CREDIT RISK MODELING AND CREDIT DEFAULT SWAP PRICING UNDER VARIANCE GAMMA PROCESS

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CREDIT RISK MODELING AND CREDIT DEFAULT SWAP PRICING UNDER VARIANCE GAMMA PROCESS

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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Abstract

CREDIT RISK MODELING AND CREDIT DEFAULT SWAP PRICING UNDER VARIANCE GAMMA PROCESS

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In this thesis, the structural model in credit risk and the credit derivatives is studied under both Black-Scholes setting and Variance Gamma (VG) setting. Using a Variance Gamma process, the distribution of the firm value process becomes asymmetric and leptokurtic. Also, the jump structure of VG processes allows random default times of the reference entities. Among structural models, the most emphasis is made on the Black-Cox model by building a relation between the survival probabilities of the Black-Cox model and the value of a binary down and out barrier option. The survival probabilities under VG setting are calculated via a Partial Integro Differential Equation (PIDE). Some applications of binary down and out barrier options, default probabilities and Credit Default Swap par spreads are also illustrated in this study.

Key words: Lévy process, Variance Gamma process, credit risk, structural model, survival probability, credit default swap, barrier option, Partial Integro Differential Equation.

Öz

VARYANS GAMA SÜRECİ ALTINDA KREDİ RİSKİ MODELLEMESİ VE KREDİ TEMERRÜT TAKASI FİYATLAMASI

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Bu tezde, kredi riskindeki yapısal modeller ve kredi türevleri hem Black-Scholes hem de Varyans Gama modelleri altında çalışılmıştır. Varyans Gama sürecinin kullanılması, firma değerinin dağılımının sağa çarpık ve sivri tepeli olmasını sağlamıştır. Ayrıca VG sürecinin sıçramalı yapısı, firmaların rastgele temerrüte düşmelerine izin vermiştir. Yapısal modeller arasında, Black-Cox model altında temerrüte düşmeme (yaşama) olasılığı ve dijital, aşağı ve iptal bariyer opsiyonu fiyatlaması arasında ilişki kurularak, en fazla önem Black-Cox modeline verilmiştir. VG altında yaşama olasılıkları, Kısmi İntegral Diferansiyel Denklemi (PIDE) yardımıyla hesaplanmıştır. Dijital, aşağı ve iptal bariyer opsiyonlar, temerrüt olasılıkları ve kredi temerrüt takaslarının bazı uygulamaları gösterilmiştir.

Anahtar Kelimeler: Lévy süreçleri, Varyans Gama süreci, kredi riski, yapısal model, yaşama olasılığı, kredi temerrüt takası, bariyer opsiyonu, Kısmi İntegral Diferansiyel Denklem.

To my family

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Chapter 1

Introduction

People in the finance sector know that risk is a reality of financial markets. All risks in financial markets are called financial risks. Among the different types of financial risks, the *credit risk* is of a great interest. Credit risk is the risk of any loss caused by credit-linked events such as changes in credit quality, variations in credit spread and the default event. All three events affect the ability of the counterparties in a financial contract to meet their obligations. For example, there can be changes in the credit quality of the counterparty, that is, an increase or a decrease in its credit ratings. In this case, the counterparty may default and cannot meet his/her obligations stated in the contract. Or the credit spread over the risk-neutral interest rate can become large, and the counterparty faces a more risky situation. Among these, modeling the default risk, that is, modeling of random time of the default event is generally the one that has been mostly researched. This is also what we will study in this work. We will model the default time of a company or a firm, called *reference entity*, which issues a bond or a loan, called *reference obligation*.

In the credit risk literature, there are mainly two types of models for the default times, *the structural (firm-valued) models* and *the reduced form (intesity based) models*. The *recovery rates*, the amount of money that is paid to the party exposed to risk in case of default, is also defined in each model.

Structural models are mainly concerned with the default risk of a reference entity. They give a model for the default time and then price the reference obligations of the firm. In such models, the default event is modeled according to whether or not the level of the firm value is below some barrier (threshold) which can either be random or non-random. This is why structural models are also regarded as firm-valued models. Hence, in structural models, the value of the firm is also investigated. Generally, the value of the firm is the total value of all its assets and is split into two parts: *equities* (shares) and *liabilities* (debts), that is,

$$V = E + D.$$

The rationale behind structural models is the following. If the asset value V is equal to or below the firm's total value of liabilities, then the firm's capital (equity) falls below zero. Thus the firm goes bankrupt and defaults. By using this rationale, Merton [16] modeled the default time of the firm as the time of bankruptcy and, by writing the debts and equities as the contingent claims on the asset value, he priced liabilities of the firm. After Merton, there have been many extentions, each of which improves the shortcomings of the Merton model. Merton assumes that the firm defaults only at maturity of the reference entity. Black and Cox [3] extended this assumption by allowing default before maturity and introduced first-passage-time models. Another assumption by Merton was to assume that asset values are modeled by a Geometric Brownian Motion (GBM). Zhou [22] modeled the firm's value as a jump-diffusion process. Introducing jumps in asset models allows the default to be an unexpected event which is not the case in GBM. Assumption about constant interest rate was improved by Shimko [20] who modeled the short term interest rate as the Vasicek Model.

In structural models, since the default time is modeled as the time when the firm's assets goes below a predetermined barrier, the default time is predictable. This was not realistic. In the reduced form models, the default time is modeled by the jump time of a jump process, so it is unpredictable and arrives as a surprise. A main issue in the reduced form models is the conditional probability of the default given no earlier default, and so is its modeling. This probability is modeled by an *intensity process* which is also called the *hazard rate process*. The preceeding papers about reduced form credit models are those of Jarrow and Turnbull [11], and Duffie and Singleton [6].

People, financial institutions and even countries are exposed to the credit risk of the counterparties so that they try to hedge themself against the risk. Due to this hedging problem, credit derivatives stepped in and have exponentially grown since then, especially in recent years. A credit derivative is an instrument which helps to trade the credit risk exposured. Among credit derivatives, the best known is Credit Default Swap (CDS). Although it is called a swap, a CDS is a kind of insurance contract. The party who is exposed to credit risk by holding a reference entity insures himself against the risk by buying a CDS contract. In a typical CDS contract, the buyer makes periodic predetermined payments to the seller until a default event occurs or until the maturity of the CDS, in case of no default. On the other hand, the seller guarantees to compensate the loss of the buyer in case of default event. This compensation can be in two forms: *physical delivery (settlement)* or *cash settlement*. If a default event occurs, the buyer delivers the reference entity to the seller and the seller pays the notional amount of it, considered as the physical settlement; or the seller pays the loss with rate $(1 - R)$ to the buyer, which is the so-called cash settlement. In a CDS contract, whether the payments will be made or not depends on the occurrence of the default. This means that the buyer makes the payments until the default event if it happens or until the maturity of CDS, and the seller compensates the loss if default happens and does not compensate otherwise. So, in short, we generally talk about expected payments (EP) and expected losses (EL). A CDS is priced by equating these EP and EL. From this equation, the amount of payment the buyer will make is looked for. This amount is defined as the par spread.

In this work, we give the modeling of the credit risk under the Black-Cox model by modeling asset values as a Lévy process. The original paper is that of Cariboni-

Shoutens [4]. As in there, we modeled the survival probability of a reference entity by assuming that the asset prices are defined via a Variance Gamma (VG) process. Then we priced the CDS with these survival probabilities. The Black-Cox model is a kind of first-passage-time model under which the default event happens when asset values fall below a predetermined barrier. Hence there is a connection between this type of model and a barrier option. So the survival probability under the Black-Cox model can be given in terms of a barrier option. Since asset values are modeled as a jump process in this work, the barrier option price is given as a solution of the Partial-Integro-Differential Equation (PIDE). Namely, the pricing of CDS is given via the solution of a related PIDE.

The outline of this work is as follows: After a short introduction in Chapter 1, the general properties of the Lévy processes are analysed in Chapter 2. Then in Chapter 3, the Variance Gamma (VG) process is defined and the asset price dynamics with VG processes is given in detail. Chapter 4 is devoted to credit risk modeling and credit derivative pricing under both the Black Scholes and the VG models. In this chapter, Barrier options are also given, due to the fact that the survival probability under the Black-Cox model is linked to the price of a barrier option. For this purpose, the pricing methods for the Binary Down and Out Barrier (BDOB) options are given. Having these prices, the BDOB option is valued via the Partial Integro Diffential Equation (PIDE) and then the Credit Default Swaps (CDS) are priced by using the BDOB option prices. In Chapter 5, after doing some application and analysis of the models, we conclude the thesis with a short summary.

Chapter 2

Lévy processes

2.1 Lévy processes and infinitely divisible distributions

In this section, we give some definitions and theorems about Lévy processes. More detailed information can be found in [1, 5, 18, 19]. Consider a probability space $(\Omega, F, \mathbb{P})$ filtered by the filtration $\{F_t, t \geqslant 0\}$.

Definition 2.1.1 (Lévy processes)**.** A stochastic process $X = \{X_t, t \geqslant 0\}$ on $(\Omega, F, \mathbb{P})$ is called a *Lévy process* if the following conditions are satisfied:

1. $X_0 = 0$, almost surely (a.s.).
2. X satisfies cadlág property, that is, X is right continuous with left limits (a.s.).
3. X has independent increments: for each $n \in \mathbb{N}$ and each $0 \leq t_0 < t_1 < \ldots < t_n < \infty$, the increments

$$X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$$

   are independent.
4. X has stationary increments, that is, the distribution of $X_{t+s} - X_s$ does not depend on s.

5. X is stochastically continuous, that is, for every $\varepsilon > 0$ and $t \geq 0$

$$\lim_{s \to t} P(\{|X_t - X_s| > \varepsilon\}) = 0 \qquad (a.s).$$

Note that the last property of Lévy processes does not say that X is a continuous process. In general, Lévy processes, except, for instance, a Brownian motion and a deterministic process, have discontinuous sample paths. So the last property means that, for a Lévy process, the probability of having a jump at a specific (given) time is zero. This makes sense for financial markets because jumps in financial markets occur randomly.

A linear process is a deterministic process; whereas a Brownian motion, a Poisson process, and a compound Poisson process are stochastic processes and all of these can be given as examples to Lévy processes. In fact, a Lévy process can be written as a combination of a linear drift, a Brownian motion and a compound Poisson process by the well-known Lévy-Itô decomposition. See for instance Theorem 2.1.9.

Definition 2.1.2 (Infinitely divisible distributions)**.** A probability distribution F of a random variable X on $(\Omega, F, \mathbb{P})$ is said to be *infinitely divisible* if for any $n \in \mathbb{N}$ with $n \geq 2$, there exist identically independent random variables $X_1, \ldots, X_n$ such that $X_1 + \cdots + X_n$ has the distribution F, that is,

$$X_1 + \cdots + X_n \sim F.$$

The definition of infinitely divisibility can also be modified in terms of the characteristic functions of the distributions.

Definition 2.1.3. A probability distribution F is said to be *infinitely divisible* if for any positive integer n, the n^{th} root of its characteristic function Φ, that is,

$$\phi_n(u) = \phi(u)^{\frac{1}{n}}$$

is also a characteristic function of a random variable.

The relation between Lévy processes and infinitely divisibility is given by the following Theorem 2.1.4.

Theorem 2.1.4. *Let* $X = \{X_t, t \geq 0\}$ *be a Lévy process on* $(\Omega, F, \mathbb{P})$. *Then, for each* t, X_t *is infinitely divisible.*

Proof. For a given $t \geq 0$ and any $n \in \mathbb{N}_+$, X_t can be written as

$$X_t = X_{t/n} + (X_{2t/n} - X_{t/n}) + \cdots + (X_t - X_{(n-1)t/n}).$$

Since X is a Lévy process, the summands in the above summation have identically independent distributions. So X is infinitely divisible. □

The characteristic function of Lévy processes has a special form given in the following definition.

Definition 2.1.5. Let $X = \{X_t, t \geq 0\}$ be a Lévy process on $(\Omega, F, \mathbb{P})$. Then the characteristic function of X is given as

$$\begin{aligned} \phi_X(u) &= E[e^{iux}] \\ &= e^{t\psi(u)}, \end{aligned}$$

where $\psi(u)$ is called the characteristic exponent.

In Definition 2.1.1, Lévy processes are defined and it is said that they have jumps at random times. However, their jump structures were not defined. A measure, the so-called *Lévy measure*, is used for this purpose. The definition of the Lévy measure may be given as follows.

Definition 2.1.6 (Levy measure)**.** The *Lévy measure* k on $\mathbb{R}$ is the expected number of jumps with certain heights per unit time and satisfies

$$k(\{0\}) = 0, \qquad \int_{\mathbb{R}} (1 \wedge |x|^2) k(\mathrm{d}x) < \infty.$$

As can be understood from the above definition, the Lévy measure of a Lévy process defines the intensity of the process as in the case of Poisson or compound Poisson processes. For example, the Lévy measure of a compound Poisson process is $k(\mathrm{d}x) = \lambda F(\mathrm{d}x)$, where λ is the intensity of the Poisson process and F is the distribution of jump sizes.

There are two important results for Lévy processes: the Lévy-Khintchine formula, which gives the characteristic function of an infinitely divisible distribution, and the Lévy-Itô decomposition, which defines the structure of paths of the processes.

Theorem 2.1.7 (Lévy-Khintchine formula)**.** *If a random variable X has an infinitely divisible distribution on $(\Omega, F, \mathbb{P})$, then its characteristic function is given by*

$$\begin{aligned}\phi_X(u) &= E[e^{iuX}] \\ &= \exp\biggl(ib_X u - \frac{1}{2}u^2\sigma_X + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}})k(\mathrm{d}x),\end{aligned}$$

where $b_X \in \mathbb{R}$, $\sigma_X \in \mathbb{R}_+$ and k is a measure satisfying

$$k(\{0\}) = 0 \qquad \int_{\mathbb{R}} (1 \wedge |x|^2)k(\mathrm{d}x) < \infty.$$

Since all Lévy processes are infinitely divisible, the characteristic function of a Lévy process can also be characterized by using the Lévy-Khintchine formula. The following corollary gives this characterization.

Corollary 2.1.8. *If $X = \{X_t, t \geqslant 0\}$ is a Lévy process with Lévy measure k, then there exist such $b_X \in \mathbb{R}$ and $\sigma_X \in \mathbb{R}_+$, such that the characteristic function of X is given by*

$$\phi_X(u) = e^{t\psi(u)},$$

where

$$\psi(u) = ib_X u - \frac{1}{2}u^2\sigma_X + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}})\, k(\mathrm{d}x).$$

The triplet (b_X, σ_X, k) is called the *characteristic triplet* or the *Lévy triplet* and is used to define a Lévy process. In other words, it is sufficient to know this triplet for a Lévy process since it defines the characteristic function of the process.

Theorem 2.1.9 (Lévy-Itô decomposition). *Let $X = \{X_t, t \geqslant 0\}$ be a Lévy process on $(\Omega, F, \mathbb{P})$ with the characteristic triplet (b_X, σ_X, k). Then X can be decomposed into four independent processes, that is,*

$$X_t = b_X t + \sigma_X W_t + \int_0^t \int_{|x| \geqslant 1} x \, J_X(ds, dx) + \int_0^t \int_{|x|<1} x \, \tilde{J}_X(ds, dx),$$

where $\tilde{J}_X(dt, dx) = J_X(dt, dx) - k(dx)dt$ is the so-called compensated jump measure.

As can be undestood from the above decomposition, a Lévy process is composed of a drift part, a Brownian motion part, which forms the continuous part of the process, and two jump parts, one of which is for larger jumps and the other is for small jumps. Since there can be infinitely many small jumps and their sum does not necessarily converge, small jumps must be compensated. There is no need to compensate the large jumps because there are only finitely many jumps that are greater than a certain height in a closed set due to the cadlág property.

Definition 2.1.6 defines the Lévy measure, but the Lévy measure is not just a definition about Lévy processes. It carries very useful information about the structure of Lévy processes. For example, by investigating the Lévy measure we may extract some properties of a Lévy process. Here are some of these collected in definitions.

Definition 2.1.10. A Lévy process $X = \{X_t, t \geq 0\}$ with characteristic triplet (b_X, σ_X, k) is said to be of finite variation if

$$\sigma_X = 0 \qquad \text{and} \qquad \int_{|x|<1} |x| \, k(\mathrm{d}x) < \infty.$$

Otherwise, it is said to be of infinite variation.

Definition 2.1.11. A Lévy process $X = \{X_t, t \geq 0\}$ with characteristic triplet (b_X, σ_X, k) has finite activity if the Lévy measure has the property

$$k(\mathbb{R}) < \infty.$$

Otherwise X has infinite activity.

Definition 2.1.12 (Subordinator)**.** A *subordinator* is a Lévy process which is non-decreasing, almost surely.

Since subordinators are increasing processes, they can be considered as stochastic models of time and used for time-changing (other Lévy) processes. This time-changing procedure is called subordination and transforms a stochastic process into a new stochastic process. The new process is then called subordinate to the original one. More information about subordination is given in Chapter 3.

2.2 Itô formula for Lévy processes

The following theorem gives the Itô formula for Lévy processes. This formula will be used to derive the PIDE of the contingent claims under VG process in Chapter 4.

