ETA-ETA PRIME MIXING IN CHIRAL PERTURBATION THEORY

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ABSTRACT

ETA-ETA PRIME MIXING IN CHIRAL PERTURBATION THEORY

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Quantum Chromodynamics (QCD) is believed to be the theory of strong interactions. At high energies, it has been successfully applied to explain the interactions in accelerators. At these energies, the method used to do the calculations is perturbation theory. But at low energies, since the strong coupling constant becomes large, perturbation theory is no longer applicable. Hence, one needs non-perturbative approaches. Some of these approaches are based on the fundamental QCD Lagrangian, such as the QCD sum rules or lattice calculations. Some others use an effective theory approach to relate experimental observables one to the other. Chiral Perturbation Theory (ChPT) is one of these approaches. In this thesis, we make a review of chiral perturbation theory and its applications to study the mixing phenomenon between the neutral pseudoscalar mesons eta and eta-prime.

Keywords: Chiral Symmetry, Symmetry Breaking, Mixing, Goldstone Bosons, U(1) Axial Anomaly.

ÖZ

KİRAL TEDİRGEME KURAMINDA ETA-ETA ÜSSÜ KARIŞIMI

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Kuvantum Renk Dinamiği'nin (QCD) kuvvetli etkileşimleri açıklayan kuram olduğuna inanılmaktadr. Yüksek enerjilerde, hızlandırıcılardaki etkileşimleri açıklamakta başarılı bir şekilde kullanılmıştır. Bu enerjilerde, hesap yapmak için tedirgeme (perturbasyon) kuramı kullanılmaktadır. Ancak düşük enerjilerde, kuvvetli etkileşme sabiti büyük değerler aldığı için, tedirgeme kuramı artık kullanılamaz. Bu yüzden tedirgemeye dayanmayan yaklaşımlara ihtiyaç vardır. Bu yaklaşımlardan bir kısmı, direk olarak QCD Lagrangian'ını kullanır, QCD toplam kuralları ve örgü (latis) hesapları gibi. Bir kısmı ise, etkin kuram yaklaşımını kullanarak, deneysel gözlemlenen nicelikler arasında ilişkiler bulur. Kiral Tedirgeme Kuramı da bunlardan biridir. Bu tezde Kiral tedirgeme kuramının genel bir tekrarını yaparak bu bağlamda eta ve eta-üssü mezonlarının karışım olayını inceleyeceğiz.

Anahtar Kelimeler: Kiral Simetri, Simetri Kırılması, Karışma, Goldstone Bozonları, U(1) Aksiyel Anomali.

AİLEME
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DOSTLARIMA...

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CHAPTER 1

INTRODUCTION

In physics, it is known that there are four fundamental interactions which describe all the physical phenomena around us. These are, gravitational, weak, electromagnetic and strong interactions. In particle physics, all we know about the nature of these interactions (except gravitational interactions) is described by the so called Standard Model. Since the influence of gravitational forces are so small at present available energies, gravitational interactions are ignored in interpreting experimental results. Moreover, from the beginnings of particle physics, universality of Fermi coupling has pushed many physicists to think about the underlying symmetry for weak interactions. Looking at the charged current nature which converts a neutron to a proton in beta decay and with other weak decays the physicists thought that the responsible symmetry is $SU(2)$. First, Glashow (1961) proposed that $SU(2) \otimes U(1)$ is the possible symmetry for weak interactions. Then, Weinberg (1967) and Salam (1968) proposed spontaneously broken $SU(2)_L \otimes U(1)_Y$ [1]. After that, quarks and the strong interactions are included in this model as well and the symmetry group $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ is supposed to describe the standard model of electro-weak and strong interactions [1]. Therefore, the weak and electromagnetic interactions were combined in a single theory called Electroweak Theory and physicists became able to describe the physics of the particles that interact weakly or electromagnetically by this single theory.

Likewise, the physics of strongly interacting particles can be described by Quantum Chromodynamics (QCD). It is a non-abelian gauge theory. That is, the generators of the theory do not commute. The dynamics of the theory is described by QCD Lagrangian. In this Lagrangian the quark and gluon fields, which carry color charges and interact with coupling strength g, are included. This Lagrangian is assumed to be renormalizable and invariant under Lorentz transformations. Charge conjugation, Parity, Time reversal etc. are some examples of accidental symmetries that QCD Lagrangian possesses.

Symmetries play a very important role in physics. Most generally, if the Lagrangian of a theory possesses an invariance, as a consequence of that conserved quantities exist. There exist many global symmetries in strong interactions. For example, $SU(2)$ isospin and $SU(3)$ flavor symmetries play an important role in QCD. Moreover, if one assume that the light quark masses m_u, m_d, m_s vanishes, QCD possesses another symmetry which is called Chiral symmetry. As we will discuss in the following chapters of this thesis, spontaneous breakdown of the chiral symmetry will create states which are called pseudo-scalar Goldstone bosons [2].

In general, mesons can be grouped according to their intrinsic properties. For example, the mesons that have a total angular momentum quantum number zero and Parity eigenvalue plus ($J^P = 0^+$) are called scalar meson fields. Similarly, the mesons with $J^P = 0^-$ are named as pseudo-scalar mesons. By similar reasoning (looking at the J and P values), one can name the remaining mesons as vectors, axial vectors, tensors etc. We gave a list of some scalar and pseudo-scalar mesons in Tab.(1.1) and Tab.(1.2) respectively. Spin, mass, parity and isospin values of these particles are given there.

Chiral perturbation theory is known as the low-energy effective theory of QCD. Since the strong coupling constant α_s gets large at small momentum transfer squared, at these energies the quarks binds to each other with strong forces [3]. Thus, in the low-energy considerations, the mesons rather than the quarks, are taken as the relevant (active) degrees of freedom. Likewise, the general method used in ChPT is that the degrees of freedom that describe the dynamics of the theory are chosen to be the eight lightest pseudo-scalar fields for the reasons

Table 1.1: Mass, Spin and Isospin values of some Scalar Mesons [4].

	Spin	Parity	Mass[Mev]	Isospin
$f_0(980)$	0	+	980 ± 10	0
$f_0(1500)$	0	+	1507 ± 5	0
$f_0(1710)$	0	+	1718 ± 6	0
$a_0(980)$	0	+	984.7 ± 0.3	1
$a_0(1450)$	0	+	1474 ± 19	1
$K^*{}_0(1430)$	0	+	1414 ± 6	1/2
$D^*{}_{s0}(2317)$	0	+	2317.8 ± 0.6	0

Table 1.2: Mass, Spin and Isospin values of some Pseudo-scalar Mesons [4].

	Spin	Parity	Mass[Mev]	Isospin
π^0	0	-	134.9 ± 0.0006	1
$\pi^{\pm}$	0	-	139.6 ± 0.0004	1
K^0	0	-	497.5 ± 0.022	1/2
$K^{\pm}$	0	-	493.6 ± 0.016	1/2
η	0	-	547.5 ± 0.2	0
η'	0	-	957.8 ± 0.14	0
$D^{\pm}$	0	-	1869 ± 0.4	1/2

that will be discussed later. Actually, this theory has been developing since 1960's. Weinberg, Gasser and Leutwyler have made important studies [5–7] in the low energy hadronic physics sector and their work brought ChPT to its improved form [8]. In fact, ChPT is a very wide subject to deal with. Hence, for this thesis we limit ourself in studying the leading order $O(p^2)$ mesonic ChPT and so we do not consider the formalism of baryons in ChPT. However, for more comprehensive studies of this theory see e.g [9–21]. For the higher order analysis one can look to the studies of Ecker [12], Meissner [18] and Bijnens [8] as well. Moreover, for the recent studies of ChPT we refer the reader to see e.g [22, 23].

The organization of this thesis is as follows. In chapter 2, we discuss some important symmetries of QCD in detail. These are $SU(2)$ isospin, $SU(3)$ flavor and Chiral symmetries. We also look at the $U(1)_A$ problem by discussing the relation of it with the mixing phenomenon in the same chapter. In chapter 3, the general features of the effective field theories and the leading order (LO) mesonic chiral effective theory are reviewed in detail. Finally, in chapter 4, making use of the low-energy expansion of the chiral effective theory, the mixing phenomenon of the pseudo-scalar particles more specifically η-$\eta^{'}$ mixing is explained. In addition, we present both the theoretical and phenomenological results for the mixing parameters of η-$\eta^{'}$ and check the consistency in between.

CHAPTER 2

SYMMETRIES IN QUANTUM CHROMODYNAMICS (QCD)

In physics, symmetries play a very important role. Symmetries are generally grouped in two: space-time symmetries and internal symmetries. Space-time symmetries are the transformations that effect the space and time coordinates. Lorentz transformations, Parity, Rotations, Time reversal etc. can be given as some examples to space-time symmetries. Second group is the internal symmetries which are the transformations that do not effect the space and time coordinates. $SU(2)$ isospin, $SU(3)$ flavor, Baryon, Lepton, Strangeness number conservations, Charge conservation, $SU(3)$ Color etc. are some examples of internal symmetries.

The existence of a symmetry in a particular theory follows from the invariance principle. According to Noether's theorem, if there is a symmetry in a system this symmetry always requires an existence of a conserved quantity in that system. For example, if we have an invariance under time translation, the total energy of our system is conserved; if the system has an invariance under space translation, the linear momentum of the system is conserved. Similarly, if we have a rotational symmetry the angular momentum is conserved [24].

In the following sections, we will examine the basic nature of some symmetries of QCD and look also at the so called the spontaneous symmetry breaking phenomenon and its consequences.

2.1 $SU(2)$ Isospin and $SU(3)$ Flavor Symmetries

Isospin symmetry has initially been introduced for hadron multiplets like the doublet of proton and neutron, triplet of pions etc., and then has been examined in quark level. It is also called the $SU(2)$ symmetry of QCD. It mainly states to treat the up and down quarks as the same constituents i.e. in an exact isospin symmetry conditions we require $m_u = m_d$. However, in the real world we know that the up and down quarks masses differ slightly ($m_u \sim 6MeV, m_d \sim 10MeV$) and thus isospin symmetry is broken. Moreover, since the up quark has a charge of $+\frac{2}{3}e$ whereas the down quark has charge $-\frac{1}{3}e$, where e is the charge of the proton, the electromagnetic interactions break the isospin symmetry as well. Nevertheless, the isospin symmetry considerations are very helpful in many applications of particle physics [25].

In $SU(2)$ symmetry, the central assumption is that the up and down quarks are associated with the same total isospin quantum number $I = \frac{1}{2}$, whereas they differ in their third component of isospin quantum number as;

$$|u\rangle = |I = \frac{1}{2}\ ,\ I_3 = \frac{1}{2}\rangle \quad , \quad |d\rangle = |I = \frac{1}{2}\ ,\ I_3 = -\frac{1}{2}\rangle$$

and for all the remaining quarks the isospin quantum number is assigned to be zero. By such a definition the up and down quarks can be treated as the two different projections of the same state. This representation of $SU(2)$ symmetry allow us to combine the particles into groups called multiplets. For example, according to this definition the proton which has a quark content of uud and a neutron with quark content udd form an $SU(2)$ doublet. Similarly, the pions π^+ , π^- and π^0 can be regarded equally and form a triplet. If we neglect the EM interactions (charge difference), there would be no way to distinguish among the particles within the same multiplet.

Unlike the weak and the EM interactions, the strong interactions are assumed to conserve the isospin symmetry. The isospin symmetry can be thought as an invariance under a rotation in the isospin space which is described by a so called isospin operator [25];

$$P_{IS} = e^{-i\vec{\sigma}.\vec{\theta}} \tag{2.1}$$

where σ^i $(i = 1, 2, 3)$ are the usual 2×2 Pauli matrices which are the generators of $SU(2)$ and θ^i's are the elements of an arbitrary constant vector rotation angle defined in the isospin space. Now, let us check the the invariance of the usual QCD Lagrangian under such a transformation;

$$\mathcal{L}_{QCD} = \bar{\Psi} i \mathcal{D}_\mu \gamma^\mu \Psi - \bar{\Psi} \mathcal{M} \Psi - \frac{1}{2} Tr(G_{\mu\nu} G^{\mu\nu}) \tag{2.2}$$

with

$$\Psi = \begin{pmatrix} u \\ d \end{pmatrix} \quad , \quad \mathcal{M} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$$

$$D_\mu = \partial_\mu - igG_\mu \quad , \quad G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu - ig[G_\mu, G_\nu] \tag{2.3}$$

The isospin transformations do not effect the gluon fields. Thus, first let us apply the isospin operator to the first term as;

$$\Psi' \rightarrow e^{-i\vec{\sigma}.\vec{\theta}} \Psi \quad , \quad \bar{\Psi}' \rightarrow \Psi^\dagger e^{i\vec{\sigma}.\vec{\theta}} \gamma^0 = \bar{\Psi} e^{i\vec{\sigma}.\vec{\theta}} \tag{2.4}$$

where, we used the fact that since the the Pauli matrices are defined in isospin space whereas the gamma matrices in spinor space, they commute and as a result of this one can interchange the places of $e^{i\vec{\sigma}.\vec{\theta}}$ and γ^0 freely. Then, if we continue we can write;

$$\bar{\Psi} i \mathcal{D}_\mu \gamma^\mu \Psi \rightarrow \bar{\Psi} e^{i\vec{\sigma}.\vec{\theta}} i \mathcal{D}_\mu \gamma^\mu e^{-i\vec{\sigma}.\vec{\theta}} \Psi = \bar{\Psi} i \mathcal{D}_\mu \gamma^\mu \Psi \tag{2.5}$$

Hence, we have seen that the first term is invariant under isospin transformation. Now, lets check whether the mass term of the Lagrangian is invariant or not. For this we can write;

$$\bar{\Psi}\mathcal{M}\Psi \rightarrow \bar{\Psi}e^{i\vec{\sigma}.\vec{\theta}}\mathcal{M}e^{-i\vec{\sigma}.\vec{\theta}}\Psi \tag{2.6}$$

where, we can write the mass matrix $\mathcal{M}$ as;

$$\begin{aligned}
\mathcal{M} &= (\frac{m_u + m_d}{2})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (\frac{m_u - m_d}{2})\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \\
&= (\frac{m_u + m_d}{2})\mathbf{1}_{2\times 2} + (\frac{m_u - m_d}{2})\sigma_3
\end{aligned} \tag{2.7}$$

Then, following our steps one can write;

$$\begin{aligned}
e^{i\vec{\sigma}.\vec{\theta}}\mathcal{M}e^{-i\vec{\sigma}.\vec{\theta}} &\rightarrow e^{i\vec{\sigma}.\vec{\theta}}[(\frac{m_u + m_d}{2})\mathbf{1}_{2\times 2} + (\frac{m_u - m_d}{2})\sigma_3]e^{-i\vec{\sigma}.\vec{\theta}} \\
&= (\frac{m_u + m_d}{2})\mathbf{1}_{2\times 2} + (\frac{m_u - m_d}{2})\underbrace{e^{i\vec{\sigma}.\vec{\theta}}\sigma_3 e^{-i\vec{\sigma}.\vec{\theta}}}
\end{aligned} \tag{2.8}$$

Now, by checking the invariance of the second term one can get;

$$\begin{aligned}
e^{i\vec{\sigma}.\vec{\theta}}\sigma_3 e^{-i\vec{\sigma}.\vec{\theta}} &= (\cos\theta + i\vec{\sigma}\hat{\theta}\sin\theta)\sigma_3(\cos\theta - i\vec{\sigma}\hat{\theta}\sin\theta) \\
&= (\cos^2\theta + \sin^2\theta)\sigma_3 - i[\sigma_3, \sigma_i]\hat{\theta}\sin\theta\cos\theta \\
&= \sigma_3\mathbf{1}_{2\times 2} - i\hat{\theta}\varepsilon_{3ik}\sigma_k\sin\theta\cos\theta \neq \sigma_3\mathbf{1}_{2\times 2}
\end{aligned} \tag{2.9}$$

As a result of this, we have seen that only when the mass difference between u and d quarks vanishes the whole QCD lagrangian becomes invariant under isospin transformations. Therefore, we can conclude that the central reason of isospin symmetry breaking is the mass difference of u and d quarks ($m_u - m_d \sim 4MeV$) and the strength of this isospin symmetry breaking can be estimated by looking at the ratio [25];

$$\frac{(m_u - m_d)}{(m_u + m_d)} \sim 0.3 \tag{2.10}$$

Furthermore, one can define the so called $U - spin$ and $V - spin$ symmetries within the realm of $SU(2)$ group. The $U - spin$ symmetry states that the strange and the down quarks can be treated equally ($d \leftrightarrow s$). It is described by the transformation [26];

$$\begin{pmatrix} d \\ s \end{pmatrix} \rightarrow e^{-i\vec{\sigma}.\vec{\theta}} \begin{pmatrix} d \\ s \end{pmatrix} \tag{2.11}$$

Since the down and strange quarks have the same charge value ($-\frac{1}{3}e$), unlike to isospin symmetry the EM interactions do not break the U-spin symmetry. In fact, this can be understood from the commutation of U-spin generators with the electric charge operator $\hat{Q}$ [26]. Letting $q^j = d, s$ and σ^i as the U-spin generators we can write;

$$\sigma^i \hat{Q}|q^j\rangle = \sigma^i e_q |q^j\rangle = e_q \underbrace{(\sigma^i |q^j\rangle)} \tag{2.12}$$

where, letting $\sigma^i|q^j\rangle = |q^{'j}\rangle$ one can write;

$$\hat{Q}(\sigma^i|q^j\rangle) = e_{q'}(\sigma^i|q^j\rangle) \tag{2.13}$$

Now, since the charge values of d and s quarks are equal, for U-spin considerations we can take $e_q = e_{q'}$ so that we obtain;

$$\hat{Q}(\sigma^i|q^j\rangle) = \sigma^i\hat{Q}|q^j\rangle \Rightarrow [\hat{Q}\sigma^i - \sigma^i\hat{Q}]|q^j\rangle = 0 \tag{2.14}$$

or

$$[\hat{Q}, \sigma^i] = 0 \tag{2.15}$$

Similarly, one can define a symmetry called V-spin which treats the up and strange quarks equally ($u \leftrightarrow s$) and has a transformation property in analogy to isospin or U-spin symmetries as;

$$\begin{pmatrix} u \\ s \end{pmatrix} \rightarrow e^{-i\vec{\sigma}.\vec{\theta}} \begin{pmatrix} u \\ s \end{pmatrix} \tag{2.16}$$

Following exactly the same procedure as we did for U-spin symmetry, we can check the commutator of the electric charge operator Q with the generators of V-spin symmetry. Because of the charge difference of u and s quarks ($e_q \neq e_{q'}$), unlike to the U-spin symmetry case, in V-spin symmetry the commutator is not zero. Therefore, EM interactions break the V-spin symmetry.

