# A THESIS SUBMITTED TO <br> THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES <br> OF MIDDLE EAST TECHNICAL UNIVERSITY 

BY

REŞAT ÖZGÜR DORUK

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
ELECTRICAL AND ELECTRONICS ENGINEERING

JUNE 2008

# NONLINEAR CONTROLLER DESIGNS FOR A REACTION WHEEL ACTUATED OBSERVATORY SATELLITE 

submitted by REŞAT ÖZGÜR DORUK in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Electrical and Electronics Engineering Department, Middle East Technical University by,

Prof. Dr. Canan Özgen
Dean, Graduate School of Natural and Applied Sciences
Prof. Dr. İsmet Erkmen
Head of the Department, Electrical \& Electronic Engineering
Prof. Dr. Erol Kocaoğlan
Supervisor, Electrical \& Electronic Engineering, METU $\qquad$

Examining committee members:

Prof. Dr. Mübeccel Demirekler (METU - EEE)

Prof. Dr. Erol Kocaoğlan (METU - EEE) $\qquad$

Prof. Dr. Kemal Leblebicioğlu (METU - EEE)

Prof. Dr. Kemal Özgören (METU - ME)

Assist. Prof. Dr. Yakup S. Özkazanç (Hacettepe U. - EEE) $\qquad$

Date: 30 / 06 / 2008

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Lastname: Reşat Özgür Doruk
Signature :

ABSTRACT<br>NONLINEAR CONTROLLER DESIGNS FOR A REACTION WHEEL ACTUATED OBSERVATORY SATELLItE<br>Doruk, Reşat Özgür<br>PhD, Department of Electrical and Electronics Engineering<br>Supervisor: Prof. Dr. Erol Kocaoğlan<br>June 2008, 143 Pages

In this research, nonlinear attitude controllers are designed for a low earth orbit satellite intended to be used in observatory missions. The attitude is represented by the Modified Rodriguez Parameters (MRP) which is a minimal representation providing a fully invertible kinematics. As a difference from the classical satellite models existent in the literature, the model of this work incorporates the dynamics of the reaction wheel (actuator) including a brushless dc motor which is armature controlled. The total model has four group of state vectors which are the attitude, body rates, actuator torque and velocity. The main control approach of this research is developed by utilizing integrator back - stepping which provides a recursive stabilization methodology to the designer. For performance comparison, a second controller based on input output feedback linearization (IOFL) is presented. Both of the approaches produce a torque demand law and this is used for generating a desired reaction wheel velocity command. A reaction wheel controller uses the motor as the actuator and produces the necessary amount of the torque according to the desired wheel velocity command. In addition for the back stepping based approach, a stability analysis against the external disturbance torques is also provided. Simulations are presented for validating the performance and robustness of the proposed controllers.

Keywords: Back - stepping, feedback linearization, attitude control, satellite, reaction wheel

## ÖZ

# tepki tekeri ile hareket eden gözlem uydulari için doğrusal olmayan DENETLEÇ TASARIMLARI 

Doruk, Reşat Özgür<br>Doktora, Elektrik ve Elektronik Mühendisliği Bölümü<br>Tez Yöneticisi: Prof. Dr. Erol Kocaoğlan<br>Haziran 2008, 143 Sayfa

Bu çalışmada, alçak yörüngeli görüntüleme uyduları için doğrusal olmayan yönelim denetleci tasarımları yapılmaktadır. Kullanılmakta olan uydu modeli Değiştirilmiş Rodriguez Parametreleri ile modellenmiş olup kaynaklarda siklıkla rastlanan klasik uydu modelinden farklı olarak motor dahil olmak üzere tepki tekerlerinin etkilerini de içermektedir. Değiştirilmiş Rodriguez Parametreleri sürekli tersinir bir kinematik sağlaması açısından önemlidir. Kullanılmakta olan motor armatür kontrolü olup tüm sistemde yönelim, gövde hızları, tork ve tepki tekeri hızı olmak üzere dört grup durum vektörü söz konusudur. Yönelim kontrolü için temel olarak seçilen yaklaşım tasarımcıya tüm durum vektörlerini teker teker inceleme olanağı sunan geri adımlamalı denetleç tasarım yöntemini kullanmaktadır. Bunun yanı sıra, benzetimlerde karşllaştırma amacı ile geri beslemeye dayalı doğrusallaştırma yöntemine dayalı ikinci bir yaklaşıma da değinilmektedir. Her iki yaklaşımda öncelikle gerekli yönelimlere ulaşlabilmesini sağlayan bir gerekli tork değeri üretecektir. Bu bilgi, ölçerlerden gelen yönelim ve açısal hız bilgileri kullanılmak suretiyle tümlenerek tepki tekeri hız komutu üretilir. Tepki tekeri sistemini kontrol eden algoritma bu değeri kullanarak motorlardan gerekli torku üretilmesini sağlar. Geri adımlamalı yaklaşım için bozucu dış etkilere karşı kararlılığı analiz eden basit bir matematiksel yaklaşımda sunulacaktır. Ayrıca her iki yaklaşım için performansların analizine yönelik benzetimler gerçekleştirilmiş olup, yaklaşımların karşılaştırımasıda bu safhada yapılmıştır.

Anahtar Kelimeler: Geri adımlamalı denetleç, geri beslemeye dayalı doğrusallaştırma, yönelim denetimi, tepki tekeri

To my family ...

## ACKNOWLEDGEMENTS

I would like gratefully to express my sincere thanks to Prof. Dr. Erol Kocaoğlan for his enthusiastic supervision, excellent guidance and kindness throughout this research.

I would also like to present my special thanks to Prof. Dr. Mübeccel Demirekler from the Electrical and Electronic Engineering Department of Middle East Technical University Ankara/Turkey and Assist. Prof. Dr. Yakup Özkazanç from the Electrical and Electronic Engineering Department of the Hacettepe University Ankara/TURKEY for their valuable comments and suggestions provided generously at all stages of my research.

Finally, I am indebted deeply to my parents for their endless support and patience throughout this work.

## TABLE OF CONTENTS

ABSTRACT ..... iv
ÖZ ..... v
ACKNOWLEDGEMENTS ..... vii
CHAPTERS

1. INTRODUCTION .....  .1
1.1 Purpose .....  1
1.2 Satellite Attitude Control: A historical perspective. ..... 1
1.3 Purpose of this Research ..... 13
1.4 Satellite Modeling ..... 13
1.5 Backstepping. ..... 15
1.6 Input Output Feedback Linearization (IOFL) ..... 16
1.7 Input to State Stability against Disturbance Torques (ISSADT) ..... 16
1.8 Outline ..... 17
1.9 Contributions ..... 17
1.10 Publications from This Work ..... 18
2. ATTITUDE DYNAMICS OF THE SATELLITE ..... 19
2.1 Introduction ..... 19
2.2 Coordinate Axes ..... 19
2.2.1 Earth Centered Inertial (ECI) Coordinate System ..... 19
2.2.2 Orbit Frame (Coordinate Axes) System ..... 19
2.2.3 Body Coordinate System ..... 21
2.3 The Kinematics. ..... 21
2.3.1 Conversion from Euler Angles to MRP ..... 22
2.3.2 Conversion from MRP to Euler Angles ..... 23
2.4 The Non - Uniqueness Problem in Attitude Kinematics ..... 23
2.5 Dynamics of Satellite Model ..... 24
2.6 A Model for the Brushless DC Motor of the Reaction Wheel ..... 30
2.7 The Usage of the Model in Attitude Control ..... 31
3. THEORETICAL BACKGROUND ..... 33
3.1 Introduction ..... 33
3.2 Theory behind Backstepping ..... 33
3.2.1 An Example for a Generic Nonlinear System ..... 33
3.2.2 Back - stepping Transformation ..... 33
3.2.3 Backstepping Control [Fossen (1997), Skjetne (2004)] ..... 34
3.3 The Basic Theory of Feedback Linearization ..... 37
3.3.1 The Notion of the Vector Relative Degree ..... 37
3.3.2 Feedback Linearization Procedure ..... 39
3.4 Input to State Stability ..... 41
3.4.1 Definition Concerning Input - To - State Stability ..... 41
CHAPTER 4 ATTITUDE CONTROL BY BACKSTEPPING ..... 44
4.1 Introduction ..... 44
4.2 Attitude Error Formulations ..... 44
4.3 The Back - Stepping Design Procedure. ..... 44
4.3.1 Step - I (Demanded Torque Generator) ..... 46
4.3.2 Step - II (Demanded Torque Generator) ..... 47
4.3.3 Step - III (Speed Control) ..... 50
4.3.4 Step - IV (Speed Control) ..... 51
4.4 Application Issues ..... 53
4.5 Final Structure of the Control ..... 55
4.6 Stability Analysis ..... 56
4.6.1 Analysis of the Demanded Torque Generator ..... 56
4.6.2 Analysis of the Speed Controller ..... 58
4.7 The Stability of Wheel Velocity Command Generator: ..... 59
4. ATTITUDE CONTROL WITH FEEDBACK LINEARIZATION ..... 60
5.1 Introduction ..... 60
5.2 Preparing for IOFL Attitude Control ..... 60
5.3 Construction of the IOFL Demanded Torque Generator ..... 62
5.4 The Reaction Wheel Velocity Command Generator ..... 66
5.5 The Wheel Speed Control System ..... 66
5.6 The Linear Quadratic Regulator ..... 68
5. NUMERICAL APPLICATIONS AND SIMULATIONS. ..... 70
6.1 Motivation. ..... 70
6.2 The Parameters of BILSAT - I ..... 70
6.3 Implementation of Back - stepping ..... 72
6.4 Simulation Outline ..... 73
6.4.1 Results of the Single Run Simulations ..... 74
6.4.2 Results of Multi - Run Simulations ..... 78
6.5 Simulation of Feedback Linearization ..... 82
6.5.1 Single Run Simulations for IOFL based Attitude Control ..... 83
6.5.2 Multi Run Simulations for IOFL based Attitude Control. ..... 86
6.6 Long Range Simulation ..... 91
6.6.1 Results for the Case 1 ..... 92
6.6.2 Results of the Case 2 : ..... 95
6.6.3 Results of the Case 3 : ..... 97
6.6.4 Comments. ..... 100
6. CONCLUSION \& FUTURE WORK ..... 101
7.1 Summary ..... 101
7.2 Simulations ..... 103
7.3 Future Work ..... 104
BIBLIOGRAPHY ..... 113
A. A SUMMARY OF ATTITUDE REPRESENTATIONS ..... 117

# CHAPTER 1 <br> INTRODUCTION 

### 1.1 Purpose

Satellites became an important application area of the new technological developments. They are used in many fields starting from telecommunications to defense technologies. For a successful operation the satellite should be stabilized at a given attitude. Thus attitude control is an important part of the space technology research. The low earth orbit satellites (LEO) are subject to some disturbances originating from earth and space environment. Because of that, the robustness characteristics of the stabilizing controller are one of the most important aspects of the attitude control approaches. In addition to that, the satellite or spacecraft models are constructed using the nominal values of parameters. In realistic conditions, the values of the model parameters may deviate from the nominals. The aim of this research is to develop nonlinear controllers for attitude stabilization of low earth orbit satellites, analyze their robustness and performances through theoretical and simulation based analyses.

### 1.2 Satellite Attitude Control: A historical perspective

There has been much research on satellite attitude control after the first artificial satellite Sputnik - I had started its mission in space in the year 1957. In this section, a literature survey will be provided for presenting the developing research in the past 50 years.

One of the oldest studies in the attitude control research is the work done by [Froelich (1959)] where reaction wheels are used for actuation. To support the operation of reaction wheels a mass ejection mechanism is utilized. The aim of this device is to compensate for the initial disturbances and removes the unwanted momentum stored in the reaction wheels. In [Sepahban (1964)] a practical attitude control approach was presented. Digital differential analyzer techniques are used in the derivation of the attitude control law. An analog computer is used for simulating the dynamics of the controlled vehicle. It is claimed that the designed digital controller is an example of true optimal control.

The study by [Showman (1967)], is an example of applying gimbaled star trackers to obtain useful attitude control signals. The amount of deviation in the gimbals from their commanded angles is used in the computation of the attitude error. In the work of [Porcelli (1967)], a simplified model of a rigid rotating body is given together with two suboptimal control laws. One of the control laws have a more complex mechanism but requires less fuel than the other. These controllers assume that limited thrust available. For the case of unlimited thrust, the approach provides two minimum fuel optimal controllers. The paper by [Dunn (1968)] provides a minimum energy controller in presence small external disturbance torques. The actuators are reaction thrusters with variable impulses. The limit cycle issues and corresponding fuel consumption rates are also discussed.