Theorem 2.2.1. *Let $X = \{X_t, t \geq 0\}$ be a Lévy process with characteristic triplet (b_X, σ_X, k) and $f : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a C^2 function. Then*

$$\begin{aligned} f(t, X_t) - f(0, X_0) &= \int_0^t \frac{\partial f}{\partial x}(s, X_{s-})\,\mathrm{d}X_s + \int \left[\frac{\partial f}{\partial s}(s, X_s) + \frac{\sigma_X^2}{2}\frac{\partial^2 f}{\partial x^2}(s, X_s)\right] ds \\ &+ \sum_{0 \leq s \leq t}^{\triangle X_s \neq 0} \left[f(s, X_{s-} + \triangle X_s) - f(s, X_s) - \triangle X_s \frac{\partial f}{\partial x}(s, X_{s-})\right]. \end{aligned}$$

Since the function of a Lévy process, $f(t, X_t)$, is a semimartingale, it can be possible to decompose $f(t, X_t)$ into a martingale part and a drift part. We conclude this chapter by the following theorem that gives the formulation of this.

Theorem 2.2.2. *Let $X = \{X_t, t \geq 0\}$ be Lévy process with Lévy triplet (b_X, σ_X, k) and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a C^2 function. Then $f(t, X_t) = M_t + V_t$ where M is the martingale part given by*

$$\begin{aligned} M_t &= f(0, X_0) + \int_0^t \sigma_X \frac{\partial f}{\partial x}(s, X_{s-}) \, \mathrm{d}W(s) \\ &\quad + \int_0^t \int_{-\infty}^{\infty} [f(s, X_{s-} + y) - f(s, X_{s-})] \, \tilde{J}_X(\mathrm{d}y, \mathrm{d}s) \end{aligned}$$

and

$$\begin{aligned} V(t) &= \int_0^t \left[\frac{\partial f}{\partial s}(s, X_s) + \frac{\sigma_X^2}{2} \frac{\partial^2 f}{x^2}(s, X_s) + b_X \frac{\partial f}{\partial x}(s, X_{s-}) \right] \mathrm{d}s \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \left[f(s, X_{s-} + y) - f(s, X_{s-}) - y \frac{\partial f}{\partial x}(s, X_{s-}) \mathbf{1}_{|y| \leq 1} \right] k(\mathrm{d}y) \, \mathrm{d}s. \end{aligned}$$

Chapter 3

The Variance Gamma Model

The Variance Gamma (VG) process is an example of a pure jump process with no continuous martingale component, finite variation and infinite arrival rate of jumps. The VG process can also be written in two different ways, either by a time-changed Brownian motion with drift or by a difference of two independent Gamma processes.

3.1 The VG process by a time-changed Brownian motion

The VG process is defined as a Brownian motion with drift, time-changed by a Gamma process which is a subordinator. Let

$$B(t;\theta,\sigma)=\theta t+\sigma W_t \tag{3.1.1}$$

be a Brownian motion with drift θ and variance σ^2, where $W(t)$ is a standard Wiener process (Brownian Motion).

Definition 3.1.1 (Gamma Process)**.** The process $G(t;\mu,\nu)$ with mean rate μ and variance rate ν is a Levy process whose increments $g=G(t+h;\mu,\nu)-G(t;\mu,\nu)$ have the gamma density with mean μh and variance νh:

$$f_h(g)=\left(\frac{\mu}{\nu}\right)^{\frac{\mu^2 h}{\nu}}\frac{g^{\frac{\mu^2 h}{\nu}-1}\exp\left(-\frac{\mu}{\nu}g\right)}{\Gamma\left(\frac{\mu^2 h}{\nu}\right)}, \tag{3.1.2}$$

where $\Gamma(x)$ is the gamma function.

The Lévy measure of the Gamma Process is then

$$k_G(x)\mathrm{d}x = \begin{cases} \frac{\mu^2 \exp\left(-\frac{\mu}{\nu}x\right)}{\nu x}\,\mathrm{d}x, & \text{for } x > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{3.1.3}$$

On the other hand, the characteristic function of the Gamma process may be calculated as follows:

$$\begin{aligned} \Phi_G(u) &= E[e^{iuG(t)}] \\ &= \int_0^\infty e^{iux} (\frac{\mu}{\nu})^{\frac{\mu^2}{\nu}t} \frac{x^{\frac{\mu^2}{\nu}t-1}\, e^{-\frac{\mu}{\nu}x}}{\Gamma(\frac{\mu^2}{\nu}t)}\,\mathrm{d}x \\ &= \frac{(\frac{\mu}{\nu})^{\frac{\mu^2}{\nu}t}}{\Gamma(\frac{\mu^2}{\nu}t)} \int_0^\infty e^{(iu-\frac{\mu}{\nu})x}\, x^{\frac{\mu^2}{\nu}t-1}\,\mathrm{d}x. \end{aligned}$$

By the transformation, $-v = (iu - \frac{\mu}{\nu})x$, we get

$$\begin{aligned} \Phi_G(u) &= \frac{(\frac{\mu}{\nu})^{\frac{\mu^2}{\nu}t}}{\Gamma(\frac{\mu^2}{\nu}t)} \int_0^\infty e^{-v} \left(\frac{v}{-(iu-\frac{\mu}{\nu})}\right)^{\frac{\mu^2}{\nu}t-1} \frac{1}{-(iu-\frac{\mu}{\nu})}\,\mathrm{d}v \\ &= \frac{(\frac{\mu}{\nu})^{\frac{\mu^2}{\nu}t}}{\Gamma(\frac{\mu^2}{\nu}t)} \frac{1}{\left(\frac{\mu}{\nu}-iu\right)}^{\frac{\mu^2}{\nu}} t \int_0^\infty e^{-v}\, v^{\frac{\mu^2}{\nu}t-1}\,\mathrm{d}v. \end{aligned}$$

Since, $\Gamma(\frac{\mu^2}{\nu}t) = \int_0^\infty e^{-v}\, v^{\frac{\mu^2}{\nu}t-1}\,\mathrm{d}v$, it follows that

$$\Phi_G(u) = \left(\frac{1}{1 - iu\frac{v}{\mu}}\right)^{\frac{\mu^2}{\nu}t}. \tag{3.1.4}$$

A sample path of the Gamma process is illustrated in Figure 3.1. It is clear that, the Gamma process is an increasing process.

The VG process $X(t;\sigma,\nu,\theta)$ is defined as the Brownian motion with drift $B(t,\theta,\sigma)$, time-changed by the Gamma process with unit mean $G(t,1,\nu)$, as

$$\begin{aligned} X(t;\sigma,\nu,\theta) &= B(G(t;1,\nu);\sigma,\theta) \\ &= \theta G(t) + \sigma W(G(t)). \end{aligned} \tag{3.1.5}$$

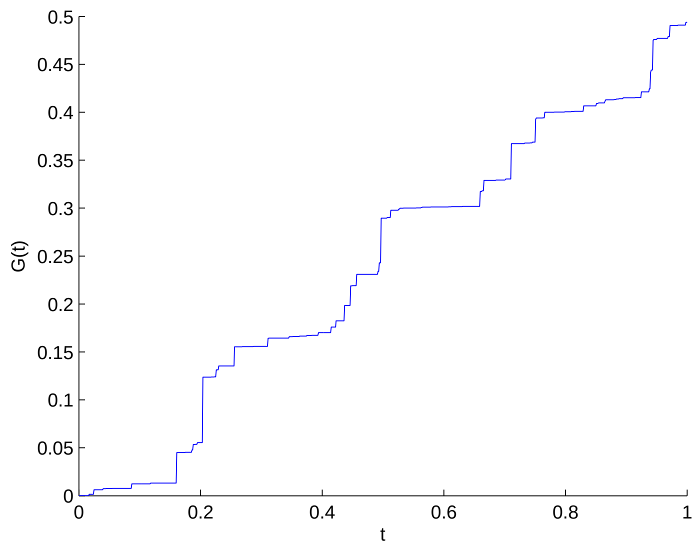

Figure 3.1: Gamma Paths

The Gamma process has unit mean rate due to normalization that a stochastic time has an expected value of the calendar time, namely, $E[G(t)] = t$.

The VG process is conditionally Gaussian so that for a given time change $g = G(t)$, the VG process $X(t) = \theta g + \sigma W(g)$ has a normal distribution. Hence, the probability density function $f_{X(t)}(x)$ can be found by first conditioning the gamma time change $G(t) = g$, and then, integrating it. That is,

$$f_{X(t)}(x) = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi g}} \exp\left(-\frac{(x-\theta g)^2}{2\sigma^2 g}\right) \frac{g^{\frac{t}{\nu}-1}\exp\left(-\frac{g}{\nu}\right)}{\nu^{\frac{t}{\nu}}\Gamma\left(\frac{t}{\nu}\right)} \mathrm{d}g. \tag{3.1.6}$$

The characteristic function $\phi_{X(t)}(u)$ of a VG process can be found similarly:

$$\phi_{X(t)}(u) = E[e^{iuX(t)}] = \left(\frac{1}{1 - i\theta\nu u + \frac{\sigma^2\nu}{2}u^2}\right)^{\frac{t}{\nu}}. \tag{3.1.7}$$

The Lévy measure of the VG process as a time-change Brownian motion is therefore

$$k_X(x)dx = \frac{\exp\left(\frac{\theta}{\sigma^2}x\right)}{\nu|x|} \exp\left(-\frac{\sqrt{\frac{2}{\nu} + \frac{\theta^2}{\sigma^2}}}{\sigma}|x|\right) dx. \tag{3.1.8}$$

This measure gives the expected number of jumps of the VG process X_t in a unit time interval.

3.2 The VG process as a difference of two independent processes

Since the VG process has a finite variation, it can also be written as a difference of two independent gamma processes as

$$X(t;\sigma,\nu,\theta) = G_p(t;\mu_p,\nu_p) - G_n(t;\mu_n,\nu_n). \tag{3.2.1}$$

Here G_p and G_n are two indepencent gamma processes with mean and variance rates μ_p and ν_p for G_p and μ_n and ν_n for G_n.

The relation between original parameters σ, ν, θ and parameters μ_p, ν_p; μ_n, ν_n of the VG process is given in the following theorem.

Theorem 3.2.1. *Let X be a VG process defined by parameters σ, ν, θ as time-changed Brownian motion and by parameters $\mu_p, \nu_p; \mu_n, \nu_n$ as the difference of two independent gamma processes. Then parameters μ_p, ν_p, μ_n and ν_n are given in terms of parameters σ, ν and θ as*

$$\mu_p = \frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}} + \frac{\theta}{2}, \tag{3.2.2}$$

$$\mu_n = \frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}} - \frac{\theta}{2}, \tag{3.2.3}$$

$$\nu_p = \left(\frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}} + \frac{\theta}{2}\right)^2 \nu, \tag{3.2.4}$$

$$\nu_n = \left(\frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}} - \frac{\theta}{2}\right)^2 \nu. \tag{3.2.5}$$

Proof. Let us define the VG process as $X(t) = G_p(t) - G_n(t)$ where $G_p(t)$ and $G_n(t)$ are two independent gamma processes with mean and variance raters μ_p

and ν_p for G_p, and μ_n and ν_n for G_n. Then the characteristic functions of these two gamma processes are

$$\Phi_{G_p}(u) = \left(\frac{1}{1 - iu\frac{\nu_p}{\mu_p}}\right)^{\frac{\mu_p^2}{\nu_p}t},$$

and

$$\Phi_{G_n}(u) = \left(\frac{1}{1 - iu\frac{\nu_n}{\mu_n}}\right)^{\frac{\mu_n^2}{\nu_n}t},$$

respectively. The characteristic function of the difference of these two gamma processes is the product of these two characteristic functions. In other words, due to the infinite divisibility of the Lévy processes, we have

$$\phi_{G_p - G_n}(u) = \Phi_{G_p}(u)\Phi_{G_n}(u)$$

and hence,

$$\phi_{G_p - G_n}(u) = \left(\frac{1}{1 - iu\nu_p/\mu_p}\right)^{\mu_p^2/\nu_p} \left(\frac{1}{1 - iu\nu_n/\mu_n}\right)^{\mu_n^2/\nu_n}.$$

Let

$$\frac{\mu_p^2}{\nu_p} = \frac{\mu_n^2}{\nu_n} = \frac{1}{\nu} \tag{3.2.6}$$

holds, so that

$$\phi_{G_p - G_n}(u) = \left(\frac{1}{1 - iu\left(\frac{\nu_n}{\mu_n} - \frac{\nu_p}{\mu_p}\right) - u^2\frac{\nu_p}{\mu_p}\frac{\nu_n}{\mu_n}}\right)^{t/\nu}. \tag{3.2.7}$$

By comparing the characteristic function in equation (3.1.7) with the characteristic function in equation (3.2.7) we deduce that,

$$\frac{\nu_p\nu_n}{\mu_p\mu_n} = \frac{\sigma^2\nu}{2} \tag{3.2.8}$$

and

$$\frac{\nu_p}{\mu_p} - \frac{\nu_n}{\mu_n} = \theta\nu. \tag{3.2.9}$$

Using (3.2.9) in (3.2.8), we obtain

$$\begin{aligned}\frac{\sigma^2\nu}{2} &= \frac{\nu_p}{\mu_p}\left(\frac{\nu_p}{\mu_p}-\theta\nu\right),\\ &= \left(\frac{\nu_p}{\mu_p}\right)^2-\theta\nu\frac{\nu_p}{\mu_p}.\end{aligned}$$

Upon multiplying by 4 and adding $\theta^2\nu^2$ to both sides yields

$$\begin{aligned}2\sigma^2\nu+\theta^2\nu^2 &= \left(2\frac{\nu_p}{\mu_p}\right)^2-4\theta\nu\frac{\nu_p}{\mu_p}+\theta^2\nu^2\\ \theta^2\nu^2+2\sigma^2\nu &= \left(2\frac{\nu_p}{\mu_p}-\theta\nu\right)^2\\ \frac{\nu_p}{\mu_p} &= \frac{1}{2}\sqrt{\theta^2\nu^2+2\sigma^2\nu}+\frac{\theta\nu}{2}.\end{aligned} \tag{3.2.10}$$

Similarly, it is found that

$$\frac{\nu_n}{\mu_n}=\frac{1}{2}\sqrt{\theta^2\nu^2+2\sigma^2\nu}-\frac{\theta\nu}{2}. \tag{3.2.11}$$

From (3.2.6), we get the parameters μ_p and μ_n as

$$\mu_p=\frac{1}{\nu}\frac{\nu_p}{\mu_p} \qquad \text{and} \qquad \mu_n=\frac{1}{\nu}\frac{\nu_n}{\mu_n}.$$

Substituting $\frac{\nu_p}{\mu_p}$ and $\frac{\nu_n}{\mu_n}$ from equations (3.2.10) and (3.2.11) we found

$$\begin{aligned}\mu_p &= \frac{1}{\nu}\left(\frac{1}{2}\sqrt{\theta^2\nu^2+2\sigma^2\nu}+\frac{\theta\nu}{2}\right)\\ &= \frac{1}{2}\sqrt{\theta^2+\frac{2\sigma^2}{\nu}}+\frac{\theta}{2},\end{aligned}$$

$$\begin{aligned}\mu_n &= \frac{1}{\nu}\left(\frac{1}{2}\sqrt{\theta^2\nu^2+2\sigma^2\nu}-\frac{\theta\nu}{2}\right)\\ &= \frac{1}{2}\sqrt{\theta^2+\frac{2\sigma^2}{\nu}}-\frac{\theta}{2}.\end{aligned}$$

Finally, from equations (3.2.10) and (3.2.11), the relations

$$\begin{aligned}\nu_p &= \left(\frac{1}{2}\sqrt{\theta^2\nu^2+2\sigma^2\nu}+\frac{\theta\nu}{2}\right)\mu_p\\ &= \nu\left(\frac{1}{2}\sqrt{\theta^2+\frac{2\sigma^2}{\nu}}+\frac{\theta}{2}\right)^2,\end{aligned}$$

and

$$\nu_n = \nu\left(\frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}} + \frac{\theta}{2}\right)^2$$

follow. This completes the proof. □

The Lévy measure of the VG process as the difference of two independent gamma processes may be written via the Lévy measure of the gamma process as

$$k_X(x)dx = \begin{cases} \frac{\mu_n^2}{\nu_n|x|}\exp\left(-\frac{\mu_n}{\nu_n}|x|\right)dx, & \text{for } x < 0 \\ \frac{\mu_p^2}{\nu_p|x|}\exp\left(-\frac{\mu_p}{\nu_p}|x|\right)dx, & \text{for } x > 0. \end{cases} \tag{3.2.12}$$

Furthermore, the following corrolary simplifies this expression.

