MONITORING HIGH QUALITY PROCESSES: A STUDY OF ESTIMATION ERRORS ON THE TIME-BETWEEN-EVENTS EXPONENTIALLY WEIGHTED MOVING AVERAGE SCHEMES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF
NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

GÜNEY ÖZSAN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
INDUSTRIAL ENGINEERING

SEPTEMBER 2008

Approval of the thesis:

MONITORING HIGH QUALITY PROCESSES: A STUDY OF ESTIMATION ERRORS ON THE TIME-BETWEEN-EVENTS EXPONENTIALLY WEIGHTED MOVING AVERAGE SCHEMES

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ABSTRACT

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September 2008, 97 pages

In some production environments the defect rates are considerably low such that measurement of fraction of *nonconforming* items reaches parts per million level. In such environments, monitoring the number of *conforming* items between consecutive *nonconforming* items, namely the *time between events* (TBE) is often suggested. However, in the design of control charts for TBE monitoring a common practice is the assumptions of known process parameters. Nevertheless, in many applications the true values of the process parameters are not known. Their estimates should be determined from a sample obtained from the process at a time when it is expected to operate in a state of statistical control. Additional variability introduced through sampling may significantly effect the performance of a control chart. In this study, the effect of parameter estimation on the performance of *Time Between Events Exponentially Weighted Moving Average* (TBE EWMA) schemes is examined. Conditional performance is evaluated to show the effect of estimation. Marginal performance is analyzed in order to make recommendations on sample size requirements. Markov chain approach is used for evaluating the results.

Keywords: Time Between Events, Exponentially Weighted Moving Average, Parameter Estimation

ÖZ

YÜKSEK KALİTELİ SÜREÇLERİN İZLENMESİ: OLAYLAR ARASI SÜRE – ÜSTEL AĞIRLIKLI HAREKETLİ ORTALAMA ŞEMALARI ÜZERİNDE PARAMETRE TAHMİNİNİN ETKİLERİ

Özsan, Güney
Yüksek Lisans, Endüstri Mühendisliği
Tez Danışmanı : Prof. Dr. Gülser Köksal
Ortak Tez Danışmanı : Doç. Dr. Murat Caner Testik

September 2008, 97 pages

Hata oranlarının çok düşük olduğu bazı üretim sistemlerinde, uygun olmayan parçaların oranı milyonda bire varabilmektedir. Bu tip sistemlerde genellikle uygun olmayan iki ardışık parça arasında üretilen uygun parçaların sayısının, yani *olaylar arası süre*'nin izlenmesi önerilir. Ancak, yaygın olarak kontrol şemalarının tasarımı sırasında gerekli işlev parametrelerin bilindiği varsayılır. Bununla birlikte, bir çok uygulamada gerçek süreç parametreleri bilinmemektedir. Sürecin istatistiksel kontrol halinde olduğu varsayılan bir anda örneklem elde edilerek bilinmeyen parametrelerin tahmin edilmesi gerekmektedir. Örnekleme sebebiyle kontrol şemalarına katılan fazladan değişkenlik, kontrol şemalarının başarısını önemli derecede etkileyebilir. Bu çalışmada, parametre tahmin işleminin *Olaylar Arası Süre - Üstel Ağırlıklı Hareketli Ortalama* şemaları üzerindeki etkileri incelenmektedir. Tahmin sürecinin etkilerini göstermek için koşullu performans çözümlemesi yapılmaktadır. Marjinal performans çözümlemesi ile de örneklem büyüklüğü önerileri getirilmektedir. Sonuçları elde edebilmek için Markov zinciri yaklaşımı kullanılmaktadır.

Anahtar Kelimeler: Olaylar Arası Süre, Üstel Ağırlıklı Hareketli Ortalama, Parametre Tahmini

…to my parents and lovely friends…

ACKNOWLEDGEMENTS

I would like to gratefully acknowlegde my co-supervisor Doç. Dr. Murat Caner Testik for his valuable contributions and I am also thankful to him for being a mentor for my life.

I would like to thank my supervisor Prof. Dr. Gülser Köksal for leading me in this thrilling path by providing essential contributions and criticism.

I would also like to thank Christian Weiss for his valuable contributions in the evaluation of Inverse Gamma distribution.

I would sure like to mention my thanks to my colleague and close friend Canan Pehlivan for her comments, stimulation and supportive friendship.

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CHAPTER 1

INTRODUCTION

1.1. Problem Definition

In high quality and high volume manufacturing, tracing the system behaviour may become challenging when the rate of defective items is considerably small. Low defect rate implies that there is not much noticeable amount of beneficial information that could be gathered from the system. Consequently, it is difficult to become aware of any change in the system behavior in a timely manner. Appropriate use of control charts would be valuable in such cases.

In order to track the system behavior in case of unexpected changes in the quality characteristics and step in timely when one occurs, several control charts are proposed in the literature. *Shewhart*, *Cumulative Sum (CUSUM)*, and *EWMA* control charts are the most commonly encountered and discussed ones in the literature. Each of these charts have some superior and inferior properties. Generally speaking, while Shewhart chart is very responsive to large shifts in the mean of a monitored quality characteristic, CUSUM and EWMA are superior in detecting small changes. Furthermore CUSUM and EWMA charts utilize past values in their control statistics, whereas the *Shewhart* chart evaluates only the most recent data available.

The traditional assumptions for these control charts are that the measurements are in a continuous scale and Normal distribution is appropriate for modeling the values of the measurements. These traditional control charts also have been extended to the case where different distribution models are also utilized.

The quality control characteristic is not always a continuous measure. In some production environments the occurrence of defects makes the item produced completely inoperable and it becomes a *nonconforming* item. In this case, instead of a measurement of a quality characteristic, the occurence of nonconforming items becomes center of interest. A common approach is to define the quality characteristic of interest as the number of nonconforming items in a sample.

Nevertheless, under circumstances where there are considerably low defect rates, the fraction of nonconforming items may reach parts per million levels and the sampling based approach that counts the number of nonconforming items observed in a sample may not provide timely information on a process change. In such cases, monitoring the time or the number of *conforming* items between consecutive nonconforming items (namely *time between events*) provides more useful information. Charts designed using this approach are called *Time Between Events* (*TBE*) charts.

Consider the Poisson distribution model as a natural choice for the counts of nonconforming items in a given time interval. Consequently, the TBE observations would follow an exponential distribution. Changes in the mean rate of nonconforming items can be tracked by monitoring the TBE observations under an exponential model. A process can be deemed *out-of-control* if the mean of TBE observations shifts. A decrease in the TBE mean would indicate more frequent nonconforming items, i.e. a decrease in quality. An increase in the mean of TBE would indicate less frequent nonconforming items, i.e. an increase in quality.

Throughout this work, our main focus would be the detection of a decrease in the TBE statistic. A lower-sided EWMA control chart for exponentially distributed TBE observations would be investigated when the true process parameters are unknown but estimated. Effects of parameter estimation and sample size requirements for better process control would be discussed. Recommendations would be provided for practitioners.

1.2. Motivation

In a control chart practice, *in-control* process parameters are required for designing the charts. It is a common but over simplified assumption in the literature that the *in-control* parameters are known. *In-control* process parameters are rarely known in practice and often their estimates should be determined from a sample obtained from the process, at a time when the process is expected to be operating in an *in-control* state. Presence of sample estimates introduces additional variability into a control chart and performance of a chart may get worse depending on the quality of the estimates.

Pehlivan (2008) investigated the robustness of TBE EWMA control charts when the observations do not follow the assumed exponential distribution. It has been shown that the chart is generally robust to the deviations from the assumed exponential model. In this thesis, conditional and marginal performance (see Testik, McCullough, and Borror, 2006; and Testik, 2007) of the TBE EWMA control chart is investigated. A Markov chain approach is used for evaluating a commonly accepted control chart performance metric, namely the *average run length* (ARL) and its various properties, where the *run length* (RL) is defined as the number of points plotted on a control chart until the first time an *out-of-control* signal is released. Conditional performance analysis is used to analyze the effect of estimation on both the *in-control* and *out-of-control* ARL's and marginal performance analysis is used to provide sample size recommendations for practitioners.

CHAPTER 2

LITERATURE REVIEW AND BACKGROUND

2.1. Statistical Process Control

Inevitably, production processes are always subject to some amount of natural variation, which is a cumulative result of several but mainly unavoidable small causes. Causes that create this type of variation are called *chance causes of variation* in the quality control literature. A process operating under chance causes of variation is called to be in a state of statistical control (*in-control*). Processes are not always in a state of statistical control. Causes of variation that are not an inherent part of the process but occur time by time, resulting in a shift of the process into an *out-of-control* state are also common. Causes that create this type of variation are called *assignable causes of variation*. When an *out-of-control* state is detected, a process can be analyzed and assignable causes of variation can be identified. As an example, for a manufacturing process, an assignable cause of variation can be a tool that went out of tuning.

Statistical Process Control (SPC) makes use of statistical techniques for analyzing the variation in a process. Control charts, which were introduced by Shewhart (1926), monitor a quality characteristic of the process and alert when the quality characteristic goes beyond some predefined control limits. The control limits are constructed according to some statistical properties of the monitored quality characteristic and usually they are defined in terms of the standard deviation of the monitored quality characteristic. A simple control chart for an *in-control* process is illustrated in Figure 2.1.

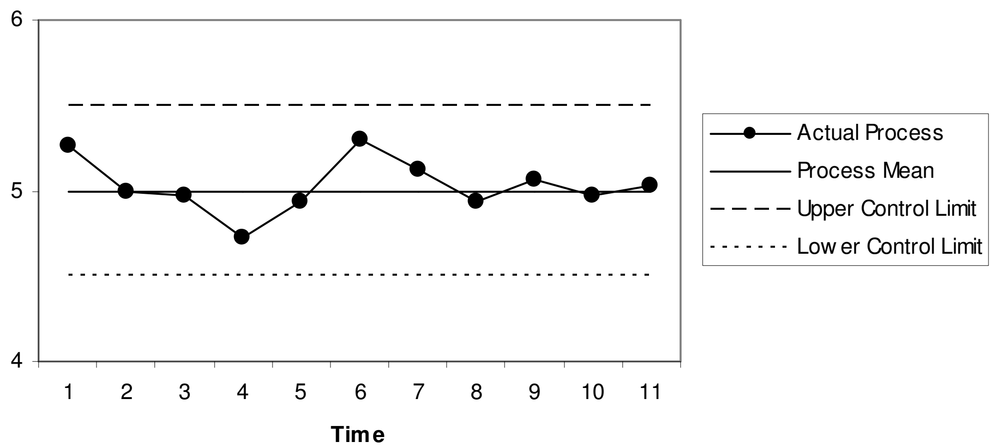

Figure 2.1 – A simple control chart for an *in-control* process.

2.2. Shewhart Control Chart

The first control chart for detection of assignable causes of variation is introduced by Shewhart (1926) and named after him. Simply, these charts plot the mean ($\bar{x}$)

$$\bar{x} = \frac{x_1 + x_2 + ... + x_n}{n},$$

of the samples of size n of a time-ordered quality characteristic X taken periodically and an *out-of-control* signal is triggered when the most recent sample mean exceeds the control limits. It is assumed that the observations are normal distributed and hence the distribution of $\bar{X}$ is also normal with mean $\mu_{\bar{x}}$ and standard deviation $\sigma_{\bar{x}}$, where,

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}}.$$

The *center line, upper control limit (UCL)*, and *lower control limit (LCL)* for Shewhart control chart are defined as,

$$\begin{aligned} \text{UCL} &= \mu_{\bar{x}} + L\sigma_{\bar{x}}, \\ \text{Center Line } &= \mu_{\bar{x}}, \\ \text{LCL} &= \mu_{\bar{x}} - L\sigma_{\bar{x}}. \end{aligned}$$

Choice of L, the distance of the control limits from the center line, is a key step in the design of a control chart.

From a hypothesis testing point of view, each point in the chart is a test of hypothesis that the process is *in-control*. Points falling within the control limits fail to reject the null hypothesis of statistical control. Points falling outside the chart limits reject the hypothesis of statistical control and process is deemed *out-of-control*.

As the hypothesis testing procedures incorporate type-I and type-II errors, a large L value decreases the probability of the type-I error (giving *out-of-control* signal although the process is *in-control*) and increase the probability of type-II error (not giving an *out-of-control* signal when the process is *out-of-control*). On the contrary, a small L value would increase the probability of a type-I error but decrease the probability of a type-II error. For a probability of type-I error being α, L can be defined as,

$$L = z_{\alpha/2}$$

In this case the limits and the center line for the chart becomes,

$$\begin{aligned} \text{UCL} &= \mu_{\bar{x}} + z_{\alpha/2}\sigma_{\bar{x}}, \\ \text{Center Line } &= \mu_{\bar{x}}, \\ \text{LCL} &= \mu_{\bar{x}} - z_{\alpha/2}\sigma_{\bar{x}}. \end{aligned}$$

However, instead of defining an L value using the approach of error handling under normal distribution, it is common to determine control limits as a multiple of the standard deviation of the plotted statistic, $\sigma_{\bar{x}}$. A value of L = 3 is common, such that (Montgomery, 2001).

Note that Shewhart charts evaluate only the last sample mean and ignore the previous information. Some rules for sensitizing the Shewhart charts are proposed in the literature to take advantage of information contained in previous observations. Some of these rules include,

- Two out of three consecutive points plot beyond the two-sigma warning limits.
- Four out of five consecutive points plot at a distance of one-sigma or beyond from the center line,
- Six points in a row steadily increasing or decreasing.

However, the efforts on showing a relationship between these rules and the occurrence of changes in quality characteristic could not end up in a statistical proof. Instead, when the process is in control, use of such rules showed an increase in false alarm rates for an *in-control* process. (Champ and Woodall, 1987).

Shewhart charts are generally effective in detecting large shifts, say more than 1.5σ, in the process mean (Montgomery, 2001). However, these may alert process shifts drastically late for small changes in the process mean. This is because the Shewhart chart only evaluates the current data point in making the decision. To overcome this deficiency, CUSUM and EWMA control charts that utilize information from past observations have been proposed. These charts are known to perform better when detecting small shifts in the process mean (Testik, 2007). Still, Shewhart charts are the most common control charts in practice due to their simplicity. Furthermore, the common belief is that small changes incur small losses and so it does not matter if they are not detected straightaway. However, a small change that persist for a long time without being noticed possibly will incur a larger total cost than a larger change that is detected and corrected quickly. It should be noted that a Shewhart chart is not an efficient way for identifying persistent small changes in mean (Hawkins and Zamba, 2003).

2.3. CUSUM and EWMA Control Charts

2.3.1. CUSUM Control Charts

In Section 2.2 it was mentioned that Shewhart charts use only the last available sample mean and as a result they are ineffective in detecting small shifts, say less than 1.5σ. As an alternative, CUSUM and EWMA control charts are effective tools when small shifts are of concern due to accumulation of information from previous samples.

The control statistic of a CUSUM chart is the cumulative sums of the difference between the sample points and a target value. Let μ_0 be an *in-control* mean or the target value and x_j be the sample average of the j^{th} sample. The CUSUM statistic is,

$$C_i = \sum_{j=1}^{i} (x_j - \mu_0) \qquad (2.1)$$

A plot of C_i against the time order or sequence i in Equation (2.1) is the CUSUM chart. Since these charts accumulate all the present and past information, they are effective in detecting small shifts in the mean of the process.

An upward shift in the mean of the process, say $\mu_1 > \mu_0$, would cause an upward drift in the cumulative sum statistic C_i in Equation (2.1), whereas a downward shift in the mean of the process, say $\mu_1 < \mu_0$ would cause a downward drift.

The CUSUM control chart give *out-of-control* signal when the CUSUM statistic takes a value out of the decision interval (*-H, H)*, while $H = 5\sigma_x$ is suggested as a reasonable value (Montgomery, 2001).

In order to obtain a CUSUM control chart, control limits should be integrated with Equation (2.1). Two methods, *tabular CUSUM* and *V-mask* are used for this purpose. Tabular CUSUM is a more convenient and recent way of representing CUSUM control charts. For detailed information on incorporating CUSUM charts with V-mask, see Gan (1991). Multivariate extensions to CUSUM control charts are investigated in Runger and Testik (2004).

2.3.2. EWMA Control Charts

EWMA control charts, introduced by Roberts (1959), use the control statistic that is a weighted sum of all previous observations.

Let x_i be the i^{th} observation, and let $0 < \lambda_q \leq 1$ be a constant. Exponentially weighted moving average, z_i, is defined as:

$$z_i = \lambda_q x_i + (1-\lambda_q) z_{i-1}, \quad (0 < \lambda_q \leq 1), \tag{2.2}$$

where λ_q is called the smoothing constant.

The starting value of the EWMA statistic, z_0, is selected as the process target value.

$$z_0 = \mu_0$$

Substituting z_{i-1} in Equation (2.2) recursively would result in the following representation,

$$z_i = \lambda_q \sum_{j=0}^{i-1} (1-\lambda_q)^j x_{i-j} + (1-\lambda_q)^i z_0 \tag{2.3}$$

In Equation (2.3), EWMA is represented as the sum of all the past and the current observations with weights $\lambda_q(1 - \lambda_q)^j$. Note that, EWMA is very insensitive to the normality assumption due to weighted averaging and hence central limit theorem. This particular property of EWMA makes it an ideal control chart to use with individual observations.

If the observations x_i are independent random variables with variance σ^2, then the variance of z_i is given as

$$\sigma_{z_i}^2 = \sigma^2 \left(\frac{\lambda_q}{2-\lambda_q} \right) [1-(1-\lambda_q)^{2i}]. \tag{2.4}$$

Plotting z_i versus the sample number i would construct the EWMA chart. The center line, upper control limit, and lower control limit for the EWMA control chart are defined as

$$\begin{aligned} \text{UCL} &= \mu_0 + L\sigma \sqrt{\frac{\lambda_q}{(2-\lambda_q)}[1-(1-\lambda_q)^{2i}]}, \\ \text{Center Line} &= \mu_0, \\ \text{LCL} &= \mu_0 - L\sigma \sqrt{\frac{\lambda_q}{(2-\lambda_q)}[1-(1-\lambda_q)^{2i}]}, \end{aligned}$$

where L is the width of the control limits.

Since the quality $[1 - (1 - \lambda_q)^{2i}]$ in (2.4) approaches unity as i gets larger, the EWMA control limits in the steady state become

$$\text{UCL} = \mu_0 + L\sigma\sqrt{\frac{\lambda_q}{(2-\lambda_q)}},$$
$$\text{Center Line } = \mu_0,$$
$$\text{LCL} = \mu_0 - L\sigma\sqrt{\frac{\lambda_q}{(2-\lambda_q)}}.$$

However, using exact limits may be suggested for detecting *out-of-control* processes that shift early at the process start-up or for processes that already start in an *out-of-control* state. Crowder (1987b, 1989) and Lucas and Saccucci (1990) provide further discussions on the EWMA control charts.

2.3.3. Some Properties of EWMA and CUSUM Control Charts

Both of the CUSUM and EWMA control charts utilize past data as well as the current. Lucas and Saccucci (1990) and Roberts (1966) showed that EWMA and CUSUM charts have very similar performances in monitoring a normal mean. Also, Gan (1991) states that the optimal CUSUM and EWMA charts have similar ARL properties.

CUSUM and EWMA charts are intended for persistent shifts. However, some special causes occur and operate for a short period of time and resolve themselves without being noticed. For such transient shifts in the mean, EWMA and CUSUM charts may not be effective in detecting mean shifts. Note that using a CUSUM chart introduces an additional benefit of estimating when the shift occurred and how large the shift is (Hawkins and Zamba, 2003).

To monitor the TBE of a process operating at close to zero-nonconformity level, Sun and Chang (2000) compared a two-stage control chart proposed by Chan et al. (1997) with the CUSUM and the EWMA charts. It is shown that the CUSUM and EWMA charts are more efficient than the two-stage control chart. Moreover, Gan (1998) suggests that EWMA and CUSUM charts also have similar performances in monitoring an exponential mean. This property makes both EWMA and CUSUM charts ideal for monitoring exponentially distributed time between events data.

A comparison of TBE CUSUM and TBE EWMA control charts is also provided by Sun and Zhang (2000).

2.4. Exponential TBE EWMA Control Chart

When the number of nonconformities is very small, the sampling based approach that counts the number of nonconforming items observed in a sample may not provide timely information on a process change. For near-zero nonconformity processes, monitoring the number of conforming items or time between consecutive nonconforming items, say *time between events* (TBE) provides more useful information compared to the number of nonconforming items.

Recall that in Section 1.1, the Poisson distribution model is defined as a natural choice for the counts of nonconforming items in a given time interval. Consequently, the TBE observations would follow an exponential distribution, $\mathrm{Exp}\left(\frac{1}{\mu}\right)$[1], with mean μ. Monitoring the *TBE* of the nonconforming items under an exponential model would provide monitoring of the mean (μ) of the nonconforming items.

[1] In order to avoid confusion with the exponential smoothing parameter λ_q, the parameter of the exponential distribution $\lambda = \frac{1}{\mu}$ is not used and only μ is used throughout the text.

A decrease in the TBE mean (μ) would indicate more frequent nonconforming items, i.e. a decrease in quality. An increase in the mean of TBE (μ) would indicate less frequent nonconforming items, i.e. an increase in quality.

EWMA charts employed for monitoring Poisson data mentioned earlier was first introduced in Borror et al. (1998) and the effect of estimation on these control charts were analyzed in Testik (2006).

In this research, the point of interest is detection of a decrease in quality and therefore a one-sided exponential TBE EWMA control chart would be used.

2.5. Construction of a TBE EWMA Control Chart

When the true process parameters are unknown, construction of a control chart involves a two phase study. In *Phase-I*, unknown process parameters are estimated from a reference sample obtained from an *in-control* process. These estimates are then used to set up the control limits and hence construct the control chart. In a *Phase-II* study, the control chart constructed in *Phase-I* is used to monitor the actual process. If the process parameters are known, *Phase-I* becomes unnecessary and may be skipped so that the control chart is constructed directly using the known parameters. However, in practice process parameters are often unknown and they should be estimated through a realization of the *Phase-I* study. The estimation process in *Phase-I* causes additional variability due to estimation error. As a result, there might be differences in the performance of the control chart using estimated parameters and the one using true process parameters.

Considering the exponential TBE EWMA charts, the parameter of interest is the mean (μ, which is also the variance) of the exponential distribution. Hence, the *in-control* mean (μ) is the only parameter needed for constructing this control chart.

A literature review by Jensen et al. (2005) examines the effects of parameter estimation on control chart properties including the Shewhart, CUSUM, and EWMA control charts. The control charts studied are generally sensitive to the parameter estimation and both their *in-control* and *out-of-control* performances are effected by parameter estimation. They concluded that in most of the cases a high number of samples or better designs to work with small number of sample sizes are required.

In a *Phase-I* study, another type of error is possible such that the actual distribution underlying the process may be different from the assumed one. The robustness of the EWMA chart to non-normality is considered in Borror et al. (1999) and it was concluded that EWMA is robust to non-normality for λ_q = 0.05 and 0.10. Regarding the TBE processes, Borror et al. (2003) examine the robustness of TBE CUSUM and Pehlivan (2008) examines the robustness of the TBE EWMA control chart. It has been shown that in both cases TBE control chart is generally robust to the deviations from the assumed exponential model.

Brook and Evans (1972) propose to use the Markov chains for studying the RL distributions of control charts. In Section 3.3, this approach will be discussed for the conditional performance analysis. Alternatively, Crowder (1987a, 1987b) introduces the integral equation approach for studying the RL distribution. Calzada and Scariano (2003) show the relation between the Markov chain approach and the integral equation approach introduced by Crowder (1987a, 1987b). Fu et al. (2001) propose a unified method based on the Markov chain approach for finding the RL distribution and ARL of a control chart. An exact procedure other than the Markov chain method is defined for calculating ARL's of an exponential EWMA control chart in Gan (1998). Moreover, a computer program based on this method for calculating ARL's is provided in Gan and Chang (2000). In addition to these, a computer program using the Markov chain approach for calculating RL distribution for the Poisson CUSUM control chart is provided by White and Keats (1996).

CHAPTER 3

METHODOLOGY

3.1. Performance Measures

In the literature, control chart performance is generally evaluated by interpreting various characteristics of the distribution of the RL. Expected value of the RL, namely *average run length* (ARL), is a widely used performance criterion. However, the distribution of the RL is quite skewed in some cases and often has large variance so that using ARL as a performance measure alone might result in inadequate or unsatisfactory results. Hence, other properties of the RL distribution such as the *standard deviation of the RL* (SDRL) and the *percentiles of the RL distribution* may be useful in evaluating the performance.

In-Control ARL and Out-of-Control ARL:

Due to the statistical nature of a process, a control chart may give *out-of-control* signals although the process is actually *in-control.* In designing a control chart, chart parameters should be set to provide an acceptable rate of false alarms. Hence, there are two types of ARL. First consider the expected value of the RL given that the process is *in-control*. Denote this by ARL_0. Although the choice of ARL_0 depends on the nature of the process, as a common design value in the literature, a value of $ARL_0 = 500$ is chosen throughout the study.

As the second case, consider the expected RL in the presence of an assignable cause, which is denoted by ARL_1. It is required to detect *out-of-control* states rapidly and at the same time not to have many false alarms when the process is *in-control*. Hence, a control chart with a large ARL_0 and a small ARL_1 is desired (Testik, 2006).

Gan (1994) investigates the decision of ARL_0 values. He suggests that cost associated with false alarms and process downtime are some of the considerations to be kept in mind when determining ARL_0. He also brings into mind that small shifts could be tolerated when some slack in the process is acceptable. Otherwise, charts that are sensitive to very small shifts might end up in an over-adjusted process with a higher associated cost. As a result, the largest shift that is not tolerable should be kept in mind when choosing the *out-of-control* mean.

3.2. Design Parameters and Standardization

3.2.1. Standardization of the Model

As it was described in Section 2.4, let the random variable X_t for representing TBE observations be distributed $\text{Exp}\left(\frac{1}{\mu}\right)$. For an *in-control* process, let the process mean be $\mu = \mu_0$ and for an *out-of-control* process let $\mu = \mu_0 \cdot \delta$. Here, μ_0 is the *in-control* process mean and δ is a shift magnitude with respect to the true *in-control* mean, i.e. $\delta = \frac{\mu}{\mu_0}$. Note that $\delta = 1$ when the process is *in-control*.

A commonly used estimator for the unknown *in-control* process mean μ_0 is,

$$\hat{\mu}_0 = \frac{1}{n}\sum_{i=1}^{n} X_i \,,$$

Without loss of generality but to simplify the designs, the following standardization is considered,

$$Y_t = \frac{X_t}{\hat{\mu}_0}\,.$$

Note that Y_t can be restated as follows:

$$Y_t = \underbrace{\frac{\mu_0}{\hat{\mu}_0}}_{w} \cdot \underbrace{\frac{\mu}{\mu_0}}_{\delta} \cdot \underbrace{\frac{X_t}{\mu}}_{Z_t},$$

i.e.

$$Y_t = w \cdot \delta \cdot Z_t.$$

Recall that the observations X_t are assumed to be distributed as $\text{Exp}\left(\frac{1}{\mu}\right)$. Hence, $Z_t = \frac{X_t}{\mu} \sim \text{Exp}(1)$ is a random variable representing the observations standardized with the true but unknown mean. The magnitude of a shift in the mean represented in terms of the true *in-control* mean is $\delta = \frac{\mu}{\mu_0}$ and the random variable W representing the ratio of the true *in-control* mean to the estimator of the *in-control* mean is $w = \frac{\mu_0}{\hat{\mu}_0}$. Note that when there is no estimation error, $w = 1$.

This representation allows us to monitor a standardized process and use a unique control chart design for various mean values.

3.2.2. Design Parameters of the EMWA control chart

In designing the exponential TBE EWMA control charts with desired ARL_0 and ARL_1 performances, the *smoothing parameter* (λ_q) of the EWMA statistic, the *boundary* (B) and the *control limit* (h_q) of the EWMA control chart, and the *sample size* (n) used in *Phase I* study for parameter estimation should be determined.

Smoothing Parameter (λ_q)

Recall that, λ_q was defined as the smoothing parameter of the TBE EWMA statistic z_i, such that the weights given to the past observations are $\lambda_q(1 - \lambda_q)^j$, ($j$ = 0, 1, 2,…, i – 1). Here λ_q is a measure of how much importance is given to the past data. In other words, how much of the past information should be carried forward. Note that for λ_q = 1, Equation (2.3) becomes

$$z_i = x_i ,$$

which corresponds to the classical Shewhart chart.

An important point in determining λ_q is that the amount of information carried from the past determines the sensitivity of the chart to the amount of shifts. Accordingly, smaller values of λ_q perform generally better in detecting smaller shifts. Hence, it is advised that at first λ_q should be determined and then the chart limit (h_q) is obtained according to the desired ARL performance. Montgomery (2001) states that the values of λ_q in the interval $0.05 \leq \lambda_q \leq 0.25$ often perform well. He also states $\lambda_q = 0.05$, $\lambda_q = 0.10$, and $\lambda_q = 0.20$ as popular choices for λ_q. Page (1954) states that a rule that is optimum for a certain magnitude of change would not be optimum for other magnitudes of change. Crowder (1989) includes a discussion on the optimal choices of smoothing parameter λ_q of an EWMA control chart.

Chart Limit (h_q)

The control limit h_q is used to make a decision for the state of a process and an *out-of-control* alarm is triggered if this limit is exceeded. In this thesis, Exponential TBE EWMA charts are considered for detecting downward shifts in the process mean. The value of the h_q should be defined according to the desired ARL performance. A large control limit h_q would result in larger ARL_0 performance and hence less false alarms when the process is *in-control*. On the other hand, defining a small h_q would result in a large ARL_0 but at the same time detection of an *out-of-control* state might be delayed. As stated above, defining an h_q to provide an acceptable ARL_0 value is a common approach. In this thesis, h_q values are determined using a binary search to obtain an ARL_0 value of 500, which is a commonly employed value in the literature.

Chart Boundary (*B*)

For one-sided EWMA control charts a boundary is used in order to ensure the EWMA statistic not exceed a certain distance from the chart limit h_q (Gan, 1998) and for finite discretization of the state space. For a lower-sided chart intended for detecting a decrease in the mean of the process, the boundary B satisfies, $h_q < B$. The control statistic takes values in the interval $[h_q, B]$, so that it does not exceed B and give *out-of-control* alarm when it goes below h_q. Then the lower-sided EWMA chart with boundary B is obtained from Equation (2.2) by plotting

$$z_i = \min\{B, \lambda_q x_i + (1-\lambda_q) z_{i-1}\}, \quad (0 < \lambda_q \leq 1).$$

Sample Size (*n*)

Sample size (n) is the sample size used in the *Phase-I* of the construction of a control chart, which was explained in Section 3.1. As a natural result of sampling variation, smaller sample sizes may result in less reliable parameters estimated. Larger sample sizes result in more accurate estimation of the parameters, while *Phase-I* may become more costly.

Initial State (z_0)

In Section 2.3.2, z_0 is defined as the starting value of the EWMA statistic. The process target value is a commonly used value for z_0 in applications. However, in some cases such as if there is a suspicion that the process is initially *out-of-control*, z_0 can be selected different from the process target value.

3.3. Calculation of Conditional Performance

Various values for the estimation error W may be considered to calculate performance measures in order to show the effect of estimation under these hypothetical cases of parameter estimation. Suppose that a reference sample of observations is obtained in *Phase-I* and μ_0 is estimated from the data. For an estimate $\hat{\mu}_0$, corresponding value for the control limit determined to obtain a desired ARL_0 is also an estimate and will be shown as $\hat{h}_q$. Hence, conditional on the specific values of design parameters $\hat{\mu}_0$, $\hat{h}_q$, z_0 and B, performance measures for RL and its properties (ARL, SDRL and percentiles) can be obtained. Since these performance measures are functions of the true process mean μ, we use the following conditional representation for the performance metrics under consideration as in Testik (2007):

$$\mathrm{ARL}_{\mathrm{conditional}} = \mathrm{ARL}(\mu \mid \hat{\mu}_0, \hat{h}_q, z_0, w), \tag{3.1}$$

$$\mathrm{SDRL}_{\mathrm{conditional}} = \mathrm{SDRL}(\mu \mid \hat{\mu}_0, \hat{h}_q, z_0, w), \tag{3.2}$$

$$\mathrm{Percentile}_{\mathrm{conditional}} = \mathrm{Percentile}(\mu, \mid \hat{\mu}_0, \hat{h}_q, z_0, w). \tag{3.3}$$

Here, " | " implies "given that" and indicates the conditional nature of the performance meausures on the specific values (estimates) of the control chart parameters.

Since the control chart can be conditioned on the estimation error (W), the effect of estimation on control chart parameters can be examined (Jones et al., 2004).

In this study, conditional performances in Equations (3.1), (3.2), and (3.3) are calculated using the Markov chain approach explained below.

3.3.1. Markov Chain Model

Brook and Evans (1972) propose to use the Markov chains for studying the RL distributions of control charts. A Markov chain representation here requires the possible values for the control statistic to be discretized to represent the states. Although representing the continuous values of the control statistic as discrete states corresponds to an approximate model, the solution to the model is exact and this makes the method favorable.