Another research on satellite attitude control was [Arnesen (1968)]. This work is an example of active magnetic coils as actuators. The laboratory demonstrations show that the best possible configuration for three axis satellite attitude control is the usage of magnetic coils for two axes and one reaction wheel or thruster for the remaining axis. [Childs (1969)] is an optimal attitude control study where necessary conditions are developed for fuel optimal attitude control laws for spin stabilized axi symmetric spacecraft. The control torques are generated by a gimbaled reaction jet system.
[Fearnsides (1970)] is another fuel mode optimal control study where the correlation between the lower bound of the fuel consumption and the structure of the disturbance torques are investigated. The study by [Harvey (1972)] discusses the limit cycle bounds on attitude control systems. The attitude control system is of the on - off type and a bounding curve that insures the cyclic behaviour on the roll dynamics is derived. In [Isley (1975)], a ground based adaptive attitude control work is presented. In that research, the attitude control algorithm is set up in a minicomputer system in the ground station. The information exchange between the ground station and the geosynchronous satellite is performed by RF command and telemetry links.

The work by [Larson (1977)] deals with an optimal Linear Quadratic Gaussian (LQG) control and estimation approach. The spacecraft has flexible appandages and the results point out the modeling errors. The spacecraft model is $15^{\text {th }}$ order where as the investigated estimators are second, fourth and sixth order. [Pande (1979)] discusses the utilization of solar pressure in attitude control in the pitch plane. The importance of the controller is that there are no mass expulsion techniques involved in the control algorithm so the lifespan of the vehicle increased.
[Joshi (1980)] proposes a model damping scheme for large space structures, which is important for achieving stability of the attitude control system. The controller uses a number of angular momentum control devices for damping enhancement. The closed loop system is claimed to be stable in the sense of Lyapunov. The work of [Crouch (1984)] studies the solution of the attitude control problem by geometric control theory. The work analyzes the controllability of the plant in the gas jet and momentum wheel actuation cases. In the case of the momentum wheels three independent actuators are needed for stable and accurate three axis attitude control.
[Sahjendra (1984)] discusses the asymptotic reproducibility (to track a given function asymptotically). The theoretical results are applied to a nonlinear satellite attitude control problem. [Dwyer (1985)] discusses an optimal solution to the attitude control problem. The spacecraft uses momentum transfer devices in actuation and pointwise minimization of sum of the squares of the norms of the states in the discretized linear plant and the linear system inputs.

An attitude control application on the satellites with flexible components is presented by [Monaco (1985)]. The control law implements a static state feedback on the linear plant obtained by feedback linearization.

An adaptive control scheme is provided by [Singh (1986)] where a model reference controller is developed which does not need any information concerning satellite model parameters and disturbance torques. The control law feeds attitude errors, the rate of change of attitude errors and compensator states. A robust control law based on nonlinear invertibility and linear feedback theory is provided by [Singh (1987)] where robustness against parametric uncertainties are obtained by incorporating a servo compensator.

Another work by [Singh (1987)] solves the large angle attitude control problem by adaptive control. The study assumes that there are unknowns in various parameters of the spacecraft and disturbance torques. The controllers enable the spacecraft to do large maneuvers in spite of the uncertainties in the system. A linear attitude control study is performed by [lyer (1987)] where matrix fraction descriptions are used to produce minimal controllable and observable linear satellite models. Observers and output feedback controllers are designed using those models and the authors claim that precise attitude control law is achieved.

The work by [Singh (1989)] presents a variable structure control law that feeds the attitudes defined in Euler angles back to the controller in presence of
uncertainty. The actuators are reaction jets. The authors claim that precise attitude control can be achieved. [Weiss (1989)] discusses a generic attitude control problem. The controller has an inner rate feedback loop and an outer attitude feedback loop. The controllers are of proportional or proportional plus integral type. An intelligent attitude control implementation incorporating artificial intelligence theories is presented by [Murgesan (1989)]. These approaches provide power of reasoning, judgement, learning, self - modification, adaptability and fault tolerance.

In [Mobasser (1990)], the combination of sun sensors and digital signal processing tools are utilized to design an attitude control system the purpose of which is to point the vehicle point towards the sun. the deviations from the sun axis are detected as attitude errors and appropriate actions are taken to correct the orientation of the spacecraft.
[lyer (1990)] has implemented an adaptive nonlinear control system for driving gyrotorquers. It is working without the necessity of the system parameters and disturbance information. The controller only uses the tracking error and its derivative for the feedback and is claimed to be successful. Some attitude control guidelines are given in the work by [Wen (1991)] including attitude representations, error definitions, kinematics, dynamics and Lyapunov function candidates. The work by [Nicosia (1992)] introduces a controlling mechanism which operates by only measuring the roll, pitch and yaw components of the attitude. A nonlinear observer is designed for reconstructing all of the necessary state variables. The nonlinear servomechanism theory is applied by [Huang (1994)] and the result is claimed to be better than the feedback linearization for sinusoidal disturbances and parametric uncertainties. In the work by [Tsiotras (1994a \&1995a)], the problem of attitude stabilization is considered. The control is actuated through two pairs of gas jets. Those actuators provide two control torques orthogonal to the axis of symmetry. [Dracopoulos (1994)] investigates the application of locally predictive networks to an adaptive attitude control problem. The network is trained by using small history of the system states up to the present time and some set of control inputs. After the training is successfully completed a genetic algorithm is used to find a suitable input from a hypothetical set of control inputs.

A system parameter independent quaternion (attitude) and angular velocity feedback control law is proposed by [Joshi (1995)] which is claimed to be robust against modeling uncertainties. The resultant control system is shown to be asymptotically stable. [Godhavn (1995)] presents a continuous feedback controller. In this work, some measurements are sampled due to the non - differentiability of the
exponential stabilizers. The design by [Greval (1995)] does not need any gyroscopic measurement and estimates the velocities from the apparent motion of the stars on the focal plane. The attitudes are computed by the usage of a square root Extended Kalman Filter (SQEKF). [Wisniewski (1996)] develops a linear control system based on the Floquet theory of periodic systems. The main clue is the periodicity of the earth's geomagnetic field. In the same text, there also exists some applications on sliding mode control. Another sliding mode application is the work by [Crassidis (1996)] in which the Modified Rodriguez Parameters (MRP) are used as the attitude representation.

An optimization study is the research of [Schaub (1996)] which focuses on near minimum time and near minimum fuel solution of the attitude control problem. An asymptotically stable nonlinear observer is designed for attitude estimation. In [Tsiotras (1994b \& 1996)] a Lyapunov function involving quadratic and logarithmic terms is utilized for obtaining linear controllers in terms of kinematical parameters. Stereographical projections are used for obtaining two additional parameters in the rotation group. Tsiotras [(1996)] has used a specific Lyapunov function which is quadratic in angular velocities and logarithmic in attitude. An optimal control scheme is obtained from solution of a quadratic cost where the overall control performance is claimed better for Modified Rodriguez Parameters. [Crassidis (1997)] applies predicitive control theory on attitude control. Another application of sliding mode attitude control is the [McDuffie (1997)]'s work which is based on decoupled sliding mode approach. It is claimed that, the controller sliding manifold is guarantees the globally stable asymptotic convergence and the full order sliding mode observer avoids the noise due to the differentiation of quaternion attitude error.

The method of State Dependent Riccati Equation is an interesting approach for designing nonlinear control laws by somehow treating the system equations as linear. By that way the properties of linear quadratic (LQR) or other optimal control methods based on the solution of Riccati equations can be approximated in certain conditions. A typical solution for attitude control problem based on that approach is shown in [Parrish (1997)]. In [Zeng (1997)] two nonlinear controllers are designed for the comparison of Modified Rodriguez Parameters and Euler angles in attitude control. The work of [Kim (1998)] provides a solution for the asymptotical instability issue of the sliding mode control in presence of external disturbance torques (Disturbance accommodating sliding mode control). Hall [(1998)] uses thrusters and momentum wheels as actuators in their attitude control approach.

An application of backstepping control is presented in [Wang (1998)]. The backstepping concept is used for partitioning the system into an inner and outer loop. The outer loop makes use of a virtual input and its stability is achieved through La Salle theorem. The inner control loop is designed by sliding - mode theory. In the study of [Tsiotras (1998)] passivity properties of the attitude modeling system are utilized to derive control laws that are linear and asymptotically stable. The comparative study of [Kim (1998)] analyzes time optimal control and sliding mode control techniques in satellite application. It is argued that the time optimal solution had a better settling time. [Charbonnel (1999)] is an implementation of the linear $H_{\infty}$ control technique where the problem is solved by LMI algorithms. They have analyzed the robustness of the achieved system by defining different structured singular value bounds and using mixed $\mu$ - analysis procedure.

A comparative study is provided by [Won (1999)] in which the implementations of $H_{2}, H_{\infty}$ and mixed $H_{2 / \infty}$ methods are presented. This is a useful resource for gaining insight about the mentioned control approaches. An implementation of the Lyapunov nonlinear control theory is done by [Long (1999)] which involves data acquisition from the sun in the generation of the control commands. Kristic [(1999)] has developed a scheme for avoiding the solution of Hamilton - Jacobi - Isaacs inequality of optimal control by using an inverse optimality approach. The required control Lyapunov function and the resultant control law are obtained through integrator backstepping. A combination of sliding mode and optimization techniques are presented by [Crassidis (1999)]. In that case, a utilization of a simple term in the control law leads to a maneuvering of the reference trajectory in the shortest available distance. The sliding surfaces are designed through optimal control theories and the stability of the resultant control system are analyzed by Lyapunov functions. In the study by [Shen (1999)], the problem of minimum - time orientation of the axisymmetric rigid spacecraft by special methods of optimization are considered. Schaub [(1999)] designs an adaptive control law for attitude tracking of the space craft. The open loop nonlinear control law leads to a linear system in the closed loop (in terms of the attitude) where the control action can be in either PD or PID form. The adaptive nature of the linear controller removes the necessity of knowing the inertia matrix and the external disturbances. Caccavale's [(1999)] paper addresses the attitude tracking problem in the case of unknown velocity information. That is solved by two approaches. In the first one a second order model based observer is adopted for estimating the angular velocity of the satellite body. In the second case, a lead filter
is used for this purpose. [Kristiansen (2000)] designs PD and LQG (linear quadratic Gaussian) controllers for satellite attitude control and applies them to different combinations of actuators. The actuators are magnetic coils and reaction wheels. In [Hull (2000)], two nonlinear controllers are designed. The first one is based on integrator backstepping and feedback linearization which is formulated as a robust nonlinear recursive algorithm. The second one is a learning control example that updates the control input iteratively without the necessity of the system parameters and inverse dynamics. Costic [(2000)] has designed a quaternion based adaptive attitude controller without the need of an angular velocity feedback. The proposed output feedback controller is proven to be asymptotically stable. In [Tsiotras (2000)], a problem similar to that of [Tsiotras (1994a \& 1995a)] is solved but in this case there is a maximum limit on the level of the input torques. The paper by [Crassidis (2000)] has combined optimal and variable structure control approaches to obtain a shortest distance attitude maneuver control.