Moreover, following the same reasoning as we did for isospin symmetry case, it is also possible to see that the U-spin symmetry is broken due to the mass difference of d and s quarks ($m_s - m_d \sim 130 MeV$). Similarly, V-spin symmetry is broken by the mass difference of u and s quarks ($m_s - m_u \sim 135 MeV$). As the ratio $(m_q - m_s)/(m_q + m_s) \simeq 1$ where $q = u, d$, the breaking of these symmetries can be large.

To continue, if we include the strange quark to our considerations of $SU(2)$ we would be able to define another symmetry, called $SU(3)$ flavor symmetry. This symmetry mainly states that the up, down and strange quarks can be treated similarly providing that their mass differences can be neglected and the electromagnetic interactions are ignored. In analogy to isospin case, the $SU(3)$ flavor transformations transform the quark fields Ψ as [26];

$$\Psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \rightarrow \Psi' = exp(-i\lambda^a.\theta^a) \begin{pmatrix} u \\ d \\ s \end{pmatrix} \tag{2.17}$$

where, λ^a with $a = 1, 2, 3...., 8$ are the usual Gell-Mann matrices which are the generators of $SU(3)$ and θ^a are arbitrary phase factors.

Like in the isospin symmetry case, $SU(3)$ flavor symmetry allow us to collect particles that have similar properties into the same groups. The important point in grouping the particles is that the particles in the same multiplet should have the same spin and masses. For example, in an exact $SU(3)$ flavor symmetry conditions the eight pseudoscalar states; $\pi^{\pm}, \pi^0, K^{\pm}, K^0$ and η, form an $SU(3)$ octet. Similarly, η' is considered as an $SU(3)$ singlet state [see Table.(1.2)]. Actually, the idea of grouping the particles into octet and a singlet state comes from the group theoretical knowledge since in group theory language one can write;

$$3 \otimes \bar{3} = 8 \oplus 1 \tag{2.18}$$

which means that the bound states of a quark (3) and an anti-quark ($\bar{3}$) can be formed either in octet(8) or singlet(1) representation. $SU(3)$ flavor symmetry does not give us any relation between the particles of the octet and the singlet groups. It only relates the parameters of the particles that are in the same multiplet. In the real world, the flavor symmetry is broken due to the large mass of strange quark compared to the masses of up and down quarks and also by EM interactions. Therefore, we do not expect the particles in the same multiplet to have exactly same masses. Because of that reason, they may differ slightly.

In order to check the strength of symmetry breaking in a theory, mostly some parameters are compared with a specified scale or limit that comes from the theory. In our case, comparing the strange quark mass ($\sim 150MeV$) with the QCD confinement scale $\Lambda_{QCD} \sim 250MeV$, one can conclude that the $SU(2)$ isospin symmetry can be regarded as a good symmetry of QCD, whereas the $SU(3)$ flavor symmetry is an approximate symmetry [27].

The final point we want to mention in this section is that the breaking of the $SU(2)$ isospin or $SU(3)$ flavor symmetries have important consequences. For example, the breaking of the $SU(2)$ isospin symmetry causes mixing between the physical particles $\pi - \eta - \eta^{'}$. Similarly, as a result of $SU(3)_F$ symmetry breaking, even if the $SU(2)$ subgroup is not broken, the physical particles η and $\eta^{'}$ can mix. We will examine the mixing phenomenon in chapter 4 in detail.

2.2 Chiral Symmetry

At relatively low energies ($E \ll 1\ GeV$) the relevant degrees of freedom of the Standard Model are the hadrons (mesons, baryons) rather than the quarks or gluons. In this energy limit the QCD Lagrangian is described only in terms of the light quarks (u,d,s) and the other quarks (c,t,b) are assumed infinitely heavy and frozen sources so that they are excluded from the theory [12]. In this limit, the fermionic part of the QCD Lagrangian has the form;

$$\mathcal{L}_{QCD} = \sum\nolimits_{q=u,d,s} \bar{q}(i\gamma_\mu \partial_\mu - m_q)q \tag{2.19}$$

One can define a left and right handed quark fields;

$$q_L = \frac{1-\gamma_5}{2}q \quad , \quad q_R = \frac{1+\gamma_5}{2}q \tag{2.20}$$

such that $q_L + q_R = q$. Then, the Lagrangian can be written as;

$$\mathcal{L}_{QCD} = \sum\nolimits_{q=u,d,s} (\bar{q_L} + \bar{q_R}) i\gamma_\mu \partial_\mu (q_L + q_R) - m_q(\bar{q_L} + \bar{q_R})(q_L + q_R)$$

$$(2.21)$$

or

$$\begin{aligned}\mathcal{L}_{QCD} &= \underbrace{\bar{q_L} i\gamma_\mu\partial_\mu q_L} + \underbrace{\bar{q_L} i\gamma_\mu\partial_\mu q_R} + \underbrace{\bar{q_R} i\gamma_\mu\partial_\mu q_L} + \underbrace{\bar{q_R} i\gamma_\mu\partial_\mu q_R} \\ &- m_q\{\bar{q_L} q_L + \bar{q_L} q_R + \bar{q_R} q_L + \bar{q_R} q_R\} \end{aligned} \tag{2.22}$$

Here, using some properties of the Dirac matrices and the quark fields as;

$$\bar{q} = q^\dagger\gamma^0 \quad , \quad \gamma^{5\dagger} = \gamma^5 \quad , \quad (\gamma^5)^2 = \mathbf{1} \quad , \quad \{\gamma^{\mathbf{5}}, \gamma^\mu\} = \mathbf{0} \tag{2.23}$$

one can write;

$$\begin{aligned}\mathcal{L}_{QCD} &= \bar{q}(\frac{1+\gamma_5}{2}) i\gamma_\mu\partial_\mu(\frac{1-\gamma_5}{2})q + \bar{q}(\frac{1-\gamma_5}{2}) i\gamma_\mu\partial_\mu(\frac{1+\gamma_5}{2})q \\ &+ \bar{q}(\frac{1+\gamma_5}{2}) i\gamma_\mu\partial_\mu(\frac{1+\gamma_5}{2})q + \bar{q}(\frac{1-\gamma_5}{2}) i\gamma_\mu\partial_\mu(\frac{1-\gamma_5}{2})q \\ &- m_q\{\bar{q}(\frac{1+\gamma_5}{2})(\frac{1-\gamma_5}{2})q + \bar{q}(\frac{1+\gamma_5}{2})(\frac{1+\gamma_5}{2})q \\ &+ \bar{q}(\frac{1-\gamma_5}{2})(\frac{1-\gamma_5}{2})q + \bar{q}(\frac{1-\gamma_5}{2})(\frac{1+\gamma_5}{2})q\} \end{aligned} \tag{2.24}$$

Besides, using $(1-\gamma_5)(1+\gamma_5) = 0$ and $(1\pm\gamma_5)^2 = 2(1\pm\gamma_5)$ the QCD Lagrangian takes the form;

$$\mathcal{L}_{QCD} = \bar{q_L} i\gamma_\mu\partial_\mu q_L + \bar{q_R} i\gamma_\mu\partial_\mu q_R - m_q\{\bar{q_L} q_R + \bar{q_R} q_L\} \tag{2.25}$$

At this point, let us check whether this Lagrangian is invariant or not under $SU(3)$ transformations which transform the left and the right handed quark fields differently [26];

$$q_L \to L q_L \quad , \quad q_R \to R q_R \tag{2.26}$$

where, $L \in SU(3)_L$ and $R \in SU(3)_R$ are defined as;

$$L = e^{i\alpha_L{}^n \lambda^n} \quad , \quad R = e^{i\alpha_R{}^n \lambda^n} \quad n = 1, ..., 8 \tag{2.27}$$

Using, Eqn.(2.23), the QCD Lagrangian (Eqn.(2.25)) transforms as;

$$\begin{aligned} \mathcal{L}_{QCD} &\to \bar{q_L} e^{-i\alpha_L{}^n\lambda^n} i\gamma_\mu \partial_\mu e^{i\alpha_L{}^n\lambda^n} q_L + \bar{q_R} e^{-i\alpha_R{}^n\lambda^n} i\gamma_\mu \partial_\mu e^{i\alpha_R{}^n\lambda^n} q_R \\ &- m\{\bar{q_L} e^{-i\alpha_L{}^n\lambda^n} i\gamma_\mu \partial_\mu e^{i\alpha_R{}^n\lambda^n} q_R + \bar{q_R} e^{-i\alpha_R{}^n\lambda^n} i\gamma_\mu \partial_\mu e^{i\alpha_L{}^n\lambda^n} q_L\} \end{aligned} \tag{2.28}$$

Making the cancelations, we can see that the kinetic terms remain invariant under these transformations whereas the mass terms do not. The mass term mixes the Left and Right handed states with each other. Thus, if we assume the up, down and strange quark masses are zero (chiral limit), the $SU(3)_L \times SU(3)_R$ transformations defined in Eqn.(2.27) leaves QCD Lagrangian invariant. The corresponding symmetry is called "Chiral Symmetry of QCD" [2]. Moreover, even in the case of finite quark masses, if we let $\alpha_L = \alpha_R = \alpha_V$ in Eqn.(2.27), one can defines a symmetry called $SU(3)$ vectorial symmetry ($SU(3)_V$) under which the Lagrangian remains invariant.

According to Noether's Theorem, to each symmetry corresponds a conserved current J^μ such that $\partial_\mu J^\mu = 0$. If one defines [24];

$$Q = \int d^3x . j_0 \tag{2.29}$$

where, Q is a conserved charge we can write;

$$\frac{dQ}{dt} = i[H, Q] = 0 \quad , \quad \Rightarrow \quad [H, Q] = 0. \tag{2.30}$$

Associated with the $SU(3)_L$ and $SU(3)_R$ algebra, we would have conserved charges Q_L^a and Q_R^a ($a = 1, 2, ..., 8$), respectively. We can define Vectorial and Axial charges as;

$$Q_V^a = Q_L^a + Q_R^a \quad , \quad Q_A^a = Q_R^a - Q_L^a \tag{2.31}$$

We can see that under parity transformation ($R \to L$, $L \to R$) they are transformed as;

$$Q_V^a \to Q_V^a \quad , \quad Q_A^a \to -Q_A^a \tag{2.32}$$

The Left-Right symmetry requires that the states $Q_V^a|\phi^a\rangle$ and $Q_A^a|\phi^a\rangle$ should have the same energy. This means that for each particle with specific spin and parity, there should exist another one with the same spin value but opposite parity [8]. However, if we look at the mass spectrum of the particles we do not see such a property. For example, comparing the masses of scalar particles with those of pseudo scalars [see Tab.(1.1) and Tab.(1.2)], we can convince ourself that this requirement does not exist at all. The answer to this problem was given by Nambu-Goldstone [28]. They have realized that the eight Axial charges do not annihilate the vacuum state while the Vectorial charges do [11].

$$Q_V^a|0\rangle = 0 \quad , \quad Q_A^a|0\rangle \neq 0 \tag{2.33}$$

In this case, since the full symmetry group of the Hamiltonian is not shared by the ground state (vacuum), the Chiral $SU(3)_L \times SU(3)_R$ symmetry is said to be broken spontaneously. In that case, there is no need for masses of scalar and pseudo scalars to be the same. Thus, the puzzle is solved. Spontaneous breaking of a continuous global symmetry is always accompanied by the existence of massless and spinless particles. This is known as the "Goldstone Theorem" and the particles appearing as a result of the spontaneous symmetry breaking are called "Goldstone Bosons". Since we have eight axial charges that do not annihilate the vacuum, eight massless and spinless particles should appear. If we let $Q_A^a|0\rangle = |\phi^a\rangle$ one can write;

$$H|\phi^a\rangle = HQ_A^a|0\rangle = Q_A^a \underbrace{H|0\rangle} = 0 \tag{2.34}$$

When we look at the particle spectrum we can see that the best candidates for the above discussions are the eight lightest hadrons $\pi^{\pm}, \pi^0, K^{\pm}, K^0$ and η [4]. In the real world, since the quark masses are finite, we expect these eight pseudo scalar particles to have masses different from zero.

Now, as a last subject to this section let us check the validity of Goldstone Theorem which states that "only when a continuous symmetry is broken massless and spinless particles appear and the number of massless bosons is equal to the number of broken symmetry generators".

Spontaneous Breaking of a Discrete Symmetry

For discrete symmetry case, let us assume we have a Lagrangian density described in terms of a scalar field ϕ as;

$$\mathcal{L} = \frac{1}{2}|\partial_\mu \phi|^2 + \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4 \tag{2.35}$$

We can see that under the discrete symmetry transformation $\phi \to -\phi$ (Reflection transformation), the Lagrangian is left invariant, since the field ϕ appears in the Lagrangian only in even powers. Now, let us first find the ground state value of the potential $V(\phi^2) = -\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4$. Here, in order for the potential energy to be bounded from below, λ should be positive [29]. However, m^2 can be both positive or negative. When $m^2 < 0$, the form of the potential $V(\phi^2)$ represent a parabola with a unique minimum at $\phi = 0$. On the other hand, if $m^2 > 0$, the ground state develops two values different from zero which can be found by differentiating the potential with respect to ϕ^2 and then equating it to zero as;

$$\begin{aligned} \frac{\partial V}{\partial \phi^2} = 0 \quad & \Rightarrow \quad -\frac{m^2}{2} + \frac{\lambda\phi^2}{2} = 0 \\ \phi \quad & = \quad \pm\sqrt{\frac{m^2}{\lambda}} \end{aligned} \tag{2.36}$$

At that point, redefining our field ϕ in terms of a new field η which is a fluctuation around this ground state as;

$$\eta \to \phi - \sqrt{\frac{m^2}{\lambda}} \quad \Rightarrow \quad \phi = \eta + \sqrt{\frac{m^2}{\lambda}} \tag{2.37}$$

and expressing our Lagrangian in terms of this new field one can write;

$$\begin{aligned} \mathcal{L} \; &= \frac{1}{2}\partial_\mu(\eta + \sqrt{\frac{m^2}{\lambda}})\partial^\mu(\eta + \sqrt{\frac{m^2}{\lambda}}) + \frac{1}{2}m^2(\eta + \sqrt{\frac{m^2}{\lambda}})^2 - \frac{\lambda}{4}(\eta + \sqrt{\frac{m^2}{\lambda}})^4 \\ \Rightarrow \; \mathcal{L}' &= \frac{1}{2}|\partial_\mu\eta|^2 + \frac{1}{2}m^2\eta^2 + \frac{m^4}{2\lambda} + m^2\sqrt{\frac{m^2}{\lambda}}\eta \\ - \; & \frac{\lambda}{4}\{\eta^4 + \frac{m^4}{\lambda^2} + \frac{2m^2}{\lambda}\eta^2 + \frac{4m^2}{\lambda}\eta^2 + 4\sqrt{\frac{m^2}{\lambda}}\eta^3 + 4\frac{m^2}{\lambda}\sqrt{\frac{m^2}{\lambda}}\eta\} \end{aligned} \tag{2.38}$$

or

$$\begin{aligned} \mathcal{L} = \frac{1}{2}|\partial_\mu\eta|^2 + \frac{1}{2}m^2\eta^2 + \frac{m^4}{2\lambda} + m^2\sqrt{\frac{m^2}{\lambda}}\eta - \frac{\lambda}{4}\eta^4 \\ -\frac{m^4}{4\lambda} - \frac{m^2}{2}\eta^2 - m^2\eta^2 - \lambda\sqrt{\frac{m^2}{\lambda}}\eta^3 - m^2\sqrt{\frac{m^2}{\lambda}}\eta \end{aligned} \tag{2.39}$$

Finally, after doing the cancelations the Lagrangian turns to be;

$$\mathcal{L} = \frac{1}{2}|\partial_\mu \eta|^2 - m^2\eta^2 + \frac{m^4}{4\lambda} - \frac{\lambda}{4}\eta^4 - \lambda\sqrt{\frac{m^2}{\lambda}}\eta^3 \tag{2.40}$$

Now, we can see that by this new parametrization, due to the presence of η^3 term in $\mathcal{L}$, our initial reflection symmetry has been spontaneously broken and η field has a mass!

