A MARKET MODEL FOR PRICING INFLATION INDEXED BONDS WITH JUMPS INCORPORATION

İBRAHİM ETHEM GÜNEY

AUGUST 2008

A MARKET MODEL FOR PRICING INFLATION INDEXED BONDS WITH JUMPS INCORPORATION

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF APPLIED MATHEMATICS
OF
THE MIDDLE EAST TECHNICAL UNIVERSITY

BY

İBRAHİM ETHEM GÜNEY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF
MASTER OF SCIENCE
IN
THE DEPARTMENT OF FINANCIAL MATHEMATICS

AUGUST 2008

Approval of the Graduate School of Applied Mathematics

Prof. Dr. Ersan AKYILDIZ
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Prof. Dr. Ersan AKYILDIZ
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Assist. Prof. Dr. Kasırga YILDIRAK Co-advisor	Assoc. Prof. Dr. Azize HAYFAVİ Supervisor

Examining Committee Members

Assoc. Prof. Dr. Azize HAYFAVİ

Prof. Dr. Gerhard Wilhelm WEBER

Assist. Prof. Dr. Kasırga YILDIRAK

Assist. Prof. Dr. Ömür UĞUR

Dr. Coşkun KÜÇÜKÖZMEN

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last name: İbrahim Ethem GÜNEY

Signature:

ABSTRACT

A MARKET MODEL FOR PRICING INFLATION INDEXED BONDS WITH JUMPS INCORPORATION

Güney, İbrahim Ethem
M.Sc., Department of Financial Mathematics
Supervisor: Assoc. Prof. Dr. Azize Hayfavi
Co-advisor: Assist. Prof. Dr. Kasırga Yıldırak

August 2008, 83 pages

Protection against inflation is an essential part of the today's financial markets, particularly in high-inflation economies. Hence, nowadays inflation indexed instruments are being increasingly popular in the world financial markets. In this thesis, we focus on pricing of the inflation-indexed bonds which are the unique inflation-indexed instruments traded in the Turkish bond market. Firstly, we review the Jarrow-Yıldırım model which deals with pricing of the inflation-indexed instruments within the HJM framework. Then, we propose a pricing model that is an extension of the Jarrow-Yıldırım model. The model allows instantaneous forward rates, inflation index and bond prices to be driven by both a standard Brownian motion and a finite number of Poisson processes. A closed-form pricing formula for an European call option on the inflation index is also derived.

Keywords: Inflation-indexed bond, HJM framework, Jarrow-Yıldırım model, Instantaneous forward rates.

ÖZ

ENFLASYONA ENDEKSLİ TAHVİLLERİ FİYATLAMAK İÇİN SIÇRAMALARI İÇEREN BİR PİYASA MODELİ

Güney, İbrahim Ethem
Yüksek Lisans, Finansal Matematik Bölümü
Tez Yöneticisi: Doç. Dr. Azize Hayfavi
Tez Yönetici Yardımcısı: Yar. Doç. Dr. Kasırga Yıldırak

Ağustos 2008, 83 sayfa

Enflasyona karşı korunmak, günümüz finansal piyasalarında, özellikle yüksek enflasyona sahip ekonomiler için oldukça hassas bir konudur. Bu nedenle, günümüzde finansal piyasalarda enflasyona endeksli enstrümanların popülaritesi gittikçe artmaktadır. Bu çalışmada, Türk tahvil piyasasında işlem gören enflasyona endeksli tek enstrüman olan tahvillerin fiyatlandırılması üzerinde çalışılmıştır. İlk olarak, Jarrow ve Yıldırım'ın geliştirmiş oldukları, enflasyona endeksli enstrümanları HJM çerçevesinde fiyatlayan model incelenmiştir. Daha sonra, Jarrow-Yıldırım modelin genişlemesi olan bir fiyatlama modeli önerilmiştir. Bu model, ileri tarihli faiz oranlarının, enflasyon endeksinin ve tahvil fiyatlarının Brown hareketi ve sonlu sayıdaki Poisson süreçlerini içerdiğini kabul etmektedir. Son olarak, enflasyon endeksi üzerine yazılmış olan Avrupa tipi bir alım opsiyonu için kapalı formda fiyatlama formülü elde edilmiştir.

Anahtar Kelimeler: Enflasyona endeksli tahvil, HJM çerçevesi, Jarrow-Yıldırım modeli, İleri tarihli anlık faiz oranları.

To my peerless family

Acknowledgements

First I would like to express my sincere gratitude to my initial supervisor Prof. Dr. Hayri Körezlioğlu, for patiently guiding and motivating me at the beginning of this study, even in his difficult times. Rest in peace my dear professor.

I would like to thank my supervisor Assoc. Prof. Dr. Azize Hayfavi, who has always support me in my graduate study, for her invaluable guidance, suggestions, comments and motivation throughout this study.

I am also grateful to my co-advisor Assist. Prof. Dr. Kasırga Yıldırak for his kindness and encouragement in the completion of this thesis.

I should not forget Prof. Dr. Gerhard Wilhelm Weber, Assist. Prof. Dr. Ömür Uğur, Jale Körezlioğlu and Dr. Coşkun Küçüközmen for helpful suggestions during the correction of the thesis.

I am indebted to Zehra Ekşi, Ayşegül İşcanoğlu Çekiç, Hüseyin Şentürk, Mehmet Ali Karadağ and Nilüfer Çalışkan for their endless support, kindness and much else.

I also would like to thank Derya Altıntan for her kind assistance, valuable encouragements and infinite patience and support.

I feel indebted to express my heartfelt thanks to Mehmet Özgür Kutlu, Ceren Karahan and Fatih Geçin for their endless support, genialness, sincerity and being with me all the way.

Finally, I would like to express my deepest gratitude to my family: to my admirable father Salim, to my lovely mother Gülsakin and to my dear sister Zehra for their love, unfailing support and patience.

TABLE OF CONTENTS

LIST OF TABLES

CHAPTER 1

INTRODUCTION

Inflation is defined as the increase over time of the prices of goods and services in the economy. Various inflation measures are in use, because there exists several price indices for different types of consumers. Two widely used indices are consumer price index (CPI) which measures the price of selection of goods purchased by the consumers and GDP deflator which measures the price of all the goods and services in gross domestic product (GDP).

An inflation indexed bond is a financial instrument which is designed to protect the purchasing power of investors' savings by indexing coupon and principal payments to inflation indices. The main difference between an inflation indexed bond and a conventional bond is that, while a conventional bond assures to pay fixed nominal coupon and principal payments, an inflation indexed bond adjusts its coupon and principal payments with respect to the inflation ratio at each time interval over its life. Therefore this instrument pays to its investors real returns. By having this property, an inflation indexed bond saves both investors and issuers from the inflation risk, over the life of the bond.

Nowadays most world markets use inflation indexed instruments including indexed bonds, swaps, options, etc. Although indexation has become increasingly popular during the $1990's$, its roots date back to the $18th$ century. Deacon, Derry and Mirfendereski [15] gives the history of indexation in detail. In 1742, the State of Massachusets issued bills of public credit related to silver prices in London Exchange. As silver prices appreciated more rapidly than general prices in the economy, the State encountered significant economic losses. Then in 1747 the State passed a law that declaring that a group of commonly consumed commodities would be used for indexation. In 1780, with the purpose of preserving the value of wages of the soldiers in the American Revolutions, as wages to soldiers, the State issued notes that were indexed to the prices of five bushels of corn, sixty eight and four-sevenths pounds of beef, ten pounds of sheep wool and sixteen pounds of sole leather. In the first half of the $19th$ century, economists published certain books about indexation of debt. In 1875, W. Stanley Jones proposed to use gold prices for indexation. In 1924, John Maynard Keynes supported the idea of using indexation of debt in his report to the British Government. Despite the early suggestions of economists, the indexation of debt only came into prominence with the high and volatile inflation levels during the Second World War. Some countries including Finland 1945 and France 1952 then issued indexed debt. In the 1950′s and $1960's$, hyperinflation was a big problem for some South American countries such as Brazil, Argentina and Mexico who also issued indexed instruments. In the last three decades several countries began to issue inflation indexed debts. For instance the United Kingdom issued them in 1981, followed by Australia (1985), Canada (1991), Sweden (1994), the United States (1997), France (1998), Greece (2003), Italy (2003) and Turkey (2007).

Recently, the number of studies on these instruments increased rapidly in the finance literature. Unfortunately, in Turkey little work has been done on this subject, leading to the observation that the pricing of these instruments may be problematic in Turkey. In the light of the forementioned, the main purposes of this study are to investigate existing literature and pricing models for inflation indexed securities, to review the Heath, Jarrow and Morton (HJM) [22] framework and the Jarrow-Yıldırım [31] model in detail, to extend the HJM framework and the Jarrow-Yıldırım model with jumps and to price options on an inflation index.

The organization of this thesis is as follows. Second chapter includes studies existing in the finance literature. In chapter 3, the basic definitions and concepts related to stochastic calculus, jump processes, bonds and interest rates are given. In the fourth chapter, the HJM framework is reviewed in detail. This is followed by the review of the Jarrow-Yıldırım model in chapter 5. In chapter 6, extensions of the HJM framework and the Jarrow-Yıldırım model with jumps are introduced. In chapter 7, a closed form formula for the price of an European call option on an inflation index is derived. Concluding remarks are given in chapter 8.

1.1 History of the Inflation Indexed Bonds in Turkey

In this section, the history of inflation indexed bonds in Turkey is given following Tekmen [47], Deacon, Derry and Mirfendereski [15] and the Public Debt Management Reports of the Turkish Treasury [19, 20, 21]. The Turkish Treasury began issuing inflation indexed bonds in July, 1994. First, bonds with one and two year maturities that pay interest semi-annually were issued. Wholesale price index was used for indexation and the issuance was performed by TAP[1]. Table 1.1 summarizes[2] TAP sales for period of $1994-1996$.

In this scope, the percentage of these WPI (Wholesale Price Index)-bonds in overall internal national debt was equal to $2,4$ in 1994, $7,5$ in 1995 and 1 in 1996. At the time, the annual increase in the wholesale price index was approximately $70-80\%$, but nominal interest rates were much above 100%, as a result with those instruments borrowing costs were decreased. However yields of those instruments were lower than nominal bonds. Coupon payments were based on WPI and a constant return. Based on the increase in inflation, the real returns of these instruments would have been decreased. Therefore there wasn't as much demand as the Treasury had hoped for.

Due to the weak demand for those WPI bonds, in March 1997 the Treasury

[1]In Tapping system, the Treasury store long term government bonds with floating interest rates in the Central Bank, investors are able to buy these bonds at any moment. The Central Bank has the ownership of these bonds until the investors buy.

[2]E. Tekmen. Enflasyona endeksli tahviller ve Türkiye uygulaması. *Hazine Müsteşarlığı Kamu Finansmanı Genel Müdürlüğü Uzmanlık Tezi.* 2005, Ankara.

Table 1.1: TAP sales for the period 1994 – 1996

Maturity (year)	*Issuance Date*	*Maturity Date*	*Sales (TL Billion)*
1	18.07.1994	18.07.1995	7820,5
1	02.01.1995	02.01.1996	20517,8
2	02.01.1995	02.01.1997	213,8
1	14.07.1995	14.07.1996	12192,7
2	14.07.1995	14.07.1997	1161
1	25.12.1995	25.12.1996	20500,9
2	25.12.1995	25.12.1997	335,9
1	12.02.1996	12.02.1997	2405,2
2	12.02.1996	12.02.1998	36,3
1	09.07.1996	09.07.1997	14166,6
2	09.07.1996	09.07.1998	301,6
Total			79652,2

decided to use the consumer price index for indexation. Those new bonds were of two-year maturity and paid interests quarterly. Their structure was based on the current pay format[3]. Unlike former bonds, CPI-bonds paid inflation adjusted coupons that fixed the real return. The second difference was that coupon payments of the latter instruments were fixed in the beginning of the coupon period. In this frame accrued interest rates could be calculated in the secondary market for CPI-indexed bonds. However, since inflation assumption was necessary in the pricing of those bonds, no transaction could be done in the secondary market. Total sales in 1997 is given[4] in Table 1.2.

Total sales in Table 1.2 includes both auction sales and sales to the public

[3] In current pay format, the inflation differences of the principal value of inflation index bonds are paid within the coupon payments. Thus, principal value is not adjusted at maturity.

[4] E. Tekmen. Enflasyona endeksli tahviller ve Türkiye uygulaması. *Hazine Müsteşarlığı Kamu Finansmanı Genel Müdürlüğü Uzmanlık Tezi.* 2005, Ankara.

Table 1.2: CPI-indexed bond sales 1997

Issuance Date	*Maturity Date*	*Real return* (%)	*Sales* (*TL Trillion*)
05.03.1997	05.03.1999	25	18,6
02.04.1997	02.04.1999	24	127
09.04.1997	09.04.1999	22	135,7
02.05.1997	02.05.1999	22	10,5
07.05.1997	07.05.1999	25	53,7
14.05.1997	14.05.1999	29,95	63
04.06.1997	04.06.1999	32	269,1
18.06.1997	18.06.1999	32	3,1
26.11.1997	26.11.1999	26	140,2
24.12.1997	24.12.1999	32	146,4
Total			967,2

institutions with a noncompetitive offer. At the end of 1997, total internal debt was equal to $6,3$ quadrillion TL and the percentage of CPI-bonds was equal to $15,5$. The method used in indexation differed from general applications. In this method index ratio which was used for adjusting coupon payments was calculated by taking the fourth root of an annual rate. For example, consider a bond issued in May, 1997. The nominal value of the first coupon was determined by the fourth root of the increase in the CPI between April 1996 and April 1997 rather than its increase between January and April 1997. The reason behind this idea was to decrease the impact of the seasonality of inflation on the cash flows and thus lessen the variability of the nominal cash flows. However using a 15 months lag for the CPI in such calculations constrained the effect of instantaneous inflation changes on the bonds' cash flows.

In 1998, the Treasury declared to change the design of these instruments following the prediction of a sharp decrease in inflation. In this frame, the Treasury started to issue CPI-bonds with 1-year maturity. In this design, all coupon and principal payments were paid at redemption. Thus, the indexation lag would have disappeared. At the end of 1998, the internal national debt was equal to $11,6$ quadrillion TL and the percentage of the CPI-bonds was to $12,3$. Table 1.3 summarizes[5] total CPI-bond sales in 1998 and 1999.

Table 1.3: Total CPI-Bond sales 1998 – 1999

Issuance Date	*Maturity Date*	*Real return* (%)	*Sales* (*TL Trillion*)
21.01.1998	22.01.1999	18,9	119
25.02.1998	24.02.1999	30,95	133,4
25.03.1998	25.03.1999	24	96
22.04.1998	21.04.1999	19	245,3
27.05.1998	26.05.1999	23	138,3
17.06.1998	16.06.1999	25	244,3
29.07.1998	28.07.1999	23	164,3
19.08.1998	18.08.1999	30	184,3
30.09.1998	29.09.1999	30	58,1
09.06.1999	09.08.2000	23,7	2,9
Total			1385,9

In September 1998, the maturities of new indexed bonds were increased from 1 year to 14 months. However, demand for the securities continued to be low and the final auction was held in June, 1999.

Within 1999 – 2007, the Treasury directly issued small quantities of indexed bonds to potential purchasers, but those were seldom traded in the secondary market.

Finally, the Treasury began issuing inflation indexed bonds again in 2007, 8 year after their last introduction. The consumer price index was used for indexation. Those instruments were of 5-year maturity with semi-annual coupon

[5]E. Tekmen. Enflasyona endeksli tahviller ve Türkiye uygulaması. *Hazine Müsteşarlığı Kamu Finansmanı Genel Müdürlüğü Uzmanlık Tezi.* 2005, Ankara.

payments. Bonds were issued with the single price auction method. Total sales of those bonds for the period January 2007-June 2008 were given[6] in Table 1.4.

Table 1.4: CPI-Bond sales in 2007 – 2008.

Issuance Date	*Maturity Date*	*Real return* (%)	*Sales* (*YTL Million*)
20.02.2007	15.02.2012	4,86	4145
29.05.2007	15.02.2012	4,85	1219
21.08.2007	15.02.2012	5,15	698
06.11.2007	15.02.2012	4,52	576
19.02.2008	15.02.2012	4,82	728
06.05.2008	15.02.2012	5,09	397
Total			7763

The percentage of CPI-bonds in the total internal debt was equal to 6 in 2007 and to $3,9$ for the period between January-June 2008. The Treasury is planning to increase this percentage in the following years.

1.2 Inflation Indexed Bond Markets

The global market for index linked bonds has established itself more firmly over the past years. We give a brief information about international indexed government bond markets[7] in Table 1.5.

Also a more detailed information for major international inflation indexed bond markets given[8] in Table 1.6.

[6]Hazine Müsteşarlığı Kamu Borç Yönetimi Raporu. No.30,32,35 Ankara
[7]www.barclayscapital.com
[8]www.barclayscapital.com

Table 1.5: Summary of International Indexed Government Bond Markets.

Country	*Market Cap* (\$ *Bn*)	*Number ofIssues*	*Average Real Yield*	*Average Life(year)*	*Average Duration*
US	499,9	24	1,38	9,58	8,45
UK	319,8	14	0,98	17,37	12,60
France	208,7	11	1,87	9,15	7,73
Brazil	143,1	11	7,10	9,06	4,82
Italy	99,7	6	2,08	10,28	8,42
Japan	74,7	14	1,02	8,42	8,03
Canada	37,3	5	1,97	22,51	15,71
Sweden	37,2	5	1,58	11,32	9,74
Greece	24,4	2	2,36	20,02	15,41
Germany	23,5	2	1,65	7,39	6,85
Argentina	20,3	6	7,93	12,12	7,21
Mexico	20,0	8	3,62	12,75	9,27
South Africa	10,6	4	2,60	12,71	9,05
Australia	8,3	3	2,64	8,39	6,95
Turkey	6,4	1	9,51	4,04	3,14
Colombia	6,4	7	5,77	5,84	4,44
Chile	4,9	10	3,12	5,68	4,84
Poland	3,9	1	2,65	8,55	7,49
South Korea	2,2	1	2,97	9,10	7,90

Table 1.6: Summary of the Major International Indexed Government Bond Markets.

	Japan	*Italy*	*Brazil*	*France*	*UK*	*US*
Market Value (US $ Bn)	74,7	99,7	143,1	208,7	319,8	499,9
Average Real Yield(%)	1,02	2,08	7,10	1,87	0,98	1,38
Coupon Frequency	semi annual	annual	semi annual	annual	semi annual	semi annual
Frequency of price index publication	one month	one month	one month	one month	one month	one month
Indexation lag(Months)	3	3	3	3	8	3
Number of issues	14	6	11	11	14	24

CHAPTER 2

LITERATURE REVIEW

In the finance literature, two types of studies stand out. Studies in the first group investigate inflation indexed securities and their working principles and studies in the second group work on pricing of such instruments.

The first study on inflation indexed securities and their working principles was done by Shen [41]. The benefits and limitations of inflation indexed Treasury bonds are expressed in detail. The paper concludes by saying that in spite of some limitations, inflation indexed Treasury bonds are very valuable innovations for financial markets.

Wrase [49] described the structure of inflation indexed bonds by explaining the advantages and disadvantages of such instruments. The reasons for issuance of these bonds by the Treasury, their importance for investors and effects of indexed bonds on the monetary policies are also explained in this work.

Wilcox [48] investigated the question of whether issuing inflation indexed debt is a good idea or not. He argues that for the following reasons it is a good idea. Firstly, it protects investors from inflation risk. Governments real expenditures become stable. Finally by these instruments, the government gets useful information on the future inflation. The author also adds that there exist some limitations of the inflation indexed debt but they are minor.

Another study in the first group is the work of Kapcke and Kimball [33]. This article analyzes inflation indexed bonds in general and Treasury inflation protected securities (TIPS) in particular to understand clearly their limited appeal to American investors. They replicate potential risk and return characteristics of TIPS by using market data and they conclude that TIPS will appeal to savers who are especially risk averse and who are especially cautious of inflation.