The report written by [Walchko (2000)] concentrates on application of fuzzy logic combined with sliding mode control. An online optimization procedure for the tuning of the controller and reinforcement learning procedure for the adaptation is developed. A nonlinear $H_{\infty}$ controller implementation is presented by [Show (2001)]. The controller is obtained by the analytical solution of the Hamilton - Jacobi - Isaacs inequality and prescribed level of $L_{2}$ gain performances are claimed to be achieved for the closed loop. The study uses quaternion as the attitude representation. In the work of [Park (2001)] a nonlinear controller having a class of relaxed feedback control laws is proposed. The implementation uses the Modified Rodriguez Parameters and the Cayley - Rodriguez Parameters for attitude representation. In [Tsiotras (2001)], a simultaneous attitude and power profile tracking action is achieved by using more than three non coplanar energy and momentum wheels. In order to prevent the singularities in energy transformations and to minimize the gyroscopic effects the total momentum of the wheels are zeroed by the usage of a specially developed momentum management algorithm. Another magnetic control research was made by [Makovec (2001)] which proposes two linear controller designs. These are a PD and a linear quadratic regulator. They are examined for different spacecraft configurations. The paper by [Wu (2001)] proposes an $H_{\infty}$ solution for the attitude control problem without the necessity of the solution of the Hamilton - Jacobi - Bellmann and Riccati equations. The research by [Akella (2001)] presents a velocity free controller by using the results from [Battiloti (1996) \& Tsiotras (1998)]. An adaptive scheme proposed by [Wong (2001)] uses an
algorithm for velocity estimation from attitude information. The authors claim that the attitude and the angular velocity tracking errors are converging in spite of the unknown spacecraft inertia. A fuzzy neural bang - bang control is proposed by [Tongchet (2001)] in which the bang - bang control scheme is implemented in fuzzly logic and the neural network serves as a support unit. In [Nam (2001)], LMl based $H_{\infty}$ output feedback control is combined with fuzzy logic for attitude stabilization of a flexible satellite. By that way, precise pointing capability is obtained in parameter varying space missions because the controller is reformulated on the fuzzy cells. [Fauske (2002)] is presents a complete discussion on satellite attitude control. The work presents operational control modes such as detumbling, spin stabilization and inverted boom recovery. It is intended for practical satellite application NCUBE (Norwegian Student Satellite). The research of [Silani (2002)] is an example of model predictive control based on linear satellite models. They use only magnetic actuators as control torque transmission. In [Myung (2002)], the nonlinear version of predictive control for three axis attitude control is discussed. The conventional predictive control approach is modified for attaining disturbance identification capability. The work of [Bang (2002)] provides a control approach which stabilizes the attitude tracking system using the moment of inertia of the satellite body. The nutational motion caused by the momentum wheel is controlled by a mechanism based on the product of inertia between the orthogonal body axes. The nutational motion is a periodic motion whose frequency is dependent upon the magnitude of angular momentum of the wheel and moment of inertia of spacecraft body. The methodology is applied to a bias momentum spacecraft which is a popular stabilizing method. In this approach a single momentum wheel is oriented along the normal orbit direction. The purpose of the wheel itself is to retain a certain level of angular momentum and its speed is varied for controlling the pitch error. The momentum wheel causes a stiffness effect on the roll - yaw plane which brings a disturbance accommodating capability.

An adaptive backstepping control approach is derived by [Singh (2002)] that utilizes the solar radiation as a torque generating source. Two large reflective surfaces are used for the generation of the solar radiation torques. Yoon [(2002)] presents an application of variable speed control moment gyroscopes (VSCMG). The difference between the conventional control moment gyroscope (CMG) application is that the VSCMG's have variable spin velocities whereas the CMG's have constant spin velocities. The gimbal rates are used for the generation of the reference tracking torques. In contrast, the wheel accelerations are used for
controlling both the attitude and power. Kim [(2001, 2002 \& 2006)] have used the model error control synthesis approach in attitude control. The latter one has employed an optimal nonlinear estimator to determine the error corrections and provide them to the nominal controller. All of the proposals make use of the approximate receding horizon control technique and optimization. The research by [Lappas (2002)] investigates the difference between the single gimbal control moment gyros and reaction wheels in satellite operation. Another optimal intelligent approach comes from [Sadati (2002)] in which case a radial basis function neural network is combined with a classical PD controller for initial stabilization of the satellite. The neural network is employed as self organizing and self learning optimal controller. The research by [Skelton (2003)] concentrates on the derivation and validation of mixed control moment gyroscope and momentum wheel control laws. Large angular maneuvers are obtained by generating the output torques through a feed forward control mechanism. The momentum wheel control laws are designed by feedback linearization and Lyapunov theories.

The thesis work by [Walchko (2003)] investigates the usage of sliding mode theory in robust nonlinear attitude control. It has a detailed discussion of disturbances such as solar snap and fuel slosh. Thienel [(2003)] designed a nonlinear control law that is combined with a nonlinear observer for estimating the unknown constant gyro biases. The estimation results converge exponentially to the true values and the rate of convergence is proven to be quite fast. A stochastic analysis is presented for investigating the effects of the measurement noises. The research by [Hall (2003)] proposes an attitude control system involving both thrusters and reaction wheels. Three control laws are proposed. The first two are derived using bang - bang control and they drive the thrusters. The reaction wheels are used for error corrections. The last one uses a linear feedback for the wheels and nonlinear feedback for the thrusters. Akella [(2003)] uses the Lyapunov's indirect method in attitude control and claims that the attitude regulation errors can be controllable by the usage of inclinometers and rate gyroscopes.

A backstepping application on attitude control is done by [Kim (2003)]. The poor properties of the simple linear backstepping control are improved by selection of a nonlinear tracking function. Careful gain selection and Lyapunov redesign outcomes a successful performance. The approach of [Wisniewski (2004)] makes use of the periodicity of the earth's geomagnetic field to implement a periodic $H_{2}$ controller that actuates the satellite only by magneto - torquer. The synthesis of the controller is performed through linear matrix inequalities (LMI).
[Topland (2004)] is another application involving classical control approaches mainly. The applications range from simple linear LQR to basic sliding mode theory. This work is intended for a practical application on ESEO (European Student Earth Orbiter) Project. The works of [Antonsen (2004), Overby (2004) and Blindheim (2004)] are again mainly classical attitude control projects. Those projects including [Topland (2004)] have applications using thrusters as actuators. [Bang (2004a)] provides a constraint based optimal control law on a linear satellite model and resulted in a controller equivalent to linear quadratic regulator (LQR) with additional robustness and disturbance rejection. The research by [Bang (2004b)] focuses on the feedback linearization technique combined with sliding mode theory to obtain an attitude and rate tracking control law. One other research by [Tsai (2004)] utilizes the method of eigenstructure assignment and linear exponential quadratic regulator with loop transfer recovery (LEQR/LTR). The combination of the two algorithms provides prescribed stability and overcomes the disadvantages of those methods. Another property of the proposed methodology is the unification of time - domain, frequency domain and robust decoupling design techniques in one procedure.

A stochastic control system design study [Won (2004)] uses parametric robust risk - sensitive control theory. It is an extension to the parametric robust linear quadratic Gaussian (PRLQG) approach of [Lin (1992)]. In Lovera's paper [(2004)], the problem of inertial pointing of a spacecraft with magnetic actuators is addressed and an almost global solution is proposed based on static attitude and rate feedback. The work of [Yamashita (2004)] introduces a robust control design. The controller is of proportional plus derivative (PD) type. There is a higher frequency vibration filter and a disturbance compensator for suppressing several disturbing effects on the satellite body. [Tafazoli $(2004,2005)]$ has investigated a flexible satellite control application in which feedback linearization is used to divide the system to a controllable and observable linear part and a nonlinear unobservable internal system. The controller design is shown to be asymptotically stable by Lyapunov theory. Sharma [(2004)] has proposed an optimal attitude controller based on Hamilton Jacobi Isaacs inequality. An infinite horizon optimal control problem that has feed forward and feedback control torques for reducing the performance index with quadratic cost functions. In [Tandale (2004)] a modified adaptive control strategy is developed for preventing parameter drift due to the trajectory tracking errors resulting from the control saturation. During the saturation the reference trajectory is modified so that the original reference trajectory is approximated closely without exceeding the control limits.

A hybrid control scheme is proposed by [Guan (2004)] that is composed of input output feedback linearization (IOFL) and fuzzy control. The main function of the adaptive fuzzy control is to compensate for the plant uncertainties so that the robustness of the overall system is increased. The projects by [Krogstad (2005) and Ruud (2005)] deals with the coordinated control of the satellite clusters (also called as formation flying). The control design projects are based on mainly classical control theories. The work by [Hegrenas (2005)] applies an approach called as explicit linear predictive control to attitude tracking. They solve the problem by multi parametric quadratic programming which provides power and CPU effective results. A robust linear control technique is presented in [Chiappa (2005)] which utilizes the $\mu$ - iteration technique to the attitude control problem. By that way the robustness of the controller can be improved. The controller designed by [Prieto (2005)] has manipulated the linear $H_{\infty}$ technique. The problem is solved by linear matrix inequalities (LMI) and the authors claimed that the resultant controller has a wide robustness margin.

Direct application of the integrator backstepping control approach is performed by [Kristiansen (2005)] and the design is proven to be asymptotically stable in the sense of Lyapunov. The actuator uses one reaction wheel and four thruster mechanisms. [Luo (2005)] has solved the optimal attitude control problem without the necessity of the analytical solution of the Hamiton - Jacobi - Isaacs inequality. This is achieved through an inverse optimality approach and division of the control process through integrator backstepping. A nonlinear adaptive scheme was presented by [Mothlag (2005)]. To increase the robustness and tracking performance they have used a mechanism for reducing the uncertainty and estimating a bound for the present uncertainties. It is claimed to be more robust then the parameter adaptive controllers. In [Bondhus (2005)] a nonlinear attitude observer is constructed by the usage of vectorial backstepping design. The function of the observer is to estimate the angular velocity. . In [Bang (2005)], sliding mode control is utilized for application on flexible satellite model. The controller initially designed for the rigid spacecraft and then extra degree of freedom is provided for reshaping the closed loop response of the system. The work by [Bajodah (2005)] uses a pseudo inversion technique for producing a pointwise - linear parameterization of the nonlinear control solutions. The pseudo inversion of the controller coefficient leads to two parts residing in the null and range spaces of the controller matrix. The part corresponding to the null space contains the pseudo control vector which parameterizes all the control variables necessary.

A velocity free control solution for general attitude tracking problem is proposed by [Gohary (2005)] in which the passive properties of the Euler dynamic equations and the structural properties of the kinematical equation are used to drive optimal control laws. Again an angular velocity estimation mechanism is used for achieving the goals. Jan [(2005)] have proposed a conceptual design framework for a practical satellite called as ROCSAT - 3. The necessary control modes are considered which are detumbling, dark sub mode, sun sub mode and normal operation. Simulation results are also presented in the proposal. The research by [Park (2005)] concentrates on the robust optimal control approach for three axis attitude control of a satellite. A worst case design is proposed initially which assumes the disturbances are at maximum level and the control torque is at the minimum level. After achieving the desired robust control law, a minimax approach is applied for investigating the optimality of the proposed control law. Gohary [(2006)] investigates the effect of friction on the control performance of the attitude controller. The attitude is controlled by a rotor system which has an internal friction and it is claimed that the form of the controller (especially linearity and nonlinearity) is affected from the characteristics of the friction as well as the selected Lyapunov functions. In [Guan (2006)] the sliding mode technique is incorporated into the framework of [Guan (2004)]. The paper by [Wang (2006)] introduces an approach in which the dynamics and kinematics of the satellite are expressed in terms of the attitude quaternion. Then a controller is designed as a proportional plus integral plus derivative type. The overall product is proclaimed to be a high precision controller. [Xi (2006)] presents a nonlinear attitude controller based on recursive passivation based on backstepping. The resultant control structure has PID like feedback control terms and some feedfonward terms that compensates for the plant dynamics. [Li (2006)] has made the modified Rodriguez version of [Show (2001)]. [Kaplan (2006) \& Karataş (2006)] are two studies based on the BILSAT - I design by Turkish Scientific and Technological Research Council. They have proposed linear and nonlinear control techniques and simulated them on BILSAT - I. The controllers are mostly of classical type. An optimization based example is given by [Lai (2007)] that uses constrained nonlinear programming. In this case, the number of the control steps is assumed to be fixed initially and the sampling period is accepted as an optimization variable. Also genetic algorithms are used for generating the initial feasible solutions.

### 1.3 Purpose of this Research

In this thesis, nonlinear attitude controllers are introduced for observatory satellites. The purpose of a satellite attitude controller is to track a given attitude trajectory with minimum error while requiring a reasonably low torque. In the literature there are numerous studies that use a sixth order nonlinear model based on newtons second law of motion. Those models take an actuation torque as the system input. As a difference from them, the one used in this research involves the actuator model. The actuator is composed of three reaction wheels each driven by a brushless dc motor. Thus the order of the complete plant is equal to twelve with four state vectors. Those are the attitude, body frame angular velocity, reaction wheel torque (or the motor torque) and reaction wheel angular velocity where each vector has a dimension equal to three. Two control laws are derived in the controller design processes. The first one is the demanded torque generator law which is the necessary amount of torque that should be applied on the satellite body to track the desired attitude. The second control law is a reaction wheel velocity control law (speed controller) that produces the necessary motor input voltage to spin the reaction wheels in the desired velocity. That leads to a torque exertion on the satellite body. The demanded torque generator law serves as a reference input to the speed controller through the reaction wheel velocity command generator. The reaction wheel velocity command is generated by integrating the reaction wheel velocity differential equation using the demanded torque, measured and estimated values of attitude, body and reaction wheel angular velocities. In the design process it is assumed that all the state variables are directly measurable. In the actual implementation those signals are estimated from the measurements of the gyroscopes, sun sensors, star tracking, earth sensor, inertial navigation systems or global positioning systems.