Spontaneous Breaking of a Continuous Symmetry

Now, let us check what happens when we spontaneously break a continuous symmetry. For this purpose, suppose we have a Lagrangian density described in terms of two real fields ϕ_1 and ϕ_2 as;

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 + \frac{1}{2}m^2({\phi_1}^2 + {\phi_2}^2) - \frac{\lambda^2}{4}({\phi_1}^2 + {\phi_2}^2)^2 \tag{2.41}$$

Initially, looking at the structure of this Lagrangian one can see that it has an invariance under the continuous $SO(2)$ transformations defined as;

$$\begin{aligned} \phi_1 &\rightarrow \phi_1 \cos\theta + \phi_2 \sin\theta \\ \phi_2 &\rightarrow -\phi_1 \sin\theta + \phi_2 \cos\theta \end{aligned} \tag{2.42}$$

Then, following the same reasoning as we did for discrete symmetry case, in order to break the symmetry we should replace the fields ϕ_1 and ϕ_2 by the new fields, which are fluctuations about the corresponding ground states. Therefore, first let us find the corresponding ground states of the potential;

$$V({\phi_1}^2 + {\phi_2}^2) = -\frac{1}{2}m^2({\phi_1}^2 + {\phi_2}^2) + \frac{\lambda^2}{4}({\phi_1}^2 + {\phi_2}^2)^2$$

by differentiating it with respect to ${\phi_1}^2 + {\phi_2}^2$ and equating it to zero as;

$$\frac{\partial V}{\partial({\phi_1}^2+{\phi_2}^2)}=0 \qquad \Rightarrow \qquad -\frac{m^2}{2}+\frac{\lambda^2}{2}({\phi_1}^2+{\phi_2}^2)=0$$
$${\phi_1}^2+{\phi_2}^2 = \frac{m^2}{\lambda^2} \tag{2.43}$$

Here, we are free to choose one of the fields to be zero. Then, we can write [24];

$$\phi_1=\frac{m}{\lambda} \quad , \quad \phi_2=0 \tag{2.44}$$

Introducing new fields that are fluctuations around these ground states as;

$$\eta=\phi_1-\frac{m}{\lambda} \quad \Rightarrow \quad \phi_1=\eta+\frac{m}{\lambda} \quad , \quad \phi_2=\phi_2 \tag{2.45}$$

the new Lagrangian turns to be;

$$\begin{aligned}\mathcal{L}' &= \frac{1}{2}(\partial_\mu\eta)^2+\frac{1}{2}(\partial_\mu\phi_2)^2+\frac{1}{2}m^2(\eta^2+\frac{2m}{\lambda}\eta+\frac{m^2}{\lambda^2}+{\phi_2}^2)\\ &- \frac{\lambda^2}{4}\{(\eta^2+\frac{2m}{\lambda}\eta+\frac{m^2}{\lambda^2}+{\phi_2}^2)^2\}\end{aligned} \tag{2.46}$$

or

$$\begin{aligned}\mathcal{L}' &= \frac{1}{2}(\partial_\mu\eta)^2+\frac{1}{2}(\partial_\mu\phi_2)^2+\frac{1}{2}m^2\eta^2+\frac{m^3}{\lambda}\eta+\frac{m^4}{4\lambda^2}+\frac{m^2}{2}{\phi_2}^2-\frac{m^2}{2}{\phi_2}^2\\ &- \frac{\lambda^2}{4}(\eta^4+2\eta^2{\phi_2}^2+{\phi_2}^4)-m^2\eta^2-m\lambda\eta^3-\frac{m^2}{2}\eta^2-\frac{m^3}{\lambda}\eta-m\lambda\eta{\phi_2}^2\end{aligned} \tag{2.47}$$

Hence, doing the simplifications we arrive at [24];

$$\begin{aligned}\mathcal{L}' = [\frac{1}{2}(\partial_\mu\eta)^2-m^2\eta^2]+[\frac{1}{2}(\partial_\mu\phi_2)^2]+\frac{m^4}{4\lambda^2}\\ +[m\lambda(\eta^3+\eta{\phi_2}^2)-\frac{\lambda^2}{4}(\eta^2+{\phi_2}^2)^2]\end{aligned} \tag{2.48}$$

Then, at that stage looking at the form of this new Lagrangian, due the presence of the terms like η^2,η^3 and $\eta{\phi_2}^2$ it is no longer invariant under the continuous $SO(2)$ transformations. The symmetry is broken spontaneously by this ground state. As a result of this symmetry breaking we see that in the new Lagrangian the mass term for the field ϕ_2 has disappeared in accordance with the Goldstone theorem.

2.3 $U(1)_A$ Anomaly

In some circumstances, a particular transformation may leaves the standard QCD Lagrangian invariant. However, one may observe that in quantum theory the divergence of the axial vector current, which is associated with the invariance of the classical Lagrangian, does not vanish ($\partial_\mu J^\mu \neq 0$). In particle physics, this problem is referred as "Anomaly".

Now, we study the anomalies in the $U(1)$ symmetries. The chiral $U(1)$ Vectorial symmetry is defined in analogy to Eqn.(2.27) but now in the $U(1)$ group as;

$$q_L \rightarrow e^{i\alpha_V} q_L \quad , \quad q_R \rightarrow e^{i\alpha_V} q_R \quad or \quad q \rightarrow e^{i\alpha_V} q \tag{2.49}$$

By similar reasoning, in the $U(1)$ group, one can define another symmetry called $U(1)$ Axial symmetry which is assumed to transform the left and right handed quark fields oppositely as;

$$q_L \rightarrow e^{i\alpha_A} q_L \quad , \quad q_R \rightarrow e^{-i\alpha_A} q_R \quad or \quad q \rightarrow e^{i\alpha_A \gamma_5} q \tag{2.50}$$

In fact, the standard QCD Lagrangian that we have written in Eqn.(2.2) has an extra term which is assumed to be connected with Anomaly. Hence, by adding this term, the usual QCD Lagrangian is modified to;

$$\mathcal{L} = \sum\nolimits_{i=u,d,s} \bar{q}_i(i\gamma_\mu D_\mu - m_q)q_i - \frac{1}{2}Tr[G_{\mu\nu}G^{\mu\nu}] - \theta\omega \tag{2.51}$$

where ω is referred as the topological charge density and it is responsible for the strong CP violation [30]. This is the most general Lagrangian which is renormalizable and invariant under the Lorentz transformations. The first term includes the kinetic and interaction terms of quarks with gluons, the second term is the usual dynamical term for the gluons and the last term is the contribution of the Anomaly and CP violating terms. In the above Lagrangian, the topological charge density ω is defined as [26];

$$\omega = \frac{\alpha_s}{4\pi} G\tilde{G} \tag{2.52}$$

where, $G\tilde{G} = \varepsilon_{\mu\nu\alpha\beta}G^{\mu\nu}G^{\alpha\beta}$ is the product of the gluon field strength tensor and its dual.

The usual QCD Lagrangian defined in Eqn.(2.2) has an invariance under the $U(1)_V$ transformations defined as $q \to e^{[i\alpha_V]}q$. This invariance is connected with the baryon number conservation. Similarly, we can check the invariance of the massless QCD Lagrangian under Axial $U(1)$ transformations ($q \to e^{[i\alpha_A\gamma_5]}q$) as;

$$\bar{q}i\gamma_\mu\partial_\mu q \to \bar{q}e^{[i\alpha_A\gamma_5]}i\gamma_\mu\partial_\mu e^{[i\alpha_A\gamma_5]}q = \bar{q}i\gamma_\mu\partial_\mu e^{[-i\alpha_A\gamma_5]}e^{[i\alpha_A\gamma_5]}q = \bar{q}i\gamma_\mu\partial_\mu q \tag{2.53}$$

where, we have used the commutation property of Dirac matrices as;

$$\{\gamma_5, \gamma_\mu\} = 0$$

We see that the $U(1)_A$ transformations leave the massless classical QCD Lagrangian invariant. Thus, according to Noether's Thm. classically we expect that the $SU(3)$ singlet axial vector current should be conserved in chiral limit. Letting $J^0{}_{\mu5} = \bar{q}\gamma_\mu\gamma_5\lambda^0 q$ one would expect to have;

$$\partial_\mu J^0{}_{\mu 5} = \frac{1}{\sqrt{3}}\{(\partial_\mu \bar{q})\gamma_\mu\gamma_5 q + \bar{q}\gamma_\mu\gamma_5(\partial_\mu q)\} = \frac{2}{\sqrt{3}} m_q \bar{q} i \gamma_5 q \to 0 \quad as \quad m_q \to 0 \tag{2.54}$$

where, we used the convention $\lambda^0 = \frac{1}{\sqrt{3}}\mathbf{1_{3\times 3}}$ and $q = u, d, s$. However, it was found that [31] in quantum theory, the divergence of the singlet axial vector current does not vanish even in the chiral limit;

$$\partial_\mu J^0{}_{\mu 5} = \frac{2}{\sqrt{3}} m_q \bar{q} i \gamma_5 q + \sqrt{3}\frac{\alpha_s}{4\pi} G\tilde{G} \quad , \quad \tilde{G}_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta} G^{\alpha\beta} \tag{2.55}$$

The second term in the above equation comes from the contribution of the Anomaly and is independent of mass [30]. As we have seen, the $U(1)_A$ symmetry that we have in classical regime is broken when we come to quantum level. This phenomenon is known as the $U(1)_A$ puzzle of particle physics [32]. If the $U(1)_A$ symmetry was a symmetry of both classical and the quantum theory, this would tell us the existence of a ninth pseudo-scalar particle with small mass value. The best candidate for this is the $\eta^{'}$ meson with a mass value of $m_{\eta^{'}} = 958MeV$ which is too heavy for a Goldstone Boson. The resolution of this puzzle is that the $U(1)_A$ symmetry is broken and the second term (Anomaly term) of Eqn.(2.55) makes $\eta^{'}$ too heavy to be accepted as the ninth Goldstone boson associated with the spontaneously broken $U(1)_A$ symmetry. For that reason, $\eta^{'}$ particle is not included in the meson ChPT analysis [2]. However, there are approaches that include $\eta^{'}$ to ChPT as the ninth GB. For example, in the so called large N_c limit [7,33] (where N_c describe the color number of quarks), the second term of Eqn.(2.55) is expended in the powers of $\frac{1}{N_c}$ and as we let the number of colors to go to infinity the anomaly contributions vanishes. Hence, in that case the extra mass of $\eta^{'}$ which comes from the Axial anomaly disappears and $\eta^{'}$ mass becomes comparable with the other Goldstone bosons' masses.

CHAPTER 3

CHIRAL PERTURBATION THEORY

In physics, an effective field theory is described as the theory that contains the appropriate degrees of freedom to describe physical phenomena occurring at a chosen energy scale, while ignores the substructure and the degrees of freedom at higher energies (or, equivalently, shorter distances). If we want to examine a specific physical system within the great features of the surrounding nature, we need to isolate some parts of the system from the others. By doing so, we would get a brief description to that particular system without trying to learn everything related to that system. The basic idea for doing that is to determine relevant variables which are assumed to be adequate to describe the physics at the chosen energy or length scale [19].

Most of the time, a specific physics problem includes energy scales that are separated from each other. For that reason, one can analyze the low-energy properties of this problem separately without a need for considering the high energy dynamics. In doing that, we should specify small parameters for the theory and neglect the parameters that reflect the high-energy details. The high energy degrees of freedom are then can be considered as the corrections to the results found in the low-energy analysis [19].

Therefore, it is believed that the effective field theories explain the low-energy physics. Actually, by low-energy it is meant that low with respect to some specified energy limit say Λ. Effective field theories only deal with the degrees of freedom which have energies lower than the energy limit Λ and they neglect the energies with $M \gg \Lambda$. The low-energy constants which occur in the expansion of the effective Lagrangian, include the information for the neglected degrees of freedom [19]. As we will examine in the next section, the so called Chiral effective theory is nothing but the low-energy effective theory of QCD which chooses an energy scale below $1GeV$ and uses the relevant degrees of freedom as the eight light pseudo-scalars ($\pi^{\pm}, \pi^0, K^{\pm}, K^0$ and η).

Now, in the next section we will examine the chiral effective theory formalism in detail and also mention about chiral power counting rule, construction methods of an effective Lagrangian, mass relations etc.

3.1 Chiral Effective Theory

As we have mentioned in the previous chapter, the main idea of ChPT is that the degrees of freedom that formulate the theory are chosen to be the eight light pseudo-scalar mesons ($\pi^{\pm}, \pi^0, K^{\pm}, K^0$) which are the Goldstone bosons that arise as a result of the spontaneous breakdown of the chiral $SU(3)_L \times SU(3)_R$ symmetry [2]. In chiral effective theory context these Goldstone bosons are represented by a so called matrix-valued field U which is assumed to be transformed under the chiral rotations as [2];

$$U \rightarrow U' = LUR^{-1} = LUR^{\dagger} \tag{3.1}$$

where, $L = e^{i\alpha_L{}^n \lambda^n} \in SU(3)_L$ and $R = e^{i\alpha_R{}^n \lambda^n} \in SU(3)_R$ $(n = 1, 2, ..., 8)$ are the usual definitions of the chiral transformations. In $SU(3)$, the matrix-valued field U can be expressed in terms of the physical fields (pions, kaons, etc.) as;

$$U = e^{\frac{i}{f}\phi} \tag{3.2}$$

where, ϕ is defined as $\phi = \phi^a.\lambda^a$ with ϕ^a's $(a = 1, 2..., 8)$ describe the eight fields which are related to the Goldstone boson fields through the relations [26];

$$\begin{aligned} \pi^{\pm} = \frac{1}{\sqrt{2}}(\phi^1 \mp i\phi^2) \quad &, \quad K^{\pm} = \frac{1}{\sqrt{2}}(\phi^4 \mp i\phi^5) \\ \pi^0 = \phi^3 \quad &, \quad \eta = \phi^8 \\ K^0 = \frac{1}{\sqrt{2}}(\phi^6 - i\phi^7) \quad &, \quad \bar{K}^0 = \frac{1}{\sqrt{2}}(\phi^6 + i\phi^7) \end{aligned} \tag{3.3}$$

and λ^a's are the usual Gell-Mann matrices which have the values;

$$\begin{aligned} \lambda^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \lambda^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \lambda^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{aligned} \tag{3.4}$$

Then, keeping these at hand, we can now determine the explicit expression for the product $\phi^a.\lambda^a$ as;

$$\begin{aligned} \phi = \phi^a.\lambda^a &= \phi^1 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \phi^2 \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \phi^3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &+ \phi^4 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \phi^5 \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} + \phi^6 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &+ \phi^7 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} + \frac{\phi^8}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{aligned} \tag{3.5}$$

which can be simplified to;

$$\phi = \begin{pmatrix} \phi^3 + \frac{\phi^8}{\sqrt{3}} & \phi^1 - i\phi^2 & \phi^4 - i\phi^5 \\ \phi^1 + i\phi^2 & -\phi^3 + \frac{\phi^8}{\sqrt{3}} & \phi^6 - i\phi^7 \\ \phi^4 + i\phi^5 & \phi^6 + i\phi^7 & -\frac{2\phi^8}{\sqrt{3}} \end{pmatrix} \tag{3.6}$$

or writing this in terms of the pseudo-scalar fields we obtain [26, 34];

$$\phi = \sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix} \tag{3.7}$$

As we have mentioned before, since our degrees of freedom are the GB's, which are combined in the matrix valued field U, the effective Lagrangian of the chiral theory will be expanded in terms of the powers of U as;

$$\mathcal{L}_{eff} = \mathcal{L}_{eff}(U, \partial U, \partial^2 U, ...) \tag{3.8}$$

In this expansion, the powers are determined by considering the momentum as the reference point. For example, the chiral powers are determined according to the number of derivatives acting on the U field. That is, each ∂U term implies one order in momenta and as we will see in the following discussions the quark masses are in the order of momentum squared $O(p^2)$. This is known as the "chiral power counting rule" in ChPT [2].

Since the effective Lagrangian should be invariant under the same transformations (Lorentz invariance, C, P, T and Chiral invariance etc.) as the usual QCD Lagrangian does, the effective Lagrangian $\mathcal{L}_{eff}$ is expanded in even chiral powers only. Because, in other case some uncontracted tensor indices remain and this would break the Lorenz invariance. Thus, we can write;

$$\mathcal{L}_{eff} = \mathcal{L}_{eff}{}^{(0)} + \mathcal{L}_{eff}{}^{(2)} + \mathcal{L}_{eff}{}^{(4)} + ... \tag{3.9}$$

In the above expansion, since we also expect each term to be invariant under chiral transformations defined in Eqn.(3.1), the choices like $Tr(U), Tr(U^3), Tr(U+U^\dagger)$...*etc.* can not be included to ${\mathcal{L}_{eff}}^{(0)}$;

$$\begin{aligned}
Tr(U^{'}) &\rightarrow Tr(LUR^\dagger) = Tr(UR^\dagger L) \neq Tr(U) \\
Tr(U^{'3}) &\rightarrow Tr(RU^\dagger L^\dagger LUR^\dagger LUR^\dagger) \neq Tr(U^3) \\
Tr(U^{'} + U^{'\dagger}) &\rightarrow Tr(U^{'}) + Tr(U^{'\dagger}) \neq Tr(U) + Tr(U^\dagger)
\end{aligned} \tag{3.10}$$

Then, the zeroth chiral power implies just a constant;

$${\mathcal{L}_{eff}}^{(0)} = cTr(UU^\dagger) = constant \tag{3.11}$$

As a result, we can omit it completely from the Lagrangian. In that circumstances, the second chiral order effective Lagrangian becomes the leading order (LO) term. The most general form that we can write for ${\mathcal{L}_{eff}}^{(2)}$ which is $O(p^2)$ is;

$${\mathcal{L}_{eff}}^{(2)} = aTr(\partial_\mu U \partial^\mu U^\dagger) + bTr(U\partial_\mu \partial^\mu U^\dagger) \tag{3.12}$$

However, one can write for the second term;

$$\partial_\mu(U^\dagger \partial^\mu U) = \partial_\mu U^\dagger \partial^\mu U + U\partial_\mu \partial^\mu U^\dagger \tag{3.13}$$

Since the term $\partial_\mu(U^\dagger \partial^\mu U)$ appearing above is a total derivative it has no effect on the physics done. Thus, we can omit it from the Lagrangian. Therefore, we see that the second term in Eqn.(3.12) can be expressed in terms of the first one so we are left with the simplest form of ${\mathcal{L}_{eff}}^{(2)}$;

$${\mathcal{L}_{eff}}^{(2)} = aTr(\partial_\mu U \partial^\mu U^\dagger) \tag{3.14}$$

Where the constant a is one of the low-energy constants of the chiral effective theory and its value can be determined by expanding the matrix-valued field U as;

$$\begin{aligned} U = e^{\frac{i}{f}\phi} &= 1 + \frac{i}{f}\phi - \frac{1}{2f^2}\phi^2 - \frac{i}{6f^3}\phi^3 + \frac{1}{24f^4}\phi^4 + ... \\ U^\dagger = e^{\frac{-i}{f}\phi} &= 1 - \frac{i}{f}\phi - \frac{1}{2f^2}\phi^2 + \frac{i}{6f^3}\phi^3 + \frac{1}{24f^4}\phi^4 + ... \end{aligned} \tag{3.15}$$

Plugging this into ${\mathcal{L}_{eff}}^{(2)}$ we get;

$$\begin{aligned} {\mathcal{L}_{eff}}^{(2)} &= aTr[(\frac{-i\partial_\mu\phi}{f} - \frac{\partial_\mu\phi^2}{2f^2} + \frac{i\partial_\mu\phi^3}{6f^3} + ...)(\frac{i\partial^\mu\phi}{f} - \frac{\partial^\mu\phi^2}{2f^2} - \frac{i\partial^\mu\phi^3}{6f^3} + ...)] \\ &= aTr[\frac{1}{f^2}\partial_\mu\phi\partial^\mu\phi + \frac{i}{2f^3}\partial_\mu\phi\partial^\mu\phi^2 - \frac{1}{6f^4}\partial_\mu\phi\partial^\mu\phi^3 - \frac{1}{6f^4}\partial_\mu\phi^3\partial^\mu\phi \\ &- \frac{i}{12f^5}\partial_\mu\phi^3\partial^\mu\phi^2 - \frac{i}{2f^3}\partial_\mu\phi^2\partial^\mu\phi + \frac{1}{4f^4}\partial_\mu\phi^2\partial^\mu\phi^2 + \frac{i}{12f^5}\partial_\mu\phi^2\partial^\mu\phi^3 \\ &+ \frac{1}{36f^6}\partial_\mu\phi^3\partial^\mu\phi^3 + ...] \end{aligned} \tag{3.16}$$

At that point, doing all the required cancelations in the above expansion, ${\mathcal{L}_{eff}}^{(2)}$ reduces to;

$${\mathcal{L}_{eff}}^{(2)} = \frac{a}{f^2}Tr(\partial_\mu\phi\partial^\mu\phi) = \frac{2a}{f^2}(\partial_\mu\phi^a\partial^\mu\phi_a) \tag{3.17}$$

where $a = 1, 2, ..., 8$ and we have used the trace property of the Gell-Mann matrices as $Tr(\lambda^a\lambda^b) = 2\delta^{ab}$. Then, comparing this result with the kinetic term of the standard Lagrangian for scalars;

$$\mathcal{L}_{kinetic} = \frac{1}{2}|\partial_\mu\phi|^2 \tag{3.18}$$

we obtain $a = \frac{f^2}{4}$. As a result, the leading order (LO) effective Lagrangian turns to be;

$$\mathcal{L}_{eff}{}^{(2)} = \frac{f^2}{4} Tr(\partial_\mu U \partial^\mu U^\dagger) \tag{3.19}$$

All of the above arguments are done by assuming that the chiral symmetry is exact i.e. $m_u = m_d = m_s = 0$. However, we know that in the real world the quark masses are small but finite. As a result of that the chiral symmetry is explicitly broken [2]. At that point, in order to add mass terms to QCD Lagrangian, we may initially think of the mass matrix of the light quarks $\mathcal{M} = diag(m_u, m_d, m_s)$ as an external field Σ which is transformed similarly with the matrix-valued field U under chiral transformations [2] ;

$$\Sigma \rightarrow L \Sigma R^\dagger \tag{3.20}$$

In that case, since the QCD Lagrangian should be invariant under the chiral transformations, the external field Σ can be added to $\mathcal{L}^{QCD}$ in chiral invariant form as [2] ;

$$\mathcal{L}^{QCD}{}_{mass} = -\bar{q_L}\mathcal{M}q_R - \bar{q_R}\mathcal{M}q_L \rightarrow -\bar{q_L}\Sigma q_R - \bar{q_R}\Sigma^\dagger q_L \tag{3.21}$$

where, $\mathcal{M} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$ is the mass matrix of u, d and s quarks.