In 2000, Taylor [46] explored the role of US inflation indexed bonds in the portfolios of expected utility maximizing investors. The findings of his work raise questions about the usefulness of US inflation indexed bonds as portfolio diversifiers. By using three different assumptions relating to the time series behavior of real yields, this paper argues that including inflation indexed bonds in an optimal portfolio does not considerably improve investor utility over and above that obtained when these bonds are excluded. Also it is observed that alternative securities with similar risk-return characteristics can easily be substituted in place of inflation indexed bonds.

Shen and Corning [42] investigated the question of whether TIPS help identifying long-term inflation expectations or not. Having an accurate measure of market inflation expectations may help policymakers assign their efficiency in controlling long term inflation, as well as their credibility among market participants. The yield difference between conventional bonds and TIPS is used as a measure of inflation expectations. It is found that yield difference is not a satisfactory measure of market inflation expectations because of the large and floating liquidity premium on TIPS.

Roll [38] analyzed the correlations of TIPS returns with the conventional bond returns and with equity returns over the period $1997 - 2003$, real and nominal effective durations and changes in volatility over time. It is observed that TIPS nominal return volatility is less than the conventional bonds, nominal effective durations are much lower for TIPS than for nominal bonds, TIPS have a small correlation with the nominal bonds with negative sign.

The most detailed work on inflation indexed securities belongs to Deacon, Derry and Mirfendereski [15]. They discuss various factors that go into the design of index based instruments such as the choice of index, the cash flow structure of the bond, the application of the index to the cash flows and the impact of tax regulations. Indexed based bonds issued by various countries are also compared in detail.

Sack and Elsasser [40] discussed the US experience with inflation indexed debt, including the development of movement in the TIPS market since its inception and valuation of these securities relative to nominal Treasury issues. They observe that in spite of the potential demand for TIPS, their yields have been surprisingly high relative to those on comparable nominal Treasury securities. Also the paper indicates that originally costs of indexed securities are higher and the liquidity of the market for these instruments is lower than the nominal ones but in time all the conditions of indexed securities improve and at the time of the writing of the article are being preferred to conventional bonds.

Kitamura [32] evaluated indexed bonds by considering the market trading records of TIPS between $1997 - 2003$ to give information to the Japanese Government. His findings show that, real interest rates are stable, expected inflation rate is more closely related to the observed CPI than to the real yield, information content of the expected inflation rates from the indexed bonds is limited and the issue conditions for the TIPS are not adequate.

Chamon and Mauro [9] described the advantages of inflation indexed bonds for financial markets, especially for emerging markets.

Hurd and Rellen [29] examined the development of inflation indexed swaps and index-linked bonds in England. They observe that by using market data of such instruments a greater range of international inflation and real interest rate forward curves are estimated. Inflation forward rate curves may be useful to raise the ability of monetary authorities to control inflation.

Garcia and Rixtel [17] gave the main reasons for and against the issuance of inflation indexed bonds and the key dynamics that effect their current development. The independence and credibility of central banks and the environment of low and stable inflation that they set up may be the most important factors for the development of inflation indexed bond markets in recent years.

One of the early studies on pricing inflation linked derivatives is the work of Hughston [27] who outlines a general theory for the pricing and hedging of inflation linked derivatives. Assumptions in his model are the completeness of the markets with no arbitrage opportunity. Methodologies for valuing foreign currency and interest rate derivatives are used in this study. The consumer price index is considered as an exchange rate between real and nominal prices. Bond price processes are also examined by using the HJM [22] model. Several inflation linked derivative pricing formulas are given in closed form. It is shown that index linked derivatives can be treated in the same way as foreign exchange derivatives.

Jarrow and Yıldırım [31] introduced a three factor HJM model in order to price Treasury inflation protected securities and options on inflation index. Foreign currency analogy, where nominal prices correspond to the domestic currency real prices correspond to the foreign currency and the inflation index corresponds to the spot exchange rate, is used. Their key assumptions are deterministic volatilities and the non-zero correlation between different factors. Bond prices are assumed to be Gaussian and forward volatility corresponds to the extended Vasicek model. The validity of their model is tested by hedging analysis and the usefulness of the model is demonstrated by pricing a European call option on the CPI-U inflation index.

Belgrade et al. [3] introduced a new market model based on inflation indexed swaps. Their model has a few parameters and is robust enough to replicate market prices. The model is only driven by the term structure of parameters, describing CPI's forwards. By this property consistent relations between zero-coupon and year-on-year swaps volatilities are obtained. They also give certain boundaries for implicit correlations between these instruments.

Mercurio [35] works on pricing inflation indexed swaps, caplets and floorlets. After reviewing the Jarrow-Yıldırım model, zero-coupon and year-on-year inflation indexed swaps are priced by this model. Two new market model approaches are introduced then. Pricing formulas for both instruments are derived with three models. Performance of models are tested in terms of calibration to market data.

Mercurio and Moreni [36] proposed a market model for pricing inflation indexed caps and floors. The key assumptions of their model are as follows: forward CPI's follow a driftless geometric Brownian motion under the corresponding forward measure and forward CPI's volatilities are stochastic that evolve according to a square-root process as in Heston [24]. In the case of zero correlations between forward rates and forward CPI's exact closed form formulas for cap and floor prices are derived. Classical drift freezing techniques are used in the non-zero correlation case.

Henrard [23] derives an explicit pricing formula for inflation bond options in the Jarrow-Yıldırım model. He defines an extra condition on the real rate volatility to get an explicit formula for bond options.

Hinnerich [25] suggested an extension of the Jarrow-Yıldırım model to price the inflation indexed swaps, swaptions and bond options. The main differences from the previous works are that, here there is no assumption that the foreign currency analogy holds and forward rate, inflation and bond price dynamics are driven by both multidimensional Wiener process and a general marked point process. Another assumption in this work is that, the intensity at the point as well as the volatilities of all asset prices and the consumer price index, with respect to both the Wiener process and the point process, are deterministic. Eventually it is proved that the foreign currency analogy is valid.

Dodgson and Kainth [16] proposed a two-process short-rate model for pricing inflation linked derivatives. The inflation rate and the short interest rate are assumed to be diffusion processes with mean reversion property. A closed form solution for inflation options is derived with constant volatility assumption. However this model may not capture the volatility smile seen in market prices for inflation options. Therefore a generalized inflation short-rate model with local volatility is defined and complex derivatives are priced with a Monte-Carlo sampling.

Hughston and Macrina [28] introduce a class of discrete time stochastic models for the pricing of inflation indexed derivatives. The main idea is that at any time, there exists imperfect information about the future values of macroeconomic factors in the market. Such partial information effects the consumption, money supply and other variables that determine interest rates and price levels. A model under this partial information is proposed in order to derive arbitrage free dynamics of real and nominal interest rates, price indices and index linked securities.

The final study on pricing inflation indexed derivatives is the work of Stewart [45]. The aim of his work is to review the framework for pricing inflation-indexed derivatives by using the two currency HJM approach introduced by Jarrow and Yıldırım and to obtain prices for the most liquid inflation indexed derivatives using the Hull and White model. He uses Mercurio and Moreni's methodology for pricing inflation indexed swaps, caps, floors and swaptions. Like previous works, performance of the model is tested by calibration to the market data. The results are consistent with the previous ones.

When we look at the Turkish financial literature only two studies stand out. A former study belongs to Balaban [2]. Balaban investigates the relevant factors that need to be considered when issuing inflation indexed securities in Turkey. These factors are system selection, exchange rate effects, inflation index selection, cash flow structure, maturity, bidding method, liquidity and amount. All the factors are examined in detail in this work.

A latter study is the work of Tekmen [47]. In this study, the benefits of inflation indexed bonds to the whole economy, design of these securities, history of inflation indexed debt in Turkey and in some developed and emerging markets are given in detail. Also a regression analysis is conducted. Strong correlation between standard deviation of expected consumer price index values and real interest rates is observed.

CHAPTER 3

PRELIMINARIES

In this chapter the main terminologies and definitions that will be used throughout this study are presented.

3.1 Basics of Stochastic Calculus

In this section a summary of stochastic calculus is given following Lamberton and Lapeyre [34], Björk [4], Shreve [44], Brigo and Mercurio [7] and Yolcu [51].

Definition 3.1.1. *Consider a complete probability space* (Ω, $\mathcal{A}$, $\mathbf{P}$)*. Let* T *be a fixed positive number and* $t \in [0,T]$*. Filtration is an increasing family* $\mathcal{F} = \{\mathcal{F}_t, t \in [0,T]\}$*. For each* $s, t \geq 0$ *, if* $s \leq t$ *then* $\mathcal{F}_s \subseteq \mathcal{F}_t$*.*

Definition 3.1.2. *Let* (Ω, $\mathcal{F}$, $\mathbf{P}$) *be a probability space. A Brownian motion is a real valued continuous stochastic process* $(X_t)_{t \geq 0}$ *with stationary and independent increments.*

- *Continuity:* $\mathbf{P}$- *a.s. the map* $s \mapsto X_s(w)$ *is continuous.*
- *Stationary increments: If* $s \leq t$ *then* $X_t - X_s$ *and* $X_{t-s} - X_0$ *have the same probability law.*
- *Independent increments: If* $s \leq t$*, then* $X_t - X_s$ *is independent of* $\mathcal{F}_s = \sigma(X_u, u \leq s)$*.*

Definition 3.1.3. *A Brownian motion is called standard if* $X_0 = 0$ $\mathbf{P}$- *a.s.,* $E(X_t) = 0$ *and* $Var(X_t) = t$*.*

Definition 3.1.4. *A stochastic process on* $(\Omega, \mathcal{A}, \mathcal{F}, \mathbf{P})$ *is said to be adapted to filtration* $\mathcal{F}$ *if* $\forall t$ X_t *is* $\mathcal{F}_t$ *- measurable.*

Definition 3.1.5. *Consider a probability space* $(\Omega, \mathcal{A}, \mathbf{P})$ *and a filtration* $(\mathcal{F}_t)_{t\geq 0}$ *on this space. An adapted family* $(M_t)_{t\geq 0}$ *of integrable random variables, i.e.,* $E(|M_t|) < +\infty$ *for any* t *is :*

- *a* **martingale** *if, for any* $s \leq t$, $E(M_t \mid \mathcal{F}_s) = M_s$,
- *a* **supermartingale** *if, for any* $s \leq t$, $E(M_t \mid \mathcal{F}_s) \leq M_s$,
- *a* **submartingale** *if, for any* $s \leq t$, $E(M_t \mid \mathcal{F}_s) \geq M_s$.

Theorem 3.1.1. *Let* $(\Omega, \mathcal{F}, \mathbf{P})$ *be a probability space. Let* $\mathbf{Q}$ *be another probability measure on* $(\Omega, \mathcal{F})$ *that is equivalent to* $\mathbf{P}$ *and let* $\mathcal{Z}$ *be almost surely positive random variable that relates* $\mathbf{P}$ *and* $\mathbf{Q}$*. Then* $\mathcal{Z}$ *is called the Radon-Nikodym derivative of* $\mathbf{Q}$*.* $\mathcal{Z}$ *is given as* $\frac{d\mathbf{Q}}{d\mathbf{P}}$*, i.e.* $\forall\ A \in \mathcal{A}$*;*

$$\mathbf{Q}(A) = \int_A \mathcal{Z}(\omega)\mathbf{dP}(\omega).$$

Theorem 3.1.2. (Girsanov Theorem) *Let* $(\Omega\ , \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbf{P})$ *be a probability space and let* $(W_t)_{0\leq t\leq T}$ *be an* $\mathcal{F}$*-Brownian motion.*
Let $(\theta_t)_{0\leq t\leq T}$ *be an adapted measurable process satisfying* $\int_0^t \theta_s^2\, ds < \infty$ *a.s. and such that the process* $(Z_t)_{0\leq t\leq T}$ *defined by*

$$Z_t \;=\; \exp\,(\int_0^t \theta_s\, dB_s - \frac{1}{2}\int_0^t \theta_s^2\, ds)$$

is a martingale. Then under the probability $\mathbf{Q}$ *with density* $Z(T)$ *relative to* $\mathbf{P}$*, the process* $(W^{\mathbf{Q}}(t))_{0\leq t\leq T}$ *defined by* $(W^{\mathbf{Q}}(t)) = (W_t) + \int_0^t \theta_s\, ds$ *, is a* $\mathcal{F}$*-Brownian motion under* $\mathbf{Q}$*.*

Definition 3.1.6. *Let* $(\Omega\ , \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ *be a probability space and let* $(W_t)_{t\geq 0}$ *be an* $\mathcal{F}$*-Brownian motion.* $(X_t)_{0\leq t\leq T}$ *is an* $\mathbf{R}$*-valued Ito process if it can be written as* $\mathbf{P}$ *a.s.* $\forall\ t \leq T$

$$X_t \;=\; X_0 + \int_0^t K_s\, ds + \int_0^t H_s\, dW_s, \tag{3.1.1}$$

where

- *X_0 is $\mathcal{F}_0$- measurable.*

- *$(K_t)_{0\leq t\leq T}$ and $(H_t)_{0\leq t\leq T}$ are $\mathcal{F}_t$ adapted processes.*

- *$\int_0^t |K_s|\, ds < \infty$ and $\int_0^t H_s^2\, ds < \infty$ $\mathbf{P}$ a.s.*

Theorem 3.1.3. *Let $(X_t)_{0\leq t\leq T}$ be an Ito process satisfying (3.1.1) and f be twice continuously differentiable function, then*

$$f(X_t) \;=\; f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle X, X\rangle_s \tag{3.1.2}$$

where $\langle X, X\rangle_t = \int_0^t H_s^2\, ds$.

Lemma 3.1.4. *The quadratic variation of the Ito process is $\langle X, X\rangle_t = \int_0^t H_s^2\, ds$.*

Theorem 3.1.5. (ITO-Deblin Formula) *Let $(X_t)_{t\geq 0}$ be an Ito process and f(t,x) be a function with well defined continuous partial derivatives, $f_t(t,x)$, $f_x(t,x)$, $f_{xx}(t,x)$. Then for every $T \geq 0$*

$$\begin{aligned} f(T, X_T) \;&=\; f(0, X_0) + \int_0^T f_t(t, X_t)dt + \int_0^T f_x(t, X_t)\, dX_t \\ &\quad + \frac{1}{2}\int_0^T f_{xx}(t, X_t)d\langle X, X\rangle_t. \end{aligned} \tag{3.1.3}$$

Proposition 3.1.6. (ITO-Integration by Parts Formula) *Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be two Ito processes such that*

$$X_t = X_0 + \int_0^t K_s\, ds + \int_0^t H_s\, dW_s$$

and

$$Y_t = Y_0 + \int_0^t K'_s\, ds + \int_0^t H'_s\, dW_s$$

then

$$X_tY_t \;=\; X_0Y_0 + \int_0^t X_s\, dY_s + \int_0^t Y_s\, dX_s + \langle X, Y\rangle_t \tag{3.1.4}$$

with $\langle X, Y\rangle_t = \int_0^t H_sH'_s\, ds$.

Definition 3.1.7. $(X_t)_{0\le t\le T}$ *is an Ito process if*

$$X_t \;=\; X_0 + \int_0^t K_s\,ds + \sum_{i=1}^{p}\int_0^t H_s^i\,dW_s^i, \tag{3.1.5}$$

where

- K_t *and all the processes* H_t^i *are adapted to* $\mathcal{F}_t$,
- $\int_0^t |K_s|\,ds < \infty$,
- $\int_0^t H_s^{i\;2}\,ds < \infty$.

Proposition 3.1.7. *Let* $(X_t^1, X_t^2, .., X_t^n)$ *be n Ito processes satisfying*

$$X_t^i \;=\; X_0^i + \int_0^t K_s^i\,ds + \sum_{j=1}^{p}\int_0^t H_s^{i,j}\,dW_s^i$$

then if f is twice differentiable with respect to x and once differentiable with respect to t with continuous partial derivatives in (t,x)

$$\begin{aligned} f(t, X_t^1, .., X_t^n) \;&=\; f(0, X_0^1, .., X_0^n) + \int_0^t \frac{\partial f}{\partial s}(s, X_s^1, .., X_s^n)ds \\ &+ \sum_{i=1}^{n}\int_0^t \frac{\partial f}{\partial X_i}(s, X_s^1, .., X_s^n)dX_i \\ &+ \frac{1}{2}\sum_{i,j=1}^{n}\int_0^t \frac{\partial^2 f}{\partial X_i \partial X_j}(s, X_s^1, .., X_s^n)d\langle X^i, X^j\rangle_s, \end{aligned} \tag{3.1.6}$$

where

- $dX_s^i \;=\; K_s^i ds + \sum_{j=1}^{p} H_s^{i,j} dW_s^j$,
- $d\langle X^i, X^j\rangle_s \;=\; \sum_{m=1}^{p} H_s^{i,m} H_s^{j,m}\,ds.$

3.2 Basics of Jump Processes

This section introduces the basics of jump processes following Shreve [44] and Cont and Tankov [11]. The Poisson process is the fundamental example of a stochastic process with discontinuous trajectories. It is the basic building block for jump processes.

3.2.1 Construction of a Poisson Process

Definition 3.2.1. *A positive random variable τ is called an exponential random variable if its probability density function is of the form:*

$$f(t) = \begin{cases} \lambda \exp(-\lambda t), & t \geq 0 \\ 0, & t < 0, \end{cases}$$

where λ is a positive constant parameter. τ has the following properties:

- *Mean of τ is equal to $\frac{1}{\lambda}$,*
- *Variance of τ is equal to $\frac{1}{\lambda^2}$,*
- *Cumulative distribution of τ is given by $\forall t \in [0, \infty]$*
  $\mathcal{F}\tau(t) = P(\tau \leq t) = 1 - \exp(-\lambda t)$,
- *τ has the memoryless property i.e. $\forall$ t,s > 0*
  $P(\tau > t + s \,|\, \tau > s) = P(\tau > s)$.

Definition 3.2.2. *An integer valued random variable N is called a Poisson random variable with parameter λ if*

$$P(N = n) = \frac{\exp(-\lambda)\lambda^n}{n!},$$

where λ is a positive constant parameter. N has the following properties:

- *Mean of N is equal to λ,*

- *Variance of N is equal to λ,*

- *Moment generating function of N is given by $M = \exp(\lambda(e^u - 1))$.*

Consider a sequence τ_1, τ_1, .. of exponential random variables. τ_i*'s* $\forall$ i=1.. have the same mean $\frac{1}{\lambda}$. Let the first jump occurs at τ_1, the second occurs τ_2 time units after the first, the third occurs τ_3 time units after the second etc. Then the time of the nth jump can be defined as

$$S_n = \sum_{k=1}^{n} \tau_k.$$

Lemma 3.2.1. *For $n \geq 1$, the random variable S_n has the gamma probability density function*

$$g_n(s) = \frac{(\lambda s)^{n-1}}{(n-1)!} \lambda e^{-\lambda s}, s \geq 0$$

Definition 3.2.3. *Let $(\tau_i)_{i \geq 0}$ be a sequence of independent random variables with parameter $\frac{1}{\lambda}$ and $S_n = \sum_{k=1}^{n} \tau_k$ be the time of the nth jump. The process $(N_t)_{t \geq 0}$ defined by*

$$N_t = \sum_{n \geq 1} 1_{t \geq S_n}$$

is called a Poisson process with intensity λ.

