AN EMPIRICAL COMPARISON OF INTEREST RATE MODELS FOR PRICING ZERO COUPON BOND OPTIONS

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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The aim of this study is to compare the performance of the four interest rate models (Vasicek Model, Cox Ingersoll Ross Model, Ho Lee Model and Black Derman Toy Model) that are commonly used in pricing zero coupon bond options. In this study, 1–5 years US Treasury Bond daily data between the dates June 1, 1976 and December 31, 2007 are used. By using the four interest rate models, estimated option prices are compared with the real observed prices for the begining work days of each months of the years 2004 and 2005. The models are then evaluated according to the sum of squared errors. Option prices are found by constructing interest rate trees for the binomial models based on Ho Lee Model and Black Derman Toy Model and by estimating the parameters for the Vasicek and the Cox Ingersoll Ross Models.

Keywords: Zero Coupon Bond Options, Interest Rate Models, Vasicek Model, Cox Ingersoll Ross Model, Ho Lee Model, Black Derman Toy Model, Arrow-Debreu Prices.

ÖZ

KUPONSUZ TAHVİL OPSİYONLARININ FİYATLAMASINDA KULLANILAN FAİZ HADDİ MODELLERİNİN AMPİRİK KARŞILAŞTIRMASI

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Bu çalışmanın amacı kuponsuz tahvillere dayalı opsiyonların fiyatlamasında kullanılan dört faiz haddi (Vasicek Model, Cox Ingersoll Ross Model, Ho Lee Model ve Black Derman Toy Model) modelinin performanslarının karşılaştırılmasıdır. Bu çalışmada ham veri olarak 1 Haziran 1976 ve 31 Aralık 2007 tarihleri arasında günlük, 1–5 yıl vadeli Amerika Birleşik Devletleri kuponsuz devlet tahvili verileri kullanılmıştır. Dört faiz haddi modeli kullanılarak, 2004 ve 2005 yılları her ayın ilk çalışma gününe ait opsiyonların tahmin edilen fiyatlarıyla gerçek gözlenen fiyatlar karşılaştırılmış, modeller hata kareleri toplamlarına göre değerlendirilmiştir. Opsiyon fiyatları, binom modeller (Ho Lee ve Black Derman Toy Modelleri) için faiz haddi ağaçları oluşturularak, Vasicek ve Cox Ingersoll Ross Modelleri için parametre tahmini yapılarak bulunmuştur.

Anahtar Kelimeler: Kuponsuz Tahvil Opsiyonları, Faiz Haddi Modelleri, Vasicek Model, Cox Ingersoll Ross Model, Ho Lee Model, Black Derman Toy Model, Arrow-Debreu Fiyatları.

To my mother and father

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CHAPTER 1

INTRODUCTION

Interest rates play an important role in our daily life even we may not realize. It extremely influences our purchase power. Moreover, the trend of interest rate has great impact on our investments. The upward or downward movements of interest rates tell us to revise our present situation as well as potential opportunities. Thus, an investor pays a great attention to this type of trends. Treasury Bills are comparison point of interest rates. For instance, an investor may expect returns of her/his money market account slide upward or downward while Treasury Bill prices begin to slip upward or downward direction. In this frame, we need interest rate models to understand the dynamics of interest rates which can be defined as a rate which is charged or paid for the use of money and often expressed as an annual percentage of the principal.

The main purpose of interest rate models can be thought as explaining the behaviour of interest rate movements. By fitting our available interest rate data to a model, we can provide both pricing and hedging interest rate derivative securities. Although, estimation of future movements of prices and rates is one of the most desirable goals; none of the interest rate models can achieve this completely. It is hard to point out which model is the best though, we may write some characteristics that a good model should have [13]:

- Accurate Valuation of Simple Market Instruments
- Ease of Calibration to the Market
- Robustness

- Extensibility to New Instruments
- Stability of Floating Parameters

It is useful to investigate some properties of interest rate models to understand them. We can define various points for distinguishing interest rate models. First, we can categorize interest rate models as discrete time and continuous time models. Compared to discrete models, continuous models have more popularity since continuous time mathematics have become more applicable in deriving formulas and proving theorems. In recent years, models that involve jumps and point process have made great contribution in order to develop discrete models.

Single and multi-factor model categorization is another way to classify interest rate models. In modelling short rate interest rates, it is important to determine how many (unknown) factors influence evolution of interest rates. Using one factor models is an important practice, since as empirical evidences show; there is more than one factor that determines the evolution of interest rate. But, working with one factor models can be helpful to understand the procedures, and hence, may be advantageous in applications of multi-factor models.

Another commonly used categorization method is dividing the interest rate models as arbitrage free and equilibrium models. This is the essential distinction from a theoretical perception. Arbitrage free models have assumptions about stochastic behaviors of interest rates, market price of risk, and also by assuming the no-arbitrage opportunities at the market, they derive the price of all contingent claims. In other words, there is no risk free strategy with zero cost that gives the possibility of positive returns.

On the contrary, equilibrium models that begin with description of economy, assume that the market is at equilibrium. However, the difference is not obvious, because equilibrium models should also be arbitrage free. If this is not the case, then the economy would not be at equilibrium.

In this study, we will use four models: two one factor equilibrium models - Vasicek and Cox Ingersoll Ross Models — and two no-arbitrage models —

Ho Lee and Black Derman Toy Models that have normal distributions (Vasicek Model and Ho Lee Model), lognormal distribution (Black Derman Toy Model) and non-central chi-square distribution (Cox Ingersoll Ross Model). One factor equilibrium models will be used to compute European call option prices on the beginning work days of months of 2004 and 2005 by using their estimated parameters. Binomial models will be applied in valuing European call options on related dates by using interest rate trees.

In this chapter, we will present some preliminaries to understand the concept that will be used throughout the study. Then we will investigate explicit solutions of some interest rate models in Chapter 2. Moreover, Chapter 3 focuses on the four models in terms of bond and option pricing. There, we will show the derivation of formulas that will be used in the applications. In Chapter 4, we will estimate the parameters of one factor equilibrium models first and then construct binomial trees. Moreover, by calculating the option prices, we will conclude the work by comparison of the models by estimating the call option prices. Finally, we give a brief conclusion in Chapter 5.

1.1 Preliminaries

In this section, we want to describe some concepts and definitions about stochastic processes in terms of both finance and mathematics that is used for modelling interest rates and interest rate options.

Definition 1.1.1. *Let* $(\Omega,\mathcal{A})$ *be a measurable space and let* $\mathbf{P}$ *be a probability measure on* $\mathcal{A}$*. Then the triple* $(\Omega,\mathcal{A},\mathbf{P})$ *is called a* ***probability space*** *[16].*

All the stochastic processes and the random variables are specified on a given probability space $(\Omega,\mathcal{A},\mathbf{P})$ throughout the thesis.

Definition 1.1.2. *A* ***filtration*** $\mathcal{F}_n$ *is an increasing sequence of* σ*-algebras in* $\mathcal{A}$*.* $\mathcal{F}_n$ *is for the information accessible at time* n*; in other words,* σ*-algebras of occurrences up to time* N*, where* N *stands for the maturity.*

Definition 1.1.3 (Martingale)**.** *An adapted sequence* $(M_n)_{0 \leq n \leq N}$ *of random variable's is said to be*

- martingale *if* $E(M_{n+1}/\mathcal{F}_n) = M_n$*,*
- submartingale *if* $E(M_{n+1}/\mathcal{F}_n) \geq M_n$*,*
- supermartingale *if* $E(M_{n+1}/\mathcal{F}_n) \leq M_n$*,*

with respect to information sets $\mathcal{F}_n$ *and probability* P*.*

Martingale concept was proposed by Paul Levy and it was then developed by Joseph Doob. The concept is very important for determining the characteristics of arbitrage free market. One of the best known Levy processes is the Brownian Motion. Brownian Motion shows the random movement of the asset prices. Since prices of zero coupon bond and options are uncertain in future, the Brownian Motion becomes the principal element in our study.

Definition 1.1.4 (Brownian Motion)**.** *A Brownian Motion is a real valued continuous process* $(X_t)_{t \geq 0}$ *with independent and stationary increments:*

- Continuity*:* $\mathbf{P}$ *almost surely the maps* $s \mapsto X_s(w)$ *is continuous (has continuous paths).*
- Independent Increments*: If* $s \leq t$*,* $X_t - X_s$ *is independent of* $\mathcal{F}_s = \sigma(X_u, u \leq s)$ *or* $X_{t_2} - X_{t_1}$*,* $X_{t_3} - X_{t_2}$*,...,*$X_{t_n} - X_{t_{n-1}}$ *independent random variables.*
- Stationary Increments*: If* $s \leq t$*,* $X_t - X_s$*,* $X_{t-s} - X_0$ *have the same probability law.*

Another important theorem in stochastic calculus is the Girsanov Theorem. It shows how to convert physical probability measure $\mathbf{P}$ to the risk neural probability measure $\mathbf{Q}$.

Theorem 1.1.1 (Girsanov Theorem [18]). *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbf{P})$ be a probability space and $(W_t^P)_{0\le t\le T}$ be an $\mathcal{F}$ Brownian motion. Let $(\theta_t)_{0\le t\le T}$ be an adapted process such that $\int_0^T \theta_s^2 ds < \infty$. Define*

$$L_t := \exp\left(\int_0^t \theta_s dW_s^P - \frac{1}{2}\int_0^t \theta_s^2 ds\right),$$

and under probability $\mathbf{Q}$,

$$W_t^Q = W_t^P - \int_0^t \theta_s ds$$

is a $\mathcal{F}$ Brownian Motion. Then, L_t is a Martingale if

$$E\left[\exp\left(\frac{1}{2}\int_0^T \theta_t^2 dt\right)\right] < \infty.$$

Lemma 1.1.2 (Ito Lemma [18]). *Let $(X_t)_{0\le t\le T}$ be an Ito process. If $f \in C^2$,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\left\langle X, X\right\rangle_s,$$

where $\left\langle X, X\right\rangle_s$ is the quadratic variation of X.

More generally, if $f(t,x) \in C^{1,2}$, then we have

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s)ds + \int_0^t \frac{\partial f}{\partial X}(s, X_s)dX_s \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial X^2}(s, X_s)d\left\langle X, X\right\rangle_s. \end{aligned}$$

In modelling interest rates, mean reversion property is one of the desirable property that the model should have. This property suggest that prices or rates move back toward average price or rate. Orstein Uhlenbeck process can be used to solve the stochastic differential equations that have mean reverting property.

Definition 1.1.5 (Ornstein Uhlenbeck Process). *The Ornstein Uhlenbeck Process is a stochastic process that satisfies*

$$dX_t = -cX_t d_t + \sigma dW_t.$$

Define $Y_t = X_t e^{ct}$, so the initial term become as $X_0 = Y_0$ and $X_0 = X$. By Ito's integration by parts formula

$$\begin{aligned} dY_t &= e^{ct} dX_t + cX_t e^{ct} dt, \\ &= e^{ct}(-cX_t dt + \sigma dW_t) + cX_t e^{ct} dt, \\ &= \sigma e^{ct} dW_t. \end{aligned}$$

Then integrating both sides gives us

$$Y_t - Y_0 = \int_0^t \sigma e^{cs} dW_s.$$

Inserting $X_t e^{ct}$ in place of Y_t yields

$$\begin{aligned} X_t e^{ct} &= X + \sigma \int_0^t e^{cs} dW_s, \\ X_t &= Xe^{-ct} + \sigma e^{-ct} \int_0^t e^{cs} dW_s. \end{aligned}$$

The expected value of X_t can be written as

$$E(X_t) = Xe^{-ct}.$$

Then, the variance of the process becomes

$$Var(X_t) = E(X_t - E(X_t))^2 = E\left(\sigma e^{-ct} \int_0^t e^{cs} dW_s\right)^2,$$

by isometry property

$$\begin{aligned} Var(X_t) &= \sigma^2 e^{-2ct} E\left(\int_0^t e^{2cs} ds\right), \\ &= \sigma^2 e^{-2ct} \int_0^t e^{2cs} ds. \end{aligned}$$

Proposition 1.1.3 (Feynman-Kac Formula)**.** *Let F be a solution to the problem*

$$\begin{aligned} \frac{\partial F}{\partial t}(t, X_t) + \mu(t, X_t)\frac{\partial F}{\partial X} + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2 F}{\partial X^2} - r_t F(t, X_t) &= 0 \\ F(T, X_T) &= \phi(X_T) \end{aligned}$$

where $\mu(t,x)$, $\sigma(t,x)$, $r(t,x)$ are $\phi(x)$ are given functions and X satisfies the SDE

$$dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s,$$

under probability measure $\mathbf{Q}$. *Then F is*

$$F(t, X_t) = E^Q\left[\exp\left(-\int_t^T r_s ds\right)\phi(X_T) \,\Big|\, \mathcal{F}_t\right].$$

Definition 1.1.6 (Arrow - Debreu Prices)**.** *Let $r(t,j)$ (or denoted as $r_{t,j}$) be the interest rate at time t and state j, at (t,j) for short, over time period $[t, t+1]$ on a binomial tree. Let* $\mathbf{P}$ *be the risk neutral probability that the interest rate will go up from $r(t,j)$ to $r(t+1, j+1)$ with probability p. (Hence, $r(t,j)$ will go down to $r(t+1,j)$ with probability $1-p$.)*

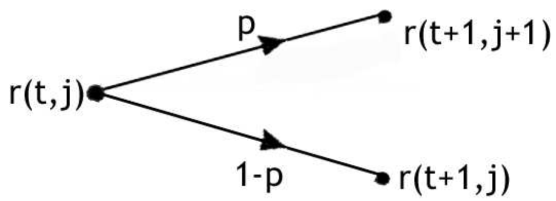

Figure 1.1: One Period Binomial Tree

For $0 \leq t_0$, $0 \leq j_0$, let $G(t_0, j_0)$ be the value of a derivative at time 0 and the payoff at $t = t_0$ is given by $\delta_{j_0 j}$ where j is the state reached at time t_0.

The $G(t,j)$'s are known as the Arrow-Debreu prices. (We also use $G(t_0, j_0)$ to denote the above defined derivative.) Note that $G(0,0)$ is 1. (Let $V(t,j)$ be the value (payoff) of an arbitrary derivative at (t,j).) It can easily be verified that $V(0,0)$, the value of the derivative at time $t = 0$ is given by

$$V(0,0) = \sum_{s=0}^{t} V(t,s)G(t,s). \tag{1.1.1}$$

If t_0 and j_0 are given, then the value of $G(t_0, j_0 + 1)$ at time $t_0 - 1$ becomes

$$G(t_0, j_0+1) = \begin{cases} (1-p)D(t_0 - 1, j+1), & \text{at state } j_0 + 1; \\ pD(t_0 - 1, j), & \text{at state } j_0; \\ 0, & \text{otherwise} \end{cases} \tag{1.1.2}$$

where $D(t, j)$ is the discount factor at (t, j) over $[t, t + 1]$. We therefore have

$$D(t) = \begin{cases} e^{-r(t,j)}, & \text{for continuously compounded interest;} \\ \dfrac{1}{1 + r(t,j)}, & \text{for simple interest.} \end{cases}$$

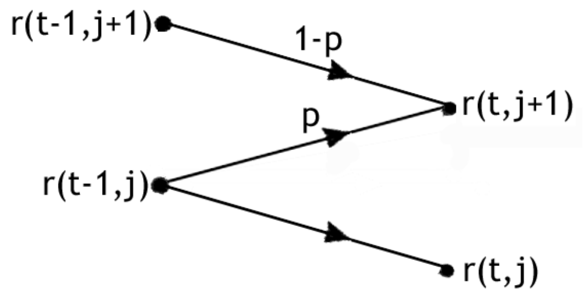

Figure 1.2: One Period Binomial Tree at Different States

Let $1 \leq t$ and $-1 \leq j \leq t - 1$ be given. Let $V(t - 1, j - 1)$ be the payoff (value) of $G(t, j + 1)$ at time $t - 1$. Note that the time zero value $V(t - 1, j + 1)$ is $G(t, j + 1)$. By equation (1.1.1) and equation (1.1.2) we have

$$G(t, j+1) = (1-p)D(t-1, j+1)G(t-1, j+1) + pD(t-1, j)G(t-1, j) \quad (1.1.3)$$

With this recursion, it is now possible to calculate $G(t, j)$ recursively [17].

In constructing interest rate trees, we need volatility estimates of the interest rate series. Instead of historical volatility and implied volatility, we prefer $GARCH$ models to estimate the volatilities of the spot interest rates.

Definition 1.1.7 (GARCH Model)**.** *Generalized autoregressive conditional heteroscedasticity (GARCH) model was claimed by Bollerslev in 1986 [4]. The GARCH (1,1) model is the simplest GARCH model, which can be written as*

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \alpha_2 \sigma_{t-1}^2.$$

In other words, at time t the conditional variance σ_t^2 of u depends not only on the squared error term u_{t-1}^2 in the previous time period but also on its conditional *variance* σ_{t-1}^2 *in the previous time period. More generally,* $GARCH$ *(p, q) model can be expressed as a generalized model in which there are p lagged terms of the squared errors and q terms of the lagged conditional variances.*

CHAPTER 2

INTEREST RATE MODELS

In this part of our work, we will present closed-form solutions of some interest rate models. Furthermore, we will give some brief descriptions about these models.

Merton (1973) and particularly Vasicek (1977) models are among the oldest methods based on modelling the evolution of the instantaneous spot interest rates. In their works they assume that the short rate followed a normal distribution; therefore, they allow negative interest rates with positive probability. Dothan (1978) and Rendleman and Bartter (1980) offered a lognormal distribution for the instantaneous spot interest rate to manage this disadvantageous. At the same time Brennan and Schwartz (1980) proposed a model by adding a mean reverting term to Dothan's model. But their models did not assume any known distribution for short rates. Ball and Torous assumed that the bond prices do not follow the original geometric Brownian motion of Black and Scholes (1973), but they follow Brownian bridges. So they included the constraints of bond price approaching its face value at maturity. Cox, Ingersoll and Ross (1985) (CIR) offered a non-central chi-square distribution instead of a lognormal distribution. All these models can be said as "endogenous term structure" models. In endogenous models, the initial term structure of interest rates is an output of the model rather than an input as observed in the market. On the other hand, Ho and Lee (1986) took the initial term structure as exogenously given at a point in time. Hull and White (1990) suggested an extension of the Vasicek (1977) Model and the Cox Ingersoll and Ross (1985) Model. In addition to Dothan (1978) and Rendleman and Bartter (1980), Black Derman Toy (1990), Black Karasinski (1991) and

Sandmann and Sondermann (1993) also offered lognormal distribution for the instantaneous spot interest rates. Heath, Jarrow and Morton (1992) considered the forward rates rather than the bond prices. Later, Longstaff and Schwartz (1992) developed an equilibrium model in which an investor has a logarithmic utility function. Moreover, the investor has alternatives of investing or consuming the only available goods in the economy. After Longstaff and Schwartz, Chen (1996) claimed his three-factor model. In Chen's model, the dynamics of short rate is related with the current short rate, stochastic mean and stochastic volatility of the short rate.

2.1 Merton Model (1973)

The Merton Model [13] can be expressed simply as

$$dr_t = \alpha dt + \sigma dW_t,$$

where α and σ are constants, respectively called the drift and the volatility. The solution r is

$$\begin{aligned} r_t &= r_0 + \int_0^t \alpha ds + \int_0^t \sigma dW_s \\ &= r_0 + \alpha t + \sigma W_t. \end{aligned}$$

This can be generalized as follows:

$$r_t = r_u + \alpha(t-u) + \sigma(W_t - W_u).$$

Since in the Merton Model, interest rates distributed normally, it is possible that r can take negative values. This is, however, unlikely observed at the interest rate markets.

2.2 Vasicek Model (1977)

The beginning devise of Vasicek's model [23] is very general: with the short term interest rate it is pronounced by a diffusion process. An arbitrage contention, having a likeness to that used to trace the Black-Scholes option pricing formula, is put to usage within this large structure to conclude that the partial differential equation is satisfied by any contingent claim. The bond price is then an outcome from the solution to this equation. Vasicek Model, however enforces more restrictive assumptions to formulate the model. The compatibility of the model description of requirements with an underlying economic equilibrium is not demonstrated. More truly, it is implicitly assumed. Vasicek uses equilibrium economy which was introduced by Merton in an analysis of price dynamics in continuous time. Equilibrium conditions indicate that interest rates are such that the demand and supply of capital are equally associated.

Assumptions of the model include the following:

Assumption 2.2.1. *The present short interest rate is known with certainty. However, the short rate values in future are not known (the assumption is made that $r(t)$ follows a stochastic process). The model also assume that $r(t)$*

- *is a continuous function of time, and*
- *conforms a Markovian process (that is, given its present value, future developments of the short rate are not influenced of past of the processes).*

Assumption 2.2.2. *Price of a discount bond $P(t, T)$ at the time t with maturity T is entirely obtained by the time t evaluation of $\{r(t^*) \mid t \leq t^* \leq T\}$. Furthermore, the progress of the short rate on $[t, T]$ is entirely determined by its present value $r(t)$. Hence, the bond price may be written as a function of the current short rate: $P(t, T) = P(r(t), t, T)$. Therefore, the whole term structure is determined by the short rate.*

Assumption 2.2.3. *It is assumed that the market is efficient. This indicates that*

- *there are no transaction costs,*
- *information is delivered to all investors at the same time,*

- *investors are rational, and*
- *riskless arbitrage is not possible.*

The main disadvantage of the Vasicek Model is that, $r(t)$ has normal distribution, and hence, it is possible that the model can generate negative rates, which is not desired in general.

