e. not to be served by a vehicle in that period) will not be visited in the associated period. If a vehicle is dispatched in any period, a fixed cost of dispatching is incurred. Transportation costs are calculated proportional to the Euclidean distances on the links between the stop points. Fixed charges are known in advance according to the links. Since Euclidean distances are used; the shortest distance going from one point to another does not include another distinct point (a third point).

Any retailer can hold inventory with the retailer specific holding cost for each unit held per period; and any retailer can backlog the end customer demand with the retailer specific backordering cost for each unsatisfied unit per period.

Table 3.1 Classification scheme of the INVROP

Component	***Characteristic***
End Point	E(1,M)
Planning Horizon	P(T)
Vehicle(s)	V(C_m,M)
Demand Structure	Dynamic, Deterministic
Inventory	I(N,Y)
Backordering	B(N,Y)
Ordering	O(N,Y)
Inventory Policy	Endogenous
Transportation Cost	Fixed (vehicle specific) + Distance
Performance Measure(s)	Minimizing total costs of inventory holding, backordering and transportation

The properties of the problem with respect to the classification scheme presented in Chapter 2 are given in Table 3.1.

This problem is similar to the problem in Chien et al. (1989), and Abdelmaguid and Dessouky (2006). The differences between our problem and its ancestors can be stated as follows.

- In our problem, the objective is to minimize system wide costs, which is the same as Abdelmaguid and Dessouky (2006); however, the objective in Chien et al. (1989) is to maximize profit while not considering inventory holding cost.

- Our problem consists of T time periods as in Abdelmaguid and Dessouky (2006); however, Chien et al. (1989) considers a single period problem.

- Since Chien et al. (1989) has a single period problem, unlike Abdelmaguid and Dessouky (2006) and ours, holding inventory makes no sense. In all three problems, backordering is allowed.

- In our problem backordering in the last period is not allowed; therefore, all the demand of end customers must be satisfied during the planning horizon. However, in Abdelmaguid and Dessouky (2006), backordering in the last period is allowed. Since Chien et al. (1989) considers a single period problem and backordering is allowed, it is different from our problem.

- Chien et al. (1989) charges transportation costs depending on the total amount of product carried on the links between the points. In Abdelmaguid and Dessouky (2006), the cost is independent of the amount carried on the links, but is based on the links' fixed usage charge. In our problem transportation cost consists of both fixed arc usage cost and variable transportation cost depending on the amount carried on these arcs.

- We assume that the supplier has unlimited inventory at its depot. However, Chien et al. (1989) assumes a predetermined amount Q in the beginning of period.

- Abdelmaguid and Dessouky (2006) assumes that a retailer is served if and only if its inventory level reaches zero. However, we do not have such a simplifying assumption which may not be the optimal allocation policy.

We use the same variable definitions in formulating the problem mathematically as it is formulated in Chien et al. (1989). Whereas, Abdelmaguid and Dessouky (2006) develops a different model to formulate the problem in consideration.

3.1 Assumptions of the INVROP

We state the assumptions of the INVROP below.

- The external demand or the demands of end customers occur at the retailers.

- Required amount to be distributed is assumed to be available at the supplier (depot) in each period.

- Depot cannot hold inventory or backorder, but decides about the vehicles to be dispatched, the retailers to be served, and the amounts to be distributed in these visits. It is actually a crossdock facility.

- We assume that there is an underlying network that hosts the system's transportation structure. In this network, nodes represent supplier and retailer sites. The arcs (links) represent connections between these nodes.

- Each vehicle of the fleet can make at most one trip in each period. Each trip starts from the depot and ends at the depot. Subtours not including the depot are not allowed.

- The amount carried by each vehicle is constrained by the vehicle capacity.

- There is no lead time for both depot and retailers. Products to be distributed to each retailer are ready at the beginning of each period and can be used to satisfy the demands of end customers at the beginning of the period. Therefore, the next period's inventory level (positive, zero, or negative) is carried from the beginning of current period.

- Backordering and keeping inventory are allowed at the retailers.

- The amount of product that can be stored at each retailer is constrained by the retailer's storage capacity.

- Initial inventory levels and initial backordered demands of all retailers are zero.

- Backordering in the last period is not allowed.

3.2 Mixed integer formulation of the INVROP

In this section, a mathematical model of the INVROP is presented. We first state indices, parameters, and definitions of the variables. Then, we explain the objective function and constraints of the model.
Indices of the model are as follows.

t : Time index (discrete time periods): 1, 2, …, T and $\overline{T} = T \cup \{0\}$.

i, j : Node index : 0, 1, …, N (i = 0 denotes depot). N denotes the set of retailers and $\overline{N} = N \cup \{0\}$.

k : Retailer index: 1, 2, …, N.

v : Vehicle index: 1, 2, …, V.

Parameters of the model are as follows.

N : Number of locations (retailers).

V : Number of vehicles.

T : Number of time periods.

K_v : Capacity of vehicle v.

$I_{\max}^{k}$: Storage capacity of retailer k.

d_{kt} : Demand of the end customer of retailer k in period t.

f_{ijvt} : Fixed cost for vehicle v in period t to use arc (i,j) for going from location i to location j.

c_{ijvt}^{k} : Variable cost of carrying one unit of product by vehicle v in period t on arc (i,j) for going from location i to location j for the designated customer k.

O_t : Fixed vehicle dispatching cost in time period t.

h_{kt} : Unit holding cost for retailer k in period t.

b_{kt} : Unit backordering cost for retailer k in period t.

Notice that parameters N, V and T will denote both index sets and the cardinality of the corresponding sets. The meaning will be clear from the context of use.

Decision variables of the model are as follows:

$$y_{ijvt} \quad : \begin{cases} 1 \text{ if vehicle } v \text{ travels from location } i \text{ to location } j \text{ using arc } (i, j) \\ \quad \text{in perod } t \\ 0 \text{ otherwise} \end{cases}$$

x_{ijvt}^{k} : Amount of product destined to retailer k, which is transported from location i to location j by vehicle v in period t.

I_{kt} : Amount of product held by retailer k in period t.

B_{kt} : Amount of product backordered by retailer k in period t.

S_{kt} : Amount of product supplied to retailer k in period t.

Note that an illustrative example for the flow variables is presented in Appendix A.

M(INVROP):

$$\text{Minimize} \quad \sum_{i=0}^{N}\sum_{\substack{j=0 \\ j\neq i}}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T} f_{ijvt}\,y_{ijvt} \quad + \quad \sum_{k=1}^{N}\sum_{t=0}^{T}\left(h_{kt}I_{kt} + b_{kt}B_{kt}\right) \quad +$$

$$\sum_{j=1}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T} O_t\,y_{0jvt} + \sum_{i=0}^{N}\sum_{\substack{j=0 \\ j\neq i}}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T}\sum_{k=1}^{N} c_{ijvt}^{k}\,x_{ijvt}^{k} \tag{3.1}$$

Subject To

$$\sum_{k=1}^{N} x_{ijvt}^{k} \le K_v y_{ijvt} \qquad \forall\, i, j \in \overline{N},\ i \ne j, v \in V, t \in T \tag{3.2}$$

$$\sum_{\substack{j=0 \\ j \ne i}}^{N} \sum_{v=1}^{V} x_{ijvt}^{k} - \sum_{\substack{j=0 \\ j \ne i}}^{N} \sum_{v=1}^{V} x_{jivt}^{k} = \begin{cases} +S_{kt} \text{ if } i = 0 \\ -S_{kt} \text{ if } i = k \end{cases} \qquad \forall\, i \in \overline{N}, t \in T, k \in N \tag{3.3}$$

$$\sum_{\substack{j=0 \\ j \ne i}}^{N} x_{ijvt}^{k} - \sum_{\substack{j=0 \\ j \ne i}}^{N} x_{jivt}^{k} = 0 \qquad \forall\ i \in N \setminus \{k\}, v \in V, t \in T, k \in N \tag{3.4}$$

$$\sum_{\substack{j=0 \\ j \ne i}}^{N} y_{ijvt} - \sum_{\substack{j=0 \\ j \ne i}}^{N} y_{jivt} = 0 \qquad \forall\, i \in \overline{N}, v \in V, t \in T \tag{3.5}$$

$$\sum_{\substack{j=0 \\ j \ne i}}^{N} y_{ijvt} \le 1 \qquad \forall\ i \in \overline{N}, v \in V, t \in T \tag{3.6}$$

$$I_{kt-1} - B_{kt-1} - I_{kt} + B_{kt} + S_{kt} = d_{kt} \qquad \forall\, t \in T, k \in N \tag{3.7}$$

$$I_{kt} \le I_{\max}^{k} \qquad \forall\, t \in \overline{T}, k \in N \tag{3.8}$$

$$x_{ijvt}^{k} \le \min\left\{ \sum_{r=1}^{t} d_{kr} + I_{\max}^{k}, K_v \right\} y_{ijvt} \qquad \forall\, i, j \in \overline{N},\ i \ne j, v \in V, t \in T,\ k \in N \tag{3.9}$$

$$B_{k0} = 0 \qquad \forall\ k \in N \tag{3.10}$$

$$I_{k0} = 0 \qquad \forall\ k \in N \tag{3.11}$$

$$B_{kT} = 0 \qquad \forall\ k \in N \tag{3.12}$$

$$\sum_{k=1}^{N} S_{kt} \le \sum_{v=1}^{V} K_v \qquad \forall\, t \in T \tag{3.13}$$

$$x_{ijvt}^{k} \le S_{kt} \qquad \forall\, i, j \in \overline{N},\ i \ne j, v \in V, t \in T,\ k \in N \tag{3.14}$$

$$S_{kt}, I_{kt}, B_{kt},\ x_{ijvt}^{k} \ge 0 \qquad \forall\, i, j \in \overline{N},\ i \ne j, v \in V, t \in T, k \in N \tag{3.15}$$

$$y_{ijvt} \in \{0,1\} \qquad \forall\, i, j \in \overline{N},\ i \ne j, v \in V, t \in T \tag{3.16}$$

The objective function (3.1) consists of fixed arc usage cost (first term), retailer specific holding cost and backordering costs (second term in summation), fixed vehicle dispatching cost (third term) and variable transportation cost depending on the amount of product carried (fourth term).

Constraint set (3.2) satisfies the vehicle capacity restriction. The total amount sent to the retailers on a specified arc should be less than or equal to the capacity of the vehicle that traverses that arc. It thus links binary variables of arc usage (y_{ijvt}) and flow variables representing the amounts carried on these arcs (x^k_{ijvt}).

Constraint set (3.3) is for the commodity flow conservation equations. The set is defined for depot and all retailers. For the depot, the cumulative product going out is equal to the total amount to be distributed to retailers by a vehicle in a period. For retailers, the difference between the amount coming into retailer k and the amount going out of retailer k is the amount supplied to retailer k with a vehicle in a period.

Constraint set (3.4) is for the commodity flow conservation equations, which is defined for the retailers that are not designated customers. The difference between the amount coming into a retailer who is not to be served and the amount going out of that retailer is equal to zero; therefore, it is ensured that a retailer that is not in the list in a period is not served in that period.

Constraint sets (3.5) and (3.6) limit the movements of vehicles. By set (3.5), it is ensured that a vehicle that visits a retailer (or depot) in a specified period must leave that retailer (or depot). By set (3.6), it is ensured that a vehicle can visit a retailer (or depot) at most once in a period. Therefore, it is assumed that a vehicle starting from the depot will turn back and each vehicle can make at most one trip in every period. Note that the formulation eliminates possible subtours that are excluding the depot.

Constraint set (3.7) is the inventory balance equations for the retailers. Incoming inventory of a retailer minus the amount backordered in the previous period minus the amount to be hold at the end of a period plus the amount backordered in that period plus the amount supplied in that period is equal to the demand of that retailer in that period. Hereby, it is obvious that in each period the system has three options: holding inventory, backordering and satisfying the demand.

Constraint set (3.8) is related with the limitation on the stocking amount at the retailers. A retailer cannot hold more inventories than its storage capacity.

Constraint set (3.9) restricts the amount carried for a designated customer on each arc with the minimum of vehicle capacity or the sum of the cumulative demand and maximum inventory level. Constraint set (3.14) also restricts the amount carried for a designated customer on each arc by with the supply amount to that customer. These two constraint sets are redundant for the original formulation but it will be helpful for developing a bounding procedure. For relaxations, these constraints help to make the formulation stronger.

Constraint set (3.10) is used not to start with backorders. Constraint set (3.11) is used to set the initial inventory levels of the retailers to zero. Constraint set (3.12) is used to prohibit backordering in the last period.

Constraint set (3.13) is the redundant supply equations. These constraints are redundant for the original model. However, they would be useful when a relaxation is applied to solve the model.

Constraint sets (3.15) and (3.16) are the non-negativity and integrality constraints, respectively.

M(INVROP) is a mega model representing possible combinations of cost applications. We can apply both flow independent and flow dependent cost components. In more specific, the formulation is able to handle realistic assumptions such as transportation cost depends not only on distance traveled and vehicles used, but also the amount carried. Moreover, in our preliminary experiments we observed that the flow dependent cost representation strengths the formulation by adding importance on the flow variables.

M(INVROP) is a huge mixed integer model. Solving the model optimally in reasonable time is not possible for even moderate size instances. The model consists of $N^3VT + 2N^2VT + NVT + 3NT + 2N$ variables in total. $N^2VT + NVT$ many of these variables are integer and the rest are continuous. Also the model has $2N^3VT + 4N^2VT + 2NVT + 4NT + 2VT + 2N + 4T + T$ many constraints. For a possible problem (taken from the literature) with {N=15, T=7, V=2} there exist 54,105 variables (3,360 integer variables) and 108,035 constraints.

3.3 Lagrangian relaxation based solution approach

Since the INVROP is hard to solve in reasonable times we propose a Lagrangian relaxation based approach in order to obtain tight lower bounds and good upper bounds (feasible solutions). In this section we give the details of the Lagrangian relaxation based solution approach applied to M(INVROP).

The Lagrangian relaxation is a strong tool used in the literature to find “good” solutions (optimal solutions are not guaranteed) for the difficult problems. Basics of the method consist of generating the original model, choosing constraints that are to be relaxed, attaching the Lagrange multipliers to these constraints and adding them to the objective function, and solving resulting (relaxed) model. Most crucial part of the method is choosing the constraint(s) to be relaxed. Beasley (1993) advices considering the following aspects in a relaxation:

- The number of Lagrange multipliers needed.
- The computational effort required to solve the relaxed problem.
- Whether the relaxed problem has integrality property or not.

While leaving the first two aspects, in this application, we relax those constraints whose removals abolish the integrality property of the relaxed problem. Therefore, we can say that our lower bounds will be better than any of others satisfying integrality property and the LP (Linear Programming) relaxation (in theory).

After choosing the constraints to be relaxed, relevant Lagrange multipliers are attached to these constraints and these constraints are added to the objective function. Multipliers can be seen as a penalty for violating the selected constraints. The model tries to minimize these violations so that the value of the objective function of the relaxed problem comes closer to the optimal value of the original problem's objective function. The solution obtained by solving relaxed problem –not necessarily feasible- gives a lower bound on the original problem's objective function. Moreover, the Lagrangian solutions are used to obtain good upper bounds for the original problem. If the solution of the relaxed problem is not feasible, for example, by applying a simple heuristic, a feasible solution can be obtained and this solution constitutes an upper bound. In order to close the gap between these two bounds and to update the Lagrangian multipliers, usually subgradient optimization is applied iteratively. Basically in each step, subgradients (differences of right-hand-sides and left-hand-sides of the relaxed constraints) are calculated, and the Lagrange multipliers are adjusted according to these subgradients. These steps will be covered in the "Subgradient Search" section.

The overall algorithm is terminated if any user defined stopping condition is satisfied. Some well-known stopping conditions are:

- Reaching a maximum iteration number (user defined).
- Upper bound = lower bound (optimal solution is found).
- The gap between the upper bound and the lower bound is below a reasonable value (user defined).
- Reaching a computation time limit (user defined).
- Reaching the minimum value of step size used in the subgradient optimization method (user defined).

For applying the Lagrangian relaxation method to M(INVROP), constraint sets (3.3), (3.4), and (3.5) are chosen since these constraints are the most complicating constraints of the problem. Recall that these constraints prohibit subtours and provide complete routes. Since the problem of finding the minimum cost tour for each period for each vehicle is a well-known NP-hard problem in the literature, relaxing these constraints simplifies the solution to the remaining problem.
Lagrange multipliers used are:

- α_{it}^{k} $\quad i = 0 \;\; or \;\; i = k$; for constraint set (3.3).
- β_{ivt}^{k} $\quad i \neq 0 \;\; and \;\; i \neq k$; for constraint set (3.4).
- γ_{ivt} $\quad$ for constraint set (3.5).

The relaxed problem (REP) is given below:

$$\text{Minimize (1)} + \sum_{k=1}^{N}\sum_{t=1}^{T}\alpha_{0t}^{k}\left(-S_{kt}+\sum_{j=1}^{N}\sum_{v=1}^{V}x_{0jvt}^{k}-\sum_{j=1}^{N}\sum_{v=1}^{V}x_{j0vt}^{k}\right)+$$

$$\sum_{k=1}^{N}\sum_{t=1}^{T}\alpha_{kt}^{k}\left(S_{kt}+\sum_{\substack{j=0\\j\neq k}}^{N}\sum_{v=1}^{V}x_{kjvt}^{k}-\sum_{\substack{j=0\\j\neq k}}^{N}\sum_{v=1}^{V}x_{jkvt}^{k}\right)+\sum_{i=1}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T}\sum_{\substack{k=1\\k\neq i}}^{N}\beta_{ivt}^{k}\left(\sum_{\substack{j=0\\j\neq i}}^{N}x_{ijvt}^{k}-\sum_{\substack{j=0\\j\neq i}}^{N}x_{jivt}^{k}\right)+$$

$$\sum_{i=0}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T}\gamma_{ivt}\left(\sum_{\substack{j=0\\j\neq i}}^{N}y_{ijvt}-\sum_{\substack{j=0\\j\neq i}}^{N}y_{jivt}\right) \tag{3.17}$$

Subject To

$$\sum_{k=1}^{N}x_{ijvt}^{k}\leq K_{v}y_{ijvt} \qquad \forall\, i,j\in\overline{N},\ i\neq j, v\in V, t\in T \tag{3.2}$$

$$\sum_{\substack{j=0\\j\neq i}}^{N}y_{ijvt}\leq 1 \qquad \forall\ i\in\overline{N}, v\in V, t\in T \tag{3.6}$$

$$I_{kt-1}-B_{kt-1}-I_{kt}+B_{kt}+S_{kt}=d_{kt} \qquad \forall\, t\in T, k\in N \tag{3.7}$$

$$I_{kt}\leq I_{\max}^{k} \qquad \forall\, t\in\overline{T}, k\in N \tag{3.8}$$

$$x_{ijvt}^{k}\leq \min\left\{\sum_{r=1}^{t}d_{kr}+I_{\max}^{k}, K_{v}\right\}y_{ijvt} \qquad \forall\, i,j\in\overline{N},\ i\neq j, v\in V, t\in T,\ k\in N \tag{3.9}$$

$$B_{k0}=0 \qquad \forall\ k\in N \tag{3.10}$$

$$I_{k0}=0 \qquad \forall\ k\in N \tag{3.11}$$

$$B_{kT}=0 \qquad \forall\ k\in N \tag{3.12}$$

$$\sum_{k=1}^{N}S_{kt}\leq\sum_{v=1}^{V}K_{v} \qquad \forall\, t\in T \tag{3.13}$$

$$x_{ijvt}^{k}\leq S_{kt} \qquad \forall\, i,j\in\overline{N},\ i\neq j, v\in V, t\in T,\ k\in N \tag{3.14}$$

$$S_{kt}, I_{kt}, B_{kt},\ x_{ijvt}^{k}\geq 0 \qquad \forall\, i,j\in\overline{N},\ i\neq j, v\in V, t\in T, k\in N \tag{3.15}$$

$$y_{ijvt}\in\{0,1\} \qquad \forall\, i,j\in\overline{N},\ i\neq j, v\in V, t\in T \tag{3.16}$$

The objective (3.17) can be written as in the form below.

$$\text{Minimize } \sum_{i=0}^{N}\sum_{\substack{j=0\\j\neq i}}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T} f_{ijvt}y_{ijvt} + \sum_{k=1}^{N}\sum_{t=0}^{T}\left(h_{kt}I_{kt}+b_{kt}B_{kt}\right) + \sum_{j=1}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T} O_t y_{0jvt} +$$

$$\sum_{i=0}^{N}\sum_{\substack{j=0\\j\neq i}}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T}\sum_{k=1}^{N} c_{ijvt}^{k}x_{ijvt}^{k} - \sum_{k=1}^{N}\sum_{t=1}^{T}\alpha_{0t}^{k}S_{kt} + \sum_{j=1}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T}\sum_{k=1}^{N}\alpha_{0t}^{k}x_{0jvt}^{k} -$$

$$\sum_{j=1}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T}\sum_{k=1}^{N}\alpha_{0t}^{k}x_{j0vt}^{k} + \sum_{k=1}^{N}\sum_{t=1}^{T}\alpha_{kt}^{k}S_{kt} + \sum_{j=0}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T}\sum_{\substack{k=1\\k\neq j}}^{N}\alpha_{kt}^{k}x_{kjvt}^{k} - \sum_{j=0}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T}\sum_{\substack{k=\\k\neq j1}}^{N}\alpha_{kt}^{k}x_{jkvt}^{k} +$$

$$\sum_{i=1}^{N}\sum_{\substack{j=0\\j\neq i}}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T}\sum_{\substack{k=1\\k\neq i}}^{N}\beta_{ivt}^{k}x_{ijvt}^{k} - \sum_{i=1}^{N}\sum_{\substack{j=0\\j\neq i}}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T}\sum_{\substack{k=1\\k\neq i}}^{N}\beta_{ivt}^{k}x_{jivt}^{k} + \sum_{i=0}^{N}\sum_{\substack{j=0\\j\neq i}}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T}\gamma_{ivt}y_{ijvt} -$$

$$\sum_{i=0}^{N}\sum_{\substack{j=0\\j\neq i}}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T}\gamma_{ivt}y_{jivt} \tag{3.18}$$

We rearrange the objective function of REP (3.18) and define new coefficients for the commonly used variables as follows:

$$\hat{f}_{\substack{0jvt\\j\neq 0}} = O_t + \gamma_{0vt} - \gamma_{jvt} + f_{0jvt} \;\rightarrow y_{\substack{0jvt\\j\neq 0}}$$

$$\hat{f}_{\substack{ijvt\\i\neq 0\\j\neq i}} = f_{ijvt} + \gamma_{ivt} - \gamma_{jvt} \;\rightarrow y_{\substack{ijvt\\i\neq 0\\j\neq i}}$$

$$p_{kt} = \alpha_{kt}^{k} - \alpha_{0t}^{k} \;\rightarrow S_{kt}$$

$$\hat{c}_{\substack{0jvt\\j\neq 0\\j\neq k}}^{k} = c_{0jvt}^{k} + \alpha_{0t}^{k} - \beta_{jvt}^{k} \;\rightarrow x_{\substack{0jvt\\j\neq 0\\j\neq k}}^{k}$$

$$\hat{c}_{\substack{0jvt\\j\neq 0\\j=k}}^{k} = c_{0jvt}^{k} + \alpha_{0t}^{k} - \alpha_{kt}^{k} \;\rightarrow x_{\substack{0jvt\\j\neq 0\\j=k}}^{k}$$

$$\hat{c}_{\substack{j0vt\\j\neq 0\\j\neq k}}^{k} = c_{j0vt}^{k} + \beta_{jvt}^{k} - \alpha_{0t}^{k} \;\rightarrow x_{\substack{j0vt\\j\neq 0\\j\neq k}}^{k}$$

$$\hat{c}^k_{\substack{j0vt\\ j\neq 0\\ j=k}} = c^k_{j0vt} + \alpha^k_{kt} - \alpha^k_{0t} \quad \rightarrow x^k_{\substack{j0vt\\ j\neq 0\\ j=k}}$$

$$\hat{c}^k_{\substack{ijt\\ i=k\\ i\neq 0\\ j\neq i}} = c^k_{ijvt} + \alpha^k_{kt} - \beta^k_{jvt} \quad \rightarrow x^k_{\substack{ijvt\\ i=k\\ i\neq 0\\ j\neq i}}$$

$$\hat{c}^k_{\substack{ijvt\\ i\neq k\\ i\neq 0\\ j=k}} = c^k_{ijvt} + \beta^k_{ivt} - \alpha^k_{kt} \quad \rightarrow x^k_{\substack{ijvt\\ i\neq k\\ i\neq 0\\ j=k}}$$

$$\hat{c}^k_{\substack{ijvt\\ i\neq j\neq k\neq 0}} = c^k_{ijvt} + \beta^k_{ivt} - \beta^k_{jvt} \quad \rightarrow x^k_{\substack{ijvt\\ i\neq j\neq k\neq 0}}$$

Let REP denote the following modified Lagrangian relaxed problem:

$$\text{Minimize} \quad \sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T} \hat{f}_{ijvt} y_{ijvt} \; + \sum_{k=1}^{N}\sum_{t=0}^{T}\left(h_{kt}I_{kt} + b_{kt}B_{kt}\right) + \sum_{k=1}^{N}\sum_{t=1}^{T} p_{kt}S_{kt} \; +$$

$$\sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T}\sum_{k=1}^{N} \hat{c}^k_{ijvt} x^k_{ijvt} \tag{3.19}$$

Subject To

$$\sum_{k=1}^{N} x^k_{ijvt} \leq K_v y_{ijvt} \qquad \forall\, i,\, j \in \overline{N},\; i \neq j, v \in V, t \in T \tag{3.2}$$

$$\sum_{\substack{j=0\\ j\neq i}}^{N} y_{ijvt} \leq 1 \qquad \forall\; i \in \overline{N}, v \in V, t \in T \tag{3.6}$$

$$I_{kt-1} - B_{kt-1} - I_{kt} + B_{kt} + S_{kt} = d_{kt} \qquad \forall\, t \in T, k \in N \tag{3.7}$$

$$I_{kt} \leq I^k_{\max} \qquad \forall\, t \in \overline{T}, k \in N \tag{3.8}$$

$$x^k_{ijvt} \leq \min\left\{\sum_{r=1}^{t} d_{kr} + I^k_{\max}, K_v\right\} y_{ijvt} \qquad \forall\, i,\, j \in \overline{N},\; i \neq j, v \in V, t \in T,\; k \in N \tag{3.9}$$

$$B_{k0} = 0 \qquad \forall\; k \in N \tag{3.10}$$

$$I_{k0} = 0 \qquad \forall\; k \in N \tag{3.11}$$

$$B_{kT} = 0 \qquad \forall\ k \in N \tag{3.12}$$

$$\sum_{k=1}^{N} S_{kt} \le \sum_{v=1}^{V} K_v \qquad \forall\, t \in T \tag{3.13}$$

$$x_{ijvt}^{k} \le S_{kt} \qquad \forall\, i, j \in \overline{N},\ i \ne j, v \in V, t \in T,\ k \in N \tag{3.14}$$

$$S_{kt}, I_{kt}, B_{kt},\ x_{ijvt}^{k} \ge 0 \qquad \forall\, i, j \in \overline{N},\ i \ne j, v \in V, t \in T, k \in N \tag{3.15}$$

$$y_{ijvt} \in \{0,1\} \qquad \forall\, i, j \in \overline{N},\ i \ne j, v \in V, t \in T \tag{3.16}$$

Note that without the constraint set (3.14), the REP actually decomposes into the following two subproblems.

- Retailer Subproblem (RESP).
- Distribution Subproblem (DISP).

The two subproblems (RESP and DISP) and the associated lower and upper bounds calculated by using these two subproblems are explained in detail in Appendix B. Since the bounds calculated by using these two subproblems are poor, we do not use this relaxation anymore.

3.4 Computation of lower bound from REPWCUT

We impose valid inequalities to the REP and transform into a stronger form REPWCUT. The valid inequalities are as follows.

$$\sum_{\substack{i=0 \\ i \ne k}}^{N} \sum_{v=1}^{V} x_{ijvt}^{k} \le S_{kt} \qquad \forall\, k \in N, t \in T \tag{3.20}$$

$$\sum_{j=1}^{N} \sum_{v=1}^{V} x_{0\,jvt}^{k} \le S_{kt} \qquad \forall\, k \in N, t \in T \tag{3.21}$$

Constraint set (3.20) limits the flow variables coming into retailer k that are designated for retailer k by its supply amount in each period t.

Constraint set (3.21) limits the flow variables leaving the depot and designated for retailer k by the supply amount of that retailer k.

Moreover, the constraint set (3.9) and (3.14) are used if they are not redundant for REP, i.e. the valid inequalities of (3.20) and (3.21) cover some of the constraint set (3.14) and they become redundant. For the periods in which vehicle capacity is more than the sum of total demand of each customer from the very beginning of the planning horizon to the current period, the maximum inventory keeping allowed constraint set (3.9) is used. For the other periods in which the flow variables are bounded by vehicle capacity constraint set (3.2) is enough. The necessary part of constraint set (3.14) after insertion of valid inequalities (3.20) and (3.21) is as follows.

$$x_{ijvt}^{k} \leq S_{kt} \qquad \forall\, i, j \in \overline{N},\ i \neq j, v \in V, t \in T,\ k \in N,\ i \neq k,\ j \neq k \tag{3.22}$$

The formulation of REPWCUT is given below, Z(REPWCUT) denotes the solution value of REPWCUT and give a lower bound on the objective function of the INVROP.

$$\text{Minimize} \sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T} \hat{f}_{ijvt}\, y_{ijvt} + \sum_{k=1}^{N}\sum_{t=0}^{T}\left(h_{kt} I_{kt} + b_{kt} B_{kt}\right) + \sum_{k=1}^{N}\sum_{t=1}^{T} p_{kt} S_{kt} +$$

$$\sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T}\sum_{k=1}^{N} \hat{c}_{ijvt}^{k}\, x_{ijvt}^{k} \tag{3.19}$$

Subject To

$$\sum_{k=1}^{N} x_{ijvt}^{k} \le K_v y_{ijvt} \qquad \forall\, i,\, j \in \overline{N},\ i \ne j, v \in V, t \in T \tag{3.2}$$

$$\sum_{\substack{j=0 \\ j \ne i}}^{N} y_{ijvt} \le 1 \qquad \forall\ i \in \overline{N}, v \in V, t \in T \tag{3.6}$$

$$I_{kt-1} - B_{kt-1} - I_{kt} + B_{kt} + S_{kt} = d_{kt} \qquad \forall\, t \in T, k \in N \tag{3.7}$$

$$I_{kt} \le I_{\max}^{k} \qquad \forall\, t \in \overline{T}, k \in N \tag{3.8}$$

$$x_{ijvt}^{k} \le \min\left\{ \sum_{r=1}^{t} d_{kr} + I_{\max}^{k}, K_v \right\} y_{ijvt} \qquad \begin{array}{l} \forall\, i,\, j \in \overline{N},\ i \ne j, v \in V, t \in T, \\ k \in N \end{array} \tag{3.9}$$

$$B_{k0} = 0 \qquad \forall\ k \in N \tag{3.10}$$

$$I_{k0} = 0 \qquad \forall\ k \in N \tag{3.11}$$

$$B_{kT} = 0 \qquad \forall\ k \in N \tag{3.12}$$

$$\sum_{k=1}^{N} S_{kt} \le \sum_{v=1}^{V} K_v \qquad \forall\, t \in T \tag{3.13}$$

$$\sum_{\substack{i=0 \\ i \ne k}}^{N} \sum_{v=1}^{V} x_{ijvt}^{k} \le S_{kt} \qquad \forall\, k \in N, t \in T \tag{3.20}$$

$$\sum_{j=1}^{N} \sum_{v=1}^{V} x_{0jvt}^{k} \le S_{kt} \qquad \forall\, k \in N, t \in T \tag{3.21}$$

$$x_{ijvt}^{k} \le S_{kt} \qquad \forall\, i,\, j \in \overline{N},\ i \ne j, v \in V, t \in T,\ k \in N,\ i \ne k,\ \ j \ne k \tag{3.22}$$

$$S_{kt}, I_{kt}, B_{kt},\ x_{ijvt}^{k} \ge 0 \qquad \forall\, i,\, j \in \overline{N},\ i \ne j, v \in V, t \in T, k \in N \tag{3.15}$$

$$y_{ijvt} \in \{0,1\} \qquad \forall\, i,\, j \in \overline{N},\ i \ne j, v \in V, t \in T \tag{3.16}$$

3.5 Computation of upper bound

After solving the REPWCUT, the values of supply variables $^{*}S_{kt}$ are known. It implies that the total amount to be shipped in each period is known. Since the amount to be shipped in a period cannot exceed the fleet capacity, a feasible schedule, i.e. allocation of shipment amount to the vehicles, is obtained by solving a capacitated vehicle routing problem (CVRP). In short, given the set of ${}^{*}S_{kt}$ variables for each time period t a CVRP$_{(t)}$ is solved. Summation of CVRP$_{(t)}$'s over all time periods is used to generate an upper bound.