Corollary 3.2.2. *The Lévy measure of the VG process can also be written as*

$$k_X(x)dx = \begin{cases} \frac{e^{-\lambda_n|x|}}{\nu|x|}dx, & \text{for } x < 0 \\ \frac{e^{-\lambda_p x}}{\nu x}dx, & \text{for } x > 0, \end{cases} \tag{3.2.13}$$

where

$$\lambda_n = \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}} + \frac{\theta}{\sigma^2} \quad \text{and} \quad \lambda_p = \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}} - \frac{\theta}{\sigma^2}.$$

Proof. The Lévy measure of the VG process as a difference of two independent gamma processes was given in (3.2.12). From the relations (3.2.6), (3.2.10) and (3.2.11) in the proof of Theorem 3.2.1, we calculate

$$\begin{aligned} \lambda_n := \frac{\mu_n}{\nu_n} &= \frac{1}{\frac{1}{2}\left(\sqrt{\theta^2\nu^2 + 2\sigma^2\nu} - \theta\nu\right)} = \frac{2\left(\sqrt{\theta^2\nu^2 + 2\sigma^2\nu} + \theta\nu\right)}{\theta^2\nu^2 + 2\sigma^2\nu - \theta\nu} \\ &= \frac{\sqrt{\theta^2\nu^2 + 2\sigma^2\nu} + \theta\nu}{\sigma^2\nu} = \sqrt{\frac{\theta^2\nu^2 + 2\sigma^2\nu}{\sigma^4\nu^2}} + \frac{\theta\nu}{\sigma^2\nu} \\ &= \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}} + \frac{\theta}{\sigma^2}. \end{aligned}$$

Similar calculation follows for λ_p:

$$\lambda_p := \frac{\mu_p}{\nu_p} = \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}} - \frac{\theta}{\sigma^2}.$$

Using these parameters in (3.2.12) proves the corollary. □

A sample path of the VG process is illustrated in Figure 3.2. This figure was plotted by simulating the VG paths as a difference of two independent Gamma process. The jump structure of the VG process can be seen clearly from this figure.

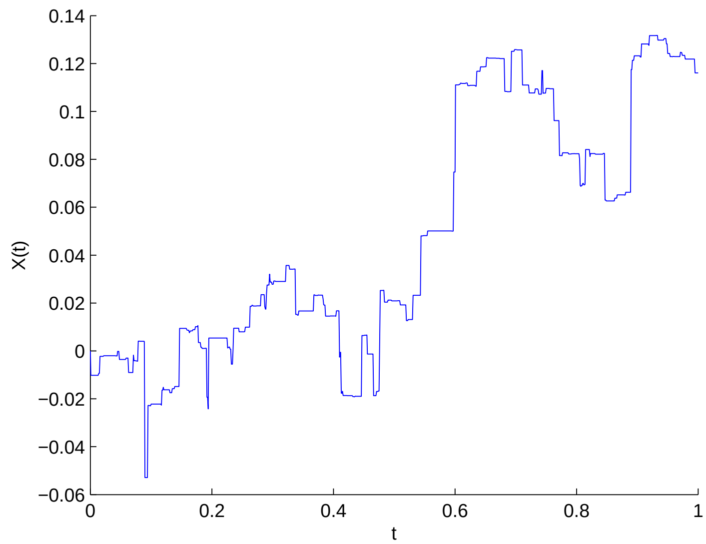

Figure 3.2: VG Paths

3.3 The Moments of the VG process

The moments of the VG process can be easily computed by differentiating its characteristic function due to the following theorem.

Theorem 3.3.1. *If the characteristic function $\Phi_X(u)$ has a derivative of n^{th} order at $u = 0$ then X has a finite moment of order n and*

$$\mathbb{E}\left[X^n\right] = i^{-n}\, \Phi_X(0). \tag{3.3.1}$$

Madan, Carr and Chang [13] showed that

$$\begin{aligned}
\mathbb{E}[X_t] &= \theta t, \\
\mathbb{E}[(X_t - \mathbb{E}[X_t])^2] &= (\theta^2\nu + \sigma^2)t, \\
\mathbb{E}[(X_t - \mathbb{E}[X_t])^3] &= (2\theta^3\nu^2 + 3\sigma^2\theta\nu)t, \\
\mathbb{E}[(X_t - \mathbb{E}[X_t])^4] &= (3\sigma^4\nu + 12\sigma^2\theta^2\nu^2 + 6\theta^4\nu^3)t \\
&\quad + (3\sigma^4 + 6\sigma^2\theta^2\nu + 3\theta^4\nu^2)t^2.
\end{aligned}$$

From the above equations, the mean, the variance, the skewness and the kurtosis of the VG process are calculated as

$$\begin{aligned}
mean[X_t] &= \mathbb{E}[X_t] = \theta t, \\
variance[X_t] &= \mathbb{E}[(X_t - \mathbb{E}[X_t])^2] \\
&= (\theta^2\nu + \sigma^2)t, \\
skewness[X_t] &= \frac{\mathbb{E}[(X_t - \mathbb{E}[X_t])^3]}{(\mathbb{E}[(X_t - \mathbb{E}[X_t])^2])^{3/2}} \\
&= \frac{\theta\nu(3\sigma^2 + 2\nu\theta^2)}{(\sigma^2 + \nu\theta^2)^{3/2}} t^{-1/2}, \\
kurtosis[X_t] &= \frac{\mathbb{E}[(X_t - \mathbb{E}[X_t])^4]}{(\mathbb{E}[(X_t - \mathbb{E}[X_t])^2])^2} \\
&= 3\left(1 + \frac{2\nu}{t} - \frac{\nu\sigma^4}{(\sigma^2 + \theta^2\nu)^2 t}\right).
\end{aligned}$$

Note that the variance of the VG process has two terms, one being $\sigma^2 t$, coming from the original Brownian motion, and the other $\theta^2\nu t$, coming from time-change. The variance of the stochastic time (the gamma process) ν, which can also be seen as a global market uncertainty, affects the variance of the VG process and θ can be seen as the rate of how much a company is exposed to global market uncertainty. When $\theta = 0$, it is obvious that time change affects only kurtosis which is $3\left(1 + \frac{\nu}{t}\right)$, where ν is the percentage excess kurtosis.

3.4 The VG stock price dynamics

The stock price in the VG model is constructed as in the Black-Sholes (BS) model, but the randomness in the VG model comes from the VG process instead of the Brownian motion. By replacing the Brownian motion in the BS model by the VG process, the VG stock price is obtained. As for the risk neutral pricing, a risk neutral measure is needed. But since jump models are incomplete models (they belong to the incomplete market), there is unique equivalent martingale measure (EMM). So we need to choose one EMM among them for pricing. In the VG model, the mean-correcting EMM will be used and the change of measure will made as in the BS model. In other words, the risk neutral VG stock price is modeled as

$$S_t = S_0 \frac{\exp((r-q)t + X_t)}{\mathbb{E}[\exp(X_t)]} \tag{3.4.1}$$

such that $\mathbb{E}[S_t] = S_0 \exp((r-q)t)$, where r is the constant continuously compounded interest rate, q is the continuously compounded dividend yield, and X_t is the VG process defined in Sections 3.1 or 3.2.

The expectation in the equation (3.4.1) is calculated as

$$\begin{aligned}
\mathbb{E}[\exp(X)] &= \Phi_X(-i) \\
&= \left(\frac{1}{1-\theta\nu-\frac{1}{2}\sigma^2\nu}\right)^{\frac{t}{\nu}} \\
&= \exp\left(\ln\left(1-\theta\nu-\frac{1}{2}\sigma^2\nu\right)^{-\frac{t}{\nu}}\right) \\
&= \exp\left(-\frac{t}{\nu}\ln\left(1-\theta\nu-\frac{1}{2}\sigma^2\nu\right)\right).
\end{aligned}$$

Defining $\omega := \frac{1}{\nu}\ln\left(1-\theta\nu-\frac{1}{2}\sigma^2\nu\right)$ gives

$$\mathbb{E}[\exp(X_t)] = \exp(-\omega t),$$

so that the risk neutral VG stock price dynamics is given by

$$S_t = S_0 \exp((r-q)t + X_t + \omega t), \tag{3.4.2}$$

where $\omega = \frac{1}{\nu}\ln\left(1-\theta\nu-\frac{1}{2}\sigma^2\nu\right)$, X_t is the VG process, r and q are the constant continuously compounded interest rate and the continuously compounded dividend yield, respectively.

The density function of the natural logarithm of the stock prices (or returns) over an interval of length t is given by the following theorem.

Theorem 3.4.1. *Let $Z_t = \ln(S_t/S_0)$ be the return process when S_t is given by the VG model defined above. Then the density function of Z_t is given by*

$$\begin{aligned} f(z) &= \frac{2e^{\theta x/\sigma^2}}{\nu^{t/\nu}\sqrt{2\pi}\Gamma\left(\frac{t}{\nu}\right)}\left(\frac{x^2}{2\sigma^2/\nu+\theta^2}\right)^{\frac{t}{2\nu}-\frac{1}{4}} \\ &\quad \times K_{\frac{t}{\nu}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{x^2(2\sigma^2/\nu)+\theta^2}\right), \end{aligned} \tag{3.4.3}$$

where $x = z-(r-q)t-\frac{t}{\nu}\ln(1-\theta\nu-\sigma^2\nu/2)$ and K is the modified Bessel function of the second kind.

Proof. Let

$$S_t = S_0\exp\left((r-q)t+X_t+\frac{t}{\nu}\ln(1-\theta\nu-\frac{1}{2}\sigma^2\nu)\right)$$

be the risk neutral stock price dynamics. From this price, the dynamics of return is found to be

$$Z_t = \ln\frac{S_t}{S_0} = (r-q)t+X_t+\frac{t}{\nu}\ln(1-\theta\nu-\frac{1}{2}\sigma^2\nu).$$

Then the VG process is given in terms of return process as

$$X_t = Z_t-(r-q)t-\frac{t}{\nu}\ln(1-\theta\nu-\frac{1}{2}\sigma^2\nu).$$

Since X_t may be written in terms of the return process Z_t, and since the probability density function of the VG process X_t is known from (3.1.6), the density function of return process Z_t can be written as

$$f(z) = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi g}}\, e^{-\frac{(x-\theta g)^2}{2\sigma^2 g}}\, \frac{g^{\frac{t}{\nu}-1}\, e^{-g/\nu}}{\nu^{t/\nu}\Gamma(t/\nu)}\,\mathrm{d}g,$$

where $x = z-(r-q)t-\frac{t}{\nu}\ln(1-\theta\nu-\frac{1}{2}\sigma^2\nu)$. The integral in this density function is calculated as follows:

$$\begin{aligned}
f(z) &= \frac{1}{\sqrt{2\pi}\sigma\nu^{t/\nu}\Gamma(t/\nu)}\int_0^\infty \frac{1}{g}\,e^{-\frac{(x-\theta g)^2}{2\sigma^2 g}}\,g^{\frac{t}{\nu}-1}\,e^{-g/\nu}\,\mathrm{d}g \\
&= \frac{1}{\sqrt{2\pi}\sigma\nu^{t/\nu}\Gamma(t/\nu)}\int_0^\infty g^{(\frac{t}{\nu}-\frac{1}{2})-1}\,e^{\frac{-(x^2-2\theta gx+\theta^2g^2)}{2\sigma^2 g}}\mathrm{d}g \\
&= \frac{1}{\sqrt{2\pi}\sigma\nu^{t/\nu}\Gamma(t/\nu)}\int_0^\infty g^{(\frac{t}{\nu}-\frac{1}{2})-1}\,e^{\frac{\theta x}{\sigma^2}}\,e^{\frac{-x^2}{2\sigma^2 g}-\frac{g}{\nu}-\frac{\theta^2 g}{2\sigma^2}}\mathrm{d}g \\
&= \frac{1}{\sqrt{2\pi}\sigma\nu^{t/\nu}\Gamma(t/\nu)}\,e^{\frac{\theta x}{\sigma^2}}\int_0^\infty g^{(\frac{t}{\nu}-\frac{1}{2})-1}\,e^{\frac{1}{2\sigma^2}\left(\frac{-x^2}{g}-\left(\frac{2\sigma^2}{\nu}+\theta^2\right)g\right)}\mathrm{d}g \\
&= \frac{e^{\theta x/\sigma^2}}{\sqrt{2\pi}\sigma\nu^{t/\nu}\Gamma(t/\nu)}\int_0^\infty g^{(\frac{t}{\nu}-\frac{1}{2})-1}\,e^{-\frac{x^2/2\sigma^2}{g}-\frac{2\sigma^2\nu+\theta^2}{2\sigma^2}g}\mathrm{d}g.
\end{aligned}$$

By using the fact that

$$\int_0^\infty x^{\alpha-1}\,e^{-\frac{\beta}{x}-\gamma x}\mathrm{d}x = 2\left(\frac{\beta}{\gamma}\right)^{\frac{\alpha}{2}}\,K_\alpha(2\sqrt{\beta\gamma}),$$

given by Gradshetyn and Ryzhik [7] at page 340, and by considering $\alpha=\frac{t}{\nu}-\frac{1}{2}$, $\beta=\frac{x^2}{2\sigma^2}$, $\gamma=\frac{2\sigma^2/\nu+\theta^2}{2\sigma^2}$, $x=g$, we obtain the density function of return process as

$$\begin{aligned}
f(z) &= \frac{e^{\theta x/\sigma^2}}{\sqrt{2\pi}\sigma\nu^{t/\nu}\Gamma(t/\nu)}\,2\left(\frac{x^2/2\sigma^2}{\frac{2\sigma/\nu+\theta^2}{2\sigma^2}}\right)^{\frac{1}{2}\left(\frac{t}{\nu}-\frac{1}{2}\right)}K_{\frac{t}{\nu}-1}\left(2\sqrt{\frac{x^2}{2\sigma^2}\,\frac{2\sigma^2/\nu+\theta^2}{2\sigma^2}}\right) \\
&= \frac{2e^{\theta x/\sigma^2}}{\sqrt{2\pi}\sigma\nu^{t/\nu}\Gamma(t/\nu)}\left(\frac{x^2}{2\sigma^2/\nu+\theta^2}\right)^{\frac{t}{2\nu}-\frac{1}{4}}K_{\frac{t}{\nu}-1}\left(\frac{1}{\sigma^2}\sqrt{x^2(2\sigma^2/\nu+\theta^2)}\right).
\end{aligned}$$

This proves the theorem. □

Similarly, the distribution function of returns can be found by conditioning on the gamma time change $G(t)=g$. The conditional distribution function of log returns is then given by

$$\mathbb{P}(Z_t<z|G_t=g)=F(z|g)=\Phi\left(\frac{z-(r-q+\omega)t-\theta g}{\sigma\sqrt{g}}\right),$$

where Φ is the cumulative distribution function of the standard normal random variable. Furthermore the unconditional distribution function is given by the following equation:

$$\begin{aligned}
\mathbb{P}\{Z_t < z\} &= \mathrm{F}(z) \\
&= \int_0^\infty F(z|g) \frac{g^{\frac{t}{\nu}-1}\, e^{-g/\nu}}{\nu^{\frac{t}{\nu}} \Gamma\left(\frac{t}{\nu}\right)} \,\mathrm{d}g \\
&= \int_0^\infty \Phi\left(\frac{z-(r-q+\omega)t-\theta g}{\sigma\sqrt{g}}\right) \frac{g^{\frac{t}{\nu}-1}\, e^{-g/\nu}}{\nu^{\frac{t}{\nu}} \Gamma\left(\frac{t}{\nu}\right)} \,\mathrm{d}g. \qquad (3.4.4)
\end{aligned}$$

3.4.1 Semimartingale decomposition of VG stock prices

This section starts with the Lévy-Itô decomposition of the VG process. This decomposition will be used to show the semimartingale decomposition of the VG process.

Definition 3.4.2 (Lévy-Itô decomposition of VG process)**.** A VG process X_t defined in Sections 3.1 or 3.2 has Lévy-Itô decomposition in the form

$$X_t = \int_0^t \int_{-\infty}^{\infty} x J(\mathrm{d}x, \mathrm{d}s), \qquad (3.4.5)$$

or in differential form

$$\mathrm{d}X_t = \int_{-\infty}^{\infty} x J(\mathrm{d}t, \mathrm{d}x). \qquad (3.4.6)$$

Above definition comes from the fact that the VG process has a finite variation $\int_{|x|\leq} k(\mathrm{d}x) < \infty$, which simply means that the sum of all small jumps is finite. Hence, there is no need to compensate these small jumps. As a consequence, the VG process can be written as the sum of all its jumps just as in the definition above.