As shown in Figure 3.1, in order to detect the downward shifts in the mean of the TBE observations, the *in-control* interval is defined as $[h_q, B]$. To use the Markov chain approach, this interval is divided into K subintervals representing the states. The j^{th} subinterval is defined by (L_j, U_j),

$$L_j = B - \left[j \cdot \frac{(B - h_q)}{K} \right],$$
$$U_j = B - \left[(j-1) \cdot \frac{(B - h_q)}{K} \right],$$

where the midpoint η_j of the j^{th} subinterval is,

$$\eta_j = B - \frac{(2j-1) \cdot (B - h_q)}{2K}. \tag{3.4}$$

The $(K + 1)^{\text{th}}$ state corresponds to the situation where the control statistic z_i is in the *out-of-control* region. Since an *out-of-control* signal results in the process being interrupted and inspected for an assignable cause, from the Markov chain point of view, the $(K + 1)^{\text{th}}$ state is the one and only absorbing state of the model. Consequently, all other remaining states 1, 2, …, K corresponding to the *in-control* region are transient states. Since there exists an absorbing state in the model, the

Markov chain is an *absorbing Markov chain*. As a result, the ARL of the control chart is the average time to absorbtion in the corresponding Markov chain model.

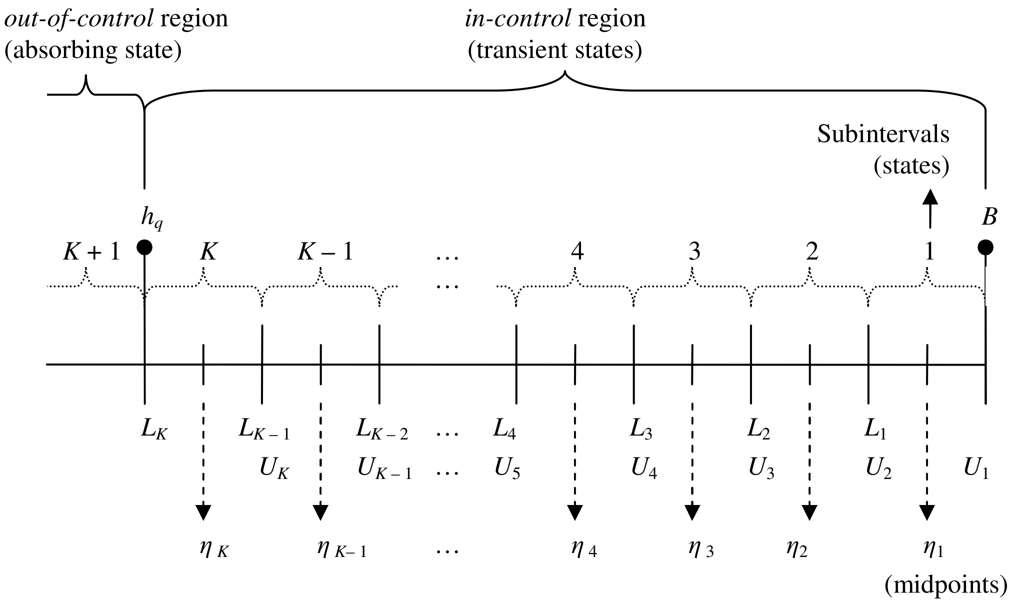

Figure 3.1 – Markov chain model

For the Markov chain model proposed, the probability p_{ij} denote the probability of transition from state i to j in one step. In other words, if the control statistic Z is in state i at time $t-1$, the probability of Z being in state j at time t is p_{ij}. Thus, the transition probability matrix $\mathbf{P}$ can be formed as,

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1j} & \cdots & p_{1K} & p_{1(K+1)} \\ p_{21} & p_{22} & \cdots & p_{2j} & \cdots & p_{2K} & p_{2(K+1)} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ p_{i1} & p_{i2} & \cdots & p_{ij} & \cdots & p_{iK} & p_{i(K+1)} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ p_{K1} & p_{K2} & \cdots & p_{Kj} & \cdots & p_{KK} & p_{K(K+1)} \\ 0 & 0 & \ldots & 0 & \ldots & 0 & 1 \end{bmatrix}. \tag{3.5}$$

Construction of Transition Probability Matrix

The transition probabilities p_{i1} in the first column of **P** in Equation (3.5) can be written as follows,

$$p_{i1} = \Pr(Z_t \geq L_1 \mid Z_{t-1} = \eta_i)\text{, for } 1 \leq i \leq K. \tag{3.6}$$

Note that, the upper limit U_1 of state 1 is not used in Equation (3.6). This ensures that all $Z_t \geq L_1$ would be treated as they were in state 1, so that the idea of the EWMA boundary B is incorporated within the Markov chain model.

The remaining transition probabilities p_{ij} of **P**, where $j > 1$, in Equation (3.5) can be rewritten as follows,

$$p_{ij} = \Pr(L_j < Z_t < U_j \mid Z_{t-1} = \eta_i)\text{, for } 1 \leq i \leq K \text{ and } j > 1. \tag{3.7}$$

Equation (3.7) represents the probabilities for Z_t to be within the boundaries (L_j, U_j) of the state j, given that $Z_{t-1} = \eta_i$ is the midpoint of state i.

The probabilities of transition from the states 1, 2, …, K to the absorbing state (K + 1) can be written as follows

$$p_{i(K+1)} = \Pr(Z_t \leq L_K \mid Z_{t-1} = \eta_i)\text{, for } 1 \leq i \leq K. \tag{3.8}$$

Equations (3.6), (3.7), and (3.8) can be evaluated by employing the probability distribution of the random variable X. Recall that, the EWMA statistic Z_t with the boundary B is

$$Z_t = \min\{B, (1-\lambda_q)Z_{t-1} + \lambda_q X_t\}. \tag{3.2}$$

Then, we may write the probabilities p_{i1} in Equation (3.6) as

$$p_{i1} = \Pr\{(1-\lambda_q)Z_{t-1} + \lambda_q X_t \geq L_1 \mid Z_{t-1} = \eta_i\}, \text{ where } 1 \leq i \leq K. \tag{3.9}$$

Substituting Z_{t-1} with η_i in Equation (3.9), we get

$$p_{i1} = \Pr\left\{(1-\lambda_q)\eta_i + \lambda_q X_t \geq B - 1 \cdot \frac{(B-h_q)}{K}\right\}, \text{ for } 1 \leq i \leq K.$$

Using Equation (3.4), we get

$$p_{i1} = \Pr\left\{X_t \geq \frac{1}{\lambda_q}\left(B - \frac{(B-h_q)}{K} - (1-\lambda_q)B + \frac{(1-\lambda_q)(2i-1)(B-h_q)}{2K}\right)\right\}$$

$$= 1 - F_{X_t}\left\{\frac{1}{\lambda_q}\left(B - \frac{(B-h_q)}{K} - (1-\lambda_q)B + \frac{(1-\lambda_q)(2i-1)(B-h_q)}{2K}\right); \mu\right\},$$

where $1 \leq i \leq K$ and $F_{X_t}(x,\mu)$ is the cumulative distribution function of exponential distribution with mean μ, defined as

$$F_{X_t}(x,\mu) = \begin{cases} 1 - e^{-x/\mu} & , \quad x \geq 0 \\ 0 & , \quad x < 0 \end{cases}$$

Hence, knowing the probability distribution of X, i.e. the exponential distribution with mean μ, the values for the probabilities p_{i1} can be easily calculated.

Equation (3.7) can also be evaluated similarly. The values for the remaining probabilities p_{ij} (for $1 \leq i \leq K$, j > 1) can be obtained as shown below:

$$p_{ij} = \Pr\left\{ B-(j-1)\cdot\frac{(B-h_q)}{K} > (1-\lambda_q)\underbrace{Z_{t-1}}_{m_i} + \lambda_q X_t > B - j\cdot\frac{(B-h_q)}{K} \mid Z_{t-1} = \eta_i \right\}$$

$$= \Pr\left\{ B-(j-1)\cdot\frac{(B-h_q)}{K} > (1-\lambda_q)B - \frac{(1-\lambda_q)(2i-1)(B-h_q)}{2K} + \lambda_q X_t > B - j\cdot\frac{(B-h_q)}{K} \right\}$$

$$= \Pr\left\{ \frac{1}{\lambda_q}\cdot\left(B-(j-1)\cdot\frac{(B-h_q)}{K} - (1-\lambda_q)B + \frac{(1-\lambda_q)(2i-1)(B-h_q)}{2K} \right) > X_t \right.$$

$$\left. > \frac{1}{\lambda_q}\left(B - j\cdot\frac{(B-h_q)}{K} - (1-\lambda_q)B + \frac{(1-\lambda_q)(2i-1)(B-h_q)}{2K} \right) \right\},$$

$$= F\left\{ \frac{1}{\lambda_q}\cdot\left(B-(j-1)\cdot\frac{(B-h_q)}{K} - (1-\lambda_q)B + \frac{(1-\lambda_q)(2i-1)(B-h_q)}{2K} \right); \mu \right\}$$
$$- F\left\{ \frac{1}{\lambda_q}\left(B - j\cdot\frac{(B-h_q)}{K} - (1-\lambda_q)B + \frac{(1-\lambda_q)(2i-1)(B-h_q)}{2K} \right); \mu \right\}$$

The number of states (K + 1) used in the model should be large enough to approximate the system behavior well. On the other side, the integration algorithms used in the marginal performance calculations can increase the computation times significantly, so that the dimensions of the matrices should be kept in a managable size. In the model constructed here, number of states is selected to be 301 gives satisfactory results, with sufficient precision level.

3.3.2. Calculation of the Performance Measures of the RL distribution

ARL of the RL distribution:

For an absorbing Markov chain it is possible to represent the transition probability matrix in *partitioned (canonical) form*. So **P** in Equation (3.5) becomes,

$$
\mathbf{P} = \begin{array}{c} \\ \text{Tr.} \\ \\ \text{Abs.} \end{array}
\overset{\begin{array}{cc} \text{Tr.} & \text{Abs.} \end{array}}{
\left[\begin{array}{cccccc|c}
p_{11} & p_{12} & \cdots & p_{1j} & \cdots & p_{1K} & p_{1(K+1)} \\
p_{21} & p_{22} & \cdots & p_{2j} & \cdots & p_{2K} & p_{2(K+1)} \\
\vdots & \vdots & & \vdots & & \vdots & \vdots \\
p_{i1} & p_{i2} & \cdots & p_{ij} & \cdots & p_{iK} & p_{i(K+1)} \\
\vdots & \vdots & & \vdots & & \vdots & \vdots \\
p_{K1} & p_{K2} & \cdots & p_{Kj} & \cdots & p_{KK} & p_{K(K+1)} \\
\hline
0 & 0 & \ldots & 0 & \ldots & 0 & 1
\end{array}\right]}
$$

$$
\mathbf{P} = \begin{array}{c} \text{Tr.} \\ \text{Abs.} \end{array}
\overset{\begin{array}{cc} \text{Tr.} & \text{Abs.} \end{array}}{
\left[\begin{array}{c|c}
\mathbf{Q} & \mathbf{R} \\
\hline
0 & \mathbf{I}
\end{array}\right]} . \tag{3.10}
$$

where **Q**, **R**, and **I** are given by

$$
\mathbf{Q} = \begin{bmatrix}
p_{11} & p_{12} & \cdots & p_{1K} \\
p_{21} & p_{22} & \cdots & p_{2K} \\
\vdots & \vdots & \ddots & \vdots \\
p_{K1} & p_{K2} & \cdots & p_{KK}
\end{bmatrix},
$$

$$
\mathbf{R} = \begin{bmatrix}
p_{1(K+1)} \\
p_{2(K+1)} \\
\vdots \\
p_{K(K+1)}
\end{bmatrix}, \tag{3.11}
$$

$$
\mathbf{I} = [1].
$$

In Equation (3.10), **Q** is a K x K matrix that represents the transition probabilities among the transient states, **I** is a 1 x 1 identity matrix that represent the absorbing state, and **R** is a nonzero K x 1 matrix that represents the escape probabilities from each transient state. In the Markov chain model constructed for TBE EWMA, the matrix **Q** can be obtained by deleting last $(K + 1)^{\text{th}}$ row and column of the matrix **P** in partitioned form.

For an absorbing Markov chain **P**, the matrix $\mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1}$ is defined as the *fundamental matrix* of **P**. In the calculation of expected time to absorbtion, which is also the ARL of the control chart model, the fundamental matrix **N** of **P** takes an important place. The entry n_{ij} of **N** is the expected number of times that the process is in the transient state j given that the process has started at transient state i. Thus, adding all the entries of the i^{th} row of **N**, $\sum_{j=1}^{K} n_{ij}$, provides the expected number of times that the process spends among the transient states before it leaves for the absorbing state, given that the process has started at transient state i. In matrix notation, we can write

$$\mathbf{T} = \mathbf{N}\mathbf{1} = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{1},$$

where **1** is a K x 1 vector of ones. The i^{th} entry, t_i, of the matrix **T** is the expected time to absorbtion (ARL) given that the process has started at state i.

SDRL of the RL distribution:

It is mentioned in Section 3.1 that ARL itself is not sufficient in defining the characteristics of the RL distribution of the control chart. With the Markov chain method it is possible to obtain higher-order moments and calculate the SDRL of the control chart. White and Keats (1996) describes how the higher-order moments can be obtained. Let M represent the ARL which is also the first-order moment of the RL distribution.

$$M = \mathbf{T} = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{1}.$$

The r^{th}-order factorial moments are given in the following recursive formula by using Equation (3.29)

$$M^{(r)} = r\{(\mathbf{I}-\mathbf{Q})^{-1} - \mathbf{I}\}M^{(r-1)}. \tag{3.12}$$

These factorial moments in Equation (3.12) can be converted into power moments by

$$M'_r = \sum_{k=1}^{r} S(r,k)M^{(r)} \tag{3.13}$$

where $S(r, k)$ is the *Stirling Number of the Second Kind*, such that

$$S(r,k) = \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{l-k}l^r,$$
$$k = 0,1,\ldots,r.$$

By using Equation (3.13), the first and second-order moments about the mean can be calculated as

$$M_1 = M'_1,$$

$$M_2 = M'_2 - (M'_1)^2 i.$$

Here, M_2, the second-order moment about the mean, defines the variance of the RL distribution. The i^{th} entry, $m_{2(\text{i})}$, of M_2 gives the variance of the RL distribution given that the process has started in state i. Accordingly the SDRL of the RL distribution is equal to the square root of the i^{th} entry, $m_{2(\text{i})}$, of M_2, i.e.

$$\text{SDRL} = \sqrt{m_{2(\mathrm{i})}}\,,$$

Here, i is the initial state of the process starts.

Percentiles of the RL distribution:

Recall the partitioned form of **P** in Equation (3.10). The last column of **P** is the probabilities $\Pr(Z_1 \leq L_K \mid Z_0 = \eta_i)$ for i = 0, 1, …, K, that is the probability of being absorbed at the first step. Accordingly, the last column of $\mathbf{P}^v$ gives the probability of being absorbed after v steps, i.e. $\Pr(Z_v \leq L_K \mid Z_0 = \eta_i)$.

Since **P** is a probability transition matrix, its row sums are equal to 1. Hence, **R** in Equation (3.11) can be defined as

$$\mathbf{R} = (\mathbf{I} - \mathbf{Q})\mathbf{1}.$$

Then $\mathbf{P}^v$ can be represented in terms of **Q**, as

$$\mathbf{P}^v = \left[\begin{array}{c|c} \mathbf{Q}^v & (\mathbf{I} - \mathbf{Q}^v)\mathbf{1} \\ \hline 0 & 1 \end{array}\right], \text{ for } v = 1, 2, \ldots$$

Let $\mathbf{F}_v$ be a K x 1 vector of length K, whose i^{th} entry is the cumulative probability function of the RL distribution corresponding to the initial state. Then from Equation (3.8),

$$(\mathbf{I}-\mathbf{Q}^{v})\mathbf{1}=\begin{bmatrix}\Pr(Z_v \le L_K \mid Z_0=\eta_1)\\ \Pr(Z_v \le L_K \mid Z_0=\eta_2)\\ \vdots \\ \Pr(Z_v \le L_K \mid Z_0=\eta_i)\\ \vdots \\ \Pr(Z_v \le L_K \mid Z_0=\eta_K)\end{bmatrix}, \text{for } v=0,1,... \qquad (3.38)$$

and

$$\mathbf{F}_v=(\mathbf{I}-\mathbf{Q}^{v})\mathbf{1}, \text{ for } (v=0,1,...). \qquad (3.14)$$

In Equation (3.14), elements of the cumulative probability vector $\mathbf{F}_v$ multiplied by 100 give the percentiles of the RL distribution, where each element corresponds to an initial state. The b^{th} percentile of the RL distribution is calculated by finding the smallest number of steps v with the first element of the cumulative probability vector $\mathbf{F}_v$ greater than or equal to $b/100$.

3.4. Calculation of Marginal Performance

In reality, it is impossible to know how the estimated mean is related with the true (or actual) *in-control* mean. Hence, it would be valuable to evaluate the marginal performance of a chart considering the distribution of the estimated parameters.

By integrating the conditional performance measures with respect to w, one can obtain the unconditional performances as

$$\text{ARL}_{\text{marginal}}(\delta, z_0) = \int_0^{+\infty} \text{ARL}_{\text{conditional}}(\mu \mid w) f_W(w) dw, \tag{3.15}$$

$$\text{SDRL}_{\text{marginal}}(\delta, z_0) \int_0^{+\infty} \text{SDRL}_{\text{conditional}}(\mu \mid w) f_W(w) dw, \tag{3.16}$$

$$\text{Percentile}_{\text{marginal}}(\delta, z_0) = \int_0^{+\infty} \text{Percentile}_{\text{conditional}}(\mu \mid w) f_W(w) dw. \tag{3.17}$$

This operation corresponds to averaging over all possible values of the random variable *W*. However, the probability density function, $f_W(w)$, of the *ratio of estimation error* (*W*) is required.

For evaluating the integrals in Equations (3.15), (3.16), and (3.17), Simpson's quadrature is used and each quadration step is calculated using the conditional performance measures given in Section 3.3.2. In order to obtain marginal performance results, six computers ran continuously for three weeks and most of the calculation time is occupied by the calculation of percentiles.

3.4.1. Probability Density Function of *W*

To define the probability density function of *W*, properties of Gamma distribution and Inverse Gamma distribution will be discussed first.

Gamma Distribution:

A random variable *X* with range (0; ∞) is distributed according to a two-parameter Gamma distribution, Gamma(*a*, *b*), where *a* is the shape and *b* is the scale parameter, if the density function is given by

$$f(x)=\frac{x^{a-1}}{b^a\Gamma(a)}\cdot\exp\left(-\frac{x}{b}\right).$$

The mean and variance of the distribution are $\mathrm{E}[X]=a\cdot b$ and $\mathrm{V}[X]=a\cdot b^2$.

Note that when a = 1, x follows the exponential distribution Exp(b) with

$$f(x)=\frac{1}{b}\cdot\exp\left(-\frac{x}{b}\right).$$

An important property of the gamma distribution is the additivity. Let X_1, X_2, …, X_n be independent random variables where $X_i \sim \mathrm{Gamma}(a_i, b)$, then the sum is distributed as

$$\sum_{i=1}^{n} X_i \sim \mathrm{Gamma}\left(\sum_{i=1}^{n} a_i, b\right).$$

Another important property of the gamma distribution is the scalability. Let X_1, X_2, …, X_n be independent random variables where $X_i \sim \mathrm{Gamma}(a_i, b)$, then

$$\alpha X_i \sim \mathrm{Gamma}(a, \alpha b).$$

Using these results, we can write

$$\frac{1}{W}=\frac{\hat{\mu}_0}{\mu_0}=\frac{\frac{1}{n}\sum_{i=1}^{n} X_i}{b} \sim \mathrm{Gamma}\left(n, \frac{1}{n}\right),$$

where a = 1, $\alpha=\frac{1}{nb}$.

In particular,

$$E\left[\frac{1}{W}\right]=1,$$

$$V\left[\frac{1}{W}\right]=\frac{1}{n},$$

Note that, $\frac{1}{W}$ is also asymptotically normal distributed according to $N\left(1,\frac{1}{n}\right)$.

Inverse Gamma (IGamma) Distribution:

Let X be a Gamma distributed random variable as Gamma(a, b^{-1}) . The inverse of this random variable, say $Y=\frac{1}{X}$ follows the Inverse-Gamma distribution IGamma(a, b) with range (0, ∞). Probability density function, cumulative distribution function, mean, and variance of the Inverse-Gamma distribution are

$$f(y)=\frac{b^a}{\Gamma(a)}\cdot y^{-a-1}\cdot\exp\left(-\frac{b}{y}\right),$$

$$F(y)=\frac{\Gamma(a,\frac{b}{y})}{\Gamma(a)},$$

$$E[Y]=\frac{b}{a-1},$$

$$V[Y]=\frac{b^2}{(a-1)^2(a-2)},$$

respectively. Hence, it can be shown that the random variable W is distributed IGamma (n, n), where,

$$f(w) = \frac{n^n}{(n-1)!} \cdot w^{-n-1} \cdot \exp\left(-\frac{n}{w}\right),$$

$$F(w) = \frac{\Gamma(n, \frac{n}{w})}{(n-1)!}, \tag{3.18}$$

$$E[W] = \frac{n}{n-1},$$

$$V[W] = \frac{n^2}{(n-1)^2 (n-2)}.$$

Note that the calculation of the terms n^n and $(n$-1)! are required in the calculation of $f(w)$, which yield quite large numbers for computations. To overcome the computational difficulties, one may first take the natural logarithm of both sides, as

$$\begin{aligned} \ln[f(w)] &= \ln\left(\frac{n^n}{(n-1)!}\right) + \ln\left(w^{-n-1}\right) + \ln\left(\exp\left(-\frac{n}{w}\right)\right) \\ &= n \cdot \ln(n) - \sum_{i=1}^{n-1} \ln(i) - (n+1) \cdot \ln(w) - \frac{n}{w} \end{aligned}$$

and then evaluate *f(w)* by taking its exponential as

$$\begin{aligned} f(w) &= \exp(\ln[f(w)]) \\ &= \exp\left[n \cdot \ln(n) - \sum_{i=1}^{n-1} \ln(i) - (n+1) \cdot \ln(w) - \frac{n}{w}\right] \end{aligned}$$

CHAPTER 4

CONDITIONAL PERFORMANCE MEASURES

4.1. Conditional Performance Results

Recall that the conditional performance measures; ARL, SDRL and percentiles of the RL distribution, are defined as:

$$\text{ARL}_{\text{conditional}} = \text{ARL}(\mu \mid \hat{\mu}_0,\ \hat{h}_q,\ z_0,\ w),$$

$$\text{SDRL}_{\text{conditional}} = \text{SDRL}(\mu \mid \hat{\mu}_0,\ \hat{h}_q,\ z_0,\ w),$$

$$\text{Percentile}_{\text{conditional}} = \text{Percentile}(\mu, b \mid \hat{\mu}_0,\ \hat{h}_q,\ z_0,\ w).$$

where " | " implies "given that" and shows the conditionality of the performance functions on the specific values of the control chart parameters and the estimation error. In order to gain insight on the effect of estimation error on the performance of the control chart, several hypothetical values for the estimation error are evaluated using the Markov chains developed in Section 3.3.1.

The values of the parameters considered in the study to evaluate the conditional performance of the RL distribution are as follows;

1. *Smoothing parameter values:* λ_q = 0.01, 0.05, 0.10, 0.20, 0.40, 0.60, and 1.00, where the values λ_q = 0.05, 0.10, and 0.20 are popular choices in practice and the case λ_q = 1 corresponds to the classical Shewhart control chart.

2. *Control chart boundary values:* B = 1, 2, and 5.

3. *Control limits:* h_q values are obtained through a binary search for each parameter combination (λ_q, B) considered above in order to obtain an ARL_0 performance of 500. Due to the precision level used in the binary search, exact ARL_0 value of 500 cannot be obtained. The h_q values obtained and the actual ARL_0 values corresponding to those values are given in Table 4.1. The h_q values does not change when there is estimation error. Because it is assumed in *Phase-I* that the estimated mean properly represents the true *in-control* mean so that in the standardized model w = 1.

Table 4.1 - The calculated h_q values and corresponding actual ARL_0 values.

λ_q	0.01		0.05		0.10		0.20		0.40		0.60		1.00	
B	h_q	**ARL_0**	h_q	**ARL_0**	h_q	**ARL_0**	h_q	**ARL_0**	h_q	**ARL_0**	h_q	**ARL_0**	h_q	**ARL_0**
1	0.8710	499.90	0.6561	499.94	0.5176	499.90	0.3577	500.34	0.1921	499.70	0.1026	500.11	0.0020	499.82
2	0.9014	500.22	0.6860	500.35	0.5450	500.24	0.3793	500.46	0.2045	500.25	0.1091	499.90	0.0020	499.82
5	0.8761	499.84	0.6853	500.11	0.5444	500.17	0.3793	500.04	0.2053	500.27	0.1098	499.97	0.0020	499.82

4. *Shift magnitude values:* δ = 0.2, 0.4, 0.6, 0.8, and 1 are considered. Note that, for an exponential TBE process, the mean (μ) of the process is also equal to the standard deviation of the process so that δ represents the decrease in the mean of the TBE observations in terms of the process standard deviation. As an example, δ = 0.8 corresponds to a 0.2 standard deviation decrease in the mean, and δ = 0.2 corresponds to a 0.8 standard deviation decrease in the mean.

   The case δ = 1 corresponds to an *in-control* process, where the corresponding ARL is the ARL_0. Among the cases considered here $\delta \leq 0.8$

corresponds to an *out-of-control* process, where the corresponding ARL is the ARL_1. Throughout the study, δ = 0.8 is the smallest and δ = 0.2 is the largest shift considered.

5. *Sample size:* n = 30, 50, 100, 200, 500, 1000, and 10000 are considered, where 10000 approximately implies the performance without an estimation error.

6. *Ratio of the true in-control mean to its estimate:* Distribution of $w = \frac{\mu_0}{\hat{\mu}_0}$ is used to obtain the hypothetical values for conditional performance evaluation as the 25th (overestimation of the mean), 50th , and the 75th (underestimation of the mean) percentiles. To obtain a value of w, the sample size n employed in *Phase-I* and the cumulative probability distribution function of W given in Equation (3.18) are used. Figure 4.1 shows the probability density function of W for various sample sizes, where 2.5% of the distribution is truncated from each tail.

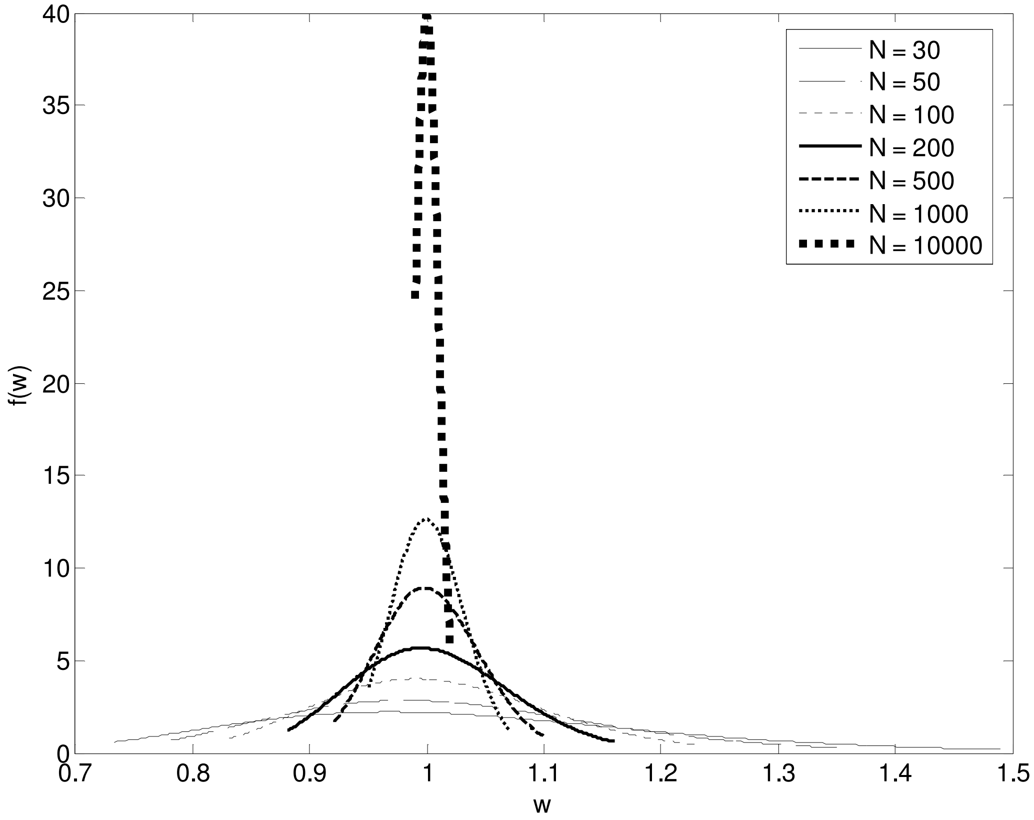

Figure 4.1 – Probability distribution function of *W* for various sample sizes (*n*).

Assuming the true mean is $\mu_0 = 1$, Table 4.2 and Figure 4.2 present the values of the estimated mean ($\hat{\mu}_0$) corresponding to the 25th, 50th and 75th percentiles of *W*. These are provided to give an idea on the magnitude of estimation error corresponding to those percentiles. Note that the value of $\hat{\mu}_0$ at the 50th percentile of *W* is less than 1 and the distribution of $\hat{\mu}_0$ is slightly left skewed. Hence, the 50th percentile of *W* does not correspond to the case that there is no estimation error (w = 1). Note also that, the value of $\hat{\mu}_0$ corresponding to the mean E(*W*) is also less than 1, which implies that underestimation is anticipated more, especially for small sample sizes. However, as expected, the values of the percentiles and E(W) converge to 1 as the sample size is increased.

Table 4.2 – Assuming the true *in-control* mean is $\mu_0 = 1$ the values of the estimated mean $\hat{\mu}_0$ corresponding to the expected value and the percentiles of *W*.

Sample Size (*n*)	$\hat{\mu}_0$ corresponding to…			
	…the E(*W*)	…the percentile of *W* 25th	50th	75th
30	0.9667	1.1163	0.9888	0.8715
50	0.9800	1.0913	0.9932	0.9013
100	0.9900	1.0654	0.9966	0.9308
200	0.9950	1.0467	0.9983	0.9514
500	0.9980	1.0298	0.9993	0.9695
1,000	0.9990	1.0211	0.9996	0.9785
10,000	0.9999	1.0066	0.9999	0.9931

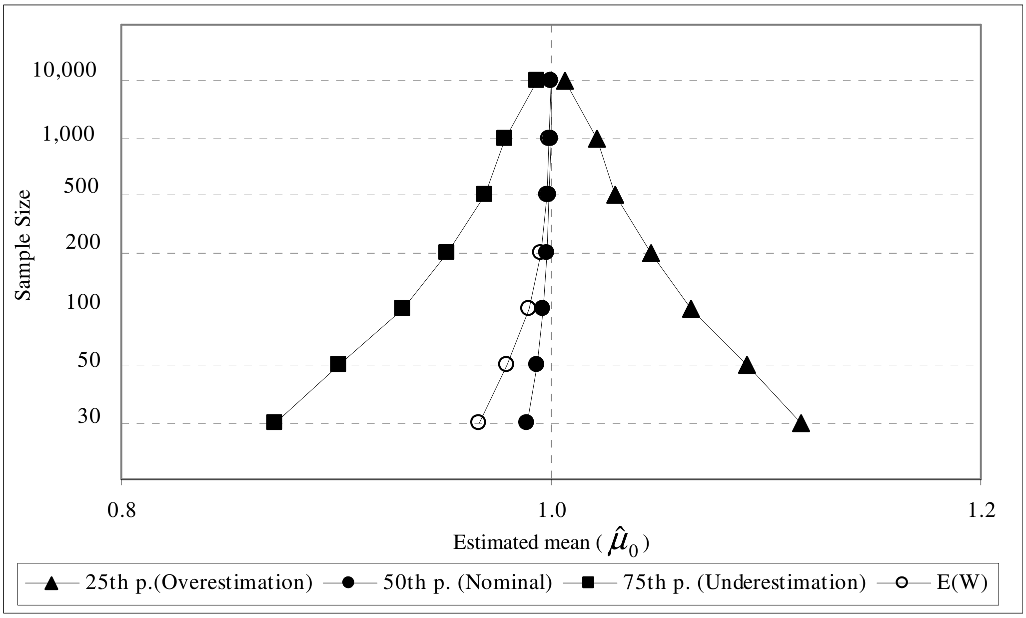

Figure 4.2 – Assuming the true *in-control* mean is $\mu_0 = 1$ the values of the estimated mean $\hat{\mu}_0$ corresponding to the expected value and the percentiles of *W*.

4.2. Known Parameter Case

To have a basis for comparing the performance and in order to understand how the control chart performs under normal operating conditions, the known parameter case ($w = 1$) is considered first. This also corresponds to the case that the sample size is infinity.