Now, since we also expect the effective Lagrangian to be invariant under the chiral transformations even in the presence of the external field Σ, we expand the $\mathcal{L}_{eff}$ by including the field Σ as [2] ;

$$\mathcal{L}_{eff} = \mathcal{L}_{eff}(U, \partial U, \partial^2 U, ...) \rightarrow \mathcal{L}_{eff} = \mathcal{L}_{eff}(U, \partial U, \partial^2 U, ...; \Sigma) \tag{3.22}$$

Therefore, the LO chiral effective Lagrangian in the presence of the external field Σ can be written as;

$$\mathcal{L}_{eff}{}^{(2)} = \frac{f^2}{4} Tr(\partial_\mu U \partial^\mu U^\dagger) + \frac{f^2}{2} BTr(\Sigma U^\dagger + U\Sigma^\dagger) \tag{3.23}$$

Where B is a constant which is assumed to be related to the explicit breakdown of the chiral symmetry and the second term is written in a chiral invariant form. Moreover, we accepted Σ to be in the chiral order of two i.e. $O(p^2)$ for the reasons that we will explain later.

Now, having constructed the $\mathcal{L}_{eff}{}^{(2)}$ in the presence of external field, we can return to our initial case and set $\Sigma = \mathcal{M}$. Hence, the LO chiral effective Lagrangian turns to be;

$$\mathcal{L}_{eff}{}^{(2)} = \frac{f^2}{4} Tr(\partial_\mu U \partial^\mu U^\dagger) + \frac{f^2}{2} BTr(\mathcal{M} U^\dagger + U\mathcal{M}) \tag{3.24}$$

Keeping these at hand, by Taylor expanding the matrix-valued field U we can now express the mass part of $\mathcal{L}_{eff}{}^{(2)}$ as;

$$\mathcal{L}_{mass}{}^{(2)} = \frac{f^2}{2} BTr(\mathcal{M} U^\dagger + U\mathcal{M}) = (m_u + m_d + m_s)Bf^2 - \frac{1}{2} BTr(\mathcal{M}\phi^2) \tag{3.25}$$

In the above expansion, the first term represent the quark condensates which reflect the quark density of the vacuum and the second term is connected with the Goldstone bosons' mass terms. The quark condensates can be found by just taking the vacuum matrix element of the derivative of QCD Hamiltonian ($H_{QCD} \supset \sum_q m_q \bar{q}q$) with respect to the masses of the quarks as;

$$\begin{aligned} \langle 0|\bar{q}q|0\rangle = \langle 0|\frac{\partial H}{\partial m_q}|0\rangle = -\langle 0|\frac{\partial \mathcal{L}}{\partial m_q}|0\rangle = -f^2 B \\ \Rightarrow \quad \langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle = \langle 0|\bar{s}s|0\rangle = -f^2 B \end{aligned} \tag{3.26}$$

Then, the second term in Eqn.(3.25) can be expanded as;

$$
\begin{aligned}
&Tr \quad \{\mathcal{M}\phi^2\} = \\
&Tr \quad \{2\mathcal{M} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} \}
\end{aligned}
\tag{3.27}
$$

Now, letting the product ϕ^2 to be equal to T we can write;

$$
\begin{aligned}
Tr(\mathcal{M}\phi^2) &= 2\{m_u T_{11} + m_d T_{22} + m_s T_{33}\} \\
&= 2[m_u(\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta)^2 + m_u\pi^+\pi^- + m_u K^- K^+ + m_d\pi^+\pi^- \\
&+ m_d K^0\bar{K}^0 + m_d(-\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta)^2 + m_s K^- K^+ + m_s K^0\bar{K}^0 + \frac{2}{3}m_s\eta^2] \\
&= 2\{(m_u + m_d)\pi^+\pi^- + (m_u + m_s)K^- K^+ + (m_d + m_s)K^0\bar{K}^0 \\
&+ (\frac{m_u + m_d}{2}){\pi^0}^2 + (\frac{m_u - m_d}{\sqrt{3}})\pi^0\eta + \frac{1}{6}(m_u + m_d + 4m_s)\eta^2\}
\end{aligned}
\tag{3.28}
$$

As we can infer from the term $(\frac{m_u - m_d}{\sqrt{3}})\pi^0\eta$ in the above expansion, unlike to other mesons' masses, the $\pi^0 - \eta$ mass matrix is non-diagonal and diagonalization of it will result in expressing the mass eigenstates of π^0 and η as a linear combination of $SU(2)$ and $SU(3)$ pure eigensates. Therefore, this will tell us that there is a mixing between $\pi^0 - \eta$. However, if we assume that the isospin symmetry is intact ($m_u = m_d = \hat{m}$), the mass term of the LO effective Lagrangian takes the form;

$$
\begin{aligned}
{\mathcal{L}_{mass}}^{(2)} &= -2B\hat{m}\pi^+\pi^- - B(\hat{m} + m_s)K^- K^+ - B(\hat{m} + m_s)K^0\bar{K}^0 \\
&- B\hat{m}{\pi^0}^2 - \frac{1}{3}B(\hat{m} + 2m_s)\eta^2
\end{aligned}
\tag{3.29}
$$

At that stage, comparing this mass Lagrangian with the mass terms of the Standard Lagrangian we obtain [2];

$$m_\pi^2 = 2\hat{m}B \quad , \quad m_{K^\pm}^2 = (m_u + m_s)B$$
$$m_{K^0}^2 = (m_d + m_s)B \quad , \quad m_\eta^2 = \frac{2}{3}B(\hat{m} + 2m_s) \tag{3.30}$$

Now, combining the results of Eqn.(3.30), we can get the so called "Gell-Mann-Okubo mass formula" ;

$$m_\eta^2 = \frac{4}{3}m_K^2 - \frac{1}{3}m_\pi^2 \tag{3.31}$$

In addition, extracting the value of B from the above relations (Eqn.3.30) and inserting it to the previously found results in Eqn.(3.26), we can get the so called "Gell-Mann-Oakes-Renner relations" ;

$$\begin{aligned} f_\pi^2 m_\pi^2 &= -2\hat{m}\langle 0|\bar{q}q|0\rangle \\ f_K^2 m_K^2 &= -(\hat{m} + m_s)\langle 0|\bar{q}q|0\rangle \\ f_\eta^2 m_\eta^2 &= -\frac{2}{3}(\hat{m} + 2m_s)\langle 0|\bar{q}q|0\rangle \end{aligned} \tag{3.32}$$

As we can see, Eqn.(3.30) provides an evidence for the assumption that the quark masses have chiral order of two. Since B is orderless and;

$$\hat{m} = \frac{m_\pi^2}{2B} \quad , \quad m_\pi^2 = {p_\pi}^2 \quad \Rightarrow \quad \hat{m} = O(p^2) \tag{3.33}$$

Another point which is worth to mention is that in lowest order ChPT analysis, the decay constant f is taken to be the decay constant of the pion which has the value [2];

$$f \simeq f_\pi \simeq 93MeV \tag{3.34}$$

Besides, the explicit chiral symmetry breaking parameter B can be calculated by using the previous formulas obtained. For which, we make use of the QCD sum rule value for the quark condensate term as $\langle 0|\bar{q}q|0\rangle = -(250MeV)^3$ [35]. Hence, one can write [2];

$$B = 1800MeV \tag{3.35}$$

Furthermore, ChPT allows us to find a relation for the ratio of the quark masses by using the known mass values of the pseudo-scalar fields, which are extracted from experiment. Because of the fact that the quark masses are dependent on the QCD renormalization scale, their values differ from one scale to another. However, when we take the ratio of the masses this scale dependency disappears [2]. Using the previously found relations we can write;

$$\frac{(m_{K^+}^2 - m_{K^0}^2) + m_\pi^2}{(m_{K^0}^2 - m_{K^+}^2) + m_\pi^2} = \frac{(m_u + m_s)B - (m_d + m_s)B + (m_u + m_d)B}{(m_d + m_s)B - (m_u + m_s)B + (m_u + m_d)B} = \frac{m_u}{m_d} \tag{3.36}$$

For this ratio, inserting the mass values of the pseudo-scalar fields from phenomenology, to lowest order one can find $\frac{m_u}{m_d} \simeq 0.55$ [36]. Following the same reasoning, to leading order we can write;

$$\frac{m_{K^0}^2 + m_{K^+}^2 - m_\pi^2}{(m_{K^0}^2 - m_{K^+}^2) + m_\pi^2} = \frac{(m_d + m_s)B + (m_u + m_s)B - (m_u + m_d)B}{(m_d + m_s)B - (m_u + m_s)B + (m_u + m_d)B} = \frac{m_s}{m_d} \tag{3.37}$$

Hence, the ratio is found to be $\frac{m_s}{m_d} \simeq 20.1$ [36].

Another point to consider here is the inclusion of the η' meson to the Chiral effective theory as the ninth pseudo-scalar field. The so-called nonet symmetry allows us to find relations between the $SU(3)$ octet and the singlet states. In addition, as we have mentioned before if the $U(1)_A$ symmetry was not broken at quantum theory, the extra mass of $\eta^{'}$ meson would disappear and it could be included to Chiral effective Lagrangian. By means of the so called $\frac{1}{N_c}$ expansion [7,33,37] $\eta^{'}$ can be accepted as the ninth GB. With the inclusion of $\eta^{'}$ (η_0), the mass part of the LO chiral effective Lagrangian improves to (in $SU(2)$ symmetric case i.e. $m_u = m_d = \hat{m}$);

$$
\begin{aligned}
\mathcal{L}^{eff\ (2)^{'}}_{mass} &= -B(m_u+m_d)\pi^+\pi^- - B(m_u+m_s)K^-K^+ - B(m_d+m_s)K^0\bar{K}^0 \\
&- B(\frac{m_u+m_d}{2})\pi^{0^2} - \frac{1}{2}m_8^2{\eta_8}^2 - \frac{1}{2}m_0^2{\eta_0}^2 - m_{08}^2\eta_0\eta_8
\end{aligned}
\tag{3.38}
$$

where, η_8 and η_0 are defined to be the $SU(3)$ octet and the singlet states, m_8 and m_0 are the masses of the octet and singlet states respectively. Similar to $\pi^0 - \eta$, the term m_{08} in the above expression implies a mixing between $\eta_8 - \eta_0$. In the $SU(3)$ symmetric case, η_8 and η_0 states correspond to the physical meson fields η and $\eta^{'}$ respectively and in this limit $m_{08} = 0$.

In the next chapter, we will give insight to this mixing phenomenon and see how to calculate these mass terms (m_0, m_8, m_{08}), which reflect the mixing here, in addition to the determination of the other mixing parameters.

CHAPTER 4

MIXING OF PSEUDOSCALAR MESONS

In the last few decades many scientists have found the mixing phenomenon of pseudo-scalar particles interesting and thus lots of studies related to this mixing have been done till now (see e.g. [38–53]).

In fact, the mixing phenomenon of pseudoscalar particles has a starting point since the invention of $SU(3)_F$ symmetry. In the real world, neither the isospin nor the $SU(3)$ flavor symmetry is exact. There is an explicit breaking of these symmetries due to the mass difference of up, down and strange quarks. Thus, in these cases some particles are not seen as pure $SU(3)_F$ eigenstates in nature. The pure $SU(3)_F$ eigenstates mix with each other and so the physical particles that are observed in nature become linear combination of these pure $SU(3)_F$ eigenstates. This phenomenon is called "Mixing of States".

At this point it is worth to mention that within pseudosacalars family, only the neutral pseudoscalars π^0, η and $\eta^{'}$ mix. This is because, when states mix, at the same time they should still obey the conservation rules. For example, K^+ can not mix to K^- because this would violate the charge conservation. Similarly, since strong interactions should conserve strangeness, K^0 can not mix to π^0, η or $\eta^{'}$. Actually, in mixing phenomenon of pseudoscalar mesons, $U(1)_A$ Anomaly has an important effect. As we have studied before, the anomalous factor in the divergence equation is not dependent upon energy or mass. As a result, one may think that mixing with heavier pseudoscalars (η_c, η_b) is also possible (provided that isospin is still conserved) [30]. In fact, it is so. However, we can assume that our basis sates are orthogonal $\langle\eta_c(\eta_b)|\eta_8(\eta_0)\rangle = 0$ so that the mixing with heavy pseudoscalars (η_c, η_b) with $I = 0$ can be neglected [38].

In this chapter, we are going to review the mixing phenomenon of pseudo-scalar mesons, more specifically the mixing of $\eta - \eta^{'}$ mesons and its consequences in detail. Making use of the low-energy expansion of ChPT and the nonet symmetry considerations, the mass terms m_8, m_0, m_{08} that appear in the improved leading order chiral effective Lagrangian (Eqn.(3.38)) will be calculated. In section 1, some basic calculational techniques which are commonly used in ChPT will be discussed. This section closely follows the paper [39]. In sections 2, we will discuss the mixing formalism of η and $\eta^{'}$ mesons in the so-called quark-flavor basis scheme in detail.

4.1 Nonet Symmetry and Matrix Element Analysis (PCAC Method)

$SU(3)$ flavor symmetry allows us to find relations among some parameters of the pseudoscalar states in the octet group. However, it does not tell us any relation between the parameters of the octet and the singlet states. On the other hand, the so called nonet symmetry allows us to relate some parameters of the octet group to the parameters of the singlet state.

The matrix elements of the Axial-vector currents taken between the vacuum and pseudo-scalar states are defined as;

$$\begin{aligned} \langle 0|\bar{s}i\gamma_\mu\gamma_5 s|\bar{s}s\rangle = if_s p_\mu \ , \ \langle 0|\bar{u}i\gamma_\mu\gamma_5 u|\bar{u}u\rangle &= if_u p_\mu \ , \ \langle 0|\bar{d}i\gamma_\mu\gamma_5 d|\bar{d}d\rangle = if_d p_\mu \\ \langle 0|\bar{u}i\gamma_\mu\gamma_5 u|\bar{d}d(\bar{s}s)\rangle &= \langle 0|\bar{s}i\gamma_\mu\gamma_5 s|\bar{d}d(\bar{u}u)\rangle = 0 \end{aligned} \tag{4.1}$$

where, $|\bar{q}q\rangle$ is the $\bar{q}q$ component of a pseudo-scalar state with a four momentum p_μ. In the above expressions, we have used the fact that since $|\bar{s}s\rangle, |\bar{u}u\rangle, |\bar{d}d\rangle$ are pseudo-scalar states, the matrix elements of the Axial-vector currents taken between them and a scalar vacuum will result in a vector. The only vector that we can relate with this matrix element is the four momenta of the pseudo-scalar fields p_μ with a proportionality constant defined as if.