3.5.1 Capacitated vehicle routing problem

The capacitated vehicle routing problem is formulated in a similar way of Chien et al. (1989). The problem which is solved for each time period $\bar{t}$ ($\bar{t}$ denotes specific time period t) is given below.

$$\text{Minimize } \mathbf{Z}\,(\text{CVRP}_{(\bar{t})}) = \sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V} f_{ijv\bar{t}}\, y_{ijv\bar{t}} + \sum_{j=1}^{N}\sum_{v=1}^{V} O_{\bar{t}}\, y_{ijv\bar{t}} +$$

$$\sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V}\sum_{k=1}^{N} c_{ijv\bar{t}}^{k}\, x_{ijv\bar{t}}^{k} \tag{3.23}$$

Subject To

$$\sum_{k=1}^{N} x_{ijv\bar{t}}^{k} \leq K_v\, y_{ijv\bar{t}} \qquad \forall\, i, j \in \overline{N},\ i \neq j, v \in V \tag{3.24}$$

$$\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V} x_{ijv\bar{t}}^{k} - \sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V} x_{jiv\bar{t}}^{k} = \begin{cases} +\ {}^{*}S_{kt} \text{ if } i = 0 \\ -\ {}^{*}S_{kt} \text{ if } i = k \end{cases} \qquad \forall\, i \in \overline{N}, k \in N \tag{3.25}$$

$$\sum_{\substack{j=0\\ j\neq i}}^{N} x_{ijv\bar{t}}^{k} - \sum_{\substack{j=0\\ j\neq i}}^{N} x_{jiv\bar{t}}^{k} = 0 \qquad \forall\ i \in N \setminus \{k\}, v \in V, k \in N \tag{3.26}$$

$$\sum_{\substack{j=0\\ j\neq i}}^{N} y_{ijv\bar{t}} - \sum_{\substack{j=0\\ j\neq i}}^{N} y_{jiv\bar{t}} = 0 \qquad \forall\, i \in \overline{N}, v \in V \tag{3.27}$$

$$\sum_{\substack{i=0\\ i\neq k}}^{N}\sum_{v=1}^{V} y_{ijv\bar{t}} = \left\{ 0 \text{ if } {}^{*}S_{kt} = 0 \right. \qquad \forall\ \ k \in N \tag{3.28}$$

$$x^{k}_{ijv\bar{t}} \geq 0 \qquad \forall\, i, j \in \overline{N},\ i \neq j, v \in V, k \in N \tag{3.29}$$

$$y_{ijv\bar{t}} \in \{0,1\} \quad \forall\, i, j \in \overline{N}, i \neq j, v \in V \tag{3.30}$$

The constraint sets (3.24), (3.25), (3.26) and (3.27) are the decompositions of the constraint sets (3.2), (3.3), (3.4) and (3.5) into time periods respectively. Constraint set (3.28) ensures that if there is no delivery planned for a particular customer *k,* there will be no shipment to that customer. For further improvements of the problem with single vehicle, we add the following constraints, in which new variables u_i's are defined.

u_i : Amount of product leaving location *i.*

$$\sum_{\substack{i=0 \\ i\neq k}}^{N} y_{ikv\bar{t}} = \begin{cases} 1 \text{ if } {}^{*}S_{k\bar{t}} > 0 \\ 0 \text{ otherwise} \end{cases} \qquad \forall \ \ k \in N \tag{3.31}$$

$$\sum_{\substack{j=0 \\ i\neq k}}^{N} y_{kjv\bar{t}} = \begin{cases} 1 \text{ if } {}^{*}S_{k\bar{t}} > 0 \\ 0 \text{ otherwise} \end{cases} \qquad \forall \ \ k \in N \tag{3.32}$$

$$\sum_{i=1}^{N} y_{i0v\bar{t}} = \begin{cases} 1 \text{ if } \sum_{k=1}^{N} {}^{*}S_{k\bar{t}} > 0 \\ 0 \text{ otherwise} \end{cases} \tag{3.33}$$

$$\sum_{j=1}^{N} y_{0jv\bar{t}} = \begin{cases} 1 \text{ if } \sum_{k=1}^{N} {}^{*}S_{k\bar{t}} > 0 \\ 0 \text{ otherwise} \end{cases} \tag{3.34}$$

$$u_i \leq \sum_{k=1}^{N} {}^{*}S_{k\bar{t}} \qquad \forall\, i \in \overline{N} \tag{3.35}$$

$$u_i - u_k + \sum_{k=1}^{N} {}^{*}S_{k\bar{t}}(1 - y_{ikv\bar{t}}) \geq {}^{*}S_{k\bar{t}} \qquad \forall\, i \in \overline{N}, k \in N, i \neq k \tag{3.36}$$

$$\sum_{\substack{j=0 \\ j \neq i}}^{N} \sum_{k=1}^{N} x_{ijv\bar{t}}^{k} \leq u_i \qquad \forall\, i \in \overline{N} \tag{3.37}$$

$$u_i \geq 0 \qquad \forall\, i \in \overline{N} \tag{3.38}$$

Constraint set (3.31) ensures that if the supply amount, which is calculated in lower bound section, of a retailer is positive, the vehicle visits that retailer. Constraint set (3.32) ensures that the vehicle must leave the customers that are visited. Constraints (3.33) and (3.34) ensure that a tour is started and ended at depot if there is any customer demand in that period. Constraint set (3.35) limits the total products leaving a location by total supply amount (which is less than or equal to the vehicle capacity). Constraint set (3.36) ensures that the amount of product leaving location *i* should cover the supply of the succeeding location *j* and the amount of product leaving location *j*. Constraint set (3.37) limits the flow variables leaving location *i* by the total demand of succeeding locations.

Given that the optimal values of y_{ijvt} (y^{*}_{ijvt}) and x^{k}_{ijvt} (x^{*k}_{ijvt}) are obtained with respect to ${}^{*}S_{kt}$ values, and using (${}^{*}I_{kt}$, ${}^{*}B_{kt}$) values that are obtained from the solution of REPWCUT, an upper bound for the original problem is computed. Note that ${}^{*}S_{kt}$, ${}^{*}I_{kt}$, ${}^{*}B_{kt}$, ${}^{*}y_{ijvt}$ and ${}^{*}x^{k}_{ijvt}$ denote the variables computed in the lower bound section, y^{*}_{ijvt} and x^{*k}_{ijvt} denote the variables computed in the upper bound section.

An algorithmic representation of upper bound computation is as follows.

Begin.

Get ${}^{*}S_{kt,}$ ${}^{*}I_{kt}$ and ${}^{*}B_{kt}$ values of REPWCUT from lower bound section;

for $k = 1$ **to** N **do**

for t = 1 **to** T **do**

if ($^{*}S_{kt} > 0$**)**

{Add customer k to the list of customers to be visited in period t;}

else

{Do not visit customer k in period t;}

endfor

endfor

for t = 1 **to** T **do**

{Solve CVRP(t`) and obtain y^{*}_{ijvt} and x^{*k}_{ijvt} values;}

endfor

Upper_Bound = Z(INVROP($^{*}S_{kt}$, $^{*}I_{kt}$, $^{*}B_{kt}$, y^{*}_{ijvt}, x^{*k}_{ijvt}));

End.

3.6 Solution of the Lagrangian dual problem

We use standard subgradient optimization algorithm to solve LADUP (Lagrangian Dual Problem) $= \underset{\alpha,\beta,\gamma}{Maximize}\, REPWCUT$. Initial values of the Lagrange multipliers are set to the optimal values of dual variables of the Linear Programming Relaxation of the INVROP (which is shown below as M(INVROPLP)), and in each iteration Lagrangian multipliers are updated.

M(INVROPLP):

Minimize $\sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T} f_{ijvt}\, y_{ijvt} \;+\; \sum_{k=1}^{N}\sum_{t=0}^{T}\left(h_{kt}I_{kt} + b_{kt}B_{kt}\right) \;+\; \sum_{j=1}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T} O_t\, y_{0jvt}$

$$+\sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T}\sum_{k=1}^{N} c_{ijvt}^{k} x_{ijvt}^{k} \tag{3.1}$$

Subject To

$$\sum_{k=1}^{N} x_{ijvt}^{k} \le K_v y_{ijvt} \qquad \forall\, i, j \in \overline{N},\ i \neq j, v \in V, t \in T \tag{3.2}$$

$$\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V} x_{ijvt}^{k} - \sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V} x_{jivt}^{k} = \begin{cases} +S_{kt} \text{ if } i=0 \\ -S_{kt} \text{ if } i=k \end{cases} \qquad \forall\, i \in \overline{N}, t \in T, k \in N \tag{3.3}$$

$$\sum_{\substack{j=0\\ j\neq i}}^{N} x_{ijvt}^{k} - \sum_{\substack{j=0\\ j\neq i}}^{N} x_{jivt}^{k} = 0 \qquad \forall\ i \in N \setminus \{k\}, v \in V, t \in T, k \in N \tag{3.4}$$

$$\sum_{\substack{j=0\\ j\neq i}}^{N} y_{ijvt} - \sum_{\substack{j=0\\ j\neq i}}^{N} y_{jivt} = 0 \qquad \forall\, i \in \overline{N}, v \in V, t \in T \tag{3.5}$$

$$\sum_{\substack{j=0\\ j\neq i}}^{N} y_{ijvt} \le 1 \qquad \forall\ i \in \overline{N}, v \in V, t \in T \tag{3.6}$$

$$I_{kt-1} - B_{kt-1} - I_{kt} + B_{kt} + S_{kt} = d_{kt} \qquad \forall\, t \in T, k \in N \tag{3.7}$$

$$I_{kt} \le I_{\max}^{k} \qquad \forall\, t \in \overline{T}, k \in N \tag{3.8}$$

$$x_{ijvt}^{k} \le \min\left\{\sum_{r=1}^{t} d_{kr} + I_{\max}^{k}, K_v\right\} y_{ijvt} \qquad \forall\, i, j \in \overline{N},\ i \neq j, v \in V, t \in T,\ k \in N \tag{3.9}$$

$$B_{k0} = 0 \qquad \forall\ k \in N \tag{3.10}$$

$$I_{k0} = 0 \qquad \forall\ k \in N \tag{3.11}$$

$$B_{kT} = 0 \qquad \forall\ k \in N \tag{3.12}$$

$$\sum_{k=1}^{N} S_{kt} \le \sum_{v=1}^{V} K_v \qquad \forall\, t \in T \tag{3.13}$$

$$x_{ijvt}^{k} \le S_{kt} \qquad \forall\, i, j \in \overline{N},\ i \neq j, v \in V, t \in T,\ k \in N \tag{3.14}$$

$$S_{kt}, I_{kt}, B_{kt}, x_{ijvt}^{k}, y_{ijvt} \ge 0 \qquad \forall\, i, j \in \overline{N},\ i \neq j, v \in V, t \in T, k \in N \tag{3.39}$$

The updating procedure of multiplier values and the step size through our iterations are as follows. Let Z^* be the best known feasible solution (the upper bound) up to the m^{th} iteration, and Z_{LR}^m be the solution of LADUP in m^{th} iteration. Let π^m is the step size scalar in the m^{th} iteration such that $0 \le \pi^m \le 2$. If the algorithm does not yield better results for a specified number of iterations, the step size scalar is halved.

Let the gradients of constraint sets (3.3) if *i*=0, (3.3) if *i*=*k*, (3.4) and (3.5) in the m^{th} iteration be g_1^m, g_2^m, g_3^m, g_4^m, respectively. These gradients are calculated by summing up the squared differences between the right hand sides and the left hand sides of the respective constraints as follows.

$$\mathrm{g}_1^{\mathrm{m}} = \sum_{k=1}^{N}\sum_{t=1}^{T}\left(-{}^*S_{kt}^m + \sum_{k=1}^{N}\sum_{v=1}^{V}{}^*x_{0jvtk}^m - \sum\sum{}^*x_{j0vtk}^m\right)^2 \tag{3.40}$$

$$\mathrm{g}_2^{\mathrm{m}} = \sum_{k=1}^{N}\sum_{t=1}^{T}\left({}^*S_{kt}^m + \sum_{\substack{j=0\\j\neq i}}^{N}\sum_{v=!}^{V}{}^*x_{kjvtk}^m - \sum_{\substack{j=0\\j\neq i}}^{N}\sum_{v=1}^{V}{}^*x_{jkvtk}^m\right)^2 \tag{3.41}$$

$$\mathrm{g}_3^{\mathrm{m}} = \sum_{\mathrm{i}=1}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T}\sum_{\substack{k=1\\k\neq i}}^{N}\left(\sum_{\substack{j=0\\j\neq i}}^{N}{}^*x_{ijvtk}^m - \sum_{\substack{j=0\\j\neq i}}^{N}{}^*x_{jivtk}^m\right)^2 \tag{3.42}$$

$$\mathrm{g}_4^{\mathrm{m}} = \sum_{i=0}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T}\left(\sum_{\substack{j=0\\j\neq i}}^{N}{}^*y_{ijvt}^m - \sum_{\substack{j=0\\j\neq i}}^{N}{}^*y_{jivt}^m\right)^2 \tag{3.43}$$

Let ρ^m be the step size in the m^{th} iteration. The step size is calculated as follows.

$$\rho^m = \frac{\pi^m (Z^* - Z_{LR}^m)}{(g_1^m + g_2^m + g_3^m + g_4^m)} \tag{3.44}$$

The new values of the Lagrangian multipliers are computed as follows.

$$\alpha_{0tk}^{m+1} = \alpha_{0tk}^m + \rho^{\mathrm{m}}(-{}^*S_{kt}^m + \sum_{j=1}^{N}\sum_{v=1}^{V} {}^*x_{0jvtk}^m - \sum_{j=1}^{N}\sum_{v=1}^{V} {}^*x_{j0vtk}^m) \quad \forall\ t \in T, k \in N \tag{3.45}$$

$$\alpha_{\mathrm{ktk}}^{\mathrm{m+1}} = \alpha_{\mathrm{ktk}}^{\mathrm{m}} + \rho^{\mathrm{m}}({}^*S_{kt}^m + \sum_{\substack{j=0 \\ j\neq k}}^{N}\sum_{v=1}^{V} {}^*x_{kjvtk}^m - \sum\sum {}^*x_{jkvtk}^m) \qquad \forall\ t \in T, k \in N \tag{3.46}$$

$$\beta_{ivtk}^{m+1} = \beta_{ivtk}^m + \rho^{\mathrm{m}}(\sum_{\substack{j=0 \\ j\neq i}}^{N} {}^*x_{ijvtk}^m - \sum_{\substack{j=0 \\ j\neq i}}^{N} {}^*x_{jivtk}^m) \ \forall\ i \in N, v \in V, t \in T, k \in N, k \neq i \tag{3.47}$$

$$\gamma_{ivt}^{m+1} = \gamma_{ivt}^m + \rho^m(\sum_{\substack{j=0 \\ j\neq i}}^{N} {}^*y_{ijvt}^m - \sum_{\substack{j=0 \\ j\neq i}}^{N} {}^*y_{jivt}^m) \qquad \forall i \in \overline{N}, v \in V, t \in T \tag{3.48}$$

Note that in order to indicate the iteration number *m*, we have defined a new index for the variables and parameters in (3.40), (3.41), (3.42) and (3.43) as follows.

${}^*S_{kt}^m = {}^*S_{kt} \quad \forall\, t \in T, k \in N$ in the m^{th} iteration.

${}^*x_{ijvtk}^m = {}^*x_{ijvt}^k \quad \forall\, i, j \in \overline{N}, i \neq j, v \in V, t \in T, k \in N$ in the m^{th} iteration.

${}^*y_{ijvt}^m = {}^*y_{ijvt} \quad \forall\, i, j \in \overline{N}, i \neq j, v \in V, t \in T$ in the m^{th} iteration.

$\alpha_{itk}^m = \alpha_{it}^k \quad \forall\, i \in \overline{N}, t \in T, k \in N$ in the m^{th} iteration.

$\beta_{ivtk}^{m} = \beta_{ivt}^{k} \quad \forall\, i \in N, v \in V, t \in T, k \in N, k \neq i$ in the m^{th} iteration.

$\gamma_{ivt}^{m} = \gamma_{ivt} \quad \forall\, i \in \overline{N}, v \in V, t \in T$ in the m^{th} iteration.

The flowchart of the algorithm applied to M(INVROP) is given in Figure 3.1.

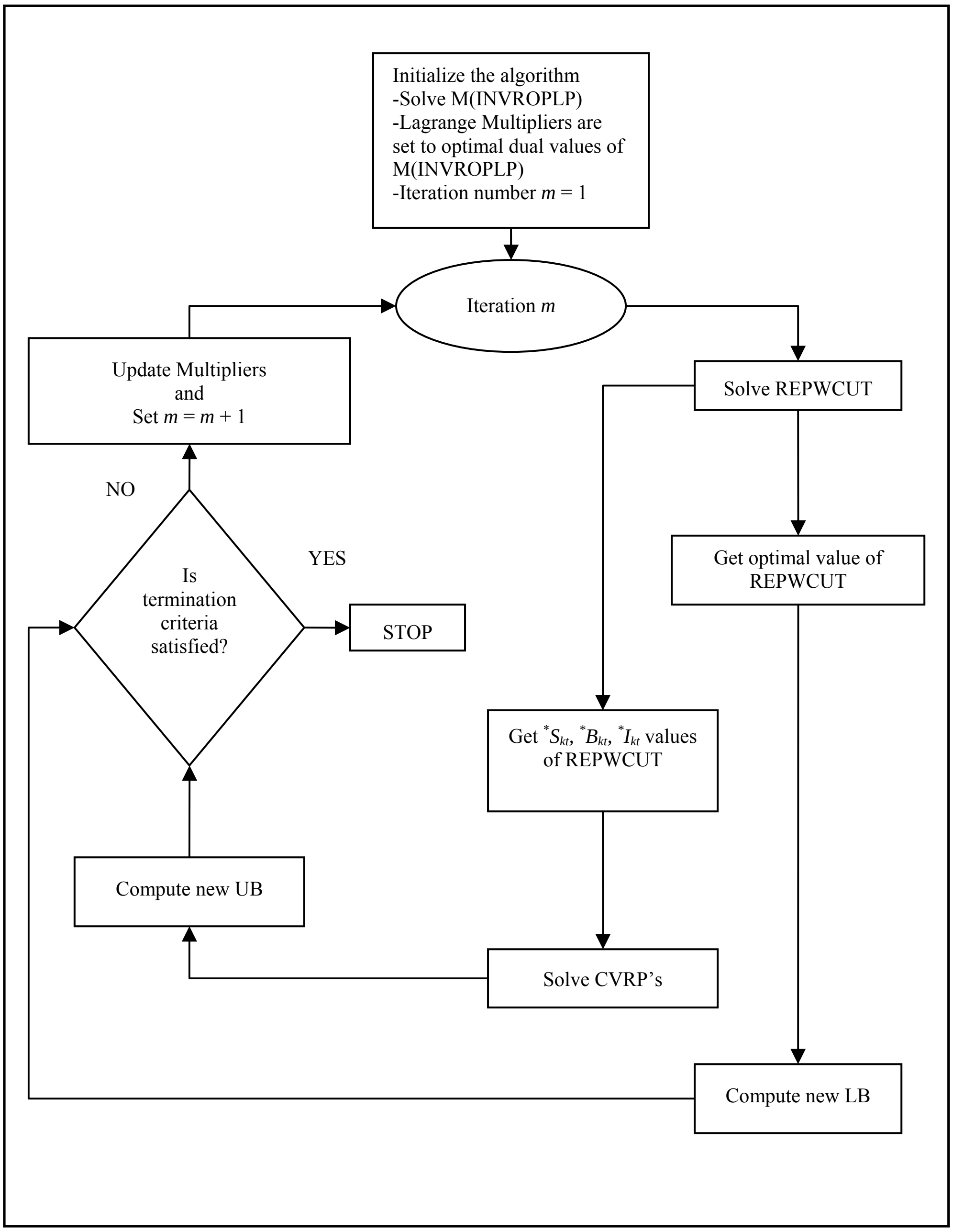

Figure 3.1 Flowchart of the Lagrangian Relaxation based algorithm

CHAPTER 4

COMPUTATIONAL RESULTS

In this chapter we present our computational results using the Lagrangian Relaxation based solution approach on different test instances taken from the literature. We first describe our computational framework. Then, present the results of preliminary experiments on small test instances. We next present the results obtained by applying Lagrangian Relaxation based solution approach on larger instances. Lastly we present benchmarking results.

4.1 Computational setting

There are three parts of our experimentation. In the first part, we use small problem instances in order to decide on best parameters that are to be set in the succeeding experiments. In this part the solution algorithm is repeated for 250 iterations and implemented for different parameters of the subgradient optimization algorithm. The parameter setting is tested as follows.

- Dividing the scalar π by two (i) after 5 consecutive non-improving iterations, or (ii) after 20 consecutive non-improving iterations.

- Initializing the lagrange multipliers by equating them (i) to zero or (ii) to the optimal dual variable values of the linear programming relaxation of the model.

The improvements that can be achieved by application of the valid inequalities that are presented in Chapter 3 are also tested by solving the problems by adding these inequalities to the problem formulation and by excluding them from the problem formulation. Moreover, the tolerance gap, which is the gap between the best integer solution found and the lower bound on the optimal solution of the relaxed mixed integer problem, at different levels are used as termination criterion for solving the relaxed problem optimally. In this part we computed the upper bound in each of the iterations of the proposed algorithm.

In the second part, we implement the best parameter values obtained in the first part and solve the larger problems with these parameters.

In the last part, for benchmarking, we revise our model and solve some of the original problems of Abdelmaguid and Dessouky (2006). The revised model is presented in the Appendix D.

While presenting our test instances we use notation *NTV*AD*k*. Here, *N* denotes the number of retailers, *T* denotes the number of time periods, *V* denotes the number of vehicles available, AD represents that the problem setting is taken from Abdelmaguid and Dessouky (2006) and *k* denotes the problem instance number.

While presenting the algorithms with different parameters we use notation "LR(a, n, l, c, d, u)", where "LR" denotes the Lagrangian Relaxation based solution algorithm, "a" denotes the number of consecutive non-improving iterations in which the subgradient optimization scalar π is halved, "n" denotes the number of iterations, "l" denotes whether tolerance gap limit or time limit is used or not, and if it is used the size of the limit is given (with "t" for time limit and "g" for gap limit, i.e. 15t denotes that 15 minutes of time limit is applied and 15g denotes 15% of gap limit is applied), "c" denotes whether the valid inequalities are used or not (1 if used, 0 otherwise), "d" denotes whether optimal dual values of linear programming relaxation is used for the initial values of lagrange multipliers or not (in the latter case all are initialized at zero) and "u" denotes whether time limit is applied in upper bounding procedure (for each vehicle routing problem) or not, and if it is used the size of the time limit is given in minutes (0 if not used). For instance, LR(20, 250, 5g, 1, 1, 15) means we use the Lagrangian Relaxation based solution approach with halving the scalar after 20 consecutive non-improving iterations, running the algorithm for 250 iterations, applying 5% gap limit, using valid inequalities, initializing multipliers by the optimal dual values, and applying 15 minutes of time limit for calculating upper bounds. The initial value of subgradient optimization scalar π is taken as two.

All the algorithms are coded in C++ programming language. For the solutions of linear programming problems as well as the mixed integer programming problems Callable Library of CPLEX 10.1 is embedded into the C++ code. Moreover, CONCORDE is called from the C++ code for solving TSPs. All the experiments are conducted on Pentium Core 2 Duo 2.33 ghz PCs with 1 GB RAM.

4.2 Basic test instances

All test instances used in this study are taken from the literature, which are developed by Abdelmaguid and Desouky (2006) with the following characteristics.

- Number of retailers (N): (5, 10, 15)

- Time horizon (T): (5, 7)

- Number of vehicles (V): (1, 2)

- Total vehicle capacity: (150, 300, 450) for N = (5, 10, 15) respectively. For the multi-vehicle settings, total vehicle capacity is allocated equally.

- Amount of product demanded from retailer k at time t (d_{kt}): Dynamic over time. Randomly generated using a uniform distribution from 5 to 50. Demand values are rounded up to the nearest integer value.

- Maximum amount of inventory per retailer k per time period t: Constant over time and is set as 120 units. We revise the maximum inventory levels of all retailers as 50 units per period.

- Beginning inventory level: Nil.

- Inventory holding cost at retailer k: Constant over time. Randomly generated using a normal distribution with a mean of 0.1 and a standard deviation of 0.02.

- Shortage cost at retailer k: Constant over time. Randomly generated with a normal distribution with a mean of 3 and a standard deviation of 0.5.

- Transportation cost per unit distance traveled: Constant over time, 2 units of cost.

- Coordinates of each retailer k: Randomly generated using a uniform distribution from 0 to 20. Coordinates are rounded to the nearest integer value.

- Coordinates of depot: (10, 10).

- Distance between two nodes (i,j): Rounded Euclidean distance between two nodes calculated with the formula:

$$\text{Dist}_{ij} = \left\lceil \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \right\rceil$$

  where (x_i, y_i) denotes the coordinates of node i on the x-axis and the y-axis, respectively.

- Fixed transportation cost between two nodes (i,j): Constant over time, 2*Dist_{ij}.

- Variable transportation cost of carrying one unit of item on arc (i,j): Constant over time, 0.05*Dist_{ij}.

- Fixed vehicle dispatching cost (O_t): Constant over time 10 units of cost per vehicle.

In the preliminary experiments we have used the settings of 551AD*k* and 552AD*k* where each setting has 5 different problems (*k*=1, 2, 3, 4 and 5).

4.3 Performance measures

In this section we present the performance measures used in the preliminary experiments applied on 10 test problems.

- **%MIP:** The percentage gap between the best feasible solution (UB) calculated by CPLEX in specified time limit and the optimal solution value of linear programming relaxation calculated by CPLEX, i.e. %(CPLEX_UB-LPR)/LPR.

- **%LGAP:** The percentage gap between the lower bound calculated with the LR and the optimal solution value, i.e. %(Opt-LB)/Opt.

- **%UGAP:** The percentage gap between the upper bound calculated with the LR and the optimal solution value, i.e. %(UB-Opt)/Opt.

- **%LRGAP:** The percentage gap between the upper bound and the lower bound calculated with the LR, i.e. %(UB-LB)/LB.

- **CPU X:** CPU time in minutes to solve X, where X will be the LR (Lagrangian Relaxation), CPUB (CPLEX upper bound) and LPR (Linear Programming Relaxation of M(INVROP)).

4.4 Part 1 (Preliminary experiments)

In this part the LR algorithm is run for 250 iterations in all settings except the settings in which we applied the valid inequalities and the relaxed problem is solved optimally for 100 iterations since in these settings too much CPU time is required to solve 250 iterations. In 250 iterations we have tried to obtain the best parameters that can be applied to the larger settings.

In Tables 4.1 - 4.7 we show the results obtained when the parameter π is halved after 5 or 20 consecutive non-improving iterations. Note that we did not give CPU LPR since CPLEX solves the LP models in less than 5 seconds. All of the CPLEX-UB values stated in this section are the optimal solution values of the respective problems. We observed that at the initial iterations -up to 100 iterations-, halving π after 5 consecutive non-improving iterations yields better results according to the performance measures other than CPU time; however, as the iteration number increased from 100 to 150 and to 250, halving π after 20 consecutive non-improving iterations yields significantly better results in all performance measures stated. For 150 iterations, on average, %LRGAP decreases from 36.54% to 33.35%, %UGAP decreases from 6.2% to 5.25%, %LGAP decreases from 22.03% to 20.9% and CPU LR decreases from 19.4 minutes to 16.8 minutes.

Table 4.1 Results of LR(5, 25, 0, 0, 0, 0) [*]

	MIP Model				LR(5, 25, 0, 0, 0, 0)					
	CPLEX UB	CPUB	LPR	%MIP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem										
551AD1*	1430.64	0.52	847.75	68.76	1594.98	990.24	2.00	30.78	11.49	61.07
551AD2	1531.68	0.09	908.02	68.68	1682.75	1156.27	1.92	24.51	9.86	45.53
551AD3	1184.78	0.08	785.64	50.80	1223.64	925.26	2.45	21.90	3.28	32.25
551AD4	1460.41	0.11	844.35	72.96	1584.43	1072.08	2.00	26.59	8.49	47.79
551AD5	1392.00	0.09	940.41	48.02	1511.54	1045.57	2.24	24.89	8.59	44.57
Average		0.18		61.85			2.12	25.74	8.34	46.24
552AD1	1145.32	0.85	868.57	31.86	1232.80	806.70	2.20	29.57	7.64	52.82
552AD2	1505.19	18.89	1194.32	26.03	1553.71	1138.82	2.25	24.34	3.22	36.43
552AD3	1138.87	11.68	918.77	23.96	1241.90	736.29	2.13	35.35	9.05	68.67
552AD4	1138.62	3.31	908.59	25.32	1221.70	769.84	2.23	32.39	7.30	58.70
552AD5	1204.92	6.15	959.35	25.60	1347.04	782.27	2.35	35.08	11.79	72.20
Average		8.18		26.55			2.23	31.34	7.80	57.76
Overall Average		4.18		44.20			2.18	28.54	8.07	52.00

[*] Note that we revised the demand figures of the setting 551AD1 in order to obtain feasibility with respect to total vehicle capacity.