The approaches presented in this research are based on integrator back stepping and input output feedback linearization (IOFL) methodologies.

### 1.4 Satellite Modeling

The mathematical model is an important issue in attitude control since it describes everything related to the motion of a satellite body. As a very complex nonlinear model they have some limitations. The main limitation comes from the attitude representations. In the literature survey of Section 1.2 four types of attitude representations are mentioned. The basic attitude representation is the Euler angle
vector that is used in some studies. It has a basic singularity at the rotations of $90^{\circ}$ in the pitch plane. This is a big obstacle for real time applications. So most of the researchers prefer the quaternion where there is no real singularity. However quaternion is not a minimal representation (it is four dimensional). Another alternative is the three dimensional Cayley - Rodriguez parameters or the so called Gibbs vector that has a singularity at $\pm 180^{\circ}$. Lastly, there is another minimal kinematical parameter called as the Modified Rodriguez Parameters (MRP). That provides a singularity at $\pm 360^{\circ}$ of rotations. One of the advantages of that representation is the fully invertible kinematical matrix in the propagation equation. This is helpful for implementing feedback linearized control laws. The details of attitude kinematics including various representations are presented in Appendix A.

Another issue is related to the actuators. The actuator is the bridge between the controller and the satellite body. In the literature, mostly noted actuators are thrusters, magnetic actuators, momentum and reaction wheels. The difference between the latter two is that reaction wheel is nominally a zero speed device and momentum wheel is an high speed device. The reaction wheel is operated when a rotation is required for the satellite body and it is a light weight body. Whereas a momentum wheel is heavier and faster operating device. Both of them are used in high precision operations however momentum wheel can also provide spin stabilization [Svartveit (2003)] during rotation about one axis. A comparison of those actuators is given in Table 1.1.

| Type of the Actuator | Advantages | Disadvantages |
| :---: | :---: | :---: |
| Thrusters | Fast | Fuel consumption <br> increases |
| Magnetic Actuators | Cheap | Low Altitude, Slow, <br> Structural Singularity |
| Reaction Wheel | Precision | Expensive \& Weight |
| Momentum Wheel | Precision | Expensive, heavier than <br> reaction wheel, high <br> speed so power <br> consumption increases. |

Table 1-1 Comparison of the Actuators [Svartveit (2003)]

As it can be understood from the table above, a precise satellite manipulation requires the usage of the reaction or momentum wheel. Even they introduce weight to the satellite body this will not be a serious problem because the observatory satellite maneuver is quite slow. As it is mentioned in Section 1.3 the reaction wheel is considered as an integral part of the satellite during mathematical modeling. A basic model is presented in [Topland (2004)] where angular velocity of the reaction wheel is included as a state vector.

### 1.5 Backstepping

Back - stepping is a special approach of control. The method stabilizes the entire plant recursively starting from the state variables that are to be tracked by the closed loop system. The order of recursion depends on the number of sub - state vectors formed as a result of partitions along the full plant state vector. This partition can be performed according to the design requirements and the physical properties of the plant dynamic variables. In the satellite model used in this research, this grouping can be done easily since the model is formed by combining four vector differential equations. Those are the dynamics of attitude represented in MRP, body coordinate angular velocity, reaction wheel motor torque and velocity. So starting from the attitude vector four steps are necessary for the completion of the attitude tracking system. For the velocity controlled approach, the distinct procedures are prepared for demanded torque generator and speed controller. The link between those are provided by the reaction wheel velocity command generator that uses the control law derived by the attitude control process. The demanded torque and corresponding motor input voltage laws are obtained in the second and fourth steps of the back - stepping procedure. The remaining steps are for the manipulation of the non - tracked states (body angular velocity and motor torque). In each step a control Lyapunov function (CLF or $V(\mathbf{x})$ ) on the corresponding state vector is proposed. The negative definiteness of the rate of change of the CLF $(\dot{V}(\mathbf{x}))$ is ensured through cancellation of the nonlinearities in the satellite model. The CLF's and their derivatives can be selected in quadratic forms. An important advantage of nonlinearity cancellation is that always negative definite CLF derivatives can be obtained. So a globally stable closed loop is obtained. For the theoretical development [Fossen (1997)] and [Skjetne (2004)] are helpful resources. An application of recursive back - stepping theory to satellite attitude control is presented in [Kristiansen (2005)] where quaternion is preferred as the attitude
representation. That work constitutes a skeleton for this research however the model and the controller structure in consideration is quite different.

### 1.6 Input Output Feedback Linearization (IOFL)

The basic principles of feedback linearization are presented in [Isidori (1989)]. Like that of the back - stepping theory IOFL also involves a nonlinearity cancellation. As a difference, this method generates a double integrator linear system. As it is known, the double integrator plant is both controllable and observable. One can apply any desired linear control methodology to the resultant linear plant to obtain the desired controller structure. There is no recursive nature of feedback linearization. Because of that, all of the system could not be linearized if the vector relative degree of the output is less than the order (size of the state vector) of the plant. In the satellite nonlinear model of this research a similar issue is existent. In this work, there are two cascaded controllers which are full relative degree systems so there is no problem of non - linearized dynamics in the IOFL based approach.

### 1.7 Input to State Stability against Disturbance Torques (ISSADT)

The closed loop stability of the approaches described in the last two sections are obtained from the results of the Lyapunov theory. However, this is only adequate for the initial design purposes since it is accepted that there are no external disturbance torques existent on the satellite body. In actual operation, there are some external torques exerted on the satellite body from the environment. The most widely known disturbance torques are gravity gradient of the earth, atmospheric drag and some photonic forces exerted by the sun rays. In normal conditions, the level of those torques could be much more smaller than the actuator torque however this does not mean that there will be no abnormal situations during the operation. So an analytical approach is provided in order to tune the control gains for attaining stability against the disturbance torques. The analysis is performed by assuming that the external disturbances are inputs to the closed loop system. By that way, input to state stability analysis techniques could be utilized to deduce information on stability against the disturbance torques. The analysis is performed by rewrititng the control Lyapunov functions in the form of inequalities. The properties of vector norms provide the basic tools required for that analysis. In this way, the analysis can be performed in the dissipative sense [Sonntag (1995)]. Some input to state stability definitions are presented in Chapter 3.

### 1.8 Outline

The organization of this research report can be summarized as shown below:
Chapter 2: In this chapter the mathematical model of the satellite that will be used throughout this work is presented. The mathematical model includes the kinematical and dynamical equations, reaction wheel and motor dynamics.

Chapter 3: The theoretical background of the back - stepping and feedback linearization methods are presented in this chapter. The back - stepping problem is discussed by solving a small example problem. In the last section, the input to state stability problem is also introduced (ISSADT).

Chapter 4: The back - stepping approach presented in Chapter 3 is applied to the model presented in Chapter 2. In this chapter, the attitude and wheel velocity control laws are presented. The stability of the closed loop against the disturbance torques are analyzed using input - to - state stability analysis methods presented in Chapter 3.

Chapter 5: In this chapter the attitude control problem is solved by input output feedback linearization and the functions to be used in the simulations are prepared.

Chapter 6: The designs presented in Chapter 4 and Chapter 5 are completed here in numerical basis. Simulations are performed in order to verify the controllers and check robustness against parametric uncertainties in the satellite and motor models. The results of both approaches are presented in forms of figures for easier comparison.

Chapter 7: The concluding remarks and future plans related to the research are presented.

Appendix A: An introduction to the general attitude kinematics is provided

### 1.9 Contributions

1. The back - stepping theory is applied to an attitude control problem where the attitude is represented in terms of the Modified Rodriguez Parameters. That constitutes a contribution to the control literature that covers the back stepping theory.
2. The satellite model of this research is different from the frequently used sixth order kinematical - dynamical pair due to its inclusion of the reaction wheel
dynamics. So one has a $12^{\text {th }}$ order model where the reaction wheel velocity and torque are included as state variables. This allows the designer to implement both an attitude controller including a motor speed control system. In reaction wheel systems manufacturers, the wheels are either torque or speed controlled. This is another contribution to the literature concerning the practical issues.
3. A method for analyzing the stability of the controlled satellite models against the external disturbance torques is proposed. The approach is based on input to state stability theory in the dissipative sense. The effectiveness of this approach is also investigated on back - stepping and IOFL attitude controllers and showed that it is too conservative for IOFL configuration.

### 1.10 Publications from This Work

1. Doruk, R.Ö., Kocaoğlan, E., Satellite attitude control by MRP based back stepping, Aircraft Engineering \& Aerospace Technology, Vol 80, Issue 1, 2008
2. Doruk, R.Ö., Kocaoğlan, E., An almost disturbance decoupling solution of the attitude control problem, Aircraft Engineering \& Aerospace Technology, Vol 80, Issue 3, 2008

The followings are in preparation:
3. Doruk, R.Ö., Kocaoğlan, E., Application of back - stepping to MRP based attitude and reaction wheel speed control in LEO satellites
4. Doruk, R.Ö., Kocaoğlan, E., Application of feedback linearization to MRP based attitude control

# CHAPTER 2 <br> ATTITUDE DYNAMICS OF THE SATELLITE 

### 2.1 Introduction

In this chapter, the nonlinear model of the satellite that is used throughout this research is introduced. The model has some differences from the classical sixth order models used in the relevant literature. The obvious difference is the addition of the reaction wheel dynamics including the model of three brushless DC motors. Secondly, the Modified Rodriguez Parameters are preferred instead of the widely used quaternion since they have some advantages over the quaternion and Euler angles considering the nature of the nonlinear control approaches of this work.

### 2.2 Coordinate Axes

There are basically three coordinate axes that are used for positional referencing in satellite attitude control. Those are earth centered inertial (ECI), orbit frame and body coordinate axes systems. Their definitions are presented below:

### 2.2.1 Earth Centered Inertial (ECI) Coordinate System

This is the primary coordinate axes system, the origin of which is on the center of mass of the earth. Its direction is fixed relative to the solar system. The $Z$ axis is directed towards the north celestial pole, X axis has a direction towards the vernal equinox and $Y$ axis forms the equatorial plane together with the $X$ axis. It is denoted by $i$ in the variables throughout this text. It is shown in Figure 2-1.

### 2.2.2 Orbit Frame (Coordinate Axes) System

The satellites orbiting around the earth uses the orbit frame as the reference point. Its $Z$ axis points the center of earth, $X$ axis directs towards the motional direction of the satellite and $Y$ axis completes the coordinate axis system according to the right hand rule. It is perpendicular to the orbital plane. The orbit coordinate system is denoted by $o$. It is shown in Figure 2-2.


Figure 2-1 Earth centered inertial (ECI) frame (courtesy of [Antonsen (2004)])


Figure 2-2 The orbit and body reference frames (courtesy of [Antonsen (2004)])

### 2.2.3 Body Coordinate System

The origin of the body frame is centered at the center of mass of the satellite body. Axes of this frame are rotating with the satellite. The rotation around the $X, Y$ and $Z$ axes of the body frame is called as roll, pitch and yaw respectively (Figure $2-3)$. It is denoted by $b$.


Figure 2-3 The roll, pitch and yaw rotations around the body axes (courtesy of [Antonsen (2004)])

### 2.3 The Kinematics

The kinematics of the satellite model is the part that is related with the attitude and angular velocities. In terms of the Modified Rodriguez Parameters the attitude differential kinematic equation is shown below [Shuster (1993)]:

$$
\begin{equation*}
\dot{\boldsymbol{\sigma}}=\mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b} \tag{2.1}
\end{equation*}
$$

In the above representation $\boldsymbol{\sigma}=\left[\begin{array}{lll}\sigma_{1} & \sigma_{2} & \sigma_{3}\end{array}\right]^{T}$ is the attitude vector in terms of the Modified Rodriguez Parameters, $\boldsymbol{\omega}_{o b}^{b}=\left[\begin{array}{lll}p & q & r\end{array}\right]^{T}$ is the body coordinate angular velocity vector represented in body frame with respect to the orbit frame [Topland (2004)]. The matrix $\mathbf{G}(\boldsymbol{\sigma})$ is a fully invertible kinematical matrix defined as:

$$
\begin{equation*}
\mathbf{G}(\boldsymbol{\sigma})=\frac{1}{2}\left(\frac{1-\boldsymbol{\sigma}^{\mathrm{T}} \boldsymbol{\sigma}}{2} \mathbf{I}_{3 \times 3}+\mathbf{S}(\boldsymbol{\sigma})+\boldsymbol{\sigma} \boldsymbol{\sigma}^{\mathrm{T}}\right) \tag{2.2}
\end{equation*}
$$

where $\mathbf{S}(\boldsymbol{\sigma})$ is the skew symmetric matrix operator defined by the following expression:

$$
\mathbf{v}=\left[\begin{array}{l}
v_{1}  \tag{2.3}\\
v_{2} \\
v_{3}
\end{array}\right], \mathbf{S}(\mathbf{v})=\left[\begin{array}{ccc}
0 & -v_{3} & v_{2} \\
v_{3} & 0 & -v_{1} \\
-v_{2} & v_{1} & 0
\end{array}\right]
$$

The conversion between the real attitude vector (more truly speaking the roll, pitch and yaw angles) and the modified Rodriguez parameters are performed through the rotation quaternion. Details concerning those transformations are presented in the Appendix A. Here, only the results are given for convenience.