Moreover, $SU(3)$ octet (η_8) and singlet (η_0) states with $I = 0$ are defined in terms of the flavor $q\bar{q}$ components as;

$$|\eta_0\rangle = (|u\bar{u} + d\bar{d} + s\bar{s}\rangle)/\sqrt{3} \tag{4.2}$$

$$|\eta_8\rangle = (|u\bar{u} + d\bar{d} - 2s\bar{s}\rangle)/\sqrt{6} \tag{4.3}$$

Now, by sandwiching the axial vector currents $qi\gamma_\mu\gamma_5\bar{q}$ and $si\gamma_\mu\gamma_5\bar{s}$, where $q = u, d$, between vacuum and octet/singlet states we can write [39]:

$$\langle 0|\bar{q}i\gamma_\mu\gamma_5 q|\eta_0\rangle \equiv if_q p_\mu/\sqrt{3} \tag{4.4}$$

$$\langle 0|\bar{q}i\gamma_\mu\gamma_5 q|\eta_8\rangle \equiv if_q p_\mu/\sqrt{6} \tag{4.5}$$

$$\langle 0|\bar{s}i\gamma_\mu\gamma_5 s|\eta_0\rangle \equiv if_s p_\mu/\sqrt{3} \tag{4.6}$$

$$\langle 0|\bar{s}i\gamma_\mu\gamma_5 s|\eta_8\rangle \equiv -2if_s p_\mu/\sqrt{6} \tag{4.7}$$

where, $f_q = f_{u,d}$ describes the decay constants related to $u\bar{u}$ and $s\bar{s}$ respectively. In addition, the matrix elements of octet/singlet axial vector currents;

$$\begin{aligned} A_{\mu 8} &= (\bar{u}i\gamma_\mu\gamma_5 u + \bar{d}i\gamma_\mu\gamma_5 d - 2\bar{s}i\gamma_\mu\gamma_5 s)/\sqrt{6} \\ A_{\mu 0} &= (\bar{u}i\gamma_\mu\gamma_5 u + \bar{d}i\gamma_\mu\gamma_5 d + \bar{s}i\gamma_\mu\gamma_5 s)/\sqrt{3} \end{aligned} \tag{4.8}$$

between vacuum and octet/singlet states give us [39];

$$\begin{aligned}
\langle 0|A_{\mu 8}|\eta_8\rangle &= (\frac{1}{\sqrt{6}})^2\{\langle 0|\bar{u}i\gamma_\mu\gamma_5 u|\bar{u}u\rangle + \langle 0|\bar{d}i\gamma_\mu\gamma_5 d|\bar{d}d\rangle + 4\langle 0|\bar{s}i\gamma_\mu\gamma_5 s|\bar{s}s\rangle\} \\
&= \frac{1}{6}(f_u + f_d + 4f_s)p_\mu \\
\langle 0|A_{\mu 8}|\eta_0\rangle &= \langle 0|A_{\mu 0}|\eta_8\rangle = \frac{1}{3\sqrt{2}}(f_u + f_d - 2f_s)p_\mu \\
\langle 0|A_{\mu 0}|\eta_0\rangle &= \frac{1}{3}(f_u + f_d + f_s)p_\mu
\end{aligned} \tag{4.9}$$

Then, because of the explicit breaking of $SU(3)_F$ symmetry due to the mass of strange quark, f_u and f_s are assumed to differ from each other by a symmetry breaking factor. With the assumption that each s-quark contributes to decay constant with a factor of ϵ, making expansion to first order in ϵ, one can write [39,40];

$$\begin{aligned}
f_{\pi^+} &= f_{u\bar{d}} \simeq f_u \ , \ f_{K^+} = f_{u\bar{s}} \simeq (1+\epsilon)f_{u\bar{d}} \simeq (1+\epsilon)f_u \\
f_s &= f_{s\bar{s}} \simeq (1+2\epsilon)f_u \simeq (1+\epsilon)^2 f_u = (1+\epsilon)\underbrace{(1+\epsilon)f_u} \simeq (1+\epsilon)\underbrace{f_K} \\
f_K &= f_{u\bar{s}} \simeq (1+\epsilon)f_{u\bar{d}} \simeq (1-\epsilon)f_{s\bar{s}}
\end{aligned} \tag{4.10}$$

Thus,

$$f_s \simeq f_K(1-\epsilon)^{-1} \simeq f_K(1+\epsilon) \simeq (1+\epsilon)^2 f_u \simeq (1+2\epsilon)f_u \tag{4.11}$$

where, $f_{u\bar{d}}, f_{u\bar{s}}...etc.$ describe the decay constants of the physical mesons $(\pi, K...)$ written in terms of the quark constituents. We also took $f_u \simeq f_d$ as a consequence of the $SU(2)$ isospin symmetry and made a Taylor expansion to first order in ϵ as;

$$\frac{1}{1-\epsilon} = 1 + \epsilon + \ ...$$

Afterwards, if we take the matrix elements of the pseudoscalar densities $\bar{u}i\gamma_5 d$, $\bar{u}i\gamma_5 s$ and $\bar{u}i\gamma_5 u - \bar{d}i\gamma_5 d$ between the vacuum and our pseudoscalar fields π^+, K^+ and π^0 respectively we get [39];

$$\begin{aligned}
\langle 0|\partial^\mu \bar{u}\gamma_\mu\gamma_5 d|\bar{u}d\rangle &= \langle 0|(\partial^\mu \bar{u})\gamma_\mu\gamma_5 d + \bar{u}\gamma_\mu\gamma_5(\partial^\mu d)|\bar{u}d\rangle \\
&= (m_u + m_d)\langle 0|\bar{u}i\gamma_5 d|\bar{u}d\rangle = f_\pi \underbrace{m_\pi^2} \\
&= f_\pi B(m_u + m_d) \qquad (4.12)
\end{aligned}$$

$$\begin{aligned}
\langle 0|\partial^\mu \bar{u}\gamma_\mu\gamma_5 s|\bar{u}s\rangle &= \langle 0|(\partial^\mu \bar{u})\gamma_\mu\gamma_5 s + \bar{u}\gamma_\mu\gamma_5(\partial^\mu s)|\bar{u}s\rangle \\
&= (m_u + m_s)\langle 0|\bar{u}i\gamma_5 s|\bar{u}s\rangle = f_K \underbrace{m_K^2} \\
&= f_K B(m_u + m_s) \qquad (4.13)
\end{aligned}$$

Similarly, for π^0 with $\pi^0 = \frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d)$ we can write [39];

$$\begin{aligned}
\langle 0|\partial^\mu(\bar{u}i\gamma_\mu\gamma_5 u - \bar{d}i\gamma_\mu\gamma_5 d)|\pi^0\rangle &= (m_u + m_d)\langle 0|\bar{u}i\gamma_5 u|\bar{u}u\rangle \\
&= f_u B(m_u + m_d) \qquad (4.14)
\end{aligned}$$

where, we took $f_u = f_d$ so that we expressed the matrix element $\langle 0|\bar{d}i\gamma_5 d|\bar{d}d\rangle$ in terms of $\langle 0|\bar{u}i\gamma_5 u|\bar{u}u\rangle$. Moreover, the B's that appear above are the usual order parameters responsible for the explicit breaking of chiral symmetry (see Eqn.(3.23)) and we have used the leading order chiral effective theory results (Eqn.(3.30)) to express the physical particles' masses in terms of B [2, 7, 26]. Looking at the quark structure of $\pi^0 = \frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d)$ one can also write [39]:

$$\langle 0|\bar{u}i\gamma_5 u|\pi^0\rangle = -\langle 0|\bar{d}i\gamma_5 d|\pi^0\rangle \qquad (4.15)$$

Now, let us calculate the matrix elements of pseudo-scalar densities taken between vacuum and octet-singlet states by adding the pole (anomaly) terms to divergence equations [39];

$$\begin{aligned} \partial_\mu A_{n\mu} &= 2(m_u\bar{u}i\gamma_5 u + m_d\bar{d}i\gamma_5 d) + 2\frac{\alpha_s}{4\pi}G_{\mu\nu}\tilde{G^{\mu\nu}} \\ \partial_\mu A_{s\mu} &= 2m_s\bar{s}i\gamma_5 s + \frac{\alpha_s}{4\pi}G_{\mu\nu}\tilde{G^{\mu\nu}} \end{aligned} \tag{4.16}$$

where, for each pseudo-scalar density term we added an anomaly contribution term $\frac{\alpha_s}{4\pi}G_{\mu\nu}\tilde{G^{\mu\nu}}$ (see Eqn.(2.55)). In the above expressions, the axial vector currents with $I = 0$ are defined as;

$$A_{n\mu} = (\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d) \quad , \quad A_{s\mu} = \bar{s}\gamma_\mu\gamma_5 s \tag{4.17}$$

For the divergence results in Eqn.(4.12), Eqn.(4.13) and Eqn.(4.14) we simply made use of the relations;

$$\begin{gathered} (i\gamma_\mu\partial_\mu - m)\Psi = 0 \quad , \quad \bar{\Psi} = \Psi^\dagger\gamma^0 \\ \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \quad , \quad \{\gamma_\mu, \gamma_5\} = 0 \quad , \quad \gamma^{5\dagger} = \gamma^5 \end{gathered} \tag{4.18}$$

For example;

$$\begin{aligned} \partial_\mu\bar{s}\gamma_\mu\gamma_5 s &= \underbrace{(\partial_\mu\bar{s})}\gamma_\mu\gamma_5 s + \bar{s}\gamma_\mu\gamma_5\underbrace{(\partial_\mu s)} \\ &= im_s\bar{s}\gamma_5 s + im_s\bar{s}\gamma_5 s = 2m_s\bar{s}i\gamma_5 s \end{aligned} \tag{4.19}$$

Similar reasoning can be applied for $\partial_\mu A_{n\mu}$. Then, having these tools at hand we can now calculate the matrix elements (by assuming $f_u = f_d$) as;

$$\begin{aligned} \langle 0|\partial^\mu A_{n\mu}|\eta_0\rangle = 2[m_u\langle 0|\bar{u}i\gamma_5 u|\eta_0\rangle + m_d\langle 0|\bar{d}i\gamma_5 d|\eta_0\rangle] \\ +2\langle 0|\frac{\alpha_s}{4\pi}G_{\mu\nu}\tilde{G^{\mu\nu}}|\eta_0\rangle \end{aligned} \tag{4.20}$$

$$\langle 0|\partial^\mu A_{s\mu}|\eta_0\rangle = 2m_s\langle 0|\bar{s}i\gamma_5 s|\eta_0\rangle + \langle 0|\frac{\alpha_s}{4\pi}G_{\mu\nu}\tilde{G^{\mu\nu}}|\eta_0\rangle \tag{4.21}$$

$$\langle 0|\partial^\mu A_{s\mu}|\eta_8\rangle = 2m_s\langle 0|\bar{s}i\gamma_5 s|\eta_8\rangle + \langle 0|\frac{\alpha_s}{4\pi}G_{\mu\nu}\tilde{G^{\mu\nu}}|\eta_8\rangle \tag{4.22}$$

$$\begin{aligned}\langle 0|\partial^\mu A_{n\mu}|\eta_8\rangle = 2[m_u\langle 0|\bar{u}i\gamma_5 u|\eta_8\rangle + m_d\langle 0|\bar{d}i\gamma_5 d|\eta_8\rangle]\\ +2\langle 0|\frac{\alpha_s}{4\pi}G_{\mu\nu}\tilde{G^{\mu\nu}}|\eta_8\rangle\end{aligned} \tag{4.23}$$

In the above equations, the left hand side of these equations is also related to the $p^2 = m_{0,8}^2$ term as;

$$\begin{aligned}\langle 0|\partial^\mu A_{n\mu}|\eta_8\rangle = f_u m_8^2 &= 2[m_u\langle 0|\bar{u}i\gamma_5 u|\eta_8\rangle + m_d\langle 0|\bar{d}i\gamma_5 d|\eta_8\rangle]\\ &+ 2\langle 0|\frac{\alpha_s}{4\pi}G_{\mu\nu}\tilde{G^{\mu\nu}}|\eta_8\rangle\end{aligned} \tag{4.24}$$

Similarly,

$$\langle 0|\partial^\mu A_{s\mu}|\eta_0\rangle = f_s m_0^2 = 2m_s\langle 0|\bar{s}i\gamma_5 s|\eta_0\rangle + \langle 0|\frac{\alpha_s}{4\pi}G_{\mu\nu}\tilde{G^{\mu\nu}}|\eta_0\rangle \tag{4.25}$$

In the limit of vanishing u and d quark masses, comparing the left and the right hand sides of Eqn.(4.24) and Eqn.(4.25), one can see that the anomaly term $\langle 0|\frac{\alpha_s}{4\pi}G_{\mu\nu}\tilde{G^{\mu\nu}}|\eta_{8,0}\rangle$ is in the order of $O(m_{8,0}^2)$.

Now, we define decay constants associated with η_0, η_8 and their mixing as f_0, f_8, f_{08} respectively. As a consequence of the nonet symmetry, taking $f_u = f_d = f_s = f_0 = f_8 = f_{08}$ and using the nonet symmetry mass relations [26] for $m_{0,8}^2$ and m_{08}^2 (see Appendix B for the detailed calculations of these relations);

$$
\begin{aligned}
m_8^2 &= B\frac{2}{3}(2m_s+\hat{m}) \\
m_0^2 &= \tilde{m_0}^2 + B\frac{2}{3}(m_s+2\hat{m}) \\
m_{08}^2 &= B\frac{2}{3}\sqrt{2}(-m_s+\hat{m})
\end{aligned}
\tag{4.26}
$$

with $\hat{m}=(m_u+m_d)/2$ and $\tilde{m_0}^2=\sqrt{3}\langle 0|\frac{\alpha_s}{4\pi}G_{\mu\nu}\tilde{G^{\mu\nu}}|\eta_0\rangle$, we can go further and calculate the pseudo-scalar densities matrix elements ($\langle 0|\bar{u}i\gamma_5 u|u\bar{u}\rangle$, $\langle 0|\bar{s}i\gamma_5 s|s\bar{s}\rangle$...etc). For this, we start from the definitions [41];

$$
|\eta_0\rangle = \frac{1}{\sqrt{3}}|u\bar{u}+d\bar{d}+s\bar{s}\rangle \quad , \quad |\eta_8\rangle = \frac{1}{\sqrt{6}}|u\bar{u}+d\bar{d}-2s\bar{s}\rangle
$$

$$
A_{\mu 5}^n = \frac{\sqrt{2}}{\sqrt{3}}(A_{\mu 5}^8+\sqrt{2}A_{\mu 5}^0) \quad , \quad A_{\mu 5}^s = \frac{1}{\sqrt{3}}(A_{\mu 5}^0-\sqrt{2}A_{\mu 5}^8) \tag{4.27}
$$

Then,

$$
\langle 0|\partial_\mu A_{\mu 5}^n|\eta_0\rangle = \frac{\sqrt{2}}{\sqrt{3}}\langle 0|\partial_\mu A_{\mu 5}^8|\eta_0\rangle + \frac{2}{\sqrt{3}}\langle 0|\partial_\mu A_{\mu 5}^0|\eta_0\rangle \tag{4.28}
$$

where we can write;

$$
\langle 0|\partial_\mu A_{\mu 5}^8|\eta_0\rangle = f_{08}m_{08}^2 \simeq f_u m_{08}^2 \quad , \quad \langle 0|\partial_\mu A_{\mu 5}^0|\eta_0\rangle = f_0 m_0^2 \simeq f_u m_0^2 \tag{4.29}
$$

Thus, inserting the nonet symmetry mass values (Eqn.(4.26)) one can get;

$$
\begin{aligned}
\langle 0|\partial_\mu A_{\mu 5}^n|\eta_0\rangle &= \frac{\sqrt{2}}{\sqrt{3}}f_u\frac{2\sqrt{2}}{3}B(\hat{m}-m_s) + \frac{2}{\sqrt{3}}f_u(\tilde{m_0}^2+B\frac{2}{3}(m_s+2\hat{m})) \\
&= \frac{4}{\sqrt{3}}f_u B\hat{m} + \frac{2}{\sqrt{3}}f_u\tilde{m_0}^2
\end{aligned}
\tag{4.30}
$$

On the other hand, $\langle 0|\partial_\mu A^n_{\mu 5}|\eta_0\rangle$ can also be written as;

$$\begin{aligned}\langle 0|\partial_\mu A^n_{\mu 5}|\eta_0\rangle &= \frac{1}{\sqrt{3}}\langle 0|4\hat{m}\bar{u}i\gamma_5 u|u\bar{u}\rangle + 2\underbrace{\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta_0\rangle} \\ &= \frac{4}{\sqrt{3}}\langle 0|\hat{m}\bar{u}i\gamma_5 u|u\bar{u}\rangle + \frac{2}{\sqrt{3}}f_u\tilde{m_0}^2\end{aligned} \tag{4.31}$$

Hence, equating both equations we obtain [39];

$$\langle 0|\bar{u}i\gamma_5 u|u\bar{u}\rangle = f_u B \tag{4.32}$$

Similarly,

$$\langle 0|\partial_\mu A^s_{\mu 5}|\eta_8\rangle = \frac{1}{\sqrt{3}}\langle 0|\partial_\mu A^0_{\mu 5}|\eta_8\rangle - \frac{\sqrt{2}}{\sqrt{3}}\langle 0|\partial_\mu A^8_{\mu 5}|\eta_8\rangle \tag{4.33}$$

where, using;

$$\langle 0|\partial_\mu A^0_{\mu 5}|\eta_8\rangle = f_{08}m^2_{08} \simeq f_s m^2_{08} \quad , \quad \langle 0|\partial_\mu A^8_{\mu 5}|\eta_8\rangle = f_8 m^2_8 \simeq f_s m^2_8 \tag{4.34}$$

with the nonet symmetry mass expressions (Eqn.(4.26)) we obtain;

$$\begin{aligned}\langle 0|\partial_\mu A^s_{\mu 5}|\eta_8\rangle &= \frac{1}{\sqrt{3}}f_s\frac{2\sqrt{2}}{3}B(\hat{m}-m_s) - \frac{\sqrt{2}}{\sqrt{3}}f_s B\frac{2}{3}(2m_s+\hat{m}) \\ &= -\frac{2\sqrt{2}}{\sqrt{3}}f_s B m_s\end{aligned} \tag{4.35}$$

At the same time, $\langle 0|\partial_\mu A^s_{\mu 5}|\eta_8\rangle$ is equal to;

$$\langle 0|\partial_\mu A^s_{\mu 5}|\eta_8\rangle = \frac{-4}{\sqrt{6}} m_s \langle 0|\bar{s} i\gamma_5 s|s\bar{s}\rangle + \underbrace{\langle 0|\frac{\alpha_s}{4\pi} G_{\mu\nu}\tilde{G^{\mu\nu}}|\eta_8\rangle} \tag{4.36}$$

Now, equating both equations and letting $\underbrace{\langle 0|\frac{\alpha_s}{4\pi} G_{\mu\nu}\tilde{G^{\mu\nu}}|\eta_8\rangle} = 0$ we obtain [39];

$$\langle 0|\bar{s} i\gamma_5 s|s\bar{s}\rangle = f_s B \tag{4.37}$$

Following the same procedure one can get the same result from the other two equations as well.

As we have seen so far, by using the nonet symmetry mass relations (Eqn.(4.26)) and adding the η_0, η_8 pole contributions (anomaly terms) to the divergence equations (Eqn.(4.31) and Eqn.(4.36)), the matrix elements for the pseudo-scalar densities can be found [39]. In the next sections, we will examine the $\eta - \eta^{'}$ mixing in detail. These sections mainly follow [42].

4.2 $\eta - \eta^{'}$ Mixing and its Consequences

As we have mentioned before, as a result of the explicit breaking of $SU(3)$ flavor symmetry, the physical particles η and $\eta^{'}$ that we observe in nature, are defined as the linear combination of the $SU(3)$ pure eigenstates η^8,η^0 with a mixing parameter θ between them as [26];

$$|\eta\rangle = \cos\theta|\eta_8\rangle - \sin\theta|\eta_0\rangle \quad , \quad |\eta^{'}\rangle = \sin\theta|\eta_8\rangle + \cos\theta|\eta_0\rangle \tag{4.38}$$

Thus, the main point here is to determine this mixing parameter θ theoretically and measure it experimentally and then to check the compatibility in between.