Table 4.2 Results of LR(5, 50, 0, 0, 0, 0) and LR(5, 75, 0, 0, 0, 0)

	LR(5, 50, 0, 0, 0, 0)						LR(5, 75, 0, 0, 0, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1552.20	1060.11	5.30	25.90	8.50	46.42	1552.20	1071.80	9.90	25.08	8.50	44.82
551AD2	1663.45	1232.00	4.85	19.57	8.60	35.02	1663.45	1242.74	8.40	18.86	8.60	33.85
551AD3	1221.24	971.53	6.31	18.00	3.08	25.70	1221.24	977.53	10.72	17.49	3.08	24.93
551AD4	1571.65	1184.43	5.09	18.90	7.62	32.69	1571.65	1196.92	8.94	18.04	7.62	31.31
551AD5	1477.90	1109.15	5.46	20.32	6.17	33.25	1474.75	1121.42	9.54	19.44	5.94	31.51
Average			5.40	20.54	6.79	34.62			9.50	19.78	6.75	33.28
552AD1	1206.80	865.20	4.49	24.46	5.37	39.48	1206.80	872.10	7.04	23.86	5.37	38.38
552AD2	1553.71	1204.14	4.49	20.00	3.22	29.03	1553.71	1208.10	6.95	19.74	3.22	28.61
552AD3	1212.01	803.34	4.41	29.46	6.42	50.87	1212.01	813.70	6.83	28.55	6.42	48.95
552AD4	1186.85	847.57	4.50	25.56	4.24	40.03	1168.04	858.55	6.96	24.60	2.58	36.05
552AD5	1341.70	877.65	4.79	27.16	11.35	52.87	1333.08	893.35	7.56	25.86	10.64	49.22
Average			4.54	25.33	6.12	42.46			7.07	24.52	5.65	40.24
Overall Average			4.97	22.93	6.46	38.54			8.29	22.15	6.20	36.76

Table 4.3 Results of LR(5, 100, 0, 0, 0, 0) and LR(5, 150, 0, 0, 0, 0)

	LR(5, 100, 0, 0, 0, 0)						LR(5, 150, 0, 0, 0, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1552.20	1073.13	15.10	24.99	8.50	44.64	1552.20	1073.27	25.77	24.98	8.50	44.62
551AD2	1663.45	1244.13	12.38	18.77	8.60	33.70	1663.45	1244.45	20.60	18.75	8.60	33.67
551AD3	1221.24	978.29	15.36	17.43	3.08	24.83	1221.24	978.47	24.54	17.41	3.08	24.81
551AD4	1571.65	1197.57	13.06	18.00	7.62	31.24	1571.65	1197.66	21.36	17.99	7.62	31.23
551AD5	1474.75	1123.53	13.86	19.29	5.94	31.26	1474.75	1124.09	23.04	19.25	5.94	31.20
Average			13.95	19.70	6.75	33.14			23.06	19.68	6.75	33.11
552AD1	1206.80	873.14	9.79	23.76	5.37	38.21	1206.80	873.31	15.83	23.75	5.37	38.19
552AD2	1553.71	1208.79	9.70	19.69	3.22	28.53	1553.71	1208.92	15.40	19.68	3.22	28.52
552AD3	1212.01	815.80	9.48	28.37	6.42	48.57	1212.01	816.04	14.86	28.35	6.42	48.52
552AD4	1168.04	860.01	9.72	24.47	2.58	35.82	1168.04	860.17	15.54	24.46	2.58	35.79
552AD5	1333.08	895.03	10.48	25.72	10.64	48.94	1333.08	895.44	17.02	25.68	10.64	48.87
Average			9.83	24.40	5.65	40.02			15.73	24.38	5.65	39.98
Overall Average			11.89	22.05	6.20	36.58			19.39	22.03	6.20	36.54

Table 4.4 Results of LR(5, 250, 0, 0, 0, 0)

	LR(5, 250, 0, 0, 0, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem						
551AD1	1552.20	1073.28	46.81	24.98	8.50	44.62
551AD2	1663.45	1244.48	36.63	18.75	8.60	33.67
551AD3	1221.24	978.47	42.92	17.41	3.08	24.81
551AD4	1571.65	1197.67	38.45	17.99	7.62	31.23
551AD5	1474.75	1124.12	41.04	19.24	5.94	31.19
Average			41.17	19.68	6.75	33.10
552AD1	1206.80	873.34	28.87	23.75	5.37	38.18
552AD2	1553.71	1208.92	26.47	19.68	3.22	28.52
552AD3	1212.01	816.07	25.60	28.34	6.42	48.52
552AD4	1168.04	860.17	27.29	24.46	2.58	35.79
552AD5	1333.08	895.44	30.04	25.68	10.64	48.87
Average			27.65	24.38	5.65	39.98
Overall Average			34.41	22.03	6.20	36.54

Table 4.5 Results of LR(20, 25, 0, 0, 0, 0) and LR(20, 50, 0, 0, 0, 0)

	LR(20, 25, 0, 0, 0, 0)						LR(20, 50, 0, 0, 0, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1526.99	694.47	1.81	51.46	6.73	119.88	1526.99	964.97	3.98	32.55	6.73	58.24
551AD2	1649.41	850.13	1.73	44.50	7.69	94.02	1649.41	1134.77	3.74	25.91	7.69	45.35
551AD3	1248.96	680.84	1.98	42.53	5.42	83.44	1219.77	924.52	4.66	21.97	2.95	31.94
551AD4	1584.43	895.76	1.83	38.66	8.49	76.88	1584.43	1116.65	4.27	23.54	8.49	41.89
551AD5	1541.92	728.08	2.17	47.70	10.77	111.78	1462.57	1040.41	5.09	25.26	5.07	40.58
Average			1.90	44.97	7.82	97.20			4.35	25.85	6.19	43.60
552AD1	1207.87	606.51	2.07	47.04	5.46	99.15	1207.87	757.01	4.22	33.90	5.46	59.56
552AD2	1589.70	843.98	2.15	43.93	5.61	88.36	1563.17	1050.86	4.25	30.18	3.85	48.75
552AD3	1274.40	584.90	2.09	48.64	11.90	117.88	1227.22	663.25	4.27	41.76	7.76	85.03
552AD4	1239.40	574.63	2.12	49.53	8.85	115.69	1191.80	732.77	4.26	35.64	4.67	62.64
552AD5	1284.69	510.09	2.16	57.67	6.62	151.86	1284.69	756.85	4.48	37.19	6.62	69.74
Average			2.12	49.36	7.69	114.59			4.30	35.74	5.67	65.14
Overall Average			2.01	47.17	7.75	105.89			4.32	30.79	5.93	54.37

Table 4.6 Results of LR(20, 75, 0, 0, 0, 0) and LR(20, 100, 0, 0, 0, 0)

	LR(20, 75, 0, 0, 0, 0)						LR(20, 100, 0, 0, 0, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1526.99	1022.03	6.66	28.56	6.73	49.41	1526.99	1063.92	10.12	25.63	6.73	43.52
551AD2	1646.07	1211.66	6.21	20.89	7.47	35.85	1646.07	1236.06	8.99	19.30	7.47	33.17
551AD3	1219.77	962.83	7.88	18.73	2.95	26.69	1219.77	985.44	11.78	16.82	2.95	23.78
551AD4	1578.65	1159.41	7.02	20.61	8.10	36.16	1578.65	1186.51	10.68	18.76	8.10	33.05
551AD5	1462.12	1123.62	8.64	19.28	5.04	30.13	1462.12	1135.26	13.12	18.44	5.04	28.79
Average			7.28	21.62	6.06	35.65			10.94	19.79	6.06	32.46
552AD1	1207.87	857.22	6.40	25.15	5.46	40.91	1202.61	875.19	8.61	23.59	5.00	37.41
552AD2	1561.30	1145.11	6.37	23.92	3.73	36.34	1561.30	1182.69	8.51	21.43	3.73	32.01
552AD3	1216.20	787.56	6.48	30.85	6.79	54.43	1216.20	814.55	8.69	28.48	6.79	49.31
552AD4	1188.39	817.39	6.48	28.21	4.37	45.39	1188.39	842.31	8.68	26.02	4.37	41.09
552AD5	1284.69	873.20	6.89	27.53	6.62	47.12	1284.69	902.75	9.29	25.08	6.62	42.31
Average			6.53	27.13	5.39	44.84			8.76	24.92	5.30	40.43
Overall Average			6.90	24.37	5.73	40.24			9.85	22.35	5.68	36.44

Table 4.7 Results of LR(20, 150, 0, 0, 0, 0) and LR(20, 250, 0, 0, 0, 0)

	LR(20, 150, 0, 0, 0, 0)						LR(20, 250, 0, 0, 0, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1526.99	1081.77	19.60	24.39	6.73	41.16	1526.99	1088.90	48.38	23.89	6.73	40.23
551AD2	1646.07	1257.87	16.30	17.88	7.47	30.86	1646.07	1262.71	38.57	17.56	7.47	30.36
551AD3	1219.77	991.16	20.42	16.34	2.95	23.07	1219.77	994.45	40.88	16.06	2.95	22.66
551AD4	1564.16	1206.68	19.77	17.37	7.10	29.63	1564.16	1211.76	43.32	17.03	7.10	29.08
551AD5	1445.32	1148.70	25.10	17.48	3.83	25.82	1445.32	1153.49	57.92	17.13	3.83	25.30
Average			20.24	18.69	5.62	30.11			45.81	18.33	5.62	29.53
552AD1	1202.61	888.46	13.49	22.43	5.00	35.36	1202.61	892.89	24.53	22.04	5.00	34.69
552AD2	1561.30	1211.82	13.13	19.49	3.73	28.84	1561.30	1221.33	24.60	18.86	3.73	27.84
552AD3	1216.20	829.14	13.36	27.20	6.79	46.68	1210.93	835.06	23.39	26.68	6.33	45.01
552AD4	1165.01	870.91	13.14	23.51	2.32	33.77	1165.01	875.23	23.15	23.13	2.32	33.11
552AD5	1284.69	928.77	14.25	22.92	6.62	38.32	1284.69	934.84	24.77	22.41	6.62	37.42
Average			13.48	23.11	4.89	36.59			24.09	22.62	4.80	35.61
Overall Average			16.86	20.90	5.25	33.35			34.95	20.48	5.21	32.57

In Tables 4.8 - 4.10, we present the test results obtained by changing the initialization. The Lagrangian multipliers are now initialized with the optimal dual values of the linear programming relaxation of the model. In these tests we use only 20 consecutive non-improving iterations to halve the scalar π.

Table 4.8 Results of LR(20, 25, 0, 0, 1, 0) and LR(20, 50, 0, 0, 1, 0)

	LR(20, 25, 0, 0, 1, 0)						LR(20, 50, 0, 0, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1565.08	951.78	1.34	33.47	9.40	64.44	1517.02	1077.58	2.78	24.68	6.04	40.78
551AD2	1637.83	1052.75	1.40	31.27	6.93	55.58	1637.83	1271.80	3.03	16.97	6.93	28.78
551AD3	1274.59	930.54	1.46	21.46	7.58	36.97	1221.24	1030.94	3.23	12.98	3.08	18.46
551AD4	1572.37	1077.60	1.49	26.21	7.67	45.91	1565.86	1234.48	3.19	15.47	7.22	26.84
551AD5	1487.56	1095.45	1.46	21.30	6.86	35.79	1462.12	1194.10	3.19	14.22	5.04	22.45
Average			1.43	26.74	7.69	47.74			3.08	16.86	5.66	27.46
552AD1	1212.56	886.94	2.15	22.56	5.87	36.71	1206.34	953.78	4.37	16.72	5.33	26.48
552AD2	1622.79	1182.32	2.17	21.45	7.81	37.25	1569.75	1261.10	4.32	16.22	4.29	24.47
552AD3	1269.06	918.77	2.17	19.33	11.43	38.13	1230.86	918.77	4.45	19.33	8.08	33.97
552AD4	1226.91	903.59	2.18	20.64	7.75	35.78	1175.24	955.98	4.35	16.04	3.22	22.94
552AD5	1312.39	961.49	2.17	20.20	8.92	36.50	1312.39	1038.72	4.48	13.79	8.92	26.35
Average			2.17	20.84	8.36	36.87			4.39	16.42	5.97	26.84
Overall Average			1.80	23.79	8.02	42.31			3.74	16.64	5.81	27.15

Table 4.9 Results of LR(20, 75, 0, 0, 1, 0) and LR(20, 100, 0, 0, 1, 0)

	LR(20, 75, 0, 0, 1, 0)						LR(20, 100, 0, 0, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1517.02	1164.84	4.61	18.58	6.04	30.23	1517.02	1185.20	6.90	17.16	6.04	28.00
551AD2	1637.83	1331.38	5.07	13.08	6.93	23.02	1637.83	1343.32	7.71	12.30	6.93	21.92
551AD3	1221.10	1060.76	5.34	10.47	3.07	15.12	1221.10	1072.77	7.71	9.45	3.07	13.83
551AD4	1560.29	1280.56	5.37	12.32	6.84	21.84	1560.29	1296.78	8.34	11.20	6.84	20.32
551AD5	1446.16	1242.45	5.30	10.74	3.89	16.40	1446.16	1255.61	7.63	9.80	3.89	15.18
Average			5.14	13.04	5.35	21.32			7.66	11.98	5.35	19.85
552AD1	1206.34	1008.35	6.98	11.96	5.33	19.64	1206.34	1018.30	10.36	11.09	5.33	18.47
552AD2	1567.35	1293.81	6.46	14.04	4.13	21.14	1567.35	1320.80	8.71	12.25	4.13	18.67
552AD3	1224.69	957.90	6.87	15.89	7.54	27.85	1201.59	976.11	10.05	14.29	5.51	23.10
552AD4	1175.24	995.49	6.59	12.57	3.22	18.06	1175.24	1007.31	8.91	11.53	3.22	16.67
552AD5	1312.39	1063.99	6.83	11.70	8.92	23.35	1312.39	1082.41	9.31	10.17	8.92	21.25
Average			6.75	13.23	5.83	22.01			9.47	11.87	5.42	19.63
Overall Average			5.94	13.13	5.59	21.66			8.56	11.92	5.39	19.74

Table 4.10 Results of LR(20, 150, 0, 0, 1, 0) and LR(20, 250, 0, 0, 1, 0)

	LR(20, 75, 0, 0, 1, 0)						LR(20, 100, 0, 0, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1517.02	1202.25	13.18	15.96	6.04	26.18	1517.02	1207.45	32.09	15.60	6.04	25.64
551AD2	1637.83	1361.61	15.84	11.10	6.93	20.29	1637.83	1365.77	37.24	10.83	6.93	19.92
551AD3	1221.10	1078.41	13.02	8.98	3.07	13.23	1221.10	1080.21	25.01	8.83	3.07	13.04
551AD4	1560.29	1309.20	15.97	10.35	6.84	19.18	1560.29	1313.01	36.78	10.09	6.84	18.83
551AD5	1446.16	1266.50	13.80	9.02	3.89	14.19	1446.16	1269.57	29.45	8.80	3.89	13.91
Average			14.36	11.08	5.35	18.61			32.11	10.83	5.35	18.27
552AD1	1206.34	1034.06	20.74	9.71	5.33	16.66	1206.34	1036.95	53.79	9.46	5.33	16.34
552AD2	1567.35	1330.40	14.07	11.61	4.13	17.81	1567.35	1333.53	36.10	11.40	4.13	17.53
552AD3	1185.12	984.86	17.59	13.52	4.06	20.33	1185.12	987.05	34.08	13.33	4.06	20.07
552AD4	1175.24	1012.92	14.07	11.04	3.22	16.02	1175.24	1015.66	28.53	10.80	3.22	15.71
552AD5	1312.39	1092.96	14.48	9.29	8.92	20.08	1312.39	1097.33	26.57	8.93	8.92	19.60
Average			16.19	11.04	5.13	18.18			35.81	10.79	5.13	17.85
Overall Average			15.28	11.06	5.24	18.40			33.96	10.81	5.24	18.06

According to the results, initializing the lagrange multipliers by equating them to the optimal dual values yields better results in all of the test instances. For instance, on average for ten instances solved with 150 iterations, %LRGAP decreases from 33.35% to 18.40%, %UGAP decreases from 5.25% to 5.24%, %LGAP decreases from 20.09% to 11.06% and the CPU LR decreases from 16.8 minutes to 15.28 minutes. The greatest improvement is achieved by closing the %LGAP, meaning that usage of dual variables at initialization increases lower bound quality, which is a desired outcome.

We present two examples of convergence graphs in which we decided on the maximum number of iterations in Figures 4.1 and 4.2. Figure 4.1 is related to 551AD1, Figure 4.2 is related to 552AD1 and both instances are solved with LR(20, 250, 0, 0, 1, 0). All convergence graphs are given in Appendix D. We observe that the algorithm mostly finishes at about 150 iterations; therefore, we will run our algorithm for 150 iterations.

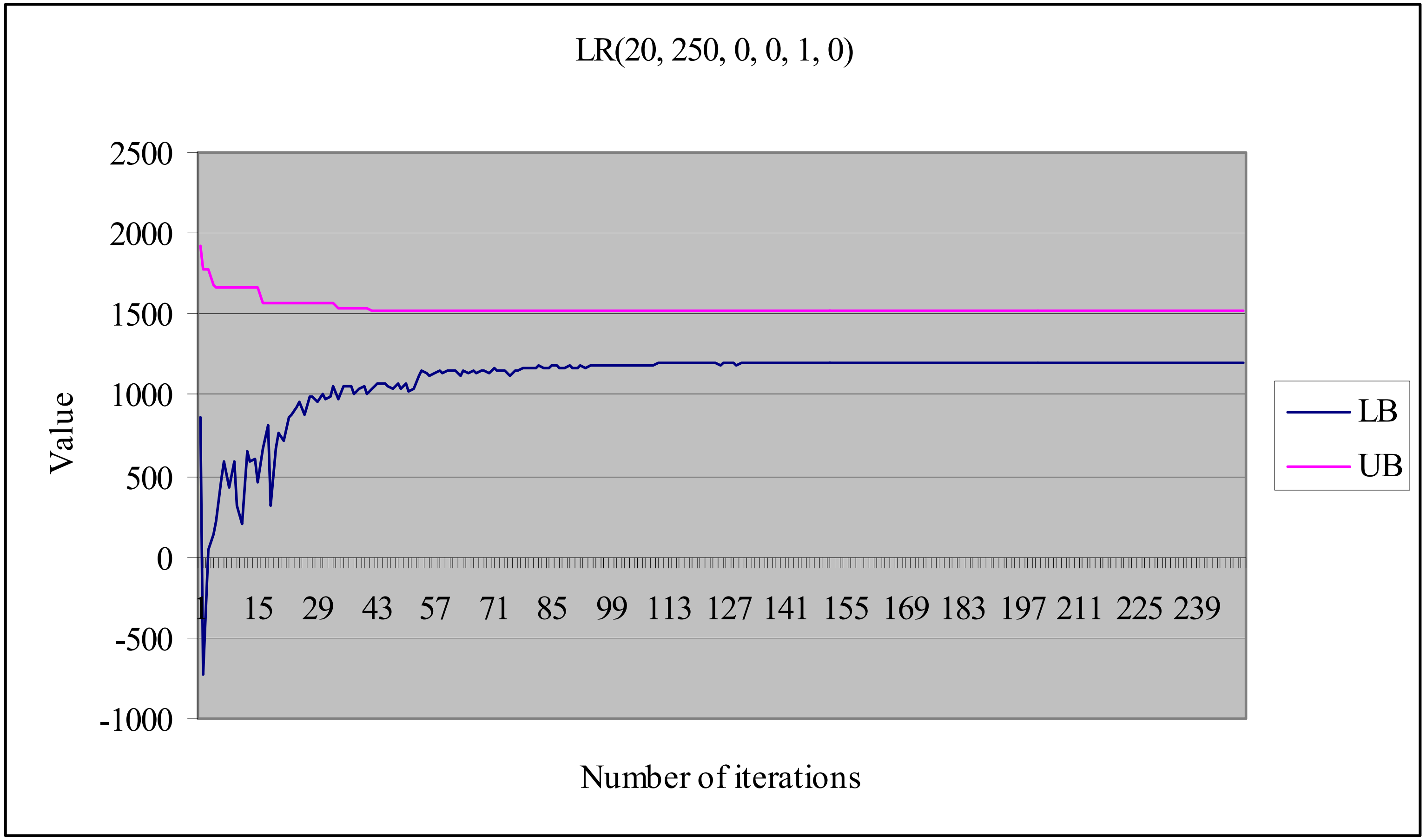

Figure 4.1 Convergence graph of 551AD1 with LR(20, 250, 0, 0, 1, 0)

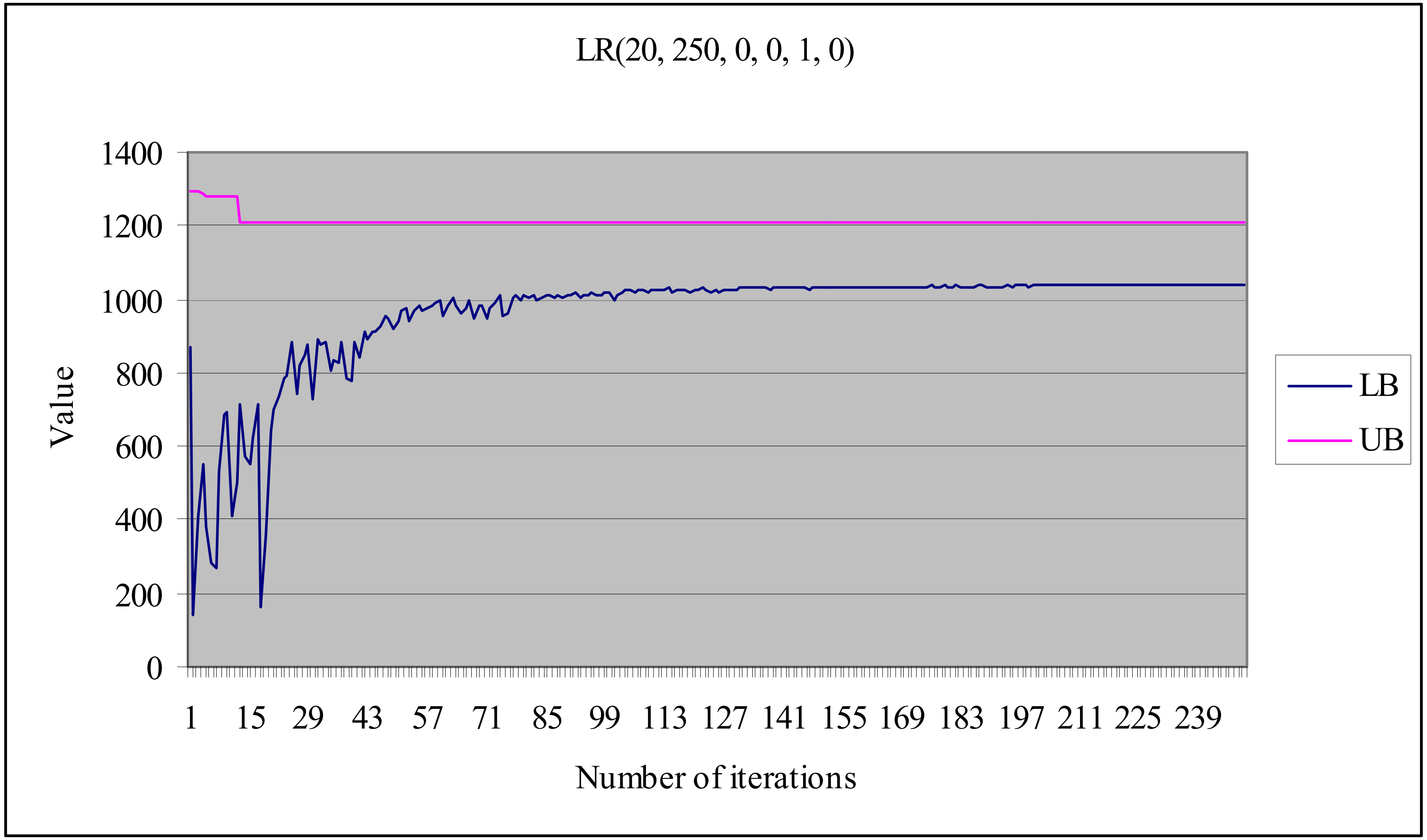

Figure 4.2 Convergence graph of 552AD1 with LR(20, 250, 0, 0, 1, 0)

In Tables 4.11 and 4.12 we give the results obtained with the insertion of valid inequalities. Due to the computation time considerations, this time we terminate our algorithm after 100 iterations. Valid inequalities greatly improve our bounds. However, it takes too much computation time. For instance, on average for ten instances solved for 100 iterations, %LRGAP decreases from 19.74% to 6.81%, %UGAP decreases from 5.39% to 2.69%, %LGAP decreases from 11.92% to 3.84%. However, CPU LR increases from 8.56 minutes to 44.74 minutes. Therefore, we also perform a test using a tolerance gap limit in solution rather than solving the relaxed problem optimally. In these settings CPLEX starts solving the relaxed problem with valid inequalities and terminates when the gap between best feasible integer solution and the lower bound drops below a specified value. We take the lower bound that CPLEX calculated as the objective function value of the relaxed problem.

Table 4.11 Results of LR(20, 25, 0, 1, 1, 0) and LR(20, 50, 0, 1, 1, 0)

	LR(20, 25, 0, 1, 1, 0)						LR(20, 50, 0, 1, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1472.77	1254.47	1.98	12.31	2.94	17.40	1472.77	1304.98	4.46	8.78	2.94	12.86
551AD2	1599.98	1423.30	1.56	7.08	4.46	12.41	1573.40	1469.21	3.52	4.08	2.72	7.09
551AD3	1227.76	1081.09	1.09	8.75	3.63	13.57	1198.27	1120.64	2.80	5.41	1.14	6.93
551AD4	1528.33	1323.41	1.49	9.38	4.65	15.48	1476.11	1376.21	3.19	5.77	1.08	7.26
551AD5	1425.51	1220.39	1.43	12.33	2.41	16.81	1425.51	1323.01	3.30	4.96	2.41	7.75
Average			1.51	9.97	3.62	15.13			3.46	5.80	2.06	8.38
552AD1	1194.07	1063.97	2.50	7.10	4.26	12.23	1185.14	1103.50	9.45	3.65	3.48	7.40
552AD2	1539.62	1407.83	2.55	6.47	2.29	9.36	1534.38	1444.71	5.55	4.02	1.94	6.21
552AD3	1212.37	1017.03	2.60	10.70	6.45	19.21	1212.37	1071.72	9.75	5.90	6.45	13.12
552AD4	1181.27	1044.29	2.49	8.28	3.75	13.12	1176.08	1082.35	7.75	4.94	3.29	8.66
552AD5	1265.93	1107.65	4.00	8.07	5.06	14.29	1265.93	1169.45	16.05	2.94	5.06	8.25
Average			2.83	8.13	4.36	13.64			9.71	4.29	4.04	8.73
Overall Average			2.17	9.05	3.99	14.39			6.58	5.04	3.05	8.55

Table 4.12 Results of LR(20, 75, 0, 1, 1, 0) and LR(20, 100, 0, 1, 1, 0)

	LR(20, 75, 0, 1, 1, 0)						LR(20, 100, 0, 1, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1472.77	1318.27	7.03	7.85	2.94	11.72	1472.77	1325.08	10.13	7.38	2.94	11.15
551AD2	1573.40	1481.94	5.68	3.25	2.72	6.17	1573.40	1485.39	7.42	3.02	2.72	5.93
551AD3	1198.27	1126.28	4.82	4.94	1.14	6.39	1198.27	1128.70	7.27	4.73	1.14	6.16
551AD4	1476.11	1388.72	5.13	4.91	1.08	6.29	1476.11	1391.39	7.47	4.73	1.08	6.09
551AD5	1425.51	1340.80	5.62	3.68	2.41	6.32	1425.51	1347.17	9.10	3.22	2.41	5.82
Average			5.65	4.93	2.06	7.38			8.28	4.62	2.06	7.03
552AD1	1180.58	1113.18	30.31	2.81	3.08	6.05	1180.58	1115.53	66.22	2.60	3.08	5.83
552AD2	1534.38	1449.49	9.19	3.70	1.94	5.86	1534.38	1452.71	13.53	3.49	1.94	5.62
552AD3	1199.03	1089.89	43.16	4.30	5.28	10.01	1199.03	1093.49	154.58	3.98	5.28	9.65
552AD4	1176.08	1093.93	24.56	3.92	3.29	7.51	1176.08	1098.18	103.66	3.55	3.29	7.09
552AD5	1265.93	1181.24	33.52	1.97	5.06	7.17	1241.30	1184.98	68.03	1.65	3.02	4.75
Average			28.15	3.34	3.73	7.32			81.20	3.06	3.32	6.59
Overall Average			16.90	4.13	2.89	7.35			44.74	3.84	2.69	6.81

In Tables 4.13 - 4.15 we present the results obtained with a gap limit of 3%. In Tables 4.16 - 4.18 we give the results with the application of 5% gap limit.

Table 4.13 Results of LR(20, 25, 3g, 1, 1, 0) and LR(20, 50, 3g, 1, 1, 0)

	LR(20, 25, 3g, 1, 1, 0)						LR(20, 50, 3g, 1, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1463.78	1209.32	1.59	15.47	2.32	21.04	1463.78	1272.36	3.30	11.06	2.32	15.04
551AD2	1608.41	1363.61	1.43	10.97	5.01	17.95	1581.52	1429.82	3.02	6.65	3.25	10.61
551AD3	1222.77	1047.15	1.46	11.62	3.21	16.77	1207.58	1088.80	2.99	8.10	1.92	10.91
551AD4	1550.75	1243.03	1.45	14.88	6.19	24.76	1523.27	1342.58	2.98	8.07	4.30	13.46
551AD5	1424.69	1206.50	1.40	13.33	2.35	18.08	1412.31	1293.60	2.99	7.07	1.46	9.18
Average			1.47	13.25	3.81	19.72			3.05	8.19	2.65	11.84
552AD1	1185.14	1040.55	2.21	9.15	3.48	13.90	1180.58	1076.64	4.58	6.00	3.08	9.65
552AD2	1560.74	1361.25	2.12	9.56	3.69	14.65	1534.64	1420.38	4.25	5.63	1.96	8.04
552AD3	1204.87	991.84	2.26	12.91	5.80	21.48	1204.87	1050.03	5.00	7.80	5.80	14.75
552AD4	1195.58	987.80	2.24	13.25	5.00	21.03	1163.68	1056.94	4.54	7.17	2.20	10.10
552AD5	1264.04	1069.48	2.44	11.24	4.91	18.19	1264.04	1132.00	5.45	6.05	4.91	11.66
Average			2.25	11.22	4.57	17.85			4.76	6.53	3.59	10.84
Overall Average			1.86	12.24	4.19	18.79			3.91	7.36	3.12	11.34

Table 4.14 Results of LR(20, 75, 3g, 1, 1, 0) and LR(20, 100, 3g, 1, 1, 0)

	LR(20, 75, 3g, 1, 1, 0)						LR(20, 100, 3g, 1, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1463.78	1288.35	5.29	9.95	2.32	13.62	1463.78	1288.35	7.35	9.95	2.32	13.62
551AD2	1581.52	1440.04	4.77	5.98	3.25	9.82	1581.52	1444.30	6.54	5.70	3.25	9.50
551AD3	1207.58	1097.65	4.61	7.35	1.92	10.02	1206.91	1101.08	6.35	7.06	1.87	9.61
551AD4	1476.11	1357.74	4.60	7.03	1.08	8.72	1476.11	1357.74	6.27	7.03	1.08	8.72
551AD5	1412.31	1310.08	4.68	5.89	1.46	7.80	1412.31	1316.01	6.42	5.46	1.46	7.32
Average			4.79	7.24	2.01	10.00			6.58	7.04	1.99	9.75
552AD1	1180.58	1085.98	7.00	5.18	3.08	8.71	1180.58	1091.39	9.46	4.71	3.08	8.17
552AD2	1534.64	1426.15	6.38	5.25	1.96	7.61	1533.59	1430.74	8.51	4.95	1.89	7.19
552AD3	1204.87	1056.20	9.01	7.26	5.80	14.08	1204.87	1061.39	15.36	6.80	5.80	13.52
552AD4	1162.95	1068.78	7.09	6.13	2.14	8.81	1162.95	1073.11	9.77	5.75	2.14	8.37
552AD5	1260.86	1145.46	8.93	4.93	4.64	10.07	1258.85	1151.11	13.37	4.47	4.48	9.36
Average			7.68	5.75	3.52	9.86			11.29	5.34	3.47	9.32
Overall Average			6.23	6.50	2.76	9.93			8.94	6.19	2.73	9.54

Table 4.15 Results of LR(20, 150, 3g, 1, 1, 0)

	LR(20, 150, 3g, 1, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem						
551AD1	1463.78	1291.35	12.00	9.74	2.32	13.35
551AD2	1556.33	1450.20	10.28	5.32	1.61	7.32
551AD3	1206.91	1105.88	9.80	6.66	1.87	9.14
551AD4	1476.11	1360.02	9.59	6.87	1.08	8.54
551AD5	1412.31	1316.45	10.16	5.43	1.46	7.28
Average			10.36	6.80	1.67	9.12
552AD1	1180.53	1093.18	14.59	4.55	3.07	7.99
552AD2	1531.34	1436.72	12.73	4.55	1.74	6.59
552AD3	1204.87	1065.99	36.01	6.40	5.80	13.03
552AD4	1162.95	1075.25	15.14	5.57	2.14	8.16
552AD5	1258.85	1154.62	23.40	4.17	4.48	9.03
Average			20.37	5.05	3.44	8.96
Overall Average			15.37	5.93	2.55	9.04

Table 4.16 Results of LR(20, 25, 5g, 1, 1, 0) and LR(20, 50, 5g, 1, 1, 0)

	LR(20, 25, 5g, 1, 1, 0)						LR(20, 50, 5g, 1, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1484.17	1178.13	1.51	17.65	3.74	25.98	1475.93	1257.00	3.08	12.14	3.17	17.42
551AD2	1583.71	1356.34	1.35	11.45	3.40	16.76	1583.71	1409.66	2.78	7.97	3.40	12.35
551AD3	1211.81	1038.51	1.41	12.35	2.28	16.69	1211.81	1076.36	2.86	9.15	2.28	12.58
551AD4	1561.76	1232.53	1.37	15.60	6.94	26.71	1495.34	1324.97	2.75	9.27	2.39	12.86
551AD5	1425.51	1149.20	1.36	17.44	2.41	24.04	1416.72	1270.46	2.82	8.73	1.78	11.51
Average			1.40	14.90	3.75	22.04			2.86	9.45	2.60	13.34
552AD1	1188.95	1023.76	2.11	10.61	3.81	16.14	1188.90	1071.83	4.24	6.42	3.81	10.92
552AD2	1539.68	1382.45	2.14	8.15	2.29	11.37	1532.25	1417.91	4.27	5.80	1.80	8.06
552AD3	1205.45	998.32	2.12	12.34	5.85	20.75	1205.45	1024.81	4.40	10.02	5.85	17.63
552AD4	1195.12	1024.04	2.21	10.06	4.96	16.71	1176.08	1056.78	4.45	7.19	3.29	11.29
552AD5	1278.93	1082.45	2.28	10.16	6.14	18.15	1278.93	1116.57	4.59	7.33	6.14	14.54
Average			2.17	10.27	4.61	16.62			4.39	7.35	4.18	12.49
Overall Average			1.79	12.58	4.18	19.33			3.62	8.40	3.39	12.92

Table 4.17 Results of LR(20, 75, 5g, 1, 1, 0) and LR(20, 100, 5g, 1, 1, 0)

	LR(20, 75, 5g, 1, 1, 0)						LR(20, 100, 5g, 1, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1474.22	1265.11	4.69	11.57	3.05	16.53	1465.44	1271.84	6.38	11.10	2.43	15.22
551AD2	1561.25	1419.51	4.23	7.32	1.93	9.99	1533.30	1432.30	5.74	6.49	0.11	7.05
551AD3	1206.91	1081.95	4.38	8.68	1.87	11.55	1206.91	1086.99	6.01	8.25	1.87	11.03
551AD4	1495.34	1343.94	4.14	7.98	2.39	11.27	1494.38	1343.94	5.58	7.98	2.33	11.19
551AD5	1411.94	1292.19	4.35	7.17	1.43	9.27	1411.94	1295.14	5.90	6.96	1.43	9.02
Average			4.36	8.54	2.13	11.72			5.92	8.16	1.63	10.70
552AD1	1188.90	1084.53	6.40	5.31	3.81	9.62	1188.86	1086.17	8.55	5.16	3.80	9.45
552AD2	1532.25	1427.16	6.36	5.18	1.80	7.36	1526.86	1428.37	8.47	5.10	1.44	6.90
552AD3	1205.45	1046.71	6.85	8.09	5.85	15.17	1205.45	1048.00	9.39	7.98	5.85	15.02
552AD4	1176.08	1058.63	6.67	7.03	3.29	11.09	1164.00	1064.70	8.86	6.49	2.23	9.33
552AD5	1262.30	1137.16	7.01	5.62	4.76	11.00	1262.30	1140.08	9.52	5.38	4.76	10.72
Average			6.66	6.25	3.90	10.85			8.96	6.02	3.62	10.28
Overall Average			5.51	7.40	3.02	11.28			7.44	7.09	2.62	10.49

Table 4.18 Results of LR(20, 150, 5g, 1, 1, 0)

	LR(20, 150, 5g, 1, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem						
551AD1	1465.44	1274.03	9.91	10.95	2.43	15.02
551AD2	1533.30	1433.80	8.81	6.39	0.11	6.94
551AD3	1199.44	1091.61	9.21	7.86	1.24	9.88
551AD4	1494.38	1348.63	8.52	7.65	2.33	10.81
551AD5	1411.94	1300.06	9.09	6.60	1.43	8.61
Average			9.11	7.89	1.51	10.25
552AD1	1188.81	1090.50	12.87	4.79	3.80	9.02
552AD2	1526.86	1431.97	12.70	4.86	1.44	6.63
552AD3	1197.74	1051.08	14.55	7.71	5.17	13.95
552AD4	1162.95	1068.06	13.26	6.20	2.14	8.88
552AD5	1230.75	1142.39	14.51	5.19	2.14	7.73
Average			13.58	5.75	2.94	9.24
Overall Average			11.34	6.82	2.22	9.75

With the valid inequalities and 5% gap limit (for the average of the ten test instances run for 100 iterations) %LRGAP is 9.75%, which was 6.81% for the case where gap limit was not applied and 19.74% where neither cuts nor the gap limit was applied. Average %UGAP of the case with 5% gap limit is the smallest among three cases with 2.22%. The average %UGAP was 2.69% for the case with valid inequalities and 5.39% for the case where neither valid inequalities nor gap limit was applied. Average %LGAP is 6.82%, which was 3.84% for the case with valid inequalities and no gap limit, 11.92% for the case where neither gap limit nor the valid inequalities were applied. Average CPU time of the case with 5% gap limit is the smallest with 8.56 minutes; the average CPU times of the former cases were 11.34 minutes and 44.74 minutes. Since we obtain good bounds in reasonable time with the case where we apply both valid inequalities and gap limit, we have decided to use is this setting for the larger instances. Note that we have chosen to start the algorithm with the optimal dual values and 20 as the number of consecutive non-improving iterations to halve the subgradient optimization scalar. All of the results obtained in the preliminary experiments are presented in Appendix E.