In what follows, $\beta > 0$ is the speed of adjustment of the interest rate towards its average long run level, $\alpha > 0$ is the long run normal interest rate and σ is the volatility. Moreover, when $r(t) > \alpha$, then the drift becomes negative, so that the rate will be forced to the level α on average. On contrary when $r(t) < \alpha$, then the drift becomes positive, so that the rate will be forced again to the level α on average. The Vasicek Model specifies that the interest rates follow the stochastic differential equation:

$$dr_t = \beta(\alpha - r_t)dt + \sigma dW_t. \tag{2.2.1}$$

The solution of equation (2.2.1) can be obtained by letting $X_t = r_t - \alpha$ so that

$$dX_t = -\beta X_t dt + \sigma dW_t. \tag{2.2.2}$$

If we further define $Y_t = e^{\beta t} X_t$, then $Y_0 = e^0 X_0 = X_0$. By Ito's integration by parts formula

$$\begin{aligned} dY_t &= \beta e^{\beta t} X_t dt + e^{\beta t} dX_t \\ &= \beta e^{\beta t} X_t dt + e^{\beta t}(-\beta X_t dt + \sigma dW_t) \\ &= e^{\beta t} \sigma dW_t. \end{aligned}$$

Then by integrating both sides, we get

$$Y_t - Y_0 = \int_0^t e^{\beta s} \sigma dW_s.$$

Putting $e^{\beta t} X_t$ for Y_t follows

$$e^{\beta t} X_t = X_0 + \int_0^t e^{\beta s} \sigma dW_s,$$

so that

$$X_t = e^{-\beta t}\left[X_0 + \int_0^t e^{\beta s}\sigma dW_s\right].$$

Returning back to r_t, we obtain

$$r_t - \alpha = e^{-\beta t}\left[r_0 - \alpha + \int_0^t e^{\beta s}\sigma dW_s\right],$$

and further,

$$\begin{aligned} r_t &= \alpha + e^{-\beta t}(r_0 - \alpha) + e^{-\beta t}\sigma \int_0^t e^{\beta s} dW_s \\ &= \alpha + e^{-\beta t}(r_0 - \alpha) + \sigma \int_0^t e^{-\beta(t-s)} dW_s \\ &= e^{-\beta t} r_0 + \alpha(1 - e^{-\beta t}) + \sigma \int_0^t e^{-\beta(t-s)} dW_s. \end{aligned}$$

More generally, we can write

$$r_t = e^{-\beta(t-u)} r_u + \alpha\left(1 - e^{-\beta(t-u)}\right) + \sigma \int_u^t e^{-\beta(t-s)} dW_s.$$

2.3 Exponential Vasicek Model (1978)

Exponential Vasicek Model assumes short rate process evolves as exponential Ornstein Uhlenbeck process with lognormal distribution. Under $\mathbf{Q}$ probability measure, the model can be expressed as

$$dr_t = r_t[\eta_t - \alpha \log r_t]dt + \sigma r_t dW_t.$$

By appling Ito Lemma for $f(x) = \log(x)$ we get:

$$\begin{aligned} \log r_t &= \log r_0 + \int_0^t \left(\frac{1}{r_s}(r_s[\eta_s - \alpha \log r_s]ds + \sigma r_s dW_s)\right) + \frac{1}{2}\int_0^t \frac{-1}{r_s^2}\sigma^2 r_s^2 ds \\ &= \log r_0 + \int_0^t (\eta_s - \alpha \log r_s) ds + \int_0^t \sigma dW_s - \frac{1}{2}\int_0^t \sigma^2 ds \\ &= \log r_0 + \int_0^t \left(\eta_s - \alpha \log r_s - \frac{1}{2}\sigma^2\right) ds + \int_0^t \sigma dW_s. \end{aligned}$$

Then, denoting $y_t := \log r_t$, we have

$$y_t = y_0 + \int_0^t \left(\eta_s - \alpha y_s - \frac{1}{2}\sigma^2\right) ds + \int_0^t \sigma dW_s.$$

In terms of stochastic differential equation, it is equivalent to

$$dy_t = \left(\eta_t - \alpha y_t - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t.$$

If we set $\theta_t : \eta_t - \dfrac{1}{2}\sigma^2$, then it can be further expressed as

$$dy_t = [\theta_t - \alpha y_t]dt + dW_t.$$

Further, we need transformations as: $\beta_t = \dfrac{\theta_t}{\alpha}$ and $X_t = y_t - \beta_t$ to obtain the Ornstein Uhlenbeck process

$$dX_t = -\alpha X_t dt + \sigma dW_t.$$

Then, the solution for X_t as follows

$$X_t = X_0 e^{-\alpha t} + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dW_s.$$

Returning back to the process y_t we have

$$y_t - \beta_t = (y_0 - \beta_t)e^{-\alpha t} + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dW_s,$$

which yields

$$y_t = y_0 e^{-\alpha t} + (1 - e^{-\alpha t})\beta_t + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dW_s.$$

Since, $y_t = \log r_t$,

$$\log r_t = \log r_0 e^{-\alpha t} + (1 - e^{-\alpha t})\beta_t + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dW_s,$$

and hence,

$$r_t = \exp\left(\log r_0 e^{-\alpha t} + \frac{\theta_t}{\alpha}(1 - e^{-\alpha t}) + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dW_s\right).$$

Since Exponential Vasicek Model has the lognormal variable, we can get non-negative interest rates, and it also follows a mean reverting Ornstein Uhlenbeck process.

2.4 Cox Ingersoll Ross Model (1985)

Cox, Ingersoll and Ross (CIR) explain the matter of interest rate modelling as one in "general equilibrium theory". Expectation of future events, such as risk, other investment and consumption choices affect the term structure. Cox Ingersoll Ross Model makes use of a general equilibrium asset pricing model to endogenously conclude the stochastic process conformed by the short term interest rate and the partial differential equation satisfied by the value of any contingent claims. Bond prices are then determined as solutions to this partial differential equation, which depends on the underlying short term interest rate. Here are some the assumptions of Cox Ingersoll Ross Model.

(i) There is a single physical good which may be assigned to investment or for consumption.

(ii) Access to all production processes is free.

(iii) There exists an immediate borrowing and loaning for the market. This take places at a rate r that is determined as section of the equilibrium in the economy.

The Cox Ingersoll Ross Model specifies that the interest rates follow the stochastic differential equation [12]:

$$dr_t = \beta(\alpha - r_t)dt + \sigma\sqrt{r_t}dW_t,$$

which is equivalent to integral equation

$$r_t - r_u = \beta \int_u^t (\alpha - r_s)ds + \sigma \int_u^t r_s^{1/2} dW_s.$$

Applying the Ito Lemma to $f(x_t) = x_t^2$ and $x_t = r_t$ as in [20] we get

$$\begin{aligned} r_t^2 &= r_u^2 + \int_u^t 2r_s \left[\beta(\alpha - r_s)ds + \sigma r_s^{1/2} dW_s\right] + \frac{1}{2}\int_u^t 2\sigma^2 r_s ds \\ &= r_u^2 + \int_u^t 2r_s\beta\alpha ds - 2r_s^2\beta ds + 2r_s^{3/2}\sigma dW_s + \int_u^t \sigma^2 r_s ds \\ &= r_u^2 + \int_u^t \left(2r_s\beta\alpha - 2r_s^2\beta + \sigma^2 r_s\right) ds + \int_u^t 2r_s^{3/2}\sigma dW_s, \end{aligned}$$

hence,

$$r_t = r_u^2 + \left(2\beta\alpha + \sigma^2\right)\int_u^t r_s ds - 2\beta\int_u^t r_s^2 ds + 2\sigma\int_u^t r_s^{3/2} dW_s.$$

If $u = 0$, we further

$$r_t = r_0 + \beta\int_0^t (\alpha - r_s)ds + \sigma\int_0^t r_s^{1/2} dW_s.$$

Although there is no explicit form for the solution to the Cox Ingersoll Ross Model, it is known that the model has unique positive solution [12].

2.5 Ho Lee Model (1986)

Ho and Lee constructed a model [10] which takes the initial interest rate term structure as input, and produces its future stochastic evolution. Hence, the theoretical zero coupon bond prices will be accurately consistent with the observed prices in the market.

Ho Lee Model uses all the information of the current observed term structure to price contingent claims by avoiding the arbitrage. The assumptions of the model include

(i) The market is frictionless, that is there are no taxes or transaction costs.

(ii) The bond market is complete.

(iii) There is a finite number of possible states of the world for each time period n. The $P_i^{(n)}(T)$ denote the equilibrium prices of a T-maturity zero coupon bond at time n, and state i. This function is called for discount function and it satisfies some certain conditions:

$$\begin{aligned} P_i^n(T) &\geq 0, \\ P_i^n(0) &= 1, \\ \lim_{T\to\infty} P_i^n(T) &= 0. \end{aligned}$$

Ho and Lee represent the changes of the discount function by a binomial lattice. The $P_i^{(n)}(\cdot)$ show the discount function for i times upstate and $(n-i)$ times downstate moves at time n. When passing from the period n to the period $(n+1)$ the discount function may depend on an upstate move or a downstate move. So, at the time n discount function $P_i^{(n)}(\cdot)$ have two possible situations when passing at time $(n+1)$: $P_{i+1}^{(n+1)}(\cdot)$ or $P_i^{(n+1)}(\cdot)$. Therefore, we have

- There are $(n+1)$ possible states at each time n.
- The discount function in each state is independent.

Each discount bond's price conforms a binomial process. This is related with the behavior of interest rates of different kinds maturities depend on to each other. This is why the binomial lattice is used to model the whole term structure rather than that of a particular bond [15].

The binomial lattice method makes the following characteristics of the bond price clear

- uncertainty is small the near maturity of the bond,
- uncertainty increases as time to maturity increases.

Following features are related with two factors:

- For longer times, the number of changes increases and therefore uncertainty connected with the term structure increases.
- As the time come nearer to the maturity of the bond, price uncertainty decreases.

The dynamics of Ho Lee Model can be expressed as

$$dr_t = \theta_t dt + \sigma dW_t, \tag{2.5.3}$$

whose solution can be computed as follows:

$$\begin{aligned} r_t &= r_0 + \int_0^t \theta_s ds + \int_0^t \sigma dW_s \\ &= r_0 + \int_0^t \theta_s ds + \sigma W_t, \end{aligned}$$

or equivalently,

$$r_t = r_u + \int_u^t \theta_s ds + \sigma[W_t - W_u]. \tag{2.5.4}$$

In the form of a binomial tree Ho and Lee represented the model of bond prices with two parameters. First is the short rate standard deviation and the second is the market price of the risk of the short rate in discrete time.

The variable θ_t in equation (2.5.3) defines the average direction that short rate moves at time t. The main drawback of this model is that it does not consider the mean reversion property of the interest rates, unlike Vasicek Model.

2.6 Hull White Extended Vasicek Model (1990)

Vasicek suggested the poor fitting of the initial term structure of interest rates. Ho and Lee made effort to construct exogenous term structure model, but their main assumption was that the whole term structure of rates follows a binomial tree although their model has continuous time limit. For the need of an exact fit, Hull and White [11] introduced a time varying parameter in the Vasicek

Model. The model then indicated a normal distribution for the short rate process. The strength of normal distribution is that it allows the derivation of analytical formulas; however, the weakness is that it also allows negative interest rates with positive probability.

The dynamics of Hull White Model can be expressed as

$$dr_t = (\beta_t - \alpha_t r_t)dt + \sigma_t dW_t, \tag{2.6.5}$$

where β_t, α_t and σ_t are deterministic functions of time. For the solution of this stochastic differential equation we define $K_t = \int_0^t \alpha_u du$. Then, applying the Ito's integration by parts formula to $e^{K_t} r_t$. We have

$$d(e^{K_t} r_t) = e^{K_t} {K_t}' r_t dt + e^{K_t} dr_t.$$

Hence, equation (2.6.5) we calculate

$$\begin{aligned}
d(e^{K_t} r_t) &= e^{K_t}\alpha_t r_t dt + e^{K_t}[(\beta_t - \alpha_t r_t)dt + \sigma_t dW_t] \\
&= e^{K_t}\alpha_t r_t dt + e^{K_t}\beta_t dt - e^{K_t}\alpha_t r_t dt + e^{K_t}\sigma_t dW_t \\
&= e^{K_t}\beta_t dt + e^{K_t}\sigma_t dW_t \\
&= e^{K_t}(\beta_t dt + \sigma_t dW_t).
\end{aligned}$$

Integrating both sides now gives

$$e^{K_t} r_t = r_0 + \int_0^t e^{K_s}\beta_s ds + \int_0^t e^{K_s}\sigma_s dW_s,$$

from which we get,

$$\begin{aligned}
r_t &= e^{-K_t}\left(r_0 + \int_0^t e^{K_s}\beta_s ds + \int_0^t e^{K_s}\sigma_s dW_s\right) \\
&= e^{-K_t} r_0 + \int_0^t e^{-(K_t - K_s)}\beta_s ds + \int_0^t e^{-(K_t-K_s)}\sigma_s dW_s.
\end{aligned}$$

To generalize the result for any $u \leq t$, we write

$$r_t = e^{-(K_t-K_s)} r_u + \int_u^t e^{-(K_t-K_s)}\beta_s ds + \int_u^t e^{-(K_t-K_s)}\sigma_s dW_s.$$

2.7 Black Derman Toy Model (1990)

For modelling interest rates in a discrete time, Black, Derman and Toy [8] use a binomial tree method. For determining all rates, the short term interest rate, the main fundamental factor is used. In order to form a binomial tree of short term interest rates in the future, both the present term structure of interest rates and the associated volatilities are used.

The essential variable that urges security prices into the model is the short term interest rate, which can be specified as the annualized one period rate of interest. The data entered to the model are a set of long-term interest rates of different maturities and their related volatilities. For this reason, to calibrate the model we need a yield curve and a volatility curve. These inputs are used to calculate means and related volatilities of future realization of the interest rate. The change in the yield and volatility curves cause changes of the means and volatilities of future short term interest rates. The changes in future volatility have an influence on the degree of mean reversion.

As with most models, the assumption of a perfect market is made. Here are the other assumptions of the model.

(i) Yields of all zero coupon bond's changes are perfectly correlated to each other.

(ii) The expected returns of one period are identical for all securities.

(iii) There are no taxes and no transaction costs.

The lognormality property enables several strengths for calibration of the model. The main advantage of lognormal distribution is that, negative interest rates are avoided and the volatility input may be given in percentages.

As mentioned before, the model produces the term structure so that it matches the observed term structure. After having calculated the price of the short term interest rate at each branch in Figure 2.1, we are able to determine the price of

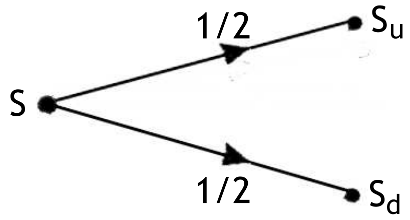

Figure 2.1: Price of Contingent Claim for One Period

any European type contingent claims. At each branch, the value is equal to the discounted expected value one time period in future. We calibrate the binomial tree to the observed risk-free rate, hence, we may price the contingent claim in a risk-neutral environment. Here we assume that the up and down movement probabilities are equal. Therefore, after one period, the expected price of our contingent claim becomes

$$S = \frac{1}{2}(S_u + S_d).$$

Here S_u stands for the price of the contingent claim after an up move and S_d is for the price of the contingent claim after a down move. If we discount the price of the contingent claim S by the current one period interest rate r, then the discounted expected price of contingent claim becomes

$$S = \frac{\frac{1}{2}(S_u + S_d)}{1 + r}.$$

The Black Derman Toy dynamics is given by the stochastic differential equation

$$d \log r_t = [\theta_t + \rho_t \log r_t] dt + \sigma_t dW_t,$$

which can also be written as

$$d \log r_t = \left[\theta_t + \frac{{\sigma_t}'}{\sigma_t} \log r_t\right] dt + \sigma_t dW_t,$$

by using $\rho_t = \dfrac{{\sigma_t}'}{\sigma_t}$.

We can choose the function σ_t to make the model consistent with the term structure of spot rate volatilities.

$$d \log r_t = \theta_t dt + \sigma dW_t.$$

By integrating both sides, we get the following result

$$\begin{aligned}
\log r_t &= \log r_0 + \int_0^t \theta_s ds + \int_0^t \sigma dW_s \\
&= \log r_0 + \int_0^t \theta_s ds + \sigma W_t \\
r_t &= r_0 \exp\left(\int_0^t \theta_s ds + \sigma W_t \right).
\end{aligned}$$

In general for $u \leq t$, the solution can be expressed as

$$r_t = r_u \exp\left(\int_u^t \theta_s ds + \sigma(W_t - W_u) \right).$$

2.8 Black Karasinski Model (1991)

Black and Karasinski [3] brings out a model, where the target rate, mean reversion rate and local volatility are time dependent, however they are deterministic functions. The future short term interest rate volatilities can be mentioned independently of the initial volatility term structure by determining the three time dependent factors. As Black Derman Toy Model assumes, the Black Karasinski Model also assumes that the short term interest rate have a lognormal distribution. In their original work, Black and Karasinski proposed a mean reverting lognormal short rate model:

$$d \log r_t = \phi_t [\log \mu_t - \log r_t] dt + \sigma_t dW_t, \tag{2.8.6}$$

where μ_t is the target rate, ϕ_t is the speed of mean reversion and σ_t is the local volatility. In solving the stochastic differential equation (2.8.6), we will assume that $\alpha_t = \phi_t$ and $\beta_t = \phi_t \log \mu_t$ following [1]. Let $Y_t = \log r_t$ and define a new deterministic function K_t as

$$K_t = \int_0^t \alpha_s ds.$$

By applying Ito's integration by parts formula to $e^{K_t} Y_t$ we obtain

$$\begin{aligned} d(e^{K_t} Y_t) &= e^{K_t} {K_t}' Y_t dt + e^{K_t} dY_t \\ &= e^{K_t} \alpha_t Y_t dt + e^{K_t} ((\beta_t - \alpha_t Y_t) dt + \sigma_t dW_t) \\ &= e^{K_t} (\beta_t + \sigma_t dW_t). \end{aligned}$$

Integrating both sides and simplifying the result follows

$$Y_t = e^{-K_t} Y_0 + \int_0^t e^{-(K_t - K_s)} \beta_s ds + \int_0^t e^{-(K_t - K_s)} \sigma_s dW_s.$$

If we further replace Y_t by $\log r_t$ then

$$\log r_t = e^{-(K_t - K_u)} \log r_u + \int_u^t e^{-(K_t - K_u)} \beta_s ds + \int_u^t e^{-(K_t - K_u)} \sigma_s dW_s,$$

where $u \leq t$. Hence,

$$r_t = \exp\left(e^{-(K_t - K_u)} \log r_u + \int_u^t e^{-(K_t - K_u)} \beta_s ds + \int_u^t e^{-(K_t - K_u)} \sigma_s dW_s \right).$$

CHAPTER 3

ZERO COUPON BOND AND OPTION PRICING

In this chapter, we will analyze the four interest rate models that are commonly used bond and option pricing. First, zero coupon bonds pricing formulas will be given for the Vasicek Model and the Cox Ingersoll Ross Model. After investigating pricing principles of these models the formulas will follow. The binomial models of Ho Lee and Black Derman Toy will specifically be treated together with some of their properties, in particular, on volatilities.

3.1 Vasicek Model for Bond Pricing

Under real world probability **P**, the model can be described by the stochastic differential equation

$$dr_t = \beta(\alpha - r_t)dt + \sigma dW_t^P. \tag{3.1.1}$$

For pricing purposes, however we need to work with the risk neutral probability measure **Q**. Hence, we use the Girsanov Theorem for the change of measure as follows: define

$$L(\tau, \lambda) = \exp\left(\int_0^\tau \lambda dW_t^P - \frac{1}{2}\int_0^\tau \lambda^2 dt\right),$$

so that

$$dQ = L(\tau, \lambda)dP.$$

Then, inserting $dW_t^P = dW_t^Q - \lambda dt$ into (3.1.1), we obtain

$$dr_t = \beta(\alpha - r_t)dt + \sigma(dW_t^Q - \lambda dt). \tag{3.1.2}$$

Therefore, under the unique probability measure **Q**, equation (3.1.1) is equivalent to the stochastic differential equation

$$dr_t = \beta\left(\alpha - \frac{\lambda\sigma}{\beta} - r_t\right)dt + \sigma dW_t^Q,$$

and hence, the bond value under the measure **Q** is

$$B(t,T) = E^Q\left(e^{-\int_t^T r_s ds} \,\middle|\, F_t\right).$$

By Ito's Lemma, the partial differential equation for bond pricing in Vasicek Model takes the form

$$\begin{aligned} \frac{dB(t,T)}{dt} + \beta\left(\alpha - \frac{\lambda\sigma}{\beta} - r_t\right)\frac{dB(t,T)}{dr} + \frac{1}{2}\sigma^2\frac{\partial^2 B(t,T)}{\partial r^2} - rB(t,T) &= 0 \\ B(T,T) &= 1. \end{aligned} \tag{3.1.3}$$

The solution of this partial differential can be formed as [2]:

$$B(r(t),t,T) = A(t,T)e^{-C(t,T)r(t)}. \tag{3.1.4}$$

Inserting (3.1.4) in (3.1.3) follows

$$A_t e^{-Cr} - rAC_t e^{-Cr} - \beta\left(\alpha - \frac{\lambda\sigma}{\beta} - r\right)ACe^{-Cr} + \frac{1}{2}\sigma^2 AC^2 e^{-Cr} - rAe^{-Cr} = 0,$$

and after simplification we find that

$$A_t - rAC_t - \beta\left(\alpha - \frac{\lambda\sigma}{\beta} - r\right)AC + \frac{1}{2}\sigma^2 AC^2 - rA = 0,$$

and

$$A_t - (\beta\alpha + \lambda\sigma)AC + \frac{1}{2}\sigma^2 AC^2 = rA + rAC_t - \beta rAC.$$

hold. Here the subscripts denote the differentiation with respect to time t. For the right hand side we assume that

$$rA + rAC_t - \beta rAC = 0,$$

so that

$$rA(1 + C_t - C\beta) = 0.$$

The solution for $C(t, T)$ can be obtained from first order linear differential equation and it is

$$C(t,T) = \frac{1}{\beta}\left(1 - e^{-\beta(T-t)}\right).$$

On the other hand, for the solution of $A(t, T)$ we have

$$A_t - (\alpha\beta + \lambda\sigma)AC + \frac{1}{2}\sigma^2 AC^2 = 0,$$

so that

$$A_t + AC\left(-(\alpha\beta + \lambda\sigma) + \frac{1}{2}\sigma^2 C\right) = 0.$$

Fortunately, we now have a separable first order equation

$$\frac{dA}{A} + C\left(\frac{1}{2}\sigma^2 C - \alpha\beta - \lambda\sigma\right) dt = 0,$$

which implies

$$\log A(T,T) - \log A(t,T) + \int_t^T \left(\frac{1}{2}\sigma^2 \frac{1}{\beta^2}\left(1 - 2e^{-\beta(T-u)} + e^{-2\beta(T-u)}\right)\right.$$
$$\left. - \left(\beta\left(\alpha + \frac{\lambda\sigma}{\beta}\right)\frac{1}{\beta}\left(1 - e^{-\beta(T-u)}\right)\right)\right) du = 0.$$

Having inserted $C(t, T)$ and taking the integral it follows that

$$\log A(T,T) - \log A(t,T) + \frac{1}{2}\frac{\sigma^2}{\beta^2}\left(u - \frac{2}{\beta}e^{-\beta(T-u)} + \frac{1}{2\beta}e^{-2\beta(T-u)}\right)\Bigg|_{u=t}^{u=T}$$
$$- \left(\alpha + \frac{\lambda\sigma}{\beta}\right)\left(u - \frac{1}{\beta}e^{-\beta(T-u)}\right)\Bigg|_{u=t}^{u=T} = 0,$$

using the fact that $A(T, T) = 1$ so that $\log A(T, T) = 0$. Then,

$$\log A(t,T) = \frac{\sigma^2}{2\beta^2}\left((T-t) - \frac{2}{\beta}\left(1 - e^{-\beta(T-t)}\right) + \frac{1}{2\beta}\left(1 - e^{-2\beta(T-t)}\right)\right)$$
$$- \left(\alpha + \frac{\lambda\sigma}{\beta}\right)\left((T-t) - \frac{1}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right).$$

By taking the exponential of both sides, we obtain the solution for $A(t,T)$ as

$$\begin{aligned} A(t,T) &= \exp\left(\frac{\sigma^2}{2\beta^2}\left((T-t) - \frac{2}{\beta}\left(1-e^{-\beta(T-t)}\right) + \frac{1}{2\beta}\left(1-e^{-2\beta(T-t)}\right)\right)\right) \\ &\quad - \left(\left(\alpha + \frac{\lambda\sigma}{\beta}\right)\left((T-t) - \frac{1}{\beta}\left(1-e^{-\beta(T-t)}\right)\right)\right), \end{aligned}$$

and some re-arrangements take the solution into the form

$$\begin{aligned} A(t,T) &= \exp\left(-(T-t)\left(\alpha - \frac{\lambda\sigma}{\beta} - \frac{\sigma^2}{2\beta^2}\right) + \frac{\sigma^2}{4\beta^2}\left(1-e^{-2\beta(T-t)}\right)\right) \\ &\quad + \left(\frac{1}{\beta}\left(1-e^{-\beta(T-t)}\right)\left(\alpha - \frac{\lambda\sigma}{\beta} - \frac{\sigma^2}{\beta^2}\right)\right). \end{aligned}$$