Actually, we will make two important assumptions while studying this mixing. In the first one, the mixing with heavier pseudo-scalars that could mix to η_0 and η_8 with $I = 0$ is neglected [38]. Next, we will assume that the mixing parameters are independent of the mass or the energy scale of the states. If they were dependent to energy, the problem would be harder to deal with [26].

For theoretical calculation of the mixing angle between these states, there are two basic schemes that are used commonly. One of them is the octet-singlet basis scheme which uses (η^8,η^0) as the basis states and the other is the quark-flavor basis scheme with (η^q,η^s) as the relevant basis states.

In either basis, owing to our knowledge from Quantum Mechanics, we can write our physical states (η,$\eta^{'}$) as a superposition of the chosen basis sates as [37, 47, 48, 54];

$$\begin{pmatrix} \eta \\ \eta^{'} \end{pmatrix} = U(\theta) \begin{pmatrix} \eta^8 \\ \eta^0 \end{pmatrix} \tag{4.39}$$

Similarly, for quark flavor basis we write [42–46];

$$\begin{pmatrix} \eta \\ \eta^{'} \end{pmatrix} = U(\phi) \begin{pmatrix} \eta^q \\ \eta^s \end{pmatrix} \tag{4.40}$$

where, U is defined to be a $2{\times}2$ unitary matrix in the form $U(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ [38] and $|\eta_q\rangle = \frac{|(u\bar{u}+d\bar{d})\rangle}{\sqrt{2}}$, $|\eta_s\rangle = |s\bar{s}\rangle$.

Now, in order to find a relation between the octet-singlet mixing angle θ and quark-flavor mixing angle ϕ we compare the mixing pattern of the states for both schemes as;

$$|\eta\rangle = \cos\theta|\eta_8\rangle - \sin\theta|\eta_0\rangle = \cos\phi|\eta_q\rangle - \sin\phi|\eta_s\rangle$$

$$|\eta^{'}\rangle = \sin\theta|\eta_8\rangle + \cos\theta|\eta_0\rangle = \sin\phi|\eta_q\rangle + \cos\phi|\eta_s\rangle \tag{4.41}$$

Using [41];

$$\begin{aligned} \eta_8 &= \frac{1}{\sqrt{3}}\eta_q + \frac{1}{\sqrt{6}}\eta_s \\ \eta_0 &= \frac{\sqrt{2}}{\sqrt{3}}\eta_q + \frac{1}{\sqrt{3}}\eta_s \end{aligned} \tag{4.42}$$

and equating both sides of Eqn.(4.41), we get;

$$\begin{aligned} \cos\phi &= \sin\theta\frac{2}{\sqrt{6}} + \cos\theta\frac{1}{\sqrt{3}} \\ \sin\phi &= \sin\theta\frac{1}{\sqrt{3}} + \cos\theta\frac{2}{\sqrt{6}} \end{aligned} \tag{4.43}$$

In the ideal limit, i.e. when there is no mixing in the octet-singlet basis ($\theta = 0$), one can define an ideal mixing angle ϕ_{ideal} for the quark-flavor basis;

$$\cos\phi_{ideal} = \frac{1}{\sqrt{3}} \quad , \quad \sin\phi_{ideal} = \frac{\sqrt{2}}{\sqrt{3}}$$

Thus, we get [42];

$$\phi_{ideal} = \arctan\sqrt{2} \tag{4.44}$$

After that, using (Eqn.(4.43)) one can also write ;

$$\sin\phi = \sin\theta\cos\phi_{ideal} + \cos\theta\sin\phi_{ideal} = \sin(\phi_{ideal} + \theta) \tag{4.45}$$

Thus,

$$\theta = \phi - \phi_{ideal} \tag{4.46}$$

This result show us how to relate the octet-singlet mixing angle to the quark-flavor basis mixing angle. Namely, once we know one of them the other can be found easily.

Then, for the calculations of the decay constants in either basis, we make use of an important and commonly used assumption that "decay constants follow the same mixing pattern as the states do" [42]. Hence, following these arguments, we can describe the octet-singlet basis decay constants as [42];

$$\begin{aligned} f_{\eta}^{8} &= f_8 \cos\theta \quad , \quad f_{\eta}^{0} = -f_0 \sin\theta \\ f_{\eta'}^{8} &= f_8 \sin\theta \quad , \quad f_{\eta'}^{0} = f_0 \cos\theta \end{aligned} \tag{4.47}$$

However, it was later understood that in octet-singlet basis scheme instead of assigning one mixing angle θ, one should assign two mixing angles θ_0 and θ_8 in order to be compatible with the theoretical study within ChPT [37, 54] and phenomenological results [55]. This requirement can be understood as follows. Initially, in octet-singlet basis if we assume that the matrix elements of the axial vector current with opposite state vanish. i.e.

$$\langle 0|A_{\mu 5}^{8}|\eta^{0}\rangle = \langle 0|A_{\mu 5}^{0}|\eta^{8}\rangle = 0 \tag{4.48}$$

Then, by using the mixing pattern of particle states;

$$\begin{aligned} |\eta\rangle &= \cos\theta|\eta_8\rangle - \sin\theta|\eta_0\rangle \\ |\eta^{'}\rangle &= \sin\theta|\eta_8\rangle + \cos\theta|\eta_0\rangle \end{aligned} \tag{4.49}$$

one can write for decay constants in terms of matrix elements (see section 4.1);

$$
\begin{aligned}
ip_\mu f_\eta^8 = \langle 0|A_{\mu 5}^8|\eta\rangle &= \cos\theta\,\underbrace{\langle 0|A_{\mu 5}^8|\eta^8\rangle} - \sin\theta\,\underbrace{\langle 0|A_{\mu 5}^8|\eta^0\rangle} \\
&= ip_\mu f_8\cos\theta
\end{aligned}
\tag{4.50}
$$

Following the same reasoning we can find for the rest [42];

$$
\begin{aligned}
f_\eta^8 = f_8\cos\theta \quad &, \quad f_\eta^0 = -f_0\sin\theta \\
f_{\eta'}^8 = f_8\sin\theta \quad &, \quad f_{\eta'}^0 = f_0\cos\theta
\end{aligned}
\tag{4.51}
$$

However, this time doing the same calculations by assuming that the matrix elements of the axial vector currents with opposite pseudo-scalar states is non-zero, i.e. $\langle 0|A_{\mu 5}^8|\eta^0\rangle = \langle 0|A_{\mu 5}^0|\eta^8\rangle \neq 0$, we can get;

$$
\begin{aligned}
ip_\mu f_\eta^8 = \langle 0|A_{\mu 5}^8|\eta\rangle &= \cos\theta\,\underbrace{\langle 0|A_{\mu 5}^8|\eta^8\rangle} - \sin\theta\,\underbrace{\langle 0|A_{\mu 5}^8|\eta^0\rangle} \\
&= ip_\mu\cos\theta f_8 - ip_\mu\sin\theta f_{08} \equiv \cos\theta_8 f_8
\end{aligned}
\tag{4.52}
$$

Performing the same procedure we obtain for the rest;

$$
\begin{aligned}
f_\eta^8 = \cos\theta_8 f_8 \quad &, \quad f_\eta^0 = -\sin\theta_0 f_0 \\
f_{\eta'}^8 = f_8\sin\theta_8 \quad &, \quad f_{\eta'}^0 = f_0\cos\theta_0
\end{aligned}
\tag{4.53}
$$

Phenomenological results [55] showed that θ_0 and θ_8 angles are distinct ($\theta_0 \simeq -9.2, \theta_8 \simeq -21.2$) and so this requirement is justified. However, in quark-flavor basis, if one initially assign two mixing angles (ϕ_q, ϕ_s) in analogy to octet-singlet scheme, by the help of the phenomenological studies done in [55], it was found that these two angles nearly coincide ($\phi_q = 39.4, \phi_s = 38.5$). Moreover, a theoretical explanation has been given to this by OZI rule which sates that "the difference between ϕ_q and ϕ_s vanishes in the leading order $1/N_c$ expansion of ChPT" [37, 54]. Therefore, in quark flavor basis one mixing angle ϕ is enough to describe the mixing between η and $\eta^{'}$. As a result, decay constants in that basis show the mixing pattern [42];

$$
\begin{aligned}
f_\eta^q = f_q \cos\phi \quad , \quad f_\eta^s = -f_s \sin\phi \\
f_{\eta'}^q = f_q \sin\phi \quad , \quad f_{\eta'}^s = f_s \cos\phi
\end{aligned} \tag{4.54}
$$

Hence, we see that the central assumption that the "decay constants follow the same mixing pattern as states do" will be valid only for quark-flavor basis mixing scheme.

As we can see from above discussions, studying in quark-flavor basis has more advantages. First, we have only three parameters to be calculated (f_q, f_s, ϕ). Moreover, it allows us to find a relation between the ratio f_q/f_s and ϕ as we will see in the next section.

4.2.1 Mixing in quark-flavor Basis

The relations we wrote in Eqn.(4.54) for the mixing pattern of quark-flavor basis decay constants can be combined in a matrix form as;

$$
\begin{pmatrix} f_\eta^q & f_\eta^s \\ f_{\eta'}^q & f_{\eta'}^s \end{pmatrix} = U(\phi)\mathcal{F} \tag{4.55}
$$

where $\mathcal{F}$ is given by;

$$
\mathcal{F} = \begin{pmatrix} f_q & 0 \\ 0 & f_s \end{pmatrix}
$$

Then, expressing the octet-singlet axial vector currents in terms of quark-flavor axial vector currents as [41];

$$\begin{aligned}
A^8_{\mu 5} &= \frac{1}{\sqrt{3}}A^q_{\mu 5} - \frac{\sqrt{2}}{\sqrt{3}}A^s_{\mu 5} = \cos\phi_{ideal}A^q_{\mu 5} - \sin\phi_{ideal}A^s_{\mu 5} \\
A^0_{\mu 5} &= \frac{\sqrt{2}}{\sqrt{3}}A^q_{\mu 5} + \frac{1}{\sqrt{3}}A^s_{\mu 5} = \sin\phi_{ideal}A^q_{\mu 5} + \cos\phi_{ideal}A^s_{\mu 5}
\end{aligned} \tag{4.56}$$

or in matrix notation;

$$\begin{pmatrix} A^8_{\mu 5} \\ A^0_{\mu 5} \end{pmatrix} = U(\phi_{ideal}) \begin{pmatrix} A^q_{\mu 5} \\ A^s_{\mu 5} \end{pmatrix} \tag{4.57}$$

one can go further and write for the decay constants;

$$\begin{aligned}
\begin{pmatrix} f^8_\eta & f^0_\eta \\ f^8_{\eta'} & f^0_{\eta'} \end{pmatrix} ip_\mu &= \begin{pmatrix} \langle 0|A^8_{\mu 5}|\eta\rangle & \langle 0|A^0_{\mu 5}|\eta\rangle \\ \langle 0|A^8_{\mu 5}|\eta'\rangle & \langle 0|A^0_{\mu 5}|\eta'\rangle \end{pmatrix} = [\langle 0| \begin{pmatrix} A^8_{\mu 5} \\ A^0_{\mu 5} \end{pmatrix} | \begin{pmatrix} \eta & \eta' \end{pmatrix} \rangle]^\dagger \\
&= [U(\phi_{ideal}). \underbrace{\langle 0| \begin{pmatrix} A^q_{\mu 5} \\ A^s_{\mu 5} \end{pmatrix} | \begin{pmatrix} \eta^q & \eta^s \end{pmatrix} \rangle} .U^\dagger(\phi)]^\dagger \\
&= [U(\phi_{ideal})\mathcal{F}U(\phi)^\dagger]^\dagger ip_\mu
\end{aligned} \tag{4.58}$$

Finally, we reach to the desired form [42];

$$\begin{pmatrix} f^8_\eta & f^0_\eta \\ f^8_{\eta'} & f^0_{\eta'} \end{pmatrix} = U(\phi)\mathcal{F}U^\dagger(\phi_{ideal}) \tag{4.59}$$

This equation shows us the relation between octet-singlet and quark-flavor basis decay constants.

Now, if we write the above results for the decay constants explicitly;

$$\begin{pmatrix} f_\eta^8 & f_\eta^0 \\ f_{\eta'}^8 & f_{\eta'}^0 \end{pmatrix} = U(\phi)\mathcal{F}U^\dagger(\phi_{ideal}) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} f_q\cos\phi_{ideal} & f_q\sin\phi_{ideal} \\ -f_s\sin\phi_{ideal} & f_s\cos\phi_{ideal} \end{pmatrix} \tag{4.60}$$

we get;

$$\begin{aligned} f_\eta^8 &= f_q\cos\phi\cos\phi_{ideal} + f_s\sin\phi\sin\phi_{ideal} \\ f_{\eta'}^8 &= f_q\sin\phi\cos\phi_{ideal} - f_s\cos\phi\sin\phi_{ideal} \\ f_\eta^0 &= f_q\cos\phi\sin\phi_{ideal} - f_s\sin\phi\cos\phi_{ideal} \\ f_{\eta'}^0 &= f_q\sin\phi\sin\phi_{ideal} + f_s\cos\phi\cos\phi_{ideal} \end{aligned} \tag{4.61}$$

and from here one can write;

$$\begin{aligned} f_8 = \sqrt{(f_\eta^8)^2 + (f_{\eta'}^8)^2} = \sqrt{f_q^2\cos\phi_{ideal}^2 + f_s^2\sin\phi_{ideal}^2} \\ f_0 = \sqrt{(f_\eta^0)^2 + (f_{\eta'}^0)^2} = \sqrt{f_q^2\sin\phi_{ideal}^2 + f_s^2\cos\phi_{ideal}^2} \end{aligned} \tag{4.62}$$

Thus, we obtain [42];

$$\begin{aligned} f_8 &= \sqrt{\frac{1}{3}f_q^2 + \frac{2}{3}f_s^2} \\ f_0 &= \sqrt{\frac{2}{3}f_q^2 + \frac{1}{3}f_s^2} \end{aligned} \tag{4.63}$$

Moreover, using the previous results found in Eqn.(4.60);

$$
\begin{aligned}
f_{\eta}^{8} &= f_8 \cos\theta_8 = f_q \cos\phi \cos\phi_{ideal} + f_s \sin\phi \sin\phi_{ideal} \\
f_{\eta'}^{8} &= f_8 \sin\theta_8 = f_q \sin\phi \cos\phi_{ideal} - f_s \cos\phi \sin\phi_{ideal}
\end{aligned} \tag{4.64}
$$

one can write;

$$
\begin{aligned}
\tan\theta_8 &= f_{\eta'}^{8}/f_{\eta}^{8} = \frac{f_q \sin\phi \cos\phi_{ideal} - f_s \cos\phi \sin\phi_{ideal}}{f_q \cos\phi \cos\phi_{ideal} + f_s \sin\phi \sin\phi_{ideal}} \\
&= \frac{(\frac{f_q}{f_s}) \sin\phi \cos\phi_{ideal} - \cos\phi \sin\phi_{ideal}}{(\frac{f_q}{f_s}) \cos\phi \cos\phi_{ideal} + \sin\phi \sin\phi_{ideal}}
\end{aligned} \tag{4.65}
$$

Similarly;

$$
\begin{aligned}
\tan\theta_0 &= -f_{\eta}^{0}/f_{\eta'}^{0} = -\frac{f_q \cos\phi \sin\phi_{ideal} - f_s \sin\phi \cos\phi_{ideal}}{f_q \sin\phi \sin\phi_{ideal} + f_s \cos\phi \cos\phi_{ideal}} \\
&= -\frac{(\frac{f_q}{f_s}) \cos\phi \sin\phi_{ideal} - \sin\phi \cos\phi_{ideal}}{(\frac{f_q}{f_s}) \sin\phi \sin\phi_{ideal} + \cos\phi \cos\phi_{ideal}}
\end{aligned} \tag{4.66}
$$

At that point, making use of some trigonometric properties and with a little algebra we can obtain the relations [42] (see Appendix D for the detailed derivation of these equations);

$$
\begin{aligned}
\theta_8 &= \phi - \arctan(f_s/f_q\sqrt{2}) \\
\theta_0 &= \phi - \arctan(f_q/f_s\sqrt{2}) \\
\tan(\theta_0 - \theta_8) &= \frac{\sqrt{2}}{3}(f_s/f_q - f_q/f_s)
\end{aligned} \tag{4.67}
$$

These relations tell us how to connect the mixing parameters occurring in octet-singlet basis ($f_0, f_8, \theta, \theta_0, \theta_8$) to the ones in the quark-flavor basis (f_q, f_s, ϕ). Now, let us look what happens to these equations in the limit of $f_q = f_s$. It is obvious from the structure of these equations that in that limit the equations above give $\theta_0 = \theta_8 = \theta$. Moreover, combining Eqn.(4.63) with Eqn.(4.67) one can also see that the octet-singlet and quark-flavor basis decay constants coincide ($f_0 = f_8 = f_q = f_s$) as expected by the nonet symmetry.