Proposition 3.4.3 (Semimartingale decomposition of VG stock prices)**.** *Let S be the VG stock price dynamics defined as*

$$S_t = S_0 \exp((r-q)t + X_t + \omega t),$$

where $\omega = \frac{1}{\nu} \ln\left(1 - \theta\nu - \frac{1}{2}\sigma^2\nu\right)$, X_t is the VG process, r and q are the constant continuously compounded interest rate and the continuously compounded divident *yield, respectively. Then S_t has semimartingale decomposition,*

$$S_t = S_0 + \int_0^t S_{u-}(r-q)\,\mathrm{d}u + \int_0^t\!\!\int_{-\infty}^{\infty} S_{u-}(e^x - 1)\,\tilde{J}_X(\mathrm{d}u, \mathrm{d}x), \tag{3.4.7}$$

where $\tilde{J}_X(du, dx)$ is the so-called compensated Lévy measure.
In the differential form, this decomposition is

$$\mathrm{d}S_t = S_{t-}(r-q)\mathrm{d}t + \int_{-\infty}^{\infty} S_{t-}\,(e^x - 1)\,\tilde{J}_X(\mathrm{d}t, \mathrm{d}x). \tag{3.4.8}$$

Proof. Applying the Itô formula for Lévy processes to the function $S_t = f(X_t, t)$ gives

$$\begin{aligned} S_t &= S_0 + \int_0^t \left(\frac{\partial S}{\partial u}(u-, X_{u-}) + \frac{\sigma_X^2}{2}\frac{\partial^2 S}{\partial x^2}(u, X_u)\right)\mathrm{d}u \\ &\quad + \int_0^t \frac{\partial S}{\partial x}(u-, X_u)\,\mathrm{d}X_u \\ &\quad + \int_0^t\!\!\int_{-\infty}^{\infty} \left(S_{u-}e^x - S_{u-} - x\frac{\partial S}{\partial x}(u-)\right) J_X(\mathrm{d}u, \mathrm{d}x). \end{aligned}$$

Since the VG process has a finite variation, it has no diffusion part, that is, $\sigma_X = 0$ by Definition 2.1.10. Then the function S_t returns to

$$\begin{aligned} S_t &= S_0 + \int_0^t S_{u-}(r-q+\omega)\,\mathrm{d}u + \int_0^t S_{u-}\,\mathrm{d}X_u \\ &\quad + \int_0^t\!\!\int_{-\infty}^{\infty} S_{u-}(e^x - 1 - x)\,J_X(\mathrm{d}u, \mathrm{d}x) \\ &= S(0) + \int_0^t S_{u-}(r-q+\omega)\,\mathrm{d}u + \int_0^t S_{u-}\int_{-\infty}^{\infty} xJ_X(\mathrm{d}u, \mathrm{d}x) \\ &\quad + \int_0^t\!\!\int_{-\infty}^{\infty} S_{u-}(e^x - 1 - x)\,J_X(\mathrm{d}u, \mathrm{d}x). \end{aligned}$$

Adding and substracting $\int_0^t\int_{-\infty}^{\infty} S(u-)(e^x-1)\,k(\mathrm{d}x)\,\mathrm{d}u$ yield

$$\begin{aligned}
S_t &= S_0 + \int_0^t S_{u-}(r-q+\omega)\,\mathrm{d}u + \int_0^t\int_{-\infty}^{\infty} S_{u-}xJ_X(\mathrm{d}u,\mathrm{d}x) \\
&\quad \mp \int_0^t\int_{-\infty}^{\infty} S_{u-}(e^x-1)\,k(\mathrm{d}x)\,\mathrm{d}u \\
&\quad + \int_0^t\int_{-\infty}^{\infty} S_{u-}(e^x-1-x)\,J_X(\mathrm{d}u,\mathrm{d}x) \\
&= S_0 + \int_0^t S_{u-}(r-q+\omega)\,\mathrm{d}u \mp \int_0^t\int_{-\infty}^{\infty} S_{u-}(e^x-1)\,k(\mathrm{d}x)\,\mathrm{d}u \\
&\quad + \int_0^t\int_{-\infty}^{\infty} S_{u-}(e^x-1-x+x)\,J_X(\mathrm{d}u,\mathrm{d}x) \\
&= S_0 + \int_0^t S_{u-}(r-q+\omega)\,\mathrm{d}u + \int_0^t\int_{-\infty}^{\infty} S_{u-}(e^x-1)\,k(\mathrm{d}x)\,\mathrm{d}u \\
&\quad + \int_0^t\int_{-\infty}^{\infty} S_{u-}(e^x-1)\,(J_X(\mathrm{d}u,\mathrm{d}x)-k(\mathrm{d}x)\,\mathrm{d}u).
\end{aligned}$$

By writing $J_X(\mathrm{d}u,\mathrm{d}x)-k(\mathrm{d}x)\mathrm{d}u$ as $\tilde{J}_X(\mathrm{d}u,\mathrm{d}x)$, we obtain

$$\begin{aligned}
S_t &= S_0 + \int_0^t S_{u-}(r-q)\,\mathrm{d}u + \int_0^t S_{u-}\omega\,\mathrm{d}u \\
&\quad + \int_0^t S_{u-}\int_{-\infty}^{\infty} (e^x-1)\,k(\mathrm{d}x)\,\mathrm{d}u \\
&\quad + \int_0^t\int_{-\infty}^{\infty} S_{u-}(e^x-1)\,\tilde{J}_X(\mathrm{d}u,\mathrm{d}x) \\
&= S_0 + \int_0^t S_{u-}(r-q)\,\mathrm{d}u + \int_0^t S_{u-}\left(\omega + \int_{-\infty}^{\infty}(e^x-1)\,k(\mathrm{d}x)\right)\mathrm{d}u \\
&\quad + \int_0^t\int_{-\infty}^{\infty} S_{u-}(e^x-1)\,\tilde{J}_X(\mathrm{d}u,\mathrm{d}x). \qquad (3.4.9)
\end{aligned}$$

At the begining, the Itô formula was applied to the price function S_t. If it had been applied to the discount price function $\tilde{S}_t = e^{-(r-q)t}S_t$, then we would have

$$\begin{aligned}
\tilde{S}_t &= \tilde{S}_0 + \int_0^t \tilde{S}_{u-}\left(\omega + \int_{-\infty}^{\infty}(e^x-1)\,k(\mathrm{d}x)\right)\mathrm{d}u \\
&\quad + \int_0^t\int_{-\infty}^{\infty} \tilde{S}_{u-}(e^x-1)\,\tilde{J}_X(\mathrm{d}u,\mathrm{d}x).
\end{aligned}$$

Since the discount prices are martingale under the risk-neutral measure, the drift part must vanish, in other words,

$$-\omega = \int_{-\infty}^{\infty} (e^x - 1)\, k(\mathrm{d}x). \tag{3.4.10}$$

By using this equation in equation (3.4.9) the semimartingale decomposition of the stock price dynamics is found to be

$$S_t = S_0 + \int_0^t S_{u-}(r-q)\,\mathrm{d}u + \int_0^t\!\!\int_{-\infty}^{\infty} S_{u-}(e^x - 1)\,\tilde{J}_X(\mathrm{d}u, \mathrm{d}x),$$

which completes the proof. □

Chapter 4

Credit Risk Modeling

In this chapter, we will analyze the credit risk of a single firm with a structural approach under the assumptions of both the BS model and the VG model, defined in the preceding chapter. We will summarize the modeling of the default probabilities of the two structural approaches, that is, the Merton and Black-Cox models. We will aso give the pricing of the contingent claims by using these default probabilities. The pricing of the contingent claims will be obtained by using *the equivalent martingale measure approach* (EMM) which is one of the two pricing approaches in literature. In the VG model setting, the Merton model will be introduced briefly and the biggest emphasis will be on the Black-Cox model as in [4]. Under this model, we will discuss the pricing of a Credit Default Swap (CDS) in detail. But contrary to the BS model, pricing will be done by using *the partial differential equation approach* (PDE) which is the other pricing approach in literature.

4.1 Credit Risk Modeling under the BS model

We assume that the firm's asset value (simply, the firm value), process V follows a Geometric Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In other words, the value V satisfies the SDE

$$dV_t = V_t\,(\mu dt + \sigma dW_t)$$

and has a solution

$$V_t = V_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right),$$

where μ and σ are the drift and volatility of the process, respectively, while W_t is the standard Brownian motion under the probability measure $\mathbb{P}$.

As said previously, contingent claims will be priced under EMM which means risk neutral pricing or pricing under no arbitrage opportunity. In markets with no arbitrage opportunity, there is a probability measure $\mathbb{P}^*$ such that the discounted prices in the market are martingale under this probability mesure. This probability measure is called the *risk neutral probability measure* and the Girsanov Theorem given below ensures the existence of this probability measure.

Theorem 4.1.1 (Girsanov Theorem)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and W_t be a standard Brownian motion under the probability measure $\mathbb{P}$. Let θ_t, $0 \leq t \leq T$, be an $\mathcal{F}_t$ measurable process satisfying $\int_0^T \theta_s^2 \,\mathrm{d}s < \infty$ a.s., such that the process*

$$L_t = \exp\left(-\int_0^t \theta_s \,\mathrm{d}W_s - \frac{1}{2}\int_0^t \theta_s^2 \,\mathrm{d}s\right)$$

is a martingale with $\mathbb{E}(L_T) = 1$. Then, under the probability measure $\mathbb{P}^*$ with density L_T with respect to $\mathbb{P}$, that is, for every $F \in \mathcal{F}$,

$$\mathbb{P}^*(F) = \int_F L(\omega)\mathrm{d}\mathbb{P}(\omega)$$

the process W_t^*, $0 \leq t \leq T$, defined by

$$W_t^* = W_t + \int_0^t \theta_s \,\mathrm{d}s$$

is a standard Brownian motion under the probability measure $\mathbb{P}^*$.

In the BS model, with $\theta_t = \frac{\mu - r}{\sigma}$, there exists a probability measure $\mathbb{P}^*$ under which W_t^* is a standard Brownian motion. Then the price process of assets is given with SDE

$$dV_t = V_t\left(rdt + \sigma dW_t^*\right)$$

with solution

$$V_t = V_0 \exp\left((r - \frac{1}{2}\sigma^2)t + \sigma W_t^*\right)$$

under the risk neutral mesure $\mathbb{P}^*$.

When the firm pays a dividend, the price process is modified to

$$V_t = V_0 \exp\left((r - q - \frac{1}{2}\sigma^2)t + \sigma W_t^*\right), \tag{4.1.1}$$

where r and q are the constant continuously compounded risk-free interest rate and the continuously compounded dividend yield, respectively and W_t^* is the standard Brownian motion under $\mathbb{P}^*$.

Finally, the price at time t of a risk-free zero coupon bond with maturity T, which pays one unit as face value, is given by

$$B(t,T) = \exp(-r(T-t)),$$

where r is the risk-free rate.

4.1.1 Merton model

In this model, the default event may only occur at the maturity T of the debts. Namely, the default time, which is generally denoted by τ, is either T or ∞, that is,

$$\tau = \begin{cases} T, & \text{if default,} \\ +\infty, & \text{if no default.} \end{cases}$$

At maturity, if the value of the firm, V, is less than the firms liabilities, L, the firm defaults and the bondholders (or debtholders) receive the value V_T. Otherwise, the firm does not default and the bondholders receive the full face value L. The default probability, which will be denoted as $P_M^d(T)$ (or $P_M^d\{\tau = T\}$), and the payoff of the bond are given by

$$P_M^d(T) = \mathbb{P}^*\{V_T < L\},$$

and

$$\text{Payoff} = \begin{cases} V_T, & V_T < L \\ L, & V_T \geq L, \end{cases}$$

respectively.

Putting the value V_T given in (4.1.1) in the expression for the default probability $P_M^d(T)$ yields

$$\begin{aligned} P_M^d(T) &= \mathbb{P}^*\{V_T < L\} \\ &= \mathbb{P}^*\left\{V_0 \exp\left((r - q - \frac{1}{2}\sigma^2)T + \sigma W_T^*\right) < L\right\} \\ &= \mathbb{P}^*\left\{(r - q - \frac{1}{2}\sigma^2)T + \sigma W_T^* < \ln(L/V_0)\right\} \\ &= \mathbb{P}^*\left\{W_T^* < \frac{\ln(L/V_0) - (r - q - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right\}. \end{aligned}$$

Since $W_T^* \sim \mathcal{N}(0, T)$, we finally get

$$P_M^d(T) = \Phi\left(\frac{\ln(L/V_0) - (r - q - \sigma^2/2)T}{\sigma\sqrt{T}}\right), \tag{4.1.2}$$

where Φ is the distribution function of a standard normal random variable.

In the EMM approach, the prices of the contingent claims are defined as the expected value of the discounted payoffs under the martingale probability $\mathbb{P}^*$, the risk-neutral probability measure. So, at any time t, the price of a zero coupon bond $D_M(t, T)$ is given by

$$\begin{aligned} D_M(t, T) &= \mathbb{E}^*[B(t, T) * \text{Payoff}|\mathcal{F}_t] \\ &= \mathbb{E}^*[B(t, T) * \left(V_T \mathbf{1}_{\{V_T < L\}} + L\mathbf{1}_{\{V_T \geq L\}}\right)|\mathcal{F}_t]. \end{aligned}$$

By computing the above expectation, we obtain the value of the firm's zero coupon bond:

$$D_M(t, T) = V_t e^{-q(T-t)}\Phi(-d_1) + LB(t, T)\Phi(d_2), \tag{4.1.3}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(V_t/L) + (r - q + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}, \\ d_2 &= \frac{\ln(V_t/L) + (r - q - \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}. \end{aligned}$$

The subscript M, in (4.1.3) for instance, means that the default probabilities and the prices are under the assumption of the Merton model.

4.1.2 Black-Cox model

In Black-Cox setting, the default event may occur not only at the debt's maturity, but also at dates prior to maturity. The default time is defined as the first-passage time (or the first hitting time) of the firm value process V to a prescribed barrier H that can be either deterministic or stochastic. Here, we assume that H is deterministic and constant. Hence, the default time τ is defined by

$$\tau := \inf\{t \in [0, T] : V_t \leq H\}.$$

If the default occurs prior to T, the bondholders receive a fraction R of the face value L. The value of R is called the *recovery rate*. We will assume that the bondholders receive the recovery at maturity T, even if the default event occurred prior to T. So the payoff function of the bond and the default probability $P^d_{BC}(T)$ are given by

$$\text{Payoff} = \begin{cases} RL, & \tau \leq T, \\ L, & \tau > T, \end{cases} \tag{4.1.4}$$

and

$$\begin{aligned} P^d_{BC}(T) &= \mathbb{P}^*\{\tau \leq T\} \\ &= \mathbb{P}^*\{\min V_t < H\} \\ &= \Phi\left(\frac{\ln(H/V_0) - (r - q - \sigma^2/2)T}{\sigma\sqrt{T}}\right) \\ &\quad + \left(\frac{H}{V_0}\right)^{\frac{2(r-q)}{\sigma^2}-1} \Phi\left(\frac{\ln(H/V_0) + (r - q - \sigma^2)T}{\sigma\sqrt{T}}\right), \end{aligned} \tag{4.1.5}$$

respectively, where Φ denotes the distribution function of a standart normal random variable.

At time t, the price $D_{BC}(t,T)$ of the zero-coupon bond with maturity T under the Black-Cox model is given as follows:

$$\begin{aligned}
D_{BC}(t,T) &= \mathbb{E}^*[B(t,T) * \text{Payoff}|\mathcal{F}_t] \\
&= \mathbb{E}^*[B(t,T) * \left(L\mathbf{1}_{\{\tau>T\}} + L \cdot R\mathbf{1}_{\{\tau\leq T\}}\right)|\mathcal{F}_t] \\
&= B(t,T)L\left[\Phi\left(\frac{\ln(H/V_0) - (r-q-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)\right. \\
&\quad \left. + \left(\frac{H}{V_0}\right)^{\frac{2(r-q)}{\sigma^2}-1}\Phi\left(\frac{\ln(H/V_0) + (r-q-\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\right].
\end{aligned}$$

The subscript BC means that the default probabilities and the prices are under the assumption of the Black-Cox model.

For more details of the Merton and Black-Cox models, see [2].

4.2 Credit Risk Modeling under the VG model

In this section, the credit risk will be studied under VG setting. So from now on the value process of the firm will evaluate under the VG model as defined in (3.4.2), that is,

$$V_t = V_0 \exp\left((r-q)t + X_t + \omega t\right),$$

where $\omega = \frac{1}{\nu}\ln(1-\theta\nu-\frac{1}{2}\sigma^2\nu)$, X_t is the VG process, r and q are the constant continuously compounded interest rate and the continuously compounded dividend yield, respectively. This pricing process of assets is assumed to be under the risk neutral probability measure $\mathbb{P}^*$ which was defined in Section 4.1. When pricing the contingent claims under the BS model, we could give a closed form solution for these prices. However, in the case of jump processes, we cannot find a closed form solution in general: even if we can, it may not be easy to work with efficiently. So, the pricing of the contingent claims under jump processes generally uses the partial differential equations (PDE) approach as an alternative. Of course another easily established method is the Monte-Carlo method.

4.2.1 Merton type VG model

The default probability $P^d_{Mvg}(T)$ is given by

$$\begin{aligned} P^d_{Mvg}(T) &= \mathbb{P}^*\{V_T < L\} \\ &= \mathrm{F}\left(\ln \frac{L}{V_0}\right), \end{aligned}$$

where F is the distribution function of the log returns Z_t which are given by

$$\begin{aligned} Z_t &= \ln\left(\frac{V_t}{V_0}\right) \\ &= (r-q)t + X_t + \frac{t}{\nu}\ln(1-\theta\nu-\frac{1}{2}\sigma^2\nu) \end{aligned} \qquad (4.2.1)$$

for all t. Therefore from (3.4.4), the default probability becomes

$$P^d_{Mvg} = \int_0^\infty \Phi\left(\frac{l-(r-q+\omega)T-\theta g}{\sigma\sqrt{g}}\right)\nu^{-T/\nu}\,\frac{g^{\frac{T}{\nu}-1}e^{-g/\nu}}{\Gamma(T/\nu)}\,\mathrm{d}g. \qquad (4.2.2)$$

4.2.2 Black-Cox type VG model

In the VG type Black-Cox model we will be interested in the survival probability instead of the default probability because we will need the survival probabilities in order to price the CDS.

If we denote by $P^s(t)$ the risk-neutral survival probability between 0 and t, then

$$\begin{aligned} P^s(t) &= \mathbb{P}^*\{V_s > H, \text{for all } 0 \le s \le t\} \\ &= \mathbb{P}^*\left\{\min_{0\le s\le t} V_s > H\right\} \\ &= \mathbb{E}^*\left[\mathbf{1}_{\{\min_{0\le s\le t} V_s > H\}}\right], \end{aligned} \qquad (4.2.3)$$

where $\mathbf{1}_{\{B\}}$ denotes the indicator function.