4.5 Part 2 (Main experiments)

In this section we present the results of the algorithm applied to the larger problem settings. Note that we will use time limit instead of gap limit because we have observed a bottleneck iteration in large instances taking too much computation time to reach the desired gap limit, and the other iterations taking relatively less amount of computation time. Therefore, we sacrifice the information gathered from the bottleneck iterations and terminate these iterations in pre-determined time limits.

We calculate an upper bound each iteration for the instances with 5 retailers, once in 5 iterations for 10 retailers and once in 20 iterations for 15 retailers.

Since we were not able to obtain the optimal solution values of the larger problem instances we use a slightly different performance measure, then the new performance measure is as follows.

- **Relative Error (RE):** The ratio of the gap of the Lagrangian Relaxation based solution approach to the gap between MIP and LP relaxation solutions, i.e. %LRGAP/%MIP

Note that in the following tables there exits a column called “Opt”. We insert “Y” for the problems that CPLEX has found an optimal solution in 60 (180) minutes for the problems with 5 (10 and 15) retailers.

In Tables 4.19 – 4.30 we present the results obtained by applying the algorithmic parameters decided in Section 4.4.

Table 4.19 Results of LR(20, 150, 5g, 1, 1, 0) for 551AD*k*

		MIP Model					LR(20, 150, 5g, 1, 1, 0)				
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	LR UB	LR LB	CPU LR	%LRGAP	RE
Problem	Opt										
551AD1	Y	1430.64	0.52	847.75	0.01	68.76	1465.44	1274.03	9.91	15.02	0.22
551AD2	Y	1531.68	0.09	908.02	0.01	68.68	1533.30	1433.80	8.81	6.94	0.10
551AD3	Y	1184.78	0.08	785.64	0.01	50.80	1199.44	1091.61	9.21	9.88	0.19
551AD4	Y	1460.41	0.11	844.35	0.01	72.96	1494.38	1348.63	8.52	10.81	0.15
551AD5	Y	1392.00	0.09	940.41	0.01	48.02	1411.94	1300.06	9.09	8.61	0.18
Average			0.18		0.01	61.85			9.11	10.25	0.17

Table 4.20 Results of LR(20, 150, 5g, 1, 1, 0) for 552AD*k*

		MIP Model					LR(20, 150, 5g, 1, 1, 0)				
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	LR UB	LR LB	CPU LR	%LRGAP	RE
Problem	Opt										
552AD1	Y	1145.32	0.85	868.57	0.01	31.86	1188.81	1090.5	12.87	9.02	0.28
552AD2	Y	1505.19	18.89	1194.32	0.01	26.03	1526.86	1431.97	12.70	6.63	0.25
552AD3	Y	1138.87	11.68	918.77	0.01	23.96	1197.74	1051.08	14.55	13.95	0.58
552AD4	Y	1138.62	3.31	908.59	0.01	25.32	1162.95	1068.06	13.26	8.88	0.35
552AD5	Y	1204.92	6.15	959.35	0.01	25.60	1230.75	1142.39	14.51	7.73	0.30
Average			8.18		0.01	26.55			13.58	9.24	0.35

Table 4.21 Results of LR(20, 150, 5g, 1, 1, 0) for 571AD*k*

		MIP Model					LR(20, 150, 5g, 1, 1, 0)				
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	LR UB	LR LB	CPU LR	%LRGAP	RE
Problem	Opt										
571AD1	Y	1723.29	0.41	1082.24	0.01	59.23	1781.61	1621.52	17.42	9.87	0.17
571AD2	Y	1431.37	0.10	1030.68	0.01	38.88	1450.15	1370.33	13.38	5.82	0.15
571AD3	Y	1199.18	0.31	779.07	0.01	53.92	1199.18	1101.22	12.75	8.90	0.16
571AD4	Y	1661.59	0.37	1043.34	0.01	59.26	1688.03	1568.87	12.77	7.60	0.13
571AD5	Y	1566.38	1.91	939.07	0.01	66.80	1607.26	1427.75	17.74	12.57	0.19
Average			0.62		0.01	55.62			14.81	8.95	0.16

Table 4.22 Results of LR(20, 150, 5g, 1, 1, 0) for 572AD*k*

		MIP Model					LR(20, 150, 5g, 1, 1, 0)				
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	LR UB	LR LB	CPU LR	%LRGAP	RE
Problem	Opt										
572AD1		1685.00	60.01	1300.60	0.01	29.56	1732.55	1593.8	19.57	8.71	0.29
572AD2		1751.34	60.03	1320.22	0.01	32.66	1816.83	1623.73	18.23	11.89	0.36
572AD3	Y	1580.88	14.76	1223.34	0.01	29.23	1618.17	1494.46	18.98	8.28	0.28
572AD4	Y	1647.73	34.23	1300.87	0.01	26.66	1694.08	1547.85	21.05	9.45	0.35
572AD5	Y	1625.45	23.17	1239.01	0.01	31.19	1687.68	1510.61	22.56	11.72	0.38
Average			38.44		0.01	29.86			20.08	10.01	0.33

Table 4.23 Results of LR(20, 100, 10g, 1, 1, 0) for 1051AD*k*

		MIP Model					LR(20, 100, 10g, 1, 1, 0)				
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	LR UB	LR LB	CPU LR	%LRGAP	RE
Problem	Opt										
1051AD1		2630.36	180.00	1289.4	0.02	104.00	2861.54	1945.99	274.55	47.05	0.45
1051AD2		2209.24	180.00	1461.27	0.04	51.19	2270.43	1873.57	53.61	21.18	0.41
1051AD3		3195.71	180.00	1626.9	0.05	96.43	3585.04	2411.66	126.46	48.65	0.50
1051AD4		2574.72	180.00	1595.81	0.04	61.34	2740.53	2012.58	215.22	36.17	0.59
1051AD5		2897.20	180.00	1594.76	0.04	81.67	3168.89	2219.18	92.07	42.80	0.52
Average			180.00		0.04	78.93			152.38	39.17	0.50

Table 4.24 Results of LR(20, 100, 10g, 1, 1, 0) for 1052AD*k*

		MIP Model					LR(20, 100, 10g, 1, 1, 0)				
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	LR UB	LR LB	CPU LR	%LRGAP	RE
Problem	Opt										
1052AD1		2505.07	180.00	1582.04	0.06	58.34	2663.46	2168.21	207.96	22.84	0.39
1052AD2		2084.02	180.00	1547.19	0.09	34.70	2206.31	1783.62	211.21	23.70	0.68
1052AD3		2326.01	180.00	1499.80	0.08	55.09	2430.18	1996.53	292.61	21.72	0.39
1052AD4		1851.85	180.00	1408.77	0.08	31.45	1918.52	1632.99	111.49	17.49	0.56
1052AD5		2456.36	180.00	1509.87	0.08	62.69	2606.97	2052.71	825.18	27.00	0.43
Average			180.00		0.08	48.45			329.69	22.55	0.49

Table 4.25 Results of LR(20, 100, 10g, 1, 1, 0) for 1071AD*k*

		MIP Model					LR(20, 100, 10g, 1, 1, 0)				
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	LR UB	LR LB	CPU LR	%LRGAP	RE
Problem	Opt										
1071AD1		4118.08	180.00	2111.37	0.05	95.04	4311.57	3096.8	215.06	39.23	0.41
1071AD2		4023.24	180.00	2263.62	0.06	77.73	4558.38	3288.11	513.99	38.63	0.50
1071AD3		3557.83	180.00	1848.37	0.06	92.48	3951.17	2599.99	239.53	51.97	0.56
1071AD4		4192.71	180.00	2346.31	0.06	78.69	4644.54	3298.35	192.67	40.81	0.52
1071AD5		3850.9	180.00	1998.42	0.06	92.70	4047.6	2811.45	804.30	43.97	0.47
Average			180.00		0.06	87.33			393.11	42.92	0.49

Table 4.26 Results of LR(20, 100, 10g, 1, 1, 0) for 1072AD*k*

		MIP Model					LR(20, 100, 10g, 1, 1, 0)				
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	LR UB	LR LB	CPU LR	%LRGAP	RE
Problem	Opt										
1072AD1		2860.50	180.00	2011.50	0.09	42.21	3143.11	2579.39	215.06	21.85	0.52
1072AD2		3344.50	180.00	2317.77	0.12	44.30	3593.82	2961.26	513.99	21.36	0.48
1072AD3		3136.73	180.00	2226.76	0.12	40.87	3487.16	2785.52	239.53	25.19	0.62
1072AD4		3263.66	180.00	2303.17	0.12	41.70	3398.79	2828.17	192.67	20.18	0.48
1072AD5		2743.09	180.00	1859.54	0.12	47.51	2916.36	2369.8	804.30	23.06	0.49
Average			180.00		0.11	43.32			393.11	22.33	0.52

Table 4.27 Results of LR(20, 75, 15t, 1, 1, 15) for 1551AD*k*[*]

		MIP Model					LR(20, 75, 15t, 1, 15)				
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	LR UB	LR LB	CPU LR	%LRGAP	RE
Problem	Opt										
1551AD1*		5581.38	180.00	2139.87	0.02	160.83	5948.05	2892.59	1529.18	105.63	0.66
1551AD2		5652.59	180.00	2096.42	0.14	169.63	5943.89	2634.75	1604.38	125.60	0.74
1551AD3		5328.09	180.00	2093.25	0.14	154.54	5533.23	2701.07	1536.29	104.85	0.68
1551AD4		4793.59	180.00	2514.65	0.15	90.63	5939.70	3318.31	1314.95	79.00	0.87
1551AD5		6163.32	180.00	2717.25	0.15	126.82	6903.48	3597.58	1496.91	91.89	0.72
Average			180.00		0.12	140.49			1496.34	101.39	0.73

[*] We revised the demand figures of the setting 1551AD1 in order to obtain feasibility with respect to total vehicle capacity

Table 4.28 Results of LR(20, 75, 15t, 1, 1, 15) for 1552AD*k*

		MIP Model					LR(20, 75, 15t, 1, 15)				
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	LR UB	LR LB	CPU LR	%LRGAP	RE
Problem	Opt										
1552AD1		3099.34	180.00	1993.20	0.20	55.50	3275.27	2332.62	1352.39	40.41	0.73
1552AD2		3039.26	180.00	1916.37	0.28	58.59	3225.74	2312.6	1248.26	39.49	0.67
1552AD3		3586.25	180.00	2384.47	0.28	50.40	3747.39	2656.49	1117.42	41.07	0.81
1552AD4		4527.98	180.00	2610.90	0.29	73.43	4779.21	3228.37	1357.56	48.04	0.65
1552AD5		3963.60	180.00	2190.01	0.28	80.99	4167.38	2633.04	1343.09	58.27	0.72
Average			180.00		0.27	63.78			1283.74	45.45	0.72

Table 4.29 Results of LR(20, 75, 15t, 1, 1, 15) for 1571AD*k*

		MIP Model					LR(20, 75, 15t, 1, 15)				
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	LR UB	LR LB	CPU LR	%LRGAP	RE
Problem	Opt										
1571AD1		6889.90	180.00	3023.01	0.18	127.92	6658.52	3641.57	1454.28	82.85	0.65
1571AD2		6305.04	180.00	2616.28	0.20	140.99	6045.42	3279.75	1506.08	84.33	0.60
1571AD3		7388.25	180.00	3180.58	0.20	132.29	6986.96	3761.78	1483.25	85.74	0.65
1571AD4		7499.02	180.00	3296.98	0.21	127.45	7825.82	4064.24	1477.80	92.55	0.73
1571AD5		7640.19	180.00	3465.09	0.20	120.49	8895.47	4651.15	1637.58	91.25	0.76
Average			180.00		0.20	129.83			1511.80	87.34	0.68

Table 4.30 Results of LR(20, 75, 15t, 1, 1, 15) for 1572AD*k*

		MIP Model					LR(20, 75, 15t, 1, 15)				
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	LR UB	LR LB	CPU LR	%LRGAP	RE
Problem	Opt										
1572AD1		4670.31	180.00	3185.86	0.29	46.59	5069.42	3415.03	1463.81	48.44	1.04
1572AD2		4867.26	180.00	2454.12	0.40	98.33	4990.38	2901.26	1766.87	72.01	0.73
1572AD3		6047.26	180.00	3094.41	0.39	95.43	5646.43	3513.25	1659.03	60.72	0.64
1572AD4		4345.58	180.00	2866.07	0.39	51.62	4908.62	3102.66	1504.80	58.21	1.13
1572AD5		6000.86	180.00	3039.68	0.40	97.42	5755.25	3843.34	1637.58	49.75	0.51
Average			180.00		0.37	77.88			1606.42	57.82	0.81

For the 5-retailer instances the average Lagrangian gap is 9.61% and MIP gap is 43.47%. The average relative error, which is the ratio of Lagrangian gap to MIP gap, is 0.25. It shows the performance of Lagrangian relaxation based algorithm over the MIP solution by CPLEX. It can be said that our algorithm closes the gap between bounds 3 times better than the MIP solution. For the 5-retailer instances with single vehicle, the average Lagrangian gap is 9.6% and it is 9.63% for multiple vehicle case.

For the 10-retailer instances, the average Lagrangian gap is 31.74%; whereas average MIP gap is 64.51% and average relative error is 0.5. The Lagrangian relaxation based algorithm is able to close half of the MIP gap. For the 10-retailer instances with single vehicle, the average Lagrangian gap is 41.05% and it is 22.44% for the multiple vehicle case.

For the 15-retailer instances, the average Lagrangian gap is 73%; whereas average MIP gap is 102.99% and average relative error is 0.73. Our algorithm is able to cover 36% of the MIP gap. The single (multiple) vehicle case yields an average Lagrangian gap of 94.37% (51.64%).

As the number of retailers in the system increases the algorithm's performance gets worse since both relaxed NP-hard problem and NP-hard CVRP need more solution times. Some iterations took days of CPU time and could not be solved optimally.

For 5-retailer instances the average gap of single and multiple vehicle cases are almost the same; whereas, for the 10-retailer and 15-retailer instances the average gap of single vehicle cases are almost the double of the multiple ones. This is due to the elimination of routing constraints while applying Lagrangian relaxation. For the multiple vehicle case the formulation is tighter than the formulation of single vehicle case. This observation is valid for the CVRP's. For the same number of retailers, without the valid inequalities presented in section 3.5.1, it takes much more time to solve single vehicle CVRP than multiple vehicle CVRP.

On overall average, the Lagrangian relaxation based solution algorithm yields 38.12% gap, which is 70.32% for CPLEX's MIP solution. The average relative error is 0.5; the Lagrangian relaxation based algorithm can cover half of the gap calculated by CPLEX. The average Lagrangian gap is 48.34% for single vehicle settings and 27.90% for multiple vehicle settings. Therefore, we can conclude that Lagrangian relaxation based algorithm yields better bounds than CPLEX solutions for all the cases, but the performance gets better for smaller instances and multiple vehicle settings. Although the overall algorithm takes too much CPU time as the size of the problems gets larger, CPLEX is not able to find even a feasible solution in compatible time limits.

4.6 Part 3 (Benchmarking)

In this section, for benchmarking purposes, we present the results of the algorithm applied on the problem instances using the revised model in Appendix D. In the revised model, backordering in the last period is allowed; therefore, supplier does not have to fulfill entire demand in the planning horizon. Moreover, transportation cost is not based on the amount supplied but on the distance traveled only. In Tables 4.31 – 4.34 we present the results obtained with our solution algorithm and the results in Abdelmaguid and Dessouky (2006).

In the last two columns of Tables 4.31 – 4.34, we present the upper bounds found by the heuristic algorithm given in Abdelmaguid and Dessouky (2006), and the gap between upper bound and the LP relaxation lower bound, namely %ABGAP (i.e. %ABGAP = %(Abdel_UB - LPR)/LPR). Note that CPLEX upper bounds are calculated in 60 minutes.

Table 4.31 Results of LR(20, 100, 5g, 1, 1, 15) for 551AD*k*

		MIP Model					LR(20, 100, 5g, 1, 1, 15)					Abdelmaguid and Dessouky	
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	LR UB	LR LB	CPU LR	%LRGAP	RE	Abdel_UB	%ABGAP
Problem	Opt												
551AD1	Y	649.80	0.67	334.22	0.01	68.76	655.95	601.45	160.60	9.06	0.13	687.83	105.80
551AD2	Y	468.00	0.05	217.33	0.01	68.68	468.36	437.03	18.17	7.17	0.10	537.27	147.22
551AD3	Y	400.00	0.13	221.76	0.01	50.80	400.44	363.64	66.18	10.12	0.20	406.85	83.47
551AD4	Y	475.29	0.15	218.12	0.01	72.96	476.03	426.52	36.11	11.61	0.16	475.95	118.21
551AD5	Y	426.01	0.22	234.77	0.01	48.02	442.67	370.82	87.64	19.38	0.40	481.87	105.25
Average			0.24		0.01	61.85			73.74	11.47	0.20		111.99

Table 4.32 Results of LR(20, 100, 5g, 1, 1, 15) for 552AD*k*

		MIP Model					LR(20, 100, 5g, 1, 1, 15)					Abdelmaguid and Dessouky	
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	LR UB	LR LB	CPU LR	%LRGAP	RE	Abdel_UB	%ABGAP
Problem	Opt												
552AD1	Y	522.82	44.68	356.40	0.01	46.69	550.25	458.081	15.83	20.12	0.43	550.13	54.36
552AD2		940.47	60.00	736.97	0.01	27.61	947.93	873.434	10.84	8.53	0.31	991.78	34.58
552AD3		512.44	60.00	370.13	0.01	38.45	532.02	435.402	18.59	22.19	0.58	578.56	56.31
552AD4		537.37	60.00	392.79	0.01	36.81	569.5	463.627	10.06	22.84	0.62	555.34	41.38
552AD5		553.20	60.00	394.59	0.01	40.20	563.26	485.063	32.65	16.12	0.40	576.96	46.22
Average			56.94		0.01	37.95			17.59	17.96	0.47		46.57

Table 4.33 Results of LR(20, 100, 5g, 1, 1, 15) for 571AD*k*

		MIP Model					LR(20, 100, 5g, 1, 1, 15)					Abdelmaguid and Dessouky	
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	LR UB	LR LB	CPU LR	%LRGAP	RE	Abdel_UB	%ABGAP
Problem	Opt												
571AD1	Y	522.97	2.17	258.34	0.01	102.43	532.47	463.739	281.11	14.82	0.14	640.65	147.98
571AD2	Y	557.89	0.09	357.51	0.01	56.05	562.03	515.443	161.08	9.04	0.16	580.81	62.46
571AD3	Y	434.86	0.71	221.09	0.01	96.69	441.8	397.765	40.30	11.07	0.11	510.92	131.09
571AD4	Y	536.42	2.68	254.67	0.01	110.64	558.42	466.331	90.01	19.75	0.18	647.07	154.08
571AD5	Y	498.08	6.64	240.51	0.01	107.10	511.2	437.994	66.86	16.71	0.16	582.22	142.08
Average			2.46		0.01	94.58			127.87	14.28	0.15		127.54

Table 4.34 Results of LR(20, 100, 5g, 1, 1, 15) for 572AD*k*

		MIP Model					LR(20, 100, 5g, 1, 1, 15)					Abdelmaguid and Dessouky	
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	LR UB	LR LB	CPU LR	%LRGAP	RE	Abdel_UB	%ABGAP
Problem	Opt												
572AD1		798.45	60.00	538.45	0.02	48.29	800.02	694.918	307.56	15.12	0.31	980.06	82.02
572AD2		855.79	60.00	572.59	0.02	49.46	878.41	742.847	18.08	18.25	0.37	1042.90	82.14
572AD3		726.68	60.00	510.06	0.02	42.47	753.85	658.832	71.86	14.42	0.34	960.04	88.22
572AD4		786.53	60.00	577.97	0.02	36.08	824.16	691.599	30.95	19.17	0.53	936.11	61.96
572AD5		771.35	60.00	516.52	0.02	49.34	800.13	682.465	89.26	17.24	0.35	930.36	80.12
Average			60.00		0.02	45.13			103.54	16.84	0.38		78.89

According to the results, on average, Lagrangian relaxation based algorithm yields 15.14% gap within 80.69 minutes; whereas the heuristic results of Abdelmaguid and Dessouky (2006) that are calculated within a minute deviates 91.25% from the solutions of LP relaxation. This gap figure is in a sense inflated because Abdelmaguid and Dessouky (2006) do not compute lower bounds and we give respective gaps with the LP relaxation results. The gap of 15.14% is larger than the average gap of (9.61%) settings presented in previous section because of the lack of variable transportation costs depending on the amount carried, but it is still plausible compared to the results of Abdelmaguid and Dessouky (2006).

CHAPTER 5

SINGLE SUPPLIER MULTIPLE RETAILER INVENTORY ROUTING PROBLEM WITH BACKORDERS

In this chapter, we first present a generalized version of our model M(INVROP). Next the Lagrangian relaxation of the model and resulting decomposed problems are specified. Then, the solution approaches of decomposed problems are discussed, and a general solution approach for the problem is given.

5.1 SSMRIRB

In this model depot is not only a coordination point or a cross-dock facility, but also an uncapacitated stock keeper. In this case the depot may hold inventory. Depot's supplier, supplies whatever needed in the beginning of each period. Retailers may hold inventory and the system may let retailers backorder the demands of end customers in order to minimize the total costs. However, all demand must be satisfied during the planning horizon. The total costs consists of fixed ordering cost and variable ordering costs at both retailers and the depot, retailer specific holding and shortage costs, supplier's holding cost, fixed vehicle dispatching cost, distance and amount dependent transportation costs.

In this setting, each vehicle distributes the specified amounts to the retailers in each period while satisfying the vehicle capacity, storage capacity and demand fulfillment limitations. Classification scheme of the single supplier, multiple retailer inventory routing problem with backorders, is given in the table 5.1 below.

Table 5.1 Classification scheme of SSMRIRB

Component	***Characteristic***
End Point	E(1,M)
Planning Horizon	P(T)
Vehicle(s)	V(C_m,M)
Demand Structure	Dynamic, Deterministic
Inventory	I(Y,Y)
Backordering	B(N,Y)
Ordering	O(Y,Y)
Inventory Policy	Endogenous
Transportation Cost	Fixed (vehicle specific) + Distance + Amount
Performance Measure(s)	Minimizing total costs

5.2 Assumptions of SSMRIRB

- The external demand or the demands of end customers occur at the retailers and the demands of retailers occur at the depot.

- Required amount to be distributed is supplied by the supplier's supplier to the depot in each period in addition to the inventory kept at the depot.

- The depot not only decides the vehicles to be dispatched, the retailers to be served and the amounts to be distributed in these visits, but also the amount of item ordered and the amount of inventory to keep in every period. In INVROP presented in Chapter 3, the depot is assumed to act as a crossdocking point that does not keep inventory; however, in SSMRIRB depot has the alternative to keep inventory for future periods.

- We assume that there is an underlying network that hosts the system's transportation structure. In this network nodes represent the supplier and the retailer sites. The arcs (links) represent the connections between these nodes.

- Each vehicle can make at most one trip in each period. Each trip starts from the depot and ends at the depot. Subtours not including the depot are not allowed.

- The amount carried by each vehicle is constrained by its capacity. Vehicle fleet is either homogeneous or heterogeneous; therefore, vehicle capacity may vary.

- There is no lead time for both the depot and the retailers. Products to be distributed to each retailer are ready at the beginning of each period and can be used to satisfy the demands of end customers at the beginning of the period. Therefore, next period's inventory level (positive, zero, or negative) is carried from the beginning of current period.

- Backordering and keeping inventory are allowed for retailers; whereas depot can only hold inventory.

- The amount of product that can be stored at each retailer is constrained with storage capacity of respective retailer; however, depot does not have storage capacity.

- Backordering in the last period is not allowed.

5.3 Mixed integer formulation of SSMRIRB

Indices of the model are as follows:

t : Time index (discrete time periods): 1, 2, …, T and $\overline{T} = T \cup \{0\}$.

i, j : Node index : 0, 1, …, N (i = 0 denotes depot). N denotes the set of retailers and $\overline{N} = N \cup \{0\}$.

k : Retailer index: 1, 2, …, N.

v : Vehicle index: 1, 2, …, V.

Parameters of the model are as follows:

N : Number of locations (retailers).

V : Number of vehicles.

T : Number of time periods.

K_v : Capacity of vehicle v.

$I_{\max}^{k}$: Storage capacity of retailer k.

d_{kt} : Demand of the end customer of retailer k at period t.

f_{ijvt} : Fixed cost for vehicle v in period t to use arc (i,j) for going from location i to location j.

c_{ijvt}^{k} : Variable cost of carrying one unit of product by vehicle v in period t on arc (i,j) for going from location i to location j for the designated customer k.

O_t : Fixed vehicle dispatching cost in time period t.

g_t : Fixed ordering cost for the depot in time period t.

p_{0t} : Unit variable cost charged to the depot in time period t.

h_{0t} : Unit holding cost of the depot in period t.

A_{kt} : Fixed ordering cost charged to retailer k in period t.

p_{kt} : Unit procurement cost of retailer k in time period t.

h_{kt} : Unit holding cost for retailer k in period t.

b_{kt} : Unit backordering cost for retailer k in period t.

M : A large number defined for the depot's fixed payment constraints.

U : A large number defined for the retailers fixed payment constraints.

Notice that parameters N, V and T denote both index sets and the cardinality of the corresponding sets.

Decision variables of the model are as follows.

$$y_{\mathrm{ijvt}} \;:\begin{cases} 1 \text{ if vehicle } v \text{ travels from location } i \text{ to location } j \text{ using arc } (i,j) \\ \quad \text{in period } t \\ 0 \text{ otherwise} \end{cases}$$

x_{ijvt}^{k} : Amount of product destined to retailer k, which is transported from location i to location j by vehicle v in time period t.

$$z_t \;:\begin{cases} 1 \text{ if depot gives an order in period } t \\ 0 \text{ otherwise} \end{cases}$$

r_{kt} : $\begin{cases} 1 \text{ if retailer } k \text{ gives an order in period } t \\ 0 \text{ otherwise} \end{cases}$

Q_t : Amount of product ordered by depot in period *t.*

W_t : Total amount to be shipped by depot to retailers in period *t.*

I_{0t} : Amount of product held by depot in period *t.*

I_{kt} : Amount of product held by retailer *k* in period *t.*

B_{kt} : Amount of demand backordered by retailer *k* in period *t.*

S_{kt} : Amount supplied to retailer *k* in period *t.*

M(SSMRIRB):

$$\text{Minimize} \sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T} f_{ijvt}y_{ijvt} + \sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T}\sum_{k=1}^{N} c_{ijvt}^{k}x_{ijvt}^{k} + \sum_{t=1}^{T} g_t z_t +$$

$$\sum_{t=1}^{T} p_{0t}Q_t + \sum_{t=0}^{T} h_{0t}I_{0t} + \sum_{k=1}^{N}\sum_{t=0}^{T}\left(h_{kt}I_{kt} + b_{kt}B_{kt}\right) + \sum_{k=1}^{N}\sum_{t=1}^{T} p_{kt}S_{kt} + \sum_{k=1}^{N}\sum_{t=1}^{T} A_{kt}r_{kt} +$$

$$\sum_{j=1}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T} O_t y_{0jvt} \tag{5.1}$$

Subject To

$$\sum_{j=1}^{N}\sum_{v=1}^{V}\sum_{k=1}^{N} x_{0jvt}^{k} = W_t \qquad \forall\, t \in T \tag{5.2}$$

$$I_{0t-1} + Q_t - I_{0t} = W_t \qquad \forall\, t \in T \tag{5.3}$$

$$Q_t \leq M\ z_t \qquad \forall\, t \in T \tag{5.4}$$

$$\sum_{k=1}^{N} x_{ijvt}^{k} \leq K_v y_{ijvt} \qquad \forall\, i, j \in \overline{N},\ i \neq j, v \in V, t \in T \tag{5.5}$$

$$\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V} x_{ijvt}^{k} - \sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V} x_{jivt}^{k} = \begin{cases} +S_{kt} \text{ if } i = 0 \\ -S_{kt} \text{ if } i = k \end{cases} \qquad \forall\, i \in \overline{N}, t \in T, k \in N \tag{5.6}$$

$$\sum_{\substack{j=0\\ j\neq i}}^{N} x_{ijvt}^{k} - \sum_{\substack{j=0\\ j\neq i}}^{N} x_{jivt}^{k} = 0 \qquad \forall\ i \in N \setminus \{k\}, v \in V, t \in T, k \in N \tag{5.7}$$

$$\sum_{\substack{j=0\\ j\neq i}}^{N} y_{ijvt} - \sum_{\substack{j=0\\ j\neq i}}^{N} y_{jivt} = 0 \qquad \forall\, i \in \overline{N}, v \in V, t \in T \tag{5.8}$$

$$\sum_{\substack{j=0\\ j\neq i}}^{N} y_{ijvt} \leq 1 \qquad \forall\ i \in \overline{N}, v \in V, t \in T \tag{5.9}$$

$$I_{kt-1} - B_{kt-1} - I_{kt} + B_{kt} + S_{kt} = d_{kt} \qquad \forall\, t \in T, k \in N \tag{5.10}$$

$$I_{kt} \leq I_{\max}^{k} \qquad \forall\, t \in \overline{T}, k \in N \tag{5.11}$$

$$B_{kT} = 0 \qquad \forall\ k \in N \tag{5.12}$$

$$x_{ijvt}^{k} \leq \min\left\{\sum_{r=1}^{t} d_{kr} + I_{\max}^{k},\ K_v\right\} y_{ijvt} \qquad \forall\, i, j \in \overline{N},\ i \neq j, v \in V, t \in T, k \in N \tag{5.13}$$

$$S_{kt} \leq U\, r_{kt} \qquad \forall\, t \in T, k \in N \tag{5.14}$$

$$\sum_{k=1}^{N} S_{kt} \leq \sum_{v=1}^{V}\sum_{j=1}^{N} K_v y_{0jvt} \qquad \forall\, t \in T \tag{5.15}$$

$$\sum_{k=1}^{N}\sum_{t=1}^{T} d_{kt} - \sum_{k=1}^{N} I_{k0} = \sum_{t=1}^{T} W_t \tag{5.16}$$

$$S_{kt},\ I_{kt},\ B_{kt},\ x_{ijvt}^{k},\ Q_t,\ W_t \geq 0 \qquad \forall\, i, j \in \overline{N},\ i \neq j, v \in V, t \in T, k \in N \tag{5.17}$$

$$y_{ijvt},\ z_t,\ r_{kt} \in \{0,1\} \qquad \forall\, i, j \in \overline{N},\ i \neq j, v \in V, t \in T, k \in N \tag{5.18}$$

The objective function (5.1) consists of fixed arc usage cost (first term), variable arc usage cost depending on the amount carried on that arc (second term), fixed ordering cost of the depot (third term), variable procurement cost of the depot (fourth term), inventory holding cost of depot (fifth term), retailer specific holding cost and backordering cost (sixth term), retailer specific procurement cost (seventh term), retailer specific fixed ordering cost (eighth term), period specific fixed vehicle dispatching cost (ninth term).

Constraint set (5.2) is used for keeping track of the flow variables initiated from the depot. The sum of the flow variables initiated from the depot is treated as the demand to the depot in each period.

Constraint set (5.3) is the inventory balance equations of the depot. Since inventory holding is possible for the depot, the amount supplied to the retailers may be different from the amount supplied to the depot.

Constraint set (5.4) forces the depot to pay fixed ordering cost if an order is made in any period.

Constraint set (5.5) satisfies the vehicle capacity restriction. The total amount sent to the retailers on a specified arc should be less than or equal to the capacity of the vehicle that traverses that arc. It thus links binary variables of arc usage (y_{ijvt}) and flow variables representing the amounts carried on these arcs (x^k_{ijvt}).

Constraint set (5.6) is for the commodity flow conservation equations. The set is defined for depot and all retailers. For the depot, the cumulative product going out is equal to the total amount to be distributed to retailers by a vehicle in a period. For retailers, the difference between the amount coming into retailer *k* and the amount going out of retailer *k* is the amount supplied to retailer *k* with a vehicle in a period.

Constraint set (5.7) is for the commodity flow conservation equations, which is defined for the retailers that are not designated customers. The difference between the amount coming into a retailer who is not to be served and the amount going out of that retailer is equal to zero; therefore, it is ensured that a retailer that is not in the list in a period is not served in that period.

Constraint sets (5.8) and (5.9) limit the movements of vehicles. By set (5.8) it is ensured that a vehicle that visits a retailer (or depot) in a specified period must leave that retailer (or depot). By set (5.9) it is ensured that a vehicle can visit a retailer (or depot) at most once in a period. Therefore, it is assumed that a vehicle starting from the depot will turn back and each vehicle can make at most one trip in every period. Note that the formulation eliminates possible subtours that are excluding the depot.