Proposition 3.2.2. *Let $(N_t)_{t \geq 0}$ be a Poisson process. Then*

1. *For any $t > 0$, (N_t) is almost surely finite.*

2. *For any ω, the sample path , $t \mapsto N_t(\omega)$ is piecewise constant and increasing.*

3. *The sample paths $t \mapsto N_t$ are right continuous with left limits (cadlag process).*

4. *For any $t > 0$, $N_{t_-} = N_t$ with probability 1.*

5. *N_t is continuous in probability, i.e. $\forall t > 0$ as $s \to t$, $N_s \to N_t$ in probability.*

6. *For any $t > 0$, N_t follows a Poisson distribution with parameter λt, i.e. $\forall k \in \mathbf{N}$*

$$P(N_t = k) = \frac{\exp(-\lambda t)(\lambda t)^k}{k!}.$$

7. *The characteristic function of N_t is given by*

$$E(e^{iuN_t}) = \exp(\lambda t(e^{iu} - 1)), \ \forall u \in \mathbf{R}.$$

8. *N_t has independent increments, i.e. for any $t_1 < t_2 < .. < t_n$; $N_{t_n} - N_{t_{n-1}}$, $N_{t_{n-1}} - N_{t_{n-2}}$,.., $N_{t_2} - N_{t_1}$, N_{t_1} are independent random variables.*

9. *The increments of N_t are stationary, i.e. for any $t > s$, $N_t - N_s$ has the same distribution of N_{t-s}.*

10. *N_t has the Markov property, i.e. $\forall t > s$*

$$E(f(N_t) \mid N_u, u \leq s) = E(f(N_t) \mid N_s).$$

Theorem 3.2.3. *Let N_t be a Poisson process with intensity $\lambda > 0$ and let $0 = t_0 < t_1 < t_2 < .. < t_n$ be given. Then the increments $N_{t_n} - N_{t_{n-1}}$, $N_{t_{n-1}} - N_{t_{n-2}}$,.., $N_{t_2} - N_{t_1}$,$N_{t_1} - N_{t_0}$ have the distribution*

$$P(N_{t_{j+1}} - N_{t_j} = k) = \frac{\exp(-\lambda(t_{j+1} - t_j))(\lambda)^k(t_{j+1} - t_j)^k}{k!},$$

where $k = 0, 1, 2..$ Then the mean and the variance of the increments are:

- *$E(N_t - N_s) = \lambda(t - s)$,*
- *$Var(N_t - N_s) = \lambda(t - s)$.*

Theorem 3.2.4. *Let N_t be a Poisson process with intensity $\lambda > 0$. Then the compensated Poisson process*

$$M_t = N_t - \lambda t$$

is a martingale.

3.2.2 Compound Poisson Processes

In this subsection, a new process that allows the jump size to be random will be introduced.

Definition 3.2.4. *A compound Poisson process with intensity $\lambda > 0$ and jump size distribution f, is a stochastic Process Q_t defined as*

$$Q_t = \sum_{i=1}^{N_t} Y_i,$$

where jump sizes are independent identically distributed random variables with distribution f and N_t is a Poisson process with intensity λ, independent from $(Y_i)_{i\geq 1}$. The jumps in Q_t occur at the same time as the jumps in N_t.

Proposition 3.2.5. *Let $(Q_t)_{t\geq 0}$ be a compound Poisson process. Then*

1. *The sample paths of Q_t are cadlag piecewise constant functions.*
2. *Q_t has independent increments, i.e. for any $t_1 < t_2 < .. < t_n$; $Q_{t_n} - Q_{t_{n-1}}$, $Q_{t_{n-1}} - Q_{t_{n-2}}$,.., $Q_{t_2} - Q_{t_1}$, Q_{t_1} are independent random variables.*
3. *The increments of Q_t are stationary, i.e. for any $t > s$, $Q_t - Q_s$ has the same distribution of Q_{t-s}.*
4. *$E(Q_t) = \beta\lambda t$ where $\beta = E(Y_i)$.*
5. *The moment generating function of Q_t is given by*

   $$\varphi_{Q_t}(u) = E(e^{uQ_t}) = \exp(\lambda t(\varphi_Y(u) - 1)),$$

   where $\varphi_Y(u) = E(e^{uY_i})$.

Theorem 3.2.6. *Let Q_t be a Poisson process with intensity $\lambda > 0$. Then the compensated Poisson process*

$$M_t = Q_t - \beta\lambda t$$

is a martingale.

3.2.3 Jump Processes and Their Integrals

Let $X(t)$ be a processes of the form:

$$X(t) = X(0) + I(t) + R(t) + J(t), \tag{3.2.7}$$

where

- $X(0)$ is a non-random initial condition,
- $I(t) = \int_0^t \Gamma(s)dW(s)$ is an Ito integral of an adapted process $\Gamma(s)$ with respect to a Brownian motion relative to the filtration,
- $R(t) = \int_0^t \Theta(s)ds$ is a Riemann integral for some adapted process Θ(s),
- $J(t)$ is an adapted right continuous pure jump process with $J(0) = 0$

The continuous part of the $X(t)$ is defined to be

$$X(t) = X(0) + I(t) + R(t) = X(0) + \int_0^t \Gamma(s)dW(s) + \int_0^t \Theta(s)ds,$$

where the quadratic variation of this process is

$$\langle X^c, X^c \rangle(t) = \int_0^t \Gamma^2(s)ds$$

or in differential form

$$dX^c(t)\, dX^c(t) = \Gamma^2(t)dt.$$

Another assumption is that $J(t)$ does not have a jump at time zero, has finitely many jumps on each finite time interval, [0,T] and is constant between jumps.

Definition 3.2.5. *A process of the form (3.2.7) with Ito integral part $I(t)$, Riemann integral part $R(t)$ and a pure jump part $J(t)$ as described above is called a jump process.*

A jump process $X(t)$ is right continuous adapted. Jump size of $X(t)$ at time T is denoted by

$$\Delta X(t) = X(t) - X(t_-)$$

Definition 3.2.6. *Let $X(t)$ be a jump process and let $\phi(t)$ be an adapted process. The stochastic integral of $\phi(t)$ with respect to X is defined as*

$$\int_0^t \phi(s)dX(s) = \int_0^t \phi(s)\Gamma(s)dW(s) + \int_0^t \phi(s)\Theta(s)ds + \sum_{0<s\leq t} \phi(s)\Delta J(s) \tag{3.2.8}$$

or in the differential form

$$\phi(t)dX(t) = \phi(t)\Gamma(t)dW(t) + \phi(t)\Theta(t)dt + \phi(t)dJ(t). \tag{3.2.9}$$

Theorem 3.2.7. *Let*

$$X_1(t) = X_1(0) + I_1(t) + R_1(t) + J_1(t)$$

be a jump process where

- $I_1(t) = \int_0^t \Gamma_1(s)dW(s)$,
- $R_1(t) = \int_0^t \Theta_1(s)ds$,
- $J_1(t)$ *is a right continuous pure jump process,*

Then,

$$\langle X_1, X_1\rangle(T) = \langle X_1^c, X_1^c\rangle(T) + \langle J_1, J_1\rangle(T) = \int_0^T \Gamma_1^2(s)ds + \sum_{0<s\leq T} (\Delta J_1(s))^2.$$

Let

$$X_2(t) = X_2(0) + I_2(t) + R_2(t) + J_2(t)$$

be another jump process, then

$$\langle X_1, X_2\rangle(T) = \langle X_1^c, X_2^c\rangle(T) + \langle J_1, J_2\rangle(T) = \int_0^T \Gamma_1(s)\Gamma_2(s)ds + \sum_{0<s\leq T} \Delta J_1(s)\Delta J_2(s).$$

Corollary 3.2.8. *Let W(t) be a Brownian motion and M(t) = N(t)- λt be a compensated Poisson process relative to the same filtration $\mathcal{F}(t)$, then*

$$\langle W, M\rangle(t) = 0, \ \ t \geq 0$$

Theorem 3.2.9. (ITO-Deblin Formula for One Jump Process) *Let $X(t)$ be a jump process and f(x) be a function with well defined continuous first and second derivatives, $f'(x)$ and $f''(x)$. Then*

$$
\begin{aligned}
f(X(t)) &= f(X_0) + \int_0^t f'(X(s))dX^c(s) + \frac{1}{2}\int_0^t f''(X(s))dX^c(s)dX^c(s) \\
&\quad + \sum_{0<s\leq T} [f(X(s)) - f(X(s_-))]. \qquad (3.2.10)
\end{aligned}
$$

Theorem 3.2.10. (Two-Dimensional ITO-Deblin Formula for Jump Processes) *Let $X_1(t)$ and $X_2(t)$ be jump processes and let $f(t, x_1, x_2)$ be a function whose first and second partial derivatives appearing in the following formula are defined and continuous. Then*

$$
\begin{aligned}
f(t, X_1(t), X_2(t)) &= f(0, X_1(0), X_2(0)) + \int_0^t f_t(s, X_1(s), X_2(s))ds \\
&\quad + \int_0^t f_{x_1}(s, X_1(s), X_2(s))dX_1^c(s) \\
&\quad + \int_0^t f_{x_2}(s, X_1(s), X_2(s))dX_2^c(s) \\
&\quad + \frac{1}{2}\int_0^t f_{x_1x_1}(s, X_1(s), X_2(s))dX_1^c(s)dX_1^c(s) \\
&\quad + \int_0^t f_{x_1x_2}(s, X_1(s), X_2(s))dX_1^c(s)dX_2^c(s) \\
&\quad + \frac{1}{2}\int_0^t f_{x_2x_2}(s, X_1(s), X_2(s))dX_2^c(s)dX_2^c(s) \\
&\quad + \sum_{0<s\leq T} [f(s, X_1(s), X_2(s)) - f(s, X_1(s_-), X_2(s_-))].
\end{aligned}
$$

(3.2.11)

Corollary 3.2.11. (ITO-Product Rule for Jump Processes) *Let $X_1(t)$ and $X_2(t)$ be jump processes. Then*

$$\begin{aligned} X_1(t)\,X_2(t) &= X_1(0)X_2(0) + \int_0^t X_2(s)\,dX_1^c(s) + \int_0^t X_1(s)\,dX_2^c(s) \\ &+\langle X_1^c, X_2^c\rangle_t + \sum_{0<s\leq T}[X_1(s)X_2(s) - X_1(s_-)X_2(s_-)]. \end{aligned} \tag{3.2.12}$$

3.2.4 Change of Measure for Jump Processes

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and W(t) is a Brownian motion defined on it. Suppose the compound Poisson process

$$Q_t = \sum_{i=1}^{N_t} Y_i$$

with intensity λ and density function of jumps f(y), is also defined on this space. There exists one filtration for both the Brownian motion and the compound Poisson process.

Let $\tilde{\lambda}$ be a positive number, let $\tilde{f}(y)$ be another density function with the property that $\tilde{f}(y) = 0$ whenever $f(y) = 0$ and $\Theta(t)$ be an adapted process. Then we define

$$Z_1(t) = \exp\,(\int_0^t \theta_s\,dB_s - \frac{1}{2}\int_0^t \theta_s^2\,ds), \tag{3.2.13}$$

$$Z_2(t) = \exp\,((\lambda - \tilde{\lambda})t)\prod_{i=1}^{N(t)} \frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}, \tag{3.2.14}$$

$$Z(t) = Z_1(t)\,Z_2(t). \tag{3.2.15}$$

Lemma 3.2.12. *The process $Z(t)$ of (3.2.15) is a martingale. In particular $E(Z(t)) = 1\ \forall t \geq 0$.*

Theorem 3.2.13. *Under the probability measure $\tilde{P}$ the process*

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(s)ds$$

is a Brownian motion , $Q(t)$ is a compound Poisson process with intensity $\tilde{\lambda}$ and independent, identically distributed jump sizes having density $\tilde{f}$, and the processes $\tilde{W}$ and $Q(t)$ are independent.

3.3 Basics of Bonds and Interest Rates

In this section we introduce basic definitions of bonds and interest rates satisfying intuition and motivation for their introduction following Altay [1], Björk [4],Brigo and Mercurio [7] and Yolcu [51].

Definition 3.3.1. *A zero-coupon bond with maturity T, is a contract which guarantees its holder the payment of one unit of currency at time T with no intermediate payments. The price of such a bond at time $t < 0$ is defined by $P(t,T)$. $P(t,T)$ is equal to one for all maturities.*

Definition 3.3.2. *A coupon bearing bond with maturity T, is a contract that ensures intermediate coupon payments at times t_i , $i = 1, .., n$ such that $0 < t_i < T$. The last cash flow includes the principal value of the bond in addition to the last coupon payment.*

Next we consider the definition of a bank account which provides a locally riskless investment in which profit accrued continuously at the market risk free rate at any moment.

Definition 3.3.3. (Bank Account) *The Bank account process is defined by*

$$B_t = \exp\left(\int_0^t r_s\, ds\right)$$

where $r(t)$ is the instantaneous short rate. $B(t)$ evolves according to the following differential equation:

$$\begin{cases} dB_t = r_t B_t \, dt \\ B_0 = 1. \end{cases}$$

Definition 3.3.4. (Discount Factor) *The discount factor $D(t,T)$ between to time instants at time t and T is the amount at time T equivalent to one unit of currency payable at time T and is given by*

$$D(t,T) = \frac{B(t)}{B(T)} = \exp(-\int_t^T r(s)ds).$$

In the following we give commonly used interest rates that have an important effect on the pricing of interest rate derivatives.

Definition 3.3.5. *The simply compounded forward rate contracted at time t for the period $[S,T]$ is denoted by*

$$L(t;S,T) = -\frac{P(t,T) - P(t,S)}{(T-S)P(t,T)}.$$

Definition 3.3.6. *The continuously compounded forward rate contracted at time t for the period $[S,T]$ is denoted by*

$$R(t;S,T) = -\frac{\log P(t,T) - \log P(t,S)}{T-S}.$$

Definition 3.3.7. *The instantaneous forward rate contracted at time t for the maturity $T > t$ is denoted by*

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}.$$

Definition 3.3.8. *The instantaneous short rate at time t is denoted by*

$$r(t) = f(t,t).$$

CHAPTER 4

HEATH-JARROW-MORTON FRAMEWORK

Short rate models are useful to clearly understand the interest rate world. These models use instantaneous short rate as the state variable and have particular advantages. The main advantages of such models are:

- Specifying $r(t)$ as the solution of the stochastic differential equation allows us to work within a partial differential equation framework.

- It is always possible to obtain tractable formulas for bond and derivative prices.

However, implementation of short rate models to the real world is a polemical subject because of the following reasons:

- An exact calibration to the initial curve of discount factors and a clear understanding the volatility structures of the forward rates are both difficult to achieve.

- The entire market is governed by one or few explanatory variables assumption is unreasonable.

- When the short rate model becomes more realistic, matching the current yield curve becomes more difficult.

- Without defining a very complicated short rate model, a realistic volatility structure can not be obtained easily, (see Björk [4]).

These facts motivated various authors to develop alternative models. In this section we focus on the Heath-Jarrow-Morton [22] framework . Heath, Jarrow and Morton proposed a continuous time general framework for modelling the entire yield curve. The key step of their approach is choosing the instantaneous forward rates as fundamental quantities to derive an arbitrage-free term structure where the forward rate dynamics are determined through their instantaneous volatility structures.

Now let us give the following example in order to better understand the Heath-Jarrow-Morton framework (see Brigo and Mercurio [7]). Let us take the following equation for the short rate under the risk neutral measure

$$dr(t) = \alpha dt + \sigma dW_t. \tag{4.0.1}$$

This is a very simple case of the Ho - Lee [26] model with constant coefficient α. For this model, the price of the zero-coupon bond can easily be computed as:

$$P(t,T) = \exp[\frac{\sigma^2}{6}(T-t)^3 - \frac{\alpha}{2}(T-t)^2 - (T-t)r(t)]. \tag{4.0.2}$$

By using the definition of instantaneous forward rate

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T} = -\frac{\sigma^2}{2}(T-t)^2 + \alpha(T-t) + r(t). \tag{4.0.3}$$

Differentiating this and substituting the short rate dynamics, the following dynamics is obtained

$$\begin{aligned} df(t,T) &= (\sigma^2(T-t) - \alpha)dt + \alpha dt + \sigma dW_t \\ &= \sigma^2(T-t)dt + \sigma dW_t. \end{aligned} \tag{4.0.4}$$

From the last equation it is seen that, the drift term is determined by a certain transformation of the volatility term σ. That is, if one wishes to model an instantaneous forward rate, the drift of its process is completely determined by the chosen volatility term. This is not a coincidence, instead it is a general fact proved by the Heath, Jarrow and Morton. An analysis of the HJM-framework both under an objective and a risk-neutral measure is given in detail.

4.1 Forward Rate and Bond Price Dynamics

Consider the $f(0,T)$, $0 \leq T \leq T'$ where T' is the longest maturity in the market. $f(0,T)$ is called the initial forward curve. For a fixed maturity T, the instantaneous forward rate $f(t,T)$ evolves under the objective probability measure $\mathbf{P}$, as follows

$$\begin{aligned} f(t,T) &= f(0,T) + \int_0^t \alpha(s,T)\,ds + \int_0^t \sigma(s,T)\,dW(s), \\ f(0,T) &= f^M(0,T), \end{aligned} \tag{4.1.5}$$

where $f^M(0,T)$, $T \geq 0$, the observed forward rate curve is used as the initial condition. As a result of this selection a perfect fit between the observed and the theoretical bond prices is satisfied. In differential form

$$df(t,T) \;=\; \alpha(t,T)\,dt + \sigma(t,T)\,dW(t) \;\;,\; 0 \leq t \leq T, \tag{4.1.6}$$

where $W(t)$ is the Brownian motion under $\mathbf{P}$, $\alpha(t,T)$ and $\sigma(t,T)$ may be random. For each fixed T, $\alpha(t,T)$ and $\sigma(t,T)$ are adapted processes in the t variable. Forward rate dynamics are driven by a single Brownian motion but the results can easily be generalized to the multiple Brownian motion case.

Firstly, bond price dynamics will be obtained by using the following equation:

$$P(t,T) \;=\; \exp(-\int_t^T f(t,s)ds). \tag{4.1.7}$$

Note that the differential of $-\int_t^T f(t,s)ds$ is given by

$$d(-\int_t^T f(t,s)ds) \;=\; f(t,t)dt - \int_t^T df(t,s)ds \tag{4.1.8}$$

since $-\int_t^T f(t,s)\,ds$ has t-variable in two places, its differential has two terms. The instantaneous short rate at time t is given by $r(t) = f(t,t)$. By using this equation, we have

$$d(-\int_t^T f(t,s)ds) \;=\; r(t)dt - \int_t^T [\alpha(t,s)\,dt + \sigma(t,s)\,dW(t)]ds. \tag{4.1.9}$$

Let us define

$$\hat{\alpha}(t,T) \;:=\; \int_t^T \alpha(t,s)ds, \tag{4.1.10}$$

$$\hat{\sigma}(t,T) \;:=\; \int_t^T \sigma(t,s)ds, \tag{4.1.11}$$

In conclusion by changing the order of the integration by the Fubini theorem and using the equations (4.1.10) and (4.1.11), the formula becomes

$$d(-\int_t^T f(t,s)\,ds) \;=\; r(t)dt - \hat{\alpha}(t,T)\,dt - \hat{\sigma}(t,T)\,dW(t). \tag{4.1.12}$$

By choosing $h(x) = e^x$, the price of the zero-coupon bond is given by

$$P(t,T) \;=\; h(-\int_t^T f(t,s)ds).$$

Then the Ito-Deblin formula implies

$$\begin{aligned} dP(t,T) &= h'(-\int_t^T f(t,s)\,ds)\,d(-\int_t^T f(t,s)ds) \\ &\quad + \frac{1}{2}h''(-\int_t^T f(t,s)\,ds)\,[\,d(-\int_t^T f(t,s)\,ds)\,]^2 \\ &= P(t,T)\,[\,r(t)dt - \hat{\alpha}(t,T)\,dt - \hat{\sigma}(t,T)\,dW(t)\,] \\ &\quad + \frac{1}{2}P(t,T)\hat{\sigma}(t,T)^2 dt. \end{aligned}$$

As a result we end up with

$$\frac{dP(t,T)}{P(t,T)} \;=\; [\,r(t) - \hat{\alpha}(t,T) + \frac{1}{2}\hat{\sigma}(t,T)^2\,]\,dt - \hat{\sigma}(t,T)\,dW(t). \tag{4.1.13}$$

The first fundamental theorem of asset pricing implies that if there exists a risk neutral probability measure in the market model, the market is arbitrage free. Therefore we should explore such a measure $\hat{\mathbf{P}}$ under which discounted asset prices are martingale. The discounted bond price is given as follows:

$$\tilde{P}(t,T) \;=\; P(t,T)\,exp(-\int_0^t r(s)\,ds) \tag{4.1.14}$$

Then Ito's integration by parts formula gives us

$$\begin{aligned} d\tilde{P}(t,T) &= -P(t,T)\, exp(-\int_0^t r(s)\, ds)\, r(t)dt \\ & +exp(-\int_0^t r(s)\, ds)\, P(t,T)\, [\, r(t) - \hat{\alpha}(t,T) + \frac{1}{2}\hat{\sigma}(t,T)^2 \,]\, dt \\ & -exp(-\int_0^t r(s)\, ds)\, P(t,T)\, \hat{\sigma}(t,T)\, dW(t). \end{aligned}$$

After small algebra,

$$\frac{d\tilde{P}(t,T)}{\tilde{P}(t,T)} = [-\hat{\alpha}(t,T) + \frac{1}{2}\hat{\sigma}(t,T)^2\,]\, dt - \hat{\sigma}(t,T)\, dW(t). \qquad (4.1.15)$$

If the right hand side of the above equation is equal to $-\hat{\sigma}(t,T)d\hat{W}(t)$ where

$$\hat{W}(t) = \int_0^t \Theta(s)ds\, + W(t) \qquad (4.1.16)$$

then the Girsanov's theorem can be applied to transfer to a risk neutral probability measure $\hat{\mathbf{P}}$, under which $\hat{W}(t)$ is a $\hat{\mathbf{P}}$-Brownian motion, $\Theta(t)$ is the market price of risk and also the dynamics of the discounted bond price is written as

$$d\tilde{P}(t,T) = -\tilde{P}(t,T)\, \hat{\sigma}(t,T)\, d\hat{W}(t). \qquad (4.1.17)$$

The next step is to solve the following equation to find a market price of risk process, $\Theta(t)$.

$$[-\hat{\alpha}(t,T) + \frac{1}{2}\hat{\sigma}(t,T)^2\,]\, dt - \hat{\sigma}(t,T)\, dW(t) = -\hat{\sigma}(t,T)\, [dW(t) + \Theta(t)\, dt].$$

Hence

$$[-\hat{\alpha}(t,T) + \frac{1}{2}\hat{\sigma}(t,T)^2\,]\, dt = -\hat{\sigma}(t,T)\, \Theta(t)\, dt,$$

where $\Theta(t)$ is the solution of the infinitely many equations above one for each maturity. However $\Theta(t)$ is a single process since the random source is due to the one-dimensional Brownian motion in our model.