Table 4.3 – Conditional RL performance of the control chart with known design parameters (w = 1)

	B	**1**					**2**					**5**				
λ_q	δ	**ARL**	**SDRL**	**10th**	**50th**	**90th**	**ARL**	**SDRL**	**10th**	**50th**	**90th**	**ARL**	**SDRL**	**10th**	**50th**	**90th**
	0.20	18.01	1.19	17	18	20	14.14	1.16	13	14	16	13.95	2.37	11	14	17
	0.40	24.61	3.67	20	24	29	19.15	3.28	15	19	23	22.14	5.61	16	21	30
	0.60	38.48	10.64	27	37	53	29.86	9.34	20	28	42	36.67	13.33	22	34	54
	0.80	83.71	43.75	41	73	141	65.16	37.11	29	56	114	80.93	47.04	35	69	142
0.01	**1.00**	**499.90**	**449.55**	**98**	**363**	**1,085**	**500.22**	**497.85**	**68**	**341**	**1,149**	**499.84**	**468.39**	**83**	**356**	**1,110**
	0.20	11.48	1.17	10	11	13	10.22	1.08	9	10	12	10.18	1.14	9	10	12
	0.40	16.97	4.06	13	16	22	14.91	3.63	11	14	20	14.87	3.71	11	14	20
	0.60	31.12	14.41	17	28	50	26.67	12.19	15	24	43	26.69	12.37	14	24	43
	0.80	91.45	71.04	27	70	184	74.89	56.92	23	58	149	75.12	57.44	23	58	150
0.05	**1.00**	**499.94**	**478.90**	**72**	**353**	**1,124**	**500.35**	**487.00**	**65**	**351**	**1,135**	**500.11**	**487.50**	**64**	**350**	**1,135**
	0.20	9.31	1.28	8	9	11	8.55	1.18	7	8	10	8.50	1.20	7	8	10
	0.40	15.07	5.05	10	14	22	13.53	4.42	9	13	19	13.47	4.45	9	12	19
	0.60	32.76	20.43	14	27	59	27.94	16.67	12	23	50	27.91	16.76	12	23	50
	0.80	110.85	97.21	24	81	237	91.11	78.42	21	67	193	91.19	78.71	21	67	194
0.10	**1.00**	**499.90**	**486.42**	**65**	**351**	**1,133**	**500.24**	**489.98**	**62**	**350**	**1,138**	**500.17**	**490.29**	**62**	**350**	**1,139**
	0.20	7.78	1.66	6	7	10	7.24	1.51	6	7	9	7.23	1.51	6	7	9
	0.40	15.29	7.85	8	13	25	13.59	6.65	7	12	22	13.59	6.66	7	12	22
	0.60	42.13	33.73	12	32	86	35.05	27.10	11	27	70	35.01	27.08	11	27	70
	0.80	145.17	136.55	23	103	323	122.23	114.13	20	87	271	121.89	113.82	20	87	270
0.20	**1.00**	**500.34**	**491.93**	**60**	**349**	**1,141**	**500.46**	**493.54**	**59**	**349**	**1,143**	**500.04**	**493.20**	**59**	**349**	**1,142**
	0.20	7.48	3.07	5	7	11	6.94	2.73	4	6	10	6.90	2.71	4	6	10
	0.40	21.30	16.32	7	16	43	18.49	13.71	6	14	36	18.32	13.57	6	14	36
	0.60	67.71	62.54	12	49	149	57.11	52.14	10	41	125	56.40	51.47	10	41	123
	0.80	197.31	192.20	25	138	448	175.51	170.69	23	123	398	173.44	168.69	23	122	393
0.40	**1.00**	**499.70**	**494.70**	**57**	**348**	**1,144**	**500.25**	**495.86**	**57**	**348**	**1,146**	**500.27**	**496.03**	**57**	**348**	**1,146**
	0.20	9.25	5.88	4	7	17	8.48	5.21	4	7	15	8.40	5.15	4	7	15
	0.40	33.08	29.50	7	24	72	28.79	25.32	6	21	62	28.32	24.87	6	21	61
	0.60	98.43	94.84	14	69	222	86.09	82.63	12	61	194	84.40	80.96	12	60	190
	0.80	239.94	236.41	28	167	548	221.33	218.00	26	154	505	217.95	214.66	26	152	498
0.60	**1.00**	**500.11**	**496.64**	**56**	**348**	**1,147**	**499.90**	**496.78**	**55**	**347**	**1,147**	**499.97**	**496.94**	**55**	**347**	**1,147**
	0.20	100.37	99.86	11	70	230	100.37	99.86	11	70	230	100.37	99.86	11	70	230
	0.40	200.23	199.73	22	139	460	200.23	199.73	22	139	460	200.23	199.73	22	139	460
	0.60	300.09	299.59	32	208	690	300.09	299.59	32	208	690	300.09	299.59	32	208	690
	0.80	399.96	399.46	43	277	920	399.96	399.46	43	277	920	399.96	399.46	43	277	920
1.00	**1.00**	**499.82**	**499.32**	**53**	**347**	**1,150**	**499.82**	**499.32**	**53**	**347**	**1,150**	**499.82**	**499.32**	**53**	**347**	**1,150**

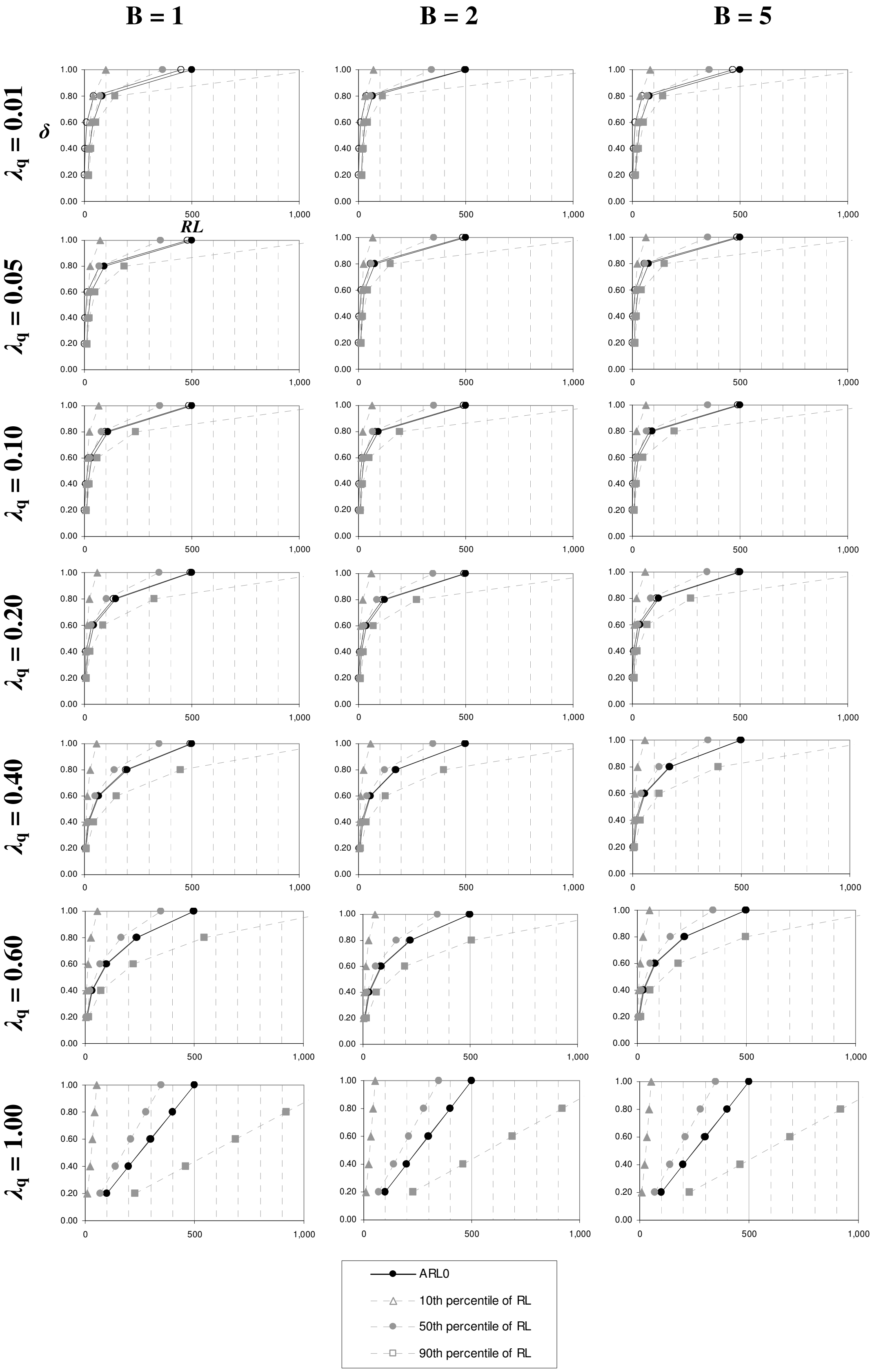

Figure 4.3 – Conditional RL performance of the control chart with known design parameters (w = 1)

Considering Table 4.3 and Figure 4.3, an inspection on the percentiles would show that the RL distribution is right skewed. However, for larger shifts in the mean, skewness of the RL distribution is less.

For $\lambda_q = 1$ (Shewhart control chart), the ARL_0 of the control chart is the highest among all. Since the SDRL values are also higher, the variation in the RL distribution is also higher. For smaller shifts smaller λ_q values, and for larger shifts larger λ_q values give better chart performance. As an example consider the case $\delta = 0.8$, which is the smallest shift considered here and note the change in the performance measures for various λ_q values.

Considering the effect of the boundary B, the performance of $B = 1$ is slightly worse than $B = 2$, and 5 for all cases. For the smallest shift $\delta = 0.8$, $B = 2$ is preferable for $\lambda_q < 0.10$ and $B = 5$ is preferable for other λ_q values. For the largest shift $\delta = 0.2$, the performance of $B = 2$ and 5 are almost identical. The parameter B has no effect for $\lambda_q = 1$ (Shewhart control chart).

4.3. In-Control Case

4.3.1. Conditional ARL performance

The results obtained for the 50th percentile of W for an *in-control* process ($\delta = 1$) are discussed in the following. Since the *in-control* case is under consideration, the ARL values here are actually the ARL_0 values.

Recall the values of $\hat{\mu}_0$ given in Table 4.2. It was mentioned in Section 4.2 that at the 50th percentile of W, underestimation is anticipated. ARL_0 values are provided in Table 4.4 and Figure 4.4. It can be seen that the ARL_0 performance is sensitive to the choice of the smoothing parameter λ_q. For $\lambda_q = 1$ (Shewhart control chart) ARL_0 values are close to the design value of 500. For smaller λ_q values, ARL_0

performance increases, especially for small sample sizes. Considering the case that represents the ideal situation, where n = 10000, the ARL_0 values converge to the value of 500.

Table 4.4 – Conditional ARL_0 performance of the control chart for an *in-control* process (δ = 1) at the 50^{th} percentile of W.

B	*n* \ λ_q	0.01	0.05	0.10	0.20	0.40	0.60	1.00
1	**30**	579.66	556.76	545.34	535.04	524.49	519.31	505.46
	50	546.17	533.35	526.78	520.97	514.50	511.60	503.22
	100	522.41	516.36	513.17	510.56	507.06	505.83	501.52
	200	511.00	508.08	506.49	505.43	503.37	502.96	500.67
	500	504.43	503.27	502.61	502.43	501.21	501.28	500.17
	1,000	502.48	501.84	501.45	501.53	500.57	500.78	500.02
	10,000	500.54	500.41	500.29	500.64	499.92	500.28	499.87
2	**30**	615.84	574.32	557.64	542.88	529.28	521.70	505.46
	50	565.90	543.44	533.97	525.57	517.54	512.93	503.22
	100	531.70	521.38	516.81	512.86	508.83	506.38	501.52
	200	515.64	510.74	508.45	506.62	504.52	503.13	500.67
	500	506.49	504.60	503.61	502.99	502.00	501.23	500.17
	1,000	503.79	502.77	502.16	501.90	501.25	500.66	500.02
	10,000	501.11	500.95	500.72	500.82	500.50	500.09	499.87
5	**30**	592.31	573.74	557.49	542.65	529.90	522.41	505.46
	50	552.94	543.00	533.85	525.25	517.91	513.38	503.22
	100	525.49	521.04	516.71	512.49	509.02	506.64	501.52
	200	512.45	510.45	508.37	506.23	504.63	503.29	500.67
	500	504.98	504.34	503.53	502.58	502.06	501.34	500.17
	1,000	502.77	502.52	502.08	501.49	501.29	500.75	500.02
	10,000	500.57	500.71	500.65	500.40	500.53	500.16	499.87

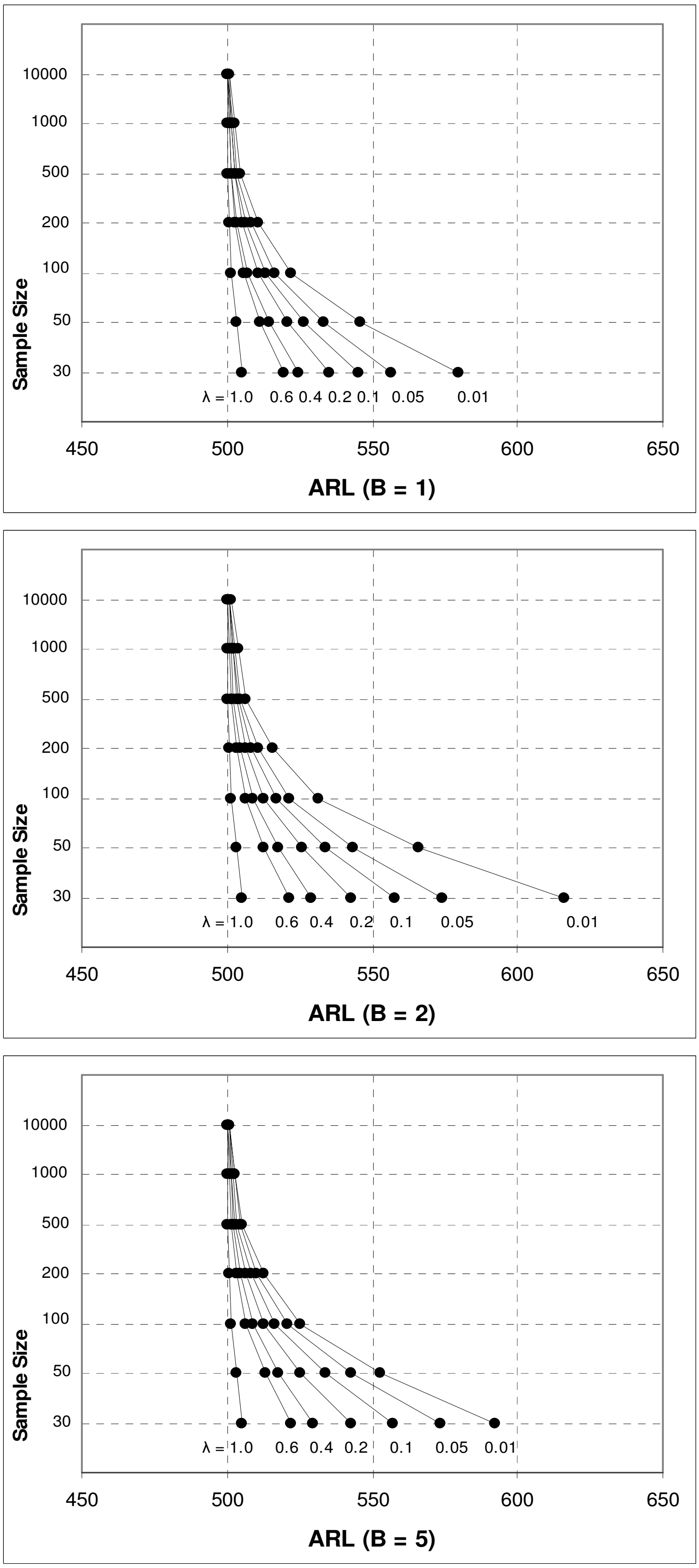

Figure 4.4 – Conditional ARL_0 performance of the control chart for an *in-control* process ($\delta = 1$) at the 50^{th} percentile of W.

Hence, at the 50th percentile of W, the smaller sample sizes chosen in *Phase-I* result in larger ARL_0 values, that is less *false alarms*, which is something favorable.

The choice of the boundary (B) had little effect on the results. For $B = 2$, the ARL_0 values are slightly larger when $\lambda_q \leq 0.20$. For higher λ_q, $B = 5$ gives slightly higher ARL_0 values. ARL_0 performance is not effected from the choice of B when $\lambda_q = 1$ (Shewhart control chart).

In this section, effects of estimation error on the ARL are examined when there is no shift in the process mean, that is the process is *in-control* ($\delta = 1$). Hypothetical estimation error values are obtained through evaluating the percentiles of the sampling distribution of W. Three cases are considered; the 25th percentile case (corresponding to *overestimation* of the mean, $w < 1$), the 50th percentile case, and the 75th percentile case (corresponding to *underestimation* of the mean $w > 1$). Letting the true mean be $\mu_0 = 1$, corresponding $\hat{\mu}_0$ values for the given percentiles of sampling distribution of W were provided in Table 4.2 and Figure 4.2.

The ARL_0 values for the 25th, 50th, and 75th percentile cases obtained from Equation (3.9) are provided in Table 4.5 and represented in Figure 4.5. It is clear that underestimation causes a larger ARL_0, overestimation causes a smaller ARL_0, and the resulting ARL_0 values corresponding to the percentiles of W are clearly right skewed. Moreover, for smaller λ_q is, the effect of estimation error on the ARL_0 is more apparent. For $\lambda_q = 1.00$ (Shewhart case) the effect of estimation error is the smallest.

The effect of the boundary B may be clearly seen. For $B = 2$, and 5, effect of estimation error on ARL_0 performance is larger than for $B = 1$. For $\lambda_q = 0.01$, $B = 2$ resulted in the highest ARL_0 and the most widespread RL distribution. Hence, unlike the known parameter case, choice of parameter B is more influential when estimation error is taken into account.

A larger sample size (n) also decreases the effect of estimation error. The RL distribution gets less spread and ARL_0 values converge to the value of 500 as the sample size is increased.

Table 4.5 – Conditional ARL_0 performance of the control chart for an *in-control* process ($\delta = 1$) at the 25th (overestimation), 50th, and 75th (underestimation) percentiles of *W*.

	λ_q	0.01			0.05			0.10			0.20			0.40			0.60			1.00		
	Sample	Percentile of *W*			Percentile of *W*			Percentile of *W*			Percentile of *W*			Percentile of *W*			Percentile of *W*			Percentile of *W*		
B	**Size (*n*)**	**25th**	**50th**	**75th**	**25th**	**50th**	**75th**	**25th**	**50th**	**75th**	**25th**	**50th**	**75th**	**25th**	**50th**	**75th**	**25th**	**50th**	**75th**	**25th**	**50th**	**75th**
1	**30**	162.88	**579.66**	4,506.27	193.87	**556.76**	2,081.78	224.65	**545.34**	1,528.09	265.22	**535.04**	1,162.70	313.37	**524.49**	910.92	346.90	**519.31**	795.82	447.79	**505.46**	573.42
	50	195.94	**546.17**	2,458.86	231.62	**533.35**	1,440.85	262.61	**526.78**	1,152.01	301.13	**520.97**	942.34	344.52	**514.50**	785.49	373.80	**511.60**	710.00	458.03	**503.22**	554.50
	100	244.36	**522.41**	1,426.43	282.61	**516.36**	1,023.00	311.54	**513.17**	883.07	345.36	**510.56**	772.01	381.28	**507.06**	681.94	404.81	**505.83**	636.56	469.16	**501.52**	536.92
	200	292.63	**511.00**	1,010.52	329.38	**508.08**	816.73	354.52	**506.49**	740.01	382.62	**505.43**	675.48	411.09	**503.37**	620.00	429.41	**502.96**	591.33	477.55	**500.67**	525.34
	500	350.07	**504.43**	763.77	380.89	**503.27**	675.83	400.09	**502.61**	636.89	420.78	**502.43**	602.60	440.66	**501.21**	571.33	453.39	**501.28**	555.00	485.39	**500.17**	515.55
	1,000	386.00	**502.48**	669.71	411.22	**501.84**	616.83	426.20	**501.45**	592.10	442.08	**501.53**	569.91	456.80	**500.57**	548.89	466.30	**500.78**	538.00	489.49	**500.02**	510.81
	10,000	459.50	**500.54**	546.89	469.62	**500.41**	533.86	475.12	**500.29**	527.18	481.04	**500.64**	521.28	485.68	**499.92**	514.72	489.14	**500.28**	511.77	496.53	**499.87**	503.27
2	**30**	129.47	**615.84**	27,389.51	163.52	**574.32**	3,750.28	194.44	**557.64**	2,206.21	237.65	**542.88**	1,443.89	293.15	**529.28**	1,019.29	331.80	**521.70**	850.82	447.79	**505.46**	573.42
	50	158.28	**565.90**	7,441.82	199.24	**543.44**	2,145.62	232.08	**533.97**	1,488.77	274.82	**525.57**	1,100.72	326.30	**517.54**	853.05	360.53	**512.93**	745.79	458.03	**503.22**	554.50
	100	203.08	**531.70**	2,628.09	250.15	**521.38**	1,307.24	282.83	**516.81**	1,039.75	322.15	**512.86**	854.66	366.23	**508.83**	720.82	394.08	**506.38**	657.91	469.16	**501.52**	536.92
	200	251.14	**515.64**	1,450.15	299.50	**510.74**	955.88	329.41	**508.45**	823.96	363.29	**506.62**	723.16	399.21	**504.52**	644.04	421.01	**503.13**	604.74	477.55	**500.67**	525.34
	500	313.05	**506.49**	927.64	356.74	**504.60**	740.43	380.80	**503.61**	678.86	406.58	**502.99**	627.91	432.44	**502.00**	584.99	447.52	**501.23**	562.60	485.39	**500.17**	515.55
	1,000	354.48	**503.79**	760.33	391.84	**502.77**	656.27	411.14	**502.16**	618.63	431.24	**501.90**	586.37	450.78	**501.25**	558.16	461.91	**500.66**	543.05	489.49	**500.02**	510.81
	10,000	445.98	**501.11**	566.95	462.30	**500.95**	544.10	469.72	**500.72**	534.48	477.25	**500.82**	525.95	483.97	**500.50**	517.8	487.52	**500.09**	513.12	496.53	**499.87**	503.27
5	**30**	155.16	**592.31**	10,750.40	163.95	**573.74**	3,711.04	194.58	**557.49**	2,204.64	236.98	**542.65**	1,457.23	290.83	**529.90**	1,039.12	328.68	**522.41**	867.27	447.79	**505.46**	573.42
	50	186.24	**552.94**	4,070.29	199.69	**543.00**	2,129.88	232.22	**533.85**	1,487.42	274.10	**525.25**	1,106.83	324.12	**517.91**	864.55	357.72	**513.38**	756.05	458.03	**503.22**	554.50
	100	232.52	**525.49**	1,850.71	250.58	**521.04**	1,301.03	282.96	**516.71**	1,038.91	321.40	**512.49**	857.10	364.34	**509.02**	726.98	391.76	**506.64**	663.81	469.16	**501.52**	536.92
	200	279.86	**512.45**	1,169.31	299.87	**510.45**	952.79	329.51	**508.37**	823.43	362.54	**506.23**	724.20	397.65	**504.63**	647.66	419.18	**503.29**	608.39	477.55	**500.67**	525.34
	500	338.00	**504.98**	822.81	356.99	**504.34**	738.89	380.86	**503.53**	678.54	405.89	**502.58**	628.19	431.31	**502.06**	586.94	446.25	**501.34**	564.66	485.39	**500.17**	515.55
	1,000	375.43	**502.77**	702.05	391.99	**502.52**	655.24	411.16	**502.08**	618.40	430.60	**501.49**	586.38	449.93	**501.29**	559.45	460.97	**500.75**	544.45	489.49	**500.02**	510.81
	10,000	454.70	**500.57**	553.78	462.21	**500.71**	543.66	469.69	**500.65**	534.36	476.74	**500.40**	525.63	483.68	**500.53**	518.18	487.24	**500.16**	513.57	496.53	**499.87**	503.27

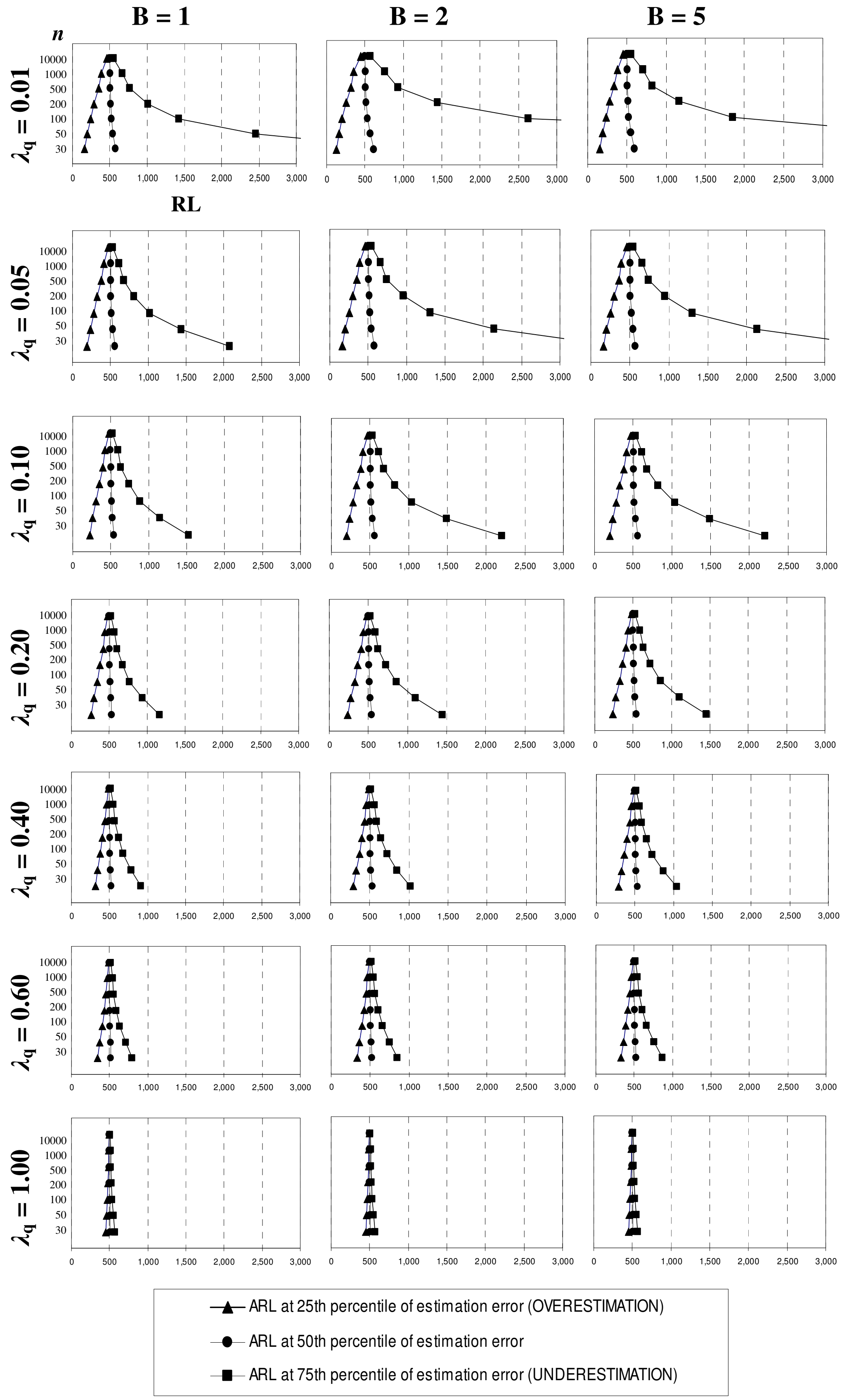

Figure 4.5 – Conditional ARL_0 performance for an *in-control* process ($\delta = 1$) at the 25th (overestimation), 50th, and 75th (underestimation) percentile of *W*.

When W assumes a value at the 75th percentile of its distribution, underestimation of the mean causes an increase in the ARL_0 values. For large λ_q, the amount of increase in the ARL_0 is small, however, for smaller λ_q and smaller n, the amount of increase can become quite large. For $B = 2$, $\lambda_q = 0.01$, and $n = 30$, ARL_0 reaches a high value such as 27,389.51.

Although the underestimation of the mean may have a large influence on the behavior of the chart, having a larger ARL_0 value is something desirable since less *false alarms* would be triggered for an *in-control* process. However, in Section 5.3, it will be presented that for small shifts in the mean such as $\delta = 0.8$, the effect of estimation error may have significant negative effect on the *out-of-control* performance of the control chart especially when λ_q and n is small.

On the other hand, when W assumes a value at the 25th percentile of its distribution, overestimation of the mean causes a decrease in the ARL_0 values. Similar to the case of underestimated mean, the amount of decrease in ARL_0 is larger for smaller λ_q and n. The lowest value of ARL_0 is 129.47 and this is obtained at $B = 2$, $n = 30$, and $\lambda_q = 0.01$.

Unlike the underestimated mean case, the consequences of overestimated mean are undesirable. A lower ARL_0 value implies that the control chart would more frequently give *false alarms*.

4.3.2. Conditional performance of the SDRL and Percentiles of RL

An extension of Table 4.5 and Figure 4.5 is provided in Table 4.8 and Figure 4.8, in which the 10^{th}, 50^{th} and the 90^{th} percentiles of the RL distribution are added to understand the effect of estimation error better. Since the behavior of the RL is similar for B = 2 and B = 5, only the case for the parameter B = 1 is included in the Figure 4.8. In order to keep the table sizes managable, results for λ_q = 0.01, 0.10, 0.20, and 1.00 are included in Table 4.8. Furthermore, Figure 4.9 represents the ARL_0 and SDRL values at the 10^{th}, 50^{th}, and 90^{th} percentiles of the RL distribution.

Similar to the results in Section 4.3.1, it can be observed from Table 4.8 and Figure 4.9 that the ARL_0 and SDRL values are close to each other in each case. Larger ARL_0 values obtained as a result of underestimation implies larger standard deviation and hence more spread RL distribution. Smaller ARL_0 values obtained as a result of overestimation imply a smaller standard deviation and hence a narrower RL distribution. Note that the RL distribution is right skewed in every case.

Let *W* assumes a value at the 50^{th} percentile of its distribution. In Table 4.6 and Figure 4.6 ARL_0 values are represented together with the 10^{th}, 50^{th}, and 90^{th} percentiles of these RL distributions. Skewness of the distribution may be observed by a comparison of the median (50^{th} percentile) with the ARL_0 value. As common for various cases in the literature, the RL distribution considered here is also quite right skewed and has a large standard deviation. Hence, ARL alone might not be a reliable measure of performance for the control charts. The structure of the percentiles of the RL distribution does not change much for different n, λ_q, and B values. The 10^{th} and 90^{th} percentiles are slightly wider for B = 2. Note that, all 50^{th} percentile values are less than 500 and roughly between 350 and 400, implying that more than half of the RL distribution would be less than 500 and yield shorter *in-control* ARL (ARL_0) values for a process designed for an ARL_0 of 500.

Table 4.6 – Conditional ARL_0 performance of the control chart and percentiles of RL distribution for an *in-control* process (δ = 1) at the 50th percentile of W.