The point here is that the default probabilities can be written in closed forms such as in (4.1.2), (4.1.5) and (4.2.2), for all the models discussed. However, the survival probability in the Black-Cox type VG model cannot be written in the closed form. However, it can be easily computed by relating it to a barrier option value and then by pricing it. The following section gives an introduction to the barrier options and pricing methods.

4.3 Barrier Options

Barrier options are a kind of path dependent options whose payoffs are determined by whether or not the extremal values of the underlying asset's prices cross a predetermined level, which is called a *barrier*. In fact, this simple definition explains the connection between structural credit risk models and barrier options.

There are mainly two categories of barrier options: *knock-out* and *knock-in*. A knock-out option ceases to exist when the underlying asset's price crosses the barrier while a knock-in option becomes active when the barrier is crossed.

These two categories are also classified according to barrier level. When the barrier is set below the initial underlying asset's price, the option is called a *down* option. On the other hand, if the barrier is set above the initial underlying asset's price, the option is called an *up* option. As a consequence there are eight types of barrier options: down-and-out, down-and-in, up-and-out, up-and-in options, each of which can be either call or put options.

Since the down-and-out option will be used in this work, its definition will be given explicitly and it will then be priced.

Definition 4.3.1. Let B denote the barrier, such that B $< S_0$. Then the down-and-out option's payoff function is given by

$$\text{Payoff} = \begin{cases} \Lambda(S_T), & \text{if } \min_{0 \leq t \leq T} S_t > B, \\ 0, & \text{otherwise}, \end{cases}$$

where $\Lambda(S_T)$ is the payoff function of the ordinary European option which is defined as $\Lambda(S_T) = (S_T - K)^+$ for the call option and $\Lambda(S_T) = (K - S_T)^+$ for the put option with strike prices K.

The above definition can also be written for call or put options:

$$\text{Payoff}^C = \begin{cases} S_T - K, & \text{if } S_T > K, \ \min_{0\le t\le T} S_t > B, \\ 0, & \text{otherwise,} \end{cases}$$

$$\text{Payoff}^P = \begin{cases} K - S_T, & \text{if } S_T < K, \ \min_{0\le t\le T} S_t > B, \\ 0, & \text{otherwise,} \end{cases}$$

respectively. Then, the prices of these options are

$$V^C = e^{-rT}\mathbb{E}^*[X^{call}] = e^{rT}(S_T - K)\mathbb{P}^*\{S_T > K, \min_{0\le t\le T} S_t > B\}$$

and

$$V^P = e^{-rT}\mathbb{E}^*[X^{call}] = e^{-rT}(K - S_T)\mathbb{P}^*\{S_T < K, \min_{0\le t\le T} S_t > B\},$$

where r is the risk-free interest rate.

Sometimes, the payoff of a barrier option can be a fixed amount of money R, just as digital options. In this case the payoff function and the price of the down-and-out barrier option become

$$\text{Payoff} = \begin{cases} R, & \min_{0\le t\le T} S_t > B, \\ 0, & \text{otherwise,} \end{cases} \tag{4.3.1}$$

and

$$\begin{aligned} \text{BDOB} &= e^{-rT}\mathbb{E}^*[\text{Payoff}] = e^{-rT}\mathbb{E}^*[\mathrm{R}\mathbf{1}_{\{\min_{0\le t\le T} S_t > B\}}] \\ &= e^{-rT}\mathrm{R}\cdot\mathbb{P}^*\left\{\min_{0\le t\le T} S_t > B\right\}. \end{aligned} \tag{4.3.2}$$

Now we will call this type of barrier option a *binary (digital) down-and-out barrier option* (BDOB) and take $R = 1$ in order to derive the survival probability in the VG type Black-Cox model. If we compare this price with the survival probability of the VG type Black-Cox model, we see that the survival probability is the same as the expectation in the BDOB option price with the underlying asset value V_t instead of S_t. So, we calculate the survival probability $P^s(T)$ from the equation

$$P^s(T) = e^{rT}\text{BDOB}. \tag{4.3.3}$$

When the asset prices are defined as in the BS model, the probability in the pricing equation, (4.3.2), has a closed form solution. However, in the VG model, there is no such closed form solution for this probability and therefore for pricing. On the other hand, it is possible to use Monte Carlo techniques or numerical solution of an appropriate PIDE. In the sequel, we will price the BDOB option under the BS and VG models via these two pricing methods.

4.3.1 Pricing BDOB option under BS Model

In order to price the BDOB option, we need the probability distribution of the minimum of a Brownian motion with drift. So this distribution will be given first.

Let Y_t be a Brownian motion with drift, i.e.,

$$Y_t = \alpha t + \sigma W_t,$$

where W_t is the standard Brownian motion. Further define

$$m(T) = \min_{0 \le t \le T} Y_t.$$

The following corollary gives the distribution of the minimum of the Brownian motion Y_t.

Corollary 4.3.2. *For every $y \le 0$, the following formula is valid:*

$$\mathbb{P}\{m(T) > y\} = \Phi\left(\frac{-y + \alpha T}{\sigma\sqrt{T}}\right) - e^{2\alpha y \sigma^{-2}} \Phi\left(\frac{y + \alpha T}{\sigma\sqrt{T}}\right), \tag{4.3.4}$$

where Φ is the distribution function of the standard normal distribution.

The detailed proof of the corollary can be found in [17].

Now, we price the BDOB option with underlying asset price S_t under BS model. The option's payoff function and the price were given in equations (4.3.1) and (4.3.2) for $R = 1$ as follows:

$$\text{Payoff} = \mathbf{1}_{\{\min_{0 \le t \le T} S_t > B\}}$$

and

$$\mathrm{V} = e^{-rT}\mathbb{E}^*[\mathbf{1}_{\{\min_{0\le t\le T} S_t > B\}}] = e^{-rT}\mathbb{P}^*\left\{\min_{0\le t\le T} S_t > B\right\}.$$

The asset prices in the BS model are given by the Geometric Brownian motion as in (4.1.1),

$$S_t = S_0 \exp\left((r - q - \frac{1}{2}\sigma^2)t + \sigma W_t^*\right),$$

where W_t^* is the standard Brownian motion under the risk neutral probability measure $\mathbb{P}^*$. If we define Y_t as a Brownian motion with drift $\alpha = r - q - \frac{1}{2}\sigma^2$, that is,

$$Y_t = \alpha t + \sigma W_t^*,$$

then the asset prices are given in terms of Y_t as

$$S_t = S_0 \exp(Y_t).$$

This helps us write the minimum value of S_t in terms of the minimum value of Y_t. Namely,

$$\min_{0\le t\le T} S_t = S_0 \exp(m(T)),$$

where $m(T) = \min_{0\le t\le T} Y_t$. Hence, the probability in the price of the BDOB option is

$$\begin{aligned}
\mathbb{P}^*\{\min_{0\le t\le T} S_t > B\} &= \mathbb{P}^*\{V_0 \exp(m(T))\} \\
&= \mathbb{P}^*\left\{m(T) > \ln\left(\frac{B}{V_0}\right)\right\}.
\end{aligned}$$

By the help of Corollary 4.3.2, the price of the BDOB option is then found to be

$$\begin{aligned}
\mathrm{BDOB} &= e^{-rT}\mathbb{P}^*\{\min_{0\le t\le T} S_t > B\} = e^{-rT}\mathbb{P}^*\left\{m(T) > \ln\left(\frac{B}{V_0}\right)\right\} \\
&= e^{-rT}\Phi\left(\frac{-y + \alpha T}{\sigma\sqrt{T}}\right) - e^{\frac{2}{\sigma^2}\alpha y}\Phi\left(\frac{y + \alpha T}{\sigma\sqrt{T}}\right),
\end{aligned}$$

where

$$\alpha = r - q - \frac{1}{2}\sigma^2 \quad \text{and} \quad y = \ln\left(\frac{B}{V_0}\right)$$

Here, the terms $\frac{-y+\alpha T}{\sigma\sqrt{T}}$, $\frac{y+\alpha T}{\sigma\sqrt{T}}$, $e^{\frac{2}{\sigma^2}\alpha y}$ are explicitly given by the initial parameters as follows:

$$\begin{aligned}
\frac{-y+\alpha T}{\sigma\sqrt{T}} &= \frac{\ln\left(\frac{V_0}{B}\right)+\left(r-q-\frac{1}{2}\sigma^2\right)}{\sigma\sqrt{T}} \\
\frac{y+\alpha T}{\sigma\sqrt{T}} &= \frac{\ln\left(\frac{B}{V_0}\right)+\left(r-q-\frac{1}{2}\sigma^2\right)}{\sigma\sqrt{T}} \\
e^{\frac{2}{\sigma^2}\alpha y} &= e^{\frac{2}{\sigma^2}\left(r-q-\frac{1}{2}\sigma^2\right)\ln\left(\frac{B}{V_0}\right)} = e^{\left(\frac{2r}{\sigma^2}-1\right)\ln\left(\frac{B}{V_0}\right)} \\
&= \left(\frac{B}{V_0}\right)^{\frac{2r}{\sigma^2}-1}.
\end{aligned}$$

Replacing them in the value of the BDOB option yields

$$\begin{aligned}
\text{BDOB} &= e^{-rT}\Phi\left(\frac{\ln\left(\frac{V_0}{B}\right)+\left(r-q-\frac{1}{2}\sigma^2\right)}{\sigma\sqrt{T}}\right) \\
&\quad + e^{-rT}\left(\frac{B}{V_0}\right)^{\frac{2r}{\sigma^2}-1}\Phi\left(\frac{\ln\left(\frac{B}{V_0}\right)+\left(r-q-\frac{1}{2}\sigma^2\right)}{\sigma\sqrt{T}}\right). \quad (4.3.5)
\end{aligned}$$

where Φ is the distribution function of the standard normal random variable.

Note that, under BS model, the price function of BDOB option is very similar to the default probabiliy of the Black-Cox model given in (4.1.5). This is because of the relation between the BDOB option price and the survival probability of the Black-Cox model. Although we have written the default probability in (4.1.5) directly, its verification can be made similarly as it has been made for the price of BDOB option above.

4.3.2 Pricing the BDOB Option under the VG Model

Pricing the BDOB option under the VG model is more difficult than that under the BS model due to the fact that asset paths have jumps because of the VG process. The two methods commonly used for pricing are the Monte Carlo simulation and the Partial Differential equation (PDE) that is satisfied by the price process. Since the VG asset paths are defined as jump processes in our case, the

PDE has an integral term which comes from jumps in asset paths. This kind of differential equation that contains also an integral term is called as the Partial Integro Differential equation (PIDE) which can be solved via some numerical methods.

The Monte Carlo method is also an alternative for pricing, but, it is considered to be more time consuming when compared to the PIDE method. However, we will price the BDOB option first via Monte Carlo, and then, via a numerical solution of the associated PIDE.

Monte Carlo Method

The Monte-Carlo method is based on the generation of samples of an interested random variable and then the estimation of its parameters. For example, we will be interested in the estimation of the parameters of a random variable X with mean μ and variance σ. In Monte-Carlo method, we construct a sequence of an independent sample, say x_i, by drawing from the same distribution as of X, then, using this sample, we calculate the sample mean

$$\hat{X}_N = \frac{1}{N}\sum_{i=1}^{N} x_i = \frac{1}{N} S_N,$$

which is an estimator for the mean μ. Since $\mathbb{E}[\hat{X}_N] = \mu$, the estimator $\hat{X}_N$ is an *unbiased* estimator. Indeed, by the *strong law of large numbers*, for the independent identically distributed random variables x_i, the random variable $\hat{X}_N = \frac{S_N}{N}$ converges to μ with probability one:

$$\mathbb{P}\left\{\lim_{N\to\infty} \frac{S_N}{N} = \mu\right\} = 1.$$

The quality of the estimator $\hat{X}_N$ is quantified by considering its variance, or the expected value of the *squared error*, that is,

$$\mathbb{E}[(\hat{X}_N - \mu)^2] = \frac{\sigma^2}{N},$$

where σ^2 may be estimated by the sample variance

$$\hat{\sigma}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \hat{X}_N)^2.$$

Clearly, the number N of replications improves the estimate although the convergence of the method is slow.

The Monte-Carlo method is used in finance because of the fundamental implication of the asset pricing theory, that is, in a risk neutral market, the price of a contingent claim is the expected value of the discounted payoff. Since pricing means finding an expected value, using the Monte-Carlo method is reasonable. Here the random variable which will be simulated are the future payoffs. Since there is a stochastic process avaliable for asset prices, we can simulate the possible future prices of the assets and then the possible payoffs of the contingent claim. In the following, we describe the simulation of the VG process both as a time-changed process and as a difference of two independent Gamma processes.

***As a time-changed Brownian motion*:** As defined in (3.1.5), the VG process with parameters θ, ν, and σ is

$$X(t;\sigma,\nu,\theta) = \theta G(t) + \sigma W(G(t)),$$

where W is the Brownian motion and $G(t)$ is the Gamma process with the shape parameter $1/\nu$ and the scale parameter ν. The simulation algorithm of the VG process at time points $\Delta t, 2\Delta t, \ldots n\Delta t$ is as follows:

1. *Given θ, ν, σ.*

2. *Simulate n independent gamma variables,*

$$\Delta G(i) \sim Gamma\left(\frac{\Delta t}{\nu}, \nu\right), \qquad i = 1, \ldots, n.$$

and n i.i.d standard normal random variables,

$$N(i) \sim \mathcal{N}(0,1), \qquad i = 1, \ldots, n.$$

3. *Set*

$$\Delta X(i) = \theta \Delta G(i) + \sigma \sqrt{\Delta G(i)} N(i), \qquad i = 1, \ldots, n.$$

4. *The VG process X_t is then*

$$X(i\Delta t) = \sum_{k=1}^{i} \Delta X(k), \qquad i = 1, \ldots, n.$$

***As a difference of two independent process*:** As defined in (3.2.1), the VG process with parameters $\mu_p, \mu_n, \nu_p, \nu_n$ is

$$X(t; \sigma, \nu, \theta) = G_p(t; \mu_p, \nu_p) - G_n(t; \mu_n, \nu_n),$$

where G_p and G_n are Gamma processes with the shape parameters $\frac{\mu_p^2}{\nu_p}\Delta t$, $\frac{\mu_n^2}{\nu_n}\Delta t$ and the scale parameters $\frac{\nu_p}{\mu_p}, \frac{\nu_n}{\mu_n}$, respectively. The simulation algorithm of the VG process at time points $\Delta t, 2\Delta t, \ldots n\Delta t$ is as follows:

1. *Given μ_p, ν_p; μ_n, νn.*

2. *Simulate n independent gamma variables*

$$\Delta G_p(i) \sim Gamma\left(\frac{\mu_p^2}{\nu_p}\Delta t, \frac{\nu_p}{\mu_p}\right), \qquad i = 1, \ldots, n.$$

and n independent gamma variables

$$\Delta G_n(i) \sim Gamma\left(\frac{\mu_n^2}{\nu_n}\Delta t, \frac{\nu_n}{\mu_n}\right), \qquad i = 1, \ldots, n.$$

3. *Set*

$$G_p(i\Delta t) = \sum_{k=1}^{i} \Delta G_p(i)$$

and

$$G_n(i\Delta t) = \sum_{k=1}^{i} \Delta G_n(i).$$

4. *The VG process X_t is then*

$$X(i\Delta t) = G_p(i\Delta t) - G_n(i\Delta t), \qquad i = 1, \ldots, n.$$

Finally, after simulating the VG process, we can now simulate VG asset priceses as in the following algorithm:

1. Simulate the VG process

$$X(i\Delta t), \qquad i = 1, \ldots, n,$$

   by using one of the above algorithms.

2. Set the asset prices

$$V(i\Delta t) = V(0)\exp\Big\{(r-q)i\Delta t + X(i\Delta t) + \frac{1}{\nu}\ln(1-\theta\nu-\frac{1}{2}\sigma^2\nu)i\Delta t\Big\},$$

   $i = 1, \ldots, n.$

PDE Method

The pricing function of any contingent claim generally depends on the underlying asset prices and time t: $V(S,t)$. The PDE method is based on giving the partial differention equation of the pricing function $V(S,t)$. In the Black Sholes model, where the assets evaluates GBM, this PDE is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{4.3.6}$$

with boundary conditions depending on the contingent claim. Under Lévy models, the PDE is, however, of the form

$$\begin{aligned} &\int_{-\infty}^{\infty}\left(V(Se^y,t) - V(S,t) - S(e^y-1)\frac{\partial V}{\partial S}(S,t)\right)k(y)\mathrm{d}y \\ &+ \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \end{aligned} \tag{4.3.7}$$

with similar boundary conditions depending on the contingent claim. The only difference between these two PDE in (4.3.6) and (4.3.7) is the integral term that appears in the latter. This integral term is due to the existence of jumps. Such a PDE is called Partial Integro Diffential Equation (PIDE).

The following proposition gives the PIDE for any contingent claim under the VG setting.