Finally, differentiating the above equation with respect to the T and using the definitions of $\hat{\alpha}(t,T)$ and $\hat{\sigma}(t,T)$ gives us

$$\alpha(t,T) \quad = \quad \sigma(t,T)\,[\,\hat{\sigma}(t,T) + \Theta(t)\,].$$

which is known as the HJM drift condition under objective probability measure. The following theorem summarizes the current study.

Theorem 4.1.1. (HJM Drift Condition) *A term structure model for a zero coupon bond prices of all $0 \le T \le T'$ is arbitrage free if there exist a process $\Theta(t)$ such that*

$$\alpha(t,T) \quad = \quad \sigma(t,T)\,[\,\hat{\sigma}(t,T) + \Theta(t)\,] \tag{4.1.18}$$

holds for all $0 \le t \le T \le T'$.

4.2 Forward Rate and Bond Price Dynamics under Martingale Measure

In this section, The HJM drift condition under the risk neutral measure will be given. Let us consider that the model satisfies the HJM no-arbitrage condition (4.1.18). Since the local rate of return should be equal to the short rate under the risk neutral measure, i.e. $\alpha = r$, then we may apply equation (4.1.18) with $\Theta(t) = 0$. The following proposition gives the HJM drift condition under the risk neutral measure.

Proposition 4.2.1. *Under martingale measure $\hat{\mathbf{P}}$, for every t and T satisfying $0 \le t \le T$, the following relation between the $\alpha(t,T)$ and $\sigma(t,T)$ processes should be satisfied*

$$\alpha(t,T) \quad = \quad \sigma(t,T)\,\hat{\sigma}(t,T). \tag{4.2.19}$$

By using the equation (4.2.19), the forward rate dynamics can be written as

$$df(t,T) \quad = \quad \sigma(t,T)\,\hat{\sigma}(t,T)\,dt + \sigma(t,T)\,d\hat{W}(t). \tag{4.2.20}$$

In addition, we have proved that the discounted zero-coupon bond price process has the following dynamics

$$d\tilde{P}(t,T) \quad = \quad -\tilde{P}(t,T)\,\hat{\sigma}(t,T)\,d\hat{W}(t).$$

Let B(t) be the bank account process. In order to reach a zero-coupon bond price process, we have to apply Ito's integration by parts formula to $d(B(t)\,\tilde{P}(t,T))$. Then

$$\begin{aligned} dP(t,T) &= d(B(t)\,\tilde{P}(t,T)) \\ &= r(t)\,P(t,T)\,dt - \hat{\sigma}(t,T)\,P(t,T)\,d\hat{W}(t). \end{aligned}$$

From the above equation it is seen that zero-coupon bonds have a risk-free return under risk neutral measure $\hat{\mathbf{P}}$.

4.3 Implementation of the HJM Framework

The key parameter of the HJM model is the volatility term $\sigma(t,T)$ of the instantaneous forward rates. Therefore, the first step is to specify the volatility structure under the actual measure. By the Girsanov Theorem, it is known that the volatility term is not affected by the change of measure. Hence, we can then reach the forward rate dynamics and bond prices of each maturity.
The following algorithm summarizes the HJM methodology.

1. Specify the volatility structure $\sigma(t,T)$.

2. Determine the drift parameters of forward rates by using the HJM drift condition:

$$\alpha(t,T) \quad = \quad \sigma(t,T)\,\hat{\sigma}(t,T).$$

3. Observe current forward rates $f^M(0,T)$ in the market.

4. Obtain forward rates by using the following formula:

$$f(t,T) \;=\; f^M(0,T) + \int_0^t \alpha(s,T)\,ds + \int_0^t \sigma(s,T)\,dW(s).$$

5. Compute the bond prices by using

$$P(t,T) \;=\; \exp(-\int_t^T f(t,s)ds).$$

Finally, the use of the HJM methodology is given with an example. Let us take volatility structure $\sigma(t,T) \;=\; \sigma$ where σ is a positive constant. Then the HJM drift condition gives us

$$\alpha(t,T) \;=\; \sigma(t,T)\,\hat{\sigma}(t,T) \;=\; \sigma \int_t^T \sigma ds \;=\; \sigma^2(T-t)$$

by substituting $\alpha(t,T)$ and $\sigma(t,T)$ processes into the forward rate equation

$$f(t,T) \;=\; f^M(0,T) + \int_0^t \sigma^2(T-s)\,ds + \int_0^t \sigma\,dW(s).$$

Then by using $f(t,t) = r(t)$

$$r(t) = f(t,t) = f^M(0,t) + \sigma^2\,\frac{t^2}{2} + \sigma\,W(t).$$

Hence the short rate dynamics are

$$dr(t) = (f_t(0,t) + \sigma^2\,t)\;dt\;+\sigma\;dW(t).$$

CHAPTER 5

JARROW-YILDIRIM MODEL

Jarrow and Yıldırım [31] propose an approach for pricing Treasury inflation protected securities (TIPS). The key assumptions in their model are the deterministic volatility and the non-zero correlation between different factors. Real prices correspond to foreign currency, nominal prices correspond to domestic currency and the inflation index corresponds to the exchange rate between nominal and real prices. This methodology is known as the foreign currency analogy. Key notations and dynamics used in their model are given in the following:

- $(\Omega, \mathcal{F}, \mathbf{P})$ is objective probability space, where Ω is a state space, $\mathcal{F}$ is the σ-algebra on Ω, $\mathbf{P}$ is the objective probability measure.
- $\{\mathcal{F}_t : \text{t} \in [0,\text{T}]\}$ is the standard filtration generated by the three Brownian motions $(W^n(t), W^r(t), W^I(t) : t \in [0,T])$ where r:real, n:nominal, I: inflation.
- Correlations between Brownian motions are given by

$$\begin{aligned} dW^n(t)dW^r(t) &= \rho_{nr}dt, \\ dW^n(t)dW^I(t) &= \rho_{nI}dt, \\ dW^r(t)dW^I(t) &= \rho_{rI}dt. \end{aligned}$$

- Nominal and real instantaneous forward rates and Consumer Price Index(CPI) dynamics under objective probability measure are given by

$$df^n(t,T) = \alpha^n(t,T)dt + \sigma^n(t,T)dW^n(t), \tag{5.0.1}$$

$$df^r(t,T) = \alpha^r(t,T)dt + \sigma^r(t,T)dW^r(t), \tag{5.0.2}$$

$$\frac{dI(t)}{I(t)} = \mu^I(t)dt + \sigma^I(t)dW^I(t), \tag{5.0.3}$$

where $\alpha^n(t,T)$, $\alpha^r(t,T)$, $\mu^I(t)$ are random, $\sigma^r(t,T)$, $\sigma^n(t,T)$, $\sigma^I(t)$ are deterministic.

- $$f^i(0,T) = f^i_M(0,T), \quad i \,\epsilon\, \{r,n\}$$

  where $f^n_M(0,T)$ and $f^r_M(0,T)$ are nominal and real instantaneous forward rates observed in the market at time 0, for maturity T.

- Nominal and real instantaneous short rate are given by

  $$\begin{aligned} r^n(t) &= f^n(t,t), \\ r^r(t) &= f^r(t,t). \end{aligned}$$

- $P^r(t,T)$ is the time t price of a real zero-coupon bond maturing at time T in CPI-U units.

- $P^n(t,T)$ is the time t price of a nominal zero-coupon bond maturing at time T in CPI-U[1] units.

- $B^n(0)$ is the time 0 price of a nominal coupon bearing bond issued at time $t_0 \leq 0$ in dollars where the coupon payment is C dollars per period, T is the maturity and F is the face value:

  $$B^n(0) = \sum_{t=1}^{T} CP^n(0,t) + FP^n(0,T). \tag{5.0.4}$$

[1]Consumer price index for all urban consumers is denoted as CPI-U in USA

- $B^{TIPS}(0)$ is the time 0 price of a coupon bearing Treasury inflation protected security in dollars issued at time $t_0 \le 0$

$$B^{TIPS}(0) \;=\; \left\{ \sum_{t=1}^{T} CI(0)P^r(0,t) + FI(0)P^r(0,T) \right\} / I(t_0). \quad (5.0.5)$$

- The price of a real zero-coupon bond in dollars without an issue date adjustment is given as

$$P^{TIPS}(t,T) = I(t)P^r(t,T). \quad (5.0.6)$$

Arbitrage-free drift restrictions in the Jarrow-Yıldırım model is given by the following proposition.

Proposition 5.1. $\frac{P^n(t,T)}{B^n(t)}$, $\frac{I(t)P^r(t,T)}{B^n(t)}$ *and* $\frac{I(t)B^r(t)}{B^n(t)}$ *are* $\hat{\mathbf{P}}$*-martingales if and only if the following conditions hold.*

$$\alpha^n(t,T) = \sigma^n(t,T)\left(\int_t^T \sigma^n(t,s)ds - \theta^n(t)\right), \quad (5.0.7)$$

$$\alpha^r(t,T) = \sigma^r(t,T)\left(\int_t^T \sigma^r(t,s)ds - \sigma^I(t)\rho_{rI} - \theta^r(t)\right), \quad (5.0.8)$$

$$\mu^I(t) = r^n(t) - r^r(t) - \sigma^I(t)\theta^I(t), \quad (5.0.9)$$

where $\hat{\mathbf{P}}$ *is an equivalent risk neutral measure to* $\mathbf{P}$, $(\theta^n(t), \theta^r(t), \theta^I(t) : t \,\epsilon\, [0,T])$ *are the risk premiums for the three risk factors in the economy,* $B^n(t)$ *and* $B^r(t)$ *are time* t *money market account values.*

Proof. First, let us prove the equation (5.0.7) which ensures that $\frac{P^n(t,T)}{B^n(t)}$ is a $\hat{\mathbf{P}}$-martingale. By equation (4.1.13),

$$\begin{aligned} dP^n(t,T) \;&=\; P^n(t,T)\left[r^n(t) - \hat{\alpha}^n(t,T) + \frac{1}{2}\left(\hat{\sigma}^n(t,T)^2\right)\right]dt \\ &\quad - P^n(t,T)\hat{\sigma}^n(t,T)dW^n(t) \quad (5.0.10) \end{aligned}$$

and the bank account

$$B^n(t) = \exp\left(\int_t^T r^n(s)ds\right)$$

or, in the differential form,

$$dB^n(t) = B^n(t)r^n(t)dt. \tag{5.0.11}$$

Then

$$B^n(t)^{-1} = \frac{1}{B^n(t)} = \exp\left(-\int_t^T r^n(s)ds\right)$$

or, in the differential form

$$dB^n(t)^{-1} = -B^n(t)^{-1}r^n(t)dt. \tag{5.0.12}$$

By Ito's integration by parts formula

$$\begin{aligned} d\left(\frac{P^n(t,T)}{B^n(t)}\right) &= d\left(P^n(t,T)B^n(t,T)^{-1}\right) \\ &= dP^n(t,T)B^n(t)^{-1} + P^n(t,T)d\left(B^n(t)^{-1}\right) + d\langle P^n, (B^n)^{-1}\rangle_t \\ &= \frac{P^n(t,T)}{B^n(t)}\left[\left(r^n(t) - \hat{\alpha}^n(t,T) + \frac{1}{2}\hat{\sigma}^n(t,T)^2\right)dt\right] \\ &\quad -\frac{P^n(t,T)}{B^n(t)}\left[\hat{\sigma}^n(t,T)dW^n(t)\right] - P^n(t,T)B^n(t)^{-1}r^n(t)dt \\ &= \frac{P^n(t,T)}{B^n(t)}\left[\left(-\hat{\alpha}^n(t,T) + \frac{1}{2}\hat{\sigma}^n(t,T)^2\right)dt - \hat{\sigma}^n(t,T)dW^n(t)\right]. \end{aligned} \tag{5.0.13}$$

In order to use Girsanov's Theorem, the right hand side of the above equation should be equal to

$$-\hat{\sigma}^n(t,T)d\hat{W}^n(t),$$

where

$$\hat{W}^n(t) = W^n(t) - \int_t^T \theta^n(s)ds$$

is the standard Brownian motion under risk neutral probability measure $\hat{\mathbf{P}}$ and the $\theta^n(t)$ is the market price of risk of the nominal prices. In the differential form

$$d\hat{W}^n(t) = dW^n(t) - \theta^n(t)dt. \tag{5.0.14}$$

Thus the right hand side of the equation (5.0.13) should be equal to

$$-\hat{\sigma}^n(t,T)\left(dW^n(t) - \theta^n(t)dt\right).$$

Therefore, the following equation has to be satisfied

$$\begin{aligned}\left(-\hat{\alpha}^n(t,T) + \frac{1}{2}\hat{\sigma}^n(t,T)^2\right)dt - \hat{\sigma}^n(t,T)dW^n(t) &= -\hat{\sigma}^n(t,T)dW^n(t) \\ &\quad +\hat{\sigma}^n(t,T)\theta^n(t)dt.\end{aligned}$$

Then

$$-\hat{\alpha}^n(t,T) + \frac{1}{2}\hat{\sigma}^n(t,T)^2 = \hat{\sigma}^n(t,T)\theta^n(t).$$

If we differentiate both sides with respect to T and use the equations (4.1.10) and (4.1.11), i.e.,

$$\begin{aligned}\hat{\alpha}^n(t,T) &= \int_t^T \alpha^n(t,s)ds, \\ \hat{\sigma}^n(t,T) &= \int_t^T \sigma^n(t,s)ds,\end{aligned}$$

we reach the following result

$$\alpha^n(t,T) = \sigma^n(t,T)\left(\int_t^T \sigma^n(t,s)ds - \theta^n(t)\right). \qquad \square$$

Secondly, we will prove the equation (5.0.9) under which $\frac{I(t)B^r(t)}{B^n(t)}$ is a $\hat{\mathbf{P}}$-martingale.

$$\frac{B^r(t)}{B^n(t)} = \exp\left(\int_t^T (r^r(s) - r^n(s))ds\right)$$

or, in the differential form

$$d\left(\frac{B^r(t)}{B^n(t)}\right) = \frac{B^r(t)}{B^n(t)}\left[r^r(t) - r^n(t)\right]dt. \tag{5.0.15}$$

Then by the integration by parts formula

$$\begin{aligned}
d\left(\frac{I(t)B^r(t)}{B^n(t)}\right) &= dI(t)B^r(t)B^n(t)^{-1} + I(t)d\left(B^r(t)B^n(t)^{-1}\right) \\
&\quad + d\langle I, B^r(B^n)^{-1}\rangle_t \\
&= \frac{I(t)B^r(t)}{B^n(t)}\left(\mu^I(t)dt + \sigma^I(t)dW^I(t)\right) \\
&\quad + \frac{I(t)B^r(t)}{B^n(t)}\left(r^r(t) - r^n(t)\right)dt \\
&= \frac{I(t)B^r(t)}{B^n(t)}\left[\left(\mu^I(t) + r^r(t) - r^n(t)\right)dt + \sigma^I(t)dW^I(t)\right].
\end{aligned} \tag{5.0.16}$$

By the Girsanov's Theorem

$$\hat{W}^I(t) = W^I(t) - \int_0^t \theta^I(s)ds$$

is the standard Brownian motion under risk neutral probability measure $\hat{\mathbf{P}}$ and the $\theta^I(t)$ is the market price of risk of the inflation. In the differential form

$$d\hat{W}^I(t) = dW^I(t) - \theta^I(t)dt.$$

Then the following equation has to be satisfied

$$\left(\mu^I(t) + r^r(t) - r^n(t)\right)dt + \sigma^I(t)dW^I(t) = \sigma^I(t)dW^I(t) - \sigma^I(t)\theta^I(t)dt.$$

Hence,

$$\mu^I(t) + r^r(t) - r^n(t) = -\sigma^I(t)\theta^I(t)$$

or, equivalently,

$$\mu^I(t) = r^n(t) - r^r(t) - \sigma^I(t)\theta^I(t),$$

which is known as the Fisher equation. $\square$

The final step is to obtain equation (5.0.8) under which $\frac{I(t)P^r(t,T)}{B^n(t)}$ is $\hat{\mathbf{P}}$-martingale by using Ito's integration by parts formula and Fisher equation. By equation (4.1.13)

$$\begin{aligned} dP^r(t,T) &= P^r(t,T)\left[r^r(t) - \hat{\alpha}^r(t,T) + \frac{1}{2}\left(\hat{\sigma}^r(t,T)^2\right)\right]dt \\ &\quad - P^r(t,T)\hat{\sigma}^r(t,T)dW^r(t) \qquad (5.0.17) \end{aligned}$$

First let us apply the integration by parts formula:

$$\begin{aligned} d\left(I(t)P^r(t,T)\right) &= dI(t)P^r(t,T) + dP^r(t,T)I(t) + d\langle I, P^r\rangle_t \\ &= P^r(t,T)I(t)\left[\mu^I(t)dt + \sigma^I(t)dW^I(t)\right] \\ &\quad + P^r(t,T)I(t)\left[\left(r^r(t) - \hat{\alpha}^r(t,T) + \frac{1}{2}\hat{\sigma}^r(t,T)^2\right)dt\right] \\ &\quad - P^r(t,T)I(t)\left[\hat{\sigma}^r(t,T)dW^r(t)\right] \\ &\quad - I(t)P^r(t,T)\sigma^I(t)\hat{\sigma}^r(t,T)\rho_{rI}dt \\ &= P^r(t,T)I(t)[\ \mu^I(t) + r^r(t) - \hat{\alpha}^r(t,T) + \frac{1}{2}\hat{\sigma}^r(t,T)^2 \\ &\quad - \sigma^I(t)\hat{\sigma}^r(t,T)\rho_{rI}\ ]dt \\ &\quad - P^r(t,T)I(t)[\ \hat{\sigma}^r(t,T)dW^r(t) - \sigma^I(t)dW^I(t)\ ]. \quad (5.0.18) \end{aligned}$$

Then, applying same formula gives

$$\begin{aligned} d\left(I(t)P^r(t,T)B^n(t,T)^{-1}\right) &= d(I(t)P^r(t,T))B^n(t)^{-1} + I(t)P^r(t,T)dB^n(t)^{-1} \\ &\quad + d\langle IP^r, (B^n)^{-1}\rangle_t \end{aligned}$$

which is equal to following in the explicit form

$$\begin{aligned} d\left(I(t)P^r(t,T)B^n(t,T)^{-1}\right) &= \frac{P^r(t,T)I(t)}{B^n(t)}[\mu^I(t) + r^r(t) - \hat{\alpha}^r(t,T) - r^n(t) \\ & -\sigma^I(t)\hat{\sigma}^r(t,T)\rho_{rI} + \frac{1}{2}\hat{\sigma}^r(t,T)^2]dt \\ & -\frac{P^r(t,T)I(t)}{B^n(t)}\left[\hat{\sigma}^r(t,T)dW^r(t) - \sigma^I(t)dW^I(t)\right] \end{aligned}$$

if we put (5.0.9) into the above equation

$$\begin{aligned} d\left(I(t)P^r(t,T)B^n(t)^{-1}\right) &= \frac{P^r(t,T)I(t)}{B^n(t)}[-\sigma^I(t)\theta^I(t) - \hat{\alpha}^r(t,T) \\ & +\frac{1}{2}\hat{\sigma}^r(t,T)^2 - \sigma^I(t)\hat{\sigma}^r(t,T)\rho_{rI}]dt \\ & -\frac{P^r(t,T)I(t)}{B^n(t)}\left[\hat{\sigma}^r(t,T)dW^r(t) - \sigma^I(t)dW^I(t)\right]. \end{aligned} \tag{5.0.19}$$

By the multi-dimensional Girsanov's Theorem

$$\begin{aligned} \hat{W}^r(t) &= W^r(t) - \int_t^T \theta^r(s)ds, \\ \hat{W}^I(t) &= W^I(t) - \int_t^T \theta^I(s)ds \end{aligned}$$

are the standard Brownian motions under risk neutral probability measure $\hat{\mathbf{P}}$ and the $\theta^r(t)$ and the $\theta^I(t)$ are the market price of risks of the real prices and inflation index respectively.