### 2.3.1 Conversion from Euler Angles to MRP

Each of the MRP can be expressed in terms of the rotation quaternion elements as shown below:

$$
\begin{equation*}
\sigma_{i}=\frac{\varepsilon_{i}}{1+\eta}, \forall i=1,2,3 \tag{2.4}
\end{equation*}
$$

In the above, the rotation quaternion is an element of the four dimensional real space and can be mathematically be expressed in several forms one of which is the vector form as $\mathbf{q}=\left[\varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3}, \eta\right]^{T}$. In this case, unit quaternion is a unity norm vector i.e. $\|\mathbf{q}\|=1$. To express a rotation, the quaternion should be normalized to unit norm. There is a possibility of instability when the scalar part $\eta=-1$ so the it should be replaced with the equivalent quaternion $\overline{\mathbf{q}}=-\mathbf{q}$ [Turner (2002)]. The quaternion can be obtained from the Euler angles according to (2.5).

$$
\mathbf{q}=\left[\begin{array}{l}
\cos \left(\frac{\phi}{2}\right) \cos \left(\frac{\theta}{2}\right) \cos \left(\frac{\psi}{2}\right)+\sin \left(\frac{\phi}{2}\right) \sin \left(\frac{\theta}{2}\right) \sin \left(\frac{\psi}{2}\right)  \tag{2.5}\\
\sin \left(\frac{\phi}{2}\right) \cos \left(\frac{\theta}{2}\right) \cos \left(\frac{\psi}{2}\right)-\cos \left(\frac{\phi}{2}\right) \sin \left(\frac{\theta}{2}\right) \sin \left(\frac{\psi}{2}\right) \\
\cos \left(\frac{\phi}{2}\right) \sin \left(\frac{\theta}{2}\right) \cos \left(\frac{\psi}{2}\right)+\sin \left(\frac{\phi}{2}\right) \cos \left(\frac{\theta}{2}\right) \sin \left(\frac{\psi}{2}\right) \\
\cos \left(\frac{\phi}{2}\right) \cos \left(\frac{\theta}{2}\right) \sin \left(\frac{\psi}{2}\right)-\sin \left(\frac{\phi}{2}\right) \sin \left(\frac{\theta}{2}\right) \cos \left(\frac{\psi}{2}\right)
\end{array}\right]
$$

### 2.3.2 Conversion from MRP to Euler Angles

This case is the reverse of the procedure defined in 2.3.1. The first step is the conversion of the MRP to the rotation quaternion. The necessary relationship for that operation is presented below:

$$
\mathbf{q}=\left[\begin{array}{c}
1-\sigma_{1}^{2}-\sigma_{2}^{2}-\sigma_{3}^{2}  \tag{2.6}\\
2 \sigma_{1} \\
2 \sigma_{2} \\
2 \sigma_{3}
\end{array}\right]
$$

In the above it is again a must to normalize $\mathbf{q}$. The second step is to convert the above formed quaternion into Euler angles as shown in the following:

$$
\left[\begin{array}{c}
\phi  \tag{2.7}\\
\theta \\
\psi
\end{array}\right]=\left[\begin{array}{c}
\tan ^{-1} \frac{2\left(\eta \varepsilon_{1}+\varepsilon_{2} \varepsilon_{3}\right)}{1-2\left(\varepsilon_{1}^{2}+\varepsilon_{3}^{2}\right)} \\
\sin ^{-1}\left(2\left[\eta \varepsilon_{2}+\varepsilon_{3} \varepsilon_{1}\right]\right) \\
\tan ^{-1} \frac{2\left(\eta \varepsilon_{3}+\varepsilon_{1} \varepsilon_{2}\right)}{1-2\left(\varepsilon_{2}^{2}+\varepsilon_{3}^{2}\right)}
\end{array}\right]
$$

The conversion given above is intended for the visualization of the satellite body motion. This restriction is due to the limitations coming from the trigonometric functions used in the transformation (2.7) as the inverse sine and tangent functions are valid in the range $-\pi / 2<\gamma<\pi / 2$.

Completion of the kinematical derivation requires an additional kinematical term which is the rotation matrix. For the rotations from the orbit frame to the body frame, it is defined in terms of the MRP as shown in below:

$$
\begin{align*}
& \mathbf{R}_{o}^{b}=\frac{1}{\Lambda^{2}}\left[\begin{array}{ccc}
4\left(\sigma_{1}^{2}-\sigma_{2}^{2}-\sigma_{3}^{2}\right)+\Sigma^{2} & 8 \sigma_{1} \sigma_{2}+4 \sigma_{3} \Sigma & 8 \sigma_{1} \sigma_{3}-4 \sigma_{2} \Sigma \\
8 \sigma_{2} \sigma_{1}-4 \sigma_{3} \Sigma & 4\left(-\sigma_{1}^{2}+\sigma_{2}^{2}-\sigma_{3}^{2}\right)+\Sigma^{2} & 8 \sigma_{2} \sigma_{3}-4 \sigma_{1} \Sigma \\
8 \sigma_{3} \sigma_{1}+4 \sigma_{2} \Sigma & 8 \sigma_{3} \sigma_{2}-4 \sigma_{1} \Sigma & 4\left(-\sigma_{1}^{2}-\sigma_{2}^{2}+\sigma_{3}^{2}\right)+\Sigma^{2}
\end{array}\right] \\
& \Sigma=\left(1-\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}\right), \Lambda=\left(1+\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}\right)  \tag{2.8}\\
& \mathbf{R}_{o}^{b}=\left[\begin{array}{lll}
\mathbf{c}_{1} & \mathbf{c}_{2} & \mathbf{c}_{3}
\end{array}\right]
\end{align*}
$$

### 2.4 The Non - Uniqueness Problem in Attitude Kinematics

Like that of the rotation quaternion the Modified Rodriguez Parameter representation is not unique. In the quaternion case, there is an equivalent set defined by $\overline{\mathbf{q}}=-\mathbf{q}$ exists. This means that the quaternions $\mathbf{q}$ and $-\mathbf{q}$ represent the
same rotations. A similar situation exists in the MRP. One can find this relation by following the conversion formulas in Section 2.3.1 (the detailed derivation is given in [Shuster (1993)]). As a result it can be obtained that

$$
\begin{equation*}
\boldsymbol{\sigma} \text { and }-\frac{1}{\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}} \boldsymbol{\sigma} \tag{2.9}
\end{equation*}
$$

represents the same rotation and equivalent to $\mathbf{q}$ and $-\mathbf{q}$ respectively. In order to solve this ambiguity many algorithms use a switching approach that is shown below:

$$
\boldsymbol{\sigma}=\left\{\begin{array}{cc}
\boldsymbol{\sigma} & \text { if } \boldsymbol{\sigma}^{T} \boldsymbol{\sigma} \leq 1  \tag{2.10}\\
-\frac{1}{\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}} \boldsymbol{\sigma} & \text { otherwise }
\end{array}\right.
$$

As a result of the above manipulation the magnitude of the attitude is limited in the region $(\|\boldsymbol{\sigma}\| \leq 1)$. That corresponds to a total rotation of $|\boldsymbol{\varphi}| \leq 180^{\circ}$. The details concerning the derivation of various attitude representations are provided as a summary in Appendix A.

### 2.5 Dynamics of Satellite Model

The dynamics of the satellite model is derived by using three dimensional Newtonian dynamics. Before going into the three dimensional derivation a one dimensional graphical depiction of the satellite body is helpful in the understanding of the interaction between the satellite body and the reaction wheel actuators. Such a representation is presented in Figure 2-4.

The satellite body is activated by the reaction wheel mounted on the center of gravity of the satellite body (the figure is provided for description purposes so it is not to scale). The reaction wheel is driven by a brushless dc motor that provides the input torque $\boldsymbol{\tau}_{a}$. This is an internal torque viewed from the satellite body. An external torque $\boldsymbol{\tau}_{e}$ is exerted on the satellite body. This can be either thrust or sum of disturbance torques sourced from the surrounding space (or both). For complete control, at least three reaction wheels are required if no other attitude control hardware is used. In this research, three reaction wheels are mounted on the center of gravity of the satellite body. The axis of rotation of each reaction wheel rotor is aligned with the satellite body fixed coordinate frame. Since the reaction wheels rotors are operating only in the axial direction no transversal components are taken into consideration.


Figure 2-4 a one dimensional depiction of the satellite body with the reaction wheel

Currently, it is possible to derive the satellite model in three dimensional space. To do that a three dimensional graphical depiction is provided in Figure 2-5

In the representation above the key variables and symbols are described below:
$O_{i}$ : is the origin of the inertial reference frame
$O_{b}$ : is the origin of the body reference frame
$F_{i}$ : is the symbol depicting the inertial reference frame
$F_{b}$ : is the symbol depicting the body reference frame
$\boldsymbol{\omega}_{i b}^{b}$ : is the angular velocity of the satellite body with respect to the inertial reference frame
$\boldsymbol{\omega}_{s}$ : is the angular velocity of the reaction wheel rotors with respect to the body reference frame
$\boldsymbol{\omega}_{r}$ : is the angular velocity of the reaction wheel rotors with respect to the inertial reference frame $\left(\boldsymbol{\omega}_{r}=\boldsymbol{\omega}_{i b}^{b}+\boldsymbol{\omega}_{s}\right)$
$\mathbf{h}_{s}$ : is the angular momentum of the reaction wheel rotors relative to the body reference frame
$\mathbf{h}$ : is the absolute angular momentum of the satellite body (with respect to the inertial reference frame)
$\mathbf{h}_{a}$ : is the absolute angular momentum of the reaction wheel rotors (with respect to the inertial reference frame)

Before proceeding it is convenient to present the assumptions used in the derivation of the satellite model used in this research:

1. Reaction wheels are mounted on the center of mass of the satellite body
2. The translational motion of the satellite along the system orbit is not taken into account.
3. The velocity of the reaction wheels are small and they have approximately zero velocity in most of the operational time. Thus they have almost zero mean velocity
4. The rate of change of the body axes angular velocities are very small.
5. Due to the above assumptions the transversal components of the reaction wheel moment of inertia are not effective on the operation of the wheels and thus neglected. So the remaining parameters of the satellite model are presented in the following:
$\mathbf{I}_{s}$ : is the mass moment of inertia of the reaction wheel rotors (transversal components are not existent as explained in the above assumptions) and shown below:

$$
\mathbf{I}_{s}=\left[\begin{array}{ccc}
i_{s} & 0 & 0  \tag{2.11}\\
0 & i_{s} & 0 \\
0 & 0 & i_{s}
\end{array}\right]
$$

I: is the mass moment of the inertia of the satellite body (symmetric matrix)

$$
\mathbf{I}=\left[\begin{array}{lll}
I_{x x} & I_{x y} & I_{x z}  \tag{2.12}\\
I_{x y} & I_{y y} & I_{y z} \\
I_{x z} & I_{y z} & I_{z z}
\end{array}\right]
$$

The derivation of the satellite model starts with the definition of the relative angularmomentum of the reaction wheel rotors:

$$
\begin{equation*}
\mathbf{h}_{s}=\mathbf{I}_{s} \boldsymbol{\omega}_{s} \tag{2.13}
\end{equation*}
$$

The body contribution to the total angular momentum is:

$$
\begin{equation*}
\mathbf{h}_{b}=\mathbf{I} \boldsymbol{\omega}_{i b}^{b} \tag{2.14}
\end{equation*}
$$