Remark 3.1. *The general form of the Ricatti equation is*

$$w'(t) + [a(t) + d(t)]w(t) + b(t)w^2(t) - c(t) = 0$$

The solution of this equation can be written as $w(t) = \dfrac{v(t)}{u(t)}$*, where* $v(t)$ *and* $u(t)$ *are solutions of the associated system of first order linear equations:*

$$\begin{aligned} -v'(t) + c(t)u(t) - d(t)v(t) &= 0 \\ u'(t) - a(t)u(t) - b(t)v(t) &= 0 \end{aligned}$$

3.2 Cox Ingersoll Ross Model for Bond Pricing

Under probability measure **P**, Cox Ingersoll Ross Model can be described as

$$dr_t = \beta(\alpha - r_t)dt + \sigma\sqrt{r_t}dW_t^P, \tag{3.2.5}$$

and Radon-Nikodym derivative in this case is

$$L(\tau,\lambda) = \exp\left(\int_0^\tau \lambda\sqrt{r_s}dW_s^P - \frac{1}{2}\int_0^\tau \lambda^2 r_s ds\right).$$

Thus for the Cox Ingersoll Ross Model, if

$$dW_t^P = dW_t^Q - \lambda\sqrt{r_t}$$

is substituted in (3.2.5) we obtain

$$dr_t = \beta(\alpha - r_t)dt + \sigma\sqrt{r_t}\left(dW_t^Q - \lambda\sqrt{r_t}dt\right). \tag{3.2.6}$$

So, under probability measure $\mathbf{Q}$ the dynamics is governed by the stochastic differential equation

$$dr_t = (\beta\alpha - (\beta + \sigma\lambda)r_t)\,dt + \sigma\sqrt{r_t}dW_t^Q.$$

By the application of the Ito's Lemma, the corresponding partial differential equation for the bond price in Cox Ingersoll Ross Model can be specified as

$$\begin{aligned} \frac{dB(t,T)}{dt} + (\beta\alpha - (\beta + \sigma\lambda)r_t)\frac{dB(t,T)}{dr} + \frac{1}{2}\sigma^2\frac{\partial^2 B(t,T)}{\partial r^2}r_t - r_t B(t,T) &= 0 \\ B(T,T) &= 1. \end{aligned}$$

The solution of this partial differential can be obtained, similarly as in Vasicek Model [2]:

$$B(t,T) = A(t,T)e^{-C(t,T)r_t}.$$

The partial differential equation, in this case becomes

$$A_t e^{-Cr} - rAC_t e^{-Cr} - (\beta\alpha - (\beta + \sigma\lambda)r)ACe^{-Cr} + \frac{1}{2}\sigma^2 rAC^2 e^{-Cr} - rAe^{-Cr} = 0,$$

from which, by dividing e^{-Cr} we find that

$$A_t - rAC_t - (\beta\alpha - (\beta + \sigma\lambda)r)AC + \frac{1}{2}\sigma^2 rAC^2 - rA = 0.$$

holds. After some re-arrangements, the equation turns to be

$$rA\left(\frac{1}{2}\sigma^2 C^2 - C_t + (\beta + \sigma\lambda)C - 1\right) = \beta\alpha AC - A_t.$$

Assuming that the right hand side vanishes,

$$A_t - \beta\alpha AC = 0, \tag{3.2.7}$$

then we have

$$C_t - (\beta + \sigma\lambda)C - \frac{1}{2}\sigma^2 C^2 + 1 = 0. \tag{3.2.8}$$

We need to solve for $A(t,T)$ and $C(t,T)$: Equation (3.2.8)is a Ricatti equation and its solution can be expressed as

$$C(t,T) = \frac{v(t,T)}{u(t,T)},$$

where $v(t,T)$ and $u(t,T)$ are solutions to the following system of equations [22]:

$$\begin{aligned} v'(t,T) + u(t,T) - \beta v(t,T) &= 0 \\ u'(t,T) + \lambda\sigma u(t,T) + \frac{1}{2}\sigma^2 v(t,T) &= 0. \end{aligned}$$

For $\tau = T - t$, where T is the bond maturity date, and $\frac{\partial}{\partial t} = -\frac{d}{d\tau}$. Thus, the above system of equations may be transformed to

$$-v'(\tau) + u(\tau) - \beta v(\tau) = 0, \tag{3.2.9}$$

and

$$-u'(\tau) + \lambda\sigma u(\tau) + \frac{1}{2}\sigma^2 v(\tau) = 0. \tag{3.2.10}$$

By taking the derivatives of both sides of (3.2.9) as

$$u'(\tau) = v''(\tau) + \beta v'(\tau), \tag{3.2.11}$$

and using it in (3.2.10) we obtain a second order linear ordinary differential equation for $v = v(\tau)$

$$-v'' - \beta v'(\tau) + \lambda\sigma v'(\tau) + \lambda\sigma\beta v(\tau) + \frac{1}{2}\sigma^2 v(\tau) = 0. \tag{3.2.12}$$

Expressing this in terms of D-operators results in a simple quadratic equation

$$\left[D^2 - (\lambda\sigma - \beta)D - \left(\lambda\sigma\beta + \frac{1}{2}\sigma^2\right)\right] v(\tau) = 0.$$

The roots of this quadratic equation are $\frac{\gamma + \lambda\sigma - \beta}{2}$ and $\frac{-\gamma + \lambda\sigma - \beta}{2}$ where $\gamma = \sqrt{(\beta + \lambda\sigma)^2 + 2\sigma^2}$ and hence the solution may be written as

$$v(\tau) = k_1 \exp((\gamma + \lambda\sigma - \beta)\tau/2) + k_2 \exp((-\gamma + \lambda\sigma - \beta)\tau/2),$$

where k_1 and k_2 are arbitrary constants. Since $C(T,T) = 0 = \dfrac{v(0)}{u(0)}$, $v(0)$ should be equal to 0 and hence, k_1 be equal to $-k_2$. By setting $k_1 = 1$ and $k_2 = -1$, $v(\tau)$ can be expressed as

$$v(\tau) = \exp((\gamma + \lambda\sigma - \beta)\tau/2) - \exp((-\gamma + \lambda\sigma - \beta)\tau/2), \tag{3.2.13}$$

Now, (3.2.9) implies that

$$v' = \frac{1}{2}(\gamma + \lambda\sigma - \beta)e^{(\gamma+\lambda\sigma-\beta)\tau/2} - \frac{1}{2}(-\gamma + \lambda\sigma - \beta)e^{(-\gamma+\lambda\sigma-\beta)\tau/2}. \tag{3.2.14}$$

Therefore we find that

$$u(\tau) = \frac{1}{2}(\gamma + \lambda\sigma + \beta)e^{(\gamma+\lambda\sigma-\beta)\tau/2} - \frac{1}{2}(-\gamma + \lambda\sigma + \beta)e^{(-\gamma+\lambda\sigma-\beta)\tau/2}. \tag{3.2.15}$$

Since $\tau = T - t$, the solution of the Ricatti equation is obtained as in (3.2.13) and equation (3.2.15). Therefore, the solution of $C(t,T)$ can be written as

$$C(t,T) = \frac{v(\tau)}{u(\tau)}.$$

Inserting $v(\tau)$ and $u(\tau)$

$$\begin{aligned} C(t,T) &= \frac{2(\exp((\gamma + \lambda\sigma - \beta)(T-t)/2) - \exp((-\gamma + \lambda\sigma - \beta)(T-t)/2))}{(\gamma + \lambda\sigma + \beta)e^{(\gamma+\lambda\sigma-\beta)(T-t)/2} - (-\gamma + \lambda\sigma + \beta)e^{(-\gamma+\lambda\sigma-\beta)(T-t)/2}} \\ &= \frac{2\left(e^{\gamma(T-t)} - 1\right)}{(\gamma + \lambda\sigma + \beta)e^{\gamma(T-t)} - (-\gamma + \lambda\sigma + \beta)}. \end{aligned}$$

Finally, simplifying the expression gives

$$C(t,T) = \frac{2\left(e^{\gamma(T-t)} - 1\right)}{(\gamma + \lambda\sigma + \beta)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma}. \tag{3.2.16}$$

Now consider $A_t - \beta\alpha AC = 0$ with fixed bond maturity T, so that the bond price is considered to be a function of t only. Hence,

$$\frac{dA}{dt} = \beta\alpha AC,$$

is a separable equation, since

$$\frac{dA}{A} = \beta\alpha C dt,$$

and integrating both sides,

$$\log A(t,T) = -\int_t^T \beta\alpha C(s,T)ds.$$

By taking the exponential, $A(t,T)$ becomes

$$A(t,T) = \exp\left(-\beta\alpha\int_t^T C(s,T)ds\right), \tag{3.2.17}$$

and by inserting equation (3.2.16) into equation (3.2.17)

$$A(t,T) = \exp\left(-2\beta\alpha\int_t^T \frac{e^{\gamma(T-s)}-1}{(\gamma+\lambda\sigma+\beta)\left(e^{\gamma(T-s)}-1\right)+2\gamma}ds\right).$$

The integral in $A(t,T)$ can be calculated by letting $y = e^{\gamma(T-s)}$ then,

$$\frac{dy}{ds} = -\gamma e^{\gamma(T-s)},$$

and

$$ds = -\frac{dy}{\gamma e^{\gamma(t-s)}} = -\frac{dy}{\gamma y}.$$

Making use of the substitution and noting that

$$(\gamma-\lambda\sigma-\beta)(\gamma+\lambda\sigma+\beta) = \gamma^2-(\beta+\lambda\sigma)^2 = 2\sigma^2,$$

the integral above can be computed

$$\begin{aligned}
\mathcal{I} \;&:= \int_t^T \frac{e^{\gamma(T-s)}-1}{(\gamma+\lambda\sigma+\beta)\left(e^{\gamma(T-s)}-1\right)+2\gamma}ds \\
&= \frac{1}{\gamma}\int_{e^{\gamma(T-t)}}^1 \frac{-(y-1)}{(\gamma+\lambda\sigma+\beta)(y-1)+2\gamma}\frac{dy}{y} \\
&= \frac{1}{\gamma}\int_{e^{\gamma(T-t)}}^1 \left[\frac{-2\gamma/(\gamma-\lambda\sigma-\beta)}{(\gamma+\lambda\sigma+\beta)(y-1)+2\gamma}+\frac{1}{(\gamma-\lambda\sigma-\beta)}\frac{1}{y}\right]dy.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathcal{I} \;&:= \frac{-2}{(\gamma-\lambda\sigma-\beta)(\gamma+\lambda\sigma+\beta)}\log\left[(\gamma+\lambda\sigma+\beta)(y-1)+2\gamma\right]\Big|_{y=e^{\gamma(T-t)}}^{y=1} \\
&+\frac{1}{\gamma(\gamma-\lambda\sigma-\beta)}\log y\Big|_{y=e^{\gamma(T-t)}}^{y=1},
\end{aligned}$$

Since $(\gamma - \lambda\sigma - \beta)(\gamma + \lambda\sigma + \beta) = 2\sigma^2$, the integral simplifies to

$$\begin{aligned}
\mathcal{I} \quad &:= \frac{1}{\sigma^2}\left[-\log((\gamma + \lambda\sigma + \beta)(y-1) + 2\gamma) + \frac{\gamma + \lambda\sigma + \beta}{2\gamma}\log y \right]\Bigg|_{y=e^{\gamma(T-t)}}^{y=1} \\
&= \frac{1}{\sigma^2}\left[\log \frac{y^{(\gamma+\lambda\sigma+\beta)/2\gamma}}{(\gamma + \beta + \lambda\sigma)(y-1) + 2\gamma} \right]\Bigg|_{y=e^{\gamma(T-t)}}^{y=1} \\
&= \frac{1}{\sigma^2}\left[-\log 2\gamma - \log \frac{e^{(\gamma+\lambda\sigma+\beta)(T-t)/2}}{(\gamma + \beta + \lambda\sigma)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma} \right].
\end{aligned}$$

Hence, the solution $A(t,T)$ is therefore, from (3.2.17),

$$A(t,T) \quad = \exp\left(\frac{2\beta\alpha}{\sigma^2} \log \frac{2\gamma e^{(\gamma+\lambda\sigma+\beta)(T-t)/2}}{(\gamma + \lambda\sigma + \beta)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma} \right),$$

which can also be expressed as

$$A(t,T) = \left(\frac{2\gamma e^{(\gamma+\lambda\sigma+\beta)(T-t)/2}}{(\gamma + \lambda\sigma + \beta)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma} \right)^{\dfrac{2\beta\alpha}{\sigma^2}}.$$

Now, we can calculate the bond prices both in Vasicek and Cox Ingersoll Ross Models by using the solutions of $A(t,T)$ and $C(t,T)$. After computing the price of bonds, we will be able to estimate their parameters. Then for comparing the models, we use the prices of European call options. In the following sections, we will show how to price a call option for Vasicek and Cox Ingersoll Ross Models.

3.3 Vasicek Model for Option Pricing

The price of a European call option on the zero coupon bond maturing at time S with strike price K and exercise date T (with $T < S$) is [14]:

$$V(t) = P(t,S)\Phi(d_1) - KP(t,T)\Phi(d_2), \tag{3.3.18}$$

where

$$d_1 = \frac{1}{\sigma_p}\log\frac{P(t,S)}{KP(t,T)} + \frac{\sigma_p}{2}, \quad d_2 = d_1 - \sigma_p,$$

and

$$\sigma_p = \frac{\sigma}{\alpha}\left(1 - e^{-\beta(S-T)}\right)\sqrt{\frac{1 - e^{-2\beta(S-T)}}{2\beta}}.$$

Here $\Phi(z)$ is the cumulative distribution function of a standard normal random variable.

In order to prove (3.3.18), we need the following lemma. This lemma establishes the joint distribution under probability measure $\mathbf{Q}$ of $\int_t^T r(s)ds$ and $r(T)$, for $r(t)$.

Lemma 3.3.1. *[6] a) The bivariate Laplace transform of* $\int_t^T r(s)ds$ *and* $r(T)$ *given* $r(t)$ *is*

$$\begin{aligned} P_L(t,T,r,v,w) &= E^Q\left[\exp\left(-v\int_t^T r(s)ds - wr(T)\right)\ \Big|\ r(t) = r\right] \\ &= \exp\left[A(t,T,v,w) - B(t,T,v,w)r\right], \end{aligned} \tag{3.3.19}$$

where

$$\tau = T - t, \quad B(t,T,v,w) = vB_1(t,T) + wB_2(t,T),$$

$$B_1(t,T) = \frac{1 - e^{-\beta\tau}}{\beta} \quad \text{and} \quad B_2(t,T) = e^{-\beta\tau}.$$

The first term in exponential is

$$\begin{aligned} A(t,T,v,w) &= -vA_1(t,T) - wA_2(t,T) + \frac{1}{2}v^2C_{11}(t,T) \\ &\quad + vwC_{12}(t,T) + \frac{1}{2}w^2C_{22}(t,T), \end{aligned}$$

where

$$A_1(t,T) = \alpha\left(\tau - \frac{1 - e^{-\beta\tau}}{\beta}\right), \quad A_2(t,T) = \alpha\left(1 - e^{-\beta\tau}\right),$$

and C_{11}, C_{12} and C_{22} stand for

$$C_{11}(t,T) = \frac{\sigma^2}{2\beta^3}\left[2\beta\tau - 3 + 4e^{-\beta\tau} - e^{-2\beta\tau}\right],$$

$$C_{12}(t,T) = \frac{\sigma^2}{2\beta^2}\left(1 - e^{-\beta\tau}\right)^2, \quad C_{22}(t,T) = \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta\tau}\right).$$

b) Hence $\int_t^T r(s)ds$ *and* $r(T)$ *given* $r(t)$ *have a bivariate normal distribution under* $\mathbf{Q}$ *with*

$$\begin{aligned} E^Q\left[r(T) \mid r(t)\right] &= B_2(t,T)r(t) + A_2(t,T) = \alpha + (r(t)-\alpha)e^{-\beta\tau}, \\ E^Q\left[\int_t^T r(s)ds \,\middle|\, r(t)\right] &= B_1(t,T)r(t) + A_1(t,T) = \alpha\tau + (r(t)-\alpha)\frac{\left(1-e^{-\beta\tau}\right)}{\beta}, \end{aligned}$$

for the expectations. Moreover, the variances of $r(T)$ and $\int_t^T r(s)ds$ given $r(t)$ are

$$\begin{aligned} Var^Q[r(T) \mid r(t)] &= C_{22}(t,T) = \frac{\sigma^2}{2\beta}\left(1-e^{-2\beta\tau}\right), \\ Var^Q\left[\int_t^T r(s)ds \,\middle|\, r(t)\right] &= C_{11}(t,T) = \frac{\sigma^2}{2\beta^3}\left[2\beta\tau - 3 + 4e^{-\beta\tau} - e^{-2\beta\tau}\right]. \end{aligned}$$

Furthermore, covariance of $r(T)$ and $\left(\int_t^T r(s)ds \,\middle|\, r(t)\right)$ is

$$Cov^Q\left[r(T),\ \int_t^T r(s)ds \,\middle|\, r(t)\right] = C_{12}(t,T) = \frac{\sigma^2}{2\beta^2}\left(1-e^{-\beta\tau}\right)^2.$$

The proof of this lemma can be found in [6]. However, below we explain the proof of (3.3.18)

Proof of Equation (3.3.18**).** The payoff on the call option at time T is

$$(P(T,S)-K)_+ := \max\{0, P(T,S)-K\}.$$

Define the indicator random variable:

$$I = \begin{cases} 1, & \text{if } P(T,S,r(T)) > K \\ 0, & \text{otherwise.} \end{cases}$$

The value of the option at time $t < T$ can be written as in [18]

$$\begin{aligned}
V(t) &= E^Q\left[e^{-\int_t^T r(s)ds}(P(T,S,r(T)) - K)_+ \,\middle|\, r(t)\right] \\
&= E^Q\left[Ie^{-\int_t^T r(s)ds}(P(T,S,r(T))) \,\middle|\, r(t)\right] - KE^Q\left[Ie^{-\int_t^T r(s)ds} \,\middle|\, r(t)\right] \\
&= E^Q\left[Ie^{-\int_t^S r(s)ds} \,\middle|\, r(t)\right] - KE^Q\left[Ie^{-\int_t^T r(s)ds} \,\middle|\, r(t)\right].
\end{aligned} \tag{3.3.20}$$

Let $\mathbf{P_1}$ and $\mathbf{P_2}$ be two new measures, equivalent to $\mathbf{Q}$ with Radon-Nikodym derivatives:

$$\frac{dP_1}{dQ} = \frac{e^{-\int_t^S r(s)ds}}{E^Q\left[e^{-\int_t^S r(s)ds} \mid r(t) = r\right]}, \quad \frac{dP_2}{dQ} = \frac{e^{-\int_t^T r(s)ds}}{E^Q\left[e^{-\int_t^T r(s)ds} \mid r(t) = r\right]}. \tag{3.3.21}$$

Then, from equation (3.3.20) we have

$$\begin{aligned}
V(t) &= E^Q\left[e^{-\int_t^S r(s)ds} \,\middle|\, r(t) = r\right] E^Q\left[\frac{dP_1}{dQ} I \,\middle|\, r(t) = r\right] \\
&\quad -KE^Q\left[e^{-\int_t^T r(s)ds} \,\middle|\, r(t) = r\right] E^Q\left[\frac{dP_2}{dQ} I \,\middle|\, r(t) = r\right] \\
&= P(t,S,r)E^{P_1}[I \mid r(t) = r] - KP(t,T,r)E^{P_2}[I \mid r(t) = r].
\end{aligned}$$

Since $E^{P_1}[I \mid r(t) = r]$ can be written as $Pr^{P_1}(P(T,S) > K \mid r(t) = r)$, the value of a call option $V(t)$ takes the form

$$\begin{aligned}
V(t) &= P(t,S,r)Pr^{P_1}(P(T,S) > K \mid r(t) = r) \\
&\quad -KP(t,T,r)Pr^{P_2}(P(T,S) > K \mid r(t) = r).
\end{aligned}$$

It remains only to establish the distribution of $r(T)$ under $\mathbf{P_1}$ and $\mathbf{P_2}$. Let us first look at $\mathbf{P_2}$:

$$\begin{aligned}
P_L(t,T,r,1,w) &= \exp[A(t,T,1,w) - B(t,T,1,w)r] \\
&= E^Q\left[e^{-\int_t^T r(s)ds - wr(T)} \,\middle|\, r(t) = r\right] \\
&= P(t,T,r)E^{P_2}\left[e^{-wr(T)} \mid r(t) = r\right],
\end{aligned}$$

hence, we have

$$E^{P_2}\left[e^{-wr(T)} \mid r(t)=r\right]$$

$$= \exp\left[A(t,T,1,w) - A(t,T,1,0) - (B(t,T,1,w) - B(t,T,1,0))r\right]$$

$$= \exp\left[-wA_2(t,T) + wC_{12}(t,T) + \frac{1}{2}w^2 C_{22}(t,T) - wB_2(t,T)r\right].$$

Comparing this equation with the moment generating function of normal distribution, $\exp\left(\mu t + \dfrac{\sigma^2 t^2}{2}\right)$, shows that $r(T)$ given $r(t)=r$ has normal distribution under $\mathbf{P_2}$ with expectation

$$\begin{aligned}
E^{P_2}[r(T) \mid r(t)=r] &= A_2(t,T) - C_{12}(t,T) + B_2(t,T)r \\
&= \alpha + (r-\alpha)e^{-\beta(T-t)} - \frac{\sigma^2}{\beta^2}\left(1 - e^{-\beta(T-t)}\right)^2 \\
&=: r_2,
\end{aligned}$$

and the variance

$$Var^{P_2}[r(T) \mid r(t)=r] = C_{22}(t,T) = \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta(T-t)}\right).$$

Hence, we obtain the standard normal distribution expressed as

$$Pr^{P_2}(r(T) < r^* \mid r(t)=r) = \Phi(d_2),$$

where

$$r^* = \frac{A(T,S) - \log K}{B(T,S)}. \tag{3.3.22}$$

Here as $\exp(A(T,S) - B(T,S)r^*) > K$, we can choose r^* given in (3.3.22). Also, from Lemma (3.3.1),

$$\begin{aligned}
P(t,S) &= E^Q\left[e^{-\int_t^T r(s)ds - \int_T^S r(s)ds} \,\Big|\, r(t)=r\right] \\
&= P(t,T)E^{P_2}[P(T,S,r(T)) \mid r(t)=r],
\end{aligned}$$

which can be expressed as

$$\begin{aligned}
P(t,S) &= P(t,T)E^{P_2}\left[e^{A(T,S)-B(T,S)r(T)}\,\middle|\, r(t)=r\right] \\
&= P(t,T)\exp\left[A(T,S)-B(T,S)r_2+\frac{1}{2}B(T,S)^2C_{22}(t,T)\right].
\end{aligned}$$

Hence, it follows that

$$\log\frac{P(t,S)}{KP(t,T)} = A(T,S)-B(T,S)r_2+\frac{1}{2}B(T,S)^2C_{22}(t,T)-\log K.$$