Proposition 4.3.3. *Let $V(S,t)$ be the price of any contingent claim at time t written on the asset S which is defined as*

$$S_t = S(0)\exp((r-q)t + X_t + \omega t),$$

where $\omega = \frac{1}{\nu}\ln\left(1-\theta\nu-\frac{1}{2}\sigma^2\nu\right)$, X_t is the VG process, r and q are the constant continuously compounded interest rate and the continuous dividend yield, respectively. Then the PIDE of the function $V(S,t)$ is given as

$$\begin{aligned}-rw(x,\tau) - \frac{\partial w}{\partial \tau}(x,\tau) + (r-q+\omega)\frac{\partial w}{\partial x}(x,\tau) & \\ + \int_{-\infty}^{\infty} \left(w(x+y,\tau) - w(x,\tau)\right) k(y)\mathrm{d}y &= 0, \end{aligned} \qquad (4.3.8)$$

where $w(x,\tau) = V(S,t)$ for $x = \log(S)$ and $\tau = T-t$.

Proof. Let $\tilde{V}(S_t,t) = e^{-rt}V(S_t,t)$ be the discounted price of a contingent claim where r is the risk free interest rate. Applying the Itô formula to the discounted price gives

$$\begin{aligned}\tilde{V}(S_t,t) &= \tilde{V}(S_0,0) \\ &+ \int_0^t e^{-ru}\left(-rV(S_{u-},u) + \frac{\partial V}{\partial u}(S_{u-},u) + \frac{\sigma_S}{2}\frac{\partial^2 V}{\partial S^2}(S_{u-},u)\mathrm{d}u\right)\mathrm{d}u \\ &+ \int_0^t e^{-ru}\frac{\partial V}{\partial S}(S_{u-},u)\,\mathrm{d}S(u) \\ &+ \int_0^t\int_{-\infty}^{\infty} e^{-ru}\Bigg(V(S_{u-}+s,u) - V(S_{u-},u) \\ &\qquad - s\frac{\partial V}{\partial S}(S_{u-},u)\Bigg) J_S(\mathrm{d}u,\mathrm{d}s). \end{aligned} \qquad (4.3.9)$$

Here the parameter σ_S is the sigma term in the Lévy triplet of the stock price dynamics S_t under the VG model. But if we look at the semimartingale decomposition of the stock price dynamics in Proposition 3.4.3, we see that the sigma term of the stock price dynamics is zero. Hence, that term vanishes.

The last summand in (4.3.9) can be written as an integral with respect to the jump measure of the VG process X_t instead of the jump measure of the stock price S_t, since the stock price process jumps whenever the VG process X_t jumps: this follows from

$$J_S(\mathrm{d}t, \mathrm{d}s) = J_X(\mathrm{d}t, \mathrm{d}y),$$

where s refers to the jumps in the price process S_t and y refers to the jumps in the VG process X_t. The jumps s in the price process can further be written in terms of the jumps y in VG process as

$$\begin{aligned}
S_{t-} + s &= S_0 e^{(r-q)t + X_{t-} + y + \omega t} \\
&= S_0 e^{(r-q)t + X_{t-} + \omega t} e^y \\
&= S_{t-} e^y \\
s &= S_{t-} e^y - S_{t-} \\
&= S_{t-}(e^y - 1).
\end{aligned}$$

Inserting $\mathrm{d}S_t$, by using the semimartingale decomposition (3.4.8) of S_t in (4.3.9) and using the jumps s above, we obtain,

$$\begin{aligned}
\tilde{V}(S_t, t) &= \tilde{V}(S_0, 0) + \int_0^t e^{-ru} \left(-rV + \frac{\partial V}{\partial u} \right) \mathrm{d}u \\
&\quad + \int_0^t e^{-ru} \frac{\partial V}{\partial S} \left(S_{u-}(r-q)\mathrm{d}u + S_{u-} \int_{-\infty}^{\infty} (e^y - 1)\, \tilde{J}_X(\mathrm{d}u, \mathrm{d}y) \right) \\
&\quad + \int_0^t \int_{-\infty}^{\infty} e^{-ru} \left(V(S_{u-}e^y, u) - V - S_{u-}(e^y - 1) \frac{\partial V}{\partial S} \right) J_X(\mathrm{d}u, \mathrm{d}y) \\
&= \tilde{V}(S_0, 0) + \int_0^t e^{-ru} \left(-rV + \frac{\partial V}{\partial u} - (r-q) S_{u-} \frac{\partial V}{\partial S} \right) \mathrm{d}u \\
&\quad + \int_0^t \int_{-\infty}^{\infty} e^{-ru} S_{u-}(e^y - 1) \frac{\partial V}{\partial S}\, \tilde{J}_X(\mathrm{d}u, \mathrm{d}y) \\
&\quad + \int_0^t \int_{-\infty}^{\infty} e^{-ru} \left(V(S_{u-}e^y, u) - V - S_{u-}(e^y - 1) \frac{\partial V}{\partial S} \right) J_X(\mathrm{d}u, \mathrm{d}y),
\end{aligned}$$

where V refers to $V(S_{u-}, u)$. Adding and substracting the compensator of the last integral, that is,

$$\mp \int_0^t \int_{-\infty}^{\infty} e^{-ru} \left(V(S_{u-}e^y, u) - V - S_{u-}(e^y - 1) \frac{\partial V}{\partial S} \right) k(y) \mathrm{d}y\, \mathrm{d}u.$$

yield

$$\begin{aligned}\tilde{V}(S_t,t) &= \tilde{V}(S_0,0)+\int_0^t e^{-ru}\left(-rV+\frac{\partial V}{\partial u}-(r-q)S_{u-}\frac{\partial V}{\partial S}\right)\,\mathrm{d}u\\
&+\int_0^t\!\!\int_{-\infty}^{\infty} e^{-ru}\left(V(S_{u-}e^y,u)-V-S_{u-}(e^y-1)\frac{\partial V}{\partial S}\right)k(y)\mathrm{d}y\,\mathrm{d}u\\
&+\int_0^t\!\!\int_{-\infty}^{\infty} e^{-ru}\left(V(S_{u-}e^y,u)-V-S_{u-}(e^y-1)\frac{\partial V}{\partial S}\right)\tilde{J}_X(\mathrm{d}u,\mathrm{d}y),\end{aligned}$$

which can be simplified to

$$\begin{aligned}\tilde{V}(S_t,t) &= \tilde{V}(S_0,0)+\int_0^t e^{-ru}\Bigg(-rV+\frac{\partial V}{\partial u}-(r-q)S_{u-}\frac{\partial V}{\partial S}\\
&+\int_{-\infty}^{\infty}\left(V(S_{u-}e^y,u)-V-S_{u-}(e^y-1)\frac{\partial V}{\partial S}\right)k(y)\mathrm{d}y\Bigg)\mathrm{d}u\\
&+\int_0^t\!\!\int_{-\infty}^{\infty} e^{-ru}\left(V(S_{u-}e^y,u)-V-S_{u-}(e^y-1)\frac{\partial V}{\partial S}\right)\tilde{J}_X(\mathrm{d}u,\mathrm{d}y).\end{aligned}$$

Since the discounted prices are martingale under the risk neutral measure, the finite variation part must vanish. In other words,

$$\begin{aligned}0 &= -rV(S_{u-},u)+\frac{\partial V}{\partial u}(S_{u-},u)-(r-q)S_{u-}\frac{\partial V}{\partial S}(S_{u-},u)\\
&+\int_{-\infty}^{\infty}\left(V(S_{u-}e^y,u)-V(S_{u-},u)-S_{u-}(e^y-1)\frac{\partial V}{\partial S}(S_{u-},u)\right)k(y)\mathrm{d}y.\end{aligned}$$

By making the change of variables $x=\ln(S)$ and $\tau=T-t$, as well as noting

$$\begin{aligned}w(x,\tau) &= V(S,t),\\
\frac{\partial w}{\partial x}(x,\tau) &= S\frac{\partial V}{\partial S}(S,t),\\
\frac{\partial w}{\partial \tau}(x,\tau) &= -\frac{\partial V}{\partial t}(S,t),\\
w(x+y,\tau) &= V(Se^y,t),\end{aligned}$$

we obtain the PIDE as a function of $w(x,\tau)$:

$$\begin{aligned}0 &= -rw(x,\tau)-\frac{\partial w}{\partial \tau}(x,\tau)+(r-q)\frac{\partial w}{\partial x}(x,\tau)\\
&+\int_{-\infty}^{\infty}\left(w(x+y,\tau)-w(x,\tau)-(e^y-1)\frac{\partial w}{\partial x}(x,\tau)\right)k(y)\mathrm{d}y.\end{aligned}$$

Since

$$\int_{-\infty}^{\infty} (e^y - 1)\, k(y)\mathrm{d}y = -\omega$$

by (3.4.10), the final equation that is to be solved returns to be

$$\begin{aligned} -rw(x,\tau) - \frac{\partial w}{\partial \tau}(x,\tau) + (r - q + \omega)\frac{\partial w}{\partial x}(x,\tau) & \\ + \int_{-\infty}^{\infty} \left(w(x+y,\tau) - w(x,\tau)\right) k(y)\mathrm{d}y &= 0. \end{aligned}$$

□

Numerical Solution of the PIDE: In order to solve the PIDE, the finite difference method will be used. In the finite difference discretization of the PIDE, a mixture of two methods is applied for the evaluation of integral term. The integrand is expanded near its singularity of $y = 0$, and this part is treated implicitly. The rest of the integrand is treated explicitly. On the other hand, the differential term of the PIDE is discretized by a fully implicit method.

Consider M equally spaced sub-intervals in the τ-direction, and N equally spaced sub-intervals on $[x_0, x_N]$. Denote

$$\Delta x = \frac{x_N - x_0}{N}, \qquad \Delta \tau = \frac{T}{M}$$

this leads to the following mesh on $[x_0, x_N] \times [0, T]$:

$$\mathcal{D} = \{(x_i, \tau_j) : x_i = x_0 + i\Delta x, i = 0, 1, \ldots, N; \tau_j = j\Delta\tau, j = 0, 1, \ldots, M\}.$$

Let $w_{i,j}$ be the discrete value of $w(x_i, \tau_j)$ on $\mathcal{D}$. Using the first order finite difference approximation for $\frac{\partial w}{\partial \tau}$ and the central difference for $\frac{\partial w}{\partial x}$, we obtain the following discrete equation at point (x_i, τ_{j+1})

$$\begin{aligned} -\frac{1}{\Delta\tau}(w_{i,j+1} - w_{i,j}) + (r - q - \omega)\frac{1}{2\Delta}(w_{i+1,j+1} - w_{i-1,j+1}) - rw_{i,j+1} & \\ + \int_{-\infty}^{\infty} \left(w(x_i + y, \tau_j) - w(x_i, \tau_j)\right) k(x)\mathrm{d}y &= 0. \end{aligned}$$

Multiplying by $\Delta\tau$ yields

$$\begin{aligned}-w_{i,j+1} + w_{i,j} + (r-q+\omega)\frac{\Delta\tau}{2\Delta x}w_{i+1,j+1} - (r-q+\omega)\frac{\Delta\tau}{2\Delta x}w_{i-1,j+1}&\\ - r\Delta\tau w_{i,j+1} + \Delta\tau\int_{-\infty}^{\infty}\left(w(x_i+y,\tau_j) - w(x_i,\tau_j)\right)k(y)\mathrm{d}y &= 0\end{aligned}$$

which can be simplified to

$$\begin{aligned}(r-q+\omega)\frac{\Delta\tau}{2\Delta x}w_{i-1,j+1} - (r-q+\omega)\frac{\Delta\tau}{2\Delta x}w_{i+1,j+1} + (1+r\Delta\tau)w_{i,j+1}&\\ = w_{i,j} + \Delta\tau\int_{-\infty}^{\infty}\left(w(x_i+y,\tau_j) - w(x_i,\tau_j)\right)k(y)\mathrm{d}y.&\end{aligned}$$

Now let us evaluate the integral part. We divide the domain of the integral into six sub-intervals. By this we obtain six integrals: $A_1, A_2, A_3, A_4, A_5, A_6$

$$\int_{-\infty}^{\infty}\left(w(x_i+y,\tau_j) - w_{i,j}\right)\, k(y)\mathrm{d}y = A_1 + A_2 + A_3 + A_4 + A_5 + A_6$$

where $k(y)\mathrm{d}y$ is the Lévy measure of the VG process defined in Corollary 3.2.2. Here, the integrals $A_1, A_2, A_3, A_4, A_5, A_6$ are defined as follows:

$$\begin{aligned}A_1 &= \int_{-\infty}^{x_0-x_i}\left[w(x_i+y,\tau_j) - w_{i,j}\right]k(y)\mathrm{d}y,\\ A_2 &= \int_{x_0-x_i}^{-\Delta x}\left(w(x_i+y,\tau_j) - w_{i,j}\right)k(y)\mathrm{d}y,\\ A_3 &= \int_{-\Delta x}^{0}\left(w(x_i+y,\tau_j) - w_{i,j}\right)\, k(y)\mathrm{d}y,\\ A_4 &= \int_{0}^{\Delta x}\left(w(x_i+y,\tau_j) - w_{i,j}\right)\, k(y)\mathrm{d}y,\\ A_5 &= \int_{\Delta x}^{x_N-x_i}\left(w(x_i+y,\tau_j) - w_{i,j}\right)k(y)\mathrm{d}y,\\ A_6 &= \int_{x_N-x_i}^{\infty}\left(w(x_i+y,\tau_j) - w_{i,j}\right)k(y)\mathrm{d}y,\end{aligned}$$

For $y \in [-\Delta x, 0]$, one can write

$$w(x_i+y,\tau_j) - w_{i,j} = \frac{w_{i-1,j} - w_{i,j}}{\Delta x}\, y + O(y^2),$$

so that the integral A_3 can be computed as

$$\begin{aligned} A_3 &= \int_{-\Delta x}^{0} \left(w(x_i+y,\tau_j) - w_{i,j}\right)\, k(y)\mathrm{d}y \\ &\cong \int_{-\Delta x}^{0} \frac{w_{i-1,j} - w_{i,j}}{\Delta x}\, y\, \frac{e^{-\lambda_n |y|}}{\nu |y|}\, \mathrm{d}y. \end{aligned}$$

This, furthermore, can be evaluated as

$$\begin{aligned} A_3 &= \int_{0}^{\Delta x} \frac{w_{i-1,j} - w_{i,j}}{\Delta x}\, y\, \frac{e^{-\lambda_n y}}{\nu y}\, \mathrm{d}y = \frac{w_{i-1,j} - w_{i,j}}{\nu \Delta x} \int_{0}^{\Delta x} e^{-\lambda_n} y\, \mathrm{d}y \\ &= \frac{1}{\nu \Delta x \lambda_n}(1 - e^{-\lambda_n \Delta x})(w_{i-1,j} - w_{i,j}). \end{aligned} \tag{4.3.10}$$

Similarly for $y \in [0, \Delta x]$, one can write

$$w(x_i + y, \tau_j) - w_{i,j} = \frac{w_{i+1,j} - w_{i,j}}{\Delta x}\, y + O(y^2),$$

so that the integral A_4 becomes

$$\begin{aligned} A_4 &= \int_{0}^{\Delta x} \left(w(x_i+y,\tau_j) - w_{i,j}\right) k(y)\mathrm{d}y \\ &\cong \frac{1}{\nu \Delta x \lambda_p}(1 - e^{-\lambda_p \Delta x})(w_{i+1,j} - w_{i,j}). \end{aligned} \tag{4.3.11}$$

For $y \in (x_0 - x_i, -\Delta x)$, since $x_i = x_0 + i\Delta x$, the interval over which the integral A_2 is taken turns into

$$(x_0 - x_i, -\Delta x) = (-i\Delta x, -\Delta x)$$

and the integral can be written as the sum of the sub-integrals over $(-(k+1)\Delta x, -k\Delta x)$. That is,

$$\begin{aligned} A_2 &= \int_{x_0 - x_i}^{-\Delta x} \left(w(x_i+y,\tau_j) - w_{i,j}\right) k(y)\mathrm{d}y \\ &= \sum_{k=1}^{i-1} \int_{-(k+1)\Delta x}^{-k\Delta x} \left(w(x_i+y,\tau_j) - w_{i,j}\right)\, \frac{e^{-\lambda_n |y|}}{\nu |y|}\, \mathrm{d}y \\ &= \sum_{k=1}^{i-1} \int_{k\Delta x}^{(k+1)\Delta x} \left(w(x_i-y,\tau_j) - w_{i,j}\right)\, \frac{e^{-\lambda_n y}}{\nu y}\, \mathrm{d}y. \end{aligned}$$

Using linear interpolation on the interval $y \in [k\Delta x, (k+1)\Delta x]$, we may write $w(x_i - y, \tau_j)$ as follows:

$$w(x_i - y, \tau_j) \cong w_{i-k,j} + \frac{w_{i-k-1,j} - w_{i-k,j}}{\Delta x}(y - k\Delta x).$$