Constraint set (5.10) is the inventory balance equations for the retailers. Incoming inventory of a retailer minus the amount backordered in the previous period minus the amount to be hold at end of a period plus the amount backordered in that period plus the amount supplied in that period is equal to the demand of that retailer in that period. Hereby, it is obvious that in each period the system has three options: holding inventory, backordering and satisfying the demand.

Constraint set (5.11) is related with the limitation on the stocking amount at the retailers. A retailer cannot hold more inventories than its buffer capacity. Constraint set (5.12) is used to prohibit backordering in the last period.

Constraint set (5.13) is the redundant supply equations for the original model. However, they would be useful for obtaining reasonable solutions when relaxation is applied to solve the model, which will be discussed later on.

Constraint set (5.14) is used to force each retailer pay fixed procurement cost if an order is made.

Constraint set (5.15) is the supply limitation equations. Total amount supplied in each period should be less than the total vehicle capacity.

Constraint set (5.16) is related with initial inventory of the system (depot and the retailers). If there is any initial inventory at the depot or at any retailers, the total amount supplied will be equal to the difference between the total demand of retailers and the initial inventory in the system, due to the assumption that dictates the total demand should be satisfied during the planning horizon.

Constraint sets (5.17) and (5.18) are the non-negativity and integrality constraints respectively.

The M(SSMRIRB) is a huge model, and since it is a generalized version of M(INVROP) it is also NP-hard. M(SSMRIRB) consists of $N^3VT + 2N^2VT + NVT + 4NT + 2N + 3T$ many variables and $N^2VT + NVT + NT + T$ many of these variables are integer and the rest are continuous. Moreover, the number of constraints is $N^3VT + 3N^2VT + 2NVT + 5NT + 2VT + N + 4T + 1$. In order to make a comparison it could be stated that for a similar setting of M(INVROP) with parameters {N=15, T=7, V=2} the number of variables is 54,231 (3,472 integer variables) and the number of constraints is 54,372.

5.4 Lagrangian relaxation based solution approach

Constraint sets (5.2), (5.6), (5.7), (5.8) and (5.15) are relaxed and added to the objective function. Lagrange multipliers used in the model are as follows:

- λ_t for constraint set (5.2),
- $\alpha_{it}^k \quad i = 0 \ \ or \ \ i = k$; for constraint set (5.6),
- $\beta_{ivt}^k \quad i \neq 0 \ \ and \ \ i \neq k$; for constraint set (5.7),
- γ_{ivt} for constraint set (5.8),
- δ_t for constraint set (5.15), $\delta_t \geq 0$.

RELAXED PROBLEM (RP)

The relaxed problem with the above Lagrangian multipliers is stated as follows.

$$\text{Minimize (5.1)} + \sum_{t=1}^{T} \lambda_t \left(-W_t + \sum_{j=1}^{N} \sum_{v=1}^{V} \sum_{k=1}^{N} x_{0jvt}^{k} \right) +$$

$$\sum_{k=1}^{N} \sum_{t=1}^{T} \alpha_{0t}^{k} \left(-S_{kt} + \sum_{j=1}^{N} \sum_{v=1}^{V} x_{0jvt}^{k} - \sum_{j=1}^{N} \sum_{v=1}^{V} x_{j0vt}^{k} \right) +$$

$$\sum_{k=1}^{N} \sum_{t=1}^{T} \alpha_{kt}^{k} \left(S_{kt} - \sum_{\substack{j=0 \\ j\neq k}}^{N} \sum_{v=1}^{V} x_{kjvt}^{k} + \sum_{\substack{j=0 \\ j\neq k}}^{N} \sum_{v=1}^{V} x_{jkvt}^{k} \right) + \sum_{i=1}^{N} \sum_{v=1}^{V} \sum_{t=1}^{T} \sum_{\substack{k=1 \\ k\neq i}}^{N} \beta_{ivt}^{k} \left(\sum_{\substack{j=0 \\ j\neq i}}^{N} x_{ijvt}^{k} - \sum_{\substack{j=0 \\ j\neq i}}^{N} x_{jivt}^{k} \right) +$$

$$\sum_{i=0}^{N} \sum_{v=1}^{V} \sum_{t=1}^{T} \gamma_{ivt} \left(\sum_{\substack{j=0 \\ j\neq i}}^{N} y_{ijvt} - \sum_{\substack{j=0 \\ j\neq i}}^{N} y_{jivt} \right) + \sum_{t=1}^{T} \delta_t \left(\sum_{k=1}^{N} S_{kt} - \sum_{j=1}^{N} \sum_{v=1}^{V} K_v y_{0jvt} \right) \tag{5.19}$$

Subject To

$$I_{0t-1} + Q_t - I_{0t} = W_t \qquad \forall\, t \in T \tag{5.3}$$

$$Q_t \leq M\ z_t \qquad \forall\, t \in T \tag{5.4}$$

$$\sum_{k=1}^{N} x_{ijvt}^{k} \leq K_v y_{ijvt} \qquad \forall\, i, j \in \overline{N},\ i \neq j, v \in V, t \in T \tag{5.5}$$

$$\sum_{\substack{j=0 \\ j\neq i}}^{N} y_{ijvt} \leq 1 \qquad \forall\ i \in \overline{N}, v \in V, t \in T \tag{5.9}$$

$$I_{kt-1} - B_{kt-1} - I_{kt} + B_{kt} + S_{kt} = d_{kt} \qquad \forall\, t \in T, k \in N \tag{5.10}$$

$$I_{kt} \leq I_{\max}^{k} \qquad \forall\, t \in \overline{T}, k \in N \tag{5.11}$$

$$B_{kT} = 0 \qquad \forall\ k \in N \tag{5.12}$$

$$x_{ijvt}^{k} \le \min\left\{\sum_{r=1}^{t} d_{kr} + I_{\max}^{k},\ K_v\right\} y_{ijvt} \qquad \forall\, i, j \in \overline{N},\ i \ne j, v \in V, t \in T, k \in N \tag{5.13}$$

$$S_{kt} \le U\, r_{kt} \qquad \forall\, t \in T, k \in N \tag{5.14}$$

$$\sum_{k=1}^{N}\sum_{t=1}^{T} d_{kt} - \sum_{k=1}^{N} I_{k0} = \sum_{t=1}^{T} W_t \tag{5.16}$$

$$S_{kt},\ I_{kt},\ B_{kt},\ x_{ijvt}^{k},\ Q_t,\ W_t \ge 0 \qquad \forall\, i, j \in \overline{N},\ i \ne j, v \in V, t \in T, k \in N \tag{5.17}$$

$$y_{ijvt},\ z_t,\ r_{kt} \in \{0,1\} \qquad \forall\, i, j \in \overline{N},\ i \ne j, v \in V, t \in T, k \in N \tag{5.18}$$

Rearranging the cost components in the objective function by redefining original parameters of the model, we come up with RPN. New parameters are defined below.

$$\hat{f}_{\substack{0jvt \\ j\ne 0}} = O_t + \gamma_{0vt} - \gamma_{jvt} + f_{0jvt} - K_v \delta_t \ \rightarrow y_{\substack{0jvt \\ j\ne 0}}$$

$$\hat{f}_{\substack{ijvt \\ i\ne 0 \\ j\ne i}} = f_{ijvt} + \gamma_{ivt} - \gamma_{jvt} \ \rightarrow y_{\substack{ijvt \\ i\ne 0 \\ j\ne i}}$$

$$\hat{p}_{kt} = p_{kt} + \alpha_{kt}^{k} - \alpha_{0t}^{k} + \delta_t \ \rightarrow S_{kt}$$

$$\hat{c}^{k}_{\substack{0jvt \\ j\ne 0 \\ j\ne k}} = c^{k}_{0jvt} + \alpha^{k}_{0t} - \beta^{k}_{jvt} + \lambda_t \ \rightarrow x^{k}_{\substack{0jvt \\ j\ne 0 \\ j\ne k}}$$

$$\hat{c}^{k}_{\substack{0jvt \\ j\ne 0 \\ j=k}} = c^{k}_{0jvt} + \alpha^{k}_{0t} - \alpha^{k}_{kt} + \lambda_t \ \rightarrow x^{k}_{\substack{0jvt \\ j\ne 0 \\ j=k}}$$

$$\hat{c}^{k}_{\substack{j0vt \\ j\ne 0 \\ j\ne k}} = c^{k}_{j0vt} + \beta^{k}_{jvt} - \alpha^{k}_{0t} \ \rightarrow x^{k}_{\substack{j0vt \\ j\ne 0 \\ j\ne k}}$$

$$\hat{c}^{k}_{\substack{j0vt \\ j\ne 0 \\ j=k}} = c^{k}_{j0vt} + \alpha^{k}_{kt} - \alpha^{k}_{0t} \ \rightarrow x^{k}_{\substack{j0vt \\ j\ne 0 \\ j=k}}$$

$$\hat{c}^{k}_{\substack{ijt\\ i=k\\ i\neq 0\\ j\neq i}} = c^{k}_{ijvt} + \alpha^{k}_{kt} - \beta^{k}_{jvt} \quad \rightarrow x^{k}_{\substack{ijvt\\ i=k\\ i\neq 0\\ j\neq i}}$$

$$\hat{c}^{k}_{\substack{ijvt\\ i\neq k\\ i\neq 0\\ j=k}} = c^{k}_{ijvt} + \beta^{k}_{ivt} - \alpha^{k}_{kt} \quad \rightarrow x^{k}_{\substack{ijvt\\ i\neq k\\ i\neq 0\\ j=k}}$$

$$\hat{c}^{k}_{\substack{ijvt\\ i\neq j\neq k\neq 0}} = c^{k}_{ijvt} + \beta^{k}_{ivt} - \beta^{k}_{jvt} \quad \rightarrow x^{k}_{\substack{ijvt\\ i\neq j\neq k\neq 0}}$$

Mathematical formulation of RPN is as follows.

$$\text{Minimize } \sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T} \hat{f}_{ijvt} y_{ijvt} + \sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T}\sum_{k=1}^{N} \hat{c}^{k}_{ijvt} x^{k}_{ijvt} + \sum_{t=1}^{T} g_t z_t + \sum_{t=1}^{T} p_{0t} Q_t +$$

$$\sum_{t=0}^{T} h_{0t} I_{0t} - \sum_{t=1}^{T} \lambda_t W_t + \sum_{k=1}^{N}\sum_{t=0}^{T} \left(h_{kt} I_{kt} + b_{kt} B_{kt} \right) + \sum_{k=1}^{N}\sum_{t=1}^{T} \hat{p}_{kt} S_{kt} + \sum_{k=1}^{N}\sum_{t=1}^{T} A_{kt} r_{kt} \qquad (5.20)$$

Subject To

$$I_{0t-1} + Q_t - I_{0t} = W_t \qquad \forall\, t \in T \qquad (5.3)$$

$$Q_t \leq M\ z_t \qquad \forall\, t \in T \qquad (5.4)$$

$$\sum_{k=1}^{N} x^{k}_{ijvt} \leq K_v y_{ijvt} \qquad \forall\, i,\, j \in \overline{N},\ i \neq j, v \in V, t \in T \qquad (5.5)$$

$$\sum_{\substack{j=0\\ j\neq i}}^{N} y_{ijvt} \leq 1 \qquad \forall\ i \in \overline{N}, v \in V, t \in T \qquad (5.9)$$

$$I_{kt-1} - B_{kt-1} - I_{kt} + B_{kt} + S_{kt} = d_{kt} \qquad \forall\, t \in T, k \in N \qquad (5.10)$$

$$I_{kt} \leq I^{k}_{\max} \qquad \forall\, t \in \overline{T}, k \in N \qquad (5.11)$$

$$B_{kT} = 0 \qquad \forall\ k \in N \qquad (5.12)$$

$$x^{k}_{ijvt} \leq \min\left\{ \sum_{r=1}^{t} d_{kr} + I^{k}_{\max},\ K_v \right\} y_{ijvt} \qquad \forall\, i,\, j \in \overline{N},\ i \neq j, v \in V, t \in T,\ k \in N \qquad (5.13)$$

$$S_{kt} \leq U\ r_{kt} \qquad \forall\, t \in T, k \in N \qquad (5.14)$$

$$\sum_{k=1}^{N}\sum_{t=1}^{T} d_{kt} - \sum_{k=1}^{N} I_{k0} = \sum_{t=1}^{T} W_t \tag{5.16}$$

$$S_{kt},\ I_{kt},\ B_{kt},\ x_{ijvt}^{k},\ Q_t,\ W_t \geq 0 \qquad \forall\, i, j \in \overline{N},\ i \neq j, v \in V, t \in T,\ k \in N \tag{5.17}$$

$$y_{ijvt},\ z_t,\ r_{kt} \in \{0,1\} \qquad \forall\, i, j \in \overline{N},\ i \neq j, v \in V, t \in T, k \in N \tag{5.18}$$

Relaxed problem RPN can be decomposed into three subproblems.

- Supplier Subproblem (SSP).
- Retailer Subproblem (RSP).
- Distribution Subproblem (DSP).

These subproblems are defined in the next section.

5.4.1 Computation of lower bound

These three subproblems are solved with the methods given below and the summation of objective function value (5.20) gives us a lower bound on the value of original objective function (5.1).

5.4.2 Supplier subproblem (SSP)

$$\text{Minimize } \sum_{t=1}^{T} g_t z_t + \sum_{t=1}^{T} p_{0t} Q_t + \sum_{t=0}^{T} h_{0t} I_{0t} - \sum_{t=1}^{T} \lambda_t W_t \tag{5.21}$$

Subject To

$$I_{0t-1} + Q_t - I_{0t} = W_t \qquad \forall\, t \in T \tag{5.3}$$

$$Q_t \leq M\ z_t \qquad \forall\, t \in T \tag{5.4}$$

$$\sum_{k=1}^{N}\sum_{t=1}^{T} d_{kt} - \sum_{k=1}^{N} I_{k0} = \sum_{t=1}^{T} W_t \tag{5.16}$$

$$I_{0t},\ Q_t,\ W_t \geq 0 \qquad \forall\, i,\, j \in \overline{N},\ i \neq j, v \in V, t \in T, k \in N \tag{5.22}$$

$$z_t \in \{0,1\} \qquad \forall\, i,\, j \in \overline{N},\ i \neq j, v \in V, t \in T, k \in N \tag{5.23}$$

SSP is a variation of standard uncapacitated lot sizing problem consisting of fixed ordering cost, variable procurement cost, inventory holding cost and sales revenue (a component added due to Lagrangian relaxation).

Observation: Depot orders in a single period and sells (distributes) the entire ordered amount in a single period in the optimal solution of the SSP. It can be formulated as a maximization problem given in (5.24).

$$\underset{r}{Max} \left\{ \begin{array}{l} \lambda_t \left(\sum_{k=1}^{N}\sum_{t=1}^{T} d_{kt} - \sum_{k=1}^{N} I_{k0} \right) - g_\eta - p_{0\eta} \left(\sum_{k=1}^{N}\sum_{t=1}^{T} d_{kt} - \sum_{k=1}^{N} I_{k0} - I_{00} \right) - \\ \sum_{l=\eta}^{r-1} h_{0l} \left(\sum_{k=1}^{N}\sum_{t=1}^{T} d_{kt} - \sum_{k=1}^{N} I_{k0} \right) \end{array} \right\} \quad r \geq \eta \tag{5.24}$$

Where;

$$\sum_{t=1}^{T} W_t = \sum_{k=1}^{N}\sum_{t=1}^{T} d_{kt} - \sum_{k=1}^{N} I_{k0}$$

$$\sum_{t=1}^{T} Q_t = \sum_{t=1}^{T} W_t - I_{00}$$

This formulation tries to find the specific period r in which depot sells the entire demand such that in the other periods depot does no sales. Note that in a single period $\eta \le r$ depot orders the entire amount. Finding maximum of such a series has a complexity of $O(T^2)$.

Proof: If the total amount is sold in two discrete periods (*t1* and *t2*) and ordered in two discrete periods ($\mu 1$ and $\mu 2$) given in Figure 5.1, the resulting optimization problem can be formulated as follows (note that $t2 \ge \mu 2 \ge t1 \ge \mu 1$ without loss of generality).

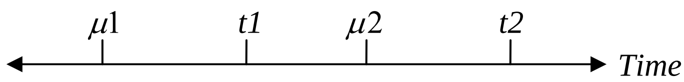

Figure 5.1 Order periods

If we assume that $\mu 1$=0, *t*2 = *T* and initial inventory level of the depot is zero, the problem can be stated in two cases.

Case 1: No inventory is carried to the second order period $\mu 2$ $\left(\sum_{k=1}^{N} I_{kt1} = 0 \right)$.

If no inventory is carried to the second order period the problem can formulated as follows.

$$\text{Maximize} \left\{ \begin{array}{l} \lambda_{t1}\left(\sum_{k=1}^{N} \sum_{t=1}^{t1} d_{kt} - \sum_{k=1}^{N} I_{k\mu 1} \right) - g_{\mu 1} - p_{0\mu 1}\left(\sum_{k=1}^{N} \sum_{t=1}^{t1} d_{kt} - \sum_{k=1}^{N} I_{k\mu 1} \right) - \\ \sum_{l=\mu 1}^{t1-1} h_{0l}\left(\sum_{k=1}^{N} \sum_{t=1}^{t1} d_{kt} \right) \end{array} \right\} +$$

$$\left\{\begin{array}{l}\lambda_{t2}\left(\sum_{k=1}^{N}\sum_{t=t1}^{t2} d_{kt}\right)-g_{\mu 2}-p_{0\mu 2}\left(\sum_{k=1}^{N}\sum_{t=t1}^{t2} d_{kt}\right)- \\ \sum_{l=t1}^{t2-1} h_{0l}\left(\sum_{k=1}^{N}\sum_{t=t1}^{t2} d_{kt}\right)\end{array}\right\}$$

Case 2: A positive amount of inventory is carried to the second order period $\mu 2$ $\left(\sum_{k=1}^{N} I_{kt1} > 0\right)$.

If a positive amount of inventory is carried to the second order period the problem can be formulated as follows.

$$\text{Maximize}\quad \left\{\begin{array}{l}\lambda_{t1}\left(\sum_{k=1}^{N}\sum_{t=1}^{t1} d_{kt}-\sum_{k=1}^{N} I_{k\mu 1}+\sum_{k=1}^{N} I_{kt1}\right)-g_{\mu 1}- \\ p_{0\mu 1}\left(\sum_{k=1}^{N}\sum_{t=1}^{t1} d_{kt}-\sum_{k=1}^{N} I_{k\mu 1}+\sum_{k=1}^{N} I_{kt1}\right)-\sum_{l=\mu 1}^{t1-1} h_{0l}\left(\sum_{k=1}^{N}\sum_{t=1}^{t1} d_{kt}+\sum_{k=1}^{N} I_{kt1}\right)\end{array}\right\} +$$

$$\left\{\begin{array}{l}\lambda_{t2}\left(\sum_{k=1}^{N}\sum_{t=t1}^{t2} d_{kt}-\sum_{k=1}^{N} I_{kt1}\right)-g_{\mu 2}-p_{0\mu 2}\left(\sum_{k=1}^{N}\sum_{t=t1}^{t2} d_{kt}-\sum_{k=1}^{N} I_{kt1}\right)- \\ \sum_{l=t1}^{t2-1} h_{0l}\left(\sum_{k=1}^{N}\sum_{t=t1}^{t2} d_{kt}\right)\end{array}\right\}$$

If we subtract the objective of Case 2 from Case 1 we come up with the following formulation.

$$\left(\left(\lambda_{t2}-p_{0\mu 2}\right)-\left(\lambda_{t1}-p_{0\mu 1}\right)+\sum_{l=t1}^{t2-1} h_{0l}\right)\sum_{k=1}^{N} I_{kt1}$$

Therefore, whatever the marginal revenues of ordering in two periods are, carrying inventory does not make sense noting that the holding costs are positive. The supplier distributes the entire amount ordered, and does not carry inventory from order period 1 to order period 2.

If we define X as the total amount ordered in period $\mu 1$, Y as the total amount ordered in period $\mu 2$, and H_t as the total holding cost up to period t from the last order period, we can write the above summation as follows.

$$\text{Maximize}(X-\varepsilon)\left(\lambda_{t1}-p_{0\mu 1}-H_{t1}-{g_{0\mu 1}}\big/{(X-\varepsilon)}\right)+$$
$$(Y+\varepsilon)\left(\lambda_{t2}-p_{0\mu 2}-H_{t2}-{g_{0\mu 2}}\big/{(Y+\varepsilon)}\right)$$

where, ε is a very small positive real number.

Replacing Y with (K-X) yields (5.25).

$$\text{Maximize}(X-\varepsilon)\left(\lambda_{t1}-p_{0\mu 1}-H_{t1}-{g_{0\mu 1}}\big/{(X-\varepsilon)}\right)$$
$$+(K-X+\varepsilon)\left(\lambda_{t2}-p_{0\mu 2}-H_{t2}-{g_{0\mu 2}}\big/{(Y+\varepsilon)}\right) \tag{5.25}$$

By rearranging the terms in (5.25) we come up with (5.26).

$$\text{Maximize }(X-\varepsilon)\left(\lambda_{t1}-p_{0\mu 1}-H_{t1}-{g_{0\mu 1}}\big/{(X-\varepsilon)}\right)-$$
$$(X-\varepsilon)\left(\lambda_{t2}-p_{0\mu 2}-H_{t2}-{g_{0\mu 2}}\big/{(Y+\varepsilon)}\right)+K\left(\lambda_{t2}-p_{0\mu 2}-H_{t2}-{g_{0\mu 2}}\big/{(Y+\varepsilon)}\right)$$

(5.26)

If marginal revenue of the first order period (second order period) is strictly greater than the marginal revenue of the second order period (first order period), (5.26) is maximized by ordering the entire amount in the first period (second period).

5.4.3 Retailer subproblem (RSP)

RSP differs from RESP (stated in Appendix B), with integer variables (r_{kt}). Fortunately, RSP can be reformulated with additional variables in strong form. The formulation of RSP is similar to the uncapacitated inventory lot sizing problem with backorders. The only difference is that there exists a fixed shipment cost, but it is equivalent to the purchasing cost in the classical model. The mixed-integer formulation of RSP is given below.

$$\text{Minimize } \sum_{k=1}^{N}\sum_{t=0}^{T}\left(h_{kt}I_{kt} + b_{kt}B_{kt}\right) + \sum_{k=1}^{N}\sum_{t=1}^{T}\hat{p}_{kt}S_{kt} + \sum_{k=1}^{N}\sum_{t=1}^{T}A_{kt}r_{kt} \tag{5.27}$$

Subject To

$$I_{kt-1} - B_{kt-1} - I_{kt} + B_{kt} + S_{kt} = d_{kt} \qquad \forall\, t \in T, k \in N \tag{5.10}$$

$$I_{kt} \le I_{\max}^{k} \qquad \forall\, t \in \overline{T}, k \in N \tag{5.11}$$

$$B_{kT} = 0 \qquad \forall\; k \in N \tag{5.12}$$

$$S_{kt} \le U\, r_{kt} \qquad \forall\, t \in T, k \in N \tag{5.14}$$

$$S_{kt},\ I_{kt},\ B_{kt} \ge 0 \qquad \forall\, t \in T, k \in N \tag{5.28}$$

$$r_{kt} \in \{0,1\} \qquad \forall\, t \in T, k \in N \tag{5.29}$$

RSP can be decomposed into k subproblems, since there is no link between retailers and no capacity limitation that binds them, each subproblem RSP-k can be represented as follows:

Minimize $\sum_{t=0}^{T}\left(h_{kt}I_{kt}+b_{kt}B_{kt}\right)+\sum_{t=1}^{T}\hat{p}_{kt}S_{kt}+\sum_{t=1}^{T}A_{kt}r_{kt}$ (5.30)

Subject To

$$I_{kt-1}-B_{kt-1}-I_{kt}+B_{kt}+S_{kt}=d_{kt} \qquad \forall\, t\in T \tag{5.31}$$

$$I_{kt}\le I_{\max}^{k} \qquad \forall\, t\in \overline{T} \tag{5.32}$$

$$B_{kT}=0 \tag{5.33}$$

$$S_{kt}\le U\, r_{kt} \qquad \forall\, t\in T \tag{5.34}$$

$$S_{kt},\ I_{kt},\ B_{kt}\ge 0 \quad \forall\, t\in T \tag{5.35}$$

$$r_{kt}\in\{0,1\} \ \ \forall\, t\in T \tag{5.36}$$

In order to solve the subproblem RSP-*k*, the shortest path reformulation given in Pochet and Wolsey (2006) is used.

Minimize $\sum_{t=0}^{T}\left(h_{kt}I_{kt}+b_{kt}B_{kt}\right)+\sum_{t=1}^{T}\hat{p}_{kt}S_{kt}+\sum_{t=1}^{T}A_{kt}r_{kt}$ (5.30)

Subject To

$$\sum_{\eta=1}^{T}\psi_{k,\eta,1}=1 \tag{5.37}$$

$$\sum_{\eta=1}^{t-1}\phi_{k,\eta,t-1}-\sum_{\eta=t}^{T}\psi_{k,\eta,t}=0 \qquad 2\le t\le T \tag{5.38}$$

$$-\sum_{l=1}^{t}\psi_{k,t,l}+\omega_{k,t,t}=0 \qquad 1\le t\le T \tag{5.39}$$

$$-\omega_{k,t,t}+\sum_{l=t}^{T}\phi_{k,t,l}=0 \qquad 1\le t\le T \tag{5.40}$$

$$\omega_{k,t,t}-r_{kt}\le 0 \qquad 1\le t\le T \tag{5.41}$$

$$S_{kt}=\sum_{\sigma=t+1}^{T}d_{k,t+1,\sigma}\phi_{k,t,\sigma}+\sum_{\sigma=1}^{t-1}d_{k,\sigma,t-1}\psi_{k,t,\sigma}+d_{kt}\omega_{k,t,t} \quad 1\le t\le T \tag{5.42}$$

$$I_{kt-1} = \sum_{\substack{\sigma,l;\sigma<t,\\ l\geq t}} d_{k,t,l}\phi_{k,\sigma,l} \qquad 1 \leq t \leq T \tag{5.43}$$

$$I_{kt} \leq I_{\max}^{k} \qquad \forall t \in \overline{T} \tag{5.32}$$

$$B_{kt} = \sum_{\substack{\sigma,l;\sigma>t,\\ l\leq t}} d_{k,l,t}\psi_{k,\sigma,l} \qquad 1 \leq t \leq T \tag{5.44}$$

$$\omega_{k,t,t}, \phi_{k,\sigma,t}, \psi_{k,\sigma,t} \geq 0 \qquad \forall\ t,\sigma \tag{5.45}$$

$$S_{kt},\ I_{kt},\ B_{kt} \geq 0 \quad \forall t \in T \tag{5.35}$$

$$r_{kt} \in \{0,1\} \quad \forall t \in T \tag{5.36}$$

Where;

$\omega_{k,t,t} = 1$ if the demand of retailer *k* of period *t* is supplied in period *t*

$\phi_{k,\sigma,t} = 1$ if the amount supplied in period σ includes the future demand up to period $t \geq \sigma$

$\psi_{k,\sigma,t} = 1$ if the amount supplied in period σ includes backlogged demand from period $t \leq \sigma$

and, $d_{k,t,l}$ is the cumulative demand of retailer *k* from period *t* to period *l*.

Pochet and Wolsey (2006) shows that if $I_{\max}^{k} \to \infty$, the strong reformulation can be solved in polynomial time. While using the shortest path reformulation, if below inequalities are added to the formulation, a tighter formulation is obtained according to Pochet and Wolsey (2006).

$\theta_{kt} = 1$ if the demand of retailer *k*, d_{kt} is satisfied from stock,

$\vartheta_{kt} = 1$ if the demand of retailer *k*, d_{kt} is satisfied from backlog,

$$\theta_{kt} + \vartheta_{kt} + r_{kt} = 1 \quad \forall t \text{ if } d_{kt} > 0 \tag{5.46}$$

$$I_{kl-1} \geq \sum_{\Theta=l}^{t} d_{k\Theta}(\theta_{k\Theta} - \sum_{\Delta=l}^{\Theta-1} r_{k\Delta}) \quad \forall l,t \qquad l \leq t \tag{5.47}$$

$$B_{kt} \geq \sum_{\Theta=t}^{l} d_{k\Theta}(\vartheta_{k\Theta} - \sum_{\Delta=\Theta+1}^{l} r_{k\Lambda}) \quad \forall l,t \qquad l \geq t \tag{5.48}$$

$$\theta_{kt}, \vartheta_{kt} \geq 0 \quad \forall\ k,t \tag{5.49}$$

The sum of the optimal solution values of RSP-k, k=1, …, N, gives us the objective function value of RSP.

5.4.4 Distribution subproblem (DSP)

The distribution subproblem is shown below.

$$\text{Minimize} \sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T} \hat{f}_{ijvt}\, y_{ijvt} + \sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T}\sum_{k=1}^{N} \hat{c}_{ijvt}^{k} x_{ijvt}^{k} \tag{5.50}$$

Subject To

$$\sum_{k=1}^{N} x_{ijvt}^{k} \leq K_v y_{ijvt} \qquad \forall\, i,\, j \in \overline{N},\ i \neq j, v \in V, t \in T \tag{5.5}$$

$$\sum_{\substack{j=0\\ j\neq i}}^{N} y_{ijvt} \leq 1 \qquad \forall\ i \in \overline{N}, v \in V, t \in T \tag{5.9}$$

$$x_{ijvt}^{k} \leq \min\left\{\sum_{r=1}^{t} d_{kr} + I_{\max}^{k},\ K_v\right\} y_{ijvt} \qquad \forall\, i,\, j \in \overline{N},\ i \neq j, v \in V, t \in T,\ k \in N \tag{5.13}$$

$$x_{ijvt}^{k} \geq 0 \qquad \forall\, i,\, j \in \overline{N},\ i \neq j, v \in V, t \in T, k \in N \tag{5.51}$$

$$y_{ijvt} \in \{0,1\} \quad \forall\, i,\, j \in \overline{N},\ i \neq j, v \in V, t \in T \tag{5.52}$$

The objective function in this subproblem consists of modified fixed cost and modified variable cost of each arc. Note that the fixed cost term includes original fixed cost of arc usage and attached Lagrange multipliers' values whereas modified variable cost is considers the amount carried on that arc and the Lagrange multipliers' values.

Although the subproblem DISP is a mixed integer problem, it can be decomposed into nodes (i). The decomposition is performed as follows. For a given node ($\underline{i}$), vehicle ($\underline{v}$), and time period ($\underline{t}$) the model reduces to DISPDEC$_{\underline{i}j^*\underline{v}\underline{t}}$ where only a single $y_{\underline{i}j^*\underline{v}\underline{t}}$ can take the value of 1 because of the constraint set (5.9). This suggests that we can fix $y_{\underline{i}j\underline{v}\underline{t}}$ to 1 for particular j, and then we easily solve a bounded continuous knapsack problem by using a greedy procedure. In this procedure, the variable costs of customers k ($\hat{c}^k_{\underline{i}j\underline{v}\underline{t}}$) are listed in a nondecreasing order. If the related cost is negative, the flow variable $x^k_{\underline{i}j\underline{v}\underline{t}}$ is set to the minimum value specified in constraint set (5.13). Otherwise, it is set to zero. This is repeated for all the variables on the list until the capacity is exhausted. By repeating the entire procedure for all j's for a given $\underline{i}$, $\underline{v}$, $\underline{t}$ triple, we determine the best $y_{\underline{i}j^*\underline{v}\underline{t}}$, as illustrated below.

$$Z_{\underline{i}j^*\underline{v}\underline{t}} = \min_{j} \left\{0, Z_{\underline{i}j\underline{v}\underline{t}}\right\} \tag{5.53}$$

The bounded continuous knapsack problem for each $j \in \overline{N}$, DISPDEC$_{\underline{i}j^*\underline{v}\underline{t}}$, is as follows.

$$\text{Minimize} \quad Z_{\underline{i}j\underline{v}\underline{t}}\left(y_{\underline{i}j\underline{v}\underline{t}} = 1\right) = \hat{f}_{\underline{i}j\underline{v}\underline{t}} + \sum_{k=1}^{N} \hat{c}^k_{\underline{i}j\underline{v}\underline{t}} x^k_{\underline{i}j\underline{v}\underline{t}} \tag{5.54}$$

Subject To

$$\sum_{k=1}^{N} x^k_{\underline{i}j\underline{v}\underline{t}} \le K_{\underline{v}} \tag{5.55}$$

$$0 \le x^k_{ij\underline{v}\underline{t}} \le \min\left\{\sum_{r=1}^{t} d_{kr} + I^k_{\max}, K_{\underline{v}}\right\} \qquad \forall\ k \in N \tag{5.56}$$

For each set of node i, vehicle v, and time period t, DISPDEC$_{\underline{i}j*\underline{v}\underline{t}}$ must be solved -meaning that $(N+1)NVT$ many problems would be solved- and the best solution value to DSP can be obtained by (5.57).

$$\mathbf{Z}\,(\text{DSP}) = \sum_{\substack{\underline{i}=0 \\ \underline{i}\neq j*}}^{N} \sum_{\underline{v}=1}^{V} \sum_{\underline{t}=1}^{T} Z_{\underline{i}j^*\underline{v}\underline{t}} \tag{5.57}$$

5.4.5 Algorithmic representation of lower bound computation

Begin:
Solve SSP;
for k=1 **to** N **do**
{Solve RSP-k;}
Get optimal values of objective functions of SSP and RSP-k's ;
Distribution_cost = 0;
for i = 0 **to** N **do**
    for v = 1 **to** V **do**
        for t = 1 **to** T **do**
            for j = 0 **to** N **do**
            Sort variable costs of customers in nondecreasing order l=1,…,N;
                for l = 1 **to** N **do**
                if (Variable cost of customer k is less than zero;**)**

{Assign the maximum possible amount to that customer according to the constraint set (5.13);
Update vehicle capacity;
Calculate cost due to delivery of the assigned amount to that customer *l*;}

else

{Assign zero to that customer;}

endfor

minimum = 0;

if ($Z_{\underline{i}j\underline{v}\underline{t}}$ < minimum)

{minimum = $Z_{\underline{i}j\underline{v}\underline{t}}$;

$j^* = j$;

Visit location *j** after location $\underline{i}$ by vehicle $\underline{v}$ in time period $\underline{t}$;}

else

{Do not visit location *j* after location $\underline{i}$ vehicle $\underline{v}$ in time period $\underline{t}$;}

endfor

{Distribution_cost = Distribution_cost + $Z_{\underline{i}j^*\underline{v}\underline{t}}$ }

endfor

endfor

endfor

Lower_Bound = Distribution_cost + Z(RSP) + Z(SSP);

End.