So the absolute angular momentum of the satellite body is:

$$
\begin{equation*}
\mathbf{h}=\mathbf{h}_{b}+\mathbf{h}_{s}=\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s} \tag{2.15}
\end{equation*}
$$

The absolute angular momentum of the reaction wheel rotors are:

$$
\begin{equation*}
\mathbf{h}_{a}=\mathbf{I}_{s}\left(\boldsymbol{\omega}_{i b}^{b}+\boldsymbol{\omega}_{s}\right)=\mathbf{I}_{s} \boldsymbol{\omega}_{r} \tag{2.16}
\end{equation*}
$$

Since the body fixed frame is rotating with an angular velocity of $\boldsymbol{\omega}_{i b}^{b}$, the angular momentum derivative is affected. Mathematically, that is expressed as shown below:

$$
\begin{equation*}
\dot{\mathbf{h}}=\boldsymbol{\tau}_{e}+\boldsymbol{\omega}_{i b}^{b} \times \mathbf{h}=\boldsymbol{\tau}_{e}+\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right) \mathbf{h} \tag{2.17}
\end{equation*}
$$



Figure 2-5 A depiction of the satellite body with three reaction wheel components

The absolute angular momentum of the satellite body is affected from the external torque $\left(\boldsymbol{\tau}_{e}\right)$. For the angular momentum of the reaction wheel rotors a similar expression is written as:

$$
\begin{equation*}
\dot{\mathbf{h}}_{a}=\boldsymbol{\tau}_{a}+\boldsymbol{\omega}_{r} \times \mathbf{h}_{a}=\boldsymbol{\tau}_{a}+\mathbf{S}\left(\boldsymbol{\omega}_{r}\right) \mathbf{h}_{a} \tag{2.18}
\end{equation*}
$$

Substituting from (2.16):

$$
\begin{equation*}
\dot{\mathbf{h}}_{a}=\boldsymbol{\tau}_{a}+\boldsymbol{\omega}_{r} \times \mathbf{h}_{a}=\boldsymbol{\tau}_{a}+\mathbf{S}\left(\boldsymbol{\omega}_{r}\right) \mathbf{I}_{s} \boldsymbol{\omega}_{r} \tag{2.19}
\end{equation*}
$$

For the $\mathbf{I}_{s}$ given in (2.11) the term $\mathbf{S}\left(\boldsymbol{\omega}_{r}\right) \mathbf{I}_{s} \boldsymbol{\omega}_{r}$ becomes zero so the above dynamics reduces to

$$
\begin{equation*}
\dot{\mathbf{h}}_{a}=\boldsymbol{\tau}_{a} \tag{2.20}
\end{equation*}
$$

Before continuing, it is convenient to rewrite the absolute angular momentum equations together as shown below:

$$
\begin{align*}
& \dot{\mathbf{h}}=\boldsymbol{\tau}_{e}+\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left\{\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right\}  \tag{2.21}\\
& \dot{\mathbf{h}}_{a}=\boldsymbol{\tau}_{a}
\end{align*}
$$

The next step is to obtain the above derivatives once again by direct differentiation of the angular momentum definitions given in (2.15) and (2.16). That is:

$$
\begin{align*}
& \dot{\mathbf{h}}=\mathbf{I} \dot{\boldsymbol{\omega}}_{i b}^{b}+\mathbf{I}_{s} \dot{\boldsymbol{\omega}}_{s}  \tag{2.22}\\
& \dot{\mathbf{h}}_{a}=\mathbf{I}_{s} \dot{\boldsymbol{\omega}}_{i b}^{b}+\mathbf{I}_{s} \dot{\boldsymbol{\omega}}_{s}
\end{align*}
$$

In order to make the derivation easier one can write the above representation in matrix form:

$$
\left[\begin{array}{c}
\dot{\mathbf{h}}  \tag{2.23}\\
\dot{\mathbf{h}}_{a}
\end{array}\right]=\left[\begin{array}{cc}
\mathbf{I} & \mathbf{I}_{s} \\
\mathbf{I}_{s} & \mathbf{I}_{s}
\end{array}\right]\left[\begin{array}{c}
\dot{\boldsymbol{\omega}}_{i b}^{b} \\
\dot{\boldsymbol{\omega}}_{s}
\end{array}\right]
$$

Taking the inverse of the above linear equation yields:

$$
\left[\begin{array}{ll}
\mathbf{I} & \mathbf{I}_{s}  \tag{2.24}\\
\mathbf{I}_{s} & \mathbf{I}_{s}
\end{array}\right]^{-1}\left[\begin{array}{c}
\dot{\mathbf{h}} \\
\dot{\mathbf{h}}_{a}
\end{array}\right]=\left[\begin{array}{c}
\dot{\boldsymbol{\omega}}_{i b}^{b} \\
\dot{\boldsymbol{\omega}}_{s}
\end{array}\right]
$$

and,

$$
\left[\begin{array}{cc}
\mathbf{I} & \mathbf{I}_{s}  \tag{2.25}\\
\mathbf{I}_{s} & \mathbf{I}_{s}
\end{array}\right]^{-1}\left[\begin{array}{c}
\boldsymbol{\tau}_{e}+\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left\{\mathbf{I}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right\} \\
\boldsymbol{\tau}_{a}
\end{array}\right]=\left[\begin{array}{l}
\boldsymbol{\omega}_{i b}^{b} \\
\dot{\boldsymbol{\omega}}_{s}
\end{array}\right]
$$

As a result:

$$
\left[\begin{array}{cc}
\mathbf{J}^{-1} & -\mathbf{J}^{-1}  \tag{2.26}\\
-\mathbf{J}^{-1} & \mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}
\end{array}\right]\left[\begin{array}{c}
\boldsymbol{\tau}_{e}+\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left\{\mathbf{I}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right\} \\
\boldsymbol{\tau}_{a}
\end{array}\right]=\left[\begin{array}{l}
\boldsymbol{\omega}_{i b}^{b} \\
\boldsymbol{\omega}_{s}
\end{array}\right]
$$

The dynamic equations of motion of the satellite in this research are obtained as:

$$
\begin{align*}
& \dot{\boldsymbol{\omega}}_{i b}^{b}=\mathbf{J}^{-1}\left[\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left\{\mathbf{\mathbf { \omega } _ { i b } ^ { b }}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right\}+\boldsymbol{\tau}_{e}\right]-\mathbf{J}^{-1} \boldsymbol{\tau}_{a}  \tag{2.27}\\
& \dot{\boldsymbol{\omega}}_{s}=-\mathbf{J}^{-1}\left[\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left\{\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right\}+\boldsymbol{\tau}_{e}\right]+\left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right) \boldsymbol{\tau}_{a}
\end{align*}
$$

As it is obviously seen, the above dynamics is based on the angular velocity of the body with respect to the earth centered inertial coordinate axes $\left(\boldsymbol{\omega}_{i b}^{b}\right)$. As one need the angular velocity in the body frame referenced at the orbit frame $\left(\boldsymbol{\omega}_{o b}^{b}\right)$ the dynamical state equations are formed as (using the fact $\boldsymbol{\omega}_{o b}^{b}=\boldsymbol{\omega}_{i b}^{b}-\mathbf{R}_{o}^{b} \boldsymbol{\omega}_{i o}^{o}$ ):

$$
\begin{align*}
& \dot{\boldsymbol{\omega}}_{o b}^{b}=\mathbf{J}^{-1}\left[-\mathbf{S}\left(\boldsymbol{\omega}_{o b}^{b}+\mathbf{R}_{o}^{b} \boldsymbol{\omega}_{i o}^{o}\right)\left(\mathbf{I}\left[\boldsymbol{\omega}_{o b}^{b}+\mathbf{R}_{o}^{b} \boldsymbol{\omega}_{i o}^{o}\right]+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\boldsymbol{\tau}_{e}\right] \\
& -\mathbf{J}^{-1} \boldsymbol{\tau}_{a}+\mathbf{S}\left(\boldsymbol{\omega}_{o b}^{b}\right) \mathbf{R}_{o}^{b} \boldsymbol{\omega}_{i o}^{o}-\mathbf{R}_{o}^{b} \boldsymbol{\omega}_{i o}^{o} \\
& \dot{\boldsymbol{\omega}}_{s}=-\mathbf{J}^{-1}\left[-\mathbf{S}\left(\boldsymbol{\omega}_{o b}^{b}+\mathbf{R}_{o}^{b} \boldsymbol{\omega}_{i o}^{o}\right)\left(\mathbf{I}\left[\boldsymbol{\omega}_{o b}^{b}+\mathbf{R}_{o}^{b} \boldsymbol{\omega}_{i o}^{o}\right]+\mathbf{I}_{s}\right)+\boldsymbol{\omega}_{s}\right]  \tag{2.28}\\
& +\left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right) \boldsymbol{\tau}_{a}
\end{align*}
$$

In this research the orbital angular velocity vector is assumed constant and equal to the $\boldsymbol{\omega}_{i o}^{o}=\left[\begin{array}{lll}0 & -\omega_{o} & 0\end{array}\right]^{T}$. So the equations are simplified to:

$$
\begin{align*}
& \dot{\boldsymbol{\omega}}_{o b}^{b}=\mathbf{J}^{-1}\left[-\mathbf{S}\left(\boldsymbol{\omega}_{o b}^{b}-\omega_{o} \mathbf{c}_{2}\right)\left(\mathbf{I}\left[\boldsymbol{\omega}_{o b}^{b}-\omega_{o} \mathbf{c}_{2}\right]+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\boldsymbol{\tau}_{e}\right]-\mathbf{J}^{-1} \boldsymbol{\tau}_{a}-\mathbf{S}\left(\boldsymbol{\omega}_{o b}^{b}\right) \omega_{o} \mathbf{c}_{2} \\
& \dot{\boldsymbol{\omega}}_{s}=-\mathbf{J}^{-1}\left[-\mathbf{S}\left(\boldsymbol{\omega}_{o b}^{b}-\omega_{o} \mathbf{c}_{2}\right)\left(\mathbf{I}\left[\boldsymbol{\omega}_{o b}^{b}-\omega_{o} \mathbf{c}_{2}\right]+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\boldsymbol{\tau}_{e}\right]+\left[\mathbf{J}^{-1}+\mathbf{I}_{s}^{-1}\right] \boldsymbol{\tau}_{a} \tag{2.29}
\end{align*}
$$

The exogenous disturbance torque includes several components. Most common ones are the gravitational $\left(\boldsymbol{\tau}_{\text {grav }}\right)$, aerodynamic $\left(\boldsymbol{\tau}_{\text {aero }}\right)$, solar pressure $\left(\boldsymbol{\tau}_{\text {sunn }}\right)$ and magnetic $\left(\boldsymbol{\tau}_{\text {mag }}\right)$ disturbance torques. Total disturbance action on the satellite is expressed as $\left(\boldsymbol{\tau}_{e}=\boldsymbol{\tau}_{g r a v}+\boldsymbol{\tau}_{\text {aero }}+\boldsymbol{\tau}_{\text {sum }}+\boldsymbol{\tau}_{\text {mag }}\right)$. In this research, the gravitational component of the disturbance is taken into consideration and expressed mathematically as:

$$
\begin{equation*}
\boldsymbol{\tau}_{g r a v}=3 \omega_{o}^{2} \mathbf{S}\left(\mathbf{c}_{3}\right) \mathbf{I} \mathbf{c}_{3} \tag{2.30}
\end{equation*}
$$

The terms $\mathbf{c}_{2} \& \mathbf{c}_{3}$ are represented in the expression of rotation matrix in the equation (2.8). The aerodynamic torque is negligible in this work since the dimension of the satellite is so small. The same is valid for the magnetic and sun pressure. So in this research the disturbance torque is modeled as the gravitational torque in (2.30) and a constant disturbance torque that models any possible disturbance that may be existent during operation.