Thus, $\frac{I(t)P^r(t,T)}{B^n(t)}$ is martingale under $\hat{\mathbf{P}}$ and the following equation is satisfied.

$$\begin{aligned} & \left[-\sigma^I(t)\theta^I(t) - \hat{\alpha}^r(t,T) + \frac{1}{2}\hat{\sigma}^r(t,T)^2 - \sigma^I(t)\hat{\sigma}^r(t,T)\rho_{rI}\right]dt \\ & -\hat{\sigma}^r(t,T)dW^r(t) + \sigma^I(t)dW^I(t) = -\hat{\sigma}^r(t,T)d\hat{W}^r(t) + \sigma^I(t)d\hat{W}^I(t) \\ & = -\hat{\sigma}^r(t,T)dW^r(t) + \hat{\sigma}^r(t,T)\theta^r(t)dt + \sigma^I(t)dW^I(t) - \sigma^I(t)\theta^I(t)dt. \end{aligned}$$

Then after some simplifications

$$-\hat{\alpha}^r(t,T) + \frac{1}{2}\hat{\sigma}^r(t,T)^2 - \sigma^I(t)\hat{\sigma}^r(t,T)\rho_{rI} = \hat{\sigma}^r(t,T)\theta^r(t)$$

or equivalently

$$\hat{\alpha}^r(t,T) = \frac{1}{2}\hat{\sigma}^r(t,T)^2 - \hat{\sigma}^r(t,T)\theta^r(t) - \hat{\sigma}^r(t,T)\sigma^I(t)\rho_{rI}.$$

By differentiating both sides with respect to T and using definitions of $\hat{\alpha}^r(t,T)$, $\hat{\sigma}^r(t,T)$, the final equation appears

$$\alpha^r(t,T) = \sigma^r(t,T)\left(\int_t^T \sigma^r(t,s)ds - \theta^r(t) - \sigma^I(t)\rho_{rI}\right). \qquad \square$$

Then under martingale measure $\hat{P}$, nominal and real forward rates, zero coupon bond prices and inflation dynamics can be restated. The next proposition gives the new dynamics.

Proposition 5.2. *The following price processes satisfy under risk-neutral measure:*

i) $$df^n(t,T) = \sigma^n(t,T)\int_t^T \sigma^n(t,s)ds + \sigma^n(t,T)d\hat{W}^n(t), \tag{5.0.20}$$

ii) $$df^r(t,T) = \sigma^r(t,T)\left[\int_t^T \sigma^r(t,s)ds - \rho_{rI}\sigma^I(t)\right]dt + \sigma^r(t,T)d\hat{W}^r(t), \tag{5.0.21}$$

iii) $$\frac{dI(t)}{I(t)} = [r^n(t) - r^r(t)]\,dt + \sigma^I(t)d\hat{W}^I(t), \tag{5.0.22}$$

iv) $$\frac{dP^n(t,T)}{P^n(t,T)} = r^n(t)dt - (\int_t^T \sigma^n(t,s)ds)d\hat{W}^n(t), \tag{5.0.23}$$

v) $$\frac{dP^{TIPS}(t,T)}{P^{TIPS}(t,T)} = r^n(t)dt + \sigma^I(t)d\hat{W}^I(t) - (\int_t^T \sigma^r(t,s)ds)d\hat{W}^r(t), \tag{5.0.24}$$

vi) $$\frac{dP^r(t,T)}{P^r(t,T)} = \left[r^r(t) + \rho_{rI}\sigma^I(t)\int_t^T \sigma^r(t,s)ds\right]dt - (\int_t^T \sigma^r(t,s)ds)d\hat{W}^r(t). \tag{5.0.25}$$

Proof. **i**) By equation (5.0.1)

$$df^n(t,T) = \alpha^n(t,T)dt + \sigma^n(t,T)dW^n(t).$$

Also using the definition of $\alpha^n(t,T)$ from proposition(5.1) gives us

$$df^n(t,T) = [\sigma^n(t,T)\hat{\sigma}^n(t,T) - \sigma^n(t,T)\theta^n(t)]\, dt + \sigma^n(t)dW^n(t).$$

Under the martingale measure the local rate of return is equal to the short rate, i.e.,

$$\theta = \frac{\alpha - r}{\sigma} = 0$$

(see Björk [4]).Thus,

$$d\hat{W}^n(t) = dW^n(t) - \theta^n(t)dt = dW^n(t),$$

which gives the result

$$df^n(t,T) = \sigma^n(t,T)\int_t^T \sigma^n(t,s)ds + \sigma^n(t,T)d\hat{W}^n(t).$$

$\square$

ii) The real forward rate dynamics under the objective probability measure is given as

$$df^r(t,T) = \alpha^r(t,T)dt + \sigma^r(t,T)dW^r(t),$$

By using the same method as in (i), putting the definition of $\alpha^r(t,T)$ into the above equation gives

$$\begin{aligned} df^r(t,T) = {} & \left(\sigma^r(t,T)\int_t^T \sigma^r(t,s)ds - \sigma^r(t,T)\sigma^I(t)\rho_{rI} - \sigma^r(t,T)\theta^r(t)\right) dt \\ & + \sigma^r(t,T)dW^r(t) \end{aligned}$$

Under the martingale measure, $\theta^r(t) = 0$ and $d\hat{W}^r(t) = dW^r(t)$. Hence,

$$df^r(t,T) = \sigma^r(t,T)[\int_t^T \sigma^r(t,s)ds - \sigma^I(t)\rho_{rI} + d\hat{W}^r(t)].$$ $\square$

iii) Inflation index follows a geometric Brownian motion process

$$dI(t) = I(t)\mu^I dt + I(t)\sigma^I(t)dW^I(t).$$

By the Fisher equation $\mu^I(t) = r^n(t) - r^r(t) - \sigma^I(t)\theta^I(t)$. Substituting $\mu^I(t)$ into the above equation provides

$$dI(t) = I(t)\left(r^n(t) - r^r(t) - \sigma^I(t)\theta^I(t)\right)dt + I(t)\sigma^I(t)dW^I(t).$$

Finally, taking $\theta^I(t) = 0$ and $d\hat{W}^I(t) = dW^I(t)$ gives the result:

$$\frac{dI(t)}{I(t)} = \left(r^n(t) - r^r(t)\right)dt + \sigma^I(t)d\hat{W}^I(t). \qquad \square$$

iv) Nominal zero coupon bond price dynamics under objective probability measure is presented as

$$dP^n(t,T) = P^n(t,T)\left[r^n(t) - \hat{\alpha}^n(t,T) + \frac{1}{2}\hat{\sigma}^n(t,T)^2\right]dt - \hat{\sigma}^n(t,T)dW^n(t).$$

By equation (5.0.7)

$$\alpha^n(t,T) = \sigma^n(t,T)\hat{\sigma}^n(t,T) - \sigma^n(t,T)\theta^n(t)$$

integrating both sides from t to T and taking market price of risk, $\theta^n(t) = 0$ gives

$$\hat{\alpha}^n(t,T) = \frac{1}{2}\hat{\sigma}^n(t,T)^2.$$

Finally, we get the following result by using above equation

$$\frac{dP^n(t,T)}{P^n(t,T)} = r^n(t)dt - (\int_t^T \sigma^n(t,s)ds)d\hat{W}^n(t).$$

$\square$

vi) Since the dynamics of $P^r(t,T)$ is required in the derivation of $P^{TIPS}(t,T)$ we will need the dynamics of $P^r(t,T)$. The dynamics of $P^r(t,T)$ under the objective probability measure is

$$\frac{dP^r(t,T)}{P^r(t,T)} = \left(r^r(t) - \hat{\alpha}^r(t,T) + \frac{1}{2}\hat{\sigma}^r(t,T)^2\right)dt - \hat{\sigma}^r(t,T)dW^r(t).$$

By equation (5.0.8)

$$\alpha^r(t,T) = \sigma^r(t,T)\hat{\sigma}^r(t,T) - \sigma^r(t,T)\sigma^I(t)\rho_{rI} - \sigma^r(t,T)\theta^r(t)$$

integrating both sides from t to T and taking market price of risk, $\theta^r(t) = 0$ gives

$$\hat{\alpha}^r(t,T) = \frac{1}{2}\hat{\sigma}^r(t,T)^2 - \hat{\sigma}^r(t,T)\sigma^I(t)\rho_{rI}.$$

Using the above equation yields

$$\frac{dP^r(t,T)}{P^r(t,T)} = \left(r^r(t) + \rho_{rI}\sigma^I(t)\int_t^T \sigma^r(t,s)ds\right)dt - (\int_t^T \sigma^r(t,s)ds)d\hat{W}^r(t). \qquad \square$$

v) The final step of the proof is to reach the dynamics of $P^{TIPS}(t,T)$ under martingale measure. By equation (5.0.6)

$$P^{TIPS}(t,T) = I(t)P^r(t,T).$$

By Ito's integration by parts formula

$$\begin{aligned}
dP^{TIPS}(t,T) &= d\left(I(t)P^r(t,T)\right) \\
&= dI(t)P^r(t,T) + I(t)dP^r(t,T) + d\langle I, P^r\rangle_t \\
&= I(t)P^r(t)\left[\left(r^n(t) - r^r(t)\right)dt + \sigma^I(t)d\hat{W}^I(t)\right] \\
&\quad + I(t)P^r(t,T)\left[\left(r^r(t) + \rho_{rI}\sigma^I(t)\hat{\sigma}^r(t,T)\right)dt - \hat{\sigma}^r(t,T)d\hat{W}^r(t)\right] \\
&\quad - \sigma^I(t)\hat{\sigma}^r(t,T)I(t)P^r(t,T)\rho_{rI}dt.
\end{aligned}$$

Then,

$$\frac{dP^{TIPS}(t,T)}{P^{TIPS}(t,T)} = r^n(t)dt + \sigma^I(t)d\hat{W}^I(t) - (\int_t^T \sigma^r(t,s)ds)d\hat{W}^r(t).$$

$\square$

After that Jarrow and Yıldırım strip real and nominal zero coupon bond prices from the observed market prices of the coupon-bearing securities. If we go back to the equation (5.0.5), the price of the coupon bearing TIPS is given as

$$B^{TIPS}(0) = \left\{ \sum_{t=1}^{T} CP^r(0,t)I(0) + FP^r(t,T)I(0) \right\} / I(t_0).$$

where C, $I(0)$, F and $I(t_0)$ are observable at time 0 in the market. $B^{TIPS}(0)$ may be known or not. If the price of the TIPS is observable at time 0 in the market, then putting all variables into the above equation easily gives the price of the real zero coupon bond at time 0. However, if it is not observable at such a time, stripping real zero coupon bond prices will be required. In Jarrow-Yıldırım model, stripping procedure is applied by using piecewise constant forward rate curve. Firstly, forward rates are estimated by nonlinear least square method. Then, by using the relation between zero coupon bonds and forward rates, real zero coupon bond prices are obtained. Nominal zero coupon bond prices are also stripped with the same method. Then the volatility parameters for the real and nominal forward rates are estimated. The validity of their model is tested by a hedging procedure.

CHAPTER 6

EXTENSIONS OF THE HJM AND THE JARROW-YILDIRIM MODELS WITH JUMPS INCORPORATION

In finance literature, most of the works propose models based on diffusion type processes and especially on the geometric Brownian motion. However, such models have same drawbacks. In financial markets, asset price processes may always have jumps and these models can not capture this property. When pricing and hedging comes to order, empirical studies show that the performances of such models are sometimes inadequate. Hence, stochastic processes with jumps have become increasingly popular in the last two decades. These new models allow prices and interest rates follow a continuous process at most of the time. On the other hand they support the fact that longer jumps may appear from time to time.

One of the early studies on the inclusion of jump components into forward rate dynamics is that of Shirakawa [43]. The framework here assumes a finite number of possible jump sizes and there exists a sufficient number of traded bonds to hedge away possible jump risks, thus guaranteeing market completeness. Björk et al. [5] and Jarrow and Madan [30] propose interest rate models driven by point processes, when the mark space is finite. Burnetas and Ritchken [8], Das [13], Das and Foresi [14] work on the pricing of interest rate derivatives in the presence of jumps. Das [12] introduce a discrete time jump diffusion version of the HJM model. Glasserman and Kou [18] derive arbitrage free dynamics of interest rates in the presence of jump diffusion process. Chiarella and Sklibosios [10] present a multifactor jump diffusion model of the term structure of interest rates under a specific volatility structure. Björk, Kabanov and Runggaldier [6] focus on interest rate models driven by point processes where the mark space is infinite. In this chapter, extensions of the HJM and Jarrow-Yıldırım models depending on Shirakawa's framework will be introduced.

6.1 An Extension of the HJM Model

Uncertainty in the financial market is characterized by (Ω, $\mathcal{F}$, $\mathbf{P}$) where Ω is the state space, $\mathcal{F}$ is the filtration on Ω, $\mathbf{P}$ is the objective probability measure on (Ω, $\mathcal{F}$). W is the standard Brownian motion, N_i's, i=1,..,n are Poisson processes which are independent of each other and of the Brownian motion.

Assumption 6.1.1. *There exists a sufficient number of traded bonds to hedge away all of the jump risks, hence the market is complete.*

Dynamics of the instantaneous forward rate $f(t,T)$, $\forall t \leq T \in \mathbf{R}^+$ is given by

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t) + \sum_{i=1}^{n} \delta_i(t,T)\left[dN_i(t) - \lambda_i dt\right], \qquad (6.1.1)$$

where $\alpha(t,T)$ and $\sigma(t,T)$ are the drift and the Brownian coefficients, δ_i's (i=1..n) are jump sizes that occur at the Poisson jump times. λ_i's are constant intensities of $N_i(t)$.

$$dN_i(t) = \begin{cases} 1 & , if\ a\ jump\ occurs\ in\ the\ time\ interval\ (t, t+dt) \\ & (with\ probability\ \lambda_i\ dt); \\ 0 & , otherwise\ (with\ probability\ 1 - \lambda_i\ dt) \end{cases} \qquad (6.1.2)$$

$\sigma(t,T)$ is a positive-valued, well defined function which depends on time and maturity. Forward rate dynamics can be expressed in stochastic integral form as

$$f(t,T) = f(0,T)+\int_0^t \alpha(s,T)ds+\int_0^t \sigma(s,T)dW(s)+\sum_{i=1}^{n}\int_0^t \delta_i(s,T)[dN_i(s)-\lambda_i ds]. \tag{6.1.3}$$

Setting t=T in the above equation gives the dynamics of the instantaneous spot rate as follows

$$r(t) = f(0,t) + \int_0^t \alpha(s,t)ds + \int_0^t \sigma(s,t)dW(s) + \sum_{i=1}^{n}\int_0^t \delta_i(s,t)[dN_i(s) - \lambda_i ds]. \tag{6.1.4}$$

Then, let us find the dynamics of $P(t,T)$ by using the following relation:

$$P(t,T) = exp(-\int_t^T f(t,s)ds)$$

also note that

$$\begin{aligned} d(-\int_t^T f(t,s)ds) &= f(t,t)dt - \int_t^T df(t,s)ds \\ &= r(t)dt - \int_t^T [\alpha(t,s)dt + \sigma(t,s)dW(t)]ds \\ &\quad - \int_t^T \sum_{i=1}^{n} \delta_i(t,s)[dN_i(t) - \lambda_i dt]ds. \end{aligned}$$

Define

$$\hat{\alpha}(t,T) = \int_t^T \alpha(t,s)ds \tag{6.1.5}$$

$$\hat{\sigma}(t,T) = \int_t^T \sigma(t,s)ds. \tag{6.1.6}$$

Then by the Fubini theorem

$$d\left(-\int_t^T f(t,s)ds\right) = r(t)dt - \hat{\alpha}(t,T)dt - \hat{\sigma}(t,T)dW(t) - \sum_{i=1}^{n}\int_t^T \delta_i(t,s)[dN_i(t) - \lambda_i dt]ds.$$

Let us apply the Ito-Deblin formula for jump processes with $g(x) = e^x$. Then,

$$\begin{aligned} dP(t,T) &= P(t^-,T)(r(t)dt - \hat{\alpha}(t,T)dt - \hat{\sigma}(t,T)dW(t) + \sum_{i=1}^{n}\lambda_i dt\int_t^T \delta_i(t,s)ds) \\ &+\frac{1}{2}P(t^-,T)\hat{\sigma}(t,T)^2 dt \\ &+\sum_{i=1}^{n}\left[e^{-\int_t^T f(t^-,s)+\delta_i(t,s)ds} - e^{-\int_t^T f(t^-,s)ds}\right]dN_i(t). \end{aligned}$$

Let us define

$$\hat{\delta}_i(t,T) = \int_t^T \delta_i(t,s)ds. \tag{6.1.7}$$

Then we get

$$\begin{aligned} \frac{dP(t,T)}{P(t^-,T)} &= \left[r(t) - \hat{\alpha}(t,T) + \frac{1}{2}\hat{\sigma}(t,T)^2 + \sum_{i=1}^{n}\lambda_i\hat{\delta}_i(t,T)\right]dt \\ &\quad - \hat{\sigma}(t,T)dW(t) + \sum_{i=1}^{n}\left(e^{-\hat{\delta}_i(t,T)} - 1\right)dN_i(t). \end{aligned} \tag{6.1.8}$$

In our model there exist $n+1$ sources of risk, 1 due to the Brownian motion, others due to the Poisson processes. For the hedging procedure we can take a suitable position in the $n+1$ bonds in order to eliminate both Poisson and Brownian motion risks.