5.4.6 Computation of upper bound

Finding a feasible solution gives us an upper bound for the P(SSMRIRB). Since it is hard to solve original problem optimally, a heuristic algorithm that yields good feasible solutions in reasonable times should be used. The heuristic algorithm that can be used in further researches should include an efficient allocation algorithm that would assign the customers to the vehicles and satisfy vehicle capacities while fulfilling the entire demand of end customers during the planning horizon.

CHAPTER 6

CONCLUSION

In this study, an inventory routing problem with backorders (INVROP) has been analyzed and a mixed integer mathematical formulation has been developed for solving the INVROP. For the small sized problem instances we have identified optimal solutions and for larger instances we have computed lower and upper bounds.

The INVROP is NP-hard because of the embedded CVRP's (capacitated vehicle routing problems) and the joint replenishment problem. Considering the difficulties in finding the optimal solutions in such cases, we have developed a Lagrangian relaxation based solution algorithm that computes both lower and upper bounds in the Lagrangian relaxation based approach, we have relaxed flow balance equations and movement restriction equations that work as subtour elimination constraints. Because of our problem characteristics we have taken test instances from the literature and revised some of them in order to achieve feasibility. We have tested our algorithm with small instances for which the optimal solutions are possibly found. In the preliminary experiments we have decided on the parameters of the algorithm and applied these parameters in the solution procedure of the larger problem instances.

The main contributions of this thesis are to develop a mathematical model for the INVROP and identify lower bounds on the optimal solution. None of the finite horizon models with deterministic demand in the literature has considered backordering as an option for the supply chain other than Chien et. al. (1989) and Abdelmaguid and Dessouky (2006). Chien et. al. (1989)

presented both lower and upper bounds for only a single period problem. Abdelmaguid and Dessouky (2006) considers multiple periods, but not compute lower bounds only upper bounds. They did not consider variable transportation costs either.

Our mathematical formulation is a mega model that could handle several cost structures such as fixed and variable transportation costs, fixed dispatching costs, inventory holding and backordering costs. For implementation any of these costs could be removed or added to the problem (in the INVROP we used all these cost structures).

We also presented an algorithm for the generalized version of INVROP, which is more complicated. Further improvements may be possible by examining the generalized version of INVROP. In the solution algorithm, we used valid inequalities that strengthen the formulation which definitely improves the computational results. We have observed that much of the CPU time was consumed by upper bounding procedure that solves CVRP's. In our knowledge the things that can be done to improve CVRP's are limited; therefore, some heuristics like cheapest insertion or using genetic algorithms to calculate upper bounds, could be helpful.

We presented a Lagrangian relaxation method without valid inequalities and compared our results with the results obtained by this method. We observed that insertion of valid inequalities significantly improves the solutions; however, the lower bounds would further be improved since the average Lagrangian gap between upper and lower bounds is 38.12% and the gap is mostly due to the lower bounds. A hybrid approach that incorporates Lagrangian relaxation and Bender's decomposition would be an alternative to find better cuts and thus better lower bounds.

During the steps of Lagrangian relaxation based algorithm we have updated the Lagrangian multipliers by general subgradient optimization technique. However, we did not apply different updating procedures which may be a way to improve the results.

The endogenous inventory policy is one part that gives room for extension. Different inventory policies may be adapted to the problem and the deterministic structure may be shifted to a stochastic case, which is more realistic.

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APPENDIX A

AN EXAMPLE ILLUSTRATING THE FLOW VARIABLES

In this section we illustrate the specifications of the flow variables ($x^k ijvt$) by the optimal solution of the test problem 551AD1. The data of this problem is given in Figure A1 and Tables A1-A3.

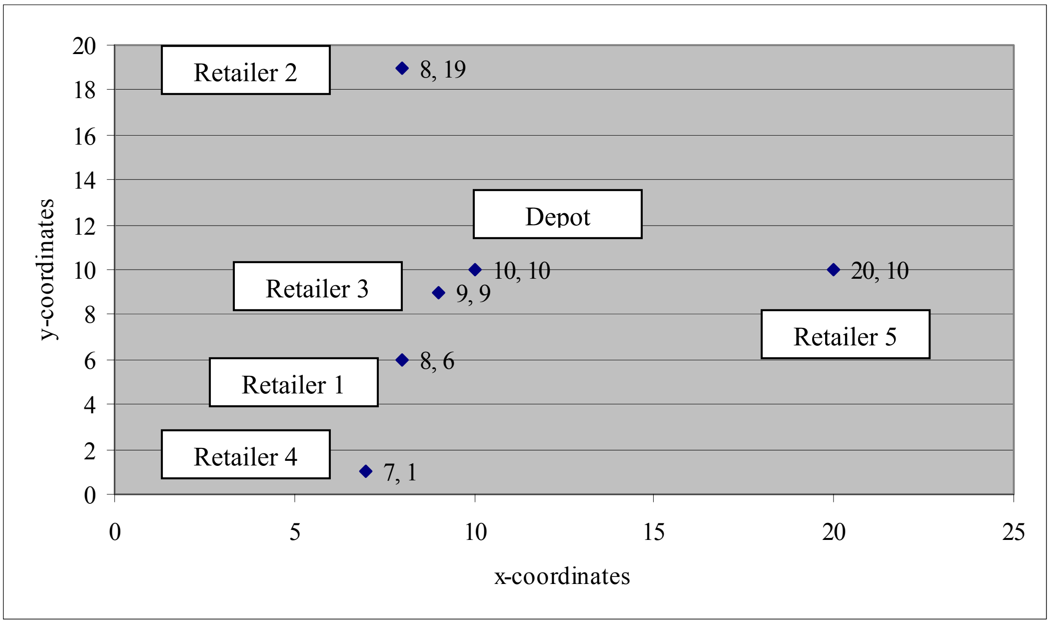

Figure A.1 The coordinates of the retailers and the depot

Table A.1 The distance matrix

	0	1	2	3	4	5
0	-					
1	9	-				
2	19	26	-			
3	3	7	21	-		
4	19	11	37	17	-	
5	20	26	30	23	32	-

Note that fixed arc usage cost is 2*Dist$_{ij}$ and variable transportation cost per unit is 0.1*Dist$_{ij}$ and fixed vehicle dispatching cost is 10 units per vehicle.

Table A.2 Demand figures of end customers observed at retailers

		Demand				
		Period				
		1	2	3	4	5
Retailer	1	20	12	13	27	38
	2	48	39	11	27	35
	3	8	34	24	49	18
	4	38	29	29	49	39
	5	47	19	16	37	40

Table A.3 Cost figures of the retailers

		Holding cost per unit per period	Backordering cost per unit per period
Retailer	1	0.13	3.35
	2	0.09	2.09
	3	0.13	2.19
	4	0.1	3.33
	5	0.12	2.51

Its optimal solution is 1430.64 and the solution values of the variables y_{ijvt}, x^{k}_{ijvt} and S_{kt} are given in Tables A.5-A.7, respectively.

Table A.4 Optimal solution values of binary variables

Variable name	Solution value
y0111	1
y0312	1
y0313	1
y0314	1
y0315	1
y1411	1
y1413	1
y1415	1
y2012	1
y2015	1
y3113	1
y3115	1
y3212	1
y3514	1
y4013	1
y4215	1
y4511	1
y5011	1
y5014	1

Table A.5 Optimal solution values of supply variables

Variable name	Solution value
S11	25
S13	47
S15	38
S22	109
S25	51
S32	41
S33	25
S34	49
S35	18
S1	67
S43	78
S45	39
S51	58
S54	101

Table A.6 Optimal solution values of flow variables

Variable name	Solution value
x^1_{0111}	25
x^4_{0111}	67
x^5_{0111}	58
x^2_{0312}	109
x^3_{0312}	41
x^1_{0313}	47
x^3_{0313}	25
x^4_{0313}	78
x^3_{0314}	49
x^5_{0314}	101
x^1_{0315}	38
x^2_{0315}	51
x^3_{0315}	18
x^4_{0315}	39
x^4_{1411}	67
x^5_{1411}	58
x^4_{1413}	78
x^2_{1415}	51
x^4_{1415}	39
x^1_{3113}	47
x^4_{3113}	78
x^1_{3115}	38
x^2_{3115}	51
x^4_{3115}	39
x^2_{3212}	109
x^5_{3514}	101
x^2_{4215}	51
x^5_{4511}	58

As defined in the M(INVROP) y_{ijvt} variables show whether an arc (i,j) is used by vehicle v, in period t. x^{k}_{ijvt} variables denote the amount of product carried on arc (i,j) for designated retailer k, by vehicle v in period t. S_{kt} corresponds to the total amount of product distributed to retailer k in period t. From the y variables given in Table A.5 we know the retailers that are visited and the order on the tour, which is given in Table A.8. Depot is indexed with 0 and is included in each tour, but is not shown in Table A.8.

Table A.7 Lists of retailers visited in each time period

Time period	The retailers that are visited
1	1, 4, 5
2	3, 2
3	3, 1, 4
4	3, 5
5	3, 1, 4, 2

The optimal tours of each of the five periods are represented in Figures A.1-A.5, respectively. Note that the arrows in the figures show the directions of the tour starting from the depot and ending at depot.

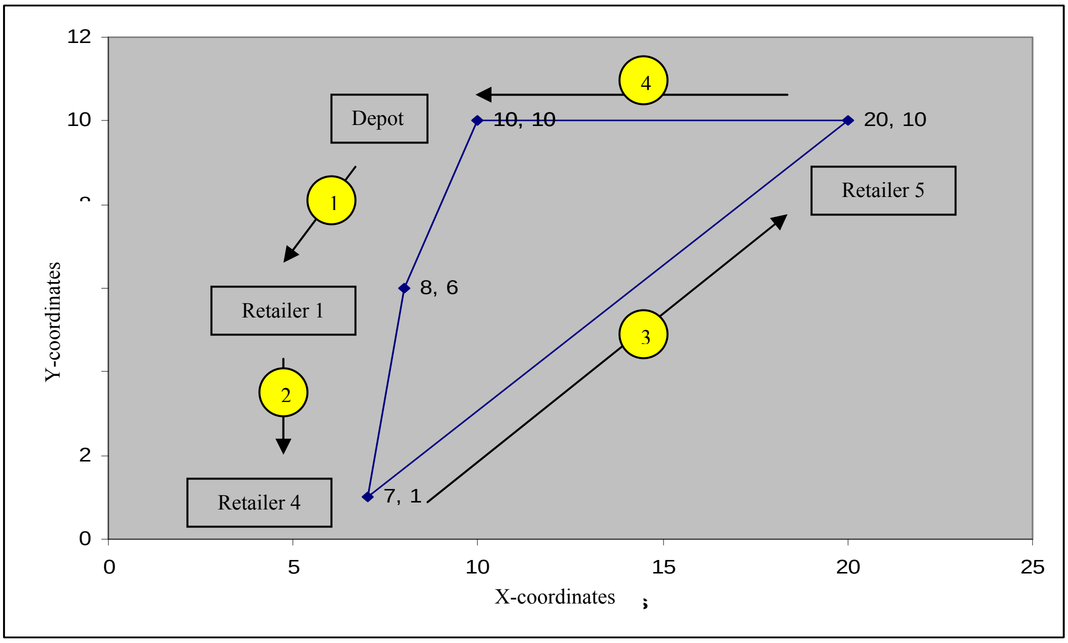

Figure A.2 The optimal tour in period 1

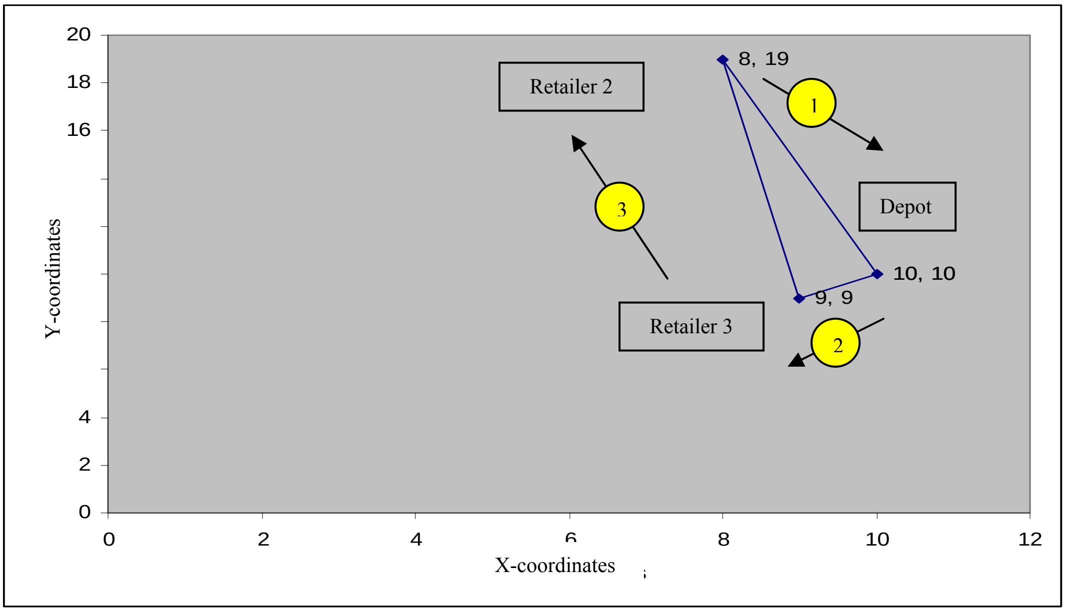

Figure A.3 The optimal tour in period 2

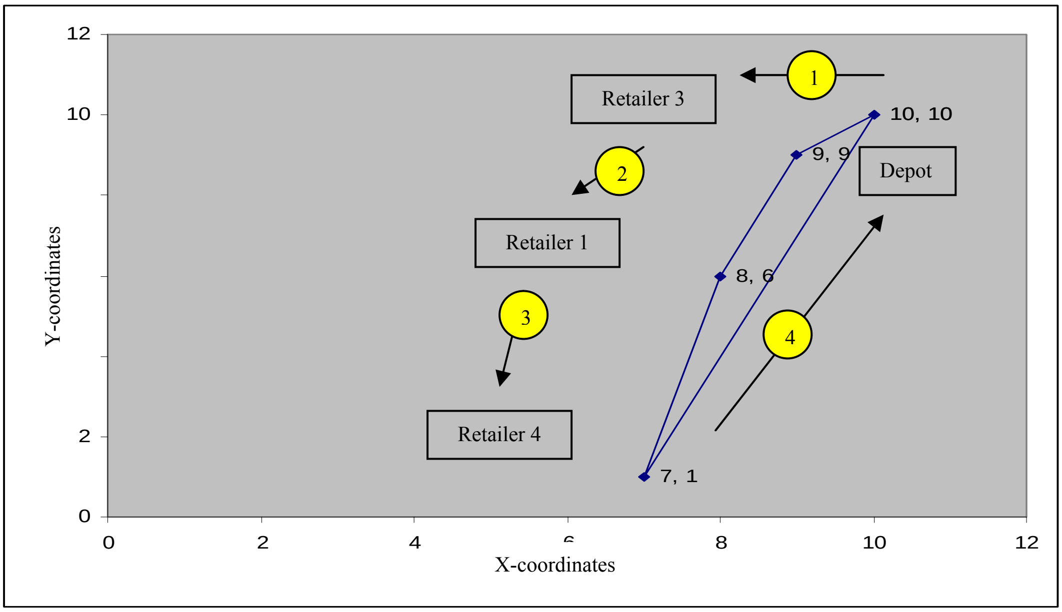

Figure A.4 The optimal tour in period 3

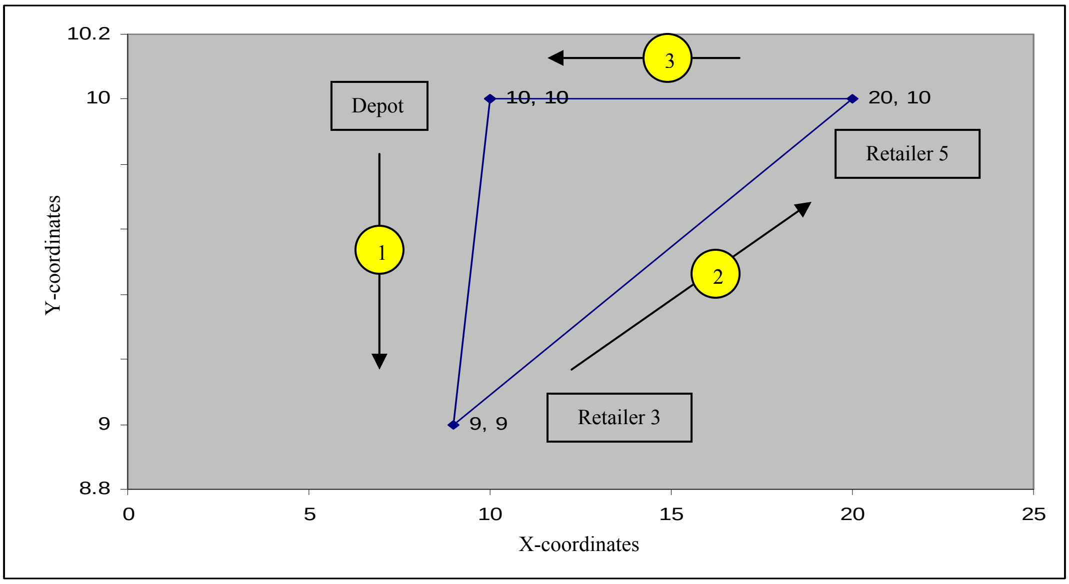

Figure A.5 The optimal tour in period 4

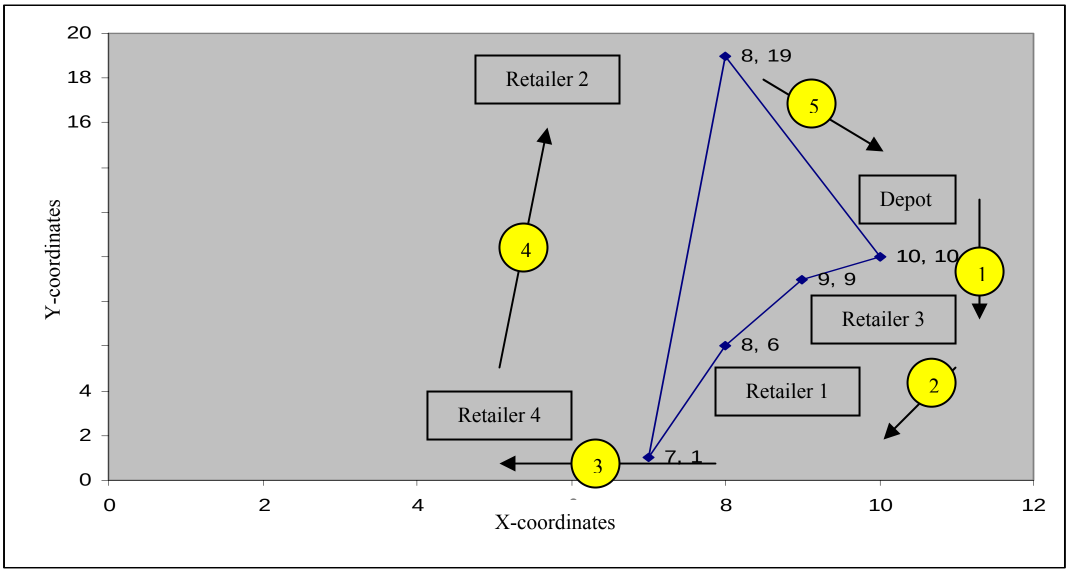

Figure A.6 The optimal tour in period 5

The flows on arcs that are labeled in Figures A.1-A.5, are shown in Tables A.9-A.13, respectively (the flows on the last arcs that are arriving at the depot are not shown since zero units are carried on these arcs). S_{kt} values denote the amounts supplied to retailer *k* in period *t*.

Table A.8 Flows on arcs in Figure A.1

Arc number					
1		2		3	
S11	25	S41	67	S51	58
x01111	25	x14114	67	x45115	58
x01114	67	x14115	58	Total	58
x01115	58	Total	125		
Total	150				

Table A.9 Flows on arcs in Figure A.2

Arc number			
1		2	
S41	67	S51	58
x14114	67	x45115	58
x14115	58	Total	58
Total	125		

Table A.10 Flows on arcs in Figure A.3

Arc number					
1		2		3	
S33	25	S13	47	S43	78
x03131	47	x31131	47	x14134	78
x03133	25	x31134	78	Total	78
x03134	78	Total	125		
Total	150				

Table A.11 Flows on arcs in Figure A.4

Arc number			
1		2	
S45	39	S25	51
x14152	51	x42152	51
x14154	39	Total	51
Total	90		

Table A.12 Flows on arcs in Figure A.5

Arc number							
1		2		3		4	
S35	18	S15	38	S45	39	S25	51
x03151	38	x31151	38	x14152	51	x42152	51
x03152	51	x31152	51	x14154	39	Total	51
x03153	18	x31154	39	Total	90		
x03154	39	Total	128				
Total	146						

It can be observed that the sum of the flow variables arriving a retailer is equal to the supply of that retailer and the supply amount of the succeeding retailers. The supply amount is left at that retailer and the vehicle arrives the retailer with the total supply of succeeding retailers.

APPENDIX B

LAGRANGIAN RELAXATION WITHOUT VALID INEQUALITIES

B.1 Lower bound computation method

In this section we provide an easy method that can be applied to M(INVROP) for calculation of lower and upper bounds on the optimal solution. Without the constraint set (3.14) presented in Chapter 3, M(INVROP) can be decomposed into two subproblems that are retailer subproblem and distribution subproblem. These subproblems are defined in the next section.

B.1.1 Retailer subproblem (RESP)

This subproblem consists of inventory balance equations and total vehicle capacity restriction.

$$\text{Minimize} \quad \sum_{k=1}^{N}\sum_{t=0}^{T}\left(h_{kt} I_{kt} + b_{kt} B_{kt}\right) + \sum_{k=1}^{N}\sum_{t=1}^{T} p_{kt} S_{kt} \tag{B.1}$$

Subject To

$$I_{kt-1} - B_{kt-1} - I_{kt} + B_{kt} + S_{kt} = d_{kt} \qquad \forall\, t \in T, k \in N \tag{B.2}$$

$$I_{kt} \le I_{\max}^{k} \qquad \forall\, t \in \overline{T}, k \in N \tag{B.3}$$

$$B_{k0} = 0 \qquad \forall\; k \in N \tag{B.4}$$

$$I_{k0} = 0 \qquad \forall\; k \in N \tag{B.5}$$

$$B_{kT} = 0 \qquad \forall\ k \in N \tag{B.6}$$

$$\sum_{k=1}^{N} S_{kt} \leq \sum_{v=1}^{V} K_v \qquad \forall\, t \in T \tag{B.7}$$

$$S_{kt}, I_{kt}, B_{kt} \geq 0 \qquad \forall\ t \in T, k \in N \tag{B.8}$$

It is a linear programming problem and it is solved in polynomial time. Several versions of this problem are studied in the literature. In McClain, Thomas and Weiss (1989), the objective of the model consists of holding, production and overtime costs. McClain et al. (1989) show that the model can be solved in polynomial time with the assumptions of no initial inventory and zero setup times and costs. In Erenguc and Tufekci (1988), the objective of the model consists of production, holding and backordering costs. Moreover, Erenguc and Tufekci (1988) have bounds on inventory as our model RESP. They show that the model has a network flow structure and can be solved in polynomial time. In Hax (1978), a multi-item linear programming formulation for aggregate production planning is given. In addition to the cost components of holding, backordering and production; overtime, hiring and firing costs are presented. This model can also be solved in polynomial time. In the M(INVROP) constraint set (B.7) is redundant. However, it is useful for RESP. Because, the solutions obtained without the total vehicle capacity limitation will possibly be far away from giving useful information. Besides, if demands and inventory limits in the RESP are integer, the model will always yields integer solutions for shipment, inventory and backorder variables.

B.1.2 Distribution subproblem (DISP)

$$\text{Minimize} \quad \sum_{i=0}^{N} \sum_{\substack{j=0 \\ j \neq i}}^{N} \sum_{v=1}^{V} \sum_{t=1}^{T} \hat{f}_{ijvt} y_{ijvt} + \sum_{i=0}^{N} \sum_{\substack{j=0 \\ j \neq i}}^{N} \sum_{v=1}^{V} \sum_{t=1}^{T} \sum_{k=1}^{N} \hat{c}_{ijvt}^{k} x_{ijvt}^{k} \tag{B.9}$$

Subject To

$$\sum_{k=1}^{N} x_{ijvt}^{k} \le K_v y_{ijvt} \qquad \forall\, i, j \in \overline{N},\ i \ne j, v \in V, t \in T \tag{B.10}$$

$$\sum_{\substack{j=0 \\ j \ne i}}^{N} y_{ijvt} \le 1 \qquad \forall\ i \in \overline{N}, v \in V, t \in T \tag{B.11}$$

$$x_{ijvt}^{k} \le \min\left\{\sum_{r=1}^{t} d_{kr} + I_{\max}^{k},\ K_v\right\} y_{ijvt} \qquad \begin{array}{l}\forall\, i, j \in \overline{N},\ i \ne j, v \in V, t \in T, \\ k \in N\end{array} \tag{B.12}$$

$$x_{ijvt}^{k} \ge 0 \qquad \forall\, i, j \in \overline{N},\ i \ne j, v \in V, t \in T, k \in N \tag{B.13}$$

$$y_{ijvt} \in \{0,1\} \quad \forall\, i, j \in \overline{N},\ i \ne j, v \in V, t \in T \tag{B.14}$$

The objective function in this subproblem consists of modified fixed costs and modified variable costs of arc usages. Note that the fixed cost term includes original fixed cost of arc usage and attached Lagrange multiplier values whereas modified variable cost is paid upon the amount carried on that arc.

Although the subproblem DISP is a mixed integer problem, it can be decomposed into nodes (i). The decomposition is performed as follows. For a given node ($\underline{i}$), vehicle ($\underline{v}$), and time period ($\underline{t}$) the model reduces to DISPDEC$_{\underline{i}j*\underline{v}\underline{t}}$ where only a single $y_{\underline{i}j^{*}\underline{v}\underline{t}}$ can take the value of 1 because of constraint set (B.11). This suggests that we can fix $y_{\underline{i}j\underline{v}\underline{t}}$ to 1 for particular j, and then we easily solve a bounded continuous knapsack problem by using a greedy procedure. In this procedure, the variable costs of customers k ($\hat{c}_{\underline{i}j\underline{v}\underline{t}}^{k}$) are listed in a nondecreasing order. If the related cost is negative, the flow variable $x_{\underline{i}j\underline{v}\underline{t}}^{k}$ can be set equal to the minimum value specified in constraint set (B.12). Otherwise, it is set to zero. This is repeated for all the variables on the list until the capacity is exhausted. By repeating the entire procedure for all *j*'s for a given $\underline{i}$, $\underline{v}$, $\underline{t}$ triple, we determine the best $y_{\underline{i}j^*\underline{v}\underline{t}}$, as illustrated below.

$$Z_{\underline{i}j^*\underline{v}\underline{t}} = \min_{j} \{0, Z_{\underline{i}j\underline{v}\underline{t}}\} \tag{B.15}$$

Chien et al. (1989) applied a similar algorithm but for a single period problem.

The bounded continuous knapsack problem for each $j \in \overline{N}$, DISPDEC$_{\underline{i}j*\underline{v}\underline{t}}$, is as follows.

$$\text{Minimize} \quad Z_{\underline{i}j\underline{v}\underline{t}}\left(y_{\underline{i}j\underline{v}\underline{t}} = 1\right) = \hat{f}_{\underline{i}j\underline{v}\underline{t}} + \sum_{k=1}^{N} \hat{c}^{k}_{\underline{i}j\underline{v}\underline{t}} x^{k}_{\underline{i}j\underline{v}\underline{t}} \tag{B.16}$$

Subject To

$$\sum_{k=1}^{N} x^{k}_{\underline{i}j\underline{v}\underline{t}} \le K_{\underline{v}} \tag{B.17}$$

$$0 \le x^{k}_{\underline{i}j\underline{v}\underline{t}} \le \min\left\{\sum_{r=1}^{t} d_{kr} + I^{k}_{\max}, K_{\underline{v}}\right\} \qquad \forall\ k \in N \tag{B.18}$$

For each set of node *i*, vehicle *v*, and time period *t*, DISPEC$_{\underline{i}j*\underline{v}\underline{t}}$ must be solved -meaning that (N+1)NVT many problems would be solved- and the best solution value to DISP can be obtained from,

$$\mathbf{Z}\ (\text{DISP}_{(\text{LowerBound})}) = \sum_{\underline{i}=0}^{N} \sum_{\underline{v}=1}^{V} \sum_{\underline{t}=1}^{T} Z_{\underline{i}j^*\underline{v}\underline{t}} \tag{B.19}$$

B.1.3 Algorithmic representation (pseudo code) of lower bound computation

Begin:
Solve RESP with CPLEX;
Get optimal objective function value and ${}^{*}S_{kt}$ values from RESP;
Distribution_cost = 0;
for $i = 0$ **to** N **do**
  for $v = 1$ **to** V **do**
    for $t = 1$ **to** T **do**
      for $j = 0$ **to** N **do**
      Sort customers according to variable costs in nondecreasing order l=1,…,N;
        for $l = 1$ **to** N **do**
        if (variable cost of customer l is less than zero;**)**

          {Assign the maximum possible amount to that customer according to the constraint set (B.18);
          Update vehicle capacity;
          Calculate cost due to delivery of assigned amount to that customer;}
        else
          {Assign zero to that customer;}
        endfor
      minimum = 0;
      if (Z_{ijvt} < minimum**)**
        {minimum = Z_{ijvt};
        $j^{*} = j$;

visit location *j** after location *i* in time period *t* by vehicle *v*; }

else

{Do not visit location *j* (customer or depot) after location *i* by vehicle *v* in period *t*;}

endfor

Distribution_cost = Distribution_cost + Z_{ij^*vt};

endfor

endfor

endfor

Lower Bound = Distribution_cost + Z(RESP);

End.

B.2 Upper bound computation method (Knapsack based heuristic)

In order to calculate upper bounds for the Lagrangian relaxation without valid inequalities we differentiated problems according to the number of vehicles available in the system. For multiple vehicles, upper bounds are calculated with the same method provided in Chapter 3. However, for the single vehicle case, we solve a Traveling Salesman Problem (TSP) in each period due to the time considerations. Each TSP is solved with CONCORDE which is an efficient program that is commercially available.

In each period we determine the customers that are included in the list of customers to be served by using ${}^*S_{kt}$ variables calculated in the lower bound section. Then we solve a TSP for each set of customers. However, M(INVROP) considers not only the fixed arc usage costs but also the costs paid upon the amount carried on each arc. Since the amounts carried on arcs are not considered in TPS formulation, we inserted carriage costs after obtaining the feasible tours by TSP. For each tour, other than the fixed arc usage costs, the greatest cost component is occurred at the arcs leaving the depot, since the full amount to be distributed has to be carried on the first arc leaving the depot. However, for the returning arcs to the depot only fixed arc usage costs are applied, sine the amount to be carried on these arcs should be zero. Therefore, the resulting problem is an Asymmetric Traveling Salesman Problem (ATSP), in which the two arcs connecting two nodes have different cost values. Fortunately, an ATSP could be formulated as a TSP by duplicating the nodes, where the arcs leaving duplicated and the original nodes represent the different cost components, and the arcs that are connecting a duplicated node and its original node having cost of zero. Therefore, the model must use the zero valued arcs. In our model we only duplicated depot, since we were not able to know the amounts carried between nodes, which constitutes a dynamic cost matrix. The duplication of depot is shown in Figure B.1

In the Figure B.1, "Cost Depot-Retailer *k*" represents the cost of carrying the whole amount to be distributed and the fixed arc usage cost; "Cost Depot`-Depot" represents the cost of using the dummy arc and it is zero; "Cost Customer *k*-Depot" represents the fixed cost of using the arc while arriving at depot.

After converting ATSP to TSP we use CONCORDE to solve each TSP, then using the feasible tours obtained, the cost of carriage on arcs are calculated according to the values of S_{kt}; then we add the backordering and inventory holding costs and obtained upper bounds.