### 2.6 A Model for the Brushless DC Motor of the Reaction Wheel

The reaction wheel dynamics in the last section is derived from momentum relationships and taking the rotor torque as a system input. However, this leads to a redundancy since the torque input affects both of the rate variables ( $\boldsymbol{\omega}_{o b}^{b}$ or $\boldsymbol{\omega}_{s}$ ). The reaction wheel dynamics present the interaction between the actuator (reaction wheel rotors) and the torque supplying motor. In order to complete the model it is convenient to add the mathematics of a brushless dc motor. Since there are three reaction wheels in the satellite, there will be three dc motors and they are modeled as [Kristiansen (2000)]:

$$
\begin{gather*}
\frac{d i_{x}}{d t}=-\frac{R_{x}}{L_{x}} i_{x}+\frac{1}{L_{x}} K_{E}^{x} \omega_{s}^{x}+\frac{1}{L_{x}} U_{a}^{x} \\
\frac{d i_{y}}{d t}=-\frac{R_{y}}{L_{y}} i_{y}+\frac{1}{L_{y}} K_{E}^{y} \omega_{s}^{y}+\frac{1}{L_{y}} U_{a}^{y}  \tag{2.31}\\
\frac{d i_{z}}{d t}=--\frac{R_{z}}{L_{z}} i_{z}+\frac{1}{L_{z}} K_{E}^{z} \omega_{s}^{z}+\frac{1}{L_{z}} U_{a}^{z} \\
\tau_{a}^{x}=K_{T}^{x} i_{x} \\
\tau_{a}^{y}=K_{T}^{y} i_{y}  \tag{2.32}\\
\tau_{a}^{z}=K_{T}^{z} i_{z}
\end{gather*}
$$

In the vectoral form, $(2.31)$ is rewritten as:

$$
\begin{aligned}
& \dot{\mathbf{I}}_{a}=-\mathbf{R}_{L} \mathbf{I}_{a}+\mathbf{K}_{E} \boldsymbol{\omega}_{s}+\mathbf{L} \mathbf{U}_{a}
\end{aligned}
$$

and (2.32) is rewritten as $\boldsymbol{\tau}_{a}=\mathbf{K}_{T} \mathbf{I}_{a}$ where $\mathbf{K}_{T}$ is:

$$
\mathbf{K}_{T}=\left[\begin{array}{ccc}
K_{T}^{x} & 0 & 0  \tag{2.34}\\
0 & K_{T}^{y} & 0 \\
0 & 0 & K_{T}^{z}
\end{array}\right]
$$

The definitions of the parameters are:
$R_{x, y, z}$ : Armature resistance
$L_{x, y, 2}$ : Armature inductance
$K_{T}^{x, y, z}$ : Torque constant
$K_{E}^{x, y, z}:$ EMF constant
$i_{x, y, z}$ : Armature current
$U_{a}^{x, y, z}$ : Input Voltage
$\boldsymbol{\omega}_{s}$ : Reaction wheel rotor velocity
$\boldsymbol{\tau}_{a}: \quad$ Reaction wheel torque (defined as a state)
After the substitution of the torque output vector (2.32) into (2.33):

$$
\begin{equation*}
\dot{\boldsymbol{\tau}}_{a}=-\mathbf{K}_{T} \mathbf{R}_{L} \mathbf{K}_{T}^{-1} \boldsymbol{\tau}_{a}+\mathbf{K}_{T} \mathbf{K}_{E} \boldsymbol{\omega}_{s}+\mathbf{K}_{T} \mathbf{L} \mathbf{U}_{a} \tag{2.35}
\end{equation*}
$$

The model assumes that the reaction wheel motors are armature controlled.

### 2.7 The Usage of the Model in Attitude Control

In the attitude control the model body angular velocity can be defined either in the ECI frame $\left(\boldsymbol{\omega}_{i b}^{b}\right)$ or in the orbit frame $\left(\boldsymbol{\omega}_{o b}^{b}\right)$. The common variables are attitude ( $\boldsymbol{\sigma}$ ) and reaction wheel velocity $\left(\boldsymbol{\omega}_{s}\right)$. In backstepping control ( $\boldsymbol{\omega}_{i b}^{b}$ ) is preferred as the state vector for an easier procedure. However, $\left(\boldsymbol{\omega}_{o b}^{b}\right)$ is fed back to the controller since the attitude information is obtained in reference to the orbit frame. In feedback linearization case, ( $\boldsymbol{\omega}_{o b}^{b}$ ) is chosen as the state vector due to the non - recursive nature of the IOFL technique. For both configurations the following nonlinear system representation is used:

$$
\begin{equation*}
\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x}) \mathbf{u}+\mathbf{p}(\mathbf{x}) \mathbf{w} \tag{2.36}
\end{equation*}
$$

The elements of the above representation are given below:
For demanded torque generator:

$$
\begin{align*}
& \mathbf{x}=\left[\begin{array}{ll}
\boldsymbol{\sigma}^{T} & \left(\boldsymbol{\omega}_{i b}^{b}\right)^{T}
\end{array}\right]^{T} \\
& \mathbf{f}(\mathbf{x})=\left[\begin{array}{c}
\mathbf{G}(\boldsymbol{\sigma})\left\{\boldsymbol{\omega}_{i b}^{b}+\mathbf{c}_{2} \omega_{o}\right\} \\
\mathbf{J}^{-1}\left[-\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)\right]
\end{array}\right]  \tag{2.37}\\
& \mathbf{g}(\mathbf{x})=\left[\begin{array}{c}
\mathbf{0}_{3 \times 3} \\
-\mathbf{J}^{-1}
\end{array}\right], \mathbf{p}(\mathbf{x})=\left[\begin{array}{c}
\mathbf{0}_{3 \times 3} \\
\mathbf{J}^{-1}
\end{array}\right] \\
& \mathbf{u}=\boldsymbol{\tau}_{a}, \mathbf{w}=\boldsymbol{\tau}_{e}
\end{align*}
$$

For the speed controller:

$$
\begin{align*}
& \mathbf{x}=\left[\begin{array}{ll}
\boldsymbol{\omega}_{s}^{T} & \boldsymbol{\tau}_{a}^{T}
\end{array}\right]^{T} \\
& \mathbf{f}(\mathbf{x})=\left[\begin{array}{c}
-\mathbf{J}^{-1}\left[-\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)\right]+\left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right) \boldsymbol{\tau}_{a} \\
\\
-\mathbf{K}_{T} \mathbf{R}_{L} \mathbf{K}_{T}^{-1} \boldsymbol{\tau}_{a}+\mathbf{K}_{T} \mathbf{K}_{E} \boldsymbol{\omega}_{s}
\end{array}\right]  \tag{2.38}\\
& \mathbf{g}(\mathbf{x})=\left[\begin{array}{c}
\mathbf{0}_{3 \times 3} \\
\mathbf{K}_{T} \mathbf{L}
\end{array}\right], \mathbf{p}(\mathbf{x})=\left[\begin{array}{c}
-\mathbf{J}^{-1} \\
\mathbf{0}_{3 \times 3}
\end{array}\right] \\
& \mathbf{u}=\mathbf{U}_{a}, \mathbf{w}=\boldsymbol{\tau}_{e}
\end{align*}
$$

If the body angular velocity is expressed in the orbit frame then the same components of the nonlinear model is:

For the demanded torque generator:

$$
\begin{align*}
& \mathbf{x}=\left[\begin{array}{ll}
\boldsymbol{\sigma}^{T} & \left(\boldsymbol{\omega}_{o b}^{b}\right)^{T}
\end{array}\right]^{T} \\
& \mathbf{f}(\mathbf{x})=\left[\begin{array}{c}
\mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b} \\
\mathbf{J}^{-1}\left[-\mathbf{S}\left(\boldsymbol{\omega}_{o b}^{b}-\mathbf{c}_{2} \omega_{o}\right)\left(\mathbf{I}\left[\boldsymbol{\omega}_{o b}^{b}-\mathbf{c}_{2} \omega_{o}\right]+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)\right]-\mathbf{S}\left(\boldsymbol{\omega}_{o b}^{b}\right) \mathbf{c}_{2} \omega_{o}
\end{array}\right]  \tag{2.39}\\
& \mathbf{g}(\mathbf{x})=\left[\begin{array}{c}
\mathbf{0}_{3 \times 3} \\
-\mathbf{J}^{-1}
\end{array}\right], \mathbf{p}(\mathbf{x})=\left[\begin{array}{c}
\mathbf{0}_{3 \times 3} \\
\mathbf{J}^{-1}
\end{array}\right] \\
& \mathbf{u}=\boldsymbol{\tau}_{a}, \mathbf{w}=\boldsymbol{\tau}_{e} \\
& \mathbf{x}=\left[\begin{array}{ll}
\boldsymbol{\omega}_{s}^{T} & \boldsymbol{\tau}_{a}^{T}
\end{array}\right]^{T} \\
& \mathbf{f}(\mathbf{x})=\left[\begin{array}{cc}
\left.-\mathbf{J}^{-1}\left[-\mathbf{S}\left(\boldsymbol{\omega}_{o b}^{b}-\omega_{o} \mathbf{c}_{2}\right)\left(\mathbf{I}\left[\boldsymbol{\omega}_{o b}^{b}-\omega_{o} \mathbf{c}_{2}\right]+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)\right]+\left[\mathbf{J}^{-1}+\mathbf{I}_{s}^{-1}\right] \boldsymbol{\tau}_{a}\right] \\
-\mathbf{K}_{T} \mathbf{R}_{L} \mathbf{K}_{T}^{-1} \boldsymbol{\tau}_{a}+\mathbf{K}_{T} \mathbf{K}_{E} \boldsymbol{\omega}_{s}
\end{array}\right]  \tag{2.40}\\
& \mathbf{g}(\mathbf{x})=\left[\begin{array}{c}
\mathbf{0}_{3 \times 3} \\
\mathbf{K}_{T} \mathbf{L}
\end{array}\right], \mathbf{p}(\mathbf{x})=\left[\begin{array}{c}
-\mathbf{J}^{-1} \\
\mathbf{0}_{3 \times 3}
\end{array}\right] \\
& \mathbf{u}=\mathbf{U}_{a}, \mathbf{w}=\boldsymbol{\tau}_{e}
\end{align*}
$$

The models given in the equations (2.37) to (2.40) are presented in separated forms i.e. the demanded torque generator and reaction wheel velocity dynamics are given in separated forms. The reason for this is that the overall attitude controller is designed in a cascaded form where the two sub - controllers are designed by using the separated models given in this chapter.

# CHAPTER 3 <br> THEORETICAL BACKGROUND 

### 3.1 Introduction

The purpose of this chapter is to present the theoretical background behind the attitude control approaches of this research. The theory of backstepping and input output feedback linearization (IOFL) are presented in a form suited for attitude control. To do that descriptive generalized system models are used. The implementation of those theoretical tools are presented in Chapters 4 and 5.

### 3.2 Theory behind Backstepping

Backstepping is a modular method of control. Descriptively, it is a recursive (or step by step) completion of the design process. This is generally performed through a partition of the state vector of the dynamical plant model. The sub - states obtained from that partition are stabilized one by one in each step.

### 3.2.1 An Example for a Generic Nonlinear System

The models used in the design procedures of this work should be in affine form such as the one shown below:

$$
\begin{align*}
& \dot{\mathbf{x}}_{1}=\mathbf{f}_{1}\left(\mathbf{x}_{1}\right) \mathbf{x}_{2} \\
& \dot{\mathbf{x}}_{2}=\mathbf{f}_{2}\left(\mathbf{x}_{1}, \mathbf{x}_{2}\right)+\mathbf{g}(\mathbf{x}) \mathbf{u} \tag{3.1}
\end{align*}
$$

where $\mathbf{x}_{1,2} \in \mathbb{R}^{m}, \mathbf{u} \in \mathbb{R}^{m}, \mathbf{x}=\left[\begin{array}{ll}\mathbf{x}_{1}^{T} & \mathbf{x}_{2}^{T}\end{array}\right]^{T} \in \mathbb{R}^{n=2 m}, \mathbf{g}(\mathbf{x}) \in \mathbb{R}^{m \times m}, \mathbf{f}_{1}(\mathbf{x}) \in \mathbb{R}^{m \times m}$ and $\mathbf{f}_{2}\left(\mathbf{x}_{1}, \mathbf{x}_{2}\right) \in \mathbb{R}^{m}$. It is assumed that $\mathbf{f}_{1}\left(\mathbf{x}_{1}\right)$ and $\mathbf{g}(\mathbf{x})$ are invertible matrices. The example is designed as a tracking problem where $\mathbf{x}_{1}(t)$ should track the desired trajectory defined by $\mathbf{x}_{1 d}(t)$.

### 3.2.2 Back - stepping Transformation

The back - stepping transformation is a state transformation mechanism to obtain the states of the closed loop after the solution of the problem. For tracking problems it will be convenient to select the new state variables as the additive errors of the states [Skjetne (2004)]. For the current example, the new state variables are obtained from the following transformation:

$$
\begin{align*}
& \mathbf{z}_{1}=\mathbf{x}_{1}-\mathbf{x}_{1 d}  \tag{3.2}\\
& \mathbf{z}_{2}=\mathbf{x}_{2}-\boldsymbol{\alpha}_{1}
\end{align*}
$$

where the new term $\boldsymbol{\alpha}_{1}$ is an artificial command input to produce an internal control loop. The next step is to differentiate them with respect to time and substitute the necessary functions from the system equation in (3.1).