Now, in order to describe the matrix elements of the axial vector current divergences taken between vacuum and meson states in terms of the physical states' parameters , we need a mass matrix of the form;

$$\mathcal{M}^2 = \begin{pmatrix} M_\eta^2 & 0 \\ 0 & M_{\eta'}^2 \end{pmatrix} \tag{4.68}$$

which satisfies the equation;

$$\langle 0|\partial^\mu A^i_{\mu 5}|P\rangle = \mathcal{F} U(\phi)^\dagger \mathcal{M}^2 \tag{4.69}$$

where, $i = q, s$ and $P = \eta, \eta'$ with $\mathcal{F} = \begin{pmatrix} f_q & 0 \\ 0 & f_s \end{pmatrix}$. On the other hand, we can also write;

$$\langle 0|\partial^\mu A^i_{\mu 5}|P\rangle = \langle 0| \begin{pmatrix} \partial^\mu A^q_{\mu 5} \\ \partial^\mu A^s_{\mu 5} \end{pmatrix} | \begin{pmatrix} \eta & \eta' \end{pmatrix} \rangle \tag{4.70}$$

Or, transforming the physical fields into the fields in quark-flavor basis as;

$$\begin{pmatrix} \eta & \eta' \end{pmatrix} = \begin{pmatrix} \eta^q & \eta^s \end{pmatrix} U(\phi)^\dagger$$

we can proceed as;

$$\langle 0|\partial^\mu A^i_{\mu 5}|P\rangle = \begin{pmatrix} \langle 0|\partial^\mu A^q_{\mu 5}|\eta^q\rangle & \langle 0|\partial^\mu A^q_{\mu 5}|\eta^s\rangle \\ \langle 0|\partial^\mu A^s_{\mu 5}|\eta^q\rangle & \langle 0|\partial^\mu A^s_{\mu 5}|\eta^s\rangle \end{pmatrix} U(\phi)^\dagger = \mathcal{F}U(\phi)^\dagger \mathcal{M}^2 \quad (4.71)$$

Thus,

$$\begin{pmatrix} \langle 0|\partial^\mu A^q_{\mu 5}|\eta^q\rangle & \langle 0|\partial^\mu A^q_{\mu 5}|\eta^s\rangle \\ \langle 0|\partial^\mu A^s_{\mu 5}|\eta^q\rangle & \langle 0|\partial^\mu A^s_{\mu 5}|\eta^s\rangle \end{pmatrix} = \mathcal{F}U(\phi)^\dagger \mathcal{M}^2 U(\phi) \quad (4.72)$$

At that point, if we define a mass matrix in quark-flavor basis as;

$$\mathcal{M}^2_{qs} = \begin{pmatrix} \frac{1}{f_q}\langle 0|\partial^\mu A^q_{\mu 5}|\eta^q\rangle & \frac{1}{f_s}\langle 0|\partial^\mu A^s_{\mu 5}|\eta^q\rangle \\ \frac{1}{f_q}\langle 0|\partial^\mu A^q_{\mu 5}|\eta^s\rangle & \frac{1}{f_s}\langle 0|\partial^\mu A^s_{\mu 5}|\eta^s\rangle \end{pmatrix} \quad (4.73)$$

one can also see that $\mathcal{M}^2_{qs}$ is equal to;

$$\begin{aligned} \mathcal{M}^2_{qs} &= \begin{pmatrix} \frac{1}{f_q}\langle 0|\partial^\mu A^q_{\mu 5}|\eta^q\rangle & \frac{1}{f_s}\langle 0|\partial^\mu A^s_{\mu 5}|\eta^q\rangle \\ \frac{1}{f_q}\langle 0|\partial^\mu A^q_{\mu 5}|\eta^s\rangle & \frac{1}{f_s}\langle 0|\partial^\mu A^s_{\mu 5}|\eta^s\rangle \end{pmatrix} = U(\phi)^\dagger \mathcal{M}^2 U(\phi) \\ &= \begin{pmatrix} M^2_\eta \cos^2\phi + M^2_{\eta'}\sin^2\phi & -M^2_\eta \cos\phi\sin\phi + M^2_{\eta'}\sin\phi\cos\phi \\ -M^2_\eta \cos\phi\sin\phi + M^2_{\eta'}\sin\phi\cos\phi & M^2_\eta \sin^2\phi + M^2_{\eta'}\cos^2\phi \end{pmatrix} \end{aligned} \quad (4.74)$$

Now, we can calculate each entry of the mass matrix (by including the anomaly contributions to axial-vector currents' divergences) as;

$$\begin{aligned} \frac{1}{f_q}\langle 0|\partial^\mu A^q_{\mu 5}|\eta^q\rangle &= \frac{\sqrt{2}}{f_q}\langle 0|m_u \bar{u} i\gamma_5 u + m_d \bar{d} i\gamma_5 d|\eta^q\rangle + \frac{\sqrt{2}}{f_q}\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta^q\rangle \\ &= m^2_{qq} + \frac{\sqrt{2}}{f_q}\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta^q\rangle \\ \frac{1}{f_s}\langle 0|\partial^\mu A^s_{\mu 5}|\eta^s\rangle &= \frac{2}{f_s}\langle 0|m_s \bar{s} i\gamma_5 s|\eta^s\rangle + \frac{1}{f_s}\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta^s\rangle = m^2_{ss} + \frac{1}{f_s}\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta^s\rangle \end{aligned}$$

$$
\begin{aligned}
\frac{1}{f_q}\langle 0|\partial^\mu A^q_{\mu 5}|\eta^s\rangle &= \frac{\sqrt{2}}{f_q}\underbrace{\langle 0|m_u\bar{u}i\gamma_5 u + m_d\bar{d}i\gamma_5 d|\eta^s\rangle} + \frac{\sqrt{2}}{f_q}\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta^s\rangle \\
&= \frac{\sqrt{2}}{f_q}\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta^s\rangle \\
\frac{1}{f_s}\langle 0|\partial^\mu A^s_{\mu 5}|\eta^q\rangle &= \frac{2}{f_s}\langle 0|m_s\bar{s}i\gamma_5 s|\eta^q\rangle + \frac{1}{f_s}\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta^q\rangle \\
&= \frac{1}{f_s}\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta^q\rangle
\end{aligned}
\tag{4.75}
$$

Therefore, our mass matrix in quark-flavor basis becomes;

$$
\mathcal{M}^2_{qs} = \begin{pmatrix} m^2_{qq} + \frac{\sqrt{2}}{f_q}\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta^q\rangle & \frac{1}{f_s}\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta^q\rangle \\ \frac{\sqrt{2}}{f_q}\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta^s\rangle & m^2_{ss} + \frac{1}{f_s}\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta^s\rangle \end{pmatrix} \tag{4.76}
$$

where, m^2_{qq} and m^2_{ss} are defined as [42];

$$
m^2_{qq} = \frac{\sqrt{2}}{f_q}\langle 0|m_u\bar{u}i\gamma_5 u + m_d\bar{d}i\gamma_5 d|\eta^q\rangle \ , \ m^2_{ss} = \frac{2}{f_s}\langle 0|m_s\bar{s}i\gamma_5 s|\eta^s\rangle \tag{4.77}
$$

As we can see, the non-diagonal elements of the mass matrix $\mathcal{M}^2_{qs}$ exist as a result of the Anomaly. Now, looking at the form of $\mathcal{M}^2_{qs}$ defined in terms of quark-flavor mixing angle ϕ (Eqn.(4.74)), one can see that the non diagonal entries are equal. Thus, by equating the non-diagonal entries appear in Eqn.(4.76), one can define the ratio of the quark-flavor basis decay constants as;

$$
y = \frac{f_q}{f_s} = \sqrt{2}\frac{\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta^s\rangle}{\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta^q\rangle} \tag{4.78}
$$

Moreover, we define another parameter a^2 as;

$$
a^2 = \frac{1}{\sqrt{2}f_q}\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta^q\rangle \tag{4.79}
$$

Now, making use of the relations found in Eqn.(4.74) we can write;

$$
\begin{aligned}
a^2 &= \frac{M_\eta^2 \cos^2\phi + M_{\eta'}^2 \sin^2\phi - m_{qq}^2}{2} \, , \\
y &= \frac{(M_{\eta'}^2 - M_\eta^2)\sin 2\phi}{2\sqrt{2}a^2}
\end{aligned} \tag{4.80}
$$

At that point, comparing the Eqn.(4.78) with Eqn.(4.80), one can obtain relation between the ratio of the decay constants ($\frac{f_q}{f_s}$) and the quark-flavor basis mixing angle ϕ. Moreover, comparing Eqn.(4.67) with Eqn.(4.78) and Eqn.(4.80), one can get a relation between the quark-flavor basis mixing angle ϕ and octet-singlet mixing angle θ_8 [42] (see Appendix C for detailed calculations) as;

$$
\cot\theta_8 = -\tan\phi(\frac{M_{\eta'}}{M_\eta})^2 \tag{4.81}
$$

Now, in order to relate the quark-flavor decay constants to the decay constants of pion and kaon, we follow the same reasoning as we did in section 4.1. To the first order in ϵ, due to the flavor symmetry breaking we can write [39, 40];

$$
\begin{aligned}
f_\pi &\simeq f_q \quad , \quad f_K = f_{u\bar{s}} \simeq (1+\epsilon)f_q \\
f_s &= f_{s\bar{s}} \simeq (1+2\epsilon)f_q \simeq (1+\epsilon)f_K
\end{aligned} \tag{4.82}
$$

Thus, combining these equations one can write [42];

$$
f_q = f_\pi \quad , \quad f_s = \sqrt{2f_K^2 - f_\pi^2} \tag{4.83}
$$

At that point, we make the theoretical estimate that the quark-flavor basis mass terms (m_{qq}^2, m_{ss}^2) are related to the physical particles' masses (m_π^2, m_K^2) in the same way as the quark-flavor basis decay constants f_q, f_s related to f_π and

f_K [38] (see Eqn.(4.83)). Then, we can write (to first order in flavor symmetry breaking) [42];

$$m_{qq}^2 = M_\pi^2 \quad , \quad m_{ss}^2 = 2M_K^2 - M_\pi^2 \tag{4.84}$$

As a result, we can see that by using all of these relations it becomes possible for us to determine all of the mixing parameters $(\phi, a^2, y, \theta, \theta_8, \theta_0)$ related to η-$\eta^{'}$ mixing provided that the physical particle' masses and their decay constants are known. Below are listed the values of these parameters which are obtained both from theoretical studies [42] and phenomenology.

Table 4.1: Values of the quark-flavor basis mixing parameters obtained from Theory and Phenomenology [42].

	f_s/f_π	f_q/f_π	ϕ	θ	$a^2[GeV^2]$	y
Theory [42]	1.41	1.00	42.4	-12.3	0.281	0.78
Phenomenology.	1.34	1.07	39.3	-15.4	0.265	0.81

Furthermore, in the $(1/N_c)$ expansion of ChPT, Leutwyler and Kaiser have found that [56];

$$f_8 = \sqrt{\frac{4}{3}f_K^2 - \frac{1}{3}f_\pi^2} \tag{4.85}$$

Actually, inserting the previously found results; $f_q = f_\pi$ and $f_s = \sqrt{2f_K^2 - f_\pi^2}$, in the above equation one gets;

$$f_8 = \sqrt{\frac{4}{3}\frac{f_s^2 + f_q^2}{2} - \frac{1}{3}f_q^2} = \sqrt{\frac{1}{3}f_q^2 + \frac{2}{3}f_s^2} \tag{4.86}$$

which is exactly the same result with Eqn.(4.63) the one that Feldmann, Kroll and Stech found [42]. Combining Eqn.(4.63), Eqn.(4.67), Eqn.(4.83) and the value of the quark-flavor basis mixing angle ϕ obtained from mass matrix, the parameters f_0, f_8, θ_0 and θ_8 can be calculated. The theoretical and Phenomenological values of these parameters are given in Table.(4.2).

Table 4.2: Values of the octet-singlet mixing parameters obtained from Theory and Phenomenology [42].

	f_0/f_π	f_8/f_π	θ_0	θ_8
Theory [42]	1.15	1.28	-2.7	-21.0
[55]	1.20	1.28	-9.1	-22.2
[56]	1.25	1.28	-4.0	-20.5
Phenomenology	1.17	1.26	-9.2	-21.2

It is important to emphasize that the calculations done by Feldmann, Kroll and Stech in this part, depend on the central assumption that the decay constants in quark-flavor basis follow the same pattern of mixing as the particle states do. Otherwise, compatibility with phenomenology could not be established. In the next part, we will check the consistency of these calculations with experimental results.

4.2.2 Phenomenological Results for ϕ

The mixing parameters can also be determined from phenomenology. For example, the quark-flavor basis mixing angle ϕ can be obtained from several decays which are discussed in references [30, 42, 45, 48, 49, 57, 58]. The main idea for the evaluation of the mixing angle ϕ from experiment is to look at the branching ratios of the processes in which η and $\eta^{'}$ are involved. Where, the results of the branching ratios only depend on the mixing angle ϕ and some factors that come from kinematics. As an example for the extraction of quark-flavor mixing angle ϕ from phenomenology, let us consider the decays of $J/\psi \to \rho\eta$ and $J/\psi \to \rho\eta^{'}$.

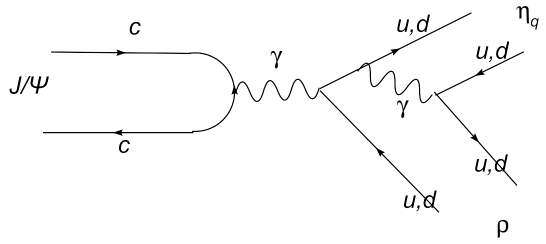

Figure 4.1: $J/\psi \rightarrow\ \rho\ \eta_q$ Decay Mode.

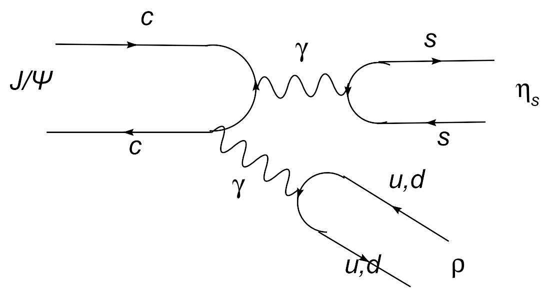

Figure 4.2: $J/\psi \rightarrow\ \rho\ \eta_s$ Decay Mode.

For these decays if the Okubo-Zweig-Iizuka (OZI) suppressed effects are ignored, which states that "the decays that correspond to disconnected quark diagrams will be suppressed compared to those in which quark lines are connected", looking at the Fig.(4.2) we see that the transition to η_s diagram includes disconnected quark lines and so we can neglect the η_s component contribution in these decays. Thus, we can say that the final state of these decays is the η_q component of η or $\eta^{'}$. In general, we can write the matrix element for these decays as;

$$\mathcal{M} = \langle \rho(k_1)\eta^{(')}(k_2)|J/\psi(q)\rangle = \varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\mu}\varepsilon^{'\nu}{k_1}^{\alpha}{k_2}^{\beta}.F_{\rho\eta^{(')}}(q^2)$$

where, ε^{μ} and $\varepsilon^{'\nu}$ are the polarizations of the vector mesons ρ and J/ψ respectively. Afterwards, making use of the transformation property of the physical states to quark-flavor basis sates one can write for the transition form factors [42];

$$F_{\rho\eta}(q^2) = \cos\phi F_{\rho\eta_q}(q^2) \quad , \quad F_{\rho\eta'}(q^2) = \sin\phi F_{\rho\eta_q}(q^2) \tag{4.87}$$

Then, using this matrix element and doing the required calculations we can write [42] (see Appendix A for detailed calculation of the decay widths);

$$R_{J/\psi} = \frac{\Gamma[J/\psi \rightarrow \rho\eta^{'}]}{\Gamma[J/\psi \rightarrow \rho\eta]} = \tan^2\phi(\frac{k_{\rho\eta'}}{k_{\rho\eta}})^3 \tag{4.88}$$

where,

$$k_{\rho P} = \frac{M_{J/\psi}}{2}[1 - (M_{\rho}^2 + M_P^2)/M_{J/\psi}^2]$$

is the magnitude of the 3-momenta of the particles that appear in the final state and $P = \eta, \eta^{'}$. As a result, we can see that by using the experimental result for this branching ratio as 0.54 ± 0.11 [59] and extracting the meson masses from experiment, it becomes possible for us to calculate the quark-flavor mixing angle ϕ. According to the study of Feldmann, Kroll and Stech [42] the value of the mixing angle has been found to be $\phi = 39.9 \pm 2.9$. Furthermore, in the all isospin-1 $J/\psi \rightarrow VP$ decays approximatively the same value for this mixing angle have been found [58]. As we have mentioned before, the value of the quark-flavor mixing angle ϕ can also be determined by looking at several decay processes [41,49,53,57,58]. Some of these experimental processes and the values obtained from them for the mixing angle ϕ are listed in Table(4.3).

Table 4.3: Values of quark-flavor mixing angle obtained from different experimental decays [38].

Decays	ϕ	Error
$J/\psi \longrightarrow \rho\eta(\eta^{'})$	39.9	± 2.9
$J/\psi \rightarrow \gamma\eta(\eta^{'})$	39.0	± 1.6
$\eta^{'} \rightarrow \rho\gamma, \rho \rightarrow \eta\gamma$	35.3	± 5.5
$D_s \rightarrow \ell\nu\eta(\eta^{'})$	41, 3	± 5.3
$p\bar{p} \rightarrow \pi(\eta,\omega)\eta(\eta^{'})$	37.4	± 1.8
$a_2 \rightarrow \pi\eta(\eta^{'})$	43.1	± 3.0
$\pi^- p \rightarrow n\eta(\eta^{'})$	39.3	± 1.2
Average	39.3	± 1.0

As can be seen from the Table.(4.3), the results obtained from all processes are compatible with each other. The average value for the quark-flavor mixing angle ϕ is found to be $\phi_{average} = 39.3 \pm 1.0$. To conclude, for recent experimental analysis of $\eta - \eta^{'}$ mixing with the decay of $J/\psi \to VP$ we refer the reader to see e.g [53].

CHAPTER 5

CONCLUSION

In this thesis, firstly some symmetries of Quantum Chromodynamics (QCD) are discussed. When discussing these symmetries, it is mentioned that in the limit of vanishing quark masses the QCD Lagrangian possess a symmetry called "Chiral symmetry". However, by the realization of Nambu and Goldstone that the QCD vacuum is not annihilated by the Axial charges , it is seen that the $SU(3)$ Chiral symmetry was broken to $SU(3)$ Vectorial symmetry. As a result of that since eight generators were broken, eight pseudo-scalar particles $(\pi^{\pm}, \pi^0, K^{\pm}, K^0, \eta)$ appear in accordance with the Goldstone Theorem. Moreover, the $U(1)$ axial anomaly is examined and it is seen that the so called anomaly occurs since even in chiral limit the the divergence of the singlet axial vector current does not vanish $(\partial_\mu {J^0}_{\mu 5} \neq 0)$.

Afterwards, the main properties of the effective field theories are investigated. As a low-energy effective field theory of QCD, the leading order (LO) mesonic ChPT is reviwed and the general construction principles for the leading order chiral effective Lagrangian are discussed. In the lowest order ChPT analysis, it is seen that there exist only one low-energy constant f and also the determination methods for the mass relations are studied in detail. It is also seen that ChPT provides an important way for the determination of the quark mass ratios by using the known mass values of the pseudo-scalar particles.