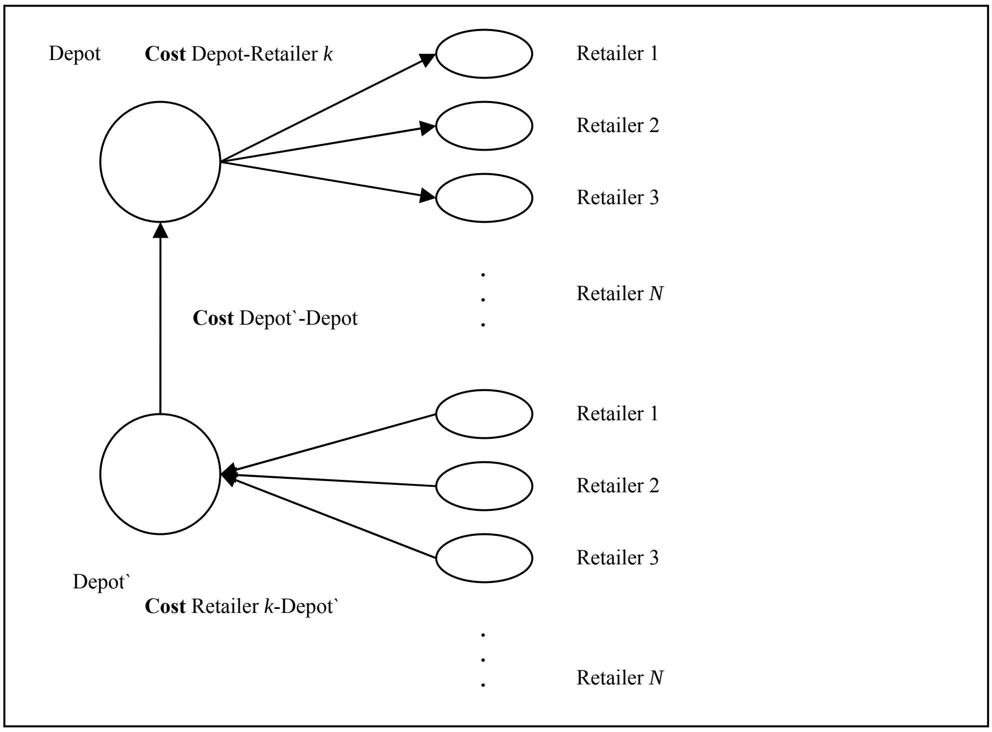

Figure B.1 Conversion of ATSP to TSP

B.2.1 Algorithmic representation of upper bound computation method

Begin:
Get ${}^{*}S_{kt,}$, ${}^{*}I_{kt}$ and ${}^{*}B_{kt}$ values of lower bound section;
for $k = 1$ **to** N **do**
    for $t = 1$ **to** T **do**
        if (${}^{*}S_{kt} > 0$**)**
            {Add customer k to the list of customers to be visited in period t;}
        else
            {Do not visit customer k in period t;}
    endfor
endfor

if (V>1)

for t = 1 **to** T **do**

{Solve CVRP(t) with CPLEXand obtain y^{*}_{ijvt} and x^{*k}_{ijvt} values;}

endfor

else

for t = 1 **to** T **do**

{Convert ATSP(t) to TSP(t);

Solve TSP(t) with CONCORDE and obtain a tour;

Obtain y^{*}_{ijvt} and x^{*k}_{ijvt} values with respect to the tour obtained;}

endfor

Upper_Bound = Z(INVROP($^{*}S_{kt}$, $^{*}I_{kt}$, $^{*}B_{kt}$, y^{*}_{ijvt}, x^{*k}_{ijvt}));

End.

The flowchart of the algorithm applied to M(INVROP) by Lagrangian relaxation without valid inequalities is given in Figure B.2.

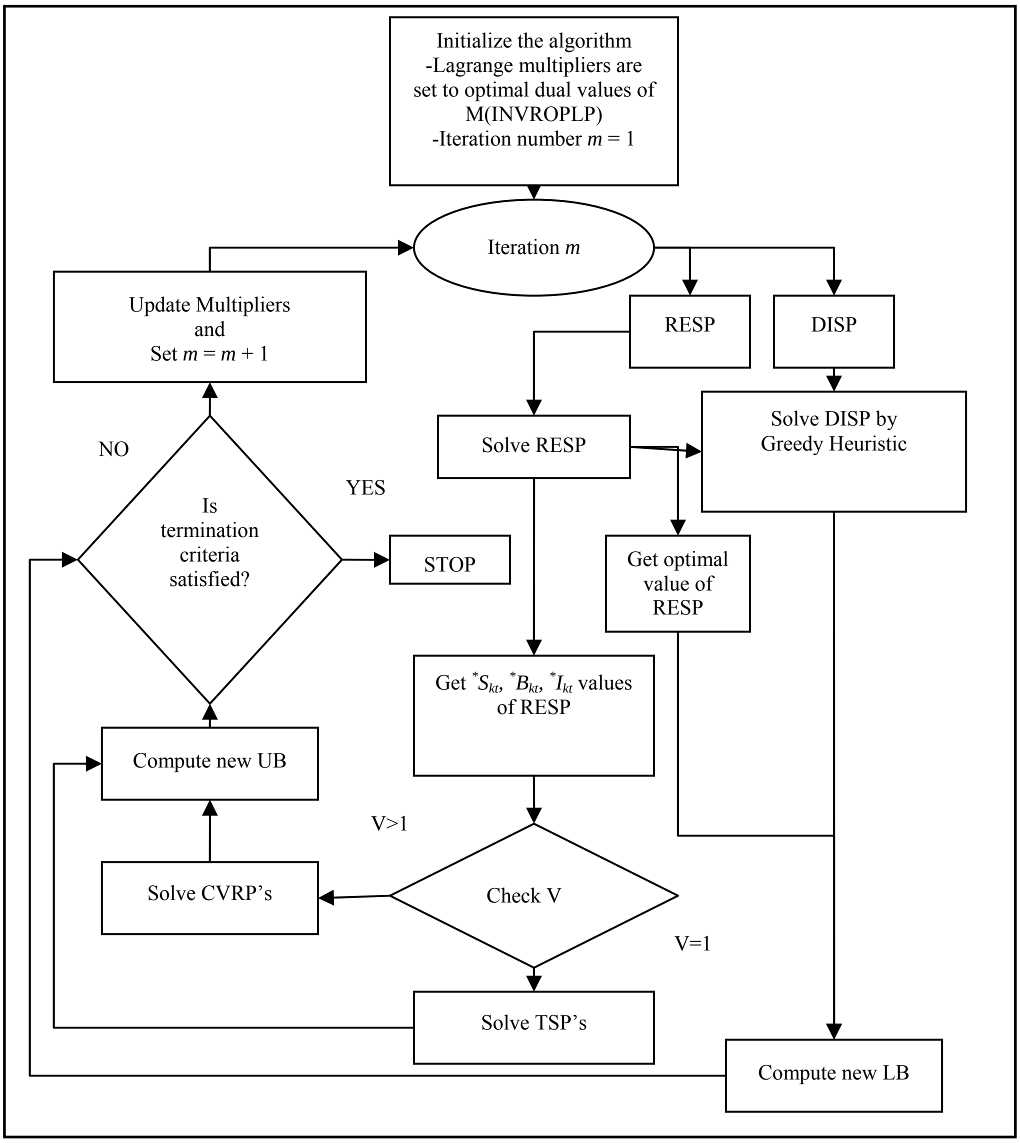

Figure B.2 Flowchart of the Lagrangian Relaxation without valid inequalities

B.3 Experimentation

In this section we present the results obtained with the knapsack problem based relaxation.

In the Tables B.1 – B.8 we used the following notations.

- KN UB denotes the upper bound calculated with knapsack problem based relaxation.
- CPU KN denotes the CPU time used by the knapsack problem based relaxation in minutes.
- %KNGAP denotes the gap between the knapsack problem based heuristic solution and linear programming relaxation (%KNGAP=%(KN UB – LPR)/LPR).
- RE is the ratio of the %KN GAP and %MIP.

where, LPR denotes the optimal solution value of the linear programming relaxation of the problems. Since lower bounds computed with the Lagrangian relaxation without valid inequalities are not better than the linear programming relaxation solutions, we used linear programming relaxation solutions as lower bounds.

Note that in the settings with 5 retailers, we calculated an upper bound in each iteration. In the settings with 10 retailers we separated the problems according to the number of vehicles. In single vehicle settings we calculated an upper bound in each iteration; whereas, in two vehicle settings we calculated an upper bound once in every five iterations.

Table B.1 Results of knapsack problem based relaxation for 551AD*k*

		MIP Model					KNAPSACK Heuristic			
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	KN UB	CPU KN	%KNGAP	RE
Problem	Opt									
551AD1	Y	1430.64	0.52	847.75	0.01	68.76	2474.11	0.75	191.84	2.79
551AD2	Y	1531.68	0.09	908.02	0.01	68.68	2191.55	0.73	141.35	2.06
551AD3	Y	1184.78	0.08	785.64	0.01	50.80	1596.59	0.74	103.22	2.03
551AD4	Y	1460.41	0.11	844.35	0.01	72.96	2111.19	0.75	150.04	2.06
551AD5	Y	1392.00	0.09	940.41	0.01	48.02	2101.05	0.73	123.42	2.57
Average			0.18		0.01	61.85		0.74	141.98	2.30

Table B.2 Results of knapsack problem based relaxation for 552AD*k*

		MIP Model					Knapsack based heuristic			
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	KN UB	CPU KN	%KNGAP	RE
Problem	Opt									
552AD1	Y	1145.32	0.85	868.57	0.01	31.86	1228.13	2.81	41.40	1.30
552AD2	Y	1505.19	18.89	1194.32	0.01	26.03	1632.4	3.93	36.68	1.41
552AD3	Y	1138.87	11.68	918.77	0.01	23.96	1257.82	2.47	36.90	1.54
552AD4	Y	1138.62	3.31	908.59	0.01	25.32	1215	3.63	33.72	1.33
552AD5	Y	1204.92	6.15	959.35	0.01	25.60	1329.34	4.36	38.57	1.51
Average			8.18		0.01	26.55		3.44	37.45	1.42

Table B.3 Results of knapsack problem based relaxation for 571AD*k*

		MIP Model					Knapsack based heuristic			
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	KN UB	CPU KN	%KNGAP	RE
Problem	Opt									
571AD1	Y	1723.29	0.41	1082.24	0.01	59.23	2040.89	1.01	88.58	1.50
571AD2	Y	1431.37	0.10	1030.68	0.01	38.88	2100.41	1.01	103.79	2.67
571AD3	Y	1199.18	0.31	779.07	0.01	53.92	1816.51	1.01	133.16	2.47
571AD4	Y	1661.59	0.37	1043.34	0.01	59.26	2416.98	1.04	131.66	2.22
571AD5	Y	1566.38	1.91	939.07	0.01	66.80	2503.4	1.03	166.58	2.49
Average			0.62		0.01	55.62		1.02	124.75	2.27

Table B.4 Results of knapsack problem based relaxation for 572AD*k*

		MIP Model					Knapsack based heuristic			
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	KN UB	CPU KN	%KNGAP	RE
Problem	Opt									
572AD1		1685.00	60.01	1300.60	0.01	29.56	1834.25	3.39	41.03	1.39
572AD2		1751.34	60.03	1320.22	0.01	32.66	1957.64	4.93	48.28	1.48
572AD3	Y	1580.88	14.76	1223.34	0.01	29.23	1887.38	6.21	54.28	1.86
572AD4	Y	1647.73	34.23	1300.87	0.01	26.66	1885.11	4.54	44.91	1.68
572AD5	Y	1625.45	23.17	1239.01	0.01	31.19	1875.43	3.98	51.37	1.65
Average			38.44		0.01	29.86		4.61	47.97	1.61

Table B.5 Results of knapsack problem based relaxation for 1051AD*k*

		MIP Model					Knapsack based heuristic			
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	KN UB	CPU KN	%KNGAP	RE
Problem	Opt									
1051AD1		2630.36	180.00	1289.4	0.02	104.00	2942.76	0.85	128.23	1.23
1051AD2		2209.24	180.00	1461.27	0.04	51.19	2949.81	0.80	101.87	1.99
1051AD3		3195.71	180.00	1626.9	0.05	96.43	3279.49	0.85	101.58	1.05
1051AD4		2574.72	180.00	1595.81	0.04	61.34	2912.78	0.82	82.53	1.35
1051AD5		2897.20	180.00	1594.76	0.04	81.67	3272.64	0.85	105.21	1.29
Average			180.00		0.04	78.93		0.83	103.88	1.38

Table B.6 Results of knapsack problem based relaxation for 1052AD*k*

		MIP Model					Knapsack based heuristic			
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	KN UB	CPU KN	%KNGAP	RE
Problem	Opt									
1052AD1		2505.07	180.00	1582.04	0.06	58.34	2832.77	227.46	79.06	1.36
1052AD2		2084.02	180.00	1547.19	0.09	34.70	2396.84	112.33	54.92	1.58
1052AD3		2326.01	180.00	1499.80	0.08	55.09	2673.4	191.08	78.25	1.42
1052AD4		1851.85	180.00	1408.77	0.08	31.45	2038.52	72.50	44.70	1.42
1052AD5		2456.36	180.00	1509.87	0.08	62.69	2737.6	399.07	81.31	1.30
Average			180.00		0.08	48.45		200.49	67.65	1.42

Table B.7 Results of knapsack problem based relaxation for 1071AD*k*

		MIP Model					Knapsack based heuristic			
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	KN UB	CPU KN	%KNGAP	RE
Problem	Opt									
1071AD1		4118.08	180.00	2111.37	0.05	95.04	4525.88	1.17	114.36	1.20
1071AD2		4023.24	180.00	2263.62	0.06	77.73	4859.29	1.13	114.67	1.48
1071AD3		3557.83	180.00	1848.37	0.06	92.48	4248.14	1.21	129.83	1.40
1071AD4		4192.71	180.00	2346.31	0.06	78.69	5547.64	1.11	136.44	1.73
1071AD5		3850.9	180.00	1998.42	0.06	92.70	3286.25	1.14	64.44	0.70
Average			180.00		0.06	87.33		1.15	111.95	1.30

Table B.8 Results of knapsack problem based relaxation for 1072AD*k*

		MIP Model					Knapsack based heuristic			
		CPLEX UB	CPUB	LPR	CPULPR	%MIP	KN UB	CPU KN	%KNGAP	RE
Problem	Opt									
1072AD1		2860.50	180.00	2011.50	0.09	42.21	3401.27	167.11	69.09	1.64
1072AD2		3344.50	180.00	2317.77	0.12	44.30	3954.41	307.26	70.61	1.59
1072AD3		3136.73	180.00	2226.76	0.12	40.87	3867.34	299.72	73.68	1.80
1072AD4		3263.66	180.00	2303.17	0.12	41.70	3901.91	260.72	69.41	1.66
1072AD5		2743.09	180.00	1859.54	0.12	47.51	3015.73	245.65	62.18	1.31
Average			180.00		0.11	43.32		256.09	68.99	1.60

Average gap of the Lagrangian relaxation without valid inequalities for the settings with single vehicle is 120.64%, CPU time is 0.94 minutes and RE (relative error) is 1.81; whereas, gap of the settings with two vehicles is 55.52%, CPU time is 116.16 minutes and RE is 1.51. The great difference between the CPU times and gaps is due to the upper bounding method. In the settings with single vehicle we use CONCORDE and it solves the TPSs in less than a second; however, in the two vehicle settings we solved CVRPs with CPLEX with five minutes time limit for each CVRP. For the 10 retailers case CPLEX used the entire time for each problem. The upper bounding procedure that uses CVRPs yields better results than the one uses TPSs with respect to the gap of upper and lower bounds, but significantly takes more computation time.

On overall average, Lagrangian relaxation without valid inequalities yields 88.08% gap and 1.7 RE (relative error). That is to say the gap between CPLEX upper bound and LP relaxation is 70% smaller than the gap between the upper bounds calculated with knapsack problem based heuristic and LP relaxation. Due to the poor results obtained, we tried to solve optimally the relaxed problem given in Chapter 3.

APPENDIX C

ADOPTED MODEL FOR BENCHMARKING

In this appendix we present an adopted version of M(INVROP) namely M(INVROPAB) in order to test our Lagrangian relaxation based algorithm on benchmarked results. The M(INVROPAB) is the same model with Abdelmaguid and Dessouky (2006) by different variable definitions.

All of the assumptions stated in Chapter 3 except the prohibition of backordering in the last period are valid for M(INVROPAB).

Indices of the model are as follows.

t : Time index (discrete time periods): 1, 2, …, T and $\overline{T} = T \cup \{0\}$.

i, j : Node index : 0, 1, …, N (i = 0 denotes depot). N denotes the set of retailers and $\overline{N} = N \cup \{0\}$.

k : Retailer index: 1, 2, …, N.

v : Vehicle index: 1, 2, …, V.

Parameters of the model are as follows.

N : Number of locations (retailers).

V : Number of vehicles.

T : Number of time periods.

K_v : Capacity of vehicle v.

$I_{\max}^{k}$: Storage capacity of retailer *k*.

d_{kt} : Demand of the end customer of retailer *k* in period *t*.

f_{ijvt} : Fixed cost for vehicle *v* in period *t* to use arc *(i,j)* for going from location *i* to location *j*.

c_{ijvt}^{k} : Variable cost of carrying one unit of product by vehicle *v* in period *t* on arc *(i,j)* for going from location *i* to location *j* for the designated customer *k*.

O_t : Fixed vehicle dispatching cost in time period *t*.

h_{kt} : Unit holding cost for retailer *k* in period *t*.

b_{kt} : Unit backordering cost for retailer *k* in period *t*.

Decision variables of the model are as follows:

$$y_{ijvt} \ : \begin{cases} 1 \text{ if vehicle } v \text{ travels from location } i \text{ to location } j \text{ using arc } (i,j) \\ \quad \text{in perod } t \\ 0 \text{ otherwise} \end{cases}$$

x_{ijvt}^{k} : Amount of product destined to retailer *k*, which is transported from location *i* to location *j* by vehicle *v* in period *t*.

I_{kt} : Amount of product held by retailer *k* in period *t*.

B_{kt} : Amount of product backordered by retailer *k* in period *t*.

S_{kt} : Amount of product supplied to retailer *k* in period *t*.

M(INVROPAB):

$$\text{Minimize} \quad \sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T} f_{ijvt}y_{ijvt} \ + \sum_{k=1}^{N}\sum_{t=0}^{T}\left(h_{kt}I_{kt} + b_{kt}B_{kt}\right) + \sum_{j=1}^{N}\sum_{v=1}^{V}\sum_{t=1}^{T} O_t y_{0jvt} \tag{C.1}$$

Subject To

$$\sum_{k=1}^{N} x_{ijvt}^{k} \le K_v y_{ijvt} \qquad \forall\, i, j \in \overline{N},\ i \ne j, v \in V, t \in T \tag{C.2}$$

$$\sum_{\substack{j=0\\ j\ne i}}^{N} \sum_{v=1}^{V} x_{ijvt}^{k} - \sum_{\substack{j=0\\ j\ne i}}^{N} \sum_{v=1}^{V} x_{jivt}^{k} = \begin{cases} +S_{kt} \text{ if } i = 0 \\ -S_{kt} \text{ if } i = k \end{cases} \qquad \forall\, i \in \overline{N}, t \in T, k \in N \tag{C.3}$$

$$\sum_{\substack{j=0\\ j\ne i}}^{N} x_{ijvt}^{k} - \sum_{\substack{j=0\\ j\ne i}}^{N} x_{jivt}^{k} = 0 \qquad \forall\ i \in N \setminus \{k\}, v \in V, t \in T, k \in N \tag{C.4}$$

$$\sum_{\substack{j=0\\ j\ne i}}^{N} y_{ijvt} - \sum_{\substack{j=0\\ j\ne i}}^{N} y_{jivt} = 0 \qquad \forall\, i \in \overline{N}, v \in V, t \in T \tag{C.5}$$

$$\sum_{\substack{j=0\\ j\ne i}}^{N} y_{ijvt} \le 1 \qquad \forall\ i \in \overline{N}, v \in V, t \in T \tag{C.6}$$

$$I_{kt-1} - B_{kt-1} - I_{kt} + B_{kt} + S_{kt} = d_{kt} \qquad \forall\, t \in T, k \in N \tag{C.7}$$

$$I_{kt} \le I_{\max}^{k} \qquad \forall\, t \in \overline{T}, k \in N \tag{C.8}$$

$$x_{ijvt}^{k} \le \min\left\{ \sum_{r=1}^{t} d_{kr} + I_{\max}^{k}, K_v \right\} y_{ijvt} \qquad \forall\, i, j \in \overline{N},\ i \ne j, v \in V, t \in T, k \in N \tag{C.9}$$

$$B_{k0} = 0 \qquad \forall\ k \in N \tag{C.10}$$

$$I_{k0} = 0 \qquad \forall\ k \in N \tag{C.11}$$

$$\sum_{k=1}^{N} S_{kt} \le \sum_{v=1}^{V} K_v \qquad \forall\, t \in T \tag{C.12}$$

$$x_{ijvt}^{k} \le S_{kt} \qquad \forall\, i, j \in \overline{N},\ i \ne j, v \in V, t \in T,\ k \in N \tag{C.13}$$

$$S_{kt}, I_{kt}, B_{kt},\ x_{ijvt}^{k} \ge 0 \qquad \forall\, i, j \in \overline{N},\ i \ne j, v \in V, t \in T, k \in N \tag{C.14}$$

$$y_{ijvt} \in \{0,1\} \qquad \forall\, i, j \in \overline{N},\ i \ne j, v \in V, t \in T \tag{C.15}$$

Note that all the constraint definitions given in Chapter 3 are valid for M(INVROPAB). The differences between M(INVROP) and M(INVROPAB) can be stated as follows.

- In M(INVROP) backordering in the last period is not allowed; however, in M(INVROPAB) backordering in the last period is allowed.

- In M(INVROP) variable transportation cost upon amount of products carried on each arc is due; however, in M(INVROPAB) variable transportation cost is not considered.

In order to apply Lagrangian relaxation based solution algorithm we relaxed constraint sets (B.3), (B.4) and (B.5) and added to the objective function. Then the same solution procedure with M(INVROP) presented in Chapter 3 is applied.

APPENDIX D

CONVERGENGENCE GRAPHS OF PRELIMINARY EXPERIMENTS

In this appendix we present the convergence graphs of preliminary experiments on the test settings 551AD*k* and 552AD*k*.