	λ_q	0.01				0.05				0.10				0.20				0.40				0.60				1.00			
	S.Size		Percentiles of RL				Percentiles of RL				Percentiles of RL				Percentiles of RL				Percentiles of RL				Percentiles of RL				Percentiles of RL		
B	**(*n*)**	**ARL**	**10th**	**50th**	**90th**	**ARL**	**10th**	**50th**	**90th**	**ARL**	**10th**	**50th**	**90th**	**ARL**	**10th**	**50th**	**90th**	**ARL**	**10th**	**50th**	**90th**	**ARL**	**10th**	**50th**	**90th**	**ARL**	**10th**	**50th**	**90th**
1	**30**	**579.66**	106	**418**	1,269	**556.76**	78	**392**	1,255	**545.34**	70	**382**	1,238	**535.04**	64	**373**	1,221	**524.49**	60	**365**	1,201	**519.31**	58	**361**	1,191	**505.46**	54	**351**	1,163
	50	**546.17**	103	**395**	1,192	**533.35**	75	**376**	1,201	**526.78**	68	**369**	1,195	**520.97**	62	**364**	1,189	**514.50**	59	**358**	1,178	**511.60**	57	**356**	1,173	**503.22**	53	**349**	1,158
	100	**522.41**	100	**378**	1,137	**516.36**	73	**364**	1,162	**513.17**	66	**360**	1,164	**510.56**	61	**356**	1,165	**507.06**	58	**353**	1,161	**505.83**	56	**352**	1,160	**501.52**	53	**348**	1,154
	200	**511.00**	99	**370**	1,111	**508.08**	72	**359**	1,142	**506.49**	65	**355**	1,149	**505.43**	61	**353**	1,153	**503.37**	58	**350**	1,153	**502.96**	56	**350**	1,154	**500.67**	53	**347**	1,152
	500	**504.43**	98	**366**	1,096	**503.27**	72	**355**	1,131	**502.61**	65	**353**	1,140	**502.43**	60	**351**	1,146	**501.21**	57	**349**	1,148	**501.28**	56	**349**	1,150	**500.17**	53	**347**	1,151
	1,000	**502.48**	98	**364**	1,091	**501.84**	72	**354**	1,128	**501.45**	65	**352**	1,137	**501.53**	60	**350**	1,144	**500.57**	57	**349**	1,146	**500.78**	56	**348**	1,149	**500.02**	53	**347**	1,151
	10,000	**500.54**	98	**363**	1,087	**500.41**	72	**353**	1,125	**500.29**	65	**351**	1,134	**500.64**	60	**350**	1,142	**499.92**	57	**348**	1,145	**500.28**	56	**348**	1,147	**499.87**	53	**347**	1,150
2	**30**	**615.84**	75	**417**	1,428	**574.32**	72	**402**	1,306	**557.64**	68	**390**	1,271	**542.88**	63	**378**	1,241	**529.28**	60	**368**	1,213	**521.70**	58	**363**	1,197	**505.46**	54	**351**	1,163
	50	**565.90**	72	**384**	1,308	**543.44**	69	**380**	1,235	**533.97**	65	**373**	1,216	**525.57**	62	**366**	1,201	**517.54**	58	**360**	1,186	**512.93**	57	**356**	1,177	**503.22**	53	**349**	1,158
	100	**531.70**	70	**361**	1,225	**521.38**	67	**365**	1,183	**516.81**	64	**361**	1,177	**512.86**	60	**358**	1,172	**508.83**	58	**354**	1,166	**506.38**	56	**352**	1,162	**501.52**	53	**348**	1,154
	200	**515.64**	69	**351**	1,186	**510.74**	66	**358**	1,159	**508.45**	63	**356**	1,157	**506.62**	60	**353**	1,158	**504.52**	57	**351**	1,156	**503.13**	56	**350**	1,154	**500.67**	53	**347**	1,152
	500	**506.49**	68	**345**	1,164	**504.60**	65	**354**	1,145	**503.61**	62	**352**	1,146	**502.99**	59	**351**	1,149	**502.00**	57	**349**	1,150	**501.23**	56	**348**	1,150	**500.17**	53	**347**	1,151
	1,000	**503.79**	68	**343**	1,158	**502.77**	65	**352**	1,140	**502.16**	62	**351**	1,143	**501.90**	59	**350**	1,147	**501.25**	57	**349**	1,148	**500.66**	56	**348**	1,149	**500.02**	53	**347**	1,151
	10,000	**501.11**	68	**341**	1,151	**500.95**	65	**351**	1,136	**500.72**	62	**350**	1,140	**500.82**	59	**349**	1,144	**500.50**	57	**348**	1,147	**500.09**	55	**348**	1,147	**499.87**	53	**347**	1,150
5	**30**	**592.31**	91	**418**	1,327	**573.74**	71	**401**	1,306	**557.49**	67	**389**	1,271	**542.65**	63	**378**	1,241	**529.90**	60	**369**	1,215	**522.41**	58	**363**	1,199	**505.46**	54	**351**	1,163
	50	**552.94**	87	**391**	1,235	**543.00**	68	**380**	1,235	**533.85**	65	**373**	1,217	**525.25**	61	**366**	1,201	**517.91**	58	**360**	1,187	**513.38**	57	**357**	1,178	**503.22**	53	**349**	1,158
	100	**525.49**	85	**373**	1,170	**521.04**	66	**365**	1,184	**516.71**	63	**361**	1,177	**512.49**	60	**357**	1,171	**509.02**	57	**354**	1,167	**506.64**	56	**352**	1,163	**501.52**	53	**348**	1,154
	200	**512.45**	84	**364**	1,140	**510.45**	65	**357**	1,159	**508.37**	62	**355**	1,158	**506.23**	59	**353**	1,157	**504.63**	57	**351**	1,156	**503.29**	56	**350**	1,155	**500.67**	53	**347**	1,152
	500	**504.98**	83	**359**	1,122	**504.34**	64	**353**	1,145	**503.53**	62	**352**	1,147	**502.58**	59	**350**	1,148	**502.06**	57	**349**	1,151	**501.34**	56	**348**	1,150	**500.17**	53	**347**	1,151
	1,000	**502.77**	83	**357**	1,117	**502.52**	64	**352**	1,141	**502.08**	62	**351**	1,143	**501.49**	59	**350**	1,146	**501.29**	57	**349**	1,149	**500.75**	55	**348**	1,149	**500.02**	53	**347**	1,151
	10,000	**500.57**	83	**356**	1,112	**500.71**	64	**351**	1,137	**500.65**	62	**350**	1,140	**500.40**	59	**349**	1,143	**500.53**	57	**348**	1,147	**500.16**	55	**348**	1,148	**499.87**	53	**347**	1,150

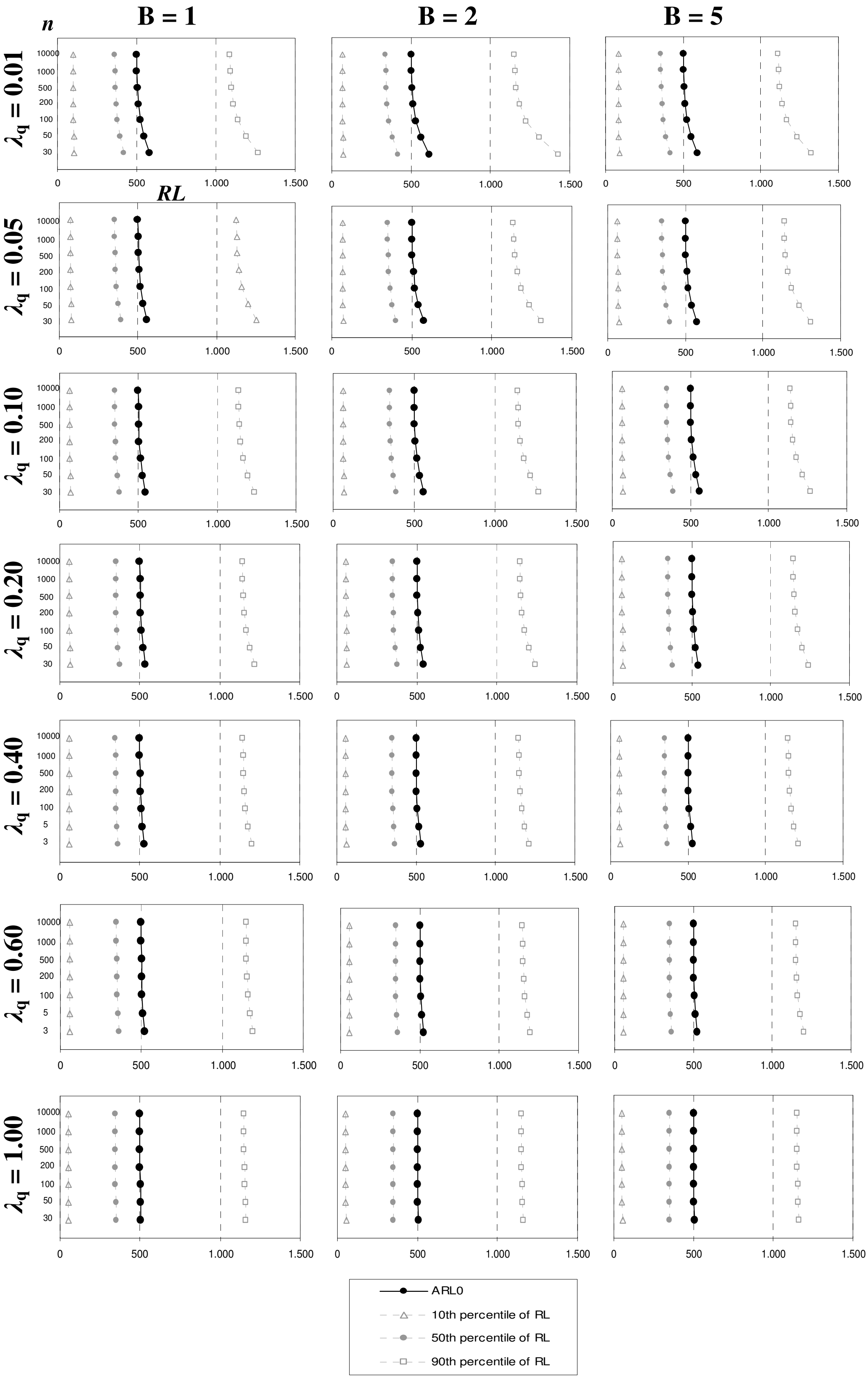

Figure 4.6 – Conditional ARL_0 performance and percentiles of RL distribution for an *in-control* process ($\delta = 1$) at the 50^{th} percentile of W.

For the value of W assumed at the 50th percentile of its distribution, Table 4.7 and Figure 4.7 provides ARL_0 values are represented together with SDRL values. Generally, SDRL values are quite close to the ARL_0 values. For larger λ_q values, the SDRL values are closer to ARL_0 values. As λ_q decreases, the difference between ARL_0 and SDRL values increase slightly since the SDRL values get smaller. However, the SDRL values are still close to the ARL_0 values. For the λ_q = 1.00 case (Shewhart control chart), ARL_0 and SDRL values are almost equal even for the small sample sizes.

The SDRL values show that the distribution of RL becomes more spread for larger ARL values and narrower for smaller ARL values. Hence, the ARL measure becomes less representative in defining the behavior of the RL distribution.

Table 4.7 – Conditional ARL_0 and SDRL performance for an *in-control* process (δ = 1) at the 50th percentile of W.

	λ_q / S.	0.01		0.05		0.10		0.20		0.40		0.60		1.00	
B	**Size (n)**	**ARL**	**SDRL**	**ARL**	**SDRL**	**ARL**	**SDRL**	**ARL**	**SDRL**	**ARL**	**SDRL**	**ARL**	**SDRL**	**ARL**	**SDRL**
1	**30**	**579.66**	529.26	**556.76**	535.78	**545.34**	531.88	**535.04**	526.64	**524.49**	519.50	**519.31**	515.85	**505.46**	504.96
	50	**546.17**	495.78	**533.35**	512.35	**526.78**	513.31	**520.97**	512.56	**514.50**	509.50	**511.60**	508.14	**503.22**	502.72
	100	**522.41**	472.04	**516.36**	495.34	**513.17**	499.69	**510.56**	502.15	**507.06**	502.06	**505.83**	502.37	**501.52**	501.02
	200	**511.00**	460.64	**508.08**	487.05	**506.49**	493.01	**505.43**	497.01	**503.37**	498.37	**502.96**	499.50	**500.67**	500.17
	500	**504.43**	454.08	**503.27**	482.24	**502.61**	489.12	**502.43**	494.01	**501.21**	496.21	**501.28**	497.82	**500.17**	499.67
	1,000	**502.48**	452.13	**501.84**	480.81	**501.45**	487.96	**501.53**	493.12	**500.57**	495.56	**500.78**	497.31	**500.02**	499.52
	10,000	**500.54**	450.19	**500.41**	479.38	**500.29**	486.80	**500.64**	492.22	**499.92**	494.92	**500.28**	496.81	**499.87**	499.37
2	**30**	**615.84**	622.65	**574.32**	561.91	**557.64**	547.71	**542.88**	536.06	**529.28**	524.92	**521.70**	518.59	**505.46**	504.96
	50	**565.90**	568.78	**543.44**	530.64	**533.97**	523.90	**525.57**	518.71	**517.54**	513.17	**512.93**	509.81	**503.22**	502.72
	100	**531.70**	531.86	**521.38**	508.29	**516.81**	506.64	**512.86**	505.97	**508.83**	504.45	**506.38**	503.26	**501.52**	501.02
	200	**515.64**	514.50	**510.74**	497.52	**508.45**	498.24	**506.62**	499.72	**504.52**	500.14	**503.13**	500.01	**500.67**	500.17
	500	**506.49**	504.62	**504.60**	491.30	**503.61**	493.36	**502.99**	496.07	**502.00**	497.62	**501.23**	498.11	**500.17**	499.67
	1,000	**503.79**	501.71	**502.77**	489.45	**502.16**	491.91	**501.90**	494.98	**501.25**	496.86	**500.66**	497.54	**500.02**	499.52
	10,000	**501.11**	498.81	**500.95**	487.61	**500.72**	490.46	**500.82**	493.90	**500.50**	496.11	**500.09**	496.97	**499.87**	499.37
5	**30**	**592.31**	564.15	**573.74**	562.11	**557.49**	547.96	**542.65**	535.92	**529.90**	525.70	**522.41**	519.40	**505.46**	504.96
	50	**552.94**	523.37	**543.00**	530.97	**533.85**	524.18	**525.25**	518.48	**517.91**	513.70	**513.38**	510.36	**503.22**	502.72
	100	**525.49**	494.94	**521.04**	508.72	**516.71**	506.93	**512.49**	505.68	**509.02**	504.79	**506.64**	503.61	**501.52**	501.02
	200	**512.45**	481.44	**510.45**	497.98	**508.37**	498.54	**506.23**	499.40	**504.63**	500.39	**503.29**	500.27	**500.67**	500.17
	500	**504.98**	473.71	**504.34**	491.78	**503.53**	493.67	**502.58**	495.74	**502.06**	497.82	**501.34**	498.31	**500.17**	499.67
	1,000	**502.77**	471.42	**502.52**	489.94	**502.08**	492.22	**501.49**	494.65	**501.29**	497.05	**500.75**	497.72	**500.02**	499.52
	10,000	**500.57**	469.15	**500.71**	488.11	**500.65**	490.77	**500.40**	493.56	**500.53**	496.29	**500.16**	497.14	**499.87**	499.37

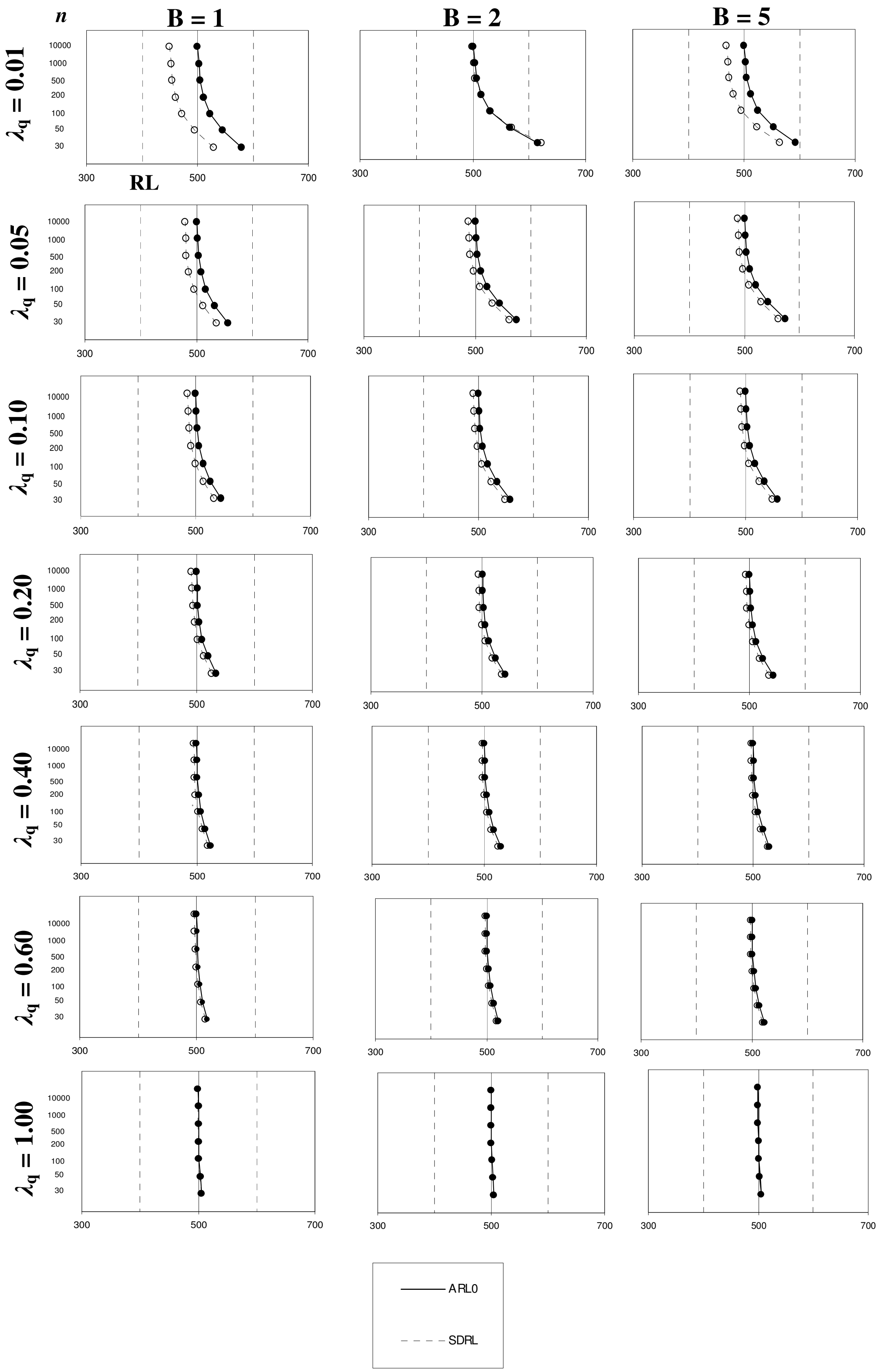

Figure 4.7 – Conditional ARL_0 and SDRL performance for an *in-control* process ($\delta = 1$) at the 50th percentile of *W*.

When W assumes a value at the 75^{th} percentile of its distribution, it was mentioned that underestimation of the mean resulted in larger ARL_0 values. Percentiles of the RL also increase in this case. In Table 4.8, the increase in 10^{th} percentile is less than the 50^{th} and 90^{th} percentiles. The 50^{th} percentile results increase more and take values close to the ARL_0 values. The SDRL values are close to the ARL values that the coefficient of correlation does not change too much. The ratios between the percentiles are very similar. Hence, the shape of the RL distribution does not change too much but only its scale is altered by estimation error or choice of design parameters.

Considering Table 4.8, when the boundary $B \in \{2, 5\}$, note also that the percentile values are larger, especially for $B = 2$ and $\lambda_q = 0.01$.

Recall that, when W assumes a value at the 25^{th} percentile of its distribution, overestimation of the mean causes a decrease in the ARL_0 values. Like the ARL_0 values, the percentiles of the RL also decrease when the mean is overestimated. When $\lambda_q \leq 0.10$, even the 90^{th} percentile of the RL distribution is under 500 for small sample sizes. This would imply frequent *false out-of control* signals. Unlike the underestimated mean case, the ratios between the percentiles are closer to 1, specifically for small λ_q, which implies that the RL distribution is less skewed.

The boundary B has a little influence on the performance measures, while B = 2 and 5 cases resulting in slightly smaller values.

Table 4.8 – Conditional ARL_0, SDRL, and the 10^{th}, 50^{th}, 90^{th} percentiles of RL for an *in-control* process (δ = 1) at 25^{th} (overestimation), 50^{th}, 75^{th} (underestimation)percentile of *W*.

75^{th} percentile of *W* (Underestimated mean)																					
	λ_q	**0.01**					**0.10**					**0.20**					**1.00**				
	Sample			**Perc. of RL**					**Perc. of RL**					**Perc. of RL**					**Perc. of RL**		
B	**Size (*n*)**	**ARL**	**SDRL**	**10^{th}**	**50^{th}**	**90^{th}**	**ARL**	**SDRL**	**10^{th}**	**50^{th}**	**90^{th}**	**ARL**	**SDRL**	**10^{th}**	**50^{th}**	**90^{th}**	**ARL**	**SDRL**	**10^{th}**	**50^{th}**	**90^{th}**
1	**30**	4,506.27	4,459.20	517	3,138	10,315	2,081.78	2,061.81	237	1,449	4,767	1,162.70	1,154.52	130	808	2,667	573.42	572.92	61	398	1,320
	50	2,458.86	2,410.35	303	1,719	5,599	1,440.85	1,420.56	170	1,005	3,291	942.34	934.10	107	656	2,159	554.50	554.00	59	385	1,276
	100	1,426.43	1,376.85	195	1,004	3,220	1,023.00	1,002.43	126	715	2,329	772.01	763.71	89	538	1,767	536.92	536.42	57	372	1,236
	200	1,010.52	960.44	152	716	2,262	816.73	795.99	105	573	1,854	675.48	667.14	79	471	1,544	525.34	524.84	56	364	1,209
	500	763.77	713.44	126	545	1,693	675.83	654.97	90	475	1,529	602.60	594.23	71	420	1,377	515.55	515.05	55	358	1,186
	1,000	669.71	619.32	116	480	1,476	616.83	595.91	84	434	1,393	569.91	561.53	68	398	1,301	510.81	510.31	54	354	1,176
	10,000	546.89	496.50	103	395	1,193	533.86	512.86	75	377	1,202	521.28	512.87	62	364	1,189	503.27	502.77	53	349	1,158
2	**30**	27,389.51	28,103.89	2,224	18,750	64,001	3,750.28	3,760.88	385	2,596	8,649	1,443.89	1,438.75	157	1,002	3,318	573.42	572.92	61	398	1,320
	50	7,441.82	7,741.00	497	5,052	17,528	2,145.62	2,147.14	224	1,486	4,943	1,100.72	1,095.05	121	765	2,527	554.50	554.00	59	385	1,276
	100	2,628.09	2,753.63	174	1,770	6,217	1,307.24	1,302.40	142	907	3,004	854.66	848.53	96	594	1,960	536.92	536.42	57	372	1,236
	200	1,450.15	1,513.72	115	974	3,424	955.88	947.75	108	665	2,190	723.16	716.76	82	503	1,657	525.34	524.84	56	364	1,209
	500	927.64	957.48	90	624	2,176	740.43	730.00	87	516	1,691	627.91	621.31	72	437	1,437	515.55	515.05	55	358	1,186
	1,000	760.33	778.12	82	512	1,775	656.27	644.86	79	458	1,496	586.37	579.67	68	409	1,341	510.81	510.31	54	354	1,176
	10,000	566.95	569.92	72	384	1,310	544.10	531.31	69	381	1,236	525.95	519.09	62	367	1,202	503.27	502.77	53	349	1,158
5	**30**	10,750.40	10,912.15	982	7,398	24,965	3,711.04	3,723.27	380	2,568	8,561	1,457.23	1,452.37	158	1,012	3,349	573.42	572.92	61	398	1,320
	50	4,070.29	4,132.80	369	2,798	9,454	2,129.88	2,132.70	221	1,475	4,908	1,106.83	1,101.35	121	769	2,541	554.50	554.00	59	385	1,276
	100	1,850.71	1,862.64	186	1,276	4,278	1,301.03	1,297.27	140	903	2,991	857.10	851.13	96	596	1,966	536.92	536.42	57	372	1,236
	200	1,169.31	1,161.00	135	811	2,682	952.79	945.62	106	662	2,185	724.20	717.93	82	504	1,659	525.34	524.84	56	364	1,209
	500	822.81	802.84	109	575	1,869	738.89	729.32	86	515	1,689	628.19	621.69	72	437	1,438	515.55	515.05	55	358	1,186
	1,000	702.05	677.82	99	493	1,585	655.24	644.66	78	457	1,495	586.38	579.77	68	408	1,342	510.81	510.31	54	354	1,176
	10,000	553.78	524.24	87	392	1,237	543.66	531.64	68	380	1,236	525.63	518.86	61	366	1,201	503.27	502.77	53	349	1,158

Table 4.8 – (*continued*)

		50th percentile of *W*																			
	λ_q	0.01					0.10					0.20					1.00				
	Sample			Percentile of RL					Percentile of RL					Percentile of RL					Percentile of RL		
B	**Size (*n*)**	**ARL**	**SDRL**	**10th**	**50th**	**90th**	**ARL**	**SDRL**	**10th**	**50th**	**90th**	**ARL**	**SDRL**	**10th**	**50th**	**90th**	**ARL**	**SDRL**	**10th**	**50th**	**90th**
1	**30**	579.66	529.26	106	418	1,269	545.34	531.88	70	382	1,238	535.04	526.64	64	373	1,221	505.46	504.96	54	351	1,163
	50	546.17	495.78	103	395	1,192	526.78	513.31	68	369	1,195	520.97	512.56	62	364	1,189	503.22	502.72	53	349	1,158
	100	522.41	472.04	100	378	1,137	513.17	499.69	66	360	1,164	510.56	502.15	61	356	1,165	501.52	501.02	53	348	1,154
	200	511.00	460.64	99	370	1,111	506.49	493.01	65	355	1,149	505.43	497.01	61	353	1,153	500.67	500.17	53	347	1,152
	500	504.43	454.08	98	366	1,096	502.61	489.12	65	353	1,140	502.43	494.01	60	351	1,146	500.17	499.67	53	347	1,151
	1,000	502.48	452.13	98	364	1,091	501.45	487.96	65	352	1,137	501.53	493.12	60	350	1,144	500.02	499.52	53	347	1,151
	10,000	500.54	450.19	98	363	1,087	500.29	486.80	65	351	1,134	500.64	492.22	60	350	1,142	499.87	499.37	53	347	1,150
2	**30**	615.84	622.65	75	417	1,428	557.64	547.71	68	390	1,271	542.88	536.06	63	378	1,241	505.46	504.96	54	351	1,163
	50	565.90	568.78	72	384	1,308	533.97	523.90	65	373	1,216	525.57	518.71	62	366	1,201	503.22	502.72	53	349	1,158
	100	531.70	531.86	70	361	1,225	516.81	506.64	64	361	1,177	512.86	505.97	60	358	1,172	501.52	501.02	53	348	1,154
	200	515.64	514.50	69	351	1,186	508.45	498.24	63	356	1,157	506.62	499.72	60	353	1,158	500.67	500.17	53	347	1,152
	500	506.49	504.62	68	345	1,164	503.61	493.36	62	352	1,146	502.99	496.07	59	351	1,149	500.17	499.67	53	347	1,151
	1,000	503.79	501.71	68	343	1,158	502.16	491.91	62	351	1,143	501.90	494.98	59	350	1,147	500.02	499.52	53	347	1,151
	10,000	501.11	498.81	68	341	1,151	500.72	490.46	62	350	1,140	500.82	493.90	59	349	1,144	499.87	499.37	53	347	1,150
5	**30**	592.31	564.15	91	418	1,327	557.49	547.96	67	389	1,271	542.65	535.92	63	378	1,241	505.46	504.96	54	351	1,163
	50	552.94	523.37	87	391	1,235	533.85	524.18	65	373	1,217	525.25	518.48	61	366	1,201	503.22	502.72	53	349	1,158
	100	525.49	494.94	85	373	1,170	516.71	506.93	63	361	1,177	512.49	505.68	60	357	1,171	501.52	501.02	53	348	1,154
	200	512.45	481.44	84	364	1,140	508.37	498.54	62	355	1,158	506.23	499.40	59	353	1,157	500.67	500.17	53	347	1,152
	500	504.98	473.71	83	359	1,122	503.53	493.67	62	352	1,147	502.58	495.74	59	350	1,148	500.17	499.67	53	347	1,151
	1,000	502.77	471.42	83	357	1,117	502.08	492.22	62	351	1,143	501.49	494.65	59	350	1,146	500.02	499.52	53	347	1,151
	10,000	500.57	469.15	83	356	1,112	500.65	490.77	62	350	1,140	500.40	493.56	59	349	1,143	499.87	499.37	53	347	1,150

Table 4.8 – (*continued*)

	λ_q	0.01					0.10					0.20					1.00				
25th percentile of *W* (Overestimated mean)																					
	Sample			**Percentile of RL**					**Percentile of RL**					**Percentile of RL**					**Percentile of RL**		
B	**Size (*n*)**	**ARL**	**SDRL**	**10th**	**50th**	**90th**	**ARL**	**SDRL**	**10th**	**50th**	**90th**	**ARL**	**SDRL**	**10th**	**50th**	**90th**	**ARL**	**SDRL**	**10th**	**50th**	**90th**
1	**30**	162.88	116.25	56	129	314	193.87	172.73	39	141	419	265.22	256.66	36	186	600	447.79	447.29	48	311	1,030
	50	195.94	148.16	61	152	389	231.62	210.43	43	167	506	301.13	292.60	39	211	682	458.03	457.53	49	318	1,054
	100	244.36	195.53	68	186	499	282.61	261.41	49	203	623	345.36	336.86	44	242	784	469.16	468.66	50	325	1,080
	200	292.63	243.18	74	219	609	329.38	308.20	54	235	731	382.62	374.14	48	268	870	477.55	477.05	51	331	1,099
	500	350.07	300.18	81	259	741	380.89	359.74	59	271	849	420.78	412.32	52	294	958	485.39	484.89	52	337	1,117
	1,000	386.00	335.93	85	284	823	411.22	390.10	62	292	919	442.08	433.63	54	309	1,007	489.49	488.99	52	339	1,126
	10,000	459.50	409.21	93	335	992	469.62	448.55	68	332	1,054	481.04	472.62	58	336	1,097	496.53	496.03	53	344	1,143
2	**30**	129.47	99.78	39	100	259	163.52	145.62	33	119	353	237.65	229.94	32	167	537	447.79	447.29	48	311	1,030
	50	158.28	129.69	43	119	327	199.24	181.76	37	144	436	274.82	267.23	36	193	623	458.03	457.53	49	318	1,054
	100	203.08	177.11	47	148	434	250.15	233.35	42	179	554	322.15	314.72	41	226	732	469.16	468.66	50	325	1,080
	200	251.14	228.61	51	179	549	299.50	283.39	46	213	669	363.29	355.98	45	254	827	477.55	477.05	51	331	1,099
	500	313.05	295.40	56	219	698	356.74	341.44	51	252	801	406.58	399.40	49	284	927	485.39	484.89	52	337	1,117
	1,000	354.48	340.19	59	245	798	391.84	377.03	55	276	883	431.24	424.13	52	301	984	489.49	488.99	52	339	1,126
	10,000	445.98	439.19	65	305	1,019	462.30	448.44	61	325	1,046	477.25	470.26	57	333	1,090	496.53	496.03	53	344	1,143
5	**30**	155.16	116.23	47	122	306	163.95	146.49	33	119	355	236.98	229.30	32	167	536	447.79	447.29	48	311	1,030
	50	186.24	146.92	52	143	377	199.69	182.70	36	144	438	274.10	266.55	36	192	621	458.03	457.53	49	318	1,054
	100	232.52	193.43	57	175	484	250.58	234.32	41	179	556	321.40	314.01	40	225	730	469.16	468.66	50	325	1,080
	200	279.86	241.61	63	207	594	299.87	284.35	46	213	670	362.54	355.29	45	254	825	477.55	477.05	51	331	1,099
	500	338.00	301.25	68	246	730	356.99	342.33	51	252	803	405.89	398.77	49	284	925	485.39	484.89	52	337	1,117
	1,000	375.43	339.80	72	271	818	391.99	377.85	54	276	884	430.60	423.56	52	301	982	489.49	488.99	52	339	1,126
	10,000	454.70	421.69	79	325	1,004	462.21	449.08	61	324	1,047	476.74	469.83	56	333	1,089	496.53	496.03	53	344	1,143

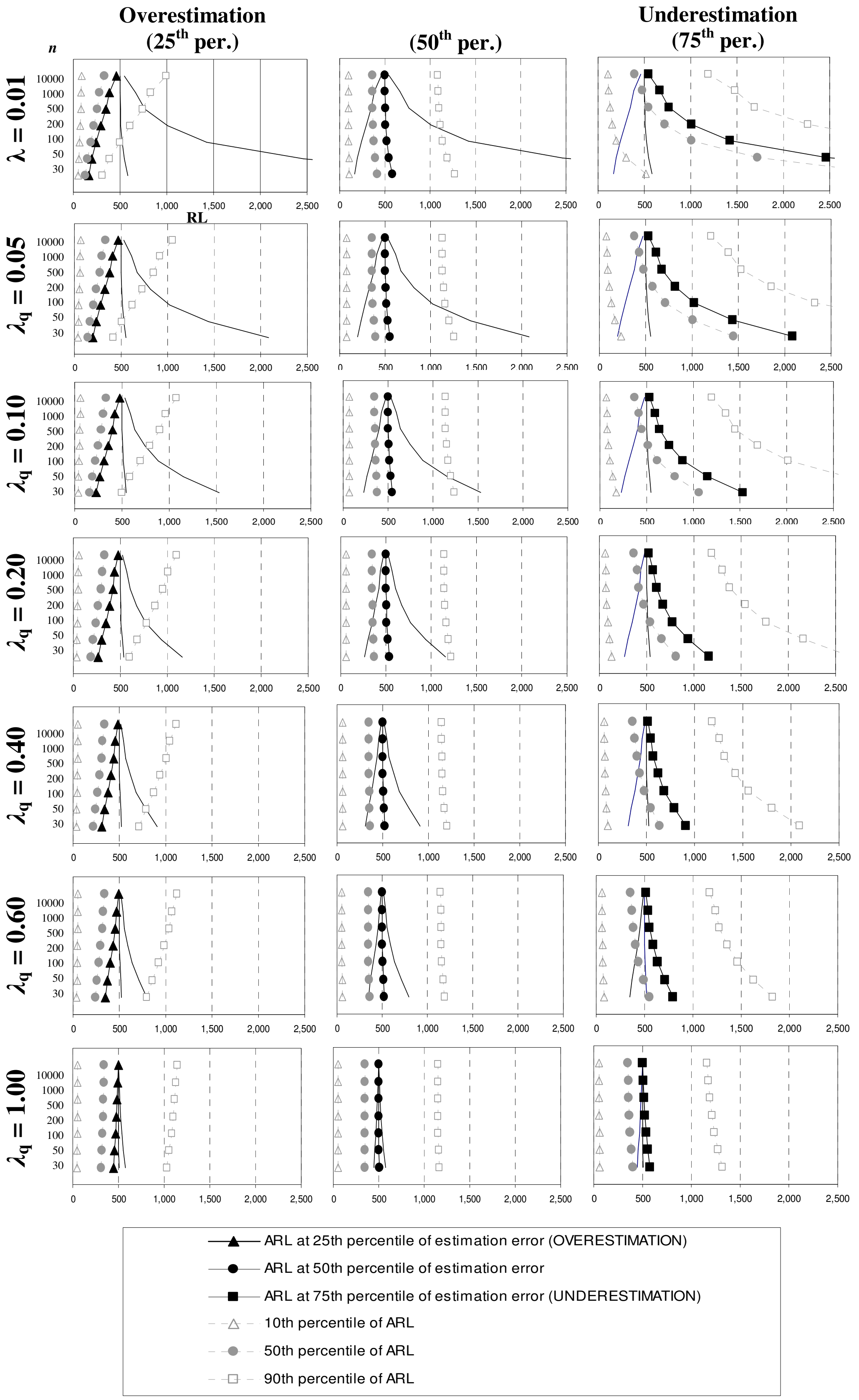

Figure 4.8 – Conditional ARL, and the 10th, 50th, 90th percentiles of RL for an in-control process ($\delta = 1$) and $B = 1$, at 25th (overestimation), 50th, 75th (underestimation) percentile of W.