Therefore,

$$\begin{aligned} A_2 &\cong \sum_{k=1}^{i-1} \int_{k\Delta x}^{(k+1)\Delta x} \left(w_{i-k,j} + \frac{w_{i-k-1,j} - w_{i-k,j}}{\Delta x}(y - k\Delta x) - w_{i,j} \right) \frac{e^{-\lambda_n y}}{\nu y} \, \mathrm{d}y \\ &= \sum_{k=1}^{i-1} \int_{k\Delta x}^{(k+1)\Delta x} \left(w_{i-k,j} - w_{i,j} - k(w_{i-k-1,j} - w_{i-k,j}) \right) \frac{e^{-\lambda_n y}}{\nu y} \, \mathrm{d}y \\ &\quad + \sum_{k=1}^{i-1} \int_{k\Delta x}^{(k+1)\Delta x} \frac{w_{i-k-1,j} - w_{i-k,j}}{\Delta x} \, y \, \frac{e^{-\lambda_n y}}{\nu y} \, \mathrm{d}y. \end{aligned}$$

Moreover,

$$\begin{aligned} A_2 &= \sum_{k=1}^{i-1} \frac{1}{\nu} \left(w_{i-k,j} - w_{i,j} - k(w_{i-k-1,j} - w_{i-k,j}) \right) \int_{k\Delta x}^{(k+1)\Delta x} \frac{e^{-\lambda_n y}}{y} \, \mathrm{d}y \\ &\quad + \sum_{k=1}^{i-1} \frac{w_{i-k-1,j} - w_{i-k,j}}{\nu \Delta x} \int_{k\Delta x}^{(k+1)\Delta x} e^{-\lambda_n y} \, \mathrm{d}y \\ &= \sum_{k=1}^{i-1} \frac{1}{\nu} \left(w_{i-k,j} - w_{i,j} - k(w_{i-k-1,j} - w_{i-k,j}) \right) \int_{k\Delta x \lambda_n}^{(k+1)\Delta x \lambda_n} \frac{e^{-u}}{u} \, \mathrm{d}u \\ &\quad + \sum_{k=1}^{i-1} \frac{w_{i-k-1,j} - w_{i-k,j}}{\nu \Delta x} \left(\frac{-e^{-\lambda_n (k+1)\Delta x} + e^{-\lambda_n k \Delta x}}{\lambda_n} \right), \end{aligned}$$

and finally we obtain

$$\begin{aligned} A_2 &= \sum_{k=1}^{i-1} \frac{1}{\nu} \left(w_{i-k,j} - w_{i,j} - k(w_{i-k-1,j} - w_{i-k,j}) \right) \\ &\qquad \left(\operatorname{expint}(k\Delta x \lambda_n) - \operatorname{expint}\left((k+1)\Delta x \lambda_n\right) \right) \\ &\quad + \sum_{k=1}^{i-1} \frac{w_{i-k-1,j} - w_{i-k,j}}{\lambda_n \nu \Delta x} \left(e^{-\lambda_n k \Delta x} - e^{-\lambda_n (k+1)\Delta x} \right). \end{aligned} \tag{4.3.12}$$

For $y \in (\Delta x, x_N - x_i)$, since $x_i = x_0 + i\Delta x$, the interval over which the integral A_5 is taken turns into

$$(\Delta x, x_N - x_i) = (\Delta x, (N - i)\Delta x)$$

and the integral can be written as the sum of the sub-integrals over $(k\Delta x, (k+1)\Delta x)$. That is,

$$\begin{aligned} A_5 &= \int_{\Delta x}^{x_N - x_i} \left(w(x_i + y, \tau_j) - w_{i,j}\right) k(y) \mathrm{d}y \\ &= \sum_{k=1}^{N-i-1} \int_{k\Delta x}^{(k+1)\Delta x} \left(w(x_i + y, \tau_j) - w_{i,j}\right) \frac{e^{-\lambda_p y}}{\nu y} \mathrm{d}y. \end{aligned}$$

Using the linear interpolation again on the interval $y \in [k\Delta x, (k+1)\Delta x]$, we may write $w(x_i + y, \tau_j)$ as

$$w(x_i + y, \tau_j) = w_{i+k,j} + \frac{w_{i+k+1,j} - w_{i+k,j}}{\Delta x}(y - k\Delta x)$$

so that the integral becomes

$$\begin{aligned} A_5 &= \sum_{k=1}^{N-i-1} \int_{k\Delta x}^{(k+1)\Delta x} \left(w_{i+k,j} + \frac{w_{i+k+1,j} - w_{i+k,j}}{\Delta x}(y - k\Delta x) - w_{i,j} \right) \frac{e^{-\lambda_p y}}{\nu y} \mathrm{d}y \\ &= \sum_{k=1}^{N-i-1} \frac{1}{\nu} \left(w_{i+k,j} - w_{i,j} - k(w_{i+k+1,j} - w_{i+k,j})\right) \\ &\qquad \left(\mathrm{expint}(k\Delta x \lambda_p) - \mathrm{expint}\left((k+1)\Delta x \lambda_p\right)\right) \\ &\quad + \sum_{k=1}^{N-i-1} \frac{w_{i+k+1,j} - w_{i+k,j}}{\lambda_p \nu \Delta x} \left(e^{-\lambda_p k\Delta x} - e^{-\lambda_p (k+1)\Delta x}\right). \end{aligned} \tag{4.3.13}$$

For $y \in (-\infty, x_0 - x_i)$, since $x_0 - x_i = -i\Delta x$, the integral A_1 is

$$\begin{aligned} A_1 &= \int_{-\infty}^{x_0 - x_i} \left(w(x_i + y, \tau_j) - w_{i,j}\right) k(y) \mathrm{d}y \\ &= \int_{-\infty}^{-i\Delta x} \left(w(x_i + y, \tau_j) - w_{i,j}\right) \frac{e^{-\lambda_n |y|}}{\nu |y|} \mathrm{d}y \\ &= \int_{i\Delta x}^{\infty} \left(w(x_i - y, \tau_j) - w_{i,j}\right) \frac{e^{-\lambda_n y}}{\nu y} \mathrm{d}y. \end{aligned}$$

In the interval $(-\infty, -i\Delta x)$, $w(x_i + y, \tau_j) = 0$ since for $y \in (-\infty, -i\Delta x)$, we have

$$x_i + y \in (-\infty, x_i - i\Delta x) \Longrightarrow x_i + y \in (-\infty, x_0).$$

So, $w(x_i + y, \tau_j) = 0$. The integral A_1 in this case becomes

$$\begin{aligned} A_1 &= \int_{i\Delta x}^{\infty} (0 - w_{i,j}) \frac{e^{-\lambda_n y}}{\nu y} \mathrm{d}y = -\frac{1}{\nu} w_{i,j} \int_{i\Delta x \lambda_n}^{\infty} \frac{e^{-u}}{u} \mathrm{d}u \\ &= -\frac{1}{\nu} w_{i,j} \, \mathrm{expint}(i\Delta x \lambda_n). \end{aligned} \tag{4.3.14}$$

For $y \in (x_N - x_i, \infty)$, since $x_N - x_i = (N-i)\Delta x$, the integral A_6 is

$$\begin{aligned} A_6 &= \int_{x_N - x_i}^{\infty} (w(x_i + y, \tau_j) - w_{i,j})\, k(y) \mathrm{d}y \\ &= \int_{(N-i)\Delta x}^{\infty} (w(x_i + y, \tau_j) - w_{i,j})\, \frac{e^{-\lambda_p y}}{\nu y} \, \mathrm{d}y. \end{aligned}$$

Finally, in interval $((N-i)\Delta x, \infty)$, $w(x_i + y, \tau_j) = e^{-r\tau_j}$ since for $y \in ((N-i)\Delta x, \infty)$, we have

$$x_i + y \in (x_i + (N-i)\Delta x, \infty) \Longrightarrow x_i + y \in (x_N, \infty).$$

So, $w(x_i + y, \tau_j) = e^{-r\tau_j}$. Then the integral A_6 becomes

$$\begin{aligned} A_6 &= \int_{(N-i)\Delta x}^{\infty} \left(e^{-r\tau_j} - w_{i,j}\right) \frac{e^{-\lambda_p y}}{\nu y} \, \mathrm{d}y = \frac{1}{\nu} \left(e^{-r\tau_j} - w_{i,j}\right) \int_{(N-i)\Delta x \lambda_p}^{\infty} \frac{e^{-u}}{u} \, \mathrm{d}u \\ &= \frac{1}{\nu} \left(e^{-r\tau_j} - w_{i,j}\right) \mathrm{expint}((N-i)\Delta x \lambda_p). \end{aligned} \tag{4.3.15}$$

Having inserted all, we obtain the following finite difference approximation at points (x_i, τ_{j+1}):

$$A w_{i-1,j+1} + B_i w_{i,j+1} - C w_{i+1,j+1} = w_{i,j} + \nu^{-1} \Delta\tau R_{i,j}, \tag{4.3.16}$$

where the coefficients are

$$\begin{aligned} A &= (r - q + \omega)\frac{\Delta\tau}{2\Delta x} - (1 - e^{-\lambda_n \Delta x})\frac{\Delta\tau}{\nu \Delta x \lambda_n}, \\ B_i &= 1 + r\Delta\tau + (1 - e^{-\lambda_n \Delta x})\frac{\Delta\tau}{\nu \Delta x \lambda_n} + (1 - e^{-\lambda_p \Delta x})\frac{\Delta\tau}{\nu \Delta x \lambda_p} \\ &\quad + \frac{\Delta\tau}{\nu} \left(\mathrm{expint}(i\Delta x \lambda_n) + \mathrm{expint}((N-i)\Delta x \lambda_p)\right) \end{aligned}$$

for each $i = 1, \ldots, N$, and

$$C = (r - q + \omega)\frac{\Delta\tau}{2\Delta x} + (1 - e^{-\lambda_p \Delta x})\frac{\Delta\tau}{\nu \Delta x \lambda_p}.$$

The $R_{i,j}$ on the right-hand-side of (4.3.16) is

$$
\begin{aligned}
R_{i,j} \;=\; & \sum_{k=1}^{i-1} \left(w_{i-k,j} - w_{i,j} - k(w_{i-k-1,j} - w_{i-k,j})\right) \\
& \qquad\qquad\qquad \left(\text{expint}(k\Delta x\lambda_n) - \text{expint}\left((k+1)\Delta x\lambda_n\right)\right) \\
& + \sum_{k=1}^{i-1} \frac{1}{\lambda_n \Delta x} \left(w_{i-k-1,j} - w_{i-k,j}\right) \left(e^{-\lambda_n k\Delta x} - e^{-\lambda_n (k+1)\Delta x}\right) \\
& + \sum_{k=1}^{N-i-1} \left(w_{i+k,j} - w_{i,j} - k(w_{i+k+1,j} - w_{i+k,j})\right) \\
& \qquad\qquad\qquad \left(\text{expint}(k\Delta x\lambda_p) - \text{expint}\left((k+1)\Delta x\lambda_p\right)\right) \\
& + \sum_{k=1}^{N-i-1} \frac{1}{\lambda_p \Delta x} \left(w_{i+k+1,j} - w_{i+k,j}\right) \left(e^{-\lambda_p k\Delta x} - e^{-\lambda_p (k+1)\Delta x}\right) \\
& + e^{-r\tau_j}\, \text{expint}((N-i)\Delta x\lambda_p).
\end{aligned}
$$

The difference equation in (4.3.16) approximates the price of the BDOB option with the initial and boundary conditions

$$
\begin{aligned}
w(x_i, 0) &= 1, \\
w(x_0, \tau_j) &= 0, \\
w(x_N, \tau_j) &= e^{-r\tau_j}.
\end{aligned}
$$

Let us denote

$$
w^{(j)} := \begin{bmatrix} w_{1,j} \\ w_{1,j} \\ \vdots \\ w_{N-1,j} \end{bmatrix}
$$

and

$$
D^{(j)} := R^{(j)} + b^{(j)},
$$

where

$$R^{(j)} = \nu^{-1}\Delta\tau \begin{bmatrix} R_{1,j} \\ R_{2,j} \\ \vdots \\ R_{N-1,j} \end{bmatrix} \quad \text{and} \quad b^{(j)} = \begin{bmatrix} -Aw_{0,j+1} \\ 0 \\ \vdots \\ 0 \\ Cw_{N,j+1} \end{bmatrix}.$$

Therefore, (4.3.16) can be written in matrix-vector form as follows:

$$\mathbf{K}w^{(j+1)} = w^{(j)} + D^{(j)}, \tag{4.3.17}$$

where the coefficient matrix $\mathbf{K}$ is

$$\mathbf{K} = \begin{pmatrix} B_1 & -C & 0 & \dots & 0 & 0 \\ A & B_2 & -C & \dots & 0 & 0 \\ 0 & A & B_3 & \dots & 0 & 0 \\ \vdots & & & \ddots & \vdots & \vdots \\ \vdots & & & & B_{N-2} & -C \\ 0 & \dots & \dots & \dots & A & B_{N-1} \end{pmatrix}.$$

Here, the parameters are θ, σ, ν of the VG process and the parameters

$$\lambda_n = \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}} + \frac{\theta}{\sigma^2}, \lambda_p = \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}} - \frac{\theta}{\sigma^2}$$

are of the Lévy measure in Corollary 3.2.2 with

$$\omega = \frac{1}{\nu}\ln\left(1 - \theta\nu - \frac{1}{2}\sigma^2\nu\right)$$

.

4.4 Credit Default Swaps

Credit Default Swaps (CDS) are some sort of insurance contracts which help to trade the credit risk exposured. The buyer of the CDS makes periodic payments to the seller until the default event occurs. Namely, each payment will be made if the reference entity survives until the payment date. Payments are generally denoted by c. On the other hand, the seller of the CDS compensates the loss if the default event takes place. He/She can make physical payments which means compensation in full or can make a cash settlement which means the compensation of some of the national amount with rate $(1-R)$. The term R is called the *recovery rate*. All these payments and compensations are expected payments (EP) and expected losses (EL); because, they depend on the occurence of a default event. Under the risk neutral probability mesure, the price of the CDS is the discounted expected cash flow. If we think from the buyer's point of view, the expected cash flow and the price of the CDS are

$$\text{Cash Flow} = -\text{EP} + \text{EL}$$

and hence,

$$\text{CDS} = e^{-rT}[\text{Cash Flow}].$$

The payment amount c^* which makes the price of CDS zero is called the *par spread*. The following corollary states the formulation of the price and the par spread of a CDS contract.

Corollary 4.4.1. *Consider a CDS contract written on a reference obligation with maturity T and assume that the buyer makes payments continuously. The price and the par spread of the CDS contact are*

$$CDS = -c\int_0^T e^{-rt}P^s(t)\,\mathrm{d}t - (1-R)\int_0^T e^{-rt}\,\mathrm{d}P^s(t) \tag{4.4.1}$$

and

$$c^* = \frac{(1-R)\left(1 - e^{-rT}P^s(T) - r\int_0^T e^{-rt}P^s(t)\,\mathrm{d}t\right)}{\int_0^T e^{-rt}P^s(t)\,\mathrm{d}t}, \tag{4.4.2}$$

respectively.

Proof. First, let us assume that the buyer makes payments at fixed discrete times $t_1 < t_2 < \ldots < t_n = T$ and let $P^s(t)$ denote the survival probability until time

t, that is, the firm will survive until time t with a probability $P^s(t)$. Then the probability $Q(t) := 1 - P^s(t)$ becomes the default probability until time t. Since we assumed discrete time payments, the default event may happen between these times, $t_1 < t_2 < \ldots < t_n = T$. So we need to find the default probabilities between these times, that is,

$$\mathbb{P}\{t_{i-1} < \tau < t_i\}, \qquad i = 1, 2, \ldots, n. \tag{4.4.3}$$

Before calculating this probability, we should give the definition of the two default probabilities which will be needed: the conditional and the unconditional default probabilities.

The probability which will be used in pricing is the unconditional default probability between times t_{i-1} and t_i. This is the probability seen at time zero. The default probability between times t_{i-1} and t_i seen at time t_{i-1} is called the conditional probability on no earlier default until time t_{i-1}. Let us denote it as $Q(t_{i-1}, t_i)$. Then the unconditional default probabilty is only the product of the probabilities $P^s(t_{i-1})$ and $Q(t_{i-1}, t_i)$:

$$\mathbb{P}\{t_{i-1} < \tau < t_i\} = P^s(t_{i-1})Q(t_{i-1}, t_i) \qquad i = 1, 2, \ldots, n. \tag{4.4.4}$$

However, in order to get a simpler price formula for CDS, we need to define the probability (4.4.3) in a different way from the right hand side of the equation (4.4.4). The unconditional default probability (4.4.3) is just the difference of the default probabilities until times t_i and t_{i-1} which are the probabilities seen at time zero. Therefore,

$$\mathbb{P}\{t_{i-1} < \tau < t_i\} = Q(t_i) - Q(t_{i-1}) \qquad i = 1, 2, \ldots, n. \tag{4.4.5}$$

Then, for $i = 1, 2, \ldots, n$

$$\begin{aligned} \mathbb{P}\{t_{i-1} < \tau < t_i\} &= Q(t_i) - Q(t_{i-1}) = [1 - P^s(t_i)] - [1 - P^s(t_{i-1})] \\ &= -P^s(t_i) + P^s(t_{i-1}) = -[P^s(t_i) - P^s(t_{i-1})] \\ &= -\Delta P^s(t_i). \end{aligned} \tag{4.4.6}$$

Since the price of the CDS is the expected value of the discounted cash flows under the risk-neutral probability measure, we have

$$\begin{aligned}\text{CDS} &= \mathbb{E}^*\Big[-\sum_{i=1}^{n} e^{-rt_i} c\mathbf{1}_{\{\tau>t_i\}} + \sum_{i=1}^{n} e^{-rt_i}(1-R)\mathbf{1}_{\{t_{i-1}<\tau<t_i\}} \\ &\qquad -\sum_{i=1}^{n} e^{-rt_i}(\tau - t_{i-1})c\mathbf{1}_{\{\tau>t_i\}}\Big].\end{aligned}$$

Here, the first two summands are the expected payments and the expected compensations, respectively. When a default event happens, the buyer must make the payment for the time passed between the last payment and the default time. This payment is called *accrued interest*. The third summand in the above CDS price represents this accrued interest. Calculation of the above expectation yields

$$\begin{aligned}\text{CDS} &= -\sum_{i=1}^{n} e^{-rt_i} c\mathbb{P}\{\tau > t_i\} + \sum_{i=1}^{n} e^{-rt_i}(1-R)\mathbb{P}\{t_{i-1} < \tau < t_i\} \\ &\quad -\sum_{i=1}^{n} e^{-rt_i}(\tau - t_{i-1})c\mathbb{P}\{t_{i-1} < \tau < t_i\} \\ &= -c\sum_{i=1}^{n} e^{-rt_i} P^s(t_i) + (1-R)\sum_{i=1}^{n} e^{-rt_i}[-\Delta P^s(t_i)] \\ &\quad - c\sum_{i=1}^{n} e^{-rt_i}(\tau - t_{i-1})[-\Delta P^s(t_i)].\end{aligned}$$

Now, let us assume that the payments are made continuously, which means as $n \to \infty$, we have

$$\text{CDS} = -c\int_0^T e^{-rt} P^s(t)\,\mathrm{d}t - (1-R)\int_0^T e^{-rt}\,\mathrm{d}P^s(t).$$

Note that as $n \to \infty$, the accrued interest vanishes. This is due to the fact that the payments begin to be made continuously.