Therefore,

$$\begin{aligned}
d_2 &= \frac{r^*-r_2}{\sqrt{C_{22}(t,T)}} = \frac{A(T,S)-\log K-B(T,S)r_2}{B(T,S)\sqrt{C_{22}(t,T)}} \\
&= \frac{\log(P(t,S)/KP(t,T))-\frac{1}{2}B(T,S)^2C_{22}(t,T)}{B(T,S)\sqrt{C_{22}(t,T)}},
\end{aligned}$$

which simplifies to

$$d_2 = \frac{1}{\sigma_p}\log\frac{P(t,S)}{KP(t,T)}-\frac{\sigma_p}{2},$$

where

$$\sigma_p = B(T,S)^2C_{22}(t,T) = \sigma^2\frac{\left(1-e^{-\beta(S-T)}\right)^2}{\beta^2}\frac{\left(1-e^{-2\beta(T-t)}\right)}{2\beta}.$$

Next, we consider the distribution of $r(T)$ under $\mathbf{P_1}$:

$$E^Q\left[e^{-\int_t^S r(s)ds-wr(T)}\,\middle|\, r(t)=r\right] = P(t,S)E^{P_1}\left[e^{-wr(T)}\,\middle|\, r(t)=r\right].$$

Here, we compute

$$\begin{aligned}
&E^Q\left[e^{-\int_t^S r(s)ds-wr(T)}\,\middle|\, r(t)=r\right] \\
&= E^Q\left[e^{-\int_t^T r(s)ds-wr(T)}E^Q\left[e^{-\int_T^S r(s)ds}\,\middle|\, r(T)\right]\,\middle|\, r(t)=r\right] \\
&= E^Q\left[e^{-\int_t^T r(s)ds-wr(T)+A(T,S,1,0)-B(T,S,1,0)r(T)}\,\middle|\, r(t)=r\right].
\end{aligned}$$

This equation can further be simplified as follows:

$$E^Q\left[e^{-\int_t^S r(s)ds - wr(T)}\,\middle|\, r(t)=r\right]$$

$$= e^{A(T,S,1,0)}E^Q\left[e^{-\int_t^T r(s)ds - w_2 r(T)}\,\middle|\, r(t)=r\right]$$

$$= \exp[A(T,S,1,0) + A(t,T,1,w_2) - B(t,T,1,w_2)r],$$

where we set w_2 as

$$w_2 = w + B(T,S,1,0) = w + \frac{1-e^{-\beta(S-T)}}{\beta}.$$

Hence, similarly as in computations for $\mathbf{P_2}$, we calculate

$$E^{P_1}\left[e^{-wr(T)} \mid r(t)=r\right] = \exp[A(T,S,1,0) + A(t,T,1,w_2)$$

$$-A(t,S,1,0) - (B(t,T,1,w_2) - B(t,S,1,0))r],$$

so that

$$E^{P_1}\left[e^{-wr(T)} \mid r(t)=r\right]$$

$$= \exp[-A_1(T,S) + \frac{1}{2}C_{11}(T,S) - A_1(t,T) - w_2 A_2(t,T)$$

$$+\frac{1}{2}C_{11}(t,T) + w_2 C_{12}(t,T) + \frac{1}{2}w_2^2 C_{22}(t,T) + A_1(t,S)$$

$$-\frac{1}{2}C_{11}(t,S) - \{B_1(t,T) + w_2 B_2(t,T) - B_1(t,S)\}r].$$

After some simplifications this turns out to be

$$E^{P_1}\left[e^{-wr(T)} \mid r(t)=r\right]$$

$$= \exp[-wA_2(t,T) + wC_{12}(t,T) + wB_1(T,S)C_{22}(t,T)$$

$$+\frac{1}{2}w^2 C_{22}(t,T) - wB_2(t,T)r].$$

Thus $r(T)$ given $r(t) = r$ is normally distributed under $\mathbf{P_1}$ with expectation

$$\begin{aligned} E^{P_1}\left[r(T) \mid r(t) = r\right] &= A_2(t,T) - C_{12}(t,T) - B_1(T,S)C_{22}(t,T) + B_2(t,T)r \\ &= r_2 - \sigma^2 \frac{\left(1 - e^{-\beta(S-T)}\right)}{\beta} \frac{\left(1 - e^{-2\beta(T-t)}\right)}{2\beta} \\ &=: r_1, \end{aligned}$$

and the variance

$$Var^{P_1}[r(T) \mid r(t) = r] = C_{22}(t,T) = \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta(T-t)}\right).$$

Therefore, it can be expressed as

$$Pr^{P_1}(r(T) < r^*) = \Phi(d_1),$$

where

$$\begin{aligned} d_1 &= \frac{r^* - r_1}{\sqrt{C_{22}(t,T)}} = \frac{r^* - r_2}{\sqrt{C_{22}(t,T)}} + \frac{B(T,S)C_{22}(t,T)}{\sqrt{C_{22}(t,T)}} \\ &= d_2 + \sigma_p. \end{aligned}$$

3.4 Cox Ingersoll Ross Model for Option Pricing

Let C be the price at time 0 of a European call option on the zero coupon bond maturing at time $U = T + \tau$ with exercise date T and an exercise price K. Then given $r(0) = r$,

$$C = P(0,U,r)\chi^2(d,\lambda_1;y_1) - KP(0,T,r)\chi^2(d,\lambda_2;y_2) \tag{3.4.23}$$

Where $\chi^2(d,\lambda;y)$ is the cumulative distribution function of the non-central chi-squared distribution with d degrees of freedom and non-centrality parameter λ.

The required inputs d, λ_1, λ_2, y_1 and y_2 are calculated as follows

$$\begin{aligned}
d &= \frac{4\beta\alpha}{\sigma^2}, \\
\lambda_1 &= \frac{8\gamma^2 e^{\gamma T} r}{\sigma^2\left(e^{\gamma T}-1\right)\left(2\gamma+(\gamma+\beta+\sigma^2\tilde{B}(U-T))(e^{\gamma T}-1)\right)}, \\
\lambda_2 &= \frac{8\gamma^2 e^{\gamma T} r}{\sigma^2(e^{\gamma T}-1)(2\gamma+(\gamma+\beta)(e^{\gamma T}-1))},
\end{aligned}$$

for the degrees of freedom and non-centrality parameter. Denote

$$\begin{aligned}
A(t,T,v,w) &= \frac{2\beta\alpha}{\sigma^2}\log\left(\frac{2\gamma(v)e^{(\gamma(v)+\beta)(T-t)/2}}{(\sigma^2 w+\gamma(v)+\beta)(e^{\gamma(v)(T-t)}-1)+2\gamma(v)}\right), \\
\gamma(v) &= \sqrt{\beta^2+2\sigma^2 v},
\end{aligned}$$

and

$$B(t,T,v,w) = \frac{w(2\gamma(v)+(\gamma(v)-\beta)(e^{\gamma(v)(T-t)}-1))+2v(e^{\gamma(v)(T-t)}-1)}{(\sigma^2 w+\gamma(v)+\beta)(e^{\gamma(v)(T-t)}-1)+2\gamma(v)}.$$

By taking $v=1$ and $w=0$, it follows that

$$\begin{aligned}
\tilde{A}(\tau) &= \frac{2\beta\alpha}{\sigma^2}\log\left(\frac{2\gamma e^{(\gamma+\beta)\tau/2}}{(\gamma+\beta)(e^{\gamma\tau}-1)+2\gamma}\right), \\
\gamma &= \sqrt{\beta^2+2\sigma^2}, \\
\tilde{B}(\tau) &= \frac{2(e^{\gamma\tau}-1)}{(\gamma+\beta)(e^{\gamma\tau}-1)+2\gamma}.
\end{aligned}$$

On the other hand, the variables of the distribution are

$$y_1 = \frac{r^*}{k_1} \quad \text{and} \quad y_2 = \frac{r^*}{k_2},$$

where

$$\begin{aligned}
k_1 &= \frac{\sigma^2(e^{\gamma T}-1)}{2(2\gamma+(\gamma+\beta+\sigma^2\tilde{B}(U-T))(e^{\gamma T}-1))}, \\
k_2 &= \frac{\sigma^2(e^{\gamma T}-1)}{2(2\gamma+(\gamma+\beta)(e^{\gamma T}-1))}, \\
r^* &= \frac{\tilde{A}(U-T)-\log K}{\tilde{B}(U-T)}.
\end{aligned}$$

Now, we will derive the equation in (3.4.23). For notational convenience and without loss of generality we will assume that $t = 0$. The price at time zero of a European call option with maturing T and strike price K with the underlying zero coupon bond maturing at time $U = T + \tau > T$ is

$$C = E^Q \left[e^{-\int_0^T r(s)ds} (P(T, U, r(T)) - K)_+ \,\Big|\, r(0) = r \right].$$

Let us consider under what circumstances will the call option be exercised; that is, if and only if

$$P(T, U, r(T)) > K \Leftrightarrow e^{\tilde{A}(U-T) - \tilde{B}(U-T)r(T)} > K$$

$$\Leftrightarrow r(T) < \frac{\tilde{A}(U-T) - \log K}{\tilde{B}(U-T)} = r^*.$$

Thus, the call option price of the Cox Ingersoll Ross Model becomes

$$\begin{aligned} C &= E^Q \left[e^{-\int_0^T r(s)ds} P(T, U, r(T)) I(r(T) < r^*) \,\Big|\, r(0) = r \right] \\ &\quad - E^Q \left[e^{-\int_0^T r(s)ds} K I(r(T) < r^*) \,\Big|\, r(0) = r \right] \\ &= E^Q \left[e^{-\int_0^U r(s)ds} I(r(T) < r^*) \,\Big|\, r(0) = r \right] \\ &\quad - E^Q \left[e^{-\int_0^T r(s)ds} K I(r(T) < r^*) \,\Big|\, r(0) = r \right]. \end{aligned} \tag{3.4.24}$$

Let $\mathbf{P_1}$ and $\mathbf{P_2}$ be two new measures, equivalent to $\mathbf{Q}$ with Radon-Nikodym derivatives:

$$\frac{dP_1}{dQ} = \frac{e^{-\int_0^U r(s)ds}}{E^Q \left[e^{-\int_0^U r(s)ds} \mid r(0) = r \right]}, \qquad \frac{dP_2}{dQ} = \frac{e^{-\int_0^T r(s)ds}}{E^Q \left[e^{-\int_0^T r(s)ds} \mid r(0) = r \right]}.$$

Then, from equation (3.4.24) we have

$$\begin{aligned} C &= E^Q\left[e^{-\int_0^U r(s)ds} \,\middle|\, r(0)=r\right] E^Q\left[\frac{dP_1}{dQ} I(r(T)<r^*) \,\middle|\, r(0)=r\right] \\ &\quad - E^Q\left[e^{-\int_0^T r(s)ds} \,\middle|\, r(0)=r\right] E^Q\left[\frac{dP_2}{dQ} K I(r(T)<r^*) \,\middle|\, r(0)=r\right] \\ &= P(0,U,r)E^{P_1}[I(r(T)<r^*) \mid r(0)=r] \\ &\quad - KP(0,T,r)E^{P_2}[I(r(T)<r^*) \mid r(0)=r]. \end{aligned}$$

Since $E^{P_1}[I(r(T)<r^*) \mid r(0)=r]$ can be written as $Pr^{P_1}(r(T)) < r^* \mid r(0)=r)$, the value of a call option C becomes

$$\begin{aligned} C &= P(0,U,r)Pr^{P_1}(r(T)<r^* \mid r(0)=r) \\ &\quad - KP(0,T,r)Pr^{P_2}(r(T)<r^* \mid r(0)=r). \end{aligned}$$

In order to establish the distribution of $r(T)$ under $\mathbf{P_1}$ and $\mathbf{P_2}$, let us first consider $\mathbf{P_2}$:

$$E^Q\left[e^{-\int_0^T r(s)ds - wr(T)} \,\middle|\, r(0)=r\right] = P(0,T,r)E^{P_2}\left[e^{-wr(T)} \,\middle|\, r(0)=r\right].$$

Thus, we ahve

$$\begin{aligned} & E^{P_2}\left[e^{-wr(T)} \mid r(0)=r\right] \\ &= \exp[A(0,T,1,w) - A(0,T,1,0) - (B(0,T,1,w) - B(0,T,1,0))r]. \end{aligned}$$

By inserting A and B for $v=1$, and using $\gamma = \sqrt{\beta^2+2\sigma^2}$, the expectation can be written as

$$\begin{aligned} E^{P_2}\left[e^{-wr(T)} \mid r(0)=r\right] &= \left(\frac{(\gamma+\beta)(e^{\gamma T}-1)+2\gamma}{(\sigma^2 w+\gamma+\beta)(e^{\gamma T}-1)+2\gamma}\right)^{2\beta\alpha/\sigma^2} \\ \exp\left[-\left(\frac{w(2\gamma+(\gamma-\beta)(e^{\gamma T}-1))+2(e^{\gamma T}-1)}{(\sigma^2 w+\gamma+\beta)(e^{\gamma T}-1)+2\gamma} - \frac{2(e^{\gamma T}-1)}{(\gamma+\beta)(e^{\gamma T}-1)+2\gamma}\right) r\right]&. \end{aligned}$$

Now we concentrate on the terms involving w to establish the form of the Laplace transform. Within the exponential term we have

$$\begin{aligned} X &:= \frac{w(2\gamma + (\gamma - \beta)(e^{\gamma T} - 1)) + 2(e^{\gamma T} - 1)}{(\sigma^2 w + \gamma + \beta)(e^{\gamma T} - 1) + 2\gamma} \\ &= \frac{w(2\gamma + (\gamma - \beta)(e^{\gamma T} - 1)) + 2(e^{\gamma T} - 1)}{(2\gamma + (\gamma + \beta)(e^{\gamma T} - 1))(1 + 2k_2 w)}, \end{aligned}$$

where

$$k_2 = \frac{\sigma^2 (e^{\gamma T} - 1)}{2(2\gamma + (\gamma + \beta)(e^{\gamma T} - 1))}.$$

This can be further simplified to

$$X = \frac{\theta(1 + 2k_2 w) + \phi}{(2\gamma + (\gamma + \beta)(e^{\gamma T} - 1))(1 + 2k_2 w)},$$

by indicating θ and ϕ as

$$\begin{aligned} \theta &= \frac{r}{\sigma^2 \left(e^{\gamma T} - 1\right)} (4\gamma^2 e^{\gamma T} + 2\sigma^2 (e^{\gamma T} - 1)^2), \\ \phi &= 2(e^{\gamma T} - 1) r - \theta = -\frac{4\gamma^2 e^{\gamma T} r}{\sigma^2 (e^{\gamma T} - 1)}. \end{aligned}$$

Similarly, the first part of expectation can be expressed as

$$Y := \left(\frac{(\gamma + \beta)(e^{\gamma T} - 1) + 2\gamma}{(\sigma^2 w + \gamma + \beta)(e^{\gamma T} - 1) + 2\gamma} \right)^{2\beta\alpha/\sigma^2} = \left(\frac{1}{1 + 2k_2 w} \right)^{d/2} \times \text{constant},$$

where

$$d = 4\beta\alpha/\sigma^2.$$

Hence,

$$E^{P_2}\left[e^{-(k_2 w)(r(T)/k_2)} \mid r(0) = r \right] = \left(\frac{1}{1 + 2k_2 w} \right)^{d/2} \exp\left(\frac{\lambda_2}{2(1 + 2k_2 w)} \right) \times \text{constant},$$

for λ_2 as

$$\lambda_2 = \frac{8\gamma^2 e^{\gamma T} r}{\sigma^2 (e^{\gamma T} - 1)(2\gamma + (\gamma + \beta)(e^{\gamma T} - 1))}.$$

Compare the Laplace transform with equation

$$\begin{aligned}
E\left[e^{-kR}\right] &= \prod_{i=1}^{d} E\left[e^{-k(W_i+\delta_i)^2}\right] \\
&= \prod_{i=1}^{d} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-k(w^2+2\delta_i w+\delta_i^2)-\frac{1}{2}w^2\right) dw \\
&= \prod_{i=1}^{d} (1+2k)^{-1/2} \exp\left(-\frac{k}{1+2k}\delta_i^2\right),
\end{aligned}$$

yields

$$E\left[e^{-kR}\right] = (1+2k)^{-d/2} e^{-\lambda/2} \exp\left(\frac{\lambda}{2(1+2k)}\right),$$

where

$$\lambda = \sum_{i=1}^{d} \delta_i^2.$$

Note that we should also mention that the transform is defined for $k > -1/2$.

Therefore, under $\mathbf{P_2}$, $r(T)/k_2$ has a non-central chi-squared distribution with d degrees of freedom and non-centrality parameter λ_2:

$$Pr^{P_2}(r(T) < r^* \mid r(0) = r) = Pr^{P_2}(r(T)/k_2 < r^*/k_2 \mid r(0) = r) = \chi^2(d, \lambda_2; y_2),$$

where $y_2 = r^*/k_2$ and $\chi^2(d, \lambda; y)$ is the cumulative distribution function of the non-central chi-squared distribution with d degrees of freedom non-centrality parameter λ.

Next we consider the distribution of $r(T)$ under $\mathbf{P_1}$:

$$E^Q\left[e^{-\int_0^U r(s)ds - wr(T)} \,\middle|\, r(0) = r\right] = P(0,U,r) E^{P_1}\left[e^{-wr(T)} \,\middle|\, r(0) = r\right].$$

The expectation in this case becomes

$$\begin{aligned}
&E^Q\left[e^{-\int_0^U r(s)ds - wr(T)} \,\middle|\, r(0) = r\right] \\
&= E^Q\left[e^{-\int_0^T r(s)ds - wr(T)} E^Q\left[e^{-\int_0^U r(s)ds} \,\middle|\, r(T)\right] \,\middle|\, r(0) = r\right] \\
&= E^Q\left[e^{-\int_0^T r(s)ds - wr(T) + A(T,U,1,0) - B(T,U,1,0)r(T)} \,\middle|\, r(0) = r\right].
\end{aligned}$$

Taking the term $A(T,U,1,0)$ outside of the expectation follows

$$
\begin{aligned}
& E^Q\left[e^{-\int_0^U r(s)ds - wr(T)} \,\middle|\, r(0)=r\right] \\
&= \exp(A(T,U,1,0))E^Q\left[e^{-\int_0^T r(s)ds - w_1 r(T)} \,\middle|\, r(0)=r\right] \\
&= \exp(A(T,U,1,0)+A(0,T,1,w_1)-B(0,T,1,w_1)r),
\end{aligned}
$$

where we have denoted w_1 as

$$
w_1 = w + B(T,U,1,0) = w + \tilde{B}(U-T).
$$

Hence,

$$
\begin{aligned}
E^{P_1}\left[e^{-wr(T)} \,\middle|\, r(0)=r\right] &= E^{P_1}\left[e^{-\int_0^U r(s)ds - wr(T)} \,\middle|\, r(0)=r\right] / P(0,U,r) \\
&= \exp[A(T,U,1,0)+A(0,T,1,w_1)-A(0,U,1,0) \\
&\quad -(B(0,T,1,w_1)-B(0,U,1,0))r].
\end{aligned}
$$

Similarly, we concentrate on the terms involving w_1 to establish the form of the Laplace transform. First, we have

$$
\begin{aligned}
\exp\left(A(0,T,1,w_1)\right) &= \left(\frac{2\gamma e^{(\gamma+\beta)T/2}}{(\sigma^2 w_1+\gamma+\beta)(e^{\gamma T}-1)+2\gamma}\right)^{2\beta\alpha/\sigma^2} \\
&= \left(\frac{2\gamma e^{(\gamma+\beta)T/2}}{(\sigma^2 w+\gamma+\beta+\sigma^2\tilde{B}(U-T))(e^{\gamma T}-1)+2\gamma}\right)^{2\beta\alpha/\sigma^2} \\
&= \left(\frac{1}{1+2k_1 w}\right)^{2\beta\alpha/\sigma^2} \times \text{constant},
\end{aligned}
$$

where

$$
k_1 = \frac{\sigma^2(e^{\gamma T}-1)}{2(2\gamma+(\gamma+\beta+\sigma^2\tilde{B}(U-T))(e^{\gamma T}-1)}.
$$

Next, consider

$$
\begin{aligned}
B(0,T,1,w_1)r &= \frac{((w+\tilde{B}(U-T))(2\gamma+(\gamma-\beta)(e^{\gamma T}-1))+2(e^{\gamma T}-1))r}{(\sigma^2 w+\gamma+\beta+\sigma^2\tilde{B}(U-T))(e^{\gamma T}-1)+2\gamma} \\
&= \frac{\theta(1+2k_1 w)+\phi}{(2\gamma+(\gamma+\beta+\sigma^2\tilde{B}(U-T))(e^{\gamma T}-1))(1+2k_1 w)}.
\end{aligned}
$$

To simplify the equation, we denote

$$\begin{aligned}\theta \;&:= \frac{(2\gamma + (\gamma-\beta)(e^{\gamma T}-1))(2\gamma + (\gamma+\beta+\sigma^2\tilde{B}(U-T))(e^{\gamma T}-1))r}{\sigma^2(e^{\gamma T})-1} \\ &= \frac{r(4\gamma^2 e^{\gamma T} + 2\sigma^2(e^{\gamma T}-1)^2 + \sigma^2\tilde{B}(U-T)(e^{\gamma T}-1)(2\gamma+(\gamma-\beta)(e^{\gamma T}-1)))}{\sigma^2(e^{\gamma T}-1)},\end{aligned}$$

and

$$\phi := 2(e^{\gamma T}-1)r + \tilde{B}(U-T)(2\gamma + (\gamma-\beta)(e^{\gamma T}-1))r - \theta = -\frac{4\gamma^2 e^{\gamma T} r}{\sigma^2(e^{\gamma T}-1)}.$$

Hence, the expectation can be expressed as

$$E^{P_1}\left[e^{-(k_1 w)(r(T)/k_1)} \,\middle|\, r(0) = r\right] = \left(\frac{1}{1+2k_1 w}\right)^{d/2} \exp\left(\frac{\lambda_1}{2(1+2k_1 w)}\right) \times \text{constant},$$

where λ_1 is given by

$$\lambda_1 = \frac{8\gamma^2 e^{\gamma T} r}{\sigma^2(e^{\gamma T}-1)(2\gamma + (\gamma+\beta+\sigma^2\tilde{B}(U-T))(e^{\gamma T}-1)}.$$

Note that if $U = T$, then $k_1 = k_2$ and $\lambda_1 = \lambda_2$, as expected.

Therefore, under $\mathbf{P_1}$, $r(T)/k_1$ has a non-central chi-squared distribution with d degrees of freedom and non-centrality parameter λ_1. Thus, we derive the expression (3.4.23) as

$$Pr^{P_1}(r(T) < r^*) = \chi^2(d, \lambda_1; y_1),$$

where

$$y_1 = \frac{r^*}{k_1}.$$

As a result, we showed how the required inputs be calculated for pricing European zero coupon bond option in Cox Ingersoll Ross Model. Now, we are able to calculate the price of options with (3.3.18) for the Vasicek Model and with (3.4.23) for the Cox Ingersoll Ross Model.

3.5 Ho Lee Binomial Tree

Ho Lee Model is given by

$$dr_t = \theta_t dt + \sigma dW_t.$$

A numerical approximation for Ho Lee Model [21] based on Euler-Maruyama method is given by

$$r_{k+1} = r_k + \theta_k \tau + \sigma_k \Delta W_k,$$

where ΔW_k is an approximation of dW_t and τ stands for the time steps. Since $dW_t = \varepsilon\sqrt{dt}$, we may write

$$r_{k+1} = r_k + \theta_k \tau + \sigma_k \varepsilon_k \sqrt{\tau}.$$

Note that ε_k is a random number drawn from a standard normal distribution. In the Ho Lee binomial tree, the general form of the expression for $r_{k,j}$ is

$$r_{k+1,j+1} = r_{k,j} + m_k \tau + \sigma_k \sqrt{\tau},$$

for an up move and

$$r_{k+1,j} = r_{k,j} + m_k \tau - \sigma_k \sqrt{\tau},$$

for a down move which can be seen in Figure 3.1.