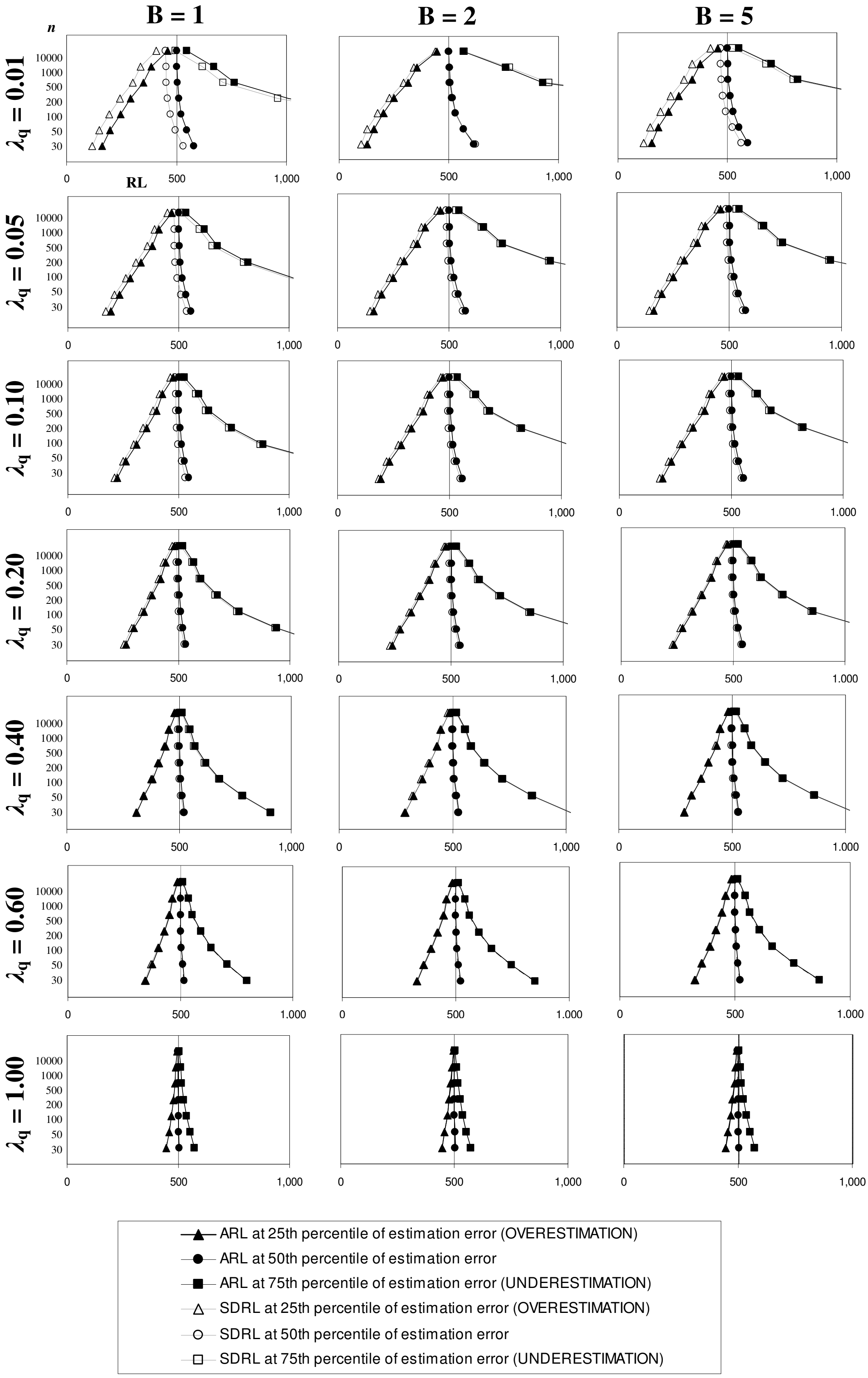

Figure 4.9 – Conditional ARL_0 and SDRL performance for an *in*-control process ($\delta = 1$) at 25^{th} (overestimation), 50^{th}, 75^{th} (underestimation) percentile of *W*.

4.4. Out-of-Control Case

In this Section, effects of parameter estimation errors are examined when the process goes *out-of-control*. Recall that shift magnitudes ($\delta = \frac{\mu}{\mu_0}$) represent the decrease in the mean of the TBE observations in terms of the process standard deviation.

4.4.1. Conditional ARL Performance

It was mentioned in Section 4.1 that the case of $\delta = 1$ corresponds to an *in-control* process, where the corresponding ARL is actually the ARL_0. On the other hand, the cases with $\delta \leq 0.8$ correspond to *out-of-control* processes, where the corresponding ARL is actually the ARL_1. In Table 4.9, ARL_0 and ARL_1 values for 25^{th}, 50^{th} and 75^{th} percentile of the sampling distribution are provided, where ARL_0 values are **bold**. The results for $B = 1$ are represented in Figure 4.10.

In Table 4.9, the cells with the most favorable cases are the ones with the smallest ARL_1 value, and the largest ARL_0 value. For each mean shift and estimation error, these combinations are highlighted with grey ( ). Note that, for the smallest shift magnitude $\delta = 0.8$, the most favorable ARL_1 values are obtained generally at $\lambda_q = 0.01$. For $\delta = 0.6$, best results are obtained for $\lambda_q = 0.05$ and 0.10. For $\delta = 0.4$, the results are close to each other for $\lambda_q = 0.05$, 0.10, and 0.20. However, the favorable ARL_1 values for the largest shift $\delta = 0.2$ are obtained when $\lambda_q = 0.20$, and 0.40. These results imply that, when small shifts are important, smaller λ_q values are required and when larger shifts are concerned, larger λ_q values become necessary.

In Table 4.9, the cells with the least favorable cases (the largest ARL_1 values and the smallest ARL_0 values) for each mean shift and estimation error combination are filled with grey diagonals (\\\ ). Note that $\lambda_q = 1$ (Shewhart control chart) resulted in the worst performance in all cases except the *in-control* ($\delta = 1$)

performance when the mean is overestimated. Even though the Shewhart control chart is the most robust one against estimation errors, its performance for alerting a decrease in a TBE process is quite slow. The small λ_q values (say $\lambda_q \leq 0.40$), whose *in-control* performances show great variability in case of estimation errors, give *out-of-control* alarms quite rapidly, even if the mean is underestimated. The effect of estimation errors vanishes as the amount of shift increases (i.e. δ decreases).

The ARL_1 performance of B = 2 is better in general and the performance of B = 1, and 5 are close to each other. The results are apparently better for B = 2 when δ = 0.6 or 0.8. For δ = 0.2 and 0.4, the effect of the parameter B becomes very small. Hence, the choice of boundary is more important when smaller shifts are of concern.

When the mean is overestimated, the ARL_1 performance of the chart is robust to estimation errors with respeoct to sample sizes. At the 25^{th} percentile of W, smaller sample sizes result even in smaller ARL_1 values. However, when the underestimation of the mean is being considered, small sample sizes reduce the ARL_1 performance of the control chart, especially for $\lambda_q = 0.01$.

Table 4.9 – Conditional ARL_0 and ARL_1 peformances at the 25th (overestimation), 50th, and 75th (underestimation) percentile of W, for B = 1.

***B* = 1**																						
	λ_q	**0.01**			**0.05**			**0.10**			**0.20**			**0.40**			**0.60**			**1.00**		
	Per. of the sampl. dist.	**25th**	**50th**	**75th**	**25th**	**50th**	**75th**	**25th**	**50th**	**75th**	**25th**	**50th**	**75th**	**25th**	**50th**	**75th**	**25th**	**50th**	**75th**	**25th**	**50th**	**75th**
n	***δ***	**ov.es.**		**und.es.**	**ov.es.**		**und.es.**	**ov.es.**		**und.es.**	**ov.es.**		**und.es.**	**ov.es.**		**und.es.**	**ov.es.**		**und.es.**	**ov.es.**		**und.es.**
30	**0.20**	17.52	18.06	18.75	11.11	11.52	12.06	8.95	9.35	9.87	7.37	7.82	8.43	6.82	7.55	8.58	8.08	9.39	11.23	89	101	115
	0.40	22.87	24.81	27.56	15.44	17.15	19.68	13.39	15.28	18.22	12.94	15.59	19.90	16.82	21.86	29.97	25.67	33.99	46.66	179	202	229
	0.60	32.76	39.22	50.90	24.85	31.97	46.36	24.60	33.89	53.18	29.65	43.83	71.86	47.33	70.35	110.45	71.63	101.76	149.25	268	303	344
	0.80	56.64	88.10	198.97	53.95	97.55	234.95	63.28	118.22	265.88	85.70	153.73	304.15	128.43	206.41	347.08	169.13	248.79	375.99	358	404	458
	1.00	**162.88**	**579.66**	**4,506.27**	**193.87**	**556.76**	**2,081.78**	**224.65**	**545.34**	**1,528.09**	**265.22**	**535.04**	**1,162.70**	**313.37**	**524.49**	**910.92**	**346.90**	**519.31**	**795.82**	**447**	**505**	**573**
50	**0.20**	17.61	18.04	18.55	11.18	11.51	11.91	9.01	9.33	9.72	7.44	7.80	8.25	6.94	7.52	8.27	8.30	9.34	10.68	92.01	101.04	111.30
	0.40	23.19	24.73	26.74	15.72	17.08	18.91	13.69	15.20	17.30	13.35	15.47	18.54	17.61	21.64	27.44	27.00	33.63	42.78	183.51	201.59	222.10
	0.60	33.75	38.93	47.03	25.90	31.62	41.41	25.94	33.43	46.54	31.70	43.14	62.45	50.81	69.29	97.61	76.38	100.43	134.58	275.02	302.13	332.90
	0.80	60.57	86.31	151.98	59.29	95.06	180.82	70.28	115.22	211.12	95.01	150.26	252.02	140.02	202.74	301.60	181.59	245.24	336.55	366.52	402.67	443.70
	1.00	**195.94**	**546.17**	**2,458.86**	**231.62**	**533.35**	**1,440.85**	**262.61**	**526.78**	**1,152.01**	**301.13**	**520.97**	**942.34**	**344.52**	**514.50**	**785.49**	**373.80**	**511.60**	**710.00**	**458.03**	**503.22**	**554.50**
100	**0.20**	17.71	18.02	18.37	11.26	11.49	11.77	9.09	9.32	9.58	7.53	7.79	8.10	7.08	7.50	8.01	8.54	9.30	10.20	94.23	100.70	107.79
	0.40	23.55	24.67	26.01	16.04	17.02	18.24	14.04	15.13	16.53	13.83	15.38	17.40	18.52	21.47	25.29	28.52	33.35	39.43	187.97	200.91	215.07
	0.60	34.90	38.70	43.91	27.13	31.37	37.54	27.53	33.10	41.34	34.14	42.63	54.91	54.88	68.49	86.90	81.82	99.43	121.99	281.70	301.11	322.35
	0.80	65.48	84.99	122.13	66.03	93.23	143.35	79.04	113.01	170.89	106.34	147.69	211.30	153.66	200.01	263.93	195.94	242.58	302.78	375.43	401.32	429.64
	1.00	**244.36**	**522.41**	**1,426.43**	**282.61**	**516.36**	**1,023.00**	**311.54**	**513.17**	**883.07**	**345.36**	**510.56**	**772.01**	**381.28**	**507.06**	**681.94**	**404.81**	**505.83**	**636.56**	**469.16**	**501.52**	**536.92**
200	**0.20**	17	18	18	11.32	11.49	11.68	9.15	9.31	9.50	7.60	7.78	7.99	7.18	7.49	7.84	8.73	9.28	9.90	95.91	100.53	105.47
	0.40	23	24	25	16.28	16.99	17.82	14.31	15.10	16.05	14.21	15.34	16.70	19.24	21.39	23.96	29.71	33.22	37.35	191.32	200.57	210.44
	0.60	35	38	42	28.12	31.24	35.30	28.82	32.93	38.34	36.13	42.38	50.49	58.14	68.10	80.43	86.13	98.93	114.20	286.73	300.60	315.40
	0.80	69	84	107	71.88	92.33	123.81	86.52	111.92	148.95	115.79	146.43	188.00	164.68	198.66	241.35	207.31	241.26	281.97	382.14	400.64	420.37
	1.00	**292**	**511**	**1,010**	**329.38**	**508.08**	**816.73**	**354.52**	**506.49**	**740.01**	**382.62**	**505.43**	**675.48**	**411.09**	**503.37**	**620.00**	**429.41**	**502.96**	**591.33**	**477.55**	**500.67**	**525.34**
500	**0.20**	17.87	18.01	18.16	11.38	11.48	11.60	9.20	9.31	9.42	7.66	7.78	7.91	7.29	7.48	7.70	8.91	9.26	9.64	97.48	100.44	103.51
	0.40	24.10	24.62	25.18	16.52	16.98	17.48	14.57	15.09	15.66	14.58	15.31	16.14	19.94	21.34	22.90	30.86	33.14	35.66	194.46	200.37	206.52
	0.60	36.71	38.53	40.62	29.12	31.17	33.59	30.12	32.83	36.05	38.12	42.23	47.08	61.35	67.87	75.31	90.31	98.64	107.93	291.44	300.30	309.53
	0.80	74.11	83.97	97.04	78.04	91.81	109.86	94.29	111.29	132.82	125.38	145.69	170.28	175.60	197.87	223.56	218.38	240.48	265.24	388.41	400.24	412.54
	1.00	**350.07**	**504.43**	**763.77**	**380.89**	**503.27**	**675.83**	**400.09**	**502.61**	**636.89**	**420.78**	**502.43**	**602.60**	**440.66**	**501.21**	**571.33**	**453.39**	**501.28**	**555.00**	**485.39**	**500.17**	**515.55**

Table 4.9 – (*continued*), for B = 1.

$B=1$																						
	λ_q	**0.01**			**0.05**			**0.10**			**0.20**			**0.40**			**0.60**			**1.00**		
Per. of the sampling dist.		25^{th}	50^{th}	75^{th}	25^{th}	50^{th}	75^{th}	25^{th}	50^{th}	75^{th}	25^{th}	50^{th}	75^{th}	25^{th}	50^{th}	75^{th}	25^{th}	50^{th}	75^{th}	25^{th}	50^{th}	75^{th}
n	δ	**ov.es.**		**und.es.**	**ov.es.**		**und.es.**	**ov.es.**		**und.es.**	**ov.es.**		**und.es.**	**ov.es.**		**und.es.**	**ov.es.**		**und.es.**	**ov.es.**		**und.es.**
1,000	**0.20**	17.91	18.01	18.11	11.41	11.48	11.56	9.23	9.31	9.39	7.69	7.78	7.87	7.34	7.48	7.63	9.01	9.26	9.53	98.30	100.41	102.56
	0.40	24.24	24.61	25.01	16.64	16.97	17.33	14.71	15.08	15.48	14.77	15.30	15.88	20.32	21.32	22.41	31.48	33.12	34.87	196.09	200.31	204.62
	0.60	37.20	38.51	39.95	29.66	31.15	32.81	30.84	32.80	35.01	39.21	42.19	45.52	63.10	67.80	72.94	92.56	98.55	105.00	293.89	300.21	306.68
	0.80	76.62	83.86	92.65	81.55	91.66	103.83	98.68	111.10	125.72	130.70	145.47	162.29	181.54	197.63	215.35	224.34	240.25	257.41	391.69	400.12	408.75
	1.00	**386.00**	**502.48**	**669.71**	**411.22**	**501.84**	**616.83**	**426.20**	**501.45**	**592.10**	**442.08**	**501.53**	**569.91**	**456.80**	**500.57**	**548.89**	**466.30**	**500.78**	**538.00**	**489.49**	**500.02**	**510.81**
10,000	**0.20**	17.97	18.01	18.04	11.46	11.48	11.51	9.28	9.31	9.33	7.75	7.78	7.81	7.43	7.48	7.52	9.18	9.26	9.34	99.71	100.38	101.05
	0.40	24.49	24.61	24.73	16.86	16.97	17.08	14.95	15.07	15.20	15.13	15.30	15.47	20.98	21.31	21.64	32.56	33.09	33.63	198.91	200.25	201.61
	0.60	38.06	38.49	38.93	30.64	31.13	31.63	32.13	32.77	33.44	41.17	42.14	43.16	66.20	67.73	69.31	96.53	98.46	100.46	298.12	300.12	302.16
	0.80	81.33	83.75	86.35	88.13	91.50	95.11	106.79	110.91	115.29	140.39	145.24	150.34	192.16	197.39	202.83	234.88	240.02	245.32	397.32	400.00	402.71
	1.00	**459.50**	**500.54**	**546.89**	**469.62**	**500.41**	**533.86**	**475.12**	**500.29**	**527.18**	**481.04**	**500.64**	**521.28**	**485.68**	**499.92**	**514.72**	**489.14**	**500.28**	**511.77**	**496.53**	**499.87**	**503.27**

Table 4.9 – (*continued*), for $B = 2$.

$B = 2$																						
	λ_q	0.01			0.05			0.10			0.20			0.40			0.60			1.00		
Per. of the sampling dist.		25th	50th	75th	25th	50th	75th	25th	50th	75th	25th	50th	75th	25th	50th	75th	25th	50th	75th	25th	50th	75th
n	δ	ov.es.		und.es.	ov.es.		und.es.	ov.es.		und.es.	ov.es.		und.es.	ov.es.		und.es.	ov.es.		und.es.	ov.es.		und.es.
30	**0.20**	13.77	14.18	14.70	9.89	10.25	10.71	8.23	8.59	9.05	6.88	7.28	7.81	6.37	7.01	7.89	7.45	8.60	10.18	89.96	101.49	115.09
	0.40	17.82	19.30	21.41	13.61	15.06	17.20	12.09	13.70	16.17	11.64	13.83	17.34	14.80	18.95	25.60	22.49	29.56	40.41	179.42	202.49	229.67
	0.60	25.42	30.44	39.53	21.53	27.36	38.91	21.40	28.84	44.10	25.13	36.41	59.08	39.95	59.37	94.44	62.17	89.10	133.02	268.88	303.48	344.25
	0.80	44.00	68.60	160.98	44.93	79.86	202.48	52.12	97.36	235.39	70.55	129.98	278.00	110.67	184.36	329.06	151.91	230.23	362.89	358.33	404.47	458.84
	1.00	**129.47**	**615.84**	**27,389.51**	**163.52**	**574.32**	**3,750.28**	**194.44**	**557.64**	**2,206.21**	**237.65**	**542.88**	**1,443.89**	**293.15**	**529.28**	**1,019.29**	**331.80**	**521.70**	**850.82**	**447.79**	**505.46**	**573.42**
50	**0.20**	13.84	14.17	14.55	9.96	10.24	10.58	8.29	8.57	8.92	6.94	7.26	7.65	6.48	6.98	7.63	7.65	8.55	9.71	92.01	101.04	111.30
	0.40	18.06	19.24	20.78	13.85	15.00	16.55	12.35	13.63	15.41	11.99	13.74	16.24	15.45	18.76	23.52	23.62	29.25	37.09	183.51	201.59	222.10
	0.60	26.19	30.21	36.51	22.39	27.08	34.96	22.48	28.48	38.84	26.76	35.86	51.39	42.85	58.46	83.04	66.36	87.89	119.27	275.02	302.13	332.90
	0.80	47.07	67.19	120.27	49.16	77.83	151.57	57.72	94.81	181.40	78.39	126.83	224.28	121.29	180.79	280.77	163.89	226.65	320.85	366.52	402.67	443.70
	1.00	**158.28**	**565.90**	**7,441.82**	**199.24**	**543.44**	**2,145.62**	**232.08**	**533.97**	**1,488.77**	**274.82**	**525.57**	**1,100.72**	**326.30**	**517.54**	**853.05**	**360.53**	**512.93**	**745.79**	**458.03**	**503.22**	**554.50**
100	**0.20**	13.92	14.15	14.42	10.02	10.23	10.46	8.36	8.56	8.80	7.02	7.25	7.52	6.60	6.96	7.40	7.86	8.51	9.30	94.23	100.70	107.79
	0.40	18.34	19.19	20.23	14.12	14.95	15.99	12.65	13.58	14.75	12.38	13.66	15.31	16.20	18.63	21.76	24.91	29.02	34.21	187.97	200.91	215.07
	0.60	27.08	30.04	34.08	23.41	26.88	31.86	23.76	28.21	34.73	28.70	35.45	45.28	46.26	57.78	73.65	71.19	86.99	107.59	281.70	301.11	322.35
	0.80	50.90	66.16	95.70	54.51	76.34	118.34	64.79	92.93	143.82	88.05	124.51	184.07	133.94	178.13	241.85	177.81	223.98	285.48	375.43	401.32	429.64
	1.00	**203.08**	**531.70**	**2,628.09**	**250.15**	**521.38**	**1,307.24**	**282.83**	**516.81**	**1,039.75**	**322.15**	**512.86**	**854.66**	**366.23**	**508.83**	**720.82**	**394.08**	**506.38**	**657.91**	**469.16**	**501.52**	**536.92**
200	**0.20**	13.98	14.15	14.33	10.08	10.22	10.38	8.41	8.56	8.72	7.08	7.24	7.43	6.69	6.95	7.25	8.02	8.50	9.03	95.91	100.53	105.47
	0.40	18.55	19.17	19.88	14.32	14.93	15.63	12.88	13.55	14.35	12.70	13.63	14.74	16.79	18.56	20.67	25.92	28.91	32.43	191.32	200.57	210.44
	0.60	27.79	29.95	32.65	24.22	26.77	30.06	24.80	28.08	32.37	30.28	35.25	41.74	49.01	57.45	68.04	75.03	86.54	100.44	286.73	300.60	315.40
	0.80	54.18	65.66	83.82	59.17	75.61	101.65	70.89	92.01	124.09	96.22	123.36	161.78	144.29	176.81	218.98	188.94	222.65	263.99	382.14	400.64	420.37
	1.00	**251.14**	**515.64**	**1,450.15**	**299.50**	**510.74**	**955.88**	**329.41**	**508.45**	**823.96**	**363.29**	**506.62**	**723.16**	**399.21**	**504.52**	**644.04**	**421.01**	**503.13**	**604.74**	**477.55**	**500.67**	**525.34**
500	**0.20**	14.04	14.14	14.26	10.13	10.22	10.32	8.46	8.56	8.65	7.13	7.24	7.35	6.78	6.95	7.13	8.18	8.49	8.82	97.48	100.44	103.51
	0.40	18.76	19.16	19.59	14.52	14.92	15.35	13.10	13.54	14.03	13.00	13.61	14.28	17.37	18.52	19.80	26.90	28.84	30.99	194.46	200.37	206.52
	0.60	28.49	29.90	31.52	25.04	26.71	28.68	25.84	28.00	30.55	31.86	35.13	39.00	51.72	57.25	63.63	78.78	86.27	94.71	291.44	300.30	309.53
	0.80	57.63	65.36	75.65	64.09	75.19	89.99	77.28	91.48	109.93	104.61	122.69	145.18	154.63	176.04	201.24	199.86	221.87	246.89	388.41	400.24	412.54
	1.00	**313.05**	**506.49**	**927.64**	**356.74**	**504.60**	**740.43**	**380.80**	**503.61**	**678.86**	**406.58**	**502.99**	**627.91**	**432.44**	**502.00**	**584.99**	**447.52**	**501.23**	**562.60**	**485.39**	**500.17**	**515.55**

Table 4.9 – (*continued*), for $B = 2$.

		B = 2																				
	λ_q	0.01			0.05			0.10			0.20			0.40			0.60			1.00		
	Per. of the sampling dist.	25th	50th	75th	25th	50th	75th	25th	50th	75th	25th	50th	75th	25th	50th	75th	25th	50th	75th	25th	50th	75th
n	δ	ov.es.		und.es.	ov.es.		und.es.	ov.es.		und.es.	ov.es.		und.es.	ov.es.		und.es.	ov.es.		und.es.	ov.es.		und.es.
1,000	**0.20**	14.07	14.14	14.22	10.15	10.22	10.29	8.49	8.55	8.62	7.16	7.24	7.32	6.82	6.95	7.07	8.26	8.48	8.71	98.30	100.41	102.56
	0.40	18.87	19.15	19.45	14.63	14.91	15.21	13.22	13.53	13.87	13.16	13.60	14.07	17.68	18.50	19.40	27.42	28.82	30.31	196.09	200.31	204.62
	0.60	28.87	29.88	31.00	25.48	26.70	28.05	26.41	27.97	29.73	32.73	35.10	37.75	53.20	57.19	61.59	80.80	86.19	92.04	293.89	300.21	306.68
	0.80	59.60	65.27	72.18	66.91	75.06	85.01	80.91	91.32	103.78	109.30	122.49	137.81	160.30	175.81	193.13	205.77	221.64	238.93	391.69	400.12	408.75
	1.00	**354.48**	**503.79**	**760.33**	**391.84**	**502.77**	**656.27**	**411.14**	**502.16**	**618.63**	**431.24**	**501.90**	**586.37**	**450.78**	**501.25**	**558.16**	**461.91**	**500.66**	**543.05**	**489.49**	**500.02**	**510.81**
10,000	**0.20**	14.12	14.14	14.17	10.20	10.22	10.24	8.53	8.55	8.58	7.21	7.24	7.26	6.90	6.94	6.98	8.41	8.48	8.55	99.71	100.38	101.05
	0.40	19.06	19.15	19.24	14.82	14.91	15.00	13.43	13.53	13.63	13.45	13.59	13.74	18.23	18.49	18.77	28.35	28.80	29.26	198.91	200.25	201.61
	0.60	29.54	29.87	30.21	26.28	26.68	27.09	27.44	27.95	28.49	34.29	35.06	35.87	55.83	57.13	58.48	84.37	86.12	87.92	298.12	300.12	302.16
	0.80	63.29	65.19	67.22	72.20	74.94	77.87	87.69	91.16	94.86	117.94	122.29	126.90	170.52	175.58	180.87	216.27	221.41	226.73	397.32	400.00	402.71
	1.00	**445.98**	**501.11**	**566.95**	**462.30**	**500.95**	**544.10**	**469.72**	**500.72**	**534.48**	**477.25**	**500.82**	**525.95**	**483.97**	**500.50**	**517.80**	**487.52**	**500.09**	**513.12**	**496.53**	**499.87**	**503.27**

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Table 4.9 – (*continued*), for $B = 5$.

$B = 5$																						
	λ_q	0.01			0.05			0.10			0.20			0.40			0.60			1.00		
Per. of the sampling dist.		25th	50th	75th	25th	50th	75th	25th	50th	75th	25th	50th	75th	25th	50th	75th	25th	50th	75th	25th	50th	75th
n	δ	ov.es.		und.es.	ov.es.		und.es.	ov.es.		und.es.	ov.es.		und.es.	ov.es.		und.es.	ov.es.		und.es.	ov.es.		und.es.
30	0.20	13.23	14.02	14.99	9.86	10.22	10.68	8.18	8.53	9.00	6.87	7.27	7.81	6.33	6.97	7.84	7.39	8.52	10.07	89.96	101.49	115.09
	0.40	20.16	22.37	25.37	13.57	15.03	17.18	12.04	13.65	16.11	11.64	13.83	17.34	14.67	18.77	25.34	22.15	29.08	39.69	179.42	202.49	229.67
	0.60	30.82	37.42	49.10	21.52	27.38	38.99	21.36	28.81	44.10	25.11	36.36	58.97	39.48	58.62	93.20	60.97	87.35	130.51	268.88	303.48	344.25
	0.80	54.76	85.11	189.11	45.04	80.10	202.92	52.13	97.45	235.54	70.40	129.61	277.28	109.23	182.24	326.90	149.13	226.81	360.11	358.33	404.47	458.84
	1.00	**155.16**	**592.31**	**10,750.40**	**163.95**	**573.74**	**3,711.04**	**194.58**	**557.49**	**2,204.64**	**236.98**	**542.65**	**1,457.23**	**290.83**	**529.90**	**1,039.12**	**328.68**	**522.41**	**867.27**	**447.79**	**505.46**	**573.42**
50	0.20	13.37	13.99	14.72	9.92	10.20	10.54	8.24	8.52	8.86	6.94	7.26	7.65	6.44	6.94	7.58	7.58	8.47	9.61	92.01	101.04	111.30
	0.40	20.54	22.28	24.48	13.81	14.97	16.52	12.30	13.58	15.35	11.98	13.73	16.24	15.32	18.59	23.29	23.25	28.77	36.44	183.51	201.59	222.10
	0.60	31.84	37.12	45.26	22.39	27.10	35.02	22.44	28.44	38.83	26.74	35.82	51.30	42.35	57.73	81.96	65.07	86.16	116.97	275.02	302.13	332.90
	0.80	58.62	83.40	144.97	49.28	78.06	151.97	57.75	94.89	181.54	78.21	126.48	223.64	119.73	178.68	278.42	160.96	223.25	317.64	366.52	402.67	443.70
	1.00	**186.24**	**552.94**	**4,070.29**	**199.69**	**543.00**	**2,129.88**	**232.22**	**533.85**	**1,487.42**	**274.10**	**525.25**	**1,106.83**	**324.12**	**517.91**	**864.55**	**357.72**	**513.38**	**756.05**	**458.03**	**503.22**	**554.50**
100	0.20	13.52	13.97	14.47	9.99	10.19	10.42	8.31	8.51	8.74	7.02	7.25	7.51	6.56	6.92	7.35	7.79	8.43	9.20	94.23	100.70	107.79
	0.40	20.95	22.21	23.69	14.08	14.92	15.96	12.59	13.52	14.70	12.38	13.66	15.31	16.06	18.45	21.55	24.52	28.55	33.63	187.97	200.91	215.07
	0.60	33.02	36.89	42.14	23.41	26.90	31.90	23.72	28.18	34.71	28.67	35.41	45.22	45.70	57.06	72.70	69.80	85.28	105.50	281.70	301.11	322.35
	0.80	63.40	82.15	117.10	54.66	76.57	118.69	64.83	93.02	143.95	87.84	124.16	183.53	132.24	176.04	239.49	174.74	220.58	282.07	375.43	401.32	429.64
	1.00	**232.52**	**525.49**	**1,850.71**	**250.58**	**521.04**	**1,301.03**	**282.96**	**516.71**	**1,038.91**	**321.40**	**512.49**	**857.10**	**364.34**	**509.02**	**726.98**	**391.76**	**506.64**	**663.81**	**469.16**	**501.52**	**536.92**
200	0.20	13.64	13.96	14.30	10.04	10.19	10.35	8.36	8.50	8.66	7.08	7.24	7.42	6.65	6.91	7.21	7.95	8.41	8.94	95.91	100.53	105.47
	0.40	21.27	22.18	23.20	14.29	14.90	15.60	12.82	13.50	14.30	12.69	13.62	14.74	16.64	18.39	20.48	25.51	28.43	31.88	191.32	200.57	210.44
	0.60	33.95	36.78	40.29	24.23	26.79	30.10	24.76	28.04	32.34	30.25	35.21	41.68	48.41	56.73	67.17	73.56	84.84	98.47	286.73	300.60	315.40
	0.80	67.48	81.54	103.25	59.33	75.84	101.96	70.94	92.09	124.20	95.97	123.02	161.31	142.48	174.74	216.68	185.76	219.26	260.53	382.14	400.64	420.37
	1.00	**279.86**	**512.45**	**1,169.31**	**299.87**	**510.45**	**952.79**	**329.51**	**508.37**	**823.43**	**362.54**	**506.23**	**724.20**	**397.65**	**504.63**	**647.66**	**419.18**	**503.29**	**608.39**	**477.55**	**500.67**	**525.34**
500	0.20	13.75	13.95	14.17	10.09	10.18	10.28	8.41	8.50	8.60	7.13	7.24	7.35	6.73	6.90	7.09	8.10	8.40	8.73	97.48	100.44	103.51
	0.40	21.57	22.16	22.78	14.49	14.88	15.32	13.04	13.48	13.97	13.00	13.60	14.28	17.21	18.35	19.62	26.47	28.37	30.47	194.46	200.37	206.52
	0.60	34.87	36.72	38.83	25.05	26.73	28.71	25.80	27.96	30.52	31.83	35.09	38.95	51.08	56.53	62.82	77.23	84.58	92.85	291.44	300.30	309.53
	0.80	71.74	81.18	93.56	64.27	75.42	90.26	77.34	91.56	110.03	104.33	122.36	144.77	152.73	173.97	199.02	196.60	218.49	243.43	388.41	400.24	412.54
	1.00	**338.00**	**504.98**	**822.81**	**356.99**	**504.34**	**738.89**	**380.86**	**503.53**	**678.54**	**405.89**	**502.58**	**628.19**	**431.31**	**502.06**	**586.94**	**446.25**	**501.34**	**564.66**	**485.39**	**500.17**	**515.55**

Table 4.9 – (*continued*), for B = 5.