In order to guarantee that the market is arbitrage free we should find risk neutral probability measure $\hat{\mathbf{P}}$ under which discounted bond prices are martingale. First let's find the dynamics of discounted bond prices

$$\tilde{P}(t,T) = P(t,T)\exp\left(-\int_0^t r(s)ds\right).$$

By the integration by parts formula for jump processes

$$\begin{aligned}
d\tilde{P}(t^-,T) &= -P(t^-,T)e^{-\int_0^t r(s)ds}r(t)dt \\
&\quad +e^{-\int_0^t r(s)ds}P(t^-,T)\left[r(t)-\hat{\alpha}(t,T)+\frac{1}{2}\hat{\sigma}(t,T)^2\right]dt \\
&\quad -e^{-\int_0^t r(s)ds}P(t^-,T)\left[\hat{\sigma}(t,T)dW(t)\right] \\
&\quad +e^{-\int_0^t r(s)ds}P(t^-,T)\left[\sum_{i=1}^n \lambda_i\hat{\delta}_i(t,T)dt+\sum_{i=1}^n\left(e^{-\hat{\delta}_i(t,T)}-1\right)dN_i(t)\right].
\end{aligned}$$

Then, finally we get

$$\begin{aligned}
\frac{d\tilde{P}(t,T)}{\tilde{P}(t^-,T)} &= \left[-\hat{\alpha}(t,T)+\frac{1}{2}\hat{\sigma}(t,T)^2+\sum_{i=1}^n \lambda_i\hat{\delta}_i(t,T)\right]dt \\
&\quad -\hat{\sigma}(t,T)dW(t)+\sum_{i=1}^n\left(e^{-\hat{\delta}_i(t,T)}-1\right)dN_i(t). \qquad (6.1.9)
\end{aligned}$$

By the Girsanov Theorem, let $\hat{\mathbf{P}}$ be a risk neutral probability measure equivalent to objective probability measure $\mathbf{P}$, and let $\phi(t)$ be the market price of diffusion risk associated with the Brownian motion sources of uncertainty W(t), and let $\psi_i(t)$'s be the intensities under $\hat{\mathbf{P}}$ defined as

$$\psi_i(t)=\lambda_i\rho_i,$$

where λ_i's are intensities under $\mathbf{P}$ and ρ_i's are the market prices of jump risk associated with the Poisson processes sources of uncertainty $N_i(t)'s$. Also note that $\rho_i \geq 0$ are predictable processes (See Oksendal [37], Runggaldier [39] for detail). Then,

$$\begin{aligned}
\hat{W}(t) &= W(t)-\int_0^t \phi(s)ds, \\
\hat{N}_i(t) &= N_i(t)-\int_0^t \psi_i(s)ds.
\end{aligned}$$

Discounted asset prices are martingale if the right hand side of equation (6.1.9) is equal to

$$-\hat{\sigma}(t,T)d\hat{W}(t)+\sum_{i=1}^{n}\left(e^{-\hat{\delta}_i(t,T)}-1\right)d\hat{N}_i(t),$$

i.e.,

$$\begin{aligned}&\left[-\hat{\alpha}(t,T)+\frac{1}{2}\hat{\sigma}(t,T)^2+\sum_{i=1}^{n}\lambda_i\hat{\delta}_i(t,T)\right]dt-\hat{\sigma}(t,T)dW(t)\\&+\sum_{i=1}^{n}\left(e^{-\hat{\delta}_i(t,T)}-1\right)dN_i(t)=-\hat{\sigma}(t,T)dW(t)+\hat{\sigma}(t)\phi(t)dt\\&+\sum_{i=1}^{n}\left(e^{-\hat{\delta}_i(t,T)}-1\right)dN_i(t)-\sum_{i=1}^{n}\left(e^{-\hat{\delta}_i(t,T)}-1\right)\psi_i(t)dt.\end{aligned}$$

After small simplifications

$$-\hat{\alpha}(t,T)+\frac{1}{2}\hat{\sigma}(t,T)^2+\sum_{i=1}^{n}\lambda_i\hat{\delta}_i(t,T)dt=\hat{\sigma}(t)\phi(t)dt-\sum_{i=1}^{n}\left(e^{-\hat{\delta}_i(t,T)}-1\right)\psi_i(t)dt.$$

Differentiating both sides with respect to T and using the definition of $\hat{\alpha}(t,T)$, $\hat{\sigma}(t,T)$, $\hat{\delta}_i(t,T)$ yields

$$\begin{aligned}-\alpha(t,T)+\sigma(t,T)\hat{\sigma}(t,T)+\sum_{i=1}^{n}\lambda_i\delta_i(t,T) &= \sigma(t)\phi(t)\\&\quad+\sum_{i=1}^{n}\delta_i(t,T)\psi_i(t)\left(e^{-\hat{\delta}_i(t,T)}-1\right).\end{aligned}$$

Finally we end up with

$$\alpha(t,T)=\sigma(t,T)\hat{\sigma}(t,T)-\sigma(t)\phi(t)+\sum_{i=1}^{n}\delta_i(t,T)\left[\lambda_i-\psi_i(t)\left(e^{-\hat{\delta}_i(t,T)}-1\right)\right], \tag{6.1.10}$$

which can be defined as the drift condition of extension of HJM model, satisfying that the financial market is arbitrage free. By substituting the drift restriction into the bond price equation we get the bond price in arbitrage free economy

$$\frac{dP(t,T)}{P(t^-,T)} = r(t)dt - \hat{\sigma}(t,T)d\hat{W}(t) - \sum_{i=1}^{n}\left(e^{-\hat{\delta}_i(t,T)} - 1\right)\left[dN_i(t) - \psi_i(t)dt\right]. \tag{6.1.11}$$

Finally, by substituting the drift condition into the short rate equation, we obtain the dynamics of spot interest rate $r(t)$ under the risk neutral measure $\hat{\mathbf{P}}$

$$\begin{aligned} r(t) &= f(0,t) + \int_0^t \sigma(s,t)\hat{\sigma}(s,t)ds + \sum_{i=1}^{n}\int_0^t \psi_i(s)\delta_i(s,t)\left[1 - e^{-\hat{\delta}_i(s,T)}\right]ds \\ &\quad + \int_0^t \sigma(s,t)d\hat{W}(s) + \sum_{i=1}^{n}\int_0^t \delta_i(s,t)[dN_i(s) - \psi_i(s)ds]. \end{aligned} \tag{6.1.12}$$

6.2 An Extension of the Jarrow-Yıldırım Model

The key assumptions in our model is consistent with the Jarrow-Yıldırım model. Volatility is assumed to be deterministic and different factors are correlated with each other. We also used foreign currency analogy. Some notations used in our model are as follows:

- $\{\mathcal{F}_t : \ t \ \epsilon \ [0.T]\}$ is the standard filtration generated by the three Brownian motions $(W^n(t),\ W^r(t),\ W^I(t) : t\epsilon\,[0,T])$ where r denotes real, n denotes nominal, I denotes inflation.
- Correlations between Brownian motions are given by

$$\begin{aligned} dW^n(t)dW^r(t) &= \rho_{nr}dt, \\ dW^n(t)dW^I(t) &= \rho_{nI}dt, \\ dW^r(t)dW^I(t) &= \rho_{rI}dt. \end{aligned}$$

• Nominal and real instantaneous forward rates and CPI dynamics under the objective probability measure are given by

$$df^n(t,T) = \alpha^n(t,T)dt + \sigma^n(t,T)dW^n(t) + \sum_{i=1}^{n} \delta_i^n(t,T)\left[dN_i^n(t) - \lambda_i^n dt\right], \tag{6.2.13}$$

$$df^r(t,T) = \alpha^r(t,T)dt + \sigma^r(t,T)dW^r(t) + \sum_{i=1}^{n} \delta_i^r(t,T)\left[dN_i^r(t) - \lambda_i^r dt\right], \tag{6.2.14}$$

$$dI(t) = I(t)\mu^I(t)dt + I(t)\sigma^I(t)dW^I(t) + \sum_{i=1}^{n} \delta_i^I(t,T)\left[dN_i^I(t) - \lambda_i^I dt\right], \tag{6.2.15}$$

where $\alpha^n(t,T)$, $\alpha^r(t,T)$, $\mu^I(t)$ are random, $\sigma^n(t,T)$, $\sigma^r(t,T)$, $\sigma^I(t)$ are deterministic, N_i^k, $k \in \{r,n,I\}$ are Poisson processes which are independent of each other and of the Brownian motions, $W^k(t)$. Although in real world, jump sizes of nominal and real rates are correlated to each other, in this study, for mathematical simplification, no correlation between Poisson processes for different factors assumption is used. δ_i^k are jump sizes that occur at the Poisson jump times, λ_i^k are constant intensities of N_i^k.

• $f^k(0,T) = f_\mu^k(0,T)$, $i \in \{r,n\}$, where $f_\mu^n(0,T)$ and $f_\mu^r(0,T)$ are nominal and real instantaneous forward rates observed in the market at time 0, for maturity T.

• All the assumptions and notations used in section 6.1 are still valid.

Assumption 6.2.1. *There are no arbitrage possibilities in the market*

By assumption 6.1.1, there exist a unique measure $\hat{\mathbf{P}}$ such that $\frac{P^n(t,T)}{B^n(t)}$, $\frac{I(t)P^r(t,T)}{B^n(t)}$, $\frac{I(t)B^r(t,T)}{B^n(t)}$ are $\hat{\mathbf{P}}$ martingales. Next proposition presents the necessary and sufficient conditions required on the bond price dynamics in order that the market be arbitrage free.

Proposition 6.2.1. *$\frac{P^n(t,T)}{B^n(t)}$, $\frac{I(t)P^r(t,T)}{B^n(t)}$, $\frac{I(t)B^r(t,T)}{B^n(t)}$ are $\hat{\mathbf{P}}$ martingales iff the following conditions hold*

$$\begin{aligned}\alpha^n(t,T) &= \sigma^n(t,T)\left(\int_t^T \sigma^n(t,s)ds - \phi^n(t)\right) \\ &\quad + \sum_{i=1}^n \delta_i^n(t,T)\left[\lambda_i^n - \psi_i^n(t)e^{-\hat{\delta}_i^n(t,T)}\right], \qquad (6.2.16)\end{aligned}$$

$$\begin{aligned}\alpha^r(t,T) &= \sigma^r(t,T)\left(\int_t^T \sigma^r(t,s)ds - \sigma^I(t)\rho_{rI} - \phi^r(t)\right) \\ &\quad + \sum_{i=1}^n \delta_i^r(t,T)\left(\lambda_i^r + e^{-\hat{\delta}_i^r(t,T)}\psi_i^r(t)\right), \qquad (6.2.17)\end{aligned}$$

$$\mu^I(t) = r^n(t) - r^r(t) - \sigma^I(t)\phi^I(t) + \sum_{i=1}^n \delta_i^I(t,T)\left(\lambda_i^I - \psi_i^I(t)\right). \qquad (6.2.18)$$

Proof. First let's find the $\hat{\mathbf{P}}$-dynamics of $\frac{P^n(t,T)}{B^n(t)}$. By equation (6.1.8), the nominal zero-coupon bond price dynamics under the objective probability measure is

$$\begin{aligned}\frac{dP^n(t,T)}{P^n(t,T)} &= \left[r^n(t) - \hat{\alpha}^n(t,T) + \frac{1}{2}\hat{\sigma}^n(t,T)^2 + \sum_{i=1}^n \lambda_i^n\hat{\delta}_i^n(t,T)\right]dt \\ &\quad -\hat{\sigma}^n(t,T)dW^n(t) + \sum_{i=1}^n\left(e^{-\hat{\delta}_i^n(t,T)} - 1\right)d\hat{N}_i^n(t). \qquad (6.2.19)\end{aligned}$$

The nominal money market account has the following dynamics

$$B^n(t) = \exp\left(\int_0^t r^n(s)ds\right)$$

or, in the differential form

$$dB^n(t) = B^n(t)r^n(t)dt.$$

Then the dynamics of $B^n(t)^{-1}$ is given

$$dB^n(t)^{-1} = -B^n(t)^{-1} r^n(t)dt.$$

By the integration by parts formula

$$\begin{aligned}
d\left(\frac{P^n(t,T)}{B^n(t)}\right) &= d\left(P^n(t,T)B^n(t)^{-1}\right) \\
&= dP^n(t,T)B^n(t)^{-1} + P^n(t,T)dB^n(t)^{-1} + d\langle P^n, (B^n)^{-1}\rangle_t \\
&= \frac{P^n(t,T)}{B^n(t)}\left[r^n(t) - \hat{\alpha}^n(t,T) + \frac{1}{2}\hat{\sigma}^n(t,T)^2 + \sum_{i=1}^n \lambda_i^n \hat{\delta}_i^n(t,T)\right] dt \\
&\quad -\frac{P^n(t,T)}{B^n(t)}\left[\hat{\sigma}^n(t,T)dW^n(t) + \sum_{i=1}^n \left(e^{-\hat{\delta}_i^n(t,T)} - 1\right) dN_i^n(t)\right] \\
&\quad -\frac{P^n(t,T)}{B^n(t)} r^n(t)dt \\
&= \frac{P^n(t,T)}{B^n(t)}\left(-\hat{\alpha}^n(t,T) + \frac{1}{2}\hat{\sigma}^n(t,T)^2 + \sum_{i=1}^n \lambda_i^n \hat{\delta}_i^n(t,T)\right) dt \\
&\quad +\frac{P^n(t,T)}{B^n(t)}\left(-\hat{\sigma}^n(t,T)dW^n(t) + \sum_{i=1}^n \left(e^{-\hat{\delta}_i^n(t,T)} - 1\right) dN_i^n(t)\right).
\end{aligned}$$

Then, let us use Girsanov Theorem to transfer to risk neutral measure with

$$\begin{aligned}
\hat{W}^n(t) &= W^n(t) - \int_0^t \phi^n(s)ds, \\
\hat{N}_i^n(t) &= N_i^n(t) - \int_0^t \psi_i^n(s)ds,
\end{aligned}$$

where $\phi^n(t)$ is the market price of diffusion risk associated with the Brownian motion sources of uncertainty $W^n(t)$, $\psi_i^n(t)$'s are the intensities under $\hat{\mathbf{P}}$ defined as

$$\psi_i^n(t) = \lambda_i^n \rho_i^n(t),$$

where λ_i^n's are intensities under $\mathbf{P}$ and $\rho_i^n(t)$'s are the market prices of jump risk associated with the Poisson processes sources of uncertainty $N_i^n(t)'s$.
Then the right hand side of the above equation should be equal to

$$-\hat{\sigma}^n(t,T)d\hat{W}^n(t) + \sum_{i=1}^{n}\left(e^{-\hat{\delta}_i^n(t,T)} - 1\right)d\hat{N}_i^n(t)$$

after some simplifications , we get

$$\begin{aligned}
&\left(-\hat{\alpha}^n(t,T) + \frac{1}{2}\hat{\sigma}^n(t,T)^2 + \sum_{i=1}^{n}\lambda_i^n\hat{\delta}_i^n(t,T)\right)dt \\
&= \left(\hat{\sigma}^n(t,T)\phi^n(t) - \sum_{i=1}^{n}\left(e^{-\hat{\delta}_i^n(t,T)} - 1\right)\psi_i^n(t)\right)dt.
\end{aligned}$$

Differentiating both sides with respect to T and using the definitions of $\hat{\alpha}^n(t,T)$, $\hat{\sigma}^n(t,T)$, $\hat{\delta}_i^n(t,T)$ presents

$$\alpha^n(t,T) \;=\; \sigma^n(t,T)\left(\int_t^T \sigma^n(t,s)ds - \phi^n(t)\right) + \sum_{i=1}^{n}\delta_i^n(t,T)\left[\lambda_i^n - \psi_i^n(t)e^{-\hat{\delta}_i^n(t,T)}\right] \tag{6.2.20}$$

which is the same as equation (6.2.16). □

Secondly, let's find the dynamics of $\frac{I(t)B^r(t)}{B^n(t)}$. By equation (6.2.15)

$$\frac{dI(t)}{I(t)} = \mu^I(t)dt + \sigma^I(t)dW^I(t) + \sum_{i=1}^{n}\delta_i^I(t,T)\left[dN_i^I(t,T) - \lambda_i^I dt\right].$$

In addition nominal and real money market account equations are

$$dB^n(t) = B^n(t)r^n(t)dt,$$

$$dB^r(t) = B^r(t)r^r(t)dt.$$

Then

$$d\left(\frac{B^r(t)}{B^n(t)}\right) = \frac{B^r(t)}{B^n(t)}\left(r^r(t) - r^n(t)\right)dt$$

Ito's integration by parts formula presents

$$
\begin{aligned}
d\left(\frac{I(t)B^r(t)}{B^n(t)}\right) &= \frac{I(t)B^r(t)}{B^n(t)}\left(\mu^I(t)dt + \sigma^I(t)dW^I(t)\right) \\
&\quad + \frac{I(t)B^r(t)}{B^n(t)}\left(\sum_{i=1}^{n} \delta_i^I(t,T)\left[dN_i^I(t) - \lambda_i^I dt\right]\right) \\
&\quad + \frac{I(t)B^r(t)}{B^n(t)}\left(r^r(t) - r^n(t)\right)dt.
\end{aligned}
$$

By Girsanov's Theorem

$$
\begin{aligned}
\hat{W}^I(t) &= W^I(t) - \int_0^t \phi^I(s)ds, \\
\hat{N}_i^I(t) &= N_i^I(t) - \int_0^t \psi_i^I(s)ds,
\end{aligned}
$$

where $\phi^I(t)$ is the market price of diffusion risk associated with the Brownian motion sources of uncertainty $W^I(t)$, $\psi_i^I(t)$'s are the intensities under $\hat{\mathbf{P}}$ defined as

$$
\psi_i^I(t) = \lambda_i^I \rho_i^I(t),
$$

where λ_i^I's are intensities under $\mathbf{P}$ and $\rho_i^I(t)$'s are the market prices of jump risk associated with the Poisson processes sources of uncertainty $N_i^I(t)'s.$ Then the right hand side of the above equation should be equal to

$$
\sigma^I(t)d\hat{W}^I(t) + \sum_{i=1}^{n} \delta_i^I d\hat{N}_i^I(t),
$$

i.e.,

$$
\begin{aligned}
&\left(\mu^I(t) + r^r(t) - r^n(t) - \sum_{i=1}^{n} \delta_i^I(t,T)\lambda_i^I\right)dt + \sigma^I(t)dW^I(t) \\
&+ \sum_{i=1}^{n} \delta_i^I(t,T)dN_i^I(t) = \sigma^I(t)dW^I(t) - \sigma^I(t)\phi^I(t)dt \\
&+ \sum_{i=1}^{n} \delta_i^I(t,T)dN_i^I(t) - \sum_{i=1}^{n} \delta_i^I(t,T)\psi_i^I dt.
\end{aligned}
$$

After some simplifications

$$\mu^I(t) + r^r(t) - r^n(t) - \sum_{i=1}^{n} \delta_i^I(t,T)\lambda_i^I = -\sigma^I(t)\phi^I(t) - \sum_{i=1}^{n} \delta_i^I(t,T)\psi_i^I(t)$$

or, equivalently,

$$\mu^I(t) = r^n(t) - r^r(t) - \sigma^I(t)\phi^I(t) + \sum_{i=1}^{n} \delta_i^I(t,T)\left(\lambda_i^I - \psi_i^I(t)\right).$$

□

The last step of the proof is to obtain equation (6.2.17). We should find the dynamics of $\frac{I(t)P^r(t,T)}{B^n(t,T)}$ first. The dynamics of I(t), $B^n(t,T)$ is obtained beforehand. By equation (6.1.8), the real zero-coupon bond price dynamics under the objective probability measure is

$$\begin{aligned}\frac{dP^r(t,T)}{P^r(t,T)} &= \left[r^r(t) - \hat{\alpha}^r(t,T) + \frac{1}{2}\hat{\sigma}^r(t,T)^2 + \sum_{i=1}^{n} \lambda_i^r \hat{\delta}_i^r(t,T)\right] dt \\ &\quad -\hat{\sigma}^r(t,T)dW^r(t) + \sum_{i=1}^{r} \left(e^{-\hat{\delta}_i^r(t,T)} - 1\right) d\hat{N}_i^r(t). \qquad (6.2.21)\end{aligned}$$

Thus, all we have to do is to apply Ito's integration by parts formula, then to make a change of measure and, finally, to use equation (6.2.18) which will give us the result. By Ito's integration by parts formula

$$d(I(t)P^r(t,T)) = I(t)dP^r(t,T) + P^r(t,T)dI(t) + d\langle I, P^r\rangle_t,$$

which is equal to the following

$$\begin{aligned} d(I(t)P^r(t,T)) &= I(t)P^r(t,T)\left(r^r(t) - \hat{\alpha}^r(t,T) + \frac{1}{2}\hat{\sigma}^r(t,T)^2\right)dt \\ &+ I(t)P^r(t,T)\left(\sum_{i=1}^{n}\lambda_i^r\hat{\delta}_i^r(t,T)\right)dt \\ &- I(t)P^r(t,T)\hat{\sigma}^r(t,T)dW^r(t) \\ &+ I(t)P^r(t,T)\sum_{i=1}^{n}\left(e^{-\hat{\delta}_i^r(t,T)} - 1\right)dN_i^r(t) \\ &+ P^r(t,T)I(t)(\mu^I(t)dt + \sigma^I(t)dW^I(t)) \\ &+ I(t)P^r(t,T)\left(\sum_{i=1}^{n}\delta_i^I(t,T)\left(dN_i^I(t) - \lambda_i^I dt\right)\right) \\ &- \left(I(t)P^r(t,T)\hat{\sigma}^r(t,T)\sigma^I(t)\rho_{rI}\right)dt \end{aligned}$$