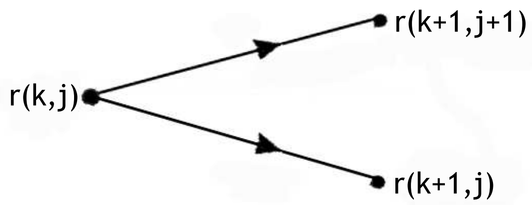

Figure 3.1: Ho Lee Model One Period Binomial Tree

In these equations m_k stands for the numerical approximation to the drift value θ_k. Now, for the first step, we have

$$r_{1,2} = r_0 + m_0 \tau + \sigma_0 \sqrt{\tau}, \tag{3.5.25}$$

and

$$r_{1,1} = r_0 + m_0\tau - \sigma_0\sqrt{\tau}. \tag{3.5.26}$$

The, rates for the second step are

$$\begin{aligned} r_{2,3} &= r_{1,2} + m_1\tau + \sigma_1\sqrt{\tau}, \\ r_{2,2} &= r_{1,2} + m_1\tau - \sigma_1\sqrt{\tau} \\ &= r_{1,1} + m_1\tau + \sigma_1\sqrt{\tau}. \end{aligned}$$

Finally the down-down scenario denoted by $r_{2,1}$, can be written as

$$r_{2,1} = r_{1,1} + m_1\tau - \sigma_1\sqrt{\tau}.$$

Since Ho Lee binomial tree is recombining lattice we can write $r_{2,2}$ as up move from $r_{1,1}$ and down move from $r_{1,2}$. A down movement from $r_{1,2}$ must be equal to an up movement from $r_{1,1}$

$$r_{1,2} + m_1\tau - \sigma_1\sqrt{\tau} = r_{1,1} + m_1\tau + \sigma_1\sqrt{\tau}. \tag{3.5.27}$$

Solving σ_1 from equation (3.5.27) yields

$$\sigma_1 = \frac{r_{1,2} - r_{1,1}}{2\sqrt{\tau}},$$

and using $r_{1,1}$ and $r_{1,2}$ from equations (3.5.25) and (3.5.26), we find that

$$\sigma_1 = \frac{2\sigma_0\sqrt{\tau}}{2\sqrt{\tau}} = \sigma_0.$$

By a simple induction, on the Ho Lee binomial tree, volatility is constant over time, that is: $\sigma_k = \sigma$.

Therefore,

$$r_{k+1,j+1} - r_{k,j} = m_k\tau + \sigma\sqrt{\tau},$$

and

$$r_{k+1,j+1} - r_{k+1,j} = 2\sigma\sqrt{\tau}.$$

Furthermore, the relation between the states 1 and $k+1$ at time k is simply

$$r_{k,k+1} - r_{k,1} = 2k\sigma\sqrt{\tau}.$$

3.6 Black Derman Toy Binomial Tree

Black Derman Toy Model is given by the stochastic differential equation

$$d \log r_t = [\theta_t + \rho_t \log r_t]dt + \sigma_t dW_t,$$

where

$$\rho_t = \frac{d}{dt} \log \sigma_t = \frac{\sigma_t{}'}{\sigma_t}.$$

Or equivalently,

$$d \log r_t = \left[\theta_t + \frac{\sigma_t{}'}{\sigma_t} \log r_t\right] dt + \sigma_t dW_t, \tag{3.6.28}$$

and setting u_t for $\log r_t$

$$du_t = \left[\theta_t + \frac{\sigma_t{}'}{\sigma_t} u_t\right] dt + \sigma_t dW_t \tag{3.6.29}$$

If σ_t is a decreasing function, then $\sigma_t{}'$ becomes smaller than zero. In this case, Black Derman Toy Model satisfies the mean reversion property. On contrary, if σ_t is an increasing function, then $\sigma_t{}'$ becomes greater than zero. In this case, Black Derman Toy will grow and it has no mean reversion effect. If σ_t is a constant function, then $\sigma_t{}'$ be equal to 0. Then the Black Derman Toy Model becomes a specific model

$$d \log r_t = \theta dt + \sigma_t dW_t,$$

which is the so-called Kalotay Williams Fabozzi Model [24].

The expected value of equation (3.6.29) can be expressed as

$$du_t = \left[\theta_t + \frac{\sigma_t{}'}{\sigma_t} u_t\right] dt. \tag{3.6.30}$$

Solution of the first order linear differential equation (3.6.30) for u_t, is given by

$$u_t = \left[\frac{u_0}{\sigma_0} + \int_0^t \frac{\theta_s}{\sigma_s} ds\right] \sigma_t,$$

and substituting $\log r_t$ for u_t gives

$$\begin{aligned} r_t &= \exp\left(\left[\frac{\log r_0}{\sigma_0} + \int_0^t \frac{\theta_s}{\sigma_s} ds\right] \sigma_t\right) \\ &= \exp\left(\frac{\sigma_t \log r_0}{\sigma_0}\right) \exp\left(\sigma_t \int_0^t \frac{\theta_s}{\sigma_s} ds\right). \end{aligned} \tag{3.6.31}$$

Multiplying and dividing by r_0, this follows that

$$r_t = r_0 \exp\left(\frac{\sigma_t - \sigma_0}{\sigma_0} \log r_0\right) \exp\left(\sigma_t \int_0^t \frac{\theta_s}{\sigma_s} ds\right).$$

From (3.6.30), it is clear that the expected value of Black Derman Toy Model depends on the volatility term σ_t. If σ_t is a decreasing function, then the first term of the equation above will have negative power and will motivate a decrease in the short rate. If σ_t is an increasing function, then the term will have positive power and will motivate an increase in the short rate. It is crucial to note that mean reversion property of Black Derman Toy Model comes from the volatility parameter.

If θ is constant, then the equation simplifies to

$$r_t = r_0 \exp\left(\frac{\sigma_t - \sigma_0}{\sigma_0} \log r_0\right) \exp\left(\theta \sigma_t \int_0^t \frac{1}{\sigma_s} ds\right).$$

In general the volatility term σ_t is very small so that the term $\dfrac{1}{\sigma_s}$ becomes large. Moreover integrating this value causes a larger value for this part. Therefore, in the second term of equation, the exponential part most probably becomes large, and the smaller volatility values, can cause the unboundness of Black Derman Toy Model. Suppose the case

$$\frac{\sigma_t{}'}{\sigma_t} = a,$$

and a is constant. In this case, the solution of (3.6.30) is simply given by

$$u_t = \left[u_0 + \frac{\theta}{a}\right] \exp(at) - \frac{\theta}{a}.$$

Since $0 < r < 1$, $u_0 = \log r_0$ will be negative, so that $u_0 + \dfrac{\theta}{a}$ could be positive or negative depending on the sign and the magnitude of the drift θ. For $a > 0$ and $u_0 + \dfrac{\theta}{a} < 0$, then $u_t \to -\infty$ and thus $r_t = \exp(u_t) \to 0$. Moreover, for $a > 0$, if $u_0 + \dfrac{\theta}{a} > 0$, then, $u_t \to \infty$ and $r_t = \exp(u_t) \to \infty$. So, we can conclude that for $a > 0$, the Black Derman Toy Model's short rate may either explode or converge to zero. If $a < 0$, then $u_t \to -\dfrac{\theta}{a}$, which indicates $r_t \to \exp\left(-\dfrac{\theta}{a}\right)$. Therefore, we predict that if volatility term decrease over time, the short rate that Black Derman Toy Model generates converge to the target rate. This target rate depends on the sign and the magnitude of θ as well as a.

Suppose the case where the volatility is linear, that is,

$$\sigma_t = mt + \sigma_0,$$

so that

$$\sigma_t{}' = m,$$

where the m is a constant. When the volatility is linear and if θ is constant then the expected value of Black Derman Toy Model becomes

$$r_t = r_0 \exp\left(\left(\frac{mt}{\sigma_0}\right)\log r_0\right)\exp\left(\left(\theta t + \frac{\theta\sigma_0}{m}\right)\log\left[\frac{mt+\sigma_0}{\sigma_0}\right]\right).$$

If m is negative, the first exponential term increases, since $\log r_0 < 0$, but the second exponential term which contains $\log\left[\frac{mt+\sigma_0}{\sigma_0}\right]$ becomes negative. So, the second term decreases if $\theta > 0$, and increases if $\theta < 0$. A similar result can be drawn for positive m. Hence, for $\sigma_t{}' = m$, the short rate that is generated by Black Derman Toy Model can grow without bound or tends to the target rate for either $\theta > 0$ or $\theta < 0$. There is also positive probability that $\sigma_t = mt + \sigma_0$ can be negative, for negative m. But for $\sigma_t < 0$, its logarithm can not be defined. As a result for linearly decreasing volatility of Black Derman Toy Model, with $m < 0$, it should satisfy

$$\begin{aligned}\frac{\sigma_T}{\sigma_0} &= \frac{mT+\sigma_0}{\sigma_0}\\ &= 1 + \frac{mT}{\sigma_0} > 0.\end{aligned}$$

A numerical approximation of Black Derman Toy Model is given by the Euler-Maruyama discretization as

$$u_{k+1} = u_k + (\theta_k + \rho_k u_k)\tau + \sigma_k \varepsilon_k \sqrt{\tau}, \tag{3.6.32}$$

where $\rho_k = \dfrac{{\sigma_k}'}{\sigma_k}$. Then, ${\sigma_k}'$ can be approximated by a forward finite difference, $(\sigma_{k+1} - \sigma_k)/\tau$, so that an approximation to ρ_k is given by

$$\rho_k = \frac{(\sigma_{k+1} - \sigma_k)/\tau}{\sigma_k}. \tag{3.6.33}$$

We now have

$$\begin{aligned} u_{k+1} &= u_k + \left(\theta_k + \frac{(\sigma_{k+1} - \sigma_k)/\tau}{\sigma_k} u_k\right) + \sigma_k \varepsilon_k \sqrt{\tau} \\ &= u_k \left(1 + \frac{(\sigma_{k+1} - \sigma_k)/\tau}{\sigma_k}\tau\right) + \theta_k \tau + \sigma_k \varepsilon_k \sqrt{\tau} \\ &= \frac{\sigma_{k+1}}{\sigma_k} u_k + \theta_k \tau + \sigma_k \varepsilon_k \sqrt{\tau}. \end{aligned}$$

The expected value of u_{k+1} is

$$u_{k+1} = \frac{\sigma_{k+1}}{\sigma_k} u_k + \theta_k \tau. \tag{3.6.34}$$

Now, in order to reach for the recurrence relation in (3.6.34), we write

$$\begin{aligned} u_1 &= \frac{\sigma_1}{\sigma_0} u_0 + \theta_0 \tau, \\ u_2 &= \frac{\sigma_2}{\sigma_1} u_1 + \theta_1 \tau = \frac{\sigma_2}{\sigma_1}\left(\frac{\sigma_1}{\sigma_0} u_0 + \theta_0 \tau\right) + \theta_1 \tau = \frac{\sigma_2}{\sigma_0} u_0 + \frac{\sigma_2}{\sigma_1} \theta_0 \tau + \theta_1 \tau, \\ &\vdots \\ u_k &= \frac{\sigma_k}{\sigma_{k-1}} u_{k-1} + \theta_{k-1} \tau = \frac{\sigma_k}{\sigma_0} u_0 + \sum_{j=1}^{k-1} \left(\frac{\sigma_k}{\sigma_j} \theta_{j-1} \tau\right) + \theta_{k-1} \tau. \end{aligned}$$

We see that u_k and thus $\log r_k$ depend on the volatility. In particular, if

$$\frac{\sigma_{k+1}}{\sigma_k} = \alpha,$$

where α is a constant, then

$$u_k = \alpha^k u_0 + \sum_{j=0}^{k-1} \alpha^j \theta_{k-j-1} \tau.$$

If $\exp(u_k)$ is replaced by r_k, then

$$r_k = r_0 \exp\left(\left(\alpha^k - 1\right)\log r_0\right) \exp\left(\sum_{j=0}^{k-1} \alpha^j \theta_{k-j-1}\tau\right).$$

In the equation if $\alpha > 1$, the first part of the equation decreases since $\log r_0 < 0$. Moreover, if $\theta < 0$, the second term also decreases and short rate of the model converges to the target rate. On the other hand, if $\theta > 0$, the second part of equation increases, hence we either reach the target rate or the second part of the equation dominates. Similar conclusion may be given if $\alpha < 1$. Therefore, to obtain more logical results, it is desirable that α be close to 1.

Returning back to equation (3.6.32), and inserting $\log r_k$ for u_k, we have

$$\log r_{k+1} = \log r_k + (\theta_k + \rho_k \log r_k)\tau + \sigma_k \varepsilon_k \sqrt{\tau},$$

so that

$$r_{k+1} = r_k \exp\left([\theta_k + \rho_k \log r_k]\,\tau + \sigma_k \varepsilon_k \sqrt{\tau}\right).$$

This expression is then used to generate r_u and r_d, respectively for up and down move, to construct the binomial Black Derman Toy tree for short rates. For the up move from r_0, it yields

$$\begin{aligned} r_u &= r_{1,2} \\ &= r_0 \exp(m_0\tau + \sigma_0\sqrt{\tau}), \end{aligned}$$

then, for the down move from r_0, we have

$$\begin{aligned} r_d &= r_{1,1} \\ &= r_0 \exp(m_0\tau - \sigma_0\sqrt{\tau}). \end{aligned}$$

We can also take m_0 from equation (3.6.28)

$$m_0 = \theta_0 - \frac{\sigma_0}{\sigma_0{}'} \log r_0,$$

and compute the ratio of r_u and r_d as

$$\frac{r_u}{r_d} = e^{2\sigma_0\sqrt{\tau}},$$

so that,

$$\log r_u - \log r_d = 2\sigma_0\sqrt{\tau}.$$

In the Black Derman Toy Model the general expressions for $r_{k,j}$ are

$$r_{k+1,j+1} = r_{k,j}\exp(m_{k,j}\tau + \sigma_k\sqrt{\tau}),$$

for the up moves, and

$$r_{k+1,j} = r_{k,j}\exp(m_{k,j}\tau - \sigma_k\sqrt{\tau}),$$

for the down moves. For example the first step can be set to

$$r_{1,2} = r_0\exp(m_0\tau + \sigma_0\sqrt{\tau}),$$

and

$$r_{1,1} = r_0\exp(m_0\tau - \sigma_0\sqrt{\tau}).$$

Then, for the second step, rates $r_{2,1}$, $r_{2,2}$ and $r_{2,3}$ can be computed as

$$\begin{aligned}
r_{2,3} &= r_{1,2}\exp(m_{1,2}\tau + \sigma_1\sqrt{\tau}), \\
r_{2,2} &= r_{1,2}\exp(m_{1,2}\tau - \sigma_1\sqrt{\tau}) = r_{1,1}\exp(m_{1,1}\tau + \sigma_1\sqrt{\tau}), \\
r_{2,1} &= r_{1,1}\exp(m_{1,1}\tau - \sigma_1\sqrt{\tau}).
\end{aligned}$$

Furthermore, by inserting $r_{1,2}$ and $r_{1,1}$ into equation of $r_{2,2}$, we obtain

$$r_0\exp(m_0\tau+\sigma_0\sqrt{\tau})\exp(m_{1,2}\tau-\sigma_1\sqrt{\tau}) = r_0\exp(m_0\tau-\sigma_0\sqrt{\tau})\exp(m_{1,1}\tau+\sigma_1\sqrt{\tau}),$$

which simplifies to

$$2\sigma_0\sqrt{\tau} + (m_{1,2} - m_{1,1})\tau = 2\sigma_1\sqrt{\tau}.$$

Hence, arranging the equation gives

$$\sigma_1 = \frac{2\sigma_0\sqrt{\tau} + (m_{1,2} - m_{1,1})\tau}{2\sqrt{\tau}},$$

which shows that only if $m_{1,2} = m_{1,1}$, σ_1 is the same as the initial volatility σ_0. Since, we want volatility change over time, the drift m should be the function of time as well as the level. So we should keep it in the form $m_{k,j}$ to denote the time and the level by subscripts.

CHAPTER 4

APPLICATION

In Chapter 3, we have shown how to price a zero coupon bond and option that is written on such a bond by using both the Vasicek and the Cox Ingersoll Ross Models. We have also mentioned some of the features of Ho Lee and Black Derman Toy Models. In this chapter, we will estimate the parameters of Vasicek and Cox Ingersoll Ross Models from a real data set for the beginning days of each month of the years 2004 and 2005. Furthermore, we will show the procedure for generating binomial trees for Ho Lee and Black Derman Toy Models. After that, we will calculate the value of options that are written on five years bond with maturity four years for different exercise prices by using the formulas obtained in Chapter 3. For binomial models we need to trace backward for pricing procedure. However, we specify some of the descriptive statistics to get some idea about our real data.

4.1 The Data

In this work, we took the data from June 1, 1976 to December 31, 2007. Since yields are already interpolated by U.S. Treasury, we do not need to do so. Here, we present some descriptive statistics for 1–5 years constant maturity U.S. Government Bond rate data in Table 4.1.

According to Table 4.1, range of the interest rate series decrease as time to maturity increase and also, all the series are skewed positively. Moreover, there is an inverse correlation between standard deviation and time to maturity of the bonds: the longer the time to maturity, the smaller the standard deviation.

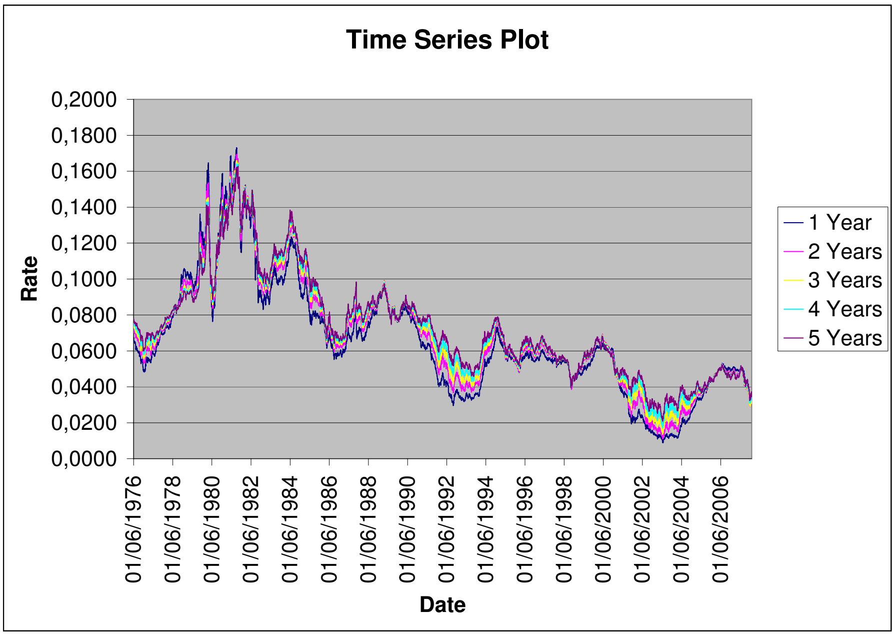

Figure 4.1: Historical Graph of Interest Rates

4.2 Parameter Estimation and Constructing Binomial Trees

In this section of the thesis, we will to calculate the price of a European call option by using Vasicek, Cox Ross Ingersoll, Ho Lee and Black Derman Toy Models. For Vasicek and Cox Ingersoll Ross Models, we will estimate the related parameters so that, we will be able to use the formulas given in the Chapter 3. On the other hand, we will investigate binomial trees of Ho Lee and Black Derman Toy Models, for the interest rates.

Table 4.1: Descriptive Statistics for Interest Rates

Statistics	1 Year	2 Years	3 Years	4 Years	5 Years
Mean	0.0656	0.0688	0.0704	0.0716	0.0729
Standard Error	0.0004	0.0004	0.0003	0.0003	0.0003
Median	0.0588	0.0627	0.0647	0.0661	0.0676
Mode	0.0546	0.0611	0.0650	0.0460	0.0455
Standard Deviation	0.0325	0.0314	0.0304	0.0296	0.0289
Sample Variance	0.0011	0.0010	0.0009	0.0009	0.0008
Kurtosis	0.5843	0.3063	0.2347	0.1862	0.1269
Skewness	0.8054	0.7227	0.7319	0.7564	0.7718
Range	0.1643	0.1585	0.1525	0.1471	0.1419
Minimum	0.0088	0.0110	0.0134	0.0171	0.0208
Maximum	0.1731	0.1695	0.1659	0.1642	0.1627
Count	7890	7890	7890	7890	7890

4.2.1 Parameter Estimation for Vasicek Model and Cox Ingersoll Ross Model

The bond price formula for Vasicek model was formed in the preceding chapter as in equation (3.1.4). The components of this equation were

$$C(t,T) = \frac{1}{\beta}(1 - e^{-\beta(T-t)}),$$

and

$$\begin{aligned} A(t,T) &= \exp\left(-(T-t)\left(\alpha - \frac{\lambda\sigma}{\beta} - \frac{\sigma^2}{2\beta^2}\right) + \frac{\sigma^2}{4\beta^2}\left(1 - e^{-2\beta(T-t)}\right)\right. \\ &\left. + \left(\frac{1}{\beta}\left(1 - e^{-\beta(T-t)}\right)\left(\alpha - \frac{\lambda\sigma}{\beta} - \frac{\sigma^2}{\beta^2}\right)\right)\right). \end{aligned}$$

Now, we need to estimate the parameters of Vasicek Model, namely α, β, σ and λ, by the method of calibration. Although, there are many calibration methods, we minimize the sum of squares of the difference between the data which generated by the models and the real observed data. By formulation

$$B_t(\alpha, \beta, \sigma, \lambda) = (B_t(\tau_1/\alpha, \beta, \sigma, \lambda), \ldots, B_t(\tau_k/\alpha, \beta, \sigma, \lambda))^T,$$

and

$$J_t(\alpha, \beta, \sigma, \lambda) = (P_t - B_t(\alpha, \beta, \sigma, \lambda)^T (P_t - B_t(\alpha, \beta, \sigma, \lambda).$$

We want minimize this calibration function $J_t(\alpha, \beta, \sigma, \lambda)$. It is important to choose initial values of the parameters, since the value of calibration function vary with respect to different initials. In this study, we tried from [0.01 0.01 0.01 0.01] to [0.1 0.1 0.1 0.1] for initials of α, β, σ and λ respectively, so we tried 10000 initial values and we got 10000 function values. Then, we took the initials that generate minimum value of calibration function $J_t(\alpha, \beta, \sigma, \lambda)$. Moreover, we took maximum number of function evaluations as 5000 and maximum number of iterations were allowed as 700 in our calculations. Based on the criteria of the smallest function value and σ should be positive, the initial values and estimated parameters are tabulated in Table 4.2.

According to results, $\hat{\alpha}$ tends to move straightly upward from 0.0496 to 0.0910. This represents the long run equilibrium value which the interest rate reverts. In other words, interest rate is expected to increase in the long run. The parameter $\hat{\beta}$ shows the speed of adjustment, the positivity of this parameter ensure stability around the long term value. Model volatility can be seen from the parameter $\hat{\sigma}$, where all the $\hat{\sigma}$'s are smaller than 0.02. The market price of risk, $\hat{\lambda}$, shows the increase in expected rate of return on a bond. It can be thought as the return of per unit risk with respect to risk free investment. In other words, it is the cost of taking the risk instead of risk free investment. According to estimated parameters, taking the risk is the least meaningful in 01/07/2007.