$B = 5$																						
	λ_q	0.01			0.05			0.10			0.20			0.40			0.60			1.00		
	Per. of the sampling dist.	25th	50th	75th	25th	50th	75th	25th	50th	75th	25th	50th	75th	25th	50th	75th	25th	50th	75th	25th	50th	75th
n	δ	ov.es.		und.es.	ov.es.		und.es.	ov.es.		und.es.	ov.es.		und.es.	ov.es.		und.es.	ov.es.		und.es.	ov.es.		und.es.
1,000	**0.20**	13.80	13.95	14.10	10.12	10.18	10.25	8.43	8.50	8.57	7.16	7.24	7.31	6.78	6.90	7.03	8.19	8.40	8.63	98.30	100.41	102.56
	0.40	21.73	22.15	22.59	14.60	14.88	15.18	13.16	13.48	13.82	13.16	13.60	14.06	17.52	18.33	19.22	26.98	28.35	29.81	196.09	200.31	204.62
	0.60	35.37	36.70	38.16	25.50	26.72	28.07	26.37	27.94	29.70	32.69	35.06	37.71	52.54	56.48	60.81	79.21	84.50	90.23	293.89	300.21	306.68
	0.80	74.15	81.08	89.41	67.10	75.29	85.27	80.98	91.40	103.88	109.01	122.16	137.42	158.35	173.74	190.95	202.47	218.26	235.49	391.69	400.12	408.75
	1.00	**375.43**	**502.77**	**702.05**	**391.99**	**502.52**	**655.24**	**411.16**	**502.08**	**618.40**	**430.60**	**501.49**	**586.38**	**449.93**	**501.29**	**559.45**	**460.97**	**500.75**	**544.45**	**489.49**	**500.02**	**510.81**
10,000	**0.20**	13.90	13.95	13.99	10.16	10.18	10.20	8.48	8.50	8.52	7.21	7.23	7.26	6.86	6.90	6.94	8.33	8.40	8.47	99.71	100.38	101.05
	0.40	22.01	22.15	22.28	14.79	14.88	14.97	13.37	13.47	13.58	13.45	13.59	13.73	18.06	18.32	18.59	27.89	28.33	28.78	198.91	200.25	201.61
	0.60	36.25	36.68	37.13	26.30	26.70	27.11	27.41	27.92	28.45	34.25	35.02	35.83	55.14	56.42	57.75	82.71	84.42	86.19	298.12	300.12	302.16
	0.80	78.66	80.97	83.44	72.42	75.16	78.11	87.76	91.24	94.95	117.61	121.96	126.55	168.49	173.52	178.76	212.91	218.02	223.33	397.32	400.00	402.71
	1.00	**454.70**	**500.57**	**553.78**	**462.21**	**500.71**	**543.66**	**469.69**	**500.65**	**534.36**	**476.74**	**500.40**	**525.63**	**483.68**	**500.53**	**518.18**	**487.24**	**500.16**	**513.57**	**496.53**	**499.87**	**503.27**

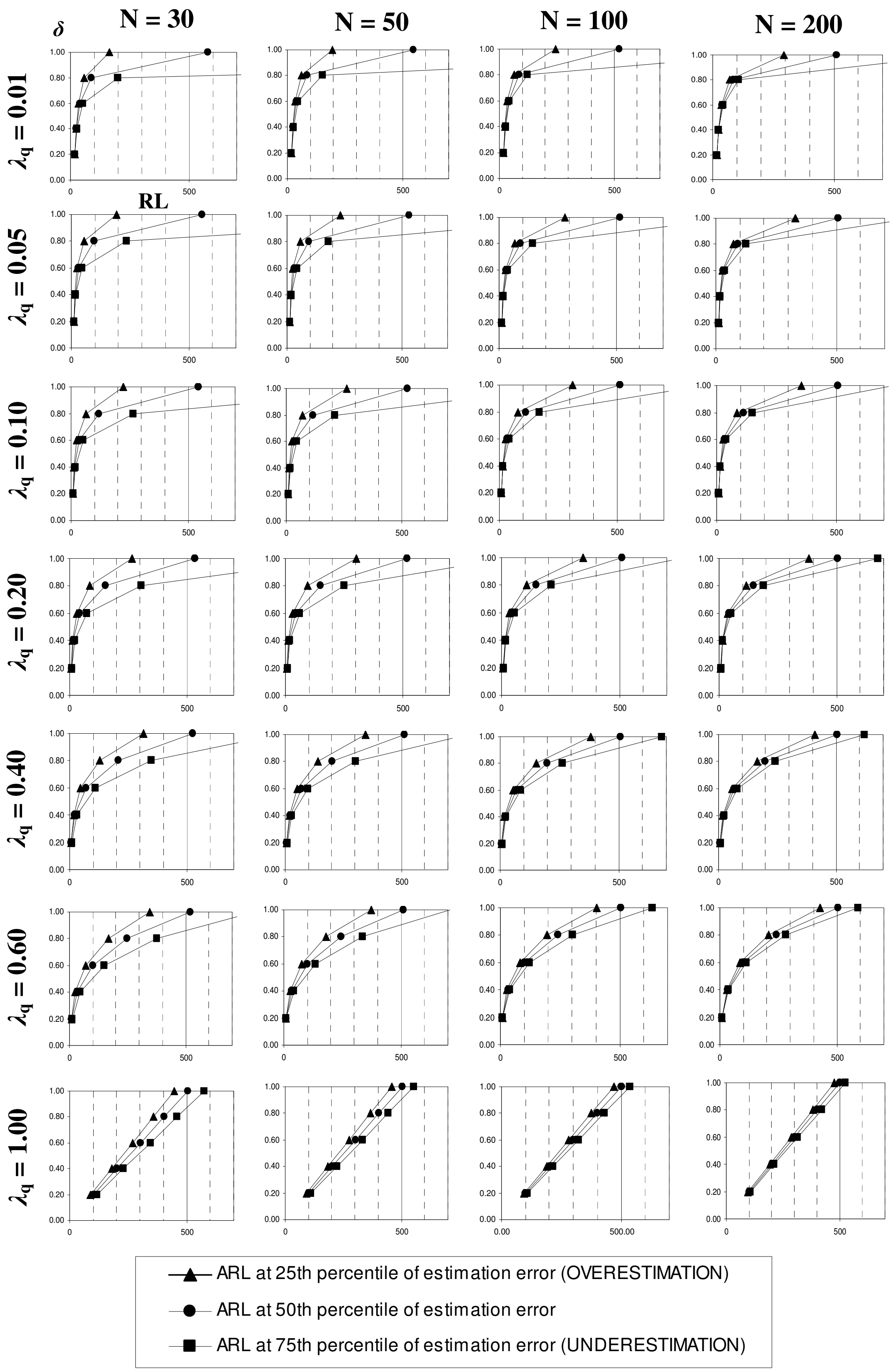


Figure 4.10 – Conditional ARL_0 and ARL_1 peformance at 25th (overestimation), 50th, and the 75th (underestimation) percentile of *W*, for $B = 1$.

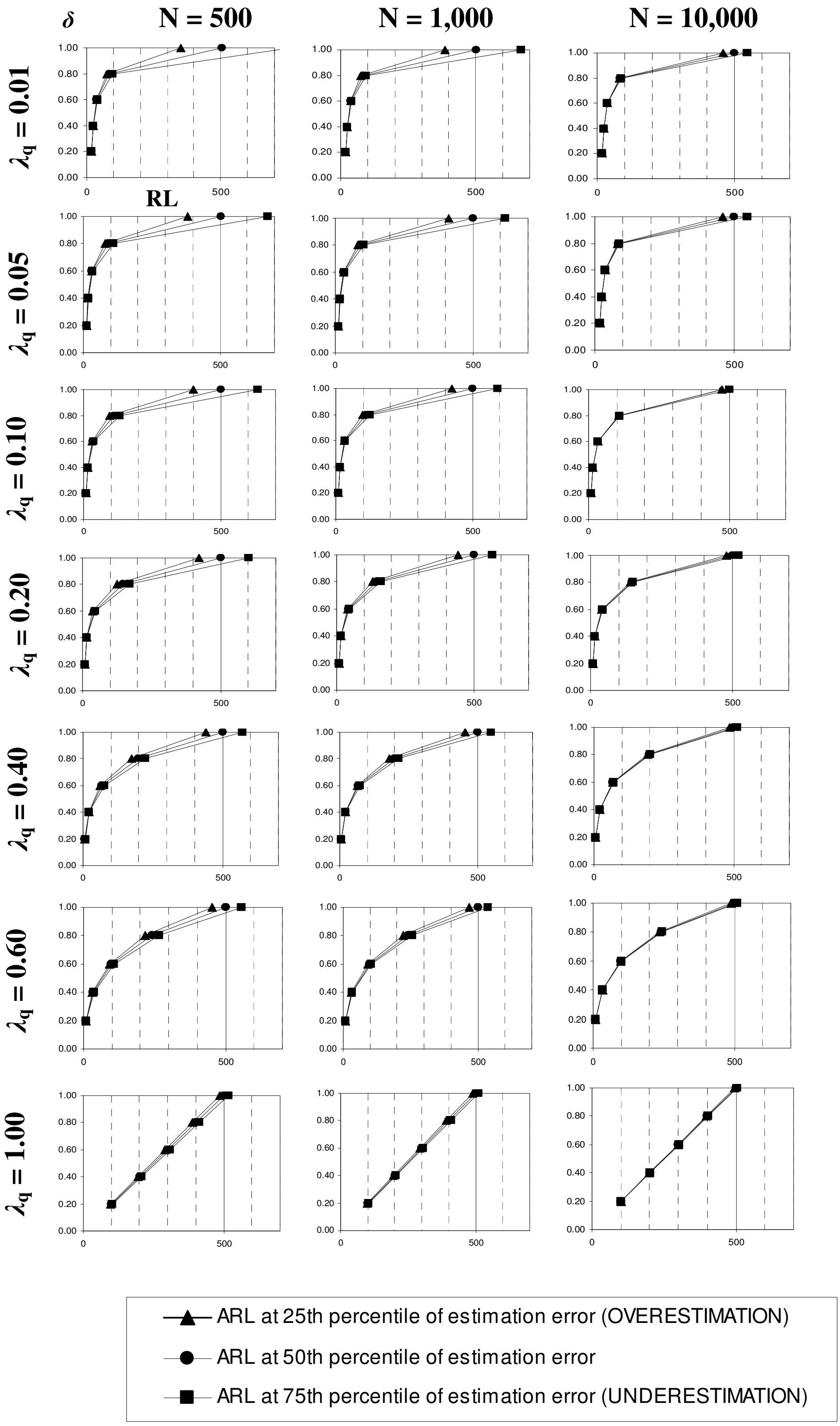


Figure 4.10 – (*continued*)

4.4.2. Conditional performances of the SDRL and the percentiles of RL

The conditional SDRL performance of the control chart is provided in Table 4.10. Since the SDRL pattern is regular, the table size is kept small by providing values for B = 1; λ_q = 0.01, 0.20, and 1.00; n = 30, 100, 500, and 10000. For λ_q = 1.00 (Shewhart control chart) the SDRL values are very close to the ARL values regardless of the magnitude of the shift (δ). For smaller values of λ_q, the variability decreases than the ARL values. When the magnitude of shift is increased, the SDRL even gets smaller. For the largest shift δ = 0.2 and $\lambda_q \leq 0.20$, SDRL is less than 2. Hence, smaller SDRL implies that the ARL values represent the chart performance better for smaller λ_q and larger shifts.

The existence of an estimation error of the mean does not change the overall behavior of SDRL explained here. The SDRL follows the ARL where underestimation increases the SDRL and overestimation decreases the SDRL.

Considering the *out-of-control* cases ($\delta \leq 0.8$), sample size is mostly effective on the SDRL performance for δ = 0.8. For smaller sample sizes, overestimation of the mean decreases the SDRL values and underestimation of the mean increases the SDRL values. For the 50th percentile of RL, SDRL values are not affected much from the sample size.

The percentiles of RL in Figure 4.11 also provide an overall understanding of the conditional SDRL performance for the control chart.

Table 4.10 – Conditional SDRL performance of the control chart peformance at the 25^{th} (overestimation), 50^{th}, and the 75^{th} (underestimation) percentile of W, for B = 1 and selected n.

	λ_q	0.01						0.05						0.10					
Percentile of the Sampling dist.		25^{th} (overestimate)		50^{th}		75^{th} (underestimate)		25^{th} (overestimate)		50^{th}		75^{th} (underestimate)		25^{th} (overestimate)		50^{th}		75^{th} (underestimate)	
n	δ	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
30	0.20	17.52	1.03	18.06	1.21	18.75	1.44	7.37	1.38	7.82	1.69	8.43	2.13	8.95	1.09	9.35	1.30	9.87	1.59
	0.40	22.87	2.96	24.81	3.76	27.56	4.98	12.94	5.76	15.59	8.12	19.90	12.11	13.39	3.83	15.28	5.20	18.22	7.50
	0.60	32.76	7.51	39.22	11.07	50.90	18.45	29.65	21.47	43.83	35.41	71.86	63.28	24.60	12.96	33.89	21.49	53.18	40.07
	0.80	56.64	22.45	88.10	47.48	198.97	151.11	85.70	77.09	153.73	145.11	304.15	295.63	63.28	49.98	118.22	104.56	265.88	252.20
	1.00	162.88	116.25	579.66	529.26	4,506.27	4,459.20	265.22	256.66	535.04	526.64	1,162.70	1,154.52	224.65	210.94	545.34	531.88	1,528.09	1,515.13
100	0.20	17.71	1.10	18.02	1.20	18.37	1.31	7.44	1.43	7.80	1.68	8.25	2.00	9.09	1.16	9.32	1.29	9.58	1.43
	0.40	23.55	3.24	24.67	3.70	26.01	4.28	13.35	6.13	15.47	8.01	18.54	10.83	14.04	4.29	15.13	5.09	16.53	6.15
	0.60	34.90	8.64	38.70	10.77	43.91	13.89	31.70	23.47	43.14	34.73	62.45	53.90	27.53	15.59	33.10	20.75	41.34	28.58
	0.80	65.48	29.02	84.99	44.83	122.13	77.86	95.01	86.40	150.26	141.64	252.02	243.46	79.04	65.57	113.01	99.36	170.89	157.17
	1.00	244.36	195.53	522.41	472.04	1,426.43	1,376.85	301.13	292.60	520.97	512.56	942.34	934.10	311.54	297.90	513.17	499.69	883.07	869.83
500	0.20	17.87	1.15	18.01	1.19	18.16	1.24	7.53	1.49	7.79	1.67	8.10	1.89	9.20	1.23	9.31	1.28	9.42	1.34
	0.40	24.10	3.46	24.62	3.68	25.18	3.92	13.83	6.54	15.38	7.93	17.40	9.77	14.57	4.67	15.09	5.06	15.66	5.49
	0.60	36.71	9.64	38.53	10.67	40.62	11.89	34.14	25.86	42.63	34.23	54.91	46.40	30.12	17.98	32.83	20.50	36.05	23.53
	0.80	74.11	35.82	83.97	43.97	97.04	55.23	106.34	97.72	147.69	139.07	211.30	202.71	94.29	80.72	111.29	97.65	132.82	119.14
	1.00	350.07	300.18	504.43	454.08	763.77	713.44	345.36	336.86	510.56	502.15	772.01	763.71	400.09	386.53	502.61	489.12	636.89	623.51
10,000	0.20	17.97	1.18	18.01	1.19	18.04	1.20	7.60	1.54	7.78	1.67	7.99	1.81	9.28	1.27	9.31	1.28	9.33	1.29
	0.40	24.49	3.63	24.61	3.68	24.73	3.73	14.21	6.88	15.34	7.89	16.70	9.13	14.95	4.96	15.07	5.05	15.20	5.14
	0.60	38.06	10.40	38.49	10.65	38.93	10.90	36.13	27.81	42.38	33.98	50.49	42.02	32.13	19.84	32.77	20.44	33.44	21.07
	0.80	81.33	41.75	83.75	43.78	86.35	45.98	115.79	107.16	146.43	137.81	188.00	179.40	106.79	93.16	110.91	97.27	115.29	101.63
	1.00	459.50	409.21	500.54	450.19	546.89	496.50	382.62	374.14	505.43	497.01	675.48	667.14	475.12	461.62	500.29	486.80	527.18	513.71

Table 4.10 – (*continued*)

	λ_q	0.20						0.40						1.00					
Percentile of the Sampling dist.		25th (overestimate)		50th		75th (underestimate)		25th (overestimate)		50th		75th (underestimate)		25th (overestimate)		50th		75th (underestimate)	
n	δ	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
30	**0.20**	**89.96**	89.46	**101.49**	100.99	**115.09**	114.58	**6.82**	2.50	**7.55**	3.14	**8.58**	4.06	**89.96**	89.46	**101.49**	100.99	**115.09**	114.58
	0.40	**179.42**	178.92	**202.49**	201.99	**229.67**	229.17	**16.82**	11.92	**21.86**	16.87	**29.97**	24.90	**179.42**	178.92	**202.49**	201.99	**229.67**	229.17
	0.60	**268.88**	268.38	**303.48**	302.98	**344.25**	343.75	**47.33**	42.19	**70.35**	65.19	**110.45**	105.29	**268.88**	268.38	**303.48**	302.98	**344.25**	343.75
	0.80	**358.33**	357.83	**404.47**	403.97	**458.84**	458.34	**128.43**	123.28	**206.41**	201.30	**347.08**	342.03	**358.33**	357.83	**404.47**	403.97	**458.84**	458.34
	1.00	**447.79**	447.29	**505.46**	504.96	**573.42**	572.92	**313.37**	308.31	**524.49**	519.50	**910.92**	906.01	**447.79**	447.29	**505.46**	504.96	**573.42**	572.92
100	**0.20**	**94.23**	93.73	**100.70**	100.20	**107.79**	107.28	**7.08**	2.72	**7.50**	3.09	**8.01**	3.54	**94.23**	93.73	**100.70**	100.20	**107.79**	107.28
	0.40	**187.97**	187.47	**200.91**	200.41	**215.07**	214.57	**18.52**	13.59	**21.47**	16.48	**25.29**	20.25	**187.97**	187.47	**200.91**	200.41	**215.07**	214.57
	0.60	**281.70**	281.20	**301.11**	300.61	**322.35**	321.85	**54.88**	49.73	**68.49**	63.33	**86.90**	81.74	**281.70**	281.20	**301.11**	300.61	**322.35**	321.85
	0.80	**375.43**	374.93	**401.32**	400.82	**429.64**	429.14	**153.66**	148.52	**200.01**	194.90	**263.93**	258.85	**375.43**	374.93	**401.32**	400.82	**429.64**	429.14
	1.00	**469.16**	468.66	**501.52**	501.02	**536.92**	536.42	**381.28**	376.24	**507.06**	502.06	**681.94**	676.98	**469.16**	468.66	**501.52**	501.02	**536.92**	536.42
500	**0.20**	**97.48**	96.98	**100.44**	99.93	**103.51**	103.01	**7.29**	2.90	**7.48**	3.08	**7.70**	3.27	**97.48**	96.98	**100.44**	99.93	**103.51**	103.01
	0.40	**194.46**	193.96	**200.37**	199.87	**206.52**	206.02	**19.94**	14.98	**21.34**	16.35	**22.90**	17.90	**194.46**	193.96	**200.37**	199.87	**206.52**	206.02
	0.60	**291.44**	290.93	**300.30**	299.80	**309.53**	309.03	**61.35**	56.20	**67.87**	62.71	**75.31**	70.15	**291.44**	290.93	**300.30**	299.80	**309.53**	309.03
	0.80	**388.41**	387.91	**400.24**	399.74	**412.54**	412.04	**175.60**	170.47	**197.87**	192.75	**223.56**	218.46	**388.41**	387.91	**400.24**	399.74	**412.54**	412.04
	1.00	**485.39**	484.89	**500.17**	499.67	**515.55**	515.05	**440.66**	435.64	**501.21**	496.21	**571.33**	566.34	**485.39**	484.89	**500.17**	499.67	**515.55**	515.05
10,000	**0.20**	**99.71**	99.20	**100.38**	99.87	**101.05**	100.55	**7.43**	3.03	**7.48**	3.07	**7.52**	3.11	**99.71**	99.20	**100.38**	99.87	**101.05**	100.55
	0.40	**198.91**	198.41	**200.25**	199.75	**201.61**	201.11	**20.98**	16.01	**21.31**	16.33	**21.64**	16.66	**198.91**	198.41	**200.25**	199.75	**201.61**	201.11
	0.60	**298.12**	297.62	**300.12**	299.62	**302.16**	301.66	**66.20**	61.04	**67.73**	62.57	**69.31**	64.15	**298.12**	297.62	**300.12**	299.62	**302.16**	301.66
	0.80	**397.32**	396.82	**400.00**	399.50	**402.71**	402.21	**192.16**	187.04	**197.39**	192.27	**202.83**	197.71	**397.32**	396.82	**400.00**	399.50	**402.71**	402.21
	1.00	**496.53**	496.03	**499.87**	499.37	**503.27**	502.77	**485.68**	480.67	**499.92**	494.92	**514.72**	509.73	**496.53**	496.03	**499.87**	499.37	**503.27**	502.77

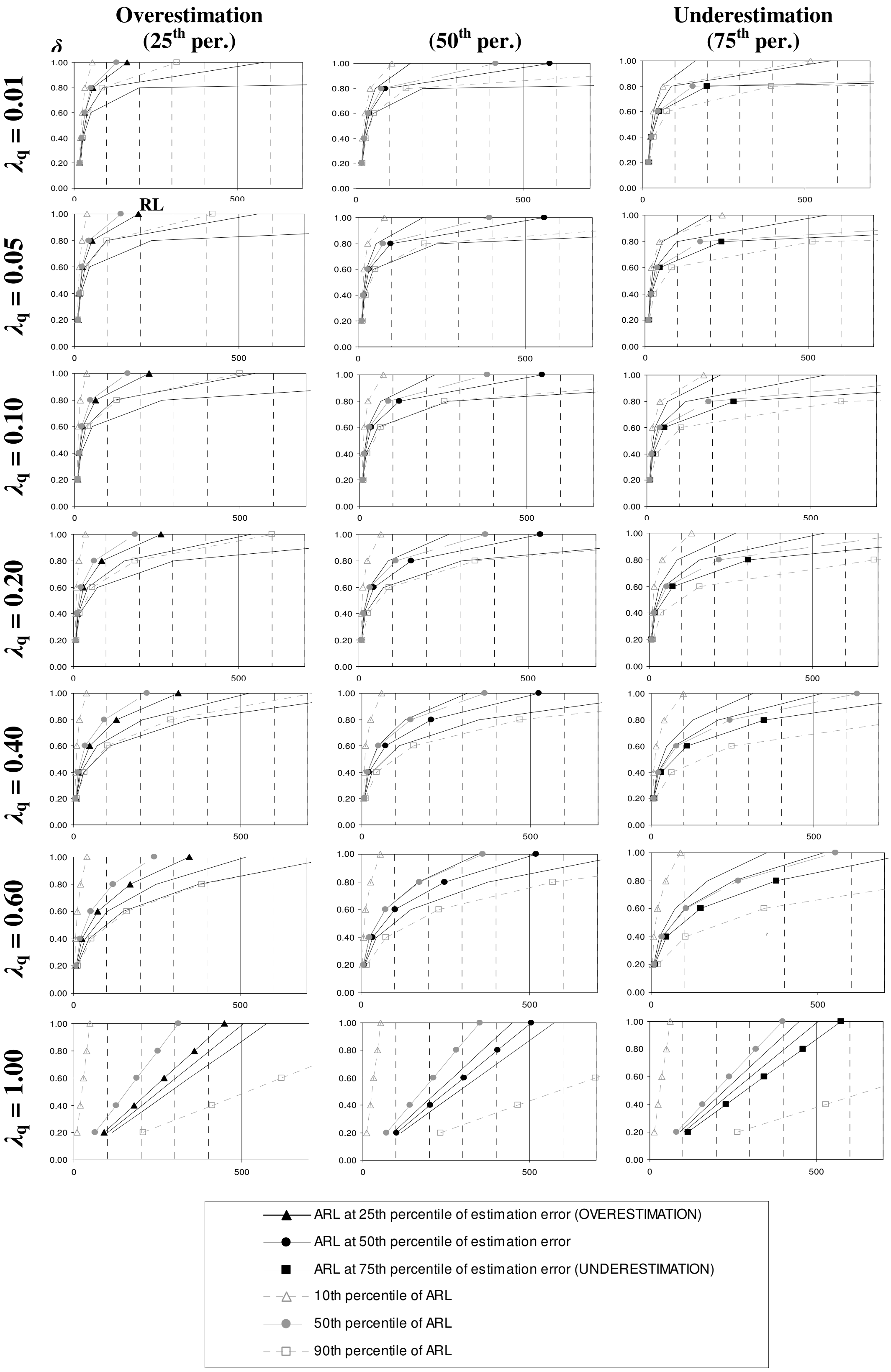


Figure 4.11 – Conditional performances of the ARL and 10th, 50th, 90th percentiles of RL distribution at 25th (overestimation), 50th, 75th (underestimation) percentile of W, for various shifts in the mean, and sample size $n = 30$.

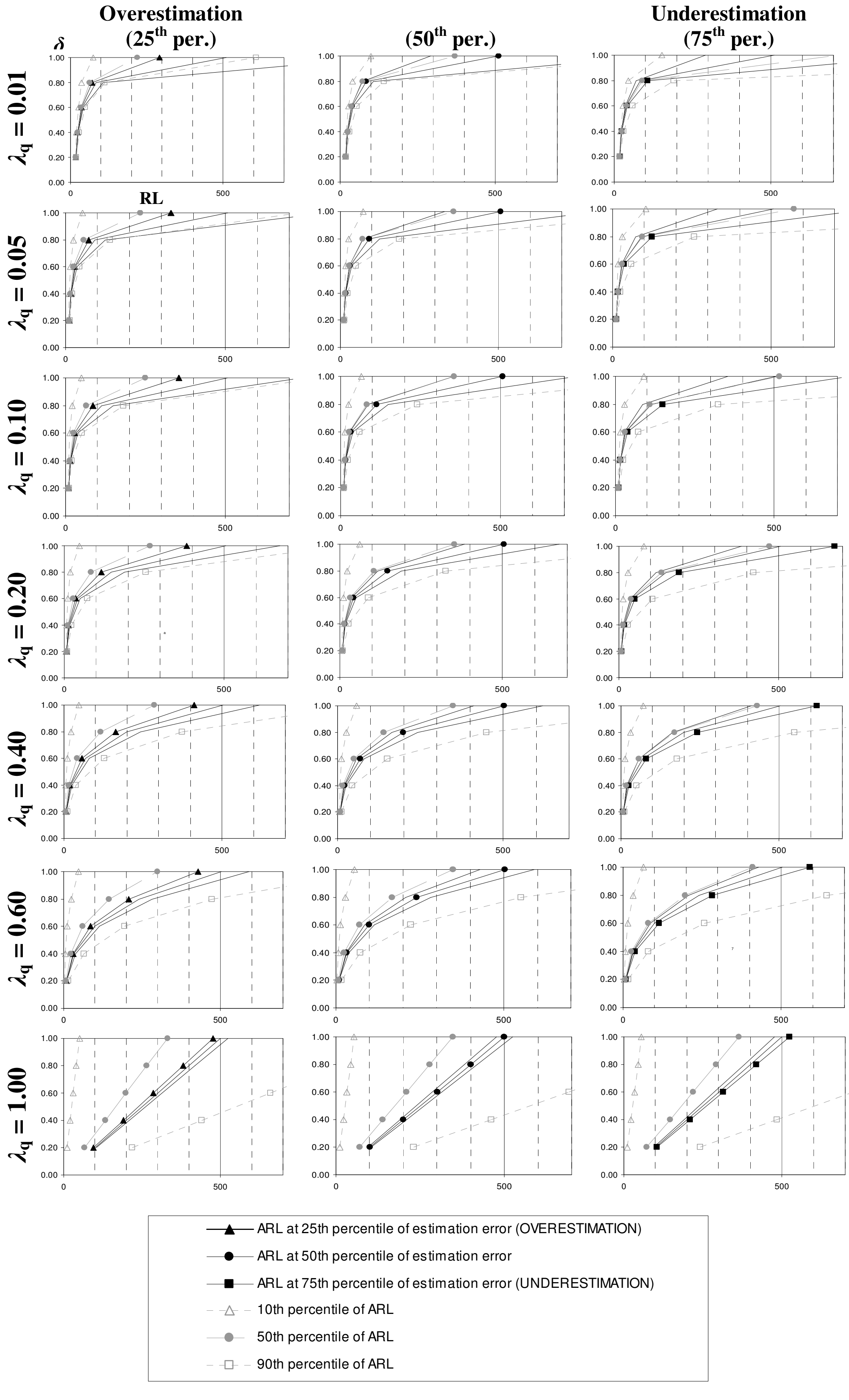


Figure 4.11 – (*continued*), sample size n = 200.

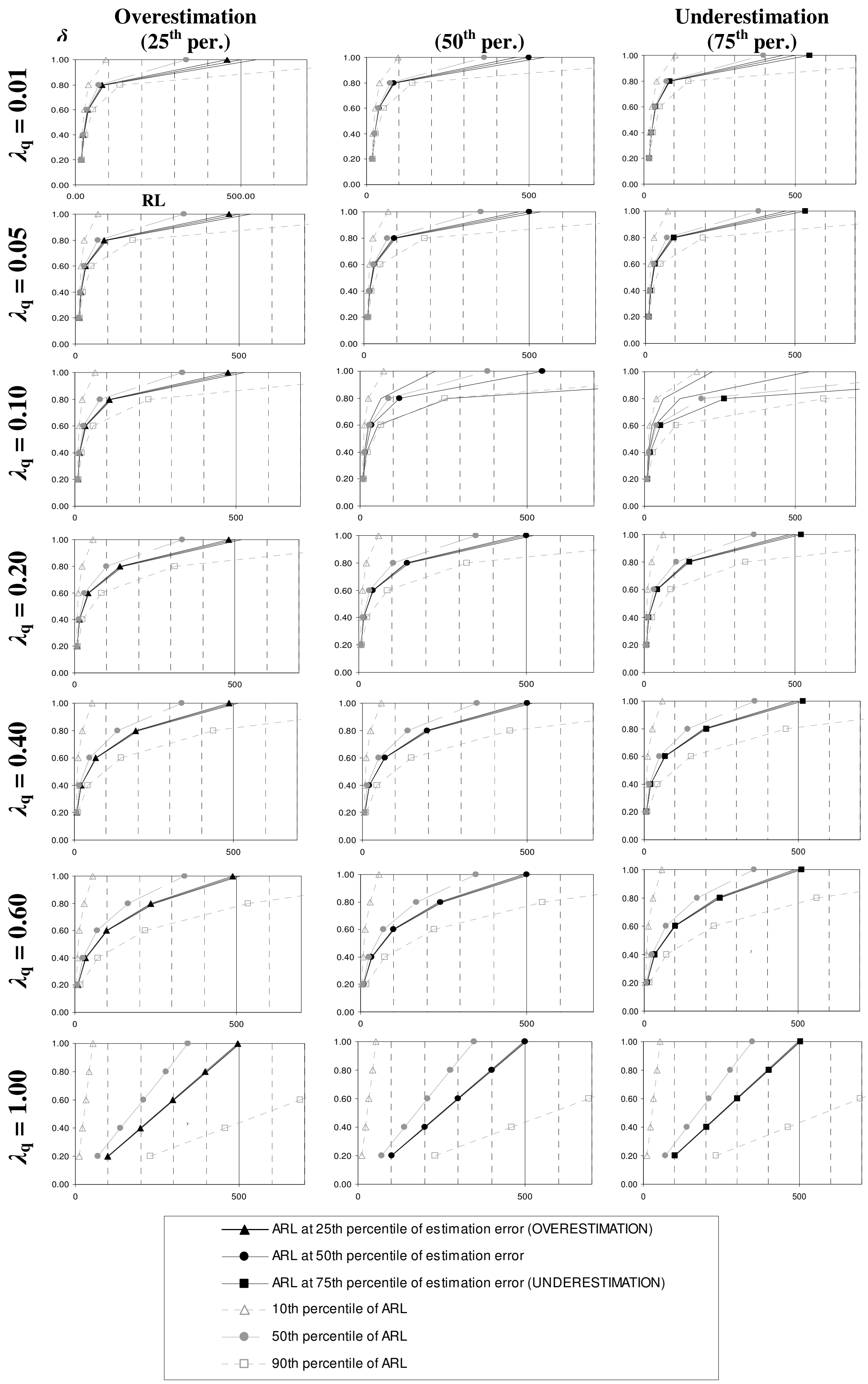


Figure 4.11 – (*continued*), sample size n = 10000.