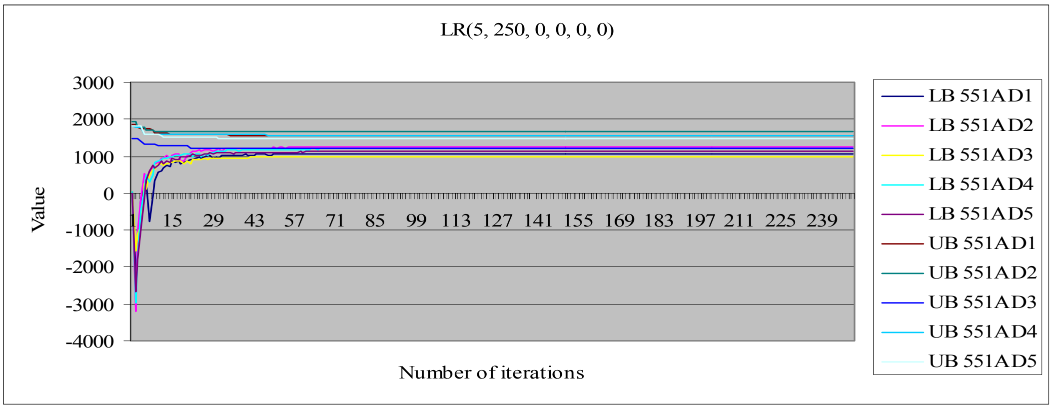

Figure D.1 Convergence graph of LR(5, 250, 0, 0, 0,0) for 551AD*k*

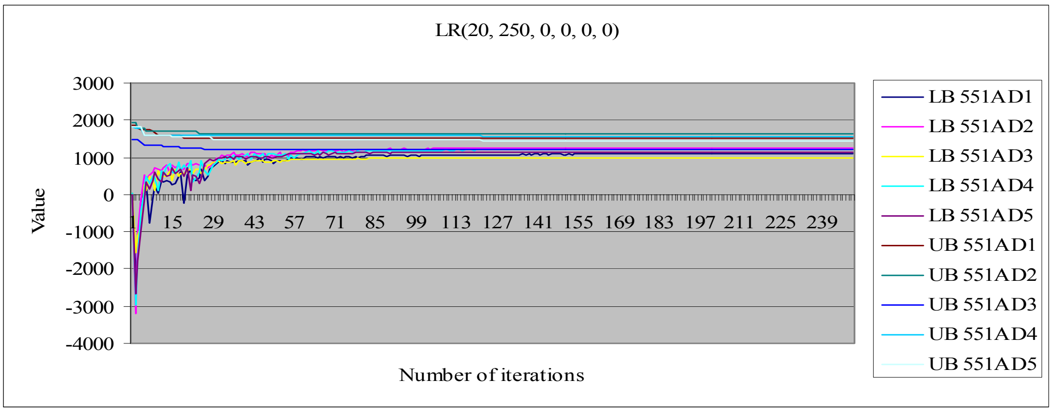

Figure D.2 Convergence graph of LR(20, 250, 0, 0, 0,0) for 551AD*k*

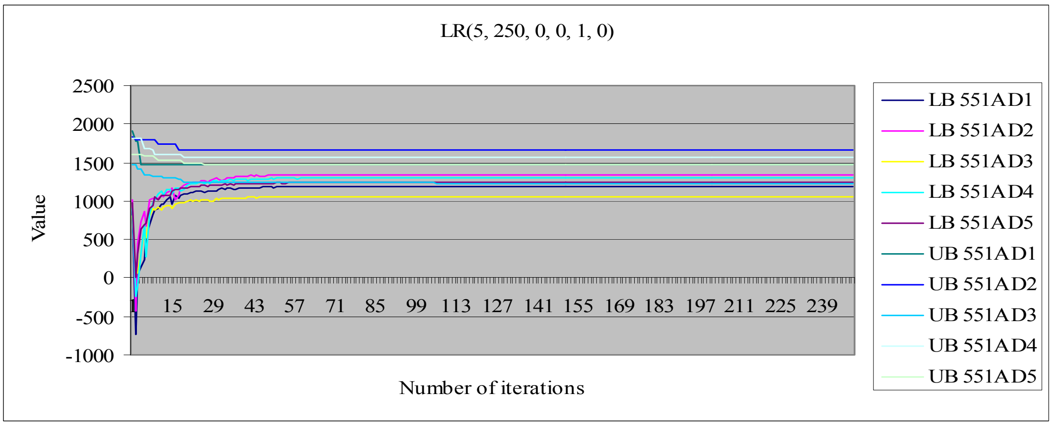

Figure D.3 Convergence graph of LR(5, 250, 0, 0, 1,0) for 551AD*k*

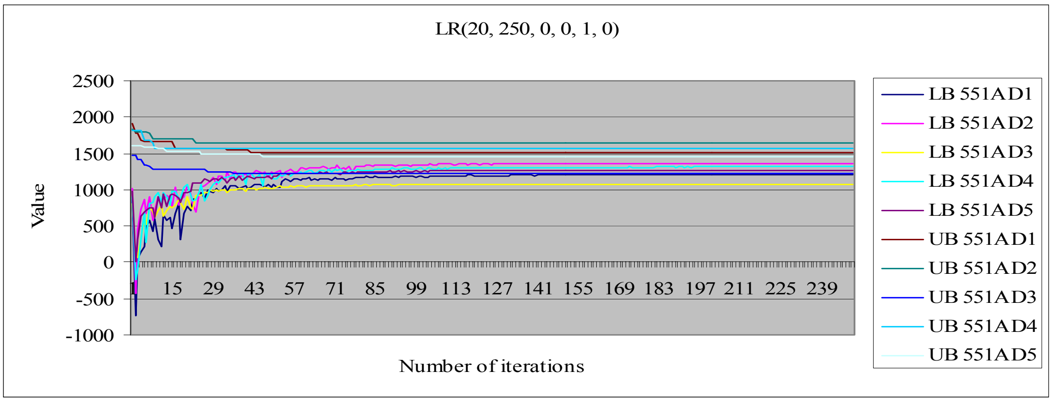

Figure D.4 Convergence graph of LR(20, 250, 0, 0, 1,0) for 551AD*k*

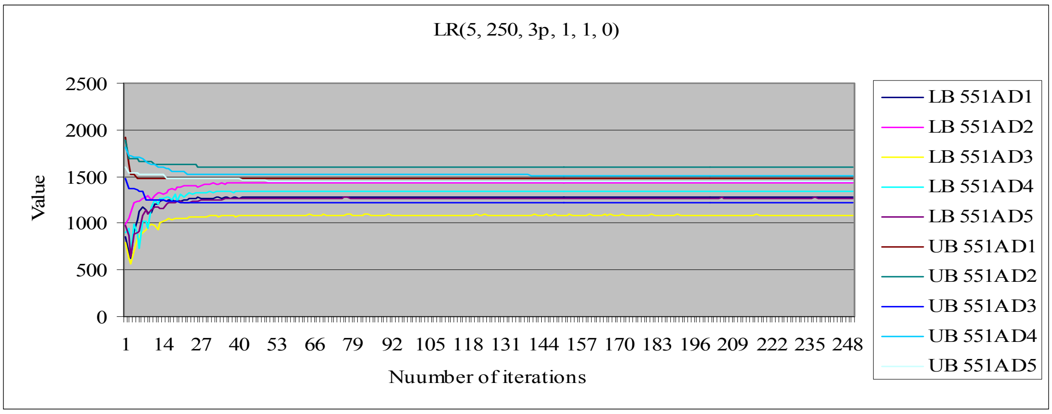

Figure D.5 Convergence graph of LR(5, 250, 3p, 1, 1,0) for 551AD*k*

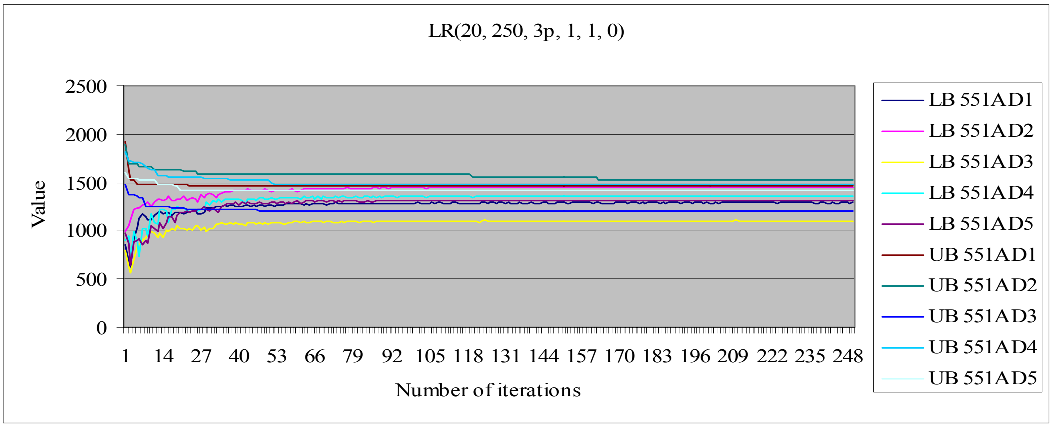

Figure D.6 Convergence graph of LR(20, 250, 3p, 1, 1,0) for 551AD*k*

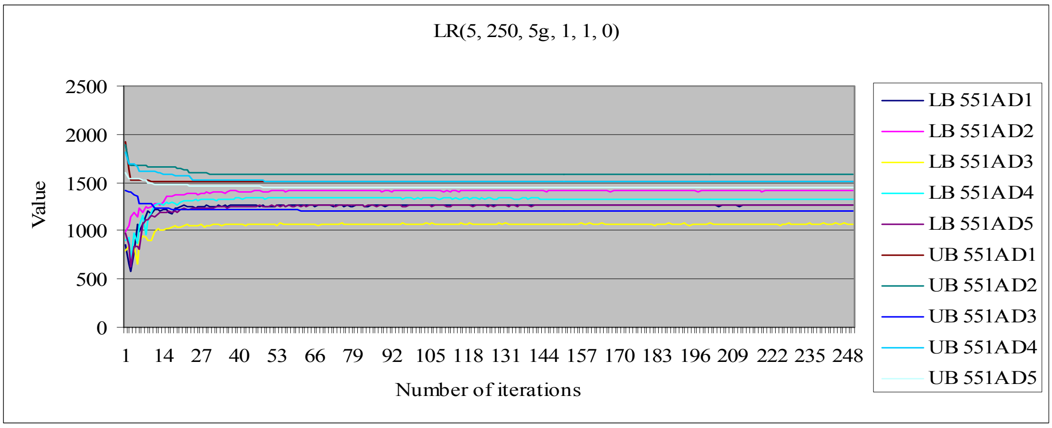

Figure D.7 Convergence graph of LR(5, 250, 5p, 1, 1,0) for 551AD*k*

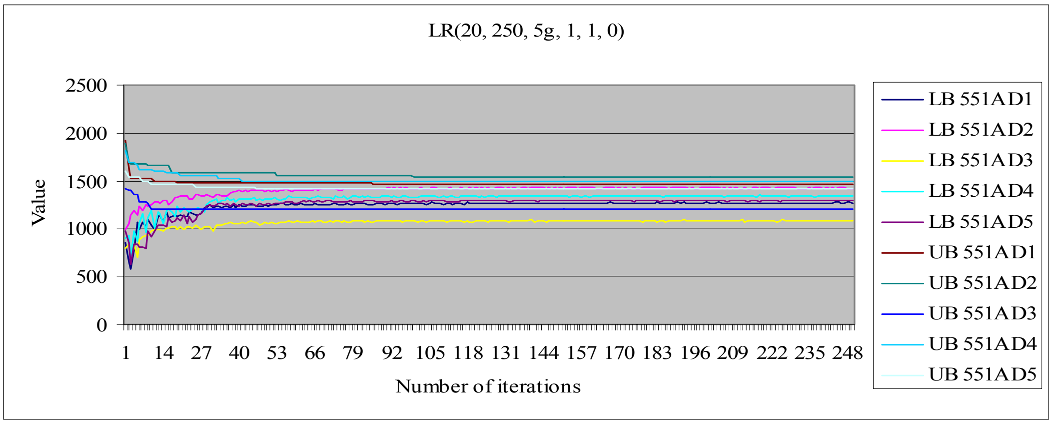

Figure D.8 Convergence graph of LR(20, 250, 5p, 1, 1,0) for 551AD*k*

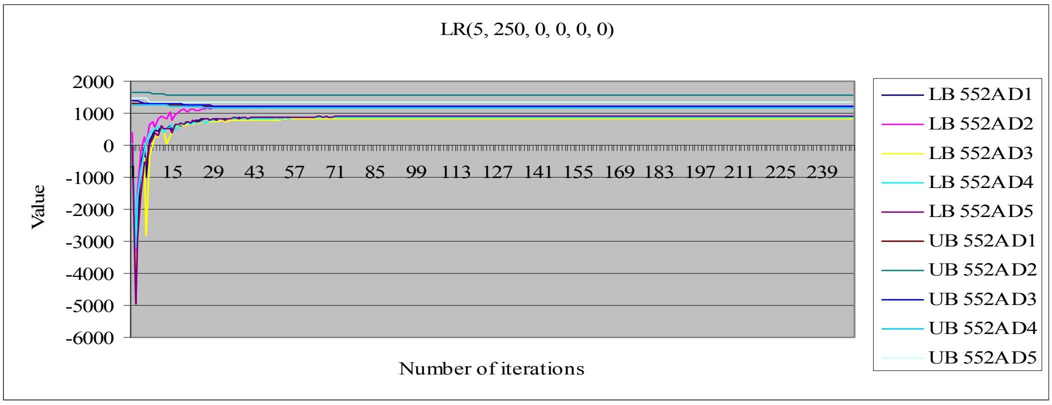

Figure D.9 Convergence graph of LR(5, 250, 0, 0, 0,0) for 552AD*k*

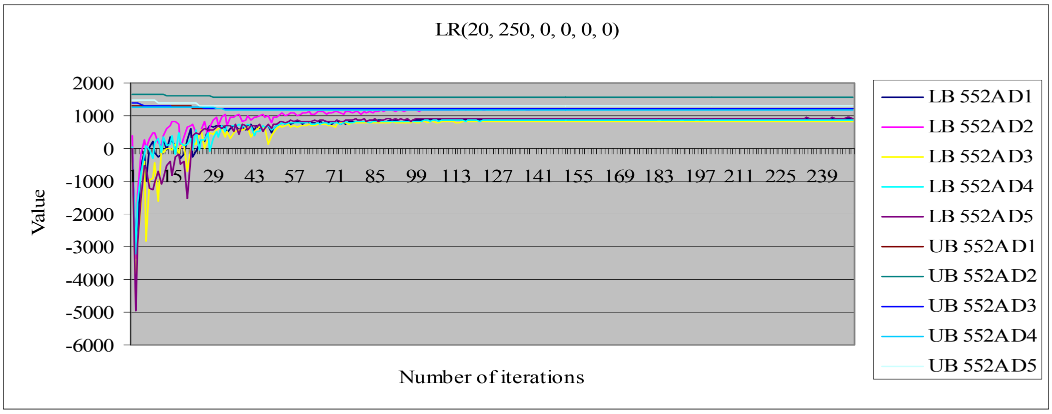

Figure D.10 Convergence graph of LR(20, 250, 0, 0, 0,0) for 552AD*k*

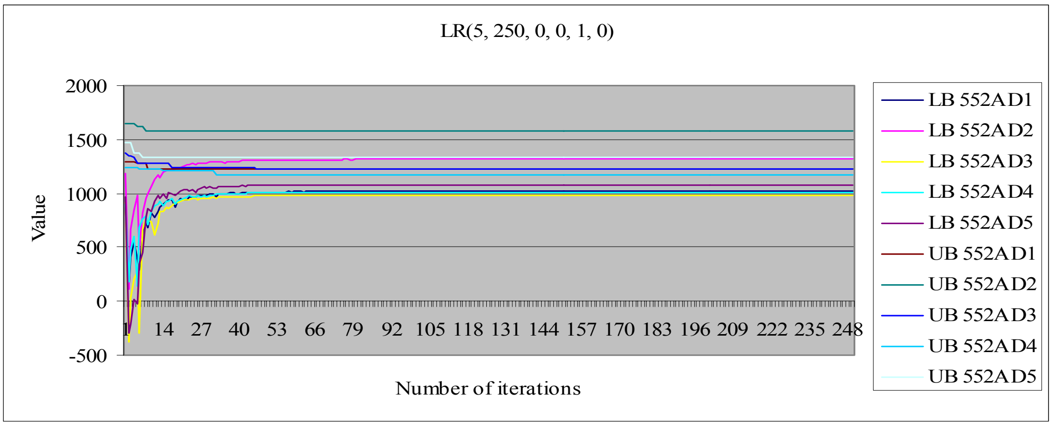

Figure D.11 Convergence graph of LR(5, 250, 0, 0, 1,0) for 552AD*k*

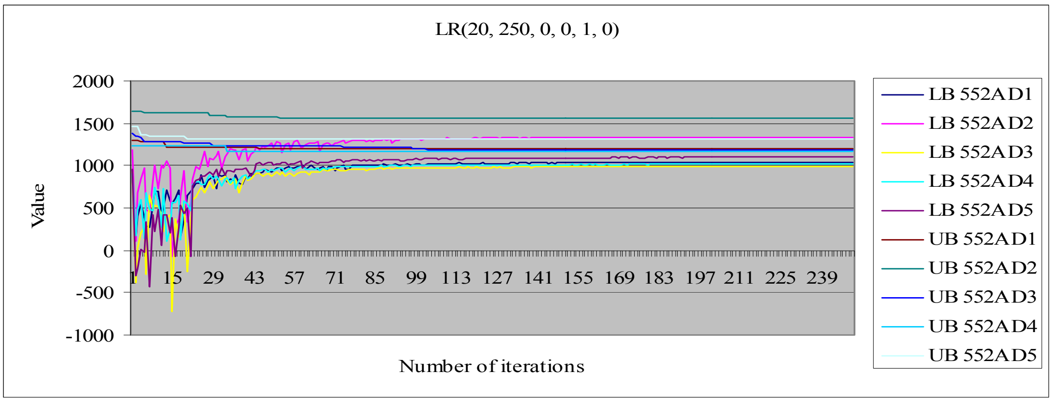

Figure D.12 Convergence graph of LR(20, 250, 0, 0, 1,0) for 552AD*k*

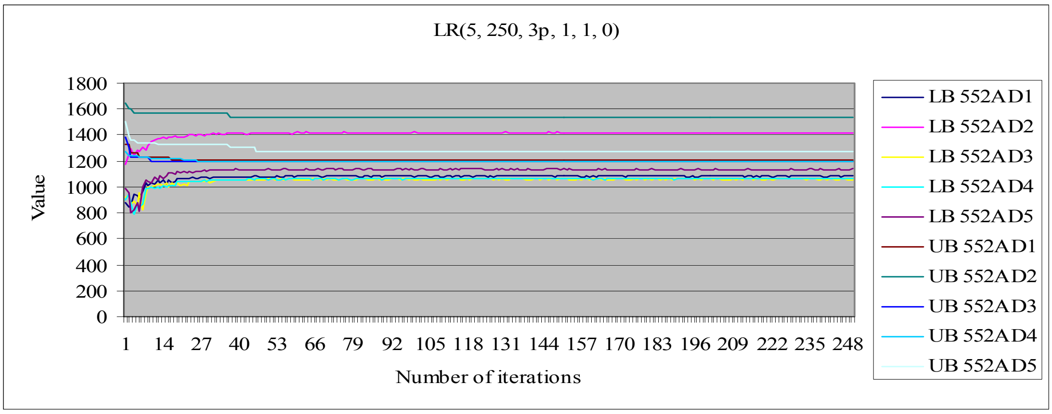

Figure D.13 Convergence graph of LR(5, 250, 3p, 1, 1,0) for 552AD*k*

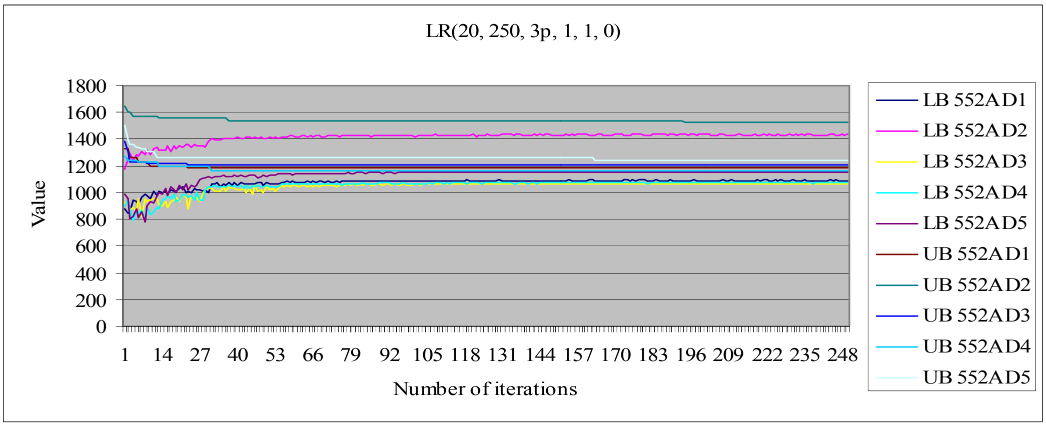

Figure D.14 Convergence graph of LR(20, 250, 3p, 1, 1,0) for 552AD*k*

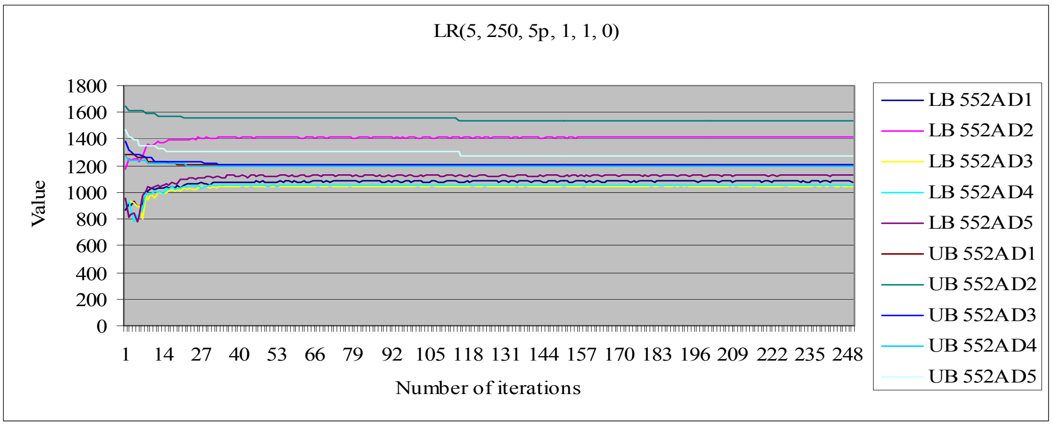

Figure D.15 Convergence graph of LR(5, 250, 5p, 1, 1,0) for 552AD*k*

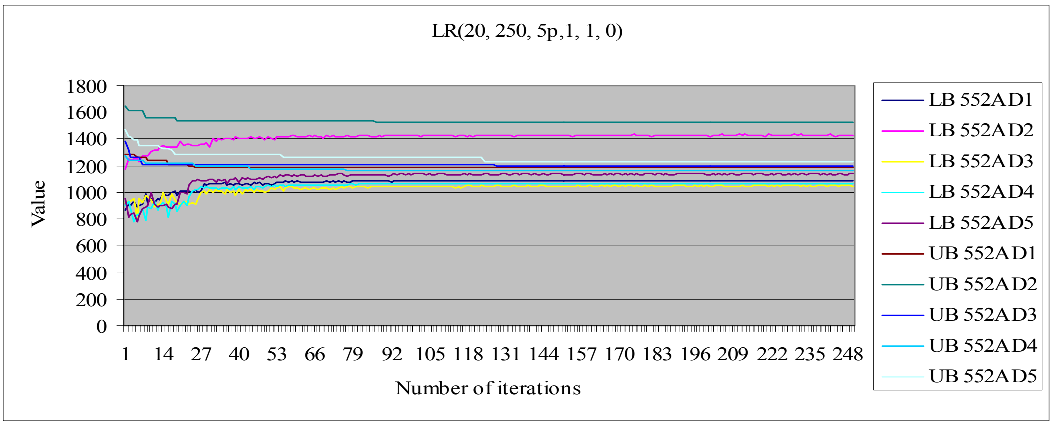

Figure D.16 Convergence graph of LR(20, 250, 5p, 1, 1,0) for 552AD*k*

APPENDIX E

DETAILED RESULTS OF PRELIMINARY EXPERIMENTS

In Section 4.4, we presented the results obtained with the parameter of halving π after 20 consecutive non-improving iterations and in this appendix we present the results obtained when π is halved after 5 consecutive non-improving iterations on the test settings 551AD*k* and 552AD*k*.

Table E.1 Results of LR(5, 25, 0, 0, 1, 0) and LR(5, 50, 0, 0, 1, 0)

	LR(5, 25, 0, 0, 1, 0)						LR(5, 50, 0, 0, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1466.75	1133.95	1.57	20.74	2.52	29.35	1466.75	1181.48	3.94	17.42	2.52	24.15
551AD2	1661.40	1256.48	1.51	17.97	8.47	32.23	1661.40	1337.66	3.86	12.67	8.47	24.20
551AD3	1236.41	1021.54	1.57	13.78	4.36	21.03	1236.41	1050.60	3.57	11.33	4.36	17.69
551AD4	1575.86	1246.53	1.65	14.65	7.91	26.42	1575.86	1293.54	3.40	11.43	7.91	21.83
551AD5	1493.62	1200.04	1.57	13.79	7.30	24.46	1474.75	1235.05	3.46	11.28	5.94	19.41
Average			1.58	16.18	6.11	26.70			3.65	12.82	5.84	21.45
552AD1	1229.34	973.53	2.25	15.00	7.34	26.28	1220.21	1012.15	4.86	11.63	6.54	20.56
552AD2	1574.00	1282.80	2.17	14.77	4.57	22.70	1574.00	1310.51	4.43	12.93	4.57	20.11
552AD3	1243.22	952.70	2.23	16.35	9.16	30.49	1220.13	977.35	4.71	14.18	7.14	24.84
552AD4	1212.83	976.80	2.22	14.21	6.52	24.16	1173.56	1005.35	4.49	11.70	3.07	16.73
552AD5	1333.06	1039.34	2.31	13.74	10.63	28.26	1333.06	1075.44	4.90	10.75	10.63	23.95
Average			2.24	14.81	7.64	26.38			4.68	12.24	6.39	21.24
Overall Average			1.91	15.50	6.88	26.54			4.16	12.53	6.11	21.35

Table E.2 Results of LR(5, 75, 0, 0, 1, 0) and LR(5, 100, 0, 0, 1, 0)

	LR(5, 75, 0, 0, 1, 0)						LR(5, 100, 0, 0, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1466.75	1186.69	7.50	17.05	2.52	23.60	1466.75	1187.28	11.21	17.01	2.52	23.54
551AD2	1661.40	1346.89	7.19	12.06	8.47	23.35	1661.40	1348.09	10.58	11.99	8.47	23.24
551AD3	1236.41	1057.75	5.94	10.72	4.36	16.89	1236.41	1058.86	8.36	10.63	4.36	16.77
551AD4	1564.16	1302.79	5.00	10.79	7.10	20.06	1564.16	1303.83	6.56	10.72	7.10	19.97
551AD5	1474.75	1239.87	5.72	10.93	5.94	18.94	1474.75	1240.83	8.09	10.86	5.94	18.85
Average			6.27	12.31	5.68	20.57			8.96	12.24	5.68	20.47
552AD1	1220.21	1017.30	8.43	11.18	6.54	19.95	1220.21	1018.16	12.56	11.10	6.54	19.84
552AD2	1574.00	1313.70	6.85	12.72	4.57	19.81	1574.00	1314.49	9.55	12.67	4.57	19.74
552AD3	1220.13	980.16	7.42	13.94	7.14	24.48	1220.13	980.44	10.12	13.91	7.14	24.45
552AD4	1173.56	1008.90	7.05	11.39	3.07	16.32	1173.56	1009.30	9.84	11.36	3.07	16.27
552AD5	1333.06	1081.03	8.05	10.28	10.63	23.31	1333.06	1081.69	12.05	10.23	10.63	23.24
Average			7.56	11.90	6.39	20.78			10.82	11.85	6.39	20.71
Overall Average			6.91	12.11	6.03	20.67			9.89	12.05	6.03	20.59

Table E.3 Results of LR(5, 150, 0, 0, 1, 0) and LR(5, 250, 0, 0, 1, 0)

	LR(5, 150, 0, 0, 1, 0)						LR(5, 250, 0, 0, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1466.75	1187.36	18.48	17.00	2.52	23.53	1466.75	1187.36	32.31	17.00	2.52	23.53
551AD2	1661.40	1348.30	18.01	11.97	8.47	23.22	1661.40	1348.30	32.30	11.97	8.47	23.22
551AD3	1221.24	1059.02	13.40	10.61	3.08	15.32	1221.24	1059.02	23.85	10.61	3.08	15.32
551AD4	1561.76	1304.05	9.63	10.71	6.94	19.76	1561.76	1304.07	15.76	10.71	6.94	19.76
551AD5	1474.75	1240.98	12.94	10.85	5.94	18.84	1474.75	1241.01	22.72	10.85	5.94	18.83
Average			14.49	12.23	5.39	20.13			25.39	12.23	5.39	20.13
552AD1	1220.21	1018.25	20.01	11.09	6.54	19.83	1220.21	1018.25	35.18	11.09	6.54	19.83
552AD2	1574.00	1314.59	15.26	12.66	4.57	19.73	1574.00	1314.62	26.69	12.66	4.57	19.73
552AD3	1220.13	980.48	15.51	13.91	7.14	24.44	1220.13	980.48	26.26	13.91	7.14	24.44
552AD4	1173.56	1009.38	15.63	11.35	3.07	16.27	1173.56	1009.39	27.23	11.35	3.07	16.26
552AD5	1333.06	1081.80	21.15	10.22	10.63	23.23	1333.06	1081.81	40.38	10.22	10.63	23.22
Average			17.51	11.85	6.39	20.70			31.15	11.85	6.39	20.70
Overall Average			16.00	12.04	5.89	20.42			28.27	12.04	5.89	20.42

Table E.4 Results of LR(5, 25, 0, 1, 1, 0) and LR(5, 50, 0, 1, 1, 0)

	LR(5, 25, 0, 1, 1, 0)						LR(5, 50, 0, 1, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1472.77	1302.90	1.94	8.93	2.94	13.04	1472.77	1320.20	5.44	7.72	2.94	11.56
551AD2	1623.99	1453.56	1.60	5.10	6.03	11.73	1597.56	1471.50	3.91	3.93	4.30	8.57
551AD3	1227.76	1098.20	1.62	7.31	3.63	11.80	1221.24	1117.56	3.77	5.67	3.08	9.28
551AD4	1553.20	1342.99	1.52	8.04	6.35	15.65	1526.10	1384.83	3.50	5.18	4.50	10.20
551AD5	1478.05	1283.80	1.45	7.77	6.18	15.13	1460.96	1307.90	3.49	6.04	4.95	11.70
Average			1.63	7.43	5.03	13.47			4.02	5.71	3.96	10.26
552AD1	1222.88	1092.81	3.48	4.58	6.77	11.90	1222.88	1105.70	16.31	3.46	6.77	10.60
552AD2	1572.36	1429.58	2.82	5.02	4.46	9.99	1572.36	1443.76	6.66	4.08	4.46	8.91
552AD3	1237.08	1077.69	7.93	5.37	8.62	14.79	1229.90	1091.17	91.18	4.19	7.99	12.71
552AD4	1192.51	1069.91	4.39	6.03	4.73	11.46	1192.51	1086.37	42.97	4.59	4.73	9.77
552AD5	1322.38	1154.89	5.90	4.15	9.75	14.50	1311.80	1169.08	29.74	2.97	8.87	12.21
Average			4.90	5.03	6.87	12.53			37.37	3.86	6.57	10.84
Overall Average			3.26	6.23	5.95	13.00			20.70	4.78	5.26	10.55

Table E.5 Results of LR(5, 75, 0, 1, 1, 0) and LR(5, 100, 0, 1, 1, 0)

	LR(5, 75, 0, 1, 1, 0)						LR(5, 100, 0, 1, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1472.77	1321.03	9.91	7.66	2.94	11.49	1472.77	1321.16	14.41	7.65	2.94	11.48
551AD2	1595.77	1472.33	6.51	3.87	4.18	8.38	1595.77	1472.44	9.17	3.87	4.18	8.38
551AD3	1221.24	1120.29	6.26	5.44	3.08	9.01	1221.24	1120.75	8.93	5.40	3.08	8.97
551AD4	1523.65	1386.51	6.18	5.06	4.33	9.89	1523.65	1386.72	9.00	5.05	4.33	9.87
551AD5	1431.64	1311.80	6.18	5.76	2.85	9.14	1431.64	1312.13	9.09	5.74	2.85	9.11
Average			7.01	5.56	3.48	9.58			10.12	5.54	3.48	9.56
552AD1	1222.88	1106.97	33.79	3.35	6.77	10.47	1222.88	1107.05	41.61	3.34	6.77	10.46
552AD2	1572.36	1444.87	11.69	4.01	4.46	8.82	1572.36	1445.00	16.86	4.00	4.46	8.81
552AD3	1229.90	1093.26	295.53	4.00	7.99	12.50	1229.90	1093.71	611.29	3.97	7.99	12.45
552AD4	1192.51	1088.11	240.55	4.44	4.73	9.59	1192.51	1088.45	594.44	4.41	4.73	9.56
552AD5	1309.85	1171.82	123.64	2.75	8.71	11.78	1288.15	1172.29	290.11	2.71	6.91	9.88
Average			141.04	3.71	6.53	10.63			310.86	3.68	6.17	10.23
Overall Average			74.02	4.63	5.01	10.11			160.49	4.61	4.83	9.90

Table E.6 Results of LR(5, 25, 3p, 1, 1, 0) and LR(5, 50, 3p, 1, 1, 0)

	LR(5, 25, 3p, 1, 1, 0)						LR(5, 50, 3p, 1, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1472.77	1263.48	1.59	11.68	2.94	16.56	1472.77	1285.32	3.50	10.16	2.94	14.58
551AD2	1623.79	1410.05	1.46	7.94	6.01	15.16	1597.56	1440.04	3.12	5.98	4.30	10.94
551AD3	1223.12	1068.68	1.50	9.80	3.24	14.45	1220.51	1087.85	3.15	8.18	3.02	12.19
551AD4	1524.74	1332.49	1.44	8.76	4.40	14.43	1524.74	1344.56	2.96	7.93	4.40	13.40
551AD5	1482.92	1241.90	1.40	10.78	6.53	19.41	1454.94	1267.93	2.94	8.91	4.52	14.75
Average			1.48	9.79	4.63	16.00			3.13	8.23	3.84	13.17
552AD1	1211.22	1073.68	2.16	6.26	5.75	12.81	1211.22	1084.08	4.42	5.35	5.75	11.73
552AD2	1568.63	1406.89	2.19	6.53	4.21	11.50	1539.10	1418.49	4.38	5.76	2.25	8.50
552AD3	1200.09	1044.98	2.30	8.24	5.38	14.84	1200.09	1059.29	6.32	6.99	5.38	13.29
552AD4	1211.12	1043.93	2.24	8.32	6.37	16.02	1194.85	1061.89	4.55	6.74	4.94	12.52
552AD5	1331.66	1122.20	2.57	6.87	10.52	18.67	1273.89	1137.30	5.72	5.61	5.72	12.01
Average			2.29	7.24	6.45	14.77			5.08	6.09	4.81	11.61
Overall Average			1.89	8.52	5.54	15.38			4.11	7.16	4.32	12.39

Table E.7 Results of LR(5, 75, 3p, 1, 1, 0) and LR(5, 100, 3p, 1, 1, 0)

	LR(5, 75, 3p, 1, 1, 0)						LR(5, 100, 3p, 1, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1472.77	1285.32	5.62	10.16	2.94	14.58	1472.77	1285.49	7.70	10.15	2.94	14.57
551AD2	1597.56	1440.27	4.80	5.97	4.30	10.92	1597.56	1440.28	6.46	5.97	4.30	10.92
551AD3	1220.51	1091.60	4.82	7.86	3.02	11.81	1220.51	1092.58	6.51	7.78	3.02	11.71
551AD4	1524.74	1347.87	4.56	7.71	4.40	13.12	1519.58	1347.87	6.16	7.71	4.05	12.74
551AD5	1454.94	1271.70	4.51	8.64	4.52	14.41	1454.94	1273.05	6.11	8.55	4.52	14.29
Average			4.86	8.07	3.84	12.97			6.59	8.03	3.77	12.84
552AD1	1211.22	1085.23	6.76	5.25	5.75	11.61	1211.22	1085.87	9.09	5.19	5.75	11.54
552AD2	1539.10	1423.36	6.56	5.44	2.25	8.13	1539.10	1423.79	8.74	5.41	2.25	8.10
552AD3	1200.09	1061.95	11.36	6.75	5.38	13.01	1200.09	1061.95	16.40	6.75	5.38	13.01
552AD4	1194.85	1062.09	6.93	6.72	4.94	12.50	1194.85	1062.52	9.32	6.68	4.94	12.45
552AD5	1273.89	1137.30	9.47	5.61	5.72	12.01	1268.70	1137.30	13.58	5.61	5.29	11.55
Average			8.21	5.95	4.81	11.45			11.43	5.93	4.72	11.33
Overall Average			6.54	7.01	4.32	12.21			9.01	6.98	4.24	12.09

Table E.8 Results of LR(5, 150, 3p, 1, 1, 0) and LR(5, 250, 3p, 1, 1, 0)

	LR(5, 150, 3p, 1, 1, 0)						LR(5, 250, 3p, 1, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1472.77	1285.49	11.96	10.15	2.94	14.57	1472.77	1285.49	20.53	10.15	2.94	14.57
551AD2	1597.56	1440.40	9.78	5.96	4.30	10.91	1597.56	1440.40	16.50	5.96	4.30	10.91
551AD3	1220.51	1092.83	9.90	7.76	3.02	11.68	1220.51	1093.87	16.66	7.67	3.02	11.58
551AD4	1503.21	1347.87	9.38	7.71	2.93	11.52	1503.21	1347.87	15.93	7.71	2.93	11.52
551AD5	1446.16	1273.05	9.30	8.55	3.89	13.60	1446.16	1274.01	15.64	8.48	3.89	13.51
Average			10.06	8.02	3.42	12.46			17.05	7.99	3.42	12.42
552AD1	1211.22	1086.08	13.71	5.17	5.75	11.52	1204.86	1087.12	22.94	5.08	5.20	10.83
552AD2	1539.10	1423.79	13.09	5.41	2.25	8.10	1539.10	1423.79	21.81	5.41	2.25	8.10
552AD3	1200.09	1061.95	26.72	6.75	5.38	13.01	1200.09	1061.95	46.74	6.75	5.38	13.01
552AD4	1194.85	1063.28	14.15	6.62	4.94	12.37	1194.85	1063.28	23.76	6.62	4.94	12.37
552AD5	1268.70	1138.97	21.83	5.47	5.29	11.39	1268.70	1139.29	42.53	5.45	5.29	11.36
Average			17.90	5.88	4.72	11.28			31.56	5.86	4.61	11.13
Overall Average			13.98	6.95	4.07	11.87			24.30	6.93	4.01	11.78

Table E.9 Results of LR(5, 25, 5p, 1, 1, 0) and LR(5, 50, 5p, 1, 1, 0)

	LR(5, 25, 5p, 1, 1, 0)						LR(5, 50, 5p, 1, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1516.03	1258.42	1.50	12.04	5.97	20.47	1504.65	1271.11	3.06	11.15	5.17	18.37
551AD2	1607.67	1385.22	1.34	9.56	4.96	16.06	1586.07	1417.75	2.81	7.44	3.55	11.87
551AD3	1223.12	1061.93	1.43	10.37	3.24	15.18	1216.84	1074.15	3.01	9.34	2.71	13.28
551AD4	1526.82	1325.00	1.37	9.27	4.55	15.23	1506.43	1346.32	2.75	7.81	3.15	11.89
551AD5	1462.66	1234.40	1.35	11.32	5.08	18.49	1450.11	1258.51	2.77	9.59	4.17	15.22
Average			1.40	10.51	4.76	17.09			2.88	9.07	3.75	14.13
552AD1	1208.32	1070.06	2.13	6.57	5.50	12.92	1208.32	1082.72	4.31	5.47	5.50	11.60
552AD2	1559.16	1405.59	2.16	6.62	3.59	10.93	1559.16	1416.64	4.31	5.88	3.59	10.06
552AD3	1227.03	1029.09	2.13	9.64	7.74	19.23	1204.18	1048.75	4.45	7.91	5.73	14.82
552AD4	1195.58	1049.53	2.24	7.82	5.00	13.92	1194.85	1054.87	4.46	7.36	4.94	13.27
552AD5	1309.45	1112.38	2.27	7.68	8.68	17.72	1309.45	1132.19	4.60	6.04	8.68	15.66
Average			2.19	7.67	6.10	14.94			4.43	6.53	5.69	13.08
Overall Average			1.79	9.09	5.43	16.01			3.65	7.80	4.72	13.61

Table E.10 Results of LR(5, 75, 5p, 1, 1, 0) and LR(5, 100, 5p, 1, 1, 0)

	LR(5, 75, 5p, 1, 1, 0)						LR(5, 100, 5p, 1, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1504.65	1271.78	4.68	11.10	5.17	18.31	1504.65	1271.78	6.28	11.10	5.17	18.31
551AD2	1586.07	1419.66	4.32	7.31	3.55	11.72	1582.62	1419.66	5.82	7.31	3.33	11.48
551AD3	1211.75	1075.67	4.60	9.21	2.28	12.65	1211.75	1075.67	6.23	9.21	2.28	12.65
551AD4	1506.43	1346.32	4.10	7.81	3.15	11.89	1506.43	1346.32	5.47	7.81	3.15	11.89
551AD5	1450.11	1266.87	4.25	8.99	4.17	14.46	1450.11	1266.87	5.74	8.99	4.17	14.46
Average			4.39	8.89	3.67	13.81			5.91	8.89	3.62	13.76
552AD1	1205.34	1083.82	6.46	5.37	5.24	11.21	1205.34	1083.82	8.60	5.37	5.24	11.21
552AD2	1559.16	1417.51	6.48	5.83	3.59	9.99	1559.16	1417.51	8.65	5.83	3.59	9.99
552AD3	1204.18	1048.75	6.83	7.91	5.73	14.82	1204.18	1050.43	9.23	7.77	5.73	14.64
552AD4	1194.85	1055.98	6.68	7.26	4.94	13.15	1194.85	1056.06	8.86	7.25	4.94	13.14
552AD5	1309.45	1134.80	6.94	5.82	8.68	15.39	1309.45	1134.80	9.26	5.82	8.68	15.39
Average			6.68	6.44	5.63	12.91			8.92	6.41	5.63	12.87
Overall Average			5.53	7.66	4.65	13.36			7.41	7.65	4.63	13.32

Table E.11 Results of LR(5, 150, 5p, 1, 1, 0) and LR(5, 250, 5p, 1, 1, 0)

	LR(5, 150, 5p, 1, 1, 0)						LR(5, 250, 5p, 1, 1, 0)					
	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP	LR UB	LR LB	CPU LR	%LGAP	%UGAP	%LRGAP
Problem												
551AD1	1504.65	1271.78	9.54	11.10	5.17	18.31	1504.65	1271.78	15.99	11.10	5.17	18.31
551AD2	1582.62	1424.27	8.79	7.01	3.33	11.12	1582.62	1424.27	14.77	7.01	3.33	11.12
551AD3	1211.75	1076.65	9.48	9.13	2.28	12.55	1211.75	1076.65	15.92	9.13	2.28	12.55
551AD4	1506.43	1346.32	8.21	7.81	3.15	11.89	1506.43	1346.32	13.45	7.81	3.15	11.89
551AD5	1450.11	1266.87	8.72	8.99	4.17	14.46	1450.11	1266.87	14.52	8.99	4.17	14.46
Average			8.95	8.81	3.62	13.67			14.93	8.81	3.62	13.67
552AD1	1205.34	1083.97	12.94	5.36	5.24	11.20	1205.34	1084.65	21.65	5.30	5.24	11.13
552AD2	1539.36	1417.51	12.97	5.83	2.27	8.60	1539.36	1417.51	21.88	5.83	2.27	8.60
552AD3	1204.03	1050.88	14.04	7.73	5.72	14.57	1204.03	1050.88	23.55	7.73	5.72	14.57
552AD4	1194.85	1056.09	13.35	7.25	4.94	13.14	1194.85	1056.09	22.32	7.25	4.94	13.14
552AD5	1269.33	1135.08	13.93	5.80	5.35	11.83	1269.33	1135.15	23.47	5.79	5.35	11.82
Average			13.45	6.39	4.70	11.87			22.57	6.38	4.70	11.85
Overall Average			11.20	7.60	4.16	12.77			18.75	7.59	4.16	12.76