The bond price for the Cox Ingersoll Ross Model had the same form as in the Vasicek Model, where the elements of (3.1.4) were

$$C(t, T) = \frac{2(e^{\gamma(T-t)} - 1)}{(\gamma + \lambda\sigma + \beta)(e^{\gamma(T-t)} - 1) + 2\gamma},$$

Table 4.2: Estimated Parameters for Vasicek Model

	Initials				Estimated Parameters			
Date	α_i	β_i	σ_i	λ_i	$\hat{\alpha}_i$	$\hat{\beta}_i$	$\hat{\sigma}_i$	$\hat{\lambda}_i$
02.01.2004	0.04	0.09	0.03	0.01	0.0496	0.1267	0.0108	0.0101
02.02.2004	0.04	0.09	0.03	0.01	0.0496	0.1266	0.0114	0.0101
01.03.2004	0.04	0.10	0.02	0.01	0.0463	0.1288	0.0112	0.0103
01.04.2004	0.04	0.09	0.01	0.04	0.0451	0.1016	0.0075	0.0412
03.05.2004	0.04	0.10	0.04	0.03	0.0497	0.1354	0.0101	0.0316
01.06.2004	0.04	0.09	0.03	0.05	0.0500	0.1224	0.0076	0.0525
01.07.2004	0.05	0.10	0.01	0.08	0.0551	0.1109	0.0079	0.0821
02.08.2004	0.05	0.10	0.04	0.01	0.0614	0.1382	0.0151	0.0104
01.09.2004	0.05	0.07	0.04	0.01	0.0523	0.1051	0.0091	0.0121
01.10.2004	0.05	0.09	0.04	0.07	0.0620	0.1288	0.0128	0.0710
01.11.2004	0.06	0.08	0.09	0.03	0.0630	0.1191	0.0128	0.0378
01.12.2004	0.06	0.08	0.01	0.07	0.0662	0.0887	0.0081	0.0710
03.01.2005	0.06	0.10	0.02	0.02	0.0670	0.1191	0.0135	0.0211
01.02.2005	0.06	0.09	0.02	0.07	0.0700	0.1132	0.0120	0.0714
01.03.2005	0.07	0.05	0.10	0.03	0.0750	0.0781	0.0080	0.0371
01.04.2005	0.07	0.07	0.01	0.02	0.0770	0.0776	0.0082	0.0205
02.05.2005	0.09	0.10	0.01	0.03	0.0770	0.0949	0.0110	0.0324
01.06.2005	0.07	0.07	0.10	0.02	0.0750	0.1036	0.0126	0.0252
01.07.2005	0.08	0.09	0.01	0.01	0.0791	0.0872	0.0104	0.0103
01.08.2005	0.10	0.10	0.01	0.03	0.0821	0.0954	0.0114	0.0318
01.09.2005	0.07	0.05	0.01	0.04	0.0810	0.0612	0.0062	0.0410
03.10.2005	0.07	0.10	0.05	0.05	0.0870	0.1393	0.0192	0.0509
01.11.2005	0.09	0.09	0.01	0.03	0.0902	0.0850	0.0103	0.0311
01.12.2005	0.08	0.09	0.02	0.07	0.0910	0.1067	0.0134	0.0726

and

$$A(t,T) = \left(\frac{2\gamma e^{(\gamma+\lambda\sigma+\beta)(T-t)/2}}{(\gamma+\lambda\sigma+\beta)(e^{\gamma(T-t)}-1)+2\gamma}\right)^{\frac{2\beta\alpha}{\sigma^2}}.$$

Here, in this case we also followed the same procedure as calibrating the Vasicek Model with the same initial values to make these two models comparable.

Unlike the Vasicek Model, estimation of the long run mean parameter, $\hat{\alpha}$ does not move straightly upward. The estimated volatility of model is greater than the Vasicek Model. Although, all estimated volatilities are smaller than 0.02 in the Vasicek Model, volatility estimate in the Cox Ingersoll Model becomes 0.2; about 10 times of the volatility in the Vasicek Model. Another lack of harmony is in the estimation market risk premium: In 01/07/2004 the market risk premium was the greatest value in Vasicek Model, but in the Cox Ingersoll Ross model, the cost of risk is the greatest in 01/09/2004. The Table 4.3 below shows the results of the estimated parameters and initial values of parameters.

4.2.2 Construction of Interest Rate Binomial Tree for Ho Lee Model and Black Derman Toy Model

For constructing Ho Lee binomial tree, Arrow-Debreu [17] prices are used. Let us define $D(t)$ as the discount factor over time period $[0,t]$. $D(t)$ can be thought of as the value at time $t=0$ of a \$1 face value default free zero bond matures at time t. Note that

$$D(t) = \begin{cases} e^{-tr(t)}, & \text{for continuously compounded interest;} \\ \dfrac{1}{(1+r(t,j))^t}, & \text{for simple interest.} \end{cases} \tag{4.2.1}$$

Therefore, $D(t,j)$ can be defined as the discount factor at time t and state j, at (t,j) for short rates, over the time period $[t,t+1]$.

$$D(t) = \begin{cases} e^{-r(t,j)}, & \text{for continuously compounded interest;} \\ \dfrac{1}{(1+r(t,j))} & \text{for simple interest.} \end{cases} \tag{4.2.2}$$

Table 4.3: Estimated Parameters for Cox Ingersoll Ross Model

	Initials				Estimated Parameters			
Date	α_i	β_i	σ_i	λ_i	$\hat{\alpha}_i$	$\hat{\beta}_i$	$\hat{\sigma}_i$	$\hat{\lambda}_i$
02.01.2004	0.04	0.09	0.03	0.01	0.0138	0.1847	0.0361	0.0049
02.02.2004	0.04	0.09	0.03	0.01	0.0132	0.1852	0.0364	0.0046
01.03.2004	0.04	0.10	0.02	0.01	0.0155	0.1994	0.0236	0.0048
01.04.2004	0.04	0.09	0.01	0.04	0.0121	0.1888	0.0121	0.0196
03.05.2004	0.04	0.10	0.04	0.03	0.0158	0.1804	0.0563	0.0266
01.06.2004	0.04	0.09	0.03	0.05	0.0146	0.1808	0.0359	0.0228
01.07.2004	0.05	0.10	0.01	0.08	0.0195	0.2000	0.0118	0.0377
02.08.2004	0.05	0.10	0.04	0.01	0.0178	0.1874	0.0547	0.0090
01.09.2004	0.05	0.07	0.04	0.01	0.0045	0.1541	0.0645	0.0085
01.10.2004	0.05	0.09	0.04	0.07	0.0143	0.1738	0.0582	0.0799
01.11.2004	0.06	0.08	0.09	0.03	0.0078	0.1205	0.1317	0.0281
01.12.2004	0.06	0.08	0.01	0.07	0.0129	0.1815	0.0122	0.0254
03.01.2005	0.06	0.10	0.02	0.02	0.0204	0.2044	0.0240	0.0093
01.02.2005	0.06	0.09	0.02	0.07	0.0175	0.1923	0.0241	0.0313
01.03.2005	0.07	0.05	0.10	0.03	0.0202	0.0753	0.1951	0.0272
01.04.2005	0.07	0.07	0.01	0.02	0.0079	0.1697	0.0122	0.0059
02.05.2005	0.09	0.10	0.01	0.03	0.0062	0.1671	0.0123	0.0282
01.06.2005	0.07	0.07	0.10	0.02	0.0049	0.0991	0.1522	0.0203
01.07.2005	0.08	0.09	0.01	0.01	0.0199	0.2001	0.0121	0.0042
01.08.2005	0.10	0.10	0.01	0.03	0.0065	0.1669	0.0124	0.0278
01.09.2005	0.07	0.05	0.01	0.04	0.0115	0.1463	0.0141	0.0009
03.10.2005	0.07	0.10	0.05	0.05	0.0112	0.1622	0.0606	0.0470
01.11.2005	0.09	0.09	0.01	0.03	0.0227	0.1988	0.0121	0.0120
01.12.2005	0.08	0.09	0.02	0.07	0.0224	0.1962	0.0239	0.0313

Note that $r(0,0) = r(1)$ at each time t, we may assume without loss of generality that $r(t,j)$ goes up $r(t+1,j+1)$ with probability $\frac{1}{2}$. Hence, $r(t,j)$ goes down to $r(t+1,j)$ with probability $\frac{1}{2}$. Suppose for $t \geq 0$ we have

$$2\sigma(t+1) = r(t+1,j+1) - r(t+1,j).$$

From now on we will assume that interests are continuously compounded, and that

$$D(t) = e^{-r(t)t} \quad \text{and} \quad D(t,j) = e^{-r(t,j)},$$

and hence,

$$e^{-2\sigma(t+1)}D(t+1,j) = D(t+1,j+1). \tag{4.2.3}$$

Given $D(1)$, $D(2)$,...$D(n)$ and $\sigma(1)$, $\sigma(2)$,...$\sigma(n)$, where $n \geq 2$. Now, we show how to find $r(t,j)$ inductively, where $1 \leq t \leq n-1$, $0 \leq j \leq t$ and under the no-arbitrage principle. At time $t = 0$ consider the following two portfolios:

- Portfolio A that consists of a zero bond which matures at time $t = 2$ with a face value of \$1.

- Portfolio B that consists of a financial derivative which pays

$$\begin{cases} D(1,0), & \text{at } (1,0); \\ D(1,1), & \text{at } (1,1). \end{cases}$$

The value of portfolio A at time $t = 0$ is $D(2)$ and the value of portfolio B at time $t = 0$ is

$$G(1,0)D(1,0) + G(1,1)D(1,1),$$

where $G(t,j)$'s are the Arrow-Debreu prices and they are known. As both portfolios have the same payoff at time $t = 1$, by the no-arbitrage argument, therefore, their value at time $t = 0$ must be the same. Hence,

$$D(2) = G(1,0)D(1,0) + G(1,1)D(1,1), \tag{4.2.4}$$

and from (4.2.3) we have

$$e^{-2\sigma(1)}D(1,0) = D(1,1).$$

Thus, equation (4.2.4) now becomes

$$D(2) = G(1,0)D(1,0) + G(1,1)e^{-2\sigma(1)}D(1,0),$$

which gives

$$D(1,0) = \frac{D(2)}{G(1,0) + G(1,1)e^{-2\sigma(1)}}.$$

Returning back to the rates we find that

$$r(1,0) = -\log\left(\frac{D(2)}{G(1,0) + G(1,1)e^{-2\sigma(1)}}\right).$$

As $r(1,0)$ is known, $r(1,1)$ could be deduced from equation (4.2.3). Now that we have worked out the spot rates at time $t = 1$, we move on to time $t = 2$. At time $t = 0$, consider (new portfolios)

- Portfolio A that consists of a zero bond which matures at time $t = 3$ with a face value of \$1.

- Portfolio B that consists of a derivative which pays

$$\begin{cases} D(2,0) & \text{at } (2,0); \\ D(2,1) & \text{at } (2,1); \\ D(2,2) & \text{at } (2,2). \end{cases}$$

Both portfolios A and B have the same payoff at time $t = 2$. By a similar no-arbitrage argument they must have the same value at time $t = 0$. This gives

$$D(3) = G(2,0)D(2,0) + G(2,1)D(2,1) + G(2,2)D(2,2). \qquad (4.2.5)$$

Again by using equation (4.2.3) wecompute

$$\begin{aligned} D(3) &= G(2,0)D(2,0) + G(2,1)e^{-2\sigma(2)}D(2,0) + G(2,2)e^{-4\sigma(2)}D(2,0), \\ D(2,0) &= \frac{D(3)}{G(2,0) + G(2,1)e^{-2\sigma(2)} + G(2,2)e^{-4\sigma(2)}}, \end{aligned}$$

and hence,

$$r(2,0) = -\log\left(\frac{D(3)}{G(2,0) + G(2,1)e^{-2\sigma(2)} + G(2,2)e^{-4\sigma(2)}}\right).$$

Generally, suppose $t \geq 0$ and let $r(t,j)$ and $G(t,j)$ are known or calculated, then writing Arrow-Debreu prices in general form as

$$G(t+1, j+1) = \frac{1}{2}D(t,j)G(t,j) + \frac{1}{2}D(t,j+1)G(t,j+1),$$

then the no-arbitrage argument leads to

$$r(t+1,0) = -\log\left(\frac{D(t+2)}{\displaystyle\sum_{j=0}^{t+1} G(t+1,j)e^{-2j\sigma(t+1)}}\right).$$

Note that $r(0,0) = r(1)$ and $G(0,0) = 1$. Since for the first five years volatility term of Ho Lee Model is constant through time we can write

$$\sigma(1) = \sigma(2) = \sigma(3) = \sigma(4) = \sigma(5).$$

In order to approximating the volatility term, first we calculated the forward rates for all the days. Then, we computed the standard deviations of all forwards rates with the beginning of our data set to the day that we want to analyze. By the concept of that knowledge of the Arrow-Debreu prices and of the short rates at time t completely determines all the Arrow-Debreu prices at time $t + \Delta t$ and knowledge of the Arrow-Debreu prices completely determines the value of a discount bond [19], so that, we can generate the binomial tree by this notion. As we mentioned before, the drift term is also important part of these trees, since it determines the slope of the curves. To calculate the drift term, we will use numerical approximation for θ, equivalently denoted by m_k. Recall that equivalently

$$m_k\tau = r_{k+1,j+1} - r_{k,j} - \sigma_k\sqrt{\tau},$$

for an up move and

$$m_k \tau = r_{k+1,j} - r_{k,j} + \sigma_k \sqrt{\tau},$$

for a down move were discussed in Chapter 3. The Figures 4.2, 4.3 and 4.4 show both the interest rate trees and the approximation of drift term θ for the days January 02, 2004, December 01, 2004 and December 01, 2005. Note that, the larger values of approximation of drifts generates the bigger changes in the interest rates. In 02/01/2004, since we have the largest drifts, the changes of interest rates are also the biggest. In that date, rates start with 1.31% and go to 4.134% for the four down scenario and 5.959% for the four up scenario. On the other hand, in 01/12/2005, since we have the smallest drifts, the changes are from 4.36% to 3.464% for the four down and 5.289% for the four up scenarios.

In the Black Derman Toy Model, we used the same procedure as in the Ho Lee Model. The discount factors were defined by equations (4.2.1) and (4.2.2). Unlike Ho Lee Model, in the Black Derman Toy Model, we have

$$r(t+1, j+1) = r(t+1, j)e^{2\sigma(t+1)}, \tag{4.2.6}$$

for $t \geq 0$. Hence,

$$D(t) = \begin{cases} e^{-r(t,0)} e^{2j\sigma(t)}, & \text{for continuously compounded interest ;} \\ \dfrac{1}{(1 + r(t,j)e^{2j\sigma(t)})}, & \text{for simple interest.} \end{cases} \tag{4.2.7}$$

As in the Ho Lee Model, two equivalent portfolios will be constructed and again we will have equations (4.2.4) and (4.2.5). The no-arbitrage argument gives

$$D(t+2) = \sum_{j=0}^{t+1} G(t+1, j) D(t+1, j),$$

and from equation (4.2.7)

$$D(t) = \begin{cases} \displaystyle\sum_{j=0}^{t+1} G(t+1, j) e^{-r(t+1,0)} e^{2j\sigma(t+1)}, & \text{for cont. compounded interest;} \\ \displaystyle\sum_{j=0}^{t+1} \frac{G(t+1, j)}{1 + r(t+1, 0)e^{2j\sigma(t+1)}}, & \text{for simple interest.} \end{cases}$$

Note that this is an equation with one unknown $r(t+1,0)$, that could be solved using a numerical method such as bisection method. Once we know $r(t+1,0)$ the $r(t+1,j)$'s could be deduced from equation (4.2.6) for $j = 1, 2, \ldots t+1$. Different by than Ho Lee Model, in the Black Derman Toy Model volatility varies through time. To estimate volatility terms as an input parameters, similar to Ho Lee Model, we computed the forward rates of bonds, then we used $GARCH$(1,1) for these forward rates. As before, the general form of expression $r_{k,j}$'s are

$$r_{k+1,j+1} = r_{k,j} \exp(m_{k,j}\tau + \sigma_k \sqrt{\tau}),$$

for an up move, and

$$r_{k+1,j} = r_{k,j} \exp(m_{k,j}\tau - \sigma_k \sqrt{\tau}),$$

for a down move, and for estimating m's

$$m_{k,j} = \log(r_{k+1,j+1}) - \log(r_{k,j}) - \sigma_k,$$

and

$$m_{k,j} = \log(r_{k+1,j}) - \log(r_{k,j}) + \sigma_k.$$

In the Ho Lee Model, m is the approximation of the drift parameter θ, whereas in the Black Derman Toy Model, the $m_{k,j}$'s involve the drift, the volatility and the initial spot rate.

Therefore, Black Derman Toy Model can be written as

$$d \log r_t = m_t dt + \sigma_t dW_t,$$

where

$$m_t = \theta_t + \rho_t \log r_t.$$

Since we approximate ρ_k by equation (3.6.33), it becomes

$$m_t = \theta_t + \frac{(\sigma_{k+1} - \sigma_k)}{\sigma_k} \log r_t.$$

The Figures 4.5, 4.6 and 4.7 show both interest rates and the drift terms at each branch for the days January 02, 2004, December 01, 2004 and December 01, 2005. The results of three analyzed days are similar to results of Ho Lee Model. As in the Ho Lee Model, drift terms are important for jump of the interest rates from one step to other. The change of interest rates from step 1 to step 5 is greater on 02/01/2004 and the smaller on the 01/12/2005 that is harmonious with their drift terms. The latest day has the smallest change, interest rates go from 4.36% to 4.884% for the four up and go to 3.904% for the four down scenarios. Moreover, the first day has the biggest change. On this day, rates start with 1.310% and go to 5.747% for the four up scenario and 4.448% for the four down scenario.

To make comparison, we need to observed values of, for instance, call options. For this purpose first we will compute the option prices then we will compare them.

4.3 Calculating Call Option Prices

In the final part of our application, we will estimate the European call option written on zero coupon bonds. For all of the four models, we use five years US Treasury Zero Coupon Bond and we estimate the four years maturity European call options written on them for each analyzed days. After computing the price of call options, we will compare the results with the real observed values. The Table 4.4 shows the estimated call prices, for all of the models.

In Chapter 3, we showed the call option prices for both Vasicek and Cox Ingersoll Ross Models, by equations (3.3.18) and (3.4.23), respectively. There is also one additional input variable in the Cox Ingersoll Ross Model that is initial interest rate. We chose initial interest rate as the rates of four year zero coupon bond rates at time t.

For the binomial models we constructed interest rate trees for the analyzed days. Then by classical binomial approaches, it is not difficult to calculate call option prices on the days that we analyze for two models. For the beginning we calculate the call prices at the end of period five by

$$C_{j,i} = \max[P_{j,i} - K, 0],$$

where K is the strike price of a call option that we chose its price same as the real strike prices. After that by backward induction we can compute all the prices at each leaves with [9]

$$C_{j,i} = \max\left[0, \frac{(0.5 \times C_{j+1,i+1} + 0.5 \times C_{j+1,i})}{(1 + r_{j,i})}\right].$$

The Figures 4.8–4.13 show the call option price trees for the days January 02, 2004, December 01, 2004 and December 01, 2005, both for Ho Lee Model and Black Derman Toy Model. Furthermore, the Table 4.4 shows all of the estimated prices for four models. We used the sum of errors criteria to compare our models

$$\min\left[\sum_{i=1}^{n}(O_i - E_i)^2\right].$$

The sum that has minimum value may be thought as the best model in terms of fitting available data.

The Table 4.4 shows the calculated European call option prices. The sum of squares of errors (SSE) is the criteria for choosing the best fit model. The model which has the minimum SSE may be thought as the best fit model. The calculates SSE's are in Table 4.5.

According to minimum SSE criteria and our available data, Black Derman Toy Model performs the best with 0.0014078, while Vasicek Model performs the worst with 0.0014450. On the other hand, after Black Derman Toy Model, Cox Ingersoll Ross Model has the minimum SSE that is 0.0014217. Among two normal distribution Model, Ho Lee Model fits the data better than the Vasicek Model. This results are coinciding with the general characteristics of desirable interest rate model [19]:

Table 4.4: Estimated Call Prices for Vasicek Model, Cox Ingersoll Ross Model, Ho Lee Model and Black Derman Toy Model

Date	Vasicek	CIR	HL	BDT
02.01.2004	0.0718	0.0716	0.0728	0.0723
02.02.2004	0.0673	0.0669	0.0679	0.0676
01.03.2004	0.0620	0.0611	0.0619	0.0616
01.04.2004	0.0626	0.0626	0.0634	0.0631
03.05.2004	0.0781	0.0780	0.0792	0.0789
01.06.2004	0.0836	0.0836	0.0849	0.0846
01.07.2004	0.0859	0.0859	0.0872	0.0868
02.08.2004	0.0800	0.0797	0.0809	0.0806
01.09.2004	0.0769	0.0769	0.0778	0.0776
01.10.2004	0.0841	0.0840	0.0851	0.0848
01.11.2004	0.0781	0.0783	0.0790	0.0788
01.12.2004	0.0894	0.0894	0.0906	0.0903
03.01.2005	0.0841	0.0841	0.0852	0.0850
01.02.2005	0.0919	0.0919	0.0930	0.0928
01.03.2005	0.0958	0.0972	0.0971	0.0969
01.04.2005	0.1021	0.1021	0.1034	0.1032
02.05.2005	0.0915	0.0915	0.0926	0.0925
01.06.2005	0.0959	0.0963	0.0968	0.0967
01.07.2005	0.0935	0.0935	0.0945	0.0944
01.08.2005	0.1040	0.1040	0.1053	0.1052
01.09.2005	0.0936	0.0937	0.0947	0.0946
03.10.2005	0.1036	0.1036	0.1049	0.1048
01.11.2005	0.1077	0.1077	0.1092	0.1091
01.12.2005	0.1083	0.1083	0.1097	0.1096

Table 4.5: Sum of Squares of Errors for Vasicek Model, Cox Ingersoll Ross Model, Ho Lee Model and Black Derman Toy Model

	Vasicek	CIR	HL	BDT
SSE	0.0014450	0.0014217	0.0014357	0.0014078

(i) Rates should not be allowed to negative interest rates

(ii) Very high values of interest rates tend to be followed by a decrease in rates, in other words a model should have mean reverting property

(iii) The level of volatility has been observed to vary with the absolute level of the rates themselves.

Among these four models, the characteristics of Black Derman Toy Model can be expected as the most fitting model while the Vasicek Model be least with its normal distribution assumption.