If we apply Ito's integration by parts formula again,

$$\begin{aligned} d\left(I(t)P^r(t)B^n(t)^{-1}\right) &= d\left(I(t)P^r(t)\right)B^n(t)^{-1} + dB_n(t)^{-1}P^r(t,T)I(t) \\ &= \frac{I(t)P^r(t,T)}{B^n(t)}\left[\left(r^r(t) - \hat{\alpha}^r(t,T) + \frac{1}{2}\hat{\sigma}^r(t,T)^2\right)dt\right] \\ &+ \frac{I(t)P^r(t,T)}{B^n(t)}\left[\sum_{i=1}^{n}\lambda_i^r\hat{\delta}_i^r(t,T)dt\right] \\ &- \sum_{i=1}^{n}\left(e^{-\hat{\delta}_i^r(t,T)} - 1\right)dN_i^r(t) + \mu^I(t) + \sigma^I(t)dW^I(t) \\ &+ \sum_{i=1}^{n}\delta_i^I(t,T)\left(dN_i^I(t) - \lambda_i^I dt\right) - \hat{\sigma}^r(t,T)\sigma^I(t)\rho_{rI}dt \\ &- B^n(t)^{-1}r^n(t)I(t)P^r(t,T)dt. \end{aligned}$$

For simplicity, let $K = \dfrac{I(t)P^r(t,T)}{B^n(t)}$. Then

$$\begin{aligned}
dK &= K\left[\left(r^r(t) - \hat{\alpha}^r(t,T) + \frac{1}{2}\hat{\sigma}^r(t,T)^2 + \sum_{i=1}^{n}\lambda_i^r\hat{\delta}_i^r(t,T)\right)dt\right] \\
&\quad + K\left[\left(\mu^I(t) - \hat{\sigma}^r(t,T)\sigma^I(t)\rho_{rI} - r^n(t) - \sum_{i=1}^{n}\delta_i^I(t,T)\lambda_i^I\right)dt\right] \\
&\quad - K\left[\hat{\sigma}^r(t,T)dW^r(t) - \sigma^I(t)dW^I(t)\right] \\
&\quad + K\left[\sum_{i=1}^{n}\left[\left(e^{-\hat{\delta}_i^r(t,T)} - 1\right)dN_i^r(t) + \delta_i^I(t,T)dN_i^I(t)\right]\right].
\end{aligned}$$

By using equation (6.2.18)

$$\begin{aligned}
\frac{dK}{K} &= \left[-\sigma^I(t)\phi^I(t) - \sum_{i=1}^{n}\delta_i^I(t,T)\psi_i^I(t) - \hat{\alpha}^r(t,T) + \frac{1}{2}\hat{\sigma}^r(t,T)^2\right]dt \\
&\quad + \left[\sum_{i=1}^{n}\lambda_i^r\hat{\delta}_i^r(t,T) - \hat{\sigma}^r(t,T)\sigma^I(t)\rho_{rI}\right]dt \\
&\quad - \hat{\sigma}^r(t,T)dW^r(t) + \sigma^I(t)dW^I(t) - \sum_{i=1}^{n}\left(e^{-\hat{\delta}_i^r(t,T)} - 1\right)dN_i^r(t) \\
&\quad + \sum_{i=1}^{n}\delta_i^I(t,T)dN_i^I(t)
\end{aligned}$$

By Girsanov's Theorem

$$\begin{aligned}
d\hat{W}^r(t) &= dW^r(t) - \phi^r(t)dt, \\
d\hat{W}^I(t) &= dW^I(t) - \phi^I(t)dt, \\
d\hat{N}_i^r(t) &= dN_i^r(t) - \psi_i^r(t)dt, \\
d\hat{N}_i^I(t) &= dN_i^I(t) - \psi_i^I(t)dt.
\end{aligned}$$

Then right hand side of the above equation should be equal to

$$-\hat{\sigma}^r(t,T)d\hat{W}^r(t) + \sigma^I(t)d\hat{W}(t) - \sum_{i=1}^{n}\left(e^{-\hat{\delta}_i^r(t,T)} - 1\right)d\hat{N}_i^r(t) + \sum_{i=1}^{n}\delta_i^I(t)d\hat{N}_i^I(t).$$

After some simplifications we have the following equation

$$\begin{aligned}\hat{\alpha}^r(t,T) &= \frac{1}{2}\hat{\alpha}^r(t,T)^2 - \hat{\sigma}^r(t,T)\sigma^I(t)\rho_{rI} + \sum_{i=1}^{n}\lambda_i^r\hat{\delta}_i^r(t,T) \\ &\quad -\hat{\sigma}^r(t,T)\phi^r(t) - \sum_{i=1}^{n}\left(e^{-\hat{\delta}_i^r(t,T)} - 1\right)\psi_i^r(t).\end{aligned}$$

Differentiating the above equation with respect to T and substituting the expressions of $\hat{\alpha}^r(t,T)$, $\hat{\sigma}^r(t,T)$, $\hat{\delta}_i^r(t,T)$, the equation for the $\alpha^r(t,T)$ is found as follows:

$$\begin{aligned}\alpha^r(t,T) &= \sigma^r(t,T)\left(\int_t^T \sigma^r(t,s)ds - \sigma^I(t)\rho_{rI} - \phi^r(t)\right) \\ &\quad +\sum_{i=1}^{n}\delta_i^r(t,T)\left(\lambda_i^r + e^{-\hat{\delta}_i^r(t,T)}\psi_i^r(t)\right).\end{aligned}$$

□

In Proposition (6.2.1) arbitrage free drift conditions have been derived. By using these equations, forward rate, inflation and bond price processes under the martingale measure will be obtained in the following proposition.

Proposition 6.2.2. *The following price processes hold under the martingale measure:*

i) $$df^n(t,T) = \sigma^n(t,T)\left(\int_t^T \sigma^n(t,s)ds + d\hat{W}^n(t)\right) + \sum_{i=1}^{n}\delta_i^n(t,T)d\hat{N}_i^n(t), \tag{6.2.22}$$

ii) $$\begin{aligned}df^r(t,T) &= \sigma^r(t,T)\left(\int_t^T \sigma^r(t,s)ds - \sigma^I(t)\rho_{rI}dt + d\hat{W}^r(t)\right) \\ &\quad +\sum_{i=1}^{n}\delta_i^r(t,T)d\hat{N}_i^r(t),\end{aligned} \tag{6.2.23}$$

$$\boldsymbol{iii}) \quad \frac{dI(t)}{I(t)} = (r^n(t) - r^r(t))\, dt + \sigma^I(t) dW^I(t) + \sum_{i=1}^{n} \delta_i^I(t,T) d\hat{N}_i^I(t), \quad (6.2.24)$$

$$\boldsymbol{iv}) \quad \frac{dP^n(t,T)}{P^n(t,T)} = r^n(t)dt - \hat{\sigma}^n(t,T) d\hat{W}^n(t) + \sum_{i=1}^{n} \left(e^{-\hat{\delta}_i^n(t,T)} - 1 \right) d\hat{N}_i^n(t), \quad (6.2.25)$$

$$\begin{aligned} \boldsymbol{v}) \quad \frac{dP^{TIPS}(t,T)}{P^{TIPS}(t,T)} &= r^n(t)dt + \sigma^I(t) d\hat{W}^I(t) + \hat{\sigma}^r(t,T) d\hat{W}^r(t) \\ &\quad + \sum_{i=1}^{n} \delta_i^I(t,T) d\hat{N}_i^I(t) + \sum_{i=1}^{n} \left(e^{-\hat{\delta}_i^r(t,T)} - 1 \right) d\hat{N}_i^r(t), \end{aligned} \quad (6.2.26)$$

$$\begin{aligned} \boldsymbol{vi}) \quad \frac{dP^r(t,T)}{P^r(t,T)} &= \left(r^r(t) + \hat{\sigma}^r(t,T) \sigma^I(t) \rho_{rI} \right) dt - \hat{\sigma}^r(t,T) d\hat{W}^r(t) \\ &\quad + \sum_{i=1}^{n} \left(e^{-\hat{\delta}_i^r(t,T)} - 1 \right) d\hat{N}_i^r(t). \end{aligned} \quad (6.2.27)$$

Proof. **i**) By equation (6.2.13) we have,

$$df^n(t,T) = \alpha^n(t,T)dt + \sigma^n(t,T)dW^n(t) + \sum_{i=1}^{n} \delta_i^n(t,T) \left[dN_i^n(t) - \lambda_i^n dt \right].$$

Substituting the definition of $\alpha^n(t,T)$ from proposition 6.2.1 into above equation presents

$$\begin{aligned} df^n(t,T) &= \left(\sigma^n(t) \hat{\sigma}^n(t,T) - \sigma^n(t,T) \phi^n(t) \right) dt \\ &\quad + \left(\sum_{i=1}^{n} \delta_i^n(t,T) \left[\lambda_i^n - \psi_i^n(t) \left(e^{-\hat{\delta}_i^n(t,T)} \right) \right] \right) dt \\ &\quad + \sigma^n(t,T) dW^n(t) + \sum_{i=1}^{n} \delta_i^n(t,T) [dN_i^n(t) - \lambda_i^n dt]. \end{aligned}$$

Under the martingale measure, market price of risks $\phi^n(t)$ and $\rho_i^n(t)$ (so $\psi_i^n(t)$) are equal to zero. Thus,

$$d\hat{W}^n(t) = dW^n(t),$$

$$d\hat{N}_i^n(t) = dN_i^n(t).$$

After some simplifications

$$df^n(t,T) = \sigma^n(t,T)\left(\int_t^T \sigma^n(t,s)ds + d\hat{W}^n(t)\right) + \sum_{i=1}^n \delta_i^n(t,T)d\hat{N}_i^n(t).$$

$\square$

ii) With the same method in (i), substituting the definition of $\alpha^r(t,T)$ into the equation (6.2.14) gives

$$\begin{aligned} df^r(t,T) &= \sigma^r(t,T)\hat{\sigma}^r(t,T) - \sigma^r(t,T)\sigma^I(t)\rho_{rI} - \sigma^r(t,T)\phi^r(t) \\ &\quad + \sum_{i=1}^n \delta_i^r(t,T)\left(\lambda_i^r + e^{-\hat{\delta}_i^r(t,T)}\psi_i^r(t)\right) + \sigma^r(t,T)dW^r(t) \\ &\quad + \sum_{i=1}^n \delta_i^r(t,T)\left[dN_i^r(t) - \lambda_i^r dt\right]. \end{aligned}$$

Since under martingale measure $\phi^r(t)$, $\rho_i^r(t)$, (so $\psi_i^r(t)$) are equal to zero, i.e.

$$d\hat{W}^r(t) = dW^r(t),$$

$$d\hat{N}_i^r(t) = dN_i^r(t).$$

After some simplifications, finally we have

$$df^r(t,T) = \sigma^r(t,T)\left(\int_t^T \sigma^r(t,s)ds - \sigma^I(t)\rho_{rI}dt + d\hat{W}^r(t)\right) + \sum_{i=1}^n \delta_i^r(t,T)d\hat{N}_i^r(t).$$

$\square$

iii) If we put the definition of $\mu^I(t)$ (6.2.18) into the equation (6.2.15) we get

$$\begin{aligned} \frac{dI(t)}{I(t)} &= \left(r^n(t) - r^r(t) - \sigma^I(t)\phi^I(t) + \sum_{i=1}^n \delta_i^I(t,T)\left(\lambda_i^I - \psi_i^I(t)\right)\right)dt \\ &\quad + \sum_{i=1}^n \delta_i^I(t,T)\left[dN_i^I(t) - \lambda_i^I dt\right] + \sigma^I(t)dW^I(t) \end{aligned}$$

where $\phi^I(t)$ and $\rho_i^I(t)$ are equal to zero under risk neutral measure. Thus,

$$\begin{aligned} d\hat{W}^I(t) &= dW^I(t), \\ d\hat{N}_i^I(t) &= dN_i^I(t). \end{aligned}$$

After some small algebra, inflation index process under martingale measure is obtained as follows:

$$\frac{dI(t)}{I(t)} = (r^n(t) - r^r(t))\, dt + \sigma^I(t) d\hat{W}^I(t) + \sum_{i=1}^{n} \delta_i^I(t,T) d\hat{N}_i^I(t).$$

□

iv) The nominal zero coupon bond price equation has been derived as (6.2.19), also by proposition (6.2.1), the definition of $\hat{\alpha}^n(t,T)$ is as follows:

$$\alpha^n(t,T) = \sigma^n(t,T)\left(\hat{\sigma}^n(t,T) - \phi^n(t)\right) + \sum_{i=1}^{n} \delta_i^n(t,T)\left[\lambda_i^n - \psi_i^n(t) e^{-\hat{\delta}_i^n(t,T)}\right].$$

Integrating both sides from t to T gives

$$\hat{\alpha}^n(t,T) = \frac{1}{2}\hat{\sigma}^n(t,T)^2 + \sum_{i=1}^{n} \hat{\delta}_i^n(t,T)\lambda_i^n - \sum_{i=1}^{n} \psi_i^n(t) e^{\hat{\delta}_i^n(t,T)} - \hat{\sigma}^n(t,T)\phi^n(t).$$

By substituting this equation into the nominal bond price equation, we have

$$\begin{aligned} \frac{dP^n(t,T)}{P^n(t,T)} &= \left(r^n(t) - \frac{1}{2}\hat{\sigma}^n(t,T)^2 - \sum_{i=1}^{n} \hat{\delta}_i^n(t,T)\lambda_i^n\right) dt \\ &\quad + \left(\sum_{i=1}^{n} \psi_i^n(t) e^{-\hat{\delta}_i^n(t,T)} + \frac{1}{2}\hat{\sigma}^n(t,T)^2 + \sum_{i=1}^{n} \lambda_i^n \hat{\delta}_i^n(t,T)\right) dt \\ &\quad - \hat{\sigma}^n(t,T) dW^n(t) + \sum_{i=1}^{n} \left(e^{-\hat{\delta}_i^n(t,T)} - 1\right) dN_i^n(t). \end{aligned}$$

Under martingale measure $\phi^n(t)$ and $\rho_i^n(t)$ are equal to zero, then some simplifications reach the following result:

$$\frac{dP^n(t,T)}{P^n(t,T)} = r^n(t)dt - \hat{\sigma}^n(t,T) d\hat{W}^n(t) + \sum_{i=1}^{n} \left(e^{-\hat{\delta}_i^n(t,T)} - 1\right) d\hat{N}_i^n(t).$$

$\square$

vi) The real zero coupon bond price equation has been derived beforehand as (6.2.21). From Proposition (6.2.1)

$$\begin{aligned} \alpha^r(t,T) &= \sigma^r(t,T)\hat{\sigma}^r(t,T) - \sigma^r(t,T)\sigma^I(t)\rho_{rI} - \sigma^n(t,T)\phi^r(t) \\ &+ \sum_{i=1}^{n} \delta_i^r(t,T)\left(\lambda_i^r + e^{-\hat{\delta}_i^r(t,T)}\psi_i^r(t)\right). \end{aligned}$$

If we integrate both sides of the above equation from t to T,

$$\begin{aligned} \hat{\alpha}^r(t,T) &= \frac{1}{2}\hat{\sigma}^r(t,T)^2 - \hat{\sigma}^r(t,T)\sigma^I(t)\rho_{rI} - \hat{\sigma}^r(t,T)\phi^r(t) \\ &+ \sum_{i=1}^{n} \hat{\delta}_i^r(t,T)\lambda_i^r - \sum_{i=1}^{n} \psi_i^r(t)e^{-\hat{\delta}_i^r(t,T)}. \end{aligned}$$

Under the martingale measure $\phi^r(t)$ and $\rho_i^r(t)$ are equal to zero. Using the expression $\hat{\alpha}^r(t,T)$ in equation (6.2.21), the final result is obtained after some simplifications:

$$\begin{aligned} \frac{dP^r(t,T)}{P^r(t,T)} &= \left(r^r(t) - \sigma^r(t,T)\sigma^I(t)\rho_{rI}\right)dt - \hat{\sigma}^r(t,T)d\hat{W}^r(t) \\ &+ \sum_{i=1}^{n}\left(e^{-\hat{\delta}_i^r(t,T)} - 1\right)d\hat{N}_i^r(t). \end{aligned}$$

$\square$

v)By equation (5.0.6)

$$P^{TIPS}(t,T) = I(t)P^r(t,T).$$

Then by the Ito's integration by parts formula

$$\begin{aligned} dP^{TIPS}(t,T) &= d\left(I(t)P^r(t,T)\right) \\ &= d(I(t))P^r(t,T) + I(t)dP^r(t,T) + d\langle I, P^r\rangle_t, \end{aligned}$$

which is equal to

$$
\begin{aligned}
dP^{TIPS}(t,T) &= I(t)P^r(t,T)\left[\left(r^n(t)-r^r(t)\right)dt+\sigma^I(t)d\hat{W}^I(t)\right] \\
&+I(t)P^r(t,T)\left[\sum_{i=1}^{n}\delta_i^I(t,T)d\hat{N}_i^I(t)\right] \\
&+I(t)P^r(t,T)\left[\left(r^r(t)-\hat{\sigma}^r(t,T)\sigma^I(t)\rho_{rI}\right)dt\right] \\
&+I(t)P^r(t,T)\left[-\hat{\sigma}^r(t,T)d\hat{W}^r(t)+\sum_{i=1}^{n}\left(e^{-\hat{\delta}_i^r(t,T)}-1\right)d\hat{N}_i^r(t)\right] \\
&+\sigma^I(t)\hat{\sigma}^r(t,T)\rho_{rI}dt.
\end{aligned}
$$

Under the martingale measure $\phi^r(t)$, $\rho_i^r(t)$, $\phi^I(t)$ and $\rho_I^r(t)$ are equal to zero. After some simplifications:

$$
\begin{aligned}
\frac{dP^{TIPS}(t,T)}{P^{TIPS}(t,T)} &= r^n(t)dt+\sigma^I(t)d\hat{W}^I(t)-\hat{\sigma}^r(t,T)d\hat{W}^r(t) \\
&+\sum_{i=1}^{n}\delta_i^I(t,T)d\hat{N}_i^I(t)+\sum_{i=1}^{n}\left(e^{-\delta_i^r(t,T)}-1\right)d\hat{N}_i^r(t).
\end{aligned}
$$

□

CHAPTER 7

PRICING EUROPEAN CALL OPTION ON THE INFLATION INDEX

In this chapter, a pricing formula for a European call option on an inflation index is derived, based on Jarrow-Yıldırım [31] model. At time T, the price of a European call option on inflation index is

$$C_T = \max[I(T) - K, 0]. \tag{7.0.1}$$

Each unit of the option is written on one CPI unit. This means $I(T)$ behaves like the nominal value of the one unit of CPI at time T. We know that real zero coupon bond pays 1 unit of CPI at maturity. Thus we can think of the payoff of the above option as

$$C_T = \max[P^r(T,T)I(T) - K, 0]. \tag{7.0.2}$$

Let

$$Z(t) = \frac{P^r(t,T)I(t)}{P^n(t,T)}.$$

By applying Ito's integration by parts formula, the dynamics of $Z(t)$ under risk neutral measure $\hat{\mathbf{P}}$ can easily be obtained. By equation (5.0.3), (5.0.10), (5.0.17),

$$\frac{dI(t)}{I(t)} = [r^n(t) - r^r(t)]dt + \sigma^I(t)d\hat{W}^I(t), \tag{7.0.3}$$

$$\frac{dP^n(t,T)}{P^n(t,T)} = r^n(t)dt - \hat{\sigma}^n(t,T)d\hat{W}^n(t), \tag{7.0.4}$$

$$\frac{dP^r(t,T)}{P^r(t,T)} = [r^r(t) - \rho_{rI}\sigma^I(t)\hat{\sigma}^r(t,T)]dt - \hat{\sigma}^r(t,T)d\hat{W}^r(t). \tag{7.0.5}$$