### 3.2.3 Backstepping Control [Fossen (1997), Skjetne (2004)]

Differentiating the first virtual state variable $\mathbf{z}_{1}$ and substituting the first equation from (3.1) leads to:

$$
\begin{equation*}
\dot{\mathbf{z}}_{1}=\dot{\mathbf{x}}_{1}-\dot{\mathbf{x}}_{1 d}=\mathbf{f}_{1}\left(\mathbf{z}_{1}+\mathbf{x}_{1 d}\right) \mathbf{x}_{2}-\dot{\mathbf{x}}_{1 d} \tag{3.3}
\end{equation*}
$$

Then substitute $\mathbf{x}_{2}$ from (3.2) to obtain:

$$
\begin{equation*}
\dot{\mathbf{z}}_{1}=\mathbf{f}_{1}\left(\mathbf{z}_{1}+\mathbf{x}_{1 d}\right)\left(\mathbf{z}_{2}+\boldsymbol{\alpha}_{1}\right)-\dot{\mathbf{x}}_{1 d} \tag{3.4}
\end{equation*}
$$

For the sake of simplicity the term $\mathbf{f}_{1}\left(\mathbf{z}_{1}+\mathbf{x}_{1 d}\right)$ will be written as $\mathbf{f}_{1}\left(\mathbf{x}_{1}\right)$ throughout this section. The actual control design procedure starts with the definition of the Lyapunov function corresponding to $\mathbf{z}_{1}$. This can be a quadratic Lyapunov function like shown below:

$$
\begin{equation*}
V_{1}=\frac{1}{2} \mathbf{z}_{1}^{T} \mathbf{z}_{1} \tag{3.5}
\end{equation*}
$$

and,

$$
\begin{equation*}
\dot{V}_{1}=\frac{1}{2} \dot{\mathbf{z}}_{1}^{T} \mathbf{z}_{1}+\frac{1}{2} \mathbf{z}_{1}^{T} \dot{\mathbf{z}}_{1} \tag{3.6}
\end{equation*}
$$

So substituting $\dot{\mathbf{z}}_{1}$ from (3.4) yields:

$$
\begin{equation*}
\dot{V}_{1}=\frac{1}{2}\left[\left(\boldsymbol{\alpha}_{1}^{T}+\mathbf{z}_{2}^{T}\right) \mathbf{f}_{1}^{T}\left(\mathbf{x}_{1}\right)-\dot{\mathbf{x}}_{1 d}^{T}\right] \mathbf{z}_{1}+\frac{1}{2} \mathbf{z}_{1}^{T}\left[\mathbf{f}_{1}\left(\mathbf{x}_{1}\right)\left(\mathbf{z}_{2}+\boldsymbol{\alpha}_{1}\right)-\dot{\mathbf{x}}_{1 d}\right] \tag{3.7}
\end{equation*}
$$

A useful expansion of the above function is shown below:
$\dot{V}_{1}=\frac{1}{2}\left[\left(\boldsymbol{\alpha}_{1}^{T}\right) \mathbf{f}_{1}^{T}\left(\mathbf{x}_{1}\right)-\dot{\mathbf{x}}_{1 d}^{T}\right] \mathbf{z}_{1}+\frac{1}{2} \mathbf{z}_{1}^{T}\left[\mathbf{f}_{1}\left(\mathbf{x}_{1}\right)\left(\boldsymbol{\alpha}_{1}\right)-\dot{\mathbf{x}}_{1 d}\right]+\frac{1}{2} \mathbf{z}_{2}^{T} \mathbf{f}_{1}^{T}\left(\mathbf{x}_{1}\right) \mathbf{z}_{1}+\frac{1}{2} \mathbf{z}_{1}^{T} \mathbf{f}_{1}\left(\mathbf{x}_{1}\right) \mathbf{z}_{2}$
To obtain a stable dynamics on variable $\mathbf{z}_{1}$ one should obtain a negative definite quadratic function of $\mathbf{z}_{1}$. The remaining part (the terms involving $\mathbf{z}_{2}$ ) is eliminated in the second step. One can do the following for this purpose:

$$
\begin{align*}
& \boldsymbol{\alpha}_{1}^{T} \mathbf{f}_{1}^{T}\left(\mathbf{x}_{1}\right)-\dot{\mathbf{x}}_{1 d}^{T}=-\mathbf{z}_{1}^{T} \mathbf{K}_{1} \\
& \&  \tag{3.9}\\
& \mathbf{f}_{1}\left(\mathbf{x}_{1}\right) \boldsymbol{\alpha}_{1}-\dot{\mathbf{x}}_{1 d}=-\mathbf{K}_{1} \mathbf{z}_{1}
\end{align*}
$$

This is the replacement of the term $\mathbf{f}_{1}\left(\mathbf{x}_{1}\right) \boldsymbol{\alpha}_{1}-\dot{\mathbf{x}}_{1 d}$ (and thus its transposed version in the other half) with $-\mathbf{K}_{1} \mathbf{z}_{1}$. One can solve for $\boldsymbol{\alpha}_{1}$ here and obtain:

$$
\begin{equation*}
\boldsymbol{\alpha}_{1}=\mathbf{f}_{1}^{-1}\left(\mathbf{x}_{1}\right) \dot{\mathbf{x}}_{1 d}-\mathbf{f}_{1}^{-1}\left(\mathbf{x}_{1}\right) \mathbf{K}_{1} \mathbf{z}_{1} \tag{3.10}
\end{equation*}
$$

As a result the Lyapunov derivative is:

$$
\begin{equation*}
\dot{V}_{1}=-\mathbf{z}_{1}^{T} \mathbf{K}_{1} \mathbf{z}_{1}+\frac{1}{2} \mathbf{z}_{2}^{T} \mathbf{f}_{1}^{T}\left(\mathbf{x}_{1}\right) \mathbf{z}_{1}+\frac{1}{2} \mathbf{z}_{1}^{T} \mathbf{f}_{1}\left(\mathbf{x}_{1}\right) \mathbf{z}_{2} \tag{3.11}
\end{equation*}
$$

As it could be understood from the above, the matrix $\mathbf{K}_{1}$ should be symmetric and positive definite. Now one can continue through the second step starting with the differentiation of the second virtual variable $\mathbf{z}_{2}$ :

$$
\begin{align*}
& \mathbf{z}_{2}=\mathbf{x}_{2}-\boldsymbol{\alpha}_{1}  \tag{3.12}\\
& \dot{\mathbf{z}}_{2}=\dot{\mathbf{x}}_{2}-\dot{\boldsymbol{\alpha}}_{1}
\end{align*}
$$

Substitute from the system equation in (3.1) and obtain:

$$
\begin{equation*}
\dot{\mathbf{z}}_{2}=\mathbf{f}_{2}\left(\mathbf{x}_{1}, \mathbf{z}_{2}+\boldsymbol{\alpha}_{1}\right)+\mathbf{g}\left(\mathbf{x}_{1}\right) \mathbf{u}-\dot{\boldsymbol{\alpha}}_{1} \tag{3.13}
\end{equation*}
$$

Similar to (3.4) the term $\mathbf{f}_{2}\left(\mathbf{x}_{1}, \mathbf{z}_{2}+\boldsymbol{\alpha}_{1}\right)$ will be written as $\mathbf{f}_{2}\left(\mathbf{x}_{1}, \mathbf{x}_{2}\right)$. The second Lyapunov function (on $\mathbf{z}_{2}$ ) can also be a quadratic Lyapunov function like that of $\mathbf{z}_{1}$ (first step) and is added to $V_{1}$. That is:

$$
\begin{equation*}
V_{2}=V_{1}+\frac{1}{2} \mathbf{z}_{2}^{T} \mathbf{P}_{2} \mathbf{z}_{2} \tag{3.14}
\end{equation*}
$$

with $\mathbf{P}_{2}$ being positive definite. Differentiating $V_{2}$ and substituting from (3.13) yields:

$$
\begin{align*}
& \dot{V}_{2}=-\mathbf{z}_{1}^{T} \mathbf{K}_{1} \mathbf{z}_{1}+\frac{1}{2} \mathbf{z}_{2}^{T} \mathbf{f}_{1}^{T}\left(\mathbf{x}_{1}\right) \mathbf{z}_{1}+\frac{1}{2} \mathbf{z}_{1}^{T} \mathbf{f}_{1}\left(\mathbf{x}_{1}\right) \mathbf{z}_{2}  \tag{3.15}\\
& +\frac{1}{2}\left\{\mathbf{u}^{T} \mathbf{g}^{T}(\mathbf{x})+\mathbf{f}_{2}^{T}\left(\mathbf{x}_{1}, \mathbf{x}_{2}\right)-\dot{\boldsymbol{\alpha}}_{1}^{T}\right\} \mathbf{P}_{2} \mathbf{z}_{2}+\frac{1}{2} \mathbf{z}_{2}^{T} \mathbf{P}_{2}\left\{\mathbf{f}_{2}\left(\mathbf{x}_{1}, \mathbf{x}_{2}\right)+\mathbf{g}(\mathbf{x}) \mathbf{u}-\dot{\mathbf{\alpha}}_{1}\right\}
\end{align*}
$$

and,

$$
\begin{align*}
& \dot{V}_{2}=-\mathbf{z}_{1}^{T} \mathbf{K}_{1} \mathbf{z}_{1}+\frac{1}{2}\left\{\mathbf{u}^{T} \mathbf{g}^{T}(\mathbf{x}) \mathbf{P}_{2}+\mathbf{f}_{2}^{T}\left(\mathbf{x}_{1}, \mathbf{x}_{2}\right) \mathbf{P}_{2}-\dot{\mathbf{\alpha}}_{1}^{T} \mathbf{P}_{2}+\mathbf{z}_{1}^{T} \mathbf{f}_{1}\left(\mathbf{x}_{1}\right)\right\} \mathbf{z}_{2}+ \\
& \frac{1}{2} \mathbf{z}_{2}^{T}\left\{\mathbf{P}_{2} \mathbf{f}_{2}\left(\mathbf{x}_{1}, \mathbf{x}_{2}\right)+\mathbf{P}_{2} \mathbf{g}(\mathbf{x}) \mathbf{u}-\mathbf{P}_{2} \dot{\mathbf{\alpha}}_{1}+\mathbf{f}_{1}^{T}\left(\mathbf{x}_{1}\right) \mathbf{z}_{1}\right\} \tag{3.16}
\end{align*}
$$

Like in the first step of the procedure it is required to replace the non quadratic terms in the above function with negative definite terms so one can do the following:

$$
\begin{align*}
& \mathbf{P}_{2} \mathbf{f}_{2}\left(\mathbf{x}_{1}, \mathbf{x}_{2}\right)+\mathbf{P}_{2} \mathbf{g}(\mathbf{x}) \mathbf{u}-\mathbf{P}_{2} \dot{\boldsymbol{\alpha}}_{1}+\mathbf{f}_{1}^{T}\left(\mathbf{x}_{1}\right) \mathbf{z}_{1}=-\mathbf{K}_{2} \mathbf{z}_{2}  \tag{3.17}\\
& \mathbf{u}^{T} \mathbf{g}^{T}(\mathbf{x}) \mathbf{P}_{2}+\mathbf{f}_{2}^{T}\left(\mathbf{x}_{1}, \mathbf{x}_{2}\right) \mathbf{P}_{2}-\dot{\boldsymbol{\alpha}}_{1}^{T} \mathbf{P}_{2}+\mathbf{z}_{1}^{T} \mathbf{f}_{1}\left(\mathbf{x}_{1}\right)=-\mathbf{z}_{2}^{T} \mathbf{K}_{2}
\end{align*}
$$

And the control input u can now be obtained as shown in below:

$$
\begin{equation*}
\mathbf{u}=-\mathbf{g}^{-1}(\mathbf{x})\left(\mathbf{P}_{2}\right)^{-1}\left\{\mathbf{K}_{2} \mathbf{z}_{2}+\mathbf{P}_{2} \mathbf{f}_{2}\left(\mathbf{x}_{1}, \mathbf{x}_{2}\right)+\mathbf{f}_{1}^{T}\left(\mathbf{x}_{1}\right) \mathbf{z}_{1}-\mathbf{P}_{2} \dot{\boldsymbol{\alpha}}_{1}\right\} \tag{3.18}
\end{equation*}
$$

So the total Lyapunov derivative is:