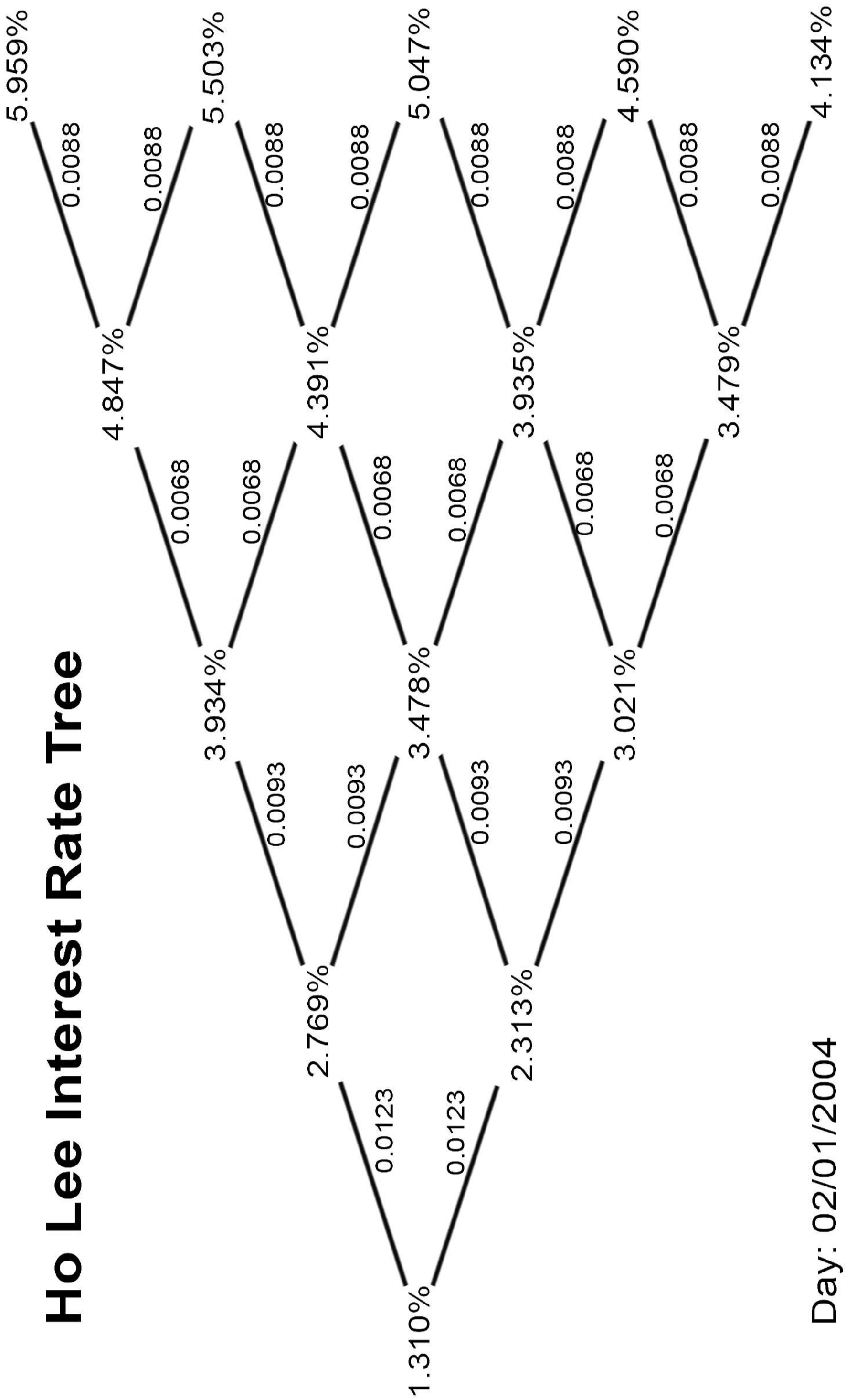

Figure 4.2: Ho Lee Model Interest Rate Tree on 02/01/2004

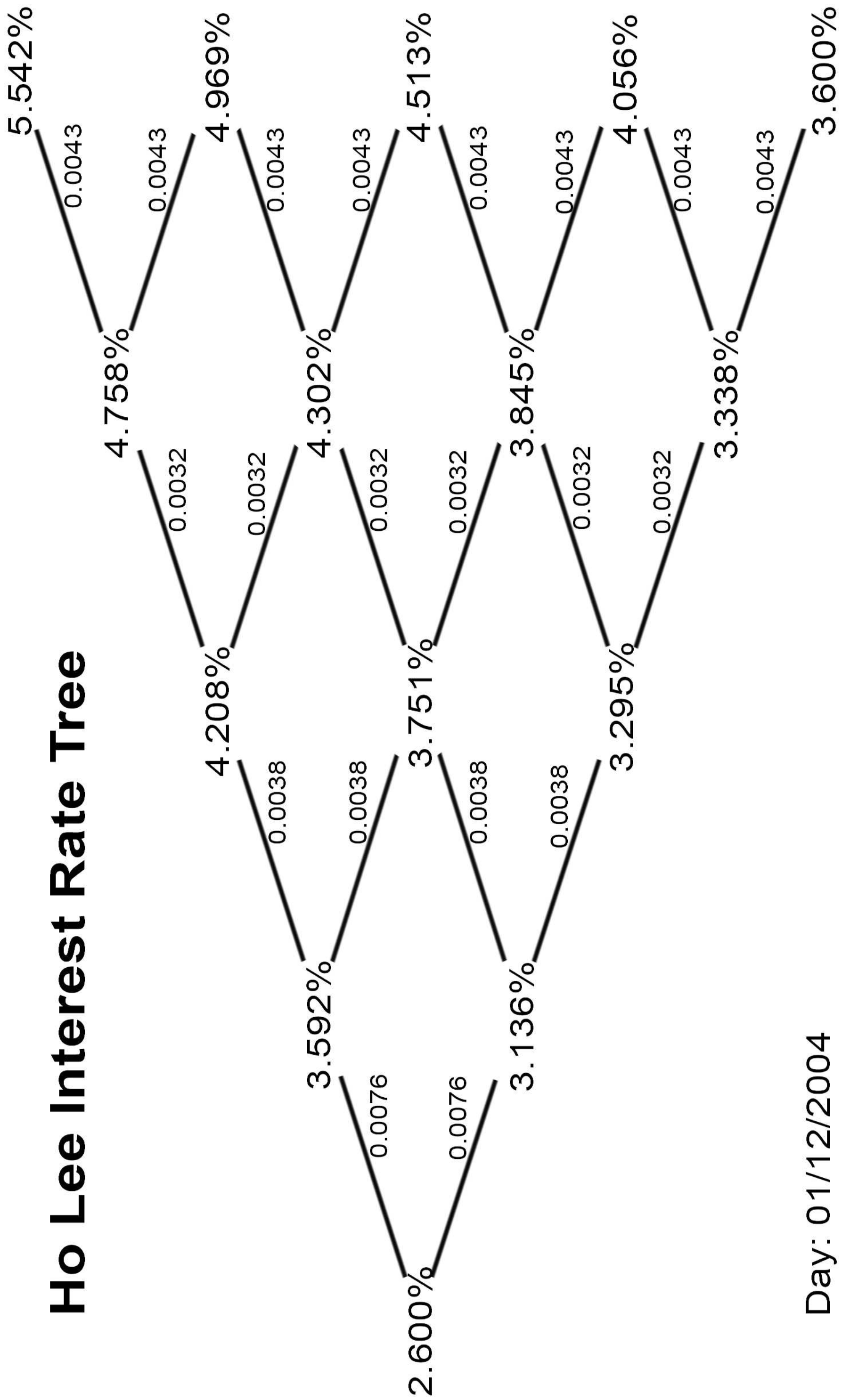

Figure 4.3: Ho Lee Model Interest Rate Tree on 01/12/2004

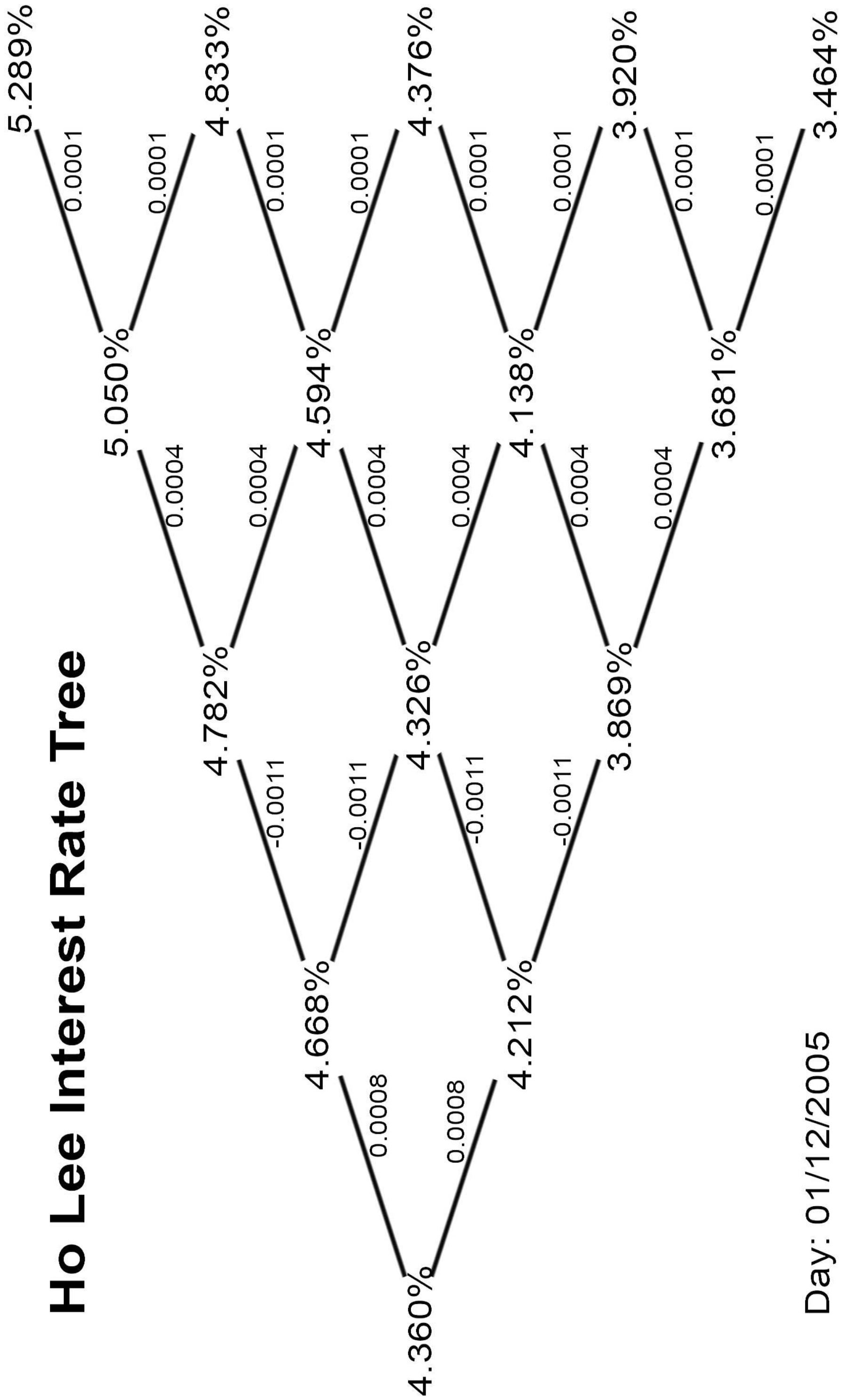

Figure 4.4: Ho Lee Model Interest Rate Tree on 01/12/2005

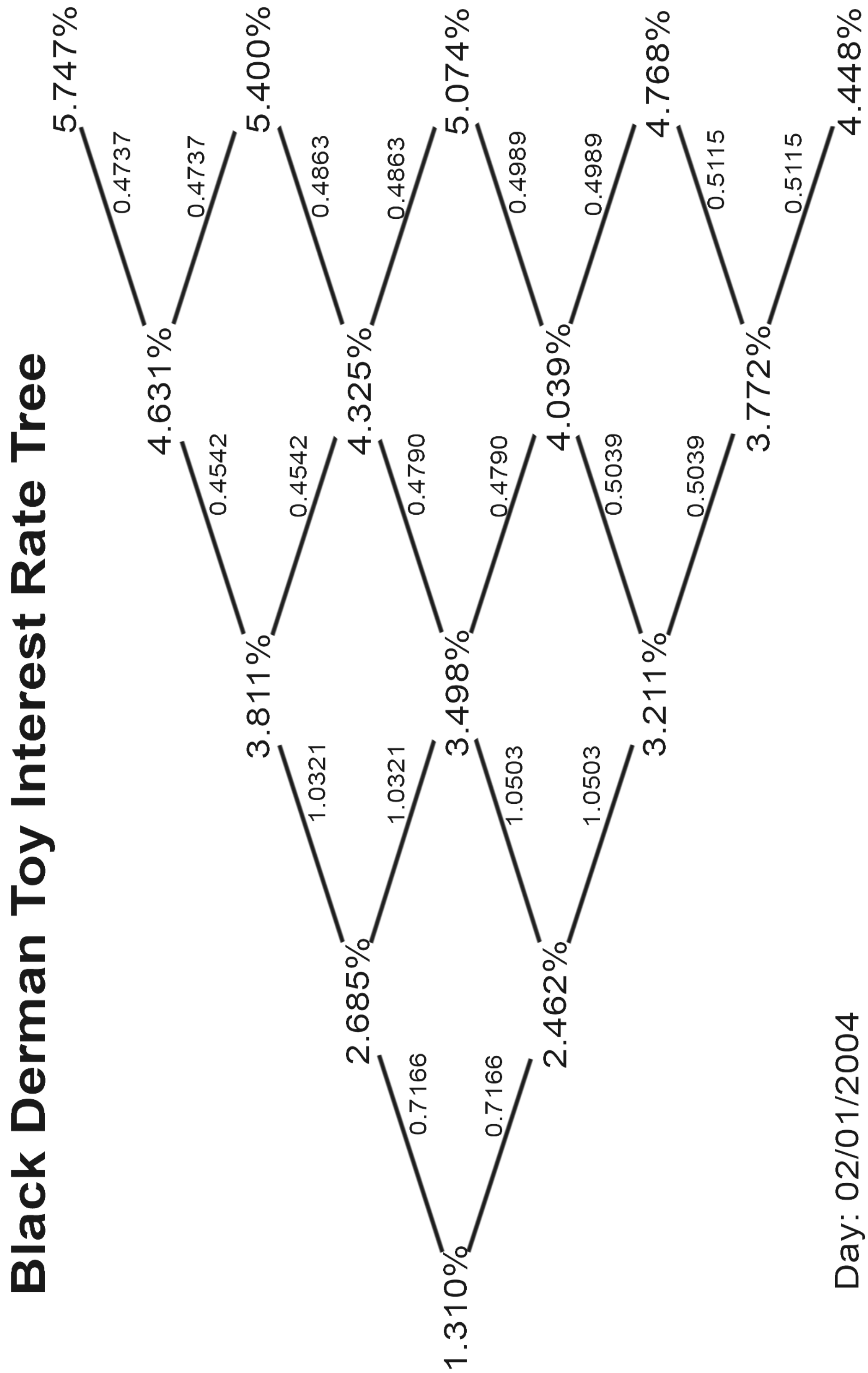

Figure 4.5: Black Derman Toy Model Interest Rate Tree on 02/01/2004

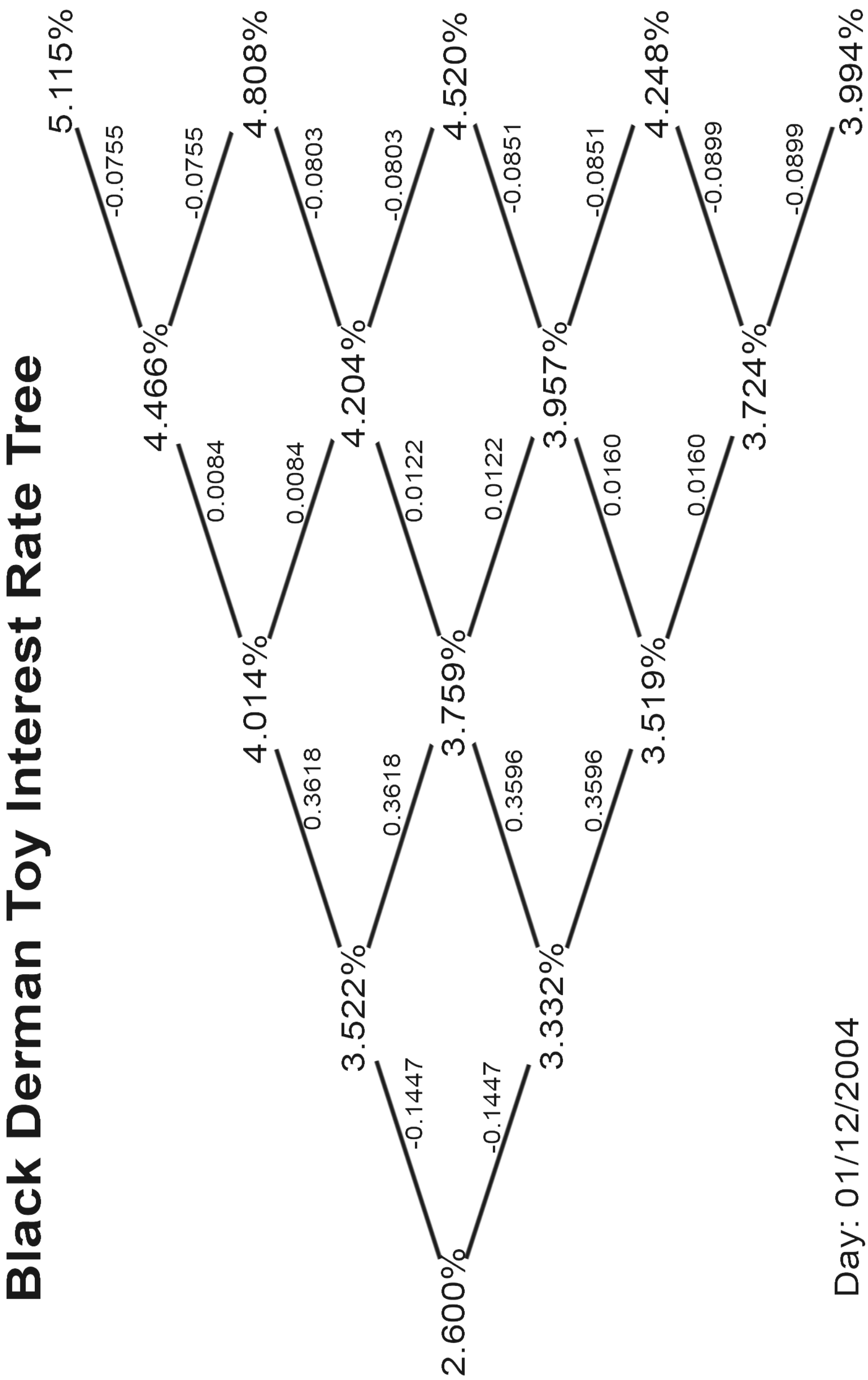

Figure 4.6: Black Derman Toy Model Interest Rate Tree on 01/12/2004

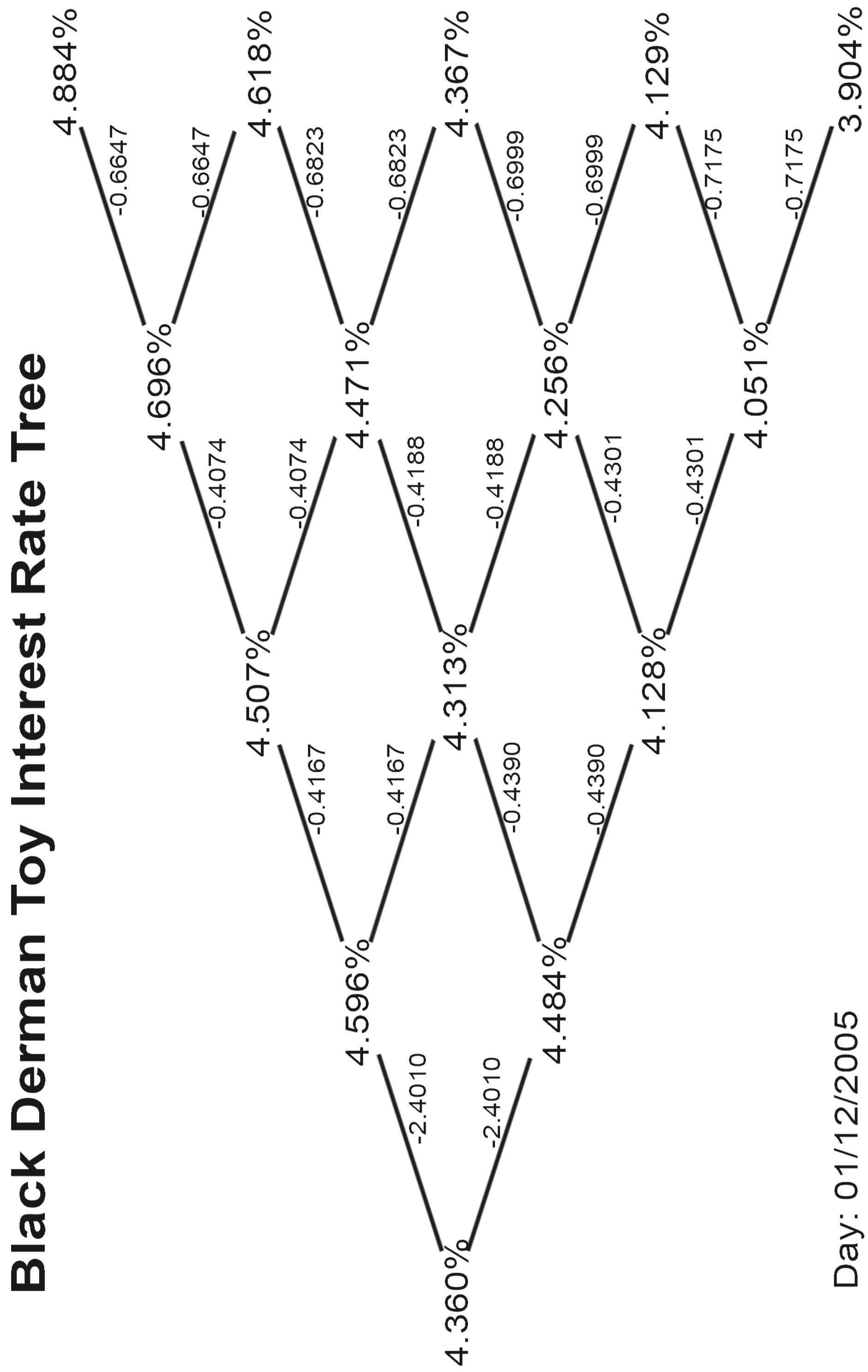

Figure 4.7: Black Derman Toy Model Interest Rate Tree on 01/12/2005

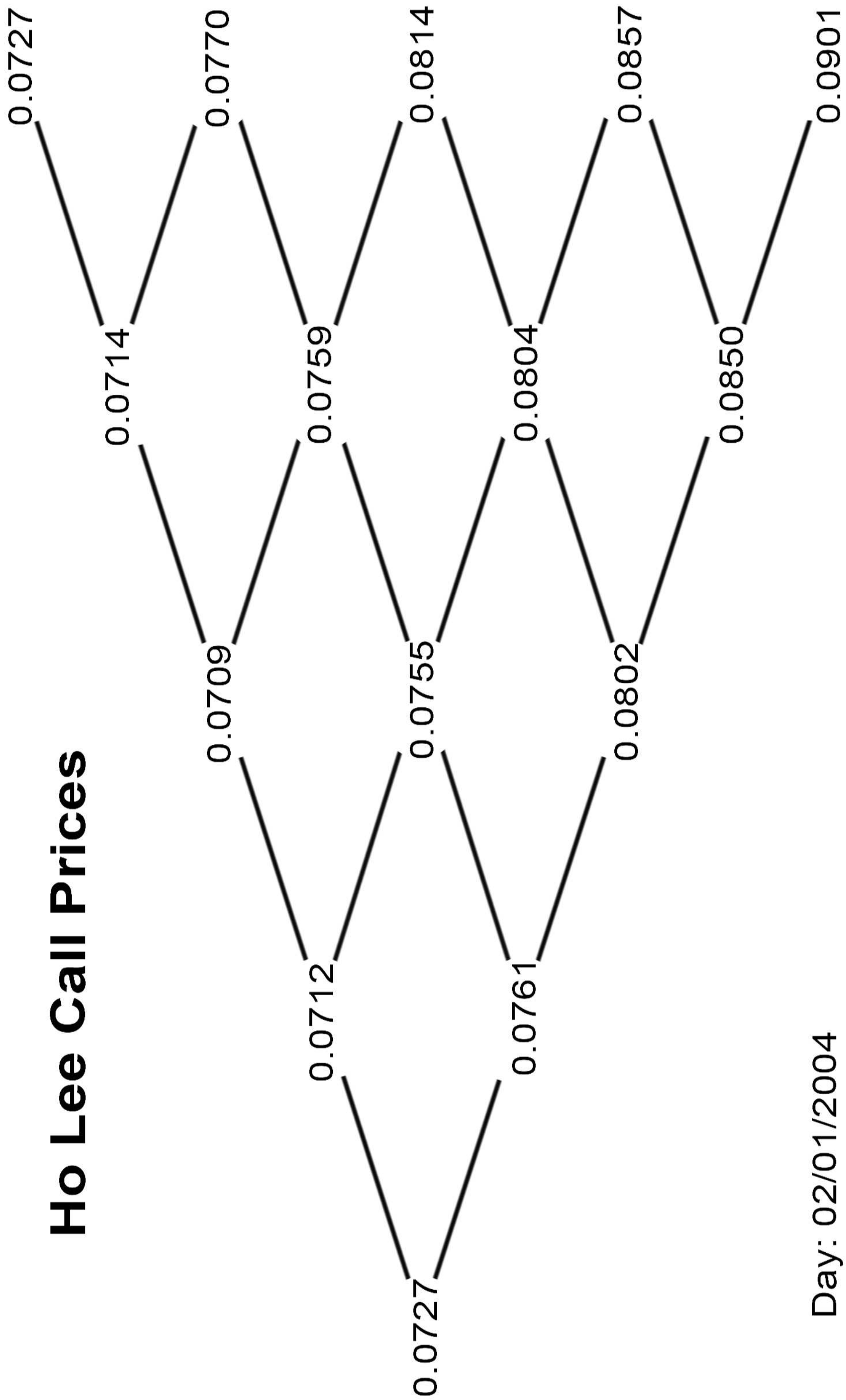

Figure 4.8: Ho Lee Model Call Option Tree on 02/01/2004

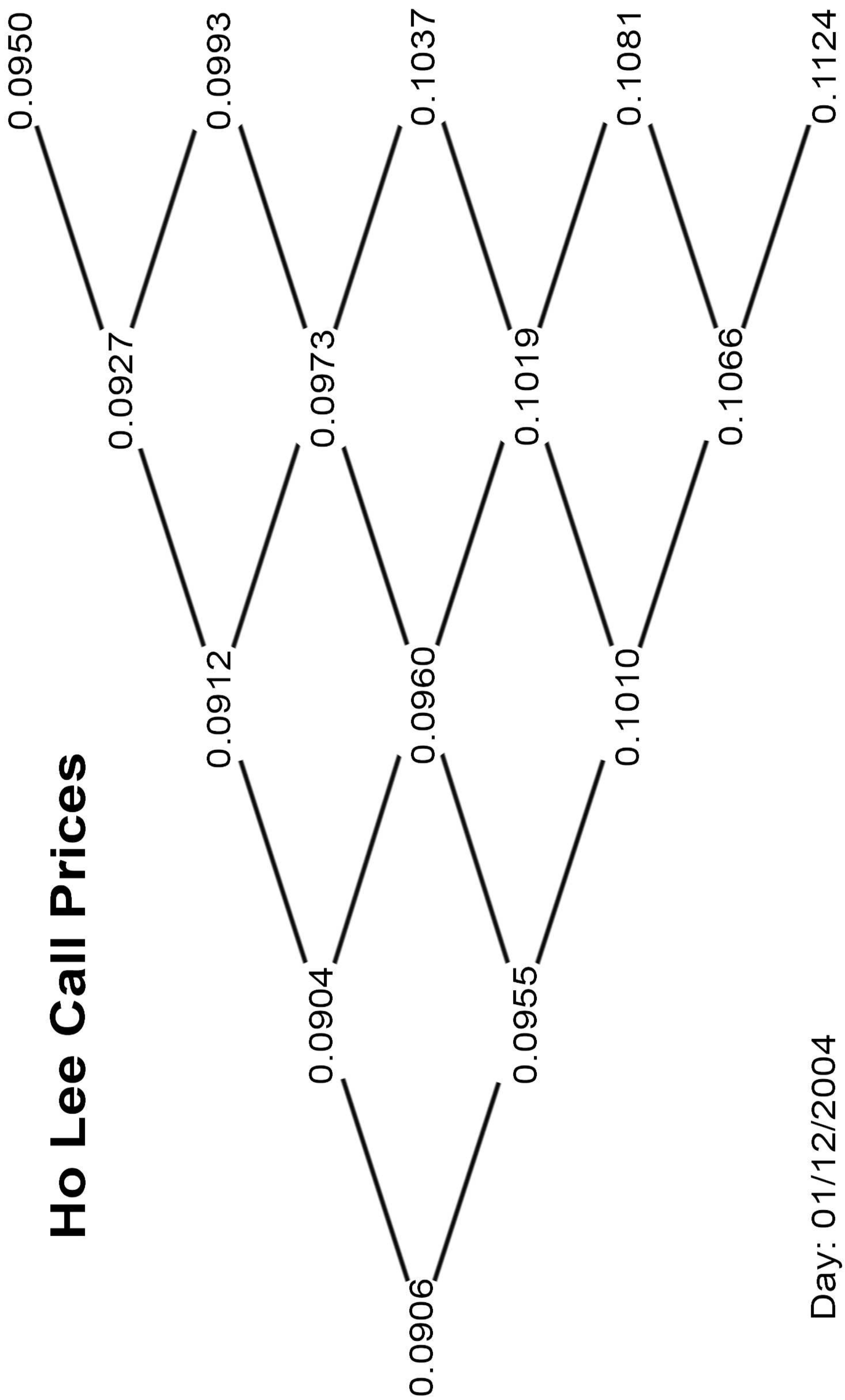

Figure 4.9: Ho Lee Model Call Option Tree on 01/12/2004

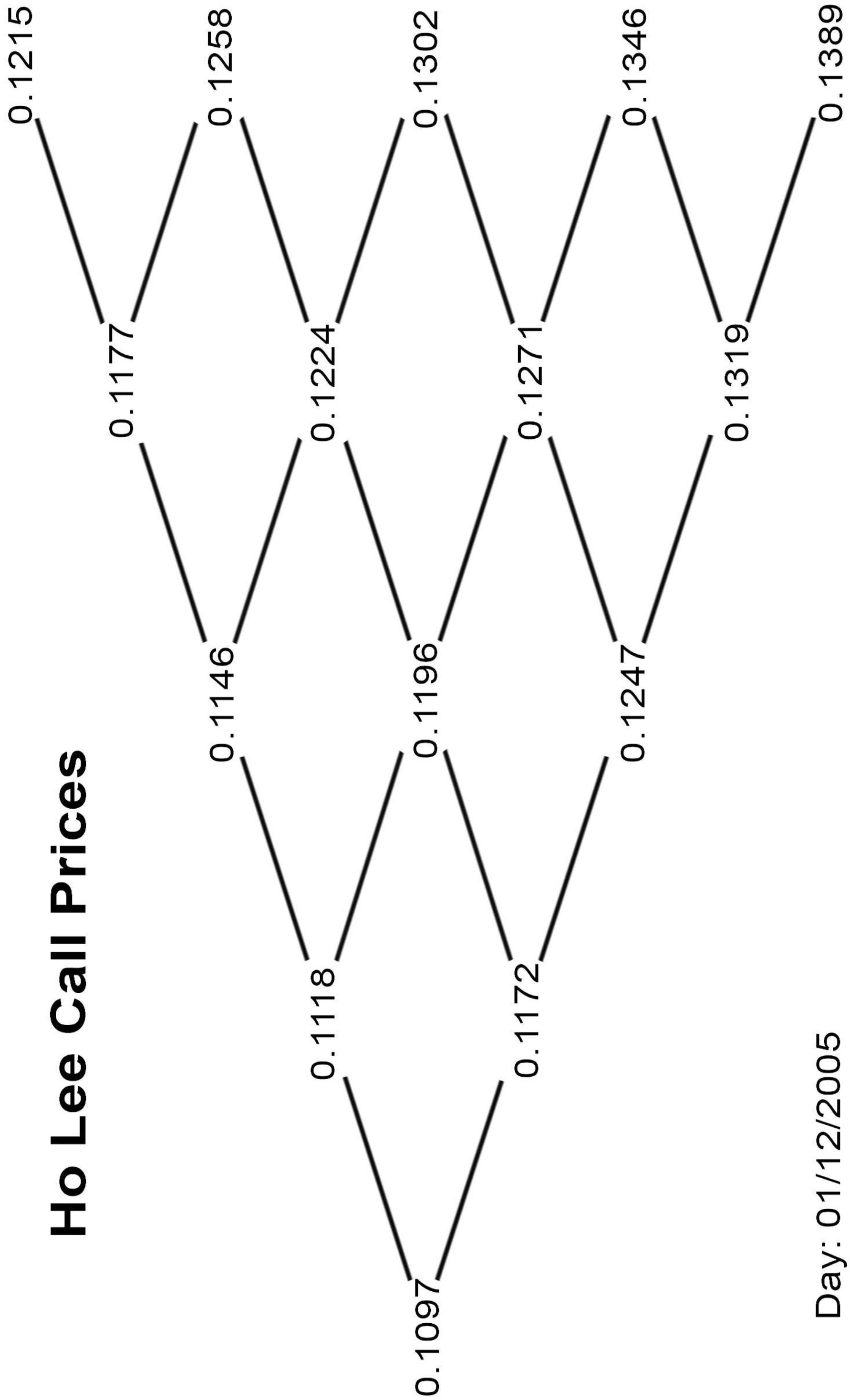

Figure 4.10: Ho Lee Model Call Option Tree on 01/12/2005

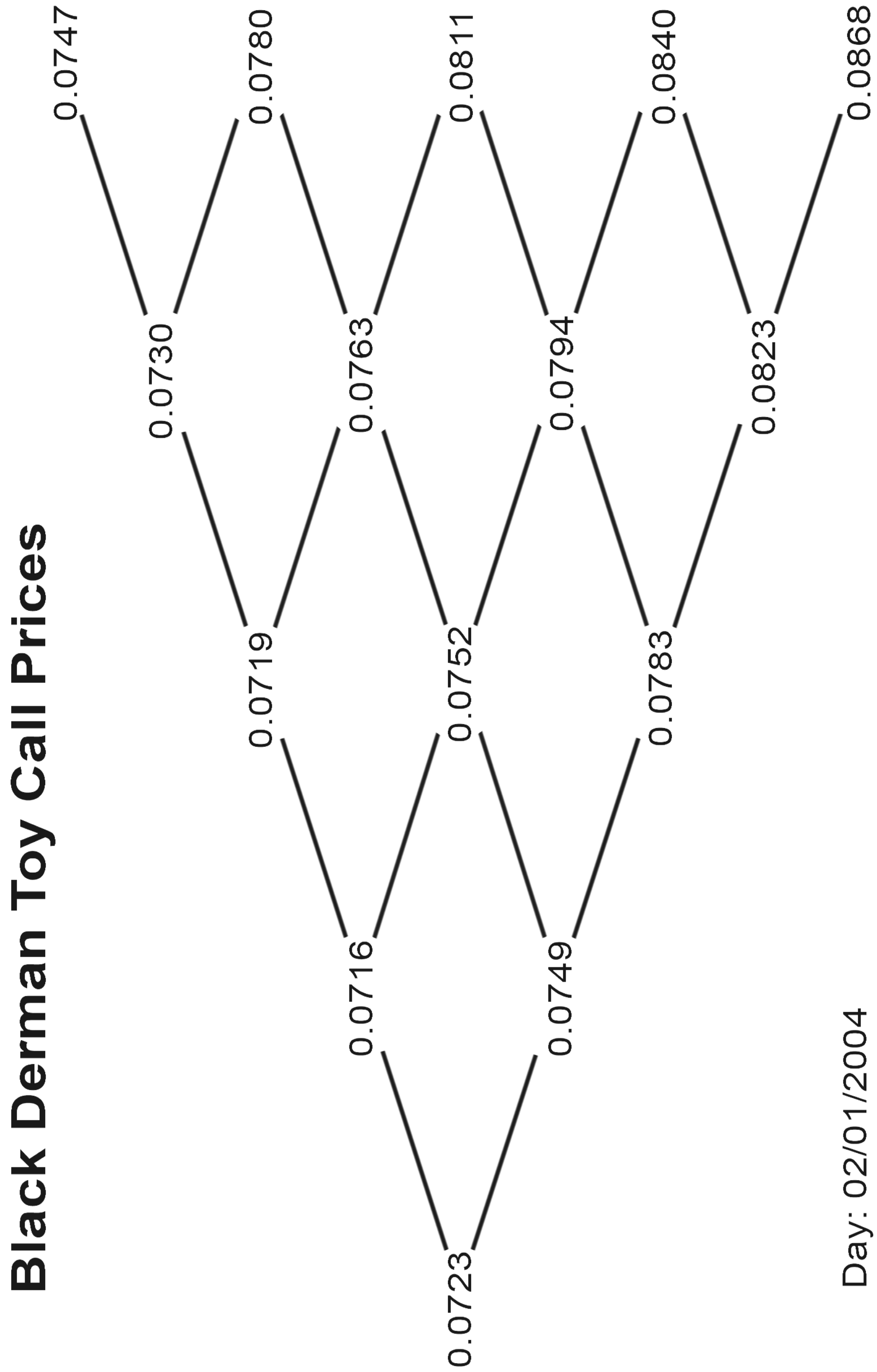

Figure 4.11: Black Derman Toy Model Call Option Tree on 02/01/2004

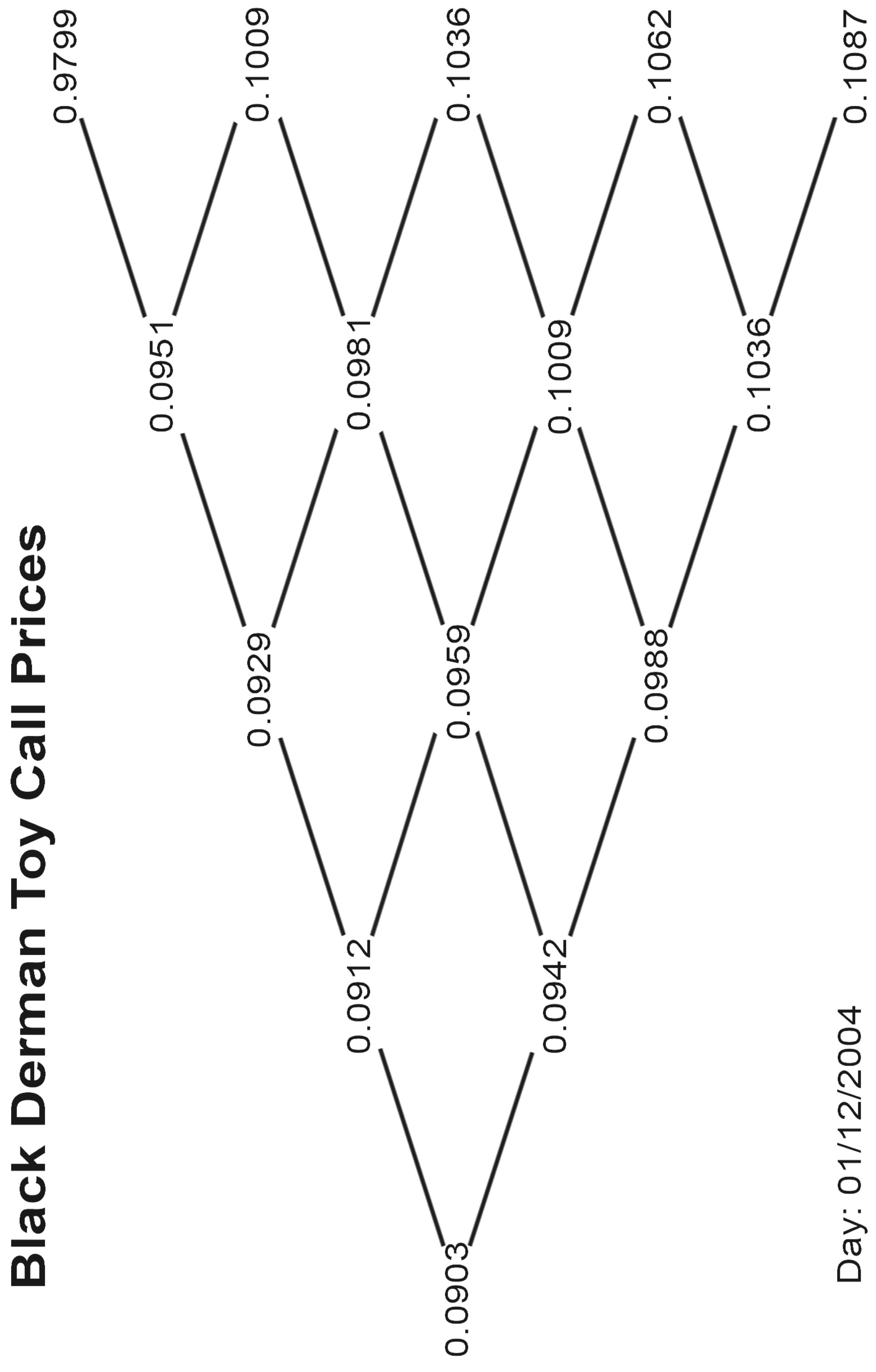

Figure 4.12: Black Derman Toy Model Call Option Tree on 01/12/2004

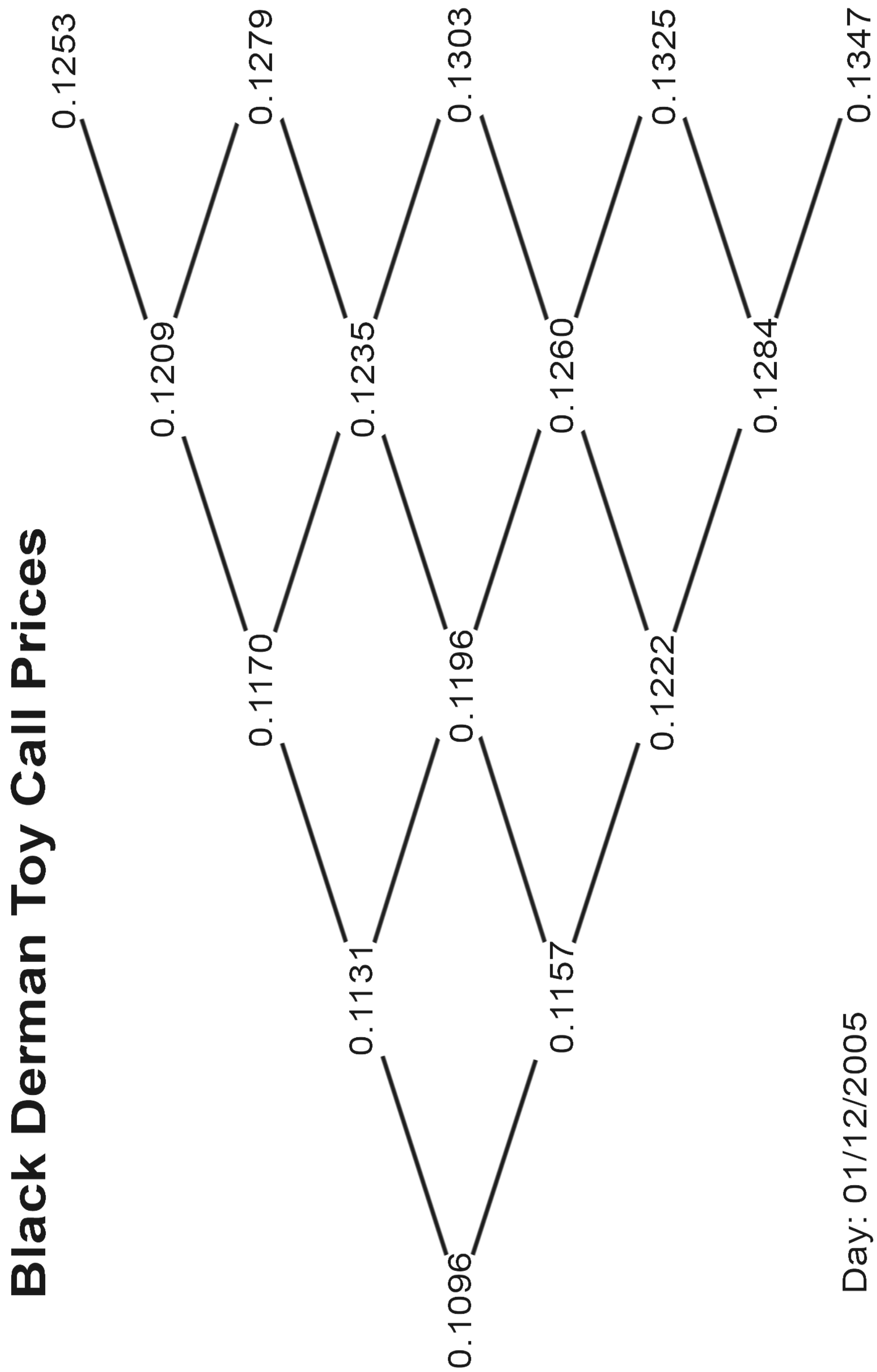

Figure 4.13: Black Derman Toy Model Call Option Tree on 01/12/2005

CHAPTER 5

CONCLUSION

In this work, we calculated the price of a European call option that is written on zero coupon bonds, by using four interest rate models: Vasicek, Cox Ingersoll Ross, Ho Lee and Black Derman Toy Models. We began our work with presenting some interest rate model's explicit solutions. Then we derived closed form solutions of bond and call option pricing of Vasicek and Cox Ingersoll Ross