CHAPTER 5

MARGINAL PERFORMANCE RESULTS

5.1. Marginal Performance Results

Recall that in Section 3.4 the marginal performance measures; ARL, SDRL and percentiles of the RL distribution, are defined as:

$$\text{ARL}_{\text{marginal}} = \int_a^b \text{ARL}(w, \delta, Z_0) f_W(w) dw,$$

$$\text{SDRL}_{\text{marginal}} = \int_a^b \text{SDRL}(w, \delta, Z_0) f_W(w) dw,$$

$$\text{Percentile}_{\text{marginal}} = \int_a^b \text{Percentile}(w, \delta, Z_0) f_W(w) dw.$$

In addition to the conditional analysis in Chapter 4, the performance measures are integrated here using the distribution of the *ratio of estimation error* ($w = \frac{\mu_0}{\hat{\mu}_0}$). Hence, marginal effect of the distribution of the estimation error on the chart performance can be obtained. The distribution function of w for various sample sizes (n) is represented in Figure 4.1.

While evaluating marginal performance of the RL distribution, the same set of parameters explained in Section 4.1 is used, except W since it is already being used in integration.

5.2. In-Control Case

Marginal performances of the ARL_0, SDRL, and percentiles of the RL distribution of the control chart for the *in-control* case (δ = 1) are provided in Table 5.1. Some extreme cases could not be calculated due to extensively long computation times and these are marked with an asterix (*). However, calculated values show a regular pattern and the ones that are not calculated are expected to follow that pattern.

When the marginal ARL_0 performance is considered, the effect of sample size on ARL_0 is less for greater choices of λ_q, and more for smaller choices of λ_q. When the sample size is 10000 (approximately the known parameter case), the marginal ARL_0 values are near 410 for $\lambda_q = 1$ and near 430 for $\lambda_q = 0.01$. The smaller sample sizes result in longer ARL_0 values. When the sample size is 30, the ARL_0 values are near 488 for $\lambda_q = 1$. For the smaller values of λ_q, ARL_0 reaches several thousands. Hence, a control chart would yield less frequent *false alarms* for a process designed for an ARL_0 of 500 for small sample sizes and small λ_q.

When $\lambda_q = 1$, there is almost no effect of parameter B and nearly the same ARL_0 values are obtained. The effect of the boundary parameter B is also more for small sample sizes and small λ_q. When $B \in \{2, 5\}$, the ARL_0 values obtained are close to each other and greater than the ones obtained from $B = 1$.

The marginal SDRL values are quite close to the ARL_0 values, which implies that the variation of the RL distribution increases for smaller choices of sample size and λ_q. The smallest ARL_0 and SDRL couple is 410.48 and 410.07, respectively; whose variation is large enough for making ARL_0 insufficient as a sole measure of performance. The effect of the SDRL can be observed on the percentiles of the RL distribution represented in Figure 5.1.

Table 5.1 – Marginal performances of the ARL_0, SDRL and the percentiles of RL distribution, for an *in-control* ($\delta = 1$) process.

	λ_q	0.01					0.05					0.10					0.20				
	S.Size			**Percentiles of RL**					**Percentiles of RL**					**Percentiles of RL**					**Percentiles of RL**		
B	**(*n*)**	**ARL**	**SDRL**	**10th**	**50th**	**90th**	**ARL**	**SDRL**	**10th**	**50th**	**90th**	**ARL**	**SDRL**	**10th**	**50th**	**90th**	**ARL**	**SDRL**	**10th**	**50th**	**90th**
1	**30**	*	*	*	*	*	2,413.11	2,393.86	271.46	1,678.69	5,531.29	1,319.83	1,307.29	150.36	918.73	3,022.66	870.21	862.31	98.81	605.66	1,993.42
	50	4,233.42	4,188.65	485.30	*	*	1,226.26	1,206.78	146.63	856.06	2,798.21	864.89	852.27	102.59	603.42	1,975.04	678.42	670.50	78.52	472.66	1,551.87
	100	1,319.61	1,273.40	179.79	929.66	2,978.13	753.27	733.51	97.02	528.20	1,708.70	637.69	624.95	78.63	445.92	1,451.64	566.91	558.94	66.91	395.37	1,294.90
	200	768.58	721.62	122.63	547.99	1,708.33	595.87	576.00	80.62	419.24	1,346.18	549.64	536.86	69.36	384.93	1,248.95	518.91	510.93	61.90	362.12	1,184.43
	500	582.35	534.74	103.75	418.85	1,278.66	526.55	506.56	73.47	371.15	1,186.38	508.90	496.06	65.25	356.61	1,155.02	496.67	488.65	59.51	346.80	1,133.16
	1,000	519.46	471.83	97.23	375.35	1,133.91	496.05	476.46	69.86	349.89	*	488.67	475.87	62.95	342.68	1,108.53	483.50	475.51	58.12	337.66	1,102.83
	10,000	424.25	382.99	81.68	307.28	922.93	419.72	402.50	59.79	296.29	944.00	417.56	406.51	53.91	292.86	947.15	415.95	409.06	49.98	290.35	948.81
2	**30**	*	*	*	*	*	19,780.08	*	*	*	*	3,284.80	3,279.11	351.06	2,278.70	7,556.08	1,257.67	1,251.88	137.68	873.47	2,888.37
	50	2,218,013.79	2,221,424.48	*	*	*	3,483.39	3,481.04	368.90	2,415.21	8,017.68	1,387.14	1,379.58	152.83	963.79	3,184.20	830.26	824.16	93.08	577.36	1,903.70
	100	13,428.10	*	*	*	*	1,145.16	1,136.71	128.10	796.33	2,625.79	783.07	774.30	90.34	545.48	1,791.60	622.53	616.17	71.24	433.52	1,425.14
	200	1,847.76	1897.67	*	*	*	727.72	717.14	86.09	507.63	1,661.85	611.31	602.09	72.70	426.66	1,395.63	546.64	540.19	63.33	380.92	1,250.24
	500	749.43	763.36	77.65	508.53	1,744.55	565.44	553.55	70.33	395.54	1,286.51	529.27	519.71	64.30	369.79	1,206.20	506.33	499.78	59.23	352.98	1,157.33
	1,000	575.26	579.75	68.95	390.81	1,331.11	514.88	502.92	65.05	360.48	1,170.00	497.65	488.00	61.08	347.90	1,133.29	487.82	481.27	57.29	340.13	1,114.74
	10,000	432.09	431.88	57.00	293.66	995.18	423.43	412.65	54.54	296.66	960.95	420.10	411.75	51.65	293.77	956.41	417.47	411.81	49.03	291.10	953.84
5	**30**	32,042,505.81	*	*	*	*	19,220.49	*	*	*	*	3,341.60	3,336.62	*	*	*	1,299.63	1,294.07	141.98	902.52	2,985.29
	50	115,846.66	*	*	*	*	3.416.92	3415.61	*	*	*	1,390.80	1,383.75	152.75	966.21	3,193.28	841.18	835.22	94.02	584.86	1,929.11
	100	3,668.01	3,681.32	*	*	*	1,135.41	1,127.80	126.31	789.25	2,604.46	782.84	774.50	89.91	545.10	1,791.72	625.17	618.91	71.44	435.28	1,431.43
	200	1,106.10	1,092.47	129.05	770.19	2,529.22	724.73	714.90	85.09	505.25	1,656.04	611.04	602.21	72.20	426.20	1,395.42	547.39	541.04	63.30	381.32	1,252.20
	500	641.72	617.38	90.96	451.66	1,445.98	564.40	553.24	69.42	394.57	1,285.02	529.11	519.92	64.03	369.59	1,206.32	506.32	499.86	59.17	352.96	1,157.36
	1,000	539.90	512.48	83.30	382.07	1,207.51	514.27	503.03	64.28	359.78	1,169.47	497.54	488.26	60.86	347.70	1,133.49	487.59	481.12	57.27	340.00	1,114.28
	10,000	426.92	401.78	69.40	303.07	950.34	423.16	413.00	53.76	296.33	961.11	420.03	411.99	51.56	293.61	956.74	417.16	411.57	48.93	290.84	953.22

Table 5.1 – (*continued*)

	λ_q	0.40					0.60					1.00				
	S.Size			**Percentiles of RL**					**Percentiles of RL**					**Percentiles of RL**		
B	**(*n*)**	**ARL**	**SDRL**	**10th**	**50th**	**90th**	**ARL**	**SDRL**	**10th**	**50th**	**90th**	**ARL**	**SDRL**	**10th**	**50th**	**90th**
1	**30**	657.32	652.60	73.43	457.11	1,507.41	582.06	578.78	64.25	404.46	1,335.97	488.43	487.95	51.85	338.71	1,123.99
	50	574.82	570.10	64.83	399.93	1,317.32	536.31	533.02	59.53	372.73	1,230.49	483.00	482.53	51.26	334.94	1,111.59
	100	522.30	517.55	59.23	363.47	1,196.42	504.86	501.57	56.12	350.92	1,158.17	479.00	478.52	50.83	332.15	1,102.32
	200	497.90	493.15	56.82	346.60	1,140.25	489.74	486.45	54.52	340.46	1,123.35	476.90	476.43	50.65	330.66	1,097.49
	500	483.84	479.08	55.22	336.91	1,107.91	480.89	477.60	53.64	334.31	1,102.97	475.65	475.17	50.55	329.90	1,094.58
	1,000	479.06	474.31	54.73	333.60	1,096.81	477.80	474.51	53.30	332.17	1,095.84	475.05	474.57	50.48	329.40	1,093.21
	10,000	413.67	409.57	47.22	287.95	947.15	413.02	410.19	46.02	287.34	947.33	410.48	410.07	43.63	284.65	944.64
2	**30**	749.08	745.05	82.52	520.47	1,719.55	620.29	617.38	67.98	430.85	1,424.49	487.64	487.17	51.78	338.16	1,122.25
	50	619.63	615.56	69.03	430.73	1,421.37	555.11	552.18	61.00	385.56	1,274.28	481.42	480.95	51.25	333.85	1,107.92
	100	541.75	537.62	60.79	376.87	1,242.04	513.40	510.45	56.67	356.78	1,178.30	479.00	478.52	50.83	332.15	1,102.32
	200	509.15	505.01	57.42	354.18	1,166.92	494.85	491.90	54.79	343.86	1,135.64	476.37	475.90	50.71	330.34	1,096.32
	500	491.60	487.43	55.55	342.12	1,126.61	485.55	482.58	53.68	337.50	1,114.19	477.58	477.10	50.71	331.24	1,098.96
	1,000	481.23	477.06	54.30	334.86	1,102.57	478.40	475.44	53.10	332.54	1,097.77	475.05	474.57	50.48	329.40	1,093.21
	10,000	414.87	411.28	46.89	288.66	950.53	413.31	410.75	45.84	287.25	948.37	410.48	410.07	43.63	284.65	944.64
5	**30**	772.58	768.77	84.77	536.70	1,774.04	634.14	631.34	69.36	440.41	1,456.61	487.64	487.17	51.78	338.16	1,122.25
	50	629.55	625.66	69.81	437.59	1,444.54	561.88	559.06	61.70	390.32	1,290.15	481.42	480.95	51.25	333.85	1,107.92
	100	545.59	541.61	61.02	379.37	1,251.12	516.33	513.48	56.94	358.73	1,185.17	479.00	478.52	50.83	332.15	1,102.32
	200	510.93	506.94	57.42	355.36	1,171.24	496.32	493.46	54.84	344.85	1,139.06	476.37	475.90	50.71	330.34	1,096.32
	500	492.31	488.28	55.45	342.61	1,128.34	486.19	483.31	53.69	337.91	1,115.69	477.58	477.10	50.71	331.24	1,098.96
	1,000	481.55	477.53	54.23	334.97	1,103.61	478.72	475.84	53.00	332.65	1,098.57	475.05	474.57	50.48	329.40	1,093.21
	10,000	415.00	411.53	46.83	288.63	951.16	413.48	411.00	45.76	287.38	948.92	410.48	410.07	43.63	284.65	944.64

In Figure 5.1 the marginal ARL_0 values, together with 10^{th}, 50^{th} (median) and 90^{th} percentiles of RL distribution are provided. As in the conditional performance of the RL distribution explained in Chapter 4, similar to the marginal RL distribution is also right skewed. Note that for smaller sample sizes and smaller λ_q, the 10^{th} and 90^{th} percentiles are a little wider for $B = 2$.

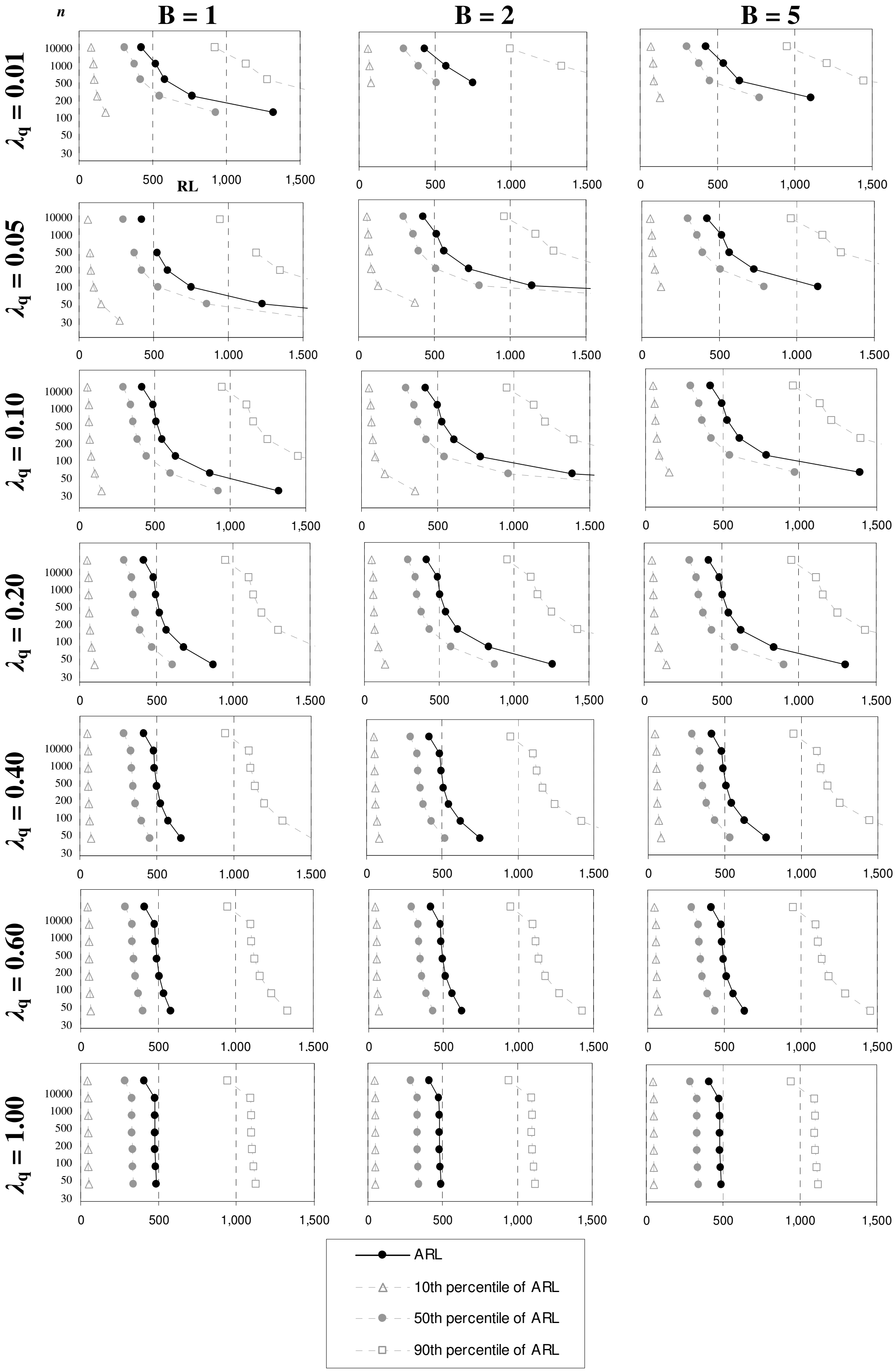


Figure 5.1 – Marginal performances of the ARL_0 and the percentiles of RL distribution.

5.3. Out-of-Control Case

The marginal performances of ARL, SDRL, and percentiles of the RL distribution are provided in Table 5.2, where the ARL_0 values are in **bold** and the results for $B = 1$ are represented in Figure 5.2.

In Table 5.2, the cells with the most favorable cases (the smallest ARL_1 values and the largest ARL_0 values) for each δ are highlighted with grey ( ). In all the cases, highest ARL_0 values are obtained for $\lambda_q = 0.01$ and for $\lambda_q \leq 0.20$ most of the ARL_0 values exceeds 500. However, the SDRL values also increase with high ARL_0 values and 10^{th} percentile of the RL distribution generally takes values between 100 and 200. So that, although a high ARL_0 is better, it does not imply that number of *false alarms* would be decreased that much.

When the mean shift is small, the lowest ARL_1 values are obtained for small λ_q. On the other hand, when the mean shift is larger, greater λ_q values are required. In general, for $\delta = 0.8$, $\lambda_q = 0.01$ or 0.05; for $\delta = 0.6$, $\lambda_q = 0.05$; for $\delta = 0.4$, $\lambda_q = 0.10$; and for $\delta = 0.2$, $\lambda_q = 0.40$ gives the lowest ARL_1 values. Note that the SDRL values the decrease more than decreasing ARL_1 values. Hence, a smaller ARL_1 implies faster detection of an *out-of-control* state and provides a more reliable measure of performance. Specifically for $\delta \leq 0.4$, the variation and skewness of the RL distribution decrease a lot so that ARL alone is enough for describing the performance.

Exceptionally, when $n = 30$, $B = 2$ and 5 yields the lowest ARL_1 values for $\lambda_q = 0.20$. In this case, the ARL_1 values increase drastically for $\lambda_q = 0.01$. For example, when $B = 1$, ARL_1 increases only up to 301.41. However for B = 2, ARL_1 reaches 1305.97, which is more than 500 and looks like a very poor performance. However, in this case ARL_0 values are extremely high that any alarm can be treated as a true *out-of-control* alarm. Hence, when small shifts such as $\delta = 0.8$ are important and the sample size is 30, the choice of parameter $B \in \{2, 5\}$

may introduce extreme behaviors far from the designed performance and using $B = 1$ yields safer results. For $\delta \leq 0.8$ or $n \geq 50$, marginal performances of $B = 2$ and 5 are not problematic and better than $B = 1$.

In general, the effect of the sample size on the marginal performance of the control chart is more for smaller choices of λ_q, and less for greater choices of λ_q. Also, the effect of the sample size is most significant for the mean shift of $\delta = 0.8$. For the mean shifts $\delta \leq 0.6$, the effect of the sample size becomes harder to distinguish.

To be more specific on small shifts, marginal ARL_1 performance for the mean shift with $\delta = 0.8$ and B = 2 is represented in Figure 5.3. Note that for $n \geq 100$, the improvement in ARL_1 is small. For n ≤ 100, there is a considerable improvement in ARL_1 especially for $\lambda_q = 0.01$, 0.05, and 0.10. For greater choices of λ_q, the effect of sample size on the marginal performance becomes smaller.

Hence, when detection of small mean shifts such as $\delta = 0.8$ are more important, choosing a sample size greater than 100 would provide a good ARL_1 performance for the control chart. For greater mean shifts such as $\delta \leq 0.6$, all sample sizes considered here provided a satisfactory performance, while, the results are slightly better for sample sizes greater than 100.

Table 5.2 – Marginal performances of the ARL, SDRL, and the percentiles of the RL distribution, for B = 1.

	λ_q	0.01					0.05					0.10					0.20				
				Per of the RL dist.					Per of the RL dist.					Per of the RL dist.					Per of the RL dist.		
n	δ	ARL	SDRL	10th	50th	90th	ARL	SDRL	10th	50th	90th	ARL	SDRL	10th	50th	90th	ARL	SDRL	10th	50th	90th
30	0.2	18.01	1.25	16.57	17.90	19.62	11.52	1.24	10.12	11.36	13.13	9.38	1.37	7.97	9.14	11.12	7.91	1.81	6.03	7.50	10.27
	0.4	25.17	4.15	20.41	24.66	30.62	17.72	4.79	12.68	16.78	23.95	16.15	6.17	10.10	14.64	24.23	17.14	9.80	8.08	14.37	29.83
	0.6	46.48	17.56	28.48	42.46	69.46	42.33	25.65	18.45	35.27	75.72	46.93	34.84	15.22	36.75	92.26	59.77	51.59	13.71	44.06	126.95
	0.8	301.41	262.96	63.69	222.55	643.87	229.06	210.17	40.55	164.94	502.84	284.89	271.69	41.64	201.76	638.74	268.16	259.73	35.77	188.62	606.46
	1.0	*	*	*	*	*	**2,413.11**	**2,393.86**	**271.46**	**1,678.69**	**5,531.29**	**1,319.83**	**1,307.29**	**150.36**	**918.73**	**3,022.66**	**870.21**	**862.31**	**98.81**	**605.66**	**1,993.42**
50	0.2	17.93	1.22	16.53	17.82	19.53	11.46	1.21	10.05	11.28	13.01	9.31	1.33	7.93	9.10	11.00	7.82	1.74	5.97	7.44	10.00
	0.4	24.98	3.96	20.46	24.42	30.20	17.43	4.49	12.67	16.50	23.32	15.72	5.70	10.01	14.40	23.09	16.39	9.00	8.03	13.86	28.14
	0.6	41.62	13.37	27.48	39.00	59.20	35.90	19.31	17.56	30.75	61.03	39.28	27.18	14.48	31.42	74.70	50.86	42.66	12.70	37.82	106.41
	0.8	141.44	103.03	46.55	111.83	275.56	145.57	126.52	31.82	107.22	351.02	159.68	146.93	28.22	114.65	351.02	184.80	176.70	26.84	130.56	415.08
	1.0	**4,233.42**	**4,188.65**	**485.30**	*	*	**1,226.26**	**1,206.78**	**146.63**	**856.06**	**2,798.21**	**864.89**	**852.27**	**102.59**	**603.42**	**1,975.04**	**678.42**	**670.50**	**78.52**	**472.66**	**1,551.87**
100	0.2	17.88	1.20	16.52	17.75	19.42	11.41	1.18	9.96	11.19	12.91	9.26	1.30	7.92	9.04	10.93	7.76	1.69	5.94	7.34	9.94
	0.4	24.65	3.79	20.25	24.11	29.65	17.09	4.23	12.57	16.25	22.64	15.29	5.32	9.95	14.06	22.20	15.72	8.34	7.93	13.40	26.59
	0.6	39.38	11.62	26.90	37.21	54.71	32.77	16.28	16.98	28.61	53.96	35.21	23.12	13.96	28.52	65.31	45.60	37.39	12.05	34.28	94.29
	0.8	99.00	60.61	41.80	82.53	177.71	109.86	90.60	28.39	82.59	227.91	128.46	115.57	24.98	93.12	279.00	159.20	151.03	24.20	112.87	355.84
	1.0	**1,319.61**	**1,273.40**	**179.79**	**929.66**	**2,978.13**	**753.27**	**733.51**	**97.02**	**528.20**	**1,708.70**	**637.69**	**624.95**	**78.63**	**445.92**	**1,451.64**	**566.91**	**558.94**	**66.91**	**395.37**	**1,294.90**
200	0.2	17.67	1.18	16.46	17.58	19.19	11.27	1.16	9.81	11.00	12.76	9.14	1.27	7.84	8.87	10.77	7.65	1.65	5.88	7.24	9.76
	0.4	24.51	3.71	20.23	24.00	29.41	16.94	4.12	12.51	16.16	22.35	15.10	5.15	9.92	13.88	21.84	15.42	8.04	7.94	13.19	25.89
	0.6	38.49	10.98	26.54	36.43	53.12	31.55	15.12	16.82	27.88	51.20	33.56	21.47	13.78	27.40	61.56	43.34	35.12	11.81	32.64	89.04
	0.8	87.98	49.79	40.16	74.95	152.68	97.68	78.39	27.24	74.12	199.71	116.62	103.73	23.67	84.95	251.67	148.66	140.50	22.97	105.49	331.63
	1.0	**768.58**	**721.62**	**122.63**	**547.99**	**1,708.33**	**595.87**	**576.00**	**80.62**	**419.24**	**1,346.18**	**549.64**	**536.86**	**69.36**	**384.93**	**1,248.95**	**518.91**	**510.93**	**61.90**	**362.12**	**1,184.43**
500	0.2	17.65	1.17	16.52	17.62	19.15	11.26	1.15	9.80	10.89	12.76	9.13	1.26	7.84	8.82	10.78	7.63	1.64	5.88	7.14	9.76
	0.4	24.17	3.63	19.97	23.67	28.99	16.68	4.01	12.40	15.96	21.93	14.84	5.00	9.81	13.67	21.37	15.09	7.79	7.84	12.95	25.20
	0.6	38.01	10.64	26.43	36.10	52.09	30.90	14.50	16.68	27.29	49.70	32.67	20.58	13.67	26.85	59.45	42.09	33.86	11.71	31.78	86.13
	0.8	83.10	44.87	39.57	71.66	141.49	91.55	72.12	26.49	69.94	185.44	110.38	97.40	23.30	80.75	237.24	143.01	134.80	22.53	101.63	318.70
	1.0	**582.35**	**534.74**	**103.75**	**418.85**	**1,278.66**	**526.55**	**506.56**	**73.47**	**371.15**	**1,186.38**	**508.90**	**496.06**	**65.25**	**356.61**	**1,155.02**	**496.67**	**488.65**	**59.51**	**346.80**	**1,133.16**
10,000	0.2	17.61	1.16	16.63	17.61	19.29	11.23	1.14	9.78	10.76	12.72	9.11	1.25	7.83	8.80	10.76	7.61	1.63	5.87	6.85	9.78
	0.4	24.07	3.60	19.67	23.48	28.83	16.60	3.97	12.68	15.65	21.77	14.75	4.94	9.78	13.69	21.19	14.97	7.68	7.83	12.73	24.93
	0.6	37.66	10.42	26.33	35.80	51.40	30.46	14.12	16.63	26.96	48.83	32.08	20.02	13.69	26.39	58.11	41.26	33.04	11.73	31.24	84.18
	0.8	69.36	36.51	33.65	60.25	116.83	76.00	59.26	22.52	58.24	153.19	92.11	80.93	19.71	67.37	197.45	120.42	113.36	19.07	85.62	268.12
	1.0	**424.25**	**382.99**	**81.68**	**307.28**	**922.93**	**419.72**	**402.50**	**59.79**	**296.29**	**944.00**	**417.56**	**406.51**	**53.91**	**292.86**	**947.15**	**415.95**	**409.06**	**49.98**	**290.35**	**948.81**

Table 5.2 – (*continued*), B = 1.

	λ_q	**0.40**					**0.60**					**1.00**				
				Per of the RL dist.					**Per of the RL dist.**					**Per of the RL dist.**		
n	δ	**ARL**	**SDRL**	**10th**	**50th**	**90th**	**ARL**	**SDRL**	**10th**	**50th**	**90th**	**ARL**	**SDRL**	**10th**	**50th**	**90th**
30	**0.2**	7.79	3.41	4.69	6.84	12.22	9.86	6.52	4.05	7.89	18.37	102.52	102.03	11.22	71.32	235.38
	0.4	24.71	19.82	6.97	18.82	50.47	37.79	34.28	7.10	27.24	82.49	202.37	201.88	21.66	140.42	465.33
	0.6	87.55	82.52	13.67	62.23	195.02	117.69	114.18	15.78	82.72	266.47	303.32	302.83	32.42	210.45	697.79
	0.8	279.51	274.47	33.96	195.28	636.96	299.66	296.17	34.74	208.74	685.40	408.60	408.11	43.51	283.36	940.15
	1.0	**657.32**	**652.6**	**73.43**	**457.11**	**1,507.41**	**582.06**	**578.78**	**64.25**	**404.46**	**1,335.97**	**488.43**	**487.95**	**51.85**	**338.71**	**1,123.99**
50	**0.2**	7.62	3.25	4.72	6.74	11.85	9.56	6.22	4.01	7.69	17.62	101.23	100.74	11.11	70.30	232.40
	0.4	23.37	18.45	6.86	17.81	47.40	36.01	32.47	7.11	26.03	78.41	201.97	201.47	21.74	140.06	464.48
	0.6	78.02	72.97	12.77	55.60	173.04	108.49	104.98	14.55	76.30	245.17	299.56	299.06	31.98	207.83	689.07
	0.8	223.32	218.49	27.85	156.22	507.96	255.76	252.42	29.97	178.34	584.54	385.23	384.76	40.99	267.21	886.39
	1.0	**574.82**	**570.1**	**64.83**	**399.93**	**1,317.32**	**536.31**	**533.02**	**59.53**	**372.73**	**1,230.49**	**483.00**	**482.53**	**51.26**	**334.94**	**1,111.59**
100	**0.2**	7.51	3.14	4.78	6.62	11.58	9.36	6.01	3.97	7.59	17.10	100.27	99.78	11.03	69.68	230.24
	0.4	22.17	17.24	6.73	16.88	44.61	34.32	30.78	6.81	24.87	74.40	200.05	199.55	21.50	138.79	460.08
	0.6	71.83	66.78	12.09	51.32	158.84	102.23	98.72	13.97	71.95	230.83	296.82	296.33	31.69	205.91	682.82
	0.8	204.54	199.68	25.86	143.24	464.73	241.56	238.20	28.42	168.52	551.74	399.60	399.11	42.53	277.12	919.42
	1.0	**522.30**	**517.55**	**59.23**	**363.47**	**1,196.42**	**504.86**	**501.57**	**56.12**	**350.92**	**1,158.17**	**479.00**	**478.52**	**50.83**	**332.15**	**1,102.32**
200	**0.2**	7.38	3.06	4.82	6.63	11.34	9.16	5.85	3.92	7.42	16.65	98.80	98.31	10.87	68.59	226.86
	0.4	21.62	16.68	6.75	16.55	43.35	33.52	29.97	6.71	24.35	72.40	199.18	198.69	21.47	138.21	457.97
	0.6	69.01	63.96	11.79	49.38	152.28	99.29	95.78	13.68	70.01	224.06	295.43	294.94	31.60	204.89	679.63
	0.8	196.28	191.43	25.03	137.52	445.62	235.01	231.66	27.76	163.94	536.73	381.19	380.72	40.59	264.36	877.10
	1.0	**497.9**	**493.15**	**56.82**	**346.6**	**1,140.25**	**489.74**	**486.45**	**54.52**	**340.46**	**1,123.35**	**476.9**	**476.43**	**50.65**	**330.66**	**1,097.49**
500	**0.2**	7.35	3.03	4.90	6.61	11.28	9.11	5.80	3.92	7.34	16.59	98.53	98.04	10.82	68.44	226.25
	0.4	21.07	16.19	6.61	16.14	42.15	32.71	29.19	6.62	23.78	70.76	196.57	196.08	21.09	136.49	452.06
	0.6	67.40	62.34	11.65	48.32	148.56	97.58	94.06	13.41	68.69	220.14	294.60	294.11	31.46	204.35	677.74
	0.8	192.04	187.17	24.57	134.58	435.78	232.01	228.64	27.49	161.79	529.86	396.17	395.68	42.14	274.79	911.54
	1.0	**483.84**	**479.08**	**55.22**	**336.91**	**1,107.91**	**480.89**	**477.6**	**53.64**	**334.31**	**1,102.97**	**475.65**	**475.17**	**50.55**	**329.9**	**1,094.58**
10,000	**0.2**	7.32	3.01	4.89	6.85	11.12	9.06	5.75	3.91	7.29	16.58	98.19	97.70	10.76	68.24	225.45
	0.4	20.85	15.98	6.85	16.04	41.65	32.38	28.88	6.85	23.55	69.96	195.89	195.40	21.11	135.96	450.39
	0.6	66.30	61.25	11.63	47.53	146.07	96.36	92.85	13.33	67.86	217.40	293.59	293.10	31.35	203.70	675.37
	0.8	163.22	159.03	20.98	114.31	370.34	198.10	195.20	23.48	138.06	452.36	328.46	328.05	34.98	227.76	755.81
	1.0	**413.67**	**409.57**	**47.22**	**287.95**	**947.15**	**413.02**	**410.19**	**46.02**	**287.34**	**947.33**	**410.48**	**410.07**	**43.63**	**284.65**	**944.64**

Table 5.2 – (*continued*), for B = 2.