Firstly, we will find the dynamics of $\frac{1}{P^n(t,T)}$. Let us apply Ito's formula with $f(x) = \frac{1}{x}$. Then,

$$d(\frac{1}{P^n(t,T)}) \quad = \quad \frac{1}{P^n(t,T)}[(-r^n(t) - \hat{\sigma}^n(t,T)^2)dt + \hat{\sigma}^n(t,T)d\hat{W}^n(t)]. \tag{7.0.6}$$

Secondly, let's obtain the dynamics of $P^r(t,T)I(t)$ by Ito's integration by parts formula,

$$\begin{aligned} d(P^r(t,T)I(t)) &= P^r(t,T)I(t)[(r^n(t) - r^r(t))dt + \sigma^I(t)d\hat{W}^I(t)] \\ &\quad +I(t)P^r(t,T)[(r^r(t) - \rho_{rI}\sigma^I(t)\hat{\sigma}^r(t,T))dt \\ &\quad -\hat{\sigma}^r(t,T)d\hat{W}^r(t)] + \rho_{rI}\hat{\sigma}^r(t,T)\sigma^I(t))dt. \end{aligned}$$

After some simplifications

$$d(P^r(t,T)I(t)) \quad = \quad P^r(t,T)I(t)[(r^n(t)dt + \sigma^I(t)d\hat{W}^I(t) - \hat{\sigma}^r(t,T)d\hat{W}^r(t)]. \tag{7.0.7}$$

Finally, let's apply Ito's integration by parts formula to obtain the dynamics of $Z(t)$,

$$\begin{aligned} d(\frac{I(t)P^r(t,T)}{P^n(t,T)}) &= \frac{P^r(t,T)I(t)}{P^n(t,T)}[(r^n(t)dt + \sigma^I(t)dW^I(t) - \hat{\sigma}^I(t)d\hat{W}^I(t)] \\ &\quad +\frac{P^r(t,T)I(t)}{P^n(t,T)}[(-r^n(t) - \hat{\sigma}^n(t,T)^2)dt + \hat{\sigma}^n(t,T)d\hat{W}^n(t)] \\ &\quad +\frac{P^r(t,T)I(t)}{P^n(t,T)}[\rho_{nI}\sigma^I(t)\hat{\sigma}^n(t,T) - \rho_{rI}\hat{\sigma}^I(t,T)\hat{\sigma}^n(t,T)]dt. \end{aligned}$$

Then, after some small algebra

$$
\begin{aligned}
d(Z(t)) \quad &= \quad Z(t)[(\hat{\sigma}^n(t,T)^2 + \rho_{nI}\sigma^I(t)\hat{\sigma}^n(t,T) - \rho_{rI}\hat{\sigma}^I(t,T)\hat{\sigma}^n(t,T))dt \\
&\quad + \hat{\sigma}^n(t,T)d\hat{W}^n(t) - \hat{\sigma}^r(t,T)d\hat{W}^r(t) + \sigma^I(t)dW^I(t)). \qquad (7.0.8)
\end{aligned}
$$

The $\hat{\mathbf{P}}$ - dynamics of $Z(t)$ is obtained in the above discussion. Also since Z is an asset price, normalized by the nominal price of a T-bond, it has zero drift (see Björk [4]) and under $\hat{\mathbf{P}}^T$(nominal T-forward measure), its $\hat{\mathbf{P}}^T$ dynamics are given by

$$
dZ(t) = Z(t)\sigma^Z(t)d\hat{W}^T(t), \qquad (7.0.9)
$$

where $\sigma^Z(t)$ is deterministic. This is basically a geometric Brownian motion, driven by multidimensional Brownian motion and the solution is given by

$$
Z(T) = Z(t)\exp(-\frac{1}{2}\int_t^T \|\sigma^Z(s)\|^2 ds + \int_t^T \sigma^Z(s)d\hat{W}^T(s)). \qquad (7.0.10)
$$

Then the stochastic integral in the above equation has a zero mean and variance

$$
\varepsilon^2(T) = \int_t^T \|\sigma^Z(s)\|^2 ds
$$

and it is known that a volatility process is not affected by change of measure. Therefore in the $\hat{\mathbf{P}}^T$ dynamics of $Z(t)$, we have

$$
\sigma^Z(t) = (\hat{\sigma}^n(t,T), -\hat{\sigma}^r(t,T), \hat{\sigma}^I(t)).
$$

So we know that under $\hat{\mathbf{P}}^T$

$$
lnZ(t) \in \mathbf{N}(lnZ(t) - \frac{1}{2}\varepsilon^2, \varepsilon^2),
$$

where

$$\begin{aligned}
\varepsilon^2 &= \int_t^T \|\sigma^n(s) - \sigma^r(s) + \sigma^I(s)\|^2 ds \\
&= \int_t^T \hat{\sigma}^n(s,T)^2 ds - 2\int_t^T \rho_{nr}\hat{\sigma}^n(s,T)\hat{\sigma}^r(s,T)ds \\
&\quad + \int_t^T \hat{\sigma}^r(s,T)^2 ds + 2\rho_{nI}\sigma^I(t)\int_t^T \hat{\sigma}^n(s,T)ds \\
&\quad -2\rho_{rI}\sigma^I(t)\int_t^T \hat{\sigma}^r(s,T)ds + (\sigma^I(t))^2(T-t). \qquad (7.0.11)
\end{aligned}$$

Now let's return to our main problem.

$$C_T = \max[P^r(T,T)I(T) - K]_+ = [Z(T) - K]_+$$

then under nominal T-forward measure $\hat{\mathbf{P}}^T$, the value at time t is

$$C_t \;=\; E_t^{\hat{\mathbf{P}}^T}[(Z(T) - K)_+ \, e^{-\int_t^T r^n(s)ds}].$$

Then by using equation (7.0.10)

$$C_t \;=\; E_t^{\hat{\mathbf{P}}^T}[(Z(t)\, e^{-\frac{1}{2}\int_t^T \|\sigma^Z(s)\|^2 ds + \int_t^T \sigma^Z(s)d\hat{W}^T(s)} - K)_+ \, e^{-\int_t^T r^n(s)ds}],$$

which is equal in distribution to the right hand side of the following formula:

$$C_t =^d E_t^{\hat{\mathbf{P}}^T}[(Z(t)\, e^{-\frac{1}{2}\int_t^T \|\sigma^Z(s)\|^2 ds + \varepsilon Y} - K)_+ \, e^{-\int_t^T r^n(s)ds}],$$

where $Y \in N(0,1)$. Now let's find the region where the above $\hat{\mathbf{P}}^T$- expectation is defined:

$$Z(t)e^{-\frac{1}{2}\int_t^T \|\sigma^Z(s)\|^2 ds + \varepsilon Y} > K.$$

By taking the logarithm of both sides of the above inequality we get

$$\log Z(t) - \frac{1}{2}\int_t^T \|\sigma^Z(s)\|^2 ds + \varepsilon Y > \log K,$$

then

$$\log\frac{Z(t)}{K}-\frac{1}{2}\varepsilon^2+\varepsilon Y>0.$$

Thus

$$Y>\frac{\log\frac{Z(t)}{K}-\frac{1}{2}\varepsilon^2}{\varepsilon}=-d_2.$$

Hence

$$\begin{aligned}C_t &=^d \; E_t^{\hat{\mathbf{P}}^T}[(Z(t)\;e^{-\frac{1}{2}\varepsilon^2+\varepsilon Y}-K)1_{(Y+d_2>0)}\;e^{-\int_t^T r^n(s)ds}]\\ &=^d \; E_t^{\hat{\mathbf{P}}^T}[(Z(t)\;e^{-\frac{1}{2}\varepsilon^2+\varepsilon Y-\int_t^T r^n(s)ds}-K)\;e^{-\int_t^T r^n(s)ds}1_{(Y>-d_2)}].\end{aligned}$$

Let us write the above $\hat{\mathbf{P}}^T$- expectation in Riemann integral form;

$$C_t=\int_{-d2}^{\infty}(Z(t)\;e^{-\frac{1}{2}\varepsilon^2+\varepsilon Y-\int_t^T r^n(s)ds}-K\;e^{-\int_t^T r^n(s)ds})\;\frac{e^{-\frac{Y^2}{2}}}{\sqrt{2\Pi}}\;dy.$$

By a small change of variable procedure, we can change the bounds of the above integral and we get

$$C_t=\int_{-\infty}^{d_2}(Z(t)\;e^{-\frac{1}{2}\varepsilon^2+\varepsilon Y-\int_t^T r^n(s)ds}-K\;e^{-\int_t^T r^n(s)ds})\;\frac{e^{-\frac{Y^2}{2}}}{\sqrt{2\Pi}}\;dy.$$

Let us define

$$I_1=\int_{-\infty}^{d_2}\frac{Z(t)}{\sqrt{2\Pi}}\;e^{-\frac{Y^2}{2}}e^{-\frac{1}{2}\varepsilon^2+\varepsilon Y-\int_t^T r^n(s)ds}dy$$

and

$$I_2=K\;e^{-\int_t^T r^n(s)ds}\;N(d_2).$$

Then the price of the option at time t is

$$C_t=I_1-I_2. \tag{7.0.12}$$

Since $e^{-\int_t^T r^n(s)ds}=P^n(t,T)$,

$$I_2=KP^n(t,T)N(d_2)$$

where N denotes a normal distribution. To find I_1, let us define $Y' = Y + \varepsilon$. Then $dY = dY'$. By using this transformation we obtain

$$I_1 = \frac{Z(t)}{\sqrt{2\Pi}}\ e^{-\int_t^T r^n(s)ds} \int_{-\infty}^{d_2+\varepsilon} e^{-\frac{(Y'-\varepsilon)^2}{2} - \frac{1}{2}\varepsilon^2 + \varepsilon(Y'-\varepsilon)} dy'.$$

Substituting the definition of $Z(t)$ into above equation and after some simplifications we have

$$I_1 = \frac{I(t)P^r(t,T)}{\sqrt{2\Pi}} \int_{-\infty}^{d_2+\varepsilon} e^{-\frac{Y'^2}{2}} dy'.$$

Defining $d_1 = d_2 + \varepsilon$ gives

$$I_1 = P^r(t,T)I(t)N(d_1).$$

Finally by substituting the definitions of I_1 and I_2 into the equation (7.0.12) we end up with

$$\begin{aligned} C_t &= I(t)P^r(t,T)N\left(\frac{\log \frac{I(t)P^r(t,T)}{KP^n(t,T)} + \frac{1}{2}\varepsilon^2}{\varepsilon}\right) \\ &\quad -KP^n(t,T)N\left(\frac{\log \frac{I(t)P^r(t,T)}{KP^n(t,T)} + \frac{1}{2}\varepsilon^2}{\varepsilon}\right), \end{aligned} \tag{7.0.13}$$

where

$$\begin{aligned} \varepsilon^2 &= \int_t^T \|\sigma^n(s) - \sigma^r(s) + \sigma^I(s)\|^2 ds \\ &= \int_t^T \hat{\sigma}^n(s,T)^2 ds - 2\int_t^T \rho_{nr}\hat{\sigma}^n(s,T)\hat{\sigma}^r(s,T)ds \\ &\quad + \int_t^T \hat{\sigma}^r(s,T)^2 ds + 2\rho_{nI}\sigma^I(t)\int_t^T \hat{\sigma}^n(s,T)ds \\ &\quad -2\rho_{rI}\sigma^I(t)\int_t^T \hat{\sigma}^r(s,T)ds + (\sigma^I(t))^2(T-t). \end{aligned}$$

CHAPTER 8

CONCLUSION

Inflation indexed instruments has become increasingly popular in financial markets during the last decade. Therefore, importance of the pricing of these instruments has increased significantly. In the finance literature, there exist certain works on this subject. However when we look at the Turkish financial literature, rare studies stand out. To close this gap, we focus on pricing of inflation indexed bonds which are the unique inflation-indexed instruments traded in the Turkish bond market.

Firstly, we analysed the history of indexation and the existing literature on inflation indexed instruments. Then, we reviewed the HJM-framework and the Jarrow-Yıldırım model in detail. After that we proposed extensions of the HJM-framework [22] and the Jarrow-Yıldırım [31] models within the framework of Shirakawa [43]. Our models differ from the Jarrow-Yıldırım model in that, the instantaneous forward rates, inflation index and bond price processes are driven by both the standard Brownian motion and the finite number of Poisson noises. Volatility functions of these processes are both time-deterministic. We assumed there exists a sufficient number of traded bonds to hedge away all of the jump risks in the market that ensures market completeness. Finally, we derived a closed-form pricing formula for the European call option on the inflation index.

Our future reseach will be on pricing of the inflation indexed swaps, swaptions, caps and floors which are relatively more liquid inflation indexed instruments in financial markets. Another idea we may use in our future research is that, the instantaneous forward rates, inflation index and bond price processes may have stochastic volatility functions.

REFERENCES

[1] S. Altay. On forward interest rate models: Via random fields and markov jump processes. *Master of Science Thesis, METU*, 2007.

[2] E. Balaban. Türkiye'de enflasyona endeksli kamu menkul kıymetlerinin çıkarılmasına ilişkin bir çalışma. *TCMB Araştırma Genel Müdürlüğü.* 97/2, Ankara.

[3] N. Belgrade, E. Benhamou and E. Koehler. A market model for inflation. *CDC IXIS*, 2004.

[4] T. Björk. *Arbitrage Theory in Continuous Time.* Oxford University Press, 1998.

[5] T. Björk, Y. Kabanov and W. Runggaldier. Bond markets where prices are driven by a general marked point process. *Working Paper, Stockholm School of Economics*, 67, 1995.

[6] T. Björk, Y. Kabanov and W. Runggaldier. Bond market structure in the presence of marked point processes. *Mathematical Finance*, 7(2): 211-239, 1997.

[7] D. Brigo, F. Mercurio. *Interest Rate Models, Theory and Practice with Smile, Inflation and Credit.* Springer-Finance, Second Edition, 2006.

[8] A. N. Burnetas, P. Ritchken. On rational jump-diffusion models in the Flesaker-Hughston paradigm. *Working Paper, Case Western Reserve University*, 1996.

[9] M. Chamon, P. Mauro. Pricing Growth-Indexed Bonds. *IMF Working Paper*, WP/05/216, 2005.

[10] C. Chiarella, C. N. Sklibosios. A class of jump-diffusion bond pricing models within the HJM-Framework. *Asia-Pacific Financial Markets*, 10: 87-127, 2003.

[11] R. Cont, P. Tankov. *Financial Modelling with Jump Processes.* Word Scientific, 2000.

[12] S. R. Das. Discrete time bond and option pricing for jump diffusion processes. *Review of Derivatives Research*, 1: 211 -244, 1997.

[13] S. R. Das. The surprise element: Jumps in the interest rates. *Journal of Econometrics*, 106: 27-65, 2002.

[14] S. R. Das, S. Foresi. Exact solutions for bond and option prices with systematic jump risk. *Review of Derivatives Research*, 1: 7 -24, 1996.

[15] M. Deacon, A. Derry and D. Mirfendereski. *Inflation-Indexed Securities, Bonds, Swaps and other Derivatives.* Wiley-Finance, 2004.

[16] M. Dodgson, D. Kainth. *Inflation-linked derivatives.* Royal Bank of Scotland Risk Training Course, Market Risk Group, 2006.

[17] J. A. Garcia, A. V. Rixtel. *Inflation-linked bonds from a central bank respective.* Bank of Spain, 0705, 2007.

[18] P. Glasserman, S. G. Kou. The term structure of simple forward rates with jump risk. *Mathematical Finance*, 13(3): 383-410 , 2003.

[19] Hazine Müsteşarlığı Kamu Borç Yönetimi Raporu. No.30, Ocak 2008, Ankara.

[20] Hazine Müsteşarlığı Kamu Borç Yönetimi Raporu. No.32, Mart 2008, Ankara.

[21] Hazine Müsteşarlığı Kamu Borç Yönetimi Raporu. No.35, Haziran 2008, Ankara.

[22] D. Heath, R. Jarrow and A. Morton. *Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation,* Econometrica, 60(1): 77-105, 1992.

[23] M. Henrard. *TIPS Options in the Jarrow-Yıldırım model.* Munich Personel RePEc Archive, 1423, 2006.

[24] S. L. Heston. A closed-form solution for optins with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies*, 6 (2): 327-343, 1993.

[25] M. Hinnerich. On the pricing of swaps and swaptions. *Licentiate Thesis in Department of Finance, Stocholm School of Economics*, 2006.

[26] T. S. Y. Ho, S. Lee. Term structure movements and the pricing of interest rate contingent claims. *The Journal of Finance*, 41: 1011-1029, 1986.

[27] L. P. Hughston. Inflation derivatives. *Working Paper, Merrill Lynch*, 1998.

[28] L. P. Hughston, A. Macrina. Information, inflation and interest. *Working Paper, King's College, London*, 2007.

[29] M. Hurd, J. Rellen. *New information from inflation swaps and index linked bonds*, Bank of England Quarterly Bulletin, 2006.

[30] R. A. Jarrow, D. B. Madan. Option pricing using the term structure of interest rates to hedge systematic discontinuities in asset returns. *Working Paper, Cornell University*, 1991.

[31] R. Jarrow, Y. Yıldırım. Pricing Treasury inflation protected securities and related derivatives using an HJM model, *Journal of Financial and Quantitative Analysis*, 38(2): 337-357,2003.

[32] Y. Kitamura. Information content of inflation indexed bond prices: Evaluation of US Treasury inflation protected securities. *Monetary and Economics Studies*, 2004.

[33] R. C. Kimball, R. W. Kapcke. Inflation indexed bonds: The dog that didn't bark. *New England Economic Review*. January- February, 1999.

[34] D. Lamberton, B. Lapeyre. *Introduction to Stochastic Calculus Applied to Finance*. Chapman & Hall/CRC, 2000.

[35] F. Mercurio. Pricing inflation-indexed derivatives. *Quantitative Finance*, 5(3): 289-302, 2005.

[36] F. Mercurio, N. Moreni. Pricing inflation-indexed option with stochastic volatility. August 2005.

[37] B. Oksendal. *Stochastic Differential Equations: An Introduction with Applications*. Springer, 1998.

[38] R. Roll. Emprical TIPS. *Financial Analysis Journal*, January/ February : 31-53, 2004.

[39] W. Runggaldier. Jump-diffusion models. *Lecture Notes University of Padova*, 2002.

[40] B. Sack, R. Elsasser. Treasury inflation-indexed debt: A review of the US experience. *FRBNY Economics Review , Third Quarter, Federal Reserve of Kansas City*, 2004.

[41] P. Shen. Benefits and limitations of inflation indexed Treasury bonds. *Economic Review , Third Quarter, Federal Reserve of Kansas City*, 41-56, 1995.

[42] P. Shen, J. Corning. Can TIPS help identify long-term inflation expectations. *Economic Review , Fourth Quarter, Federal Reserve of Kansas City*, 61-87, 2001.

[43] H. Shirakawa. Interest rate option pricing with Poisson-Gaussian forward rate curve processes. *Mathematical Finance*, 1: 77-94, 1991.

[44] S. Shreve. *Stochastic Calculus for Finance II* . Springer, 2004.

[45] A. Stewart. Pricing inflation-indexed derivatives using the extended Vasicek model of Hull and White. *Master Thesis in Mathematical Finance, Exeter College, University of Oxford*, 2007.

[46] N. Taylor. US inflation-indexed bonds in the long run: A hypothetical view. *Applied Financial Economics*, 10: 667-677, 2000.

[47] E. Tekmen. Enflasyona endeksli tahviller ve Türkiye uygulaması. *Hazine Müsteşarlığı Kamu Finansmanı Genel Müdürlüğü Uzmanlık Tezi.* 2005, Ankara

[48] D. M. Wilcox. Policy watch the introduction of indexed government debt in the United States. *Journal of Economic Perspectives*, 12(1): 219-227, 1998.

[49] J. M. Wrase. Inflation-indexed bonds: How do they work?. *Business Review, July-August, Federal Reserve of Philadelphia* , 3-16, 1997.

[50] www.barclayscapital.com

[51] Y. Yolcu. One factor interest rate models: Analytic solutions and approximations. *Master of Science Thesis, METU*, 2005.