Models and also we mentioned some characteristics about Ho Lee and Black Derman Toy Models. Finally, we put into practice our models with United States Zero Coupon Bond with maturity time from one years to five years. Since our data set have constant maturity, we did not need to make any interpolation. To calculate call option prices with Vasicek and Cox Ingersoll Ross Models, we estimated their parameters, by using calibration method. We calibrated these models 10000 times with different initial values for each day that we analyzed. We choose the initial values that made the calibration function value the smallest. By this time, we constructed binomial trees for Ho Lee and Black Derman Toy Models for the days. The drift terms and slope of curves were also approximated while constructing binomial trees. Finally, we computed European call option prices by the closed form formulas for the Vasicek and Cox Ingersoll Ross Models and by backward induction for Ho Lee and Black Derman Toy Models.

In this study, we compared our models with respect to sum of squares errors of fitted results. According to results of SSE, it can be inferred that the Black Derman Toy Model fits the data best, while the Vasicek Model fits worst. Moreover, the Cox Ingersoll Ross Model performs better than the Ho Lee Model. As a result, the normal distributed models performed poorer than others. Moreover, the binomial models better fitted than the one factor equilibrium models.

In this thesis, we used only one factor equilibrium and no-arbitrage models. It might be advantageous to use multi-factor models for the further research.

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