The par spread c^* that makes the CDS price equal to zero is found to be

$$c^* = \frac{-(1-R)\int_0^T e^{-rt}\,\mathrm{d}P^s(t)}{\int_0^T e^{-rt} P^s(t)\,\mathrm{d}t}.$$

By the integration by parts, we obtain

$$\begin{aligned}
\int_0^T e^{-rt}\,\mathrm{d}P^s(t) &= e^{-rt}P^s(t)\Big|_0^T - \int_0^T -r\,e^{-rt}P^s(t)\,\mathrm{d}t \\
&= e^{-rT}P^s(T) - 1 + r\int_0^T e^{-rt}P^s(t)\,\mathrm{d}t,
\end{aligned}$$

so that

$$c^* = \frac{(1-R)\left(1 - e^{-rT}P^s(T) - r\int_0^T e^{-rt}P^s(t)\,\mathrm{d}t\right)}{\int_0^T e^{-rt}P^s(t)\,\mathrm{d}t}.$$

This completes the proof. □

In the formula (4.4.2) for the par spread c^*, the survival probability $P^s(t)$ can be obtained from the price of the BDOB option as in (4.3.3),

$$P^s(t) = e^{rt}BDOB(t, B),$$

with maturity t and barrier B. The par spread in terms of the BDOB option price is, therefore,

$$c^* = \frac{(1-R)\left(1 - BDOB(T,B) - r\int_0^T BDOB(t,B)\,\mathrm{d}t\right)}{\int_0^T BDOB(t,B)\,\mathrm{d}t}. \tag{4.4.7}$$

Chapter 5

Analysis and Conclusion

This chapter is devoted to some applications and analysis of the models described in this thesis, and conclude the work by a short summary.

5.1 Application and Analysis

In this section, some applications will be shown for the given parameters. These parameters are

$B = 40$, $S_0 = 80$, $T = 1$, $r = 0.05$, $q = 0.0133$, $\theta = -0.1851$, $\sigma = 0.2041$ and $\nu = 0.4199$.

In Section 4.3, we mentioned Barrier options and gave the definition of the Binary Down and Out Barrier option (BDOB) under both BS and VG setting. The prices of the BDOB option at times $t = 0$, $t = 0.25$, $t = 0.50$, $t = 1$ under the BS setting is given in Figure 5.1. In this figure, the graphics are constructed by using the numerical solution of the PDE for the BS model. In fact, these prices may also be obtained via the closed form solution in (4.3.5).

On the other hand, Figure 5.2 shows the prices under the VG setting. These prices are calculated via the numerical solution of the PIDE defined in Section 4.3.2. As we stated before, there is no closed form solution of the BDOB option under the VG setting due to the jump structure of the VG process.

When we compare Figures 5.1 and 5.2, we see that the prices under the VG setting are not smooth whereas they are smooth under the BS setting. The main reason is that, in the BS setting, the resulting PDE is converted to a diffusion equation (heat equation), which is parabolic and smooths the initial data (terminal condition) as well as boundaries. However, in the VG setting, since the PDE is of degree one contrary to BS setting, it does not smooth the initial data and boundaries. So, when the underlying asset price is very close to the barrier, the PIDE gives numerical errors which causes the prices not to be smooth. If we look at figure (5.3), the difference between the BS and VG models can be seen clearly. This figure gives the BDOB option prices under both BS and VG settings at time $t = 0$. The numerical solution of the PIDE is a mixture of the implicit and explicit methods, and its stability needs to be investigated as an outlook as well.

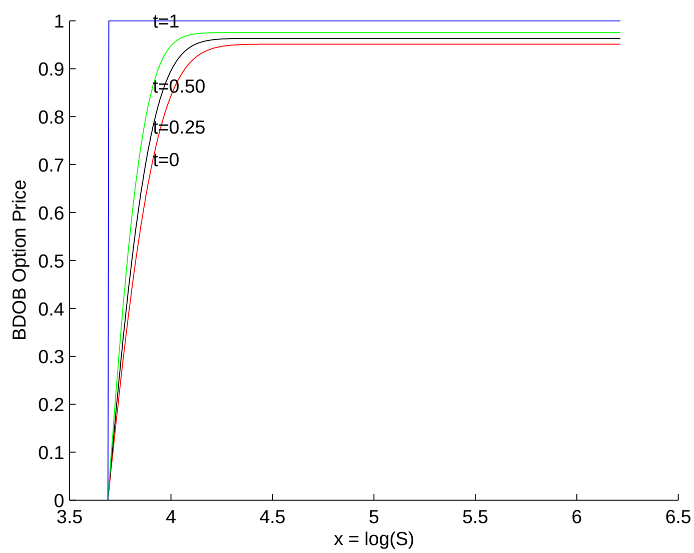

Figure 5.1: BDOB option under BS model

In Figure 5.2, the discretization of the domain $\mathcal{D} = [0, 1] \times [x_{min}, x_{max}]$ were taken as $M = 500$, $N = 500$. For various discretizations, the results can be seen in Table 5.1. The PIDE solution gives very accurate results with discretization

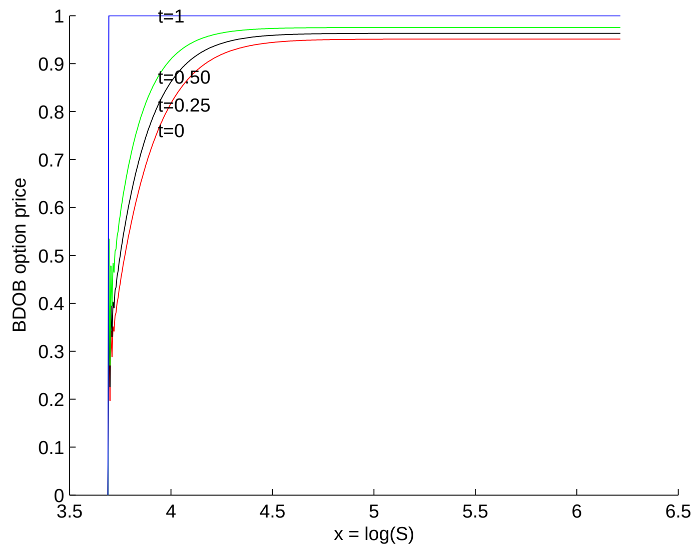

Figure 5.2: BDOB option under VS model

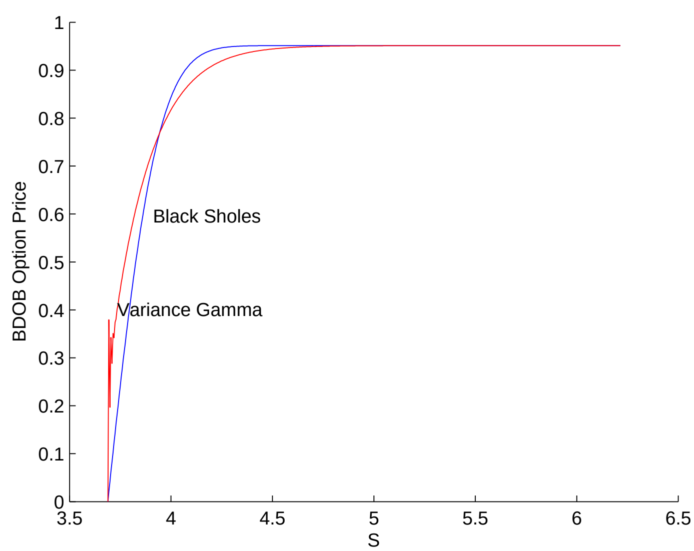

Figure 5.3: BDOB option at $t = 0$

of the domain $\mathcal{D}$ greater than $M = 150$, $N = 150$.

A BDOB option can be priced easily by the Monte-Carlo (MC) method in addition to the PIDE method. However, the Monte-Carlo method is in general too time consuming. Table 5.2 gives the results of the MC method for various iterations and time discretization M. The MC method needs at least 100000 iterations with $M = 250$ time discretization in order to get accurate results. If we compare the cpu times in the Tables 5.1 and 5.2, we see that the MC method takes much longer.

Table 5.1: The outcomes of PIDE solution

M	N	c^*	BDOB	cpu time
50	50	106	0.9344	1.0938
100	100	96	0.9360	1.4688
150	150	93	0.9364	2.9375
200	200	92	0.9366	5.3438
250	250	92	0.9366	8.5313
500	250	93	0.9365	16.4844
250	500	91	0.9368	21.2813
500	500	91	0.9367	42.0625
750	750	91	0.9367	111.1094
1000	1000	91	0.9367	230.5625

In Section 3.3, we showed that the parameters θ and ν affects the skewness and the kurtosis of the distribution of the proceses, respectively. These parameters cause the distribution to be more asymmetric and leptokurtic. Figures 5.4 and 5.5 give the sensitivity analysis of the default probabilities with respect to these parameters ν and θ. The figures indicate that higher kurtosis and more negative skewness result in higher default probabilities. Finally, Figures 5.6 and 5.7 give the sensitivity of CDS par spreads with respect to kurtosis and skewness, respectively. Higher kurtosis and more negative skewness result in higher CDS

Table 5.2: The outcomes of the MC solution

Iterations	M	c^*	BDOB	cpu time
1000	100	78	0.9389	1259
10000	100	102	0.9351	2982
10000	250	101	0.9352	4581
100000	250	91	0.9368	52452
100000	500	92	0.9367	92505
1000000	500	91	0.9367	336330

par spreads.

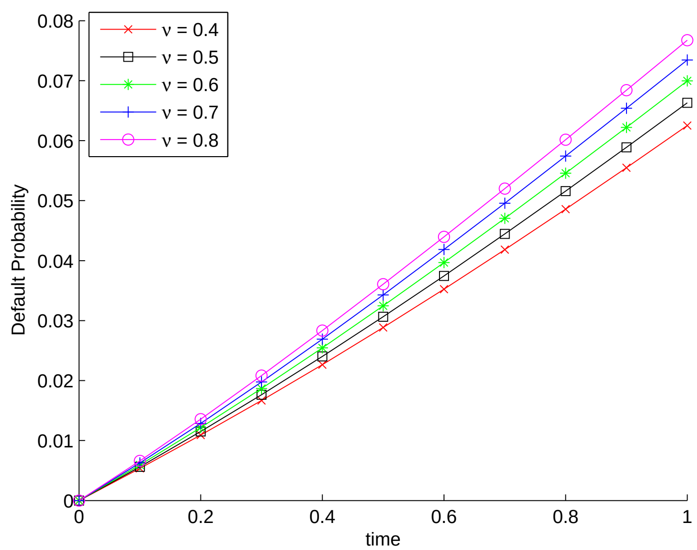

Figure 5.4: Kurtosis sensitivity in default probabilities

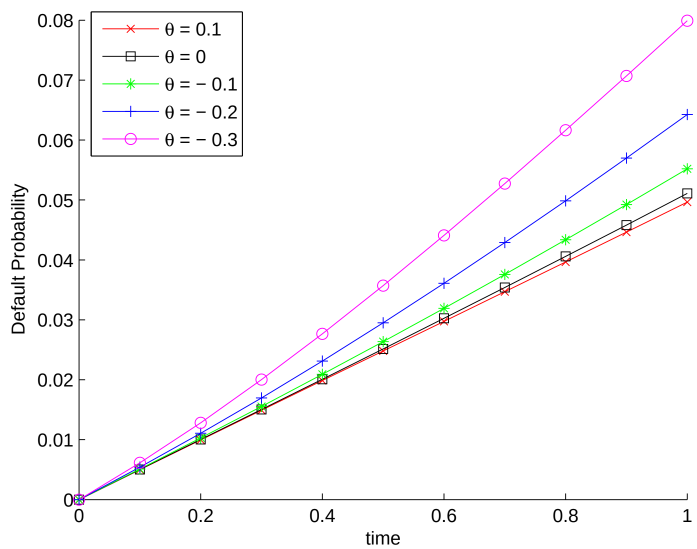

Figure 5.5: Skewness sensitivity in default probabilities

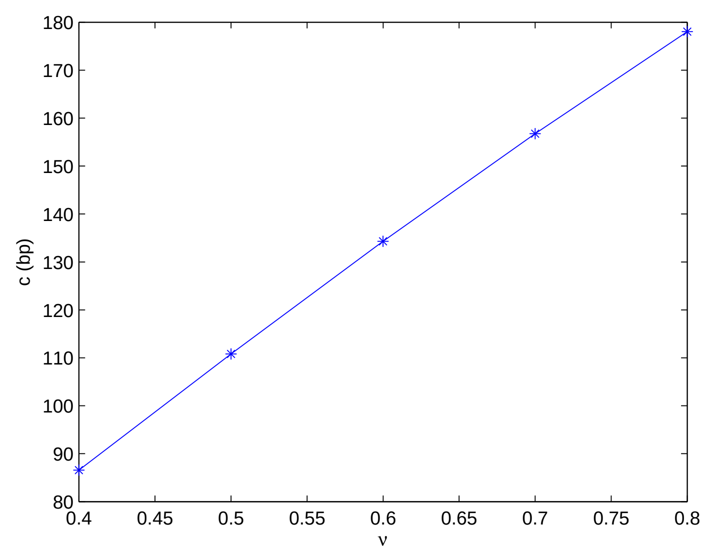

Figure 5.6: Kurtosis sensitivity in CDS par spreads

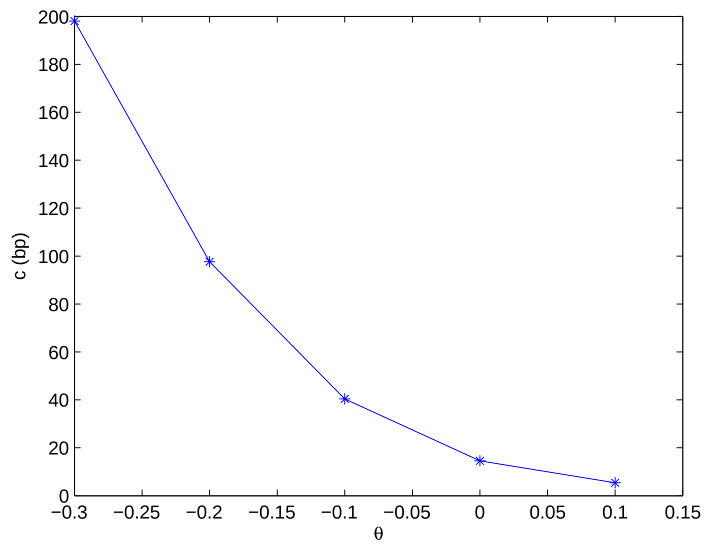

Figure 5.7: Skewness sensitivity in CDS par spreads

5.2 Conclusion

In this work, the structural models in credit risk and credit derivatives have been studied. After summarizing the structural models under the Black-Scholes setting, the modeling under the Variance Gamma setting was given. Since the VG process is a pure jump process, it allows random default of the reference entities which is not the case under the BS setting.

Among structural models in the VG setting, the emphasis has been on the Black-Cox model. Survival probabilities under this model have been calculated. By taking advantage of the relation between the survival probability of the Black-Cox model and the Binary Down and Out Barrier option, survival probabilities have been calculated via the BDOB option price. The BDOB option price, however, have been given by the numerical solution of a Partial Integro Differential Equation. An explicit finite difference methods have been applied to this PIDE in order to solve it numerically. Besides PIDE, the Monte-Carlo method has been used to price the BDOB option. However, the Monte-Carlo method has been shown to be much slower than the solution of PIDE.

Finaly, sensitivity analysis on the parameters θ and ν has shown that we have high default probabilities and CDS spread

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