In the final part of this thesis, the mixing phenomenon of the pseudo-scalar states are discussed. It is shown that the main reason for the mixing of the pseudo-scalar particles is the explicit breakdown of $SU(2)$ isospin or $SU(3)$ flavor symmetries. Namely, the mass differences of u, d and s quarks cause the mixing. While studying the mixing phenomenon of η-$\eta^{'}$ mesons, a brief explanation to the general methods used to determine the mixing parameters in the matrix element analysis section was given. Afterwards, following the work of Feldmann, Kroll and Stech, the determination methods for the quark-flavor basis mixing parameters f_q, f_s, ϕ are reviewed and in the last part the consistency of these results with the values obtained from phenomenology is checked. Looking at the results obtained from different experimental processes which involve η and $\eta^{'}$, it is seen that all the values obtained for the quark-flavor mixing angle ϕ were compatible with each other and the average value for it was found to be $\phi_{average} = 39.3 \pm 1.0$.

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APPENDIX A

BRANCHING RATIO CALCULATIONS

For the decay of $J/\psi \to \rho\eta'$, the decay width can be written as;

$$\Gamma = \frac{1}{2M_{J/\psi}} \int |\mathcal{M}|^2 \frac{d^3k_1}{(2\pi)^3 2E_1} \frac{d^3k_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^4(q - k_1 - k_2) \tag{A.1}$$

where, q, k_1 and k_2 are the 3 momenta of J/ψ, ρ and η' particles respectively. Applying the conservation rules in the rest frame of J/ψ and using the properties of the dirac delta functions we can write;

$$\begin{gathered} q = (M_{J/\psi}, 0) \\ k_1 = (E_1, \overrightarrow{k_1}) \\ k_2 = (E_2, \overrightarrow{k_2}) \\ E_1 + E_2 = M_{J/\psi} \\ \overrightarrow{k_1} + \overrightarrow{k_2} = 0 \end{gathered}$$

Moreover, one can separate the 4-dimensional delta function as;

$$\delta^4(q - k_1 - k_2) = \delta(M_{J/\psi} - E_1 - E_2)\delta^3(0 - \overrightarrow{k_1} - \overrightarrow{k_2})$$

Then, using these properties and taking the k_1 integral we get;

$$\Gamma = \frac{1}{2M_{J/\psi}} \int |\mathcal{M}|^2 \frac{1}{(2\pi)^3 2E_1} \frac{d^3k_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta(M_{J/\psi} - E_1 - E_2) \tag{A.2}$$

where, $\mathcal{M}$ can be defined as;

$$\mathcal{M} = \langle \rho(k_1)\eta(k_2)|J/\psi(q)\rangle = \varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\mu}\varepsilon'^{\nu}k_1{}^{\alpha}k_2{}^{\beta}.F(q^2) \tag{A.3}$$

Taking square of this we get;

$$\mathcal{M}^2 = \varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\mu}\varepsilon'^{\nu}k_1{}^{\alpha}k_2{}^{\beta}F(q^2) \times \varepsilon_{\mu'\nu'\alpha'\beta'}\varepsilon^{\mu'}\varepsilon'^{\nu'}k_1{}^{\alpha'}k_2{}^{\beta'}F(q^2)$$

Now, using the relations;

$$\sum \varepsilon^{\mu}\varepsilon^{\mu'} = -g^{\mu\mu'} + \frac{k_1{}^{\mu}k_1{}^{\mu'}}{M_{\rho}{}^2}$$

$$\sum \varepsilon'^{\nu}\varepsilon'^{\nu'} = -g^{\nu\nu'} + \frac{k_2{}^{\nu}k_2{}^{\nu'}}{M_{\eta'}{}^2}$$

and inserting the product of two Levi-Civita tensors in $|\mathcal{M}|^2$;

$$\begin{aligned}\underbrace{\varepsilon_{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\alpha'\beta'}}\, k_1{}^{\alpha}k_2{}^{\beta}k_1{}^{\alpha'}k_2{}^{\beta'} &= 2(g_{\alpha\alpha'}g_{\beta\beta'} - g_{\alpha\beta'}g_{\beta\alpha'})k_1{}^{\alpha}k_2{}^{\beta}k_1{}^{\alpha'}k_2{}^{\beta'} \\ = 2[k_1^2k_2^2 - (k_1k_2)^2] &= 2[k_1^2k_2^2 - \frac{1}{4}[(k_1+k_2)^2 - k_1^2 - k_2^2]^2] \\ &= 2[m_1^2m_2^2 - \frac{1}{4}(M_{J\psi}^2 - m_1^2 - m_2^2)^2]\end{aligned} \tag{A.4}$$

we obtain;

$$|\mathcal{M}|^2 = 8M_{J/\psi}.[\frac{(M_{J/\psi}^2 + m_1^2 - m_2^2)^2}{4M_{J/\psi}^2} - m_1^2]F_{\rho\eta'}^2(q^2) = 8M_{J/\psi}k_{\rho\eta'}^2F_{\rho\eta'}^2(q^2) \tag{A.5}$$

where, since $q^2 = M_{J/\psi}{}^2 = constant$, $F(q^2)$ becomes a constant too. Moreover, from the structure of $\mathcal{M}^2$ (Eqn.(A.5)), it follows that the matrix element squared is a constant so that in the decay width relation it can be taken out of the integral freely. Now, going back to the decay width expression (Eqn.(A.3)) and in the rest frame of J/ψ inserting

$$\overrightarrow{k_1} = -\overrightarrow{k_2}$$

$$E_1 = \sqrt{{k_1}^2 + {m_1}^2} = \sqrt{{k_2}^2 + {m_1}^2}$$
$$E_2 = \sqrt{{k_2}^2 + {m_2}^2}$$

one can get;

$$\Gamma = \frac{1}{2M_{J/\psi}}|\mathcal{M}|^2 \int \frac{1}{2\sqrt{{k_2}^2 + {m_1}^2}(2\pi)^3} \frac{4\pi {k_2}^2 dk_2}{(2\pi)^3 2\sqrt{{k_2}^2 + {m_2}^2}}(2\pi)^4$$
$$\delta(M_{J/\psi} - \sqrt{{k_2}^2 + {m_1}^2} - \sqrt{{k_2}^2 + {m_2}^2}) \tag{A.6}$$

and

$$\Gamma = \frac{1}{8M_{J/\psi}\pi}|\mathcal{M}|^2 \int \frac{1}{\sqrt{{k_2}^2 + {m_1}^2}} \frac{{k_2}^2 dk_2}{\sqrt{{k_2}^2 + {m_2}^2}}\delta(M_{J/\psi} - \sqrt{{k_2}^2 + {m_1}^2} - \sqrt{{k_2}^2 + {m_2}^2}) \tag{A.7}$$

In order to take this integral, we make use of the properties of delta function;

$$\int_{-\infty}^{\infty} dx f(x)\delta(g(x)) = \Sigma_i \frac{f(x_i)}{|g'(x_i)|}$$

where, x_i's are defined to be the zeros of the function g. Thus, in order to find x_i values one can write;

$$\begin{aligned} [M_{J/\psi} &- \sqrt{{k_2}^2 + {m_1}^2}]^2 = [\sqrt{{k_2}^2 + {m_2}^2}]^2 \\ \Rightarrow \quad {M_{J/\psi}}^2 &- 2M_{J/\psi}\sqrt{{k_2}^2 + {m_1}^2} + {k_2}^2 + {m_1}^2 = {k_2}^2 + {m_2}^2 \end{aligned} \tag{A.8}$$

so that we get the relation;

$$k_2^2 = k_{\rho\eta'}{}^2 = \frac{(M_{J/\psi}{}^2 - M_\rho{}^2 - M_{\eta'}{}^2)^2}{4M_{J/\psi}{}^2}$$

or

$$k_{\rho\eta'} = \frac{M_{J/\psi}}{2}[1 - \frac{M_\rho{}^2 + M_{\eta'}{}^2}{M_{J/\psi}{}^2}]$$

After these, we can take k_2 integral and the result is;

$$\Gamma = \frac{1}{8\pi M_{J/\psi}} |\mathcal{M}|^2 \frac{M_{J/\psi}}{2}[1 - \frac{M_\rho{}^2 + M_{\eta'}{}^2}{M_{J/\psi}{}^2}] \tag{A.9}$$

or

$$\Gamma = \frac{1}{8\pi M_{J/\psi}} |\mathcal{M}|^2 k_{\rho\eta'} \tag{A.10}$$

At that point, inserting the value of $|\mathcal{M}|^2$ from Eqn.(A.5), one can conclude that

$$\Gamma = \frac{1}{\pi M_{J/\psi}} M_{J/\psi} k_{\rho\eta'}^2 k_{\rho\eta'} F_{\rho\eta'}^2(q^2) = \frac{1}{\pi} k_{\rho\eta'}^3 F_{\rho\eta'}^2(q^2) \tag{A.11}$$

Finally, following the same calculations for $J/\psi \to \rho\eta$ one can obtain;

$$\Gamma \propto k_{\rho\eta}^3 F_{\rho\eta}^2(q^2) \tag{A.12}$$

Therefore, the ratio of the decay widths is;

$$R_{J/\psi} = \frac{\Gamma[J/\psi \to \rho\eta']}{\Gamma[J/\psi \to \rho\eta]} = \tan^2\phi(\frac{k_{\rho\eta'}}{k_{\rho\eta}})^3 \tag{A.13}$$

where, $\tan^2\phi$ factor comes from the definitions of the form factors as [42];

$$F_{\rho\eta'}(q^2) = \sin\phi F_{\rho\eta_q}(q^2)$$

$$F_{\rho\eta}(q^2) = \cos\phi F_{\rho\eta_q}(q^2).$$

APPENDIX B

CALCULATION OF THE NONET SYMMETRY MASS EXPRESSIONS

In the nonet symmetry conditions letting $f_u = f_d = f_s = f_8 = f_0 = f_{08}$ and using the relations

$$\langle 0|\bar{u}i\gamma_5 u|\bar{u}u\rangle = Bf_u \quad , \quad \langle 0|\bar{u}i\gamma_5 u|\eta_8\rangle = \frac{B}{\sqrt{6}}f_u$$
$$\langle 0|\bar{s}i\gamma_5 s|\eta_8\rangle = \frac{-2}{\sqrt{6}}Bf_s \tag{B.1}$$

one can write;

$$\begin{aligned}
\langle 0|\partial_\mu A^8_\mu|\eta^8\rangle &= f_8 m_8^2 = \langle 0|\frac{1}{\sqrt{6}}\partial_\mu A^n_\mu - \frac{2}{\sqrt{6}}\partial_\mu A^s_\mu|\eta^8\rangle \\
&= \frac{2}{\sqrt{6}}\{m_u\langle 0|\bar{u}i\gamma_5 u|\eta^8\rangle + m_d\langle 0|\bar{d}i\gamma_5 d|\eta^8\rangle - 2m_s\langle 0|\bar{s}i\gamma_5 s|\eta^8\rangle\} \\
&= \frac{2}{\sqrt{6}}\{\frac{m_u f_u B}{\sqrt{6}} + \frac{m_d f_d B}{\sqrt{6}} - 2\frac{-2m_s f_s B}{\sqrt{6}}\} \\
&= \frac{2}{6}f_u B(\underbrace{m_u + m_d} + 4m_s)
\end{aligned} \tag{B.2}$$

Thus, we get;

$$\langle 0|\partial_\mu A^8_\mu|\eta^8\rangle = f_8 m_8^2 = f_u m_8^2 = \frac{2}{3}f_u B(\hat{m} + 2m_s)$$

$$\Rightarrow \quad m_8^2 = \frac{2}{3}B(\hat{m} + 2m_s) \tag{B.3}$$

Similarly, for m_0^2 mass we write;

$$\begin{aligned}
\langle 0|\partial_\mu A_\mu^0|\eta^0\rangle &= f_0 m_0^2 = \langle 0|\frac{1}{\sqrt{3}}\partial_\mu A_\mu^n + \frac{1}{\sqrt{3}}\partial_\mu A_\mu^s|\eta^0\rangle \\
&= \frac{2}{\sqrt{3}}m_u\langle 0|\bar{u}i\gamma_5 u|\eta^0\rangle + \frac{2}{\sqrt{3}}m_d\langle 0|\bar{d}i\gamma_5 d|\eta^0\rangle \\
&+ \frac{2}{\sqrt{3}}m_s\langle 0|\bar{s}i\gamma_5 s|\eta^0\rangle + \sqrt{3}\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta^0\rangle \\
&= \frac{2}{\sqrt{3}}\{\frac{m_u f_u B}{\sqrt{3}} + \frac{m_d f_d B}{\sqrt{3}} + \frac{m_s f_s B}{\sqrt{3}}\} + \sqrt{3}\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta^0\rangle
\end{aligned} \tag{B.4}$$

Thus, at that point letting $\sqrt{3}\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta^0\rangle = \tilde{m_0}^2$ we can write;

$$m_0^2 = \tilde{m_0}^2 + \frac{2}{3}B(2\hat{m} + m_s) \tag{B.5}$$

Finally, for m_{08}^2 we can write;

$$\begin{aligned}
\langle 0|\partial_\mu A_\mu^8|\eta^0\rangle &= f_{08} m_{08}^2 = \langle 0|\frac{1}{\sqrt{6}}[\partial_\mu A_\mu^n - 2\partial_\mu A_\mu^s]|\eta^0\rangle \\
&= \frac{1}{\sqrt{6}}[2m_u\langle 0|\bar{u}i\gamma_5 u|\eta^0\rangle + 2m_d\langle 0|\bar{d}i\gamma_5 d|\eta^0\rangle + 2\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta^0\rangle \\
&- 4m_s\langle 0|\bar{s}i\gamma_5 s|\eta^0\rangle - 2\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta^0\rangle] \\
&= \frac{2}{\sqrt{6}}\{\frac{m_u f_u B}{\sqrt{3}} + \frac{m_d f_d B}{\sqrt{3}} - 2\frac{m_s f_s B}{\sqrt{3}}\}
\end{aligned} \tag{B.6}$$

Therefore, we obtain;

$$m_{08}^2 = \frac{2\sqrt{2}}{3}B(\hat{m} - m_s) \tag{B.7}$$

APPENDIX C

THE DERIVATION OF THE RELATION BETWEEN THE MIXING ANGLES ϕ AND θ_8

Using the previously found relations;

$$f_\eta^8 = f_8 \cos\theta_8 \quad , \quad f_{\eta'}^8 = f_8 \sin\theta_8$$

we can write;

$$\cot\theta_8 = \frac{f_\eta^8}{f_{\eta'}^8} = \frac{\frac{1}{M_\eta^2}\langle 0|\partial_\mu A_{\mu 5}^8|\eta\rangle}{\frac{1}{M_{\eta'}^2}\langle 0|\partial_\mu A_{\mu 5}^8|\eta'\rangle} \tag{C.1}$$

Now, transforming the octet-singlet Axial vector currents to quark-flavor basis Axial vector currents, and the physical states to quark-flavor basis;

$$\begin{pmatrix} A_{\mu 5}^8 \\ A_{\mu 5}^0 \end{pmatrix} = U(\phi_{ideal}) \begin{pmatrix} A_{\mu 5}^q \\ A_{\mu 5}^s \end{pmatrix} \quad , \quad \begin{pmatrix} \eta \\ \eta' \end{pmatrix} = U(\phi) \begin{pmatrix} \eta^q \\ \eta^s \end{pmatrix} \tag{C.2}$$

we write;

$$\cot\theta_8 = \frac{M_{\eta'}^2}{M_\eta^2} . \frac{\langle 0|\frac{1}{3}\partial_\mu A_{\mu 5}^q - \frac{2}{3}\partial_\mu A_{\mu 5}^s|\cos\phi\eta^q - \sin\phi\eta^s\rangle}{\langle 0|\frac{1}{3}\partial_\mu A_{\mu 5}^q - \frac{2}{3}\partial_\mu A_{\mu 5}^s|\sin\phi\eta^q + \cos\phi\eta^s\rangle} \tag{C.3}$$

Neglecting the masses of up and down quarks, this relation reduces to;

$$\cot\theta_8 = \frac{M^2_{\eta'}}{M^2_{\eta}} . \frac{\langle 0| -\frac{2}{3}\partial_\mu A^s_{\mu 5}| -\sin\phi\eta^s\rangle}{\langle 0| -\frac{2}{3}\partial_\mu A^s_{\mu 5}|\cos\phi\eta^s\rangle} \tag{C.4}$$

where, we used the orthogonality conditions that

$$\langle 0|\partial_\mu A^q_{\mu 5}|\eta^s\rangle = \langle 0|\partial_\mu A^s_{\mu 5}|\eta^q\rangle = 0$$

Finally, we obtain the desired result;

$$\cot\theta_8 = -(\frac{M_{\eta'}}{M_\eta})^2 \tan\phi. \tag{C.5}$$

APPENDIX D

THE DERIVATION OF THE TRIGONOMETRIC RELATION OF θ_0 AND θ_8

In order to obtain the last part of the Eqn.(4.67), we can use the first two relations of the same equation;

$$\begin{aligned} \theta_8 &= \phi - \arctan(f_s/f_q\sqrt{2}) \\ \theta_0 &= \phi - \arctan(f_q/f_s\sqrt{2}) \end{aligned} \tag{D.1}$$

or writing in different form as;

$$\begin{aligned} \arctan(f_s/f_q\sqrt{2}) &= \phi - \theta_8 \\ \arctan(f_q/f_s\sqrt{2}) &= \phi - \theta_0 \end{aligned} \tag{D.2}$$

Now, using the trigonometric properties for the tangent functions as;

$$\tan(\theta_0 - \theta_8) = \frac{\tan\theta_0 - \tan\theta_8}{1 + \tan\theta_0 \tan\theta_8} \tag{D.3}$$

Therefore, inserting the values in Eqn.(D.2) into the last equation we get;

$$\begin{aligned}
\tan(\theta_0 - \theta_8) &= \tan[(\phi - \theta_8) - (\phi - \theta_0)] = \frac{\tan(\phi - \theta_8) - \tan(\phi - \theta_0)}{1 + \tan(\phi - \theta_8)\tan(\phi - \theta_0)} \\
&= \frac{\tan(\arctan(f_s/f_q\sqrt{2})) - \tan(\arctan(f_q/f_s\sqrt{2}))}{1 + \tan(\arctan(f_s/f_q\sqrt{2}))\tan(\arctan(f_q/f_s\sqrt{2}))} \\
&= \frac{f_s/f_q\sqrt{2} - f_q/f_s\sqrt{2}}{1 + f_s/f_q\sqrt{2}f_q/f_s\sqrt{2}} \qquad \text{(D.4)}
\end{aligned}$$

Hence, doing the simplifications at the end we obtain;

$$\tan(\theta_0 - \theta_8) = \frac{\sqrt{2}}{3}(f_s/f_q - f_q/f_s) \qquad \text{(D.5)}$$