	λ_q	0.01					0.05					0.10					0.20				
				Per of the RL dist.					Per of the RL dist.					Per of the RL dist.					Per of the RL dist.		
n	***δ***	**ARL**	**SDRL**	**10th**	**50th**	**90th**	**ARL**	**SDRL**	**10th**	**50th**	**90th**	**ARL**	**SDRL**	**10th**	**50th**	**90th**	**ARL**	**SDRL**	**10th**	**50th**	**90th**
30	**0.2**	14.00	1.20	12.63	13.93	15.52	10.15	1.12	8.92	9.99	11.62	8.52	1.24	7.10	8.30	10.13	7.26	1.61	5.78	6.90	9.35
	0.4	19.82	3.76	15.60	19.20	24.83	15.73	4.33	11.16	14.94	21.44	14.61	5.44	9.18	13.25	21.72	15.30	8.39	7.48	13.01	26.16
	0.6	36.25	15.26	20.95	32.59	56.20	35.82	21.32	15.72	30.01	63.57	39.72	28.63	13.50	31.34	77.03	50.46	42.75	11.95	37.62	106.19
	0.8	1,305.97	1,312.86	130.78	902.58	3,016.27	279.03	264.45	41.78	198.41	623.39	232.94	221.93	34.32	165.05	521.96	224.61	217.30	30.16	158.02	507.78
	1.0	*	*	*	*	*	**19,780.08**	*	*	*	*	**3,284.80**	**3,279.11**	**351.06**	**2,278.70**	**7,556.08**	**1,257.67**	**1,251.88**	**137.68**	**873.47**	**2,888.37**
50	**0.2**	13.94	1.17	12.65	13.85	15.44	10.09	1.09	8.88	9.92	11.49	8.46	1.21	7.02	8.24	10.02	7.19	1.55	5.84	6.84	9.21
	0.4	19.43	3.53	15.47	18.86	24.12	15.29	3.99	11.03	14.50	20.55	14.05	4.94	9.09	12.88	20.60	14.45	7.55	7.36	12.49	24.26
	0.6	32.32	11.62	20.11	29.86	47.70	30.42	15.99	15.04	26.33	51.38	33.11	22.01	12.78	26.77	61.72	42.38	34.64	11.24	32.02	87.52
	0.8	136.87	115.19	33.37	102.03	286.97	134.03	118.08	27.57	98.37	287.85	145.20	133.72	25.33	104.38	319.39	167.96	160.51	24.30	118.74	377.02
	1.0	**2,218,013.79**	**2,221,424.48**	*	*	*	**3,483.39**	**3,481.04**	**368.90**	**2,415.21**	**8,017.68**	**1,387.14**	**1,379.58**	**152.83**	**963.79**	**3,184.20**	**830.26**	**824.16**	**93.08**	**577.36**	**1,903.70**
100	**0.2**	13.90	1.15	12.69	13.79	15.49	10.05	1.07	8.82	9.89	11.42	8.42	1.18	6.95	8.15	9.94	7.14	1.51	5.88	6.85	9.09
	0.4	19.18	3.38	15.34	18.59	23.62	15.00	3.77	10.98	14.23	20.01	13.70	4.64	9.03	12.60	19.68	13.92	7.03	7.27	12.00	23.00
	0.6	30.57	10.16	19.78	28.54	44.13	27.93	13.61	14.57	24.53	45.60	29.82	18.74	12.32	24.63	54.17	37.90	30.13	10.78	28.80	77.06
	0.8	78.81	52.93	29.70	63.76	147.73	92.56	75.88	23.83	69.95	191.39	109.39	97.57	21.94	79.64	236.44	138.49	130.90	21.39	98.31	308.96
	1.0	**13,428.10**	*	*	*	*	**1,145.16**	**1,136.71**	**128.10**	**796.33**	**2,625.79**	**783.07**	**774.30**	**90.34**	**545.48**	**1,791.60**	**622.53**	**616.17**	**71.24**	**433.52**	**1,425.14**
200	**0.2**	13.88	1.14	12.74	13.75	15.32	10.03	1.06	8.82	9.82	11.40	8.40	1.17	6.88	8.06	9.90	7.11	1.49	5.88	6.86	9.04
	0.4	19.07	3.31	15.32	18.61	23.44	14.88	3.68	10.93	14.19	19.74	13.54	4.50	8.97	12.54	19.40	13.68	6.80	7.19	11.84	22.49
	0.6	29.88	9.62	19.60	28.00	42.53	26.97	12.71	14.41	23.92	43.63	28.52	17.45	12.20	23.67	51.17	36.02	28.25	10.57	27.46	72.74
	0.8	68.84	42.40	28.54	57.31	124.00	80.74	63.87	22.70	61.82	163.90	97.31	85.40	20.65	71.44	208.48	127.15	119.52	20.29	90.41	282.81
	1.0	**1,847.76**	**1,897.67**	*	*	*	**727.72**	**717.14**	**86.09**	**507.63**	**1,661.85**	**611.31**	**602.09**	**72.70**	**426.66**	**1,395.63**	**546.64**	**540.19**	**63.33**	**380.92**	**1,250.24**
500	**0.2**	13.86	1.14	12.74	13.72	15.38	10.02	1.06	8.82	9.80	11.44	8.39	1.16	6.86	7.95	9.84	7.10	1.48	5.88	6.86	8.98
	0.4	18.97	3.27	15.23	18.48	23.29	14.79	3.62	10.89	14.10	19.51	13.43	4.42	8.91	12.43	19.18	13.53	6.66	7.09	11.74	22.19
	0.6	29.50	9.33	19.50	27.65	41.81	26.46	12.24	14.32	23.45	42.42	27.82	16.76	12.11	23.21	49.53	35.00	27.21	10.47	27.00	70.36
	0.8	64.77	38.05	28.27	54.84	114.37	75.18	58.11	22.27	58.07	150.82	91.22	79.18	20.17	67.10	194.35	121.16	113.46	19.72	86.40	268.97
	1.0	**749.43**	**763.36**	**77.65**	**508.53**	**1,744.55**	**565.44**	**553.55**	**70.33**	**395.54**	**1,286.51**	**529.27**	**519.71**	**64.30**	**369.79**	**1,206.20**	**506.33**	**499.78**	**59.23**	**352.98**	**1,157.33**
10,000	**0.2**	13.83	1.13	12.72	13.69	15.65	9.99	1.05	8.80	9.78	11.73	8.37	1.15	6.85	7.83	9.78	7.08	1.47	5.87	6.85	8.80
	0.4	15.71	2.70	12.40	15.57	19.31	12.24	2.99	9.01	11.47	16.33	11.11	3.64	7.37	10.43	15.79	11.18	5.49	5.74	9.83	18.28
	0.6	29.23	9.15	19.55	27.43	41.34	26.11	11.94	14.32	23.22	41.70	27.36	16.33	11.81	22.88	48.54	34.33	26.55	10.54	26.36	68.91
	0.8	53.99	30.95	24.06	46.03	94.36	62.24	47.50	18.94	48.24	123.95	75.74	65.35	17.11	55.91	160.83	101.49	94.86	16.68	72.42	225.04
	1.0	**432.09**	**431.88**	**57.00**	**293.66**	**995.18**	**423.43**	**412.65**	**54.54**	**296.66**	**960.95**	**420.10**	**411.75**	**51.65**	**293.77**	**956.41**	**417.47**	**411.81**	**49.03**	**291.10**	**953.84**

Table 5.2 – (*continued*), for B = 2.

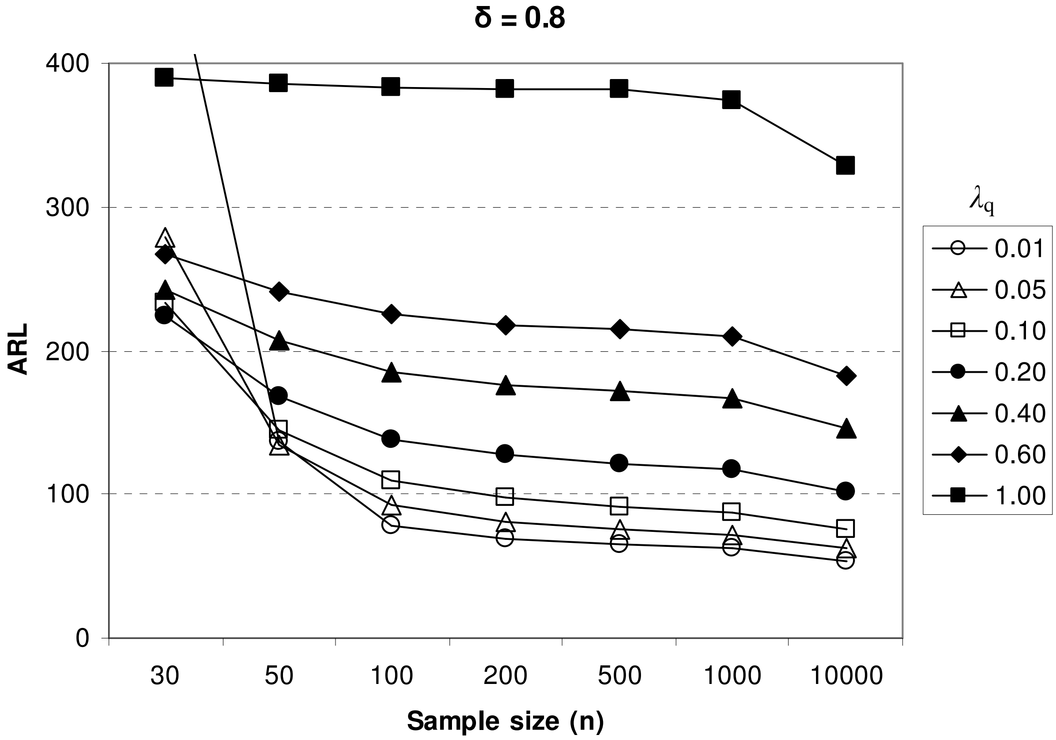
	λ_q	0.40					0.60					1.00				
				Per of the RL dist.					Per of the RL dist.					Per of the RL dist.		
n	δ	ARL	SDRL	10^{th}	50^{th}	90^{th}	ARL	SDRL	10^{th}	50^{th}	90^{th}	ARL	SDRL	10^{th}	50^{th}	90^{th}
30	0.2	7.11	2.96	4.27	6.32	10.90	8.87	5.67	3.89	7.16	16.24	101.43	100.94	11.09	70.49	232.93
	0.4	21.71	16.99	6.44	16.55	43.80	33.53	30.10	6.58	24.32	72.67	204.55	204.05	22.00	142.00	470.29
	0.6	75.97	71.15	12.37	54.12	168.57	105.36	101.99	14.14	74.08	238.26	303.32	302.83	32.42	210.45	697.79
	0.8	242.96	238.49	29.70	169.77	553.65	267.16	264.05	30.89	186.07	611.06	390.21	389.73	41.64	270.71	897.93
	1.0	**749.08**	**745.05**	**82.52**	**520.47**	**1,719.55**	**620.29**	**617.38**	**67.98**	**430.85**	**1,424.49**	**487.64**	**487.17**	**51.78**	**338.16**	**1,122.25**
50	0.2	6.98	2.84	4.20	6.19	10.91	8.63	5.43	3.90	7.05	15.73	100.18	99.69	11.04	69.62	230.01
	0.4	20.18	15.45	6.33	15.44	40.31	31.33	27.90	6.39	22.83	67.73	201.97	201.47	21.74	140.06	464.48
	0.6	66.72	61.88	11.37	47.80	147.36	96.10	92.72	13.14	67.68	216.84	299.56	299.06	31.98	207.83	689.07
	0.8	207.26	202.76	25.80	145.04	471.47	241.69	238.56	28.25	168.56	552.48	385.23	384.76	40.99	267.21	886.39
	1.0	**619.63**	**615.56**	**69.03**	**430.73**	**1,421.37**	**555.11**	**552.18**	**61.00**	**385.56**	**1,274.28**	**481.42**	**480.95**	**51.25**	**333.85**	**1,107.92**
100	0.2	6.89	2.75	4.12	6.14	10.45	8.47	5.26	3.92	6.91	15.33	99.27	98.78	10.89	69.05	227.93
	0.4	19.19	14.46	6.23	14.80	37.99	29.86	26.42	6.21	21.72	64.22	200.05	199.55	21.50	138.79	460.08
	0.6	60.94	56.08	10.81	43.74	134.02	89.94	86.55	12.41	63.39	202.74	296.82	296.33	31.69	205.91	682.82
	0.8	185.58	181.04	23.65	130.07	421.43	225.42	222.27	26.61	157.11	514.88	383.29	382.82	40.90	265.93	881.90
	1.0	**541.75**	**537.62**	**60.79**	**376.87**	**1,242.04**	**513.40**	**510.45**	**56.67**	**356.78**	**1,178.30**	**479.00**	**478.52**	**50.83**	**332.15**	**1,102.32**
200	0.2	6.85	2.71	4.03	6.08	10.31	8.39	5.18	3.92	6.83	15.06	98.80	98.31	10.87	68.59	226.86
	0.4	18.73	14.00	6.13	14.50	37.01	29.17	25.73	6.19	21.32	62.70	199.18	198.69	21.47	138.21	457.97
	0.6	58.37	53.50	10.51	42.01	128.01	87.09	83.70	12.24	61.35	196.14	295.43	294.94	31.60	204.89	679.63
	0.8	176.36	171.81	22.68	123.67	400.19	218.10	214.95	25.77	152.15	498.05	381.19	380.72	40.59	264.36	877.10
	1.0	**509.15**	**505.01**	**57.42**	**354.18**	**1,166.92**	**494.85**	**491.90**	**54.79**	**343.86**	**1,135.64**	**476.37**	**475.90**	**50.71**	**330.34**	**1,096.32**
500	0.2	6.82	2.69	3.94	5.96	10.28	8.34	5.14	3.92	6.83	15.00	98.53	98.04	10.82	68.44	226.25
	0.4	18.45	13.73	6.01	14.34	36.32	28.74	25.31	6.10	21.01	61.65	198.33	197.84	21.31	137.69	455.83
	0.6	56.91	52.04	10.38	41.03	124.68	85.44	82.05	12.05	60.28	192.25	294.60	294.11	31.46	204.35	677.74
	0.8	171.53	166.95	22.18	120.34	388.99	214.56	211.39	25.43	149.81	489.87	382.16	381.68	40.70	265.07	879.33
	1.0	**491.60**	**487.43**	**55.55**	**342.12**	**1,126.61**	**485.55**	**482.58**	**53.68**	**337.50**	**1,114.19**	**477.58**	**477.10**	**50.71**	**331.24**	**1,098.96**
10,000	0.2	6.79	2.67	3.91	5.87	10.16	8.30	5.10	3.91	6.85	14.80	98.19	97.70	10.76	68.24	225.45
	0.4	15.23	11.31	4.92	11.75	29.98	23.72	20.87	4.92	17.34	50.92	164.44	164.03	17.77	114.11	378.10
	0.6	55.93	51.06	10.24	40.25	122.41	84.28	80.90	11.85	59.42	189.71	293.59	293.10	31.35	203.70	675.37
	0.8	145.32	141.38	18.83	102.01	329.48	182.86	180.14	21.74	127.41	417.46	328.46	328.05	34.98	227.76	755.81
	1.0	**414.87**	**411.28**	**46.89**	**288.66**	**950.53**	**413.31**	**410.75**	**45.84**	**287.25**	**948.37**	**410.48**	**410.07**	**43.63**	**284.65**	**944.64**

Table 5.2 – (*continued*), for $B = 5$.

	λ_q	0.01					0.05					0.10					0.20				
				Per of the RL dist.					Per of the RL dist.					Per of the RL dist.					Per of the RL dist.		
S. size (n)	δ	ARL	SDRL	10^{th}	50^{th}	90^{th}	ARL	SDRL	10^{th}	50^{th}	90^{th}	ARL	SDRL	10^{th}	50^{th}	90^{th}	ARL	SDRL	10^{th}	50^{th}	90^{th}
30	0.2	13.92	2.42	11.12	13.71	17.11	10.11	1.19	8.88	9.97	11.62	8.47	1.26	7.04	8.25	10.11	7.26	1.62	5.78	6.90	9.35
	0.4	23.05	6.19	15.94	22.17	31.27	15.70	4.42	11.04	14.87	21.48	14.55	5.48	9.07	13.21	21.72	15.30	8.39	7.47	13.01	26.16
	0.6	44.35	20.16	23.54	40.02	70.72	35.88	21.55	15.43	30.00	64.01	39.70	28.76	13.43	31.24	77.14	50.36	42.67	11.95	37.50	105.92
	0.8	559.22	536.15	*	*	*	277.70	263.51	41.27	197.32	620.90	232.91	222.14	33.87	164.88	522.28	224.80	217.53	30.18	158.10	508.19
	1.0	**32,042,505.81**	*	*	*	*	**19,220.49**	*	*	*	*	**3,341.60**	**3,336.62**	*	*	*	**1,299.63**	**1,294.07**	**141.98**	**902.52**	**2,985.29**
50	0.2	13.81	2.38	11.04	13.51	16.99	10.06	1.16	8.83	9.90	11.55	8.41	1.22	6.95	8.16	10.00	7.19	1.56	5.82	6.84	9.21
	0.4	22.55	5.90	15.76	21.76	30.44	15.26	4.07	10.91	14.48	20.56	13.99	4.98	8.98	12.82	20.58	14.45	7.55	7.36	12.49	24.26
	0.6	39.73	16.06	22.54	36.63	60.79	30.46	16.20	14.82	26.33	51.63	33.08	22.12	12.62	26.75	61.86	42.32	34.58	11.21	31.98	87.33
	0.8	142.73	111.94	*	*	*	134.14	118.53	27.35	98.30	288.48	145.24	133.99	25.32	104.25	319.79	167.65	160.23	24.28	118.54	376.45
	1.0	**115,846.66**	*	*	*	*	**3.416.92**	**3,415.61**	*	*	*	**1,390.80**	**1,383.75**	**152.75**	**966.21**	**3,193.28**	**841.18**	**835.22**	**94.02**	**584.86**	**1,929.11**
100	0.2	13.74	2.35	10.98	13.54	16.86	10.02	1.13	8.82	9.87	11.47	8.37	1.20	6.89	8.09	9.92	7.14	1.52	5.88	6.85	9.11
	0.4	22.21	5.72	15.57	21.43	29.79	14.97	3.86	10.87	14.12	20.06	13.64	4.67	8.92	12.54	19.68	13.91	7.04	7.25	12.00	23.00
	0.6	38.07	14.55	22.19	35.41	57.25	28.38	14.08	14.58	24.86	46.89	30.28	19.22	12.39	24.92	55.27	38.49	30.67	10.86	29.15	78.41
	0.8	95.15	62.91	35.13	78.25	177.03	92.79	76.42	23.64	70.01	192.25	109.46	97.86	21.65	79.62	236.84	138.13	130.56	21.32	98.04	308.24
	1.0	**3,668.01**	**3,681.32**	*	*	*	**1,135.41**	**1,127.80**	**126.31**	**789.25**	**2,604.46**	**782.84**	**774.50**	**89.91**	**545.10**	**1,791.72**	**625.17**	**618.91**	**71.44**	**435.28**	**1,431.43**
200	0.2	13.70	2.34	10.89	13.39	16.84	10.00	1.12	8.82	9.81	11.44	8.35	1.19	6.86	8.00	9.86	7.11	1.50	5.88	6.86	9.04
	0.4	22.07	5.63	15.56	21.32	29.49	14.85	3.76	10.86	14.13	19.82	13.49	4.53	8.93	12.46	19.40	13.68	6.81	7.19	11.83	22.49
	0.6	37.13	13.81	22.07	34.65	55.36	27.32	13.07	14.44	24.13	44.35	28.85	17.79	12.21	23.89	51.91	36.45	28.60	10.67	27.86	73.67
	0.8	84.70	52.41	33.91	71.16	153.03	80.97	64.39	22.53	61.89	164.84	97.38	85.68	20.58	71.37	208.94	126.80	119.20	20.25	90.27	282.04
	1.0	**1,106.10**	**1,092.47**	**129.05**	**770.19**	**2,529.22**	**724.73**	**714.90**	**85.09**	**505.25**	**1,656.04**	**611.04**	**602.21**	**72.20**	**426.20**	**1,395.42**	**547.39**	**541.04**	**63.30**	**381.32**	**1,252.20**
500	0.2	13.68	2.33	10.80	13.40	16.81	9.98	1.12	8.82	9.80	11.50	8.33	1.18	6.86	7.89	9.83	7.10	1.49	5.88	6.86	8.98
	0.4	21.95	5.58	15.47	21.17	29.37	14.75	3.71	10.84	14.08	19.63	13.38	4.45	8.90	12.37	19.16	13.52	6.67	7.06	11.71	22.19
	0.6	36.58	13.42	21.92	34.19	54.32	26.74	12.56	14.31	23.64	43.07	28.08	17.05	12.11	23.41	50.31	35.33	27.50	10.49	27.11	71.03
	0.8	80.21	47.80	33.35	68.17	142.52	75.40	58.62	22.02	58.17	151.72	91.30	79.46	20.05	67.08	194.78	120.82	113.15	19.60	86.13	268.26
	1.0	**641.72**	**617.38**	**90.96**	**451.66**	**1,445.98**	**564.40**	**553.24**	**69.42**	**394.57**	**1,285.02**	**529.11**	**519.92**	**64.03**	**369.59**	**1,206.32**	**506.32**	**499.86**	**59.17**	**352.96**	**1,157.36**
10,000	0.2	13.64	2.32	10.76	13.69	16.63	9.96	1.11	8.80	9.78	11.74	8.31	1.17	6.85	7.83	9.78	7.08	1.48	5.87	6.85	8.80
	0.4	18.18	4.61	13.11	17.50	24.35	12.21	3.06	9.01	11.47	16.39	11.07	3.67	7.37	10.24	15.77	11.17	5.49	5.74	9.83	18.28
	0.6	35.88	13.05	21.54	33.55	53.10	26.13	12.12	13.94	23.23	41.94	27.33	16.43	11.74	22.78	48.66	34.30	26.55	10.50	26.36	68.80
	0.8	67.05	39.18	28.48	57.32	118.14	62.43	47.93	18.72	48.29	124.77	75.81	65.59	16.96	55.91	161.22	101.21	94.60	16.66	72.27	224.39
	1.0	**426.92**	**401.78**	**69.40**	**303.07**	**950.34**	**423.16**	**413.00**	**53.76**	**296.33**	**961.11**	**420.03**	**411.99**	**51.56**	**293.61**	**956.74**	**417.16**	**411.57**	**48.93**	**290.84**	**953.22**

Table 5.2 – (*continued*), for B = 5.

	λ_q	0.40					0.60					1.00				
				Per of the RL dist.					Per of the RL dist.					Per of the RL dist.		
n	***δ***	**ARL**	**SDRL**	**10^{th}**	**50^{th}**	**90^{th}**	**ARL**	**SDRL**	**10^{th}**	**50^{th}**	**90^{th}**	**ARL**	**SDRL**	**10^{th}**	**50^{th}**	**90^{th}**
30	**0.2**	7.07	2.95	4.22	6.28	10.89	8.78	5.60	3.87	7.11	16.08	101.43	100.94	11.09	70.49	232.93
	0.4	21.49	16.81	6.40	16.38	43.41	32.95	29.54	6.54	23.72	71.40	204.55	204.05	22.00	142.00	470.29
	0.6	75.08	70.30	12.22	53.58	166.70	103.53	100.18	14.01	72.81	233.94	303.32	302.83	32.42	210.45	697.79
	0.8	242.39	238.01	29.40	169.38	552.36	265.59	262.53	30.72	185.06	607.47	390.21	389.73	41.64	270.71	897.93
	1.0	**772.58**	**768.77**	**84.77**	**536.70**	**1,774.04**	**634.14**	**631.34**	**69.36**	**440.41**	**1,456.61**	**487.64**	**487.17**	**51.78**	**338.16**	**1,122.25**
50	**0.2**	6.94	2.82	4.16	6.19	10.60	8.55	5.36	3.88	6.95	15.37	100.18	99.69	11.04	69.62	230.01
	0.4	19.98	15.29	6.25	15.45	39.92	30.80	27.39	6.29	22.57	66.55	201.97	201.47	21.74	140.06	464.48
	0.6	65.89	61.09	11.15	47.17	145.45	94.30	90.95	12.90	66.28	212.82	299.56	299.06	31.98	207.83	689.07
	0.8	205.75	201.32	25.65	143.93	467.95	239.19	236.11	28.05	166.69	546.68	385.23	384.76	40.99	267.21	886.39
	1.0	**629.55**	**625.66**	**69.81**	**437.59**	**1,444.54**	**561.88**	**559.06**	**61.70**	**390.32**	**1,290.15**	**481.42**	**480.95**	**51.25**	**333.85**	**1,107.92**
100	**0.2**	6.85	2.74	4.07	6.12	10.33	8.38	5.20	3.92	6.80	15.18	99.27	98.78	10.89	69.05	227.93
	0.4	19.01	14.31	6.15	14.69	37.64	29.36	25.95	6.17	21.41	63.25	200.05	199.55	21.50	138.79	460.08
	0.6	61.13	56.27	10.81	43.92	134.40	89.45	86.06	12.62	63.07	201.59	299.83	299.33	32.11	208.07	689.73
	0.8	183.74	179.26	23.34	128.73	417.30	222.47	219.37	26.26	155.19	508.16	383.29	382.82	40.90	265.93	881.90
	1.0	**545.59**	**541.61**	**61.02**	**379.37**	**1,251.12**	**516.33**	**513.48**	**56.94**	**358.73**	**1,185.17**	**479.00**	**478.52**	**50,828.00**	**332.15**	**1,102.32**
200	**0.2**	6.80	2.70	3.99	6.03	10.28	8.31	5.12	3.92	6.76	14.95	98.80	98.31	10.87	68.59	226.86
	0.4	18.56	13.86	6.08	14.41	36.41	28.68	25.26	6.09	20.96	61.59	199.18	198.69	21.47	138.21	457.97
	0.6	58.36	53.49	10.55	41.96	127.98	86.41	83.01	12.13	60.86	194.56	298.53	298.03	31.93	207.03	686.73
	0.8	174.44	169.96	22.43	122.33	395.84	215.01	211.90	25.47	150.01	491.00	381.19	380.72	40.59	264.36	877.10
	1.0	**510.93**	**506.94**	**57.42**	**355.36**	**1,171.24**	**496.32**	**493.46**	**54.84**	**344.85**	**1,139.06**	**476.37**	**475.90**	**50.71**	**330.34**	**1,096.32**
500	**0.2**	6.78	2.67	3.92	5.92	10.24	8.26	5.08	3.92	6.79	14.84	98.53	98.04	10.82	68.44	226.25
	0.4	18.29	13.60	5.99	14.19	35.93	28.27	24.85	6.01	20.65	60.53	198.33	197.84	21.31	137.69	455.83
	0.6	56.80	51.93	10.37	41.01	124.43	84.63	81.23	11.96	59.75	190.47	297.25	296.76	31.72	206.17	683.77
	0.8	169.57	165.06	21.90	119.02	384.61	211.37	208.25	25.13	147.38	482.68	382.16	381.68	40.70	265.07	879.33
	1.0	**492.31**	**488.28**	**55.45**	**342.61**	**1,128.34**	**486.19**	**483.31**	**53.69**	**337.91**	**1,115.69**	**477.58**	**477.10**	**50.71**	**331.24**	**1,098.96**
10,000	**0.2**	6.75	2.65	3.91	5.87	10.00	8.22	5.04	3.91	6.85	14.71	98.19	97.70	10.76	68.24	225.45
	0.4	15.09	11.20	4.92	11.61	29.67	23.33	20.50	4.92	17.05	50.06	164.44	164.03	17.77	114.11	378.10
	0.6	55.24	50.43	10.05	39.79	120.93	82.64	79.28	11.75	58.32	185.84	293.41	292.92	31.31	203.48	675.11
	0.8	143.62	139.73	18.60	100.80	325.59	180.09	177.40	21.38	125.77	411.23	328.46	328.05	34.98	227.76	755.81
	1.0	**415.00**	**411.53**	**46.83**	**288.63**	**951.16**	**413.48**	**411.00**	**45.76**	**287.38**	**948.92**	**410.48**	**410.07**	**43.63**	**284.65**	**944.64**



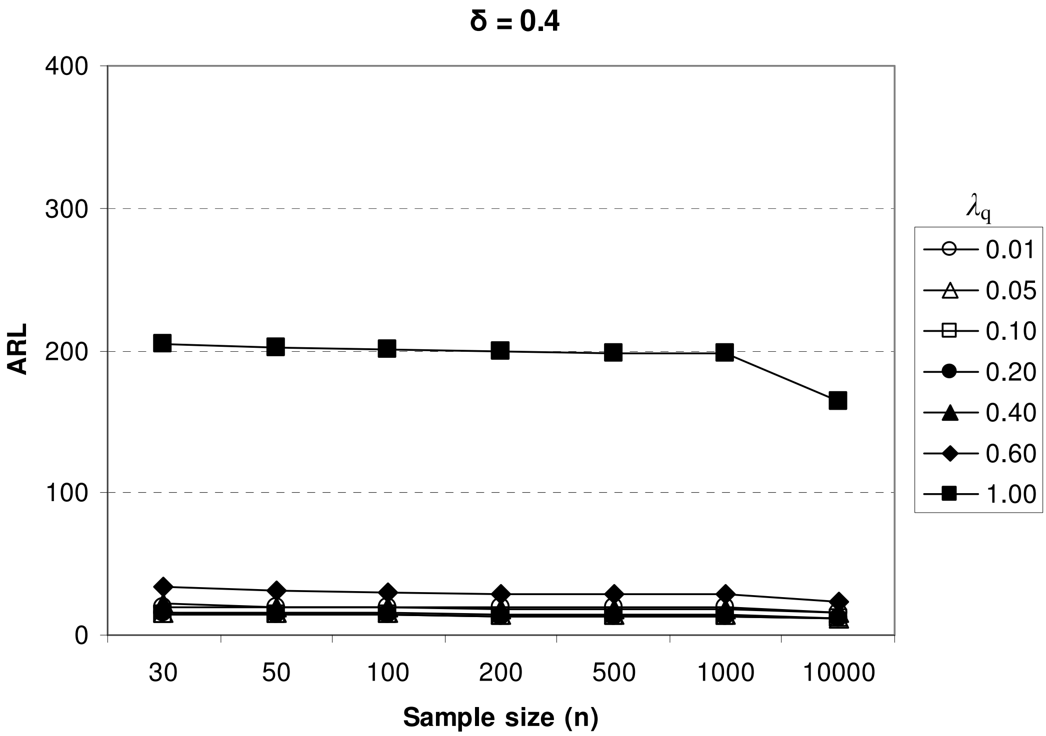


Figure 5.3 – The effect of sample size on the marginal ARL performance, for $\delta = 0.8$ and $\delta = 0.4$ (B = 2).

CHAPTER 6

CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

6.1. Conclusions

In this study, the conditional and marginal performance of the TBE exponential EWMA control charts have been studied for the effect of parameter estimation introduced in the *Phase I* of the control chart design process. The effect of parameter estimation on the targeted performance is examined in the conditional analysis. Sample size recommendations were given in marginal analysis.

Because of the high variation and skewness of the RL distribution, ARL alone is an unreliable measure of performance. Hence additional performance measures like SDRL, and 10^{th}, 50^{th} (median) and 90^{th} percentiles of the RL distribution were used. It is observed that the ARL and SDRL are nearly equal so that the variability of the RL increases with the ARL. Consequently, higher ARL values resulted in higher variations. Furthermore, control chart performance becomes harder to fit expectations. On the other side, for smaller ARL values, SDRL decreases more than the ARL, so that the chart performance fits well with the expectations. It is also observed that the RL distribution is right skewed.

In the conditional analysis, it is observed that for higher smoothing parameter (λ_q) values and smaller sample sizes (n), the control chart performance is lower when detecting small shifts. Estimation error is most effective on small shifts (δ = 0.8). Overestimation resulted in a slightly better ARL_1 performance but worse ARL_0

performance. Underestimation resulted in a worse ARL_1 performance but better ARL_0 performance with exceptionally high ARL_0 and SDRL values. Effect of estimation error becomes very small for larger shifts ($\delta \leq 0.6$).

The effect of the choice of boundary B was less than the other parameters. $B = 2$ and 5 performed close to each other and better than $B = 1$. However, in some cases $B = 2$ and 5 yield exceptionally high ARL and SDRL values for small λ_q values, so employing $B = 1$ might provide a more reliable chart performance.

In the marginal analysis, it is observed that when small shifts ($\delta = 0.8$) are concerned, especially for $\lambda_q \leq 0.20$, increasing the sample size (n) up to 100 improves the *in*-control and *out-of*-control marginal performance significantly. Beyond 100, the improvement is steady but less. For $\lambda_q = 0.01$, the ARL_0 and ARL_1 performance show some extreme values so it is suggested to choose a $\lambda_q \geq 0.05$ for a consistent performance. For larger shifts in the mean ($\delta \leq 0.6$), although the more is better, all sample sizes showed satisfactory results.

The effect of sample size also diminishes with increasing λ_q. The sample size is almost ineffective for the Shewhart case ($\lambda_q = 1.00$) and it is most effective for $\lambda_q = 0.01$. Together with the conditional analysis results, for a better control chart performance, using a smaller λ_q with a sample sizes more than 100 or 200 are recommended.

6.2. Suggestions for Future Work

In this study effects of parameter estimation on TBE exponential EWMA is considered. Note also that Pehlivan (2008) also showed that although the exponential TBE EWMA is generally robust to exponential distribution assumption, it is non-robust for some cases. In a future research, effects of parameter estimation on some other possible distributions may also be studied.

Here, sample size 10,000 is considered to approximately represent the known parameter case. However, for the smaller sample sizes considered here, there may be situations that sample size of even 30 may be hard to obtain. Considering the fact that, the chart performance may be effected significantly with small λ_q and some *B* values for small shifts in the process mean, performance may be analyzed for small sample sizes.

As another future research, sensitivity study may be developed in order to compare the results and see the effect of discretization introduced in the Markov chain model, and also the effect of quadrature used in the integration of the marginal performance analysis.

Since a decrease in the mean of a TBE process implies an increasing number of defects, a one-sided chart is employed. Analysis for a two-sided chart may be performed in order to investigate the effects of increases in TBE means.

Note also that the control chart considered here is a univariate control chart. Some processes require more than one quality characteristic to be monitored. Simultaneous monitoring of correlated quality characteristic involve the use of multivariate control charts. Effects of parameter estimation on multivariate control charts may also be suggested as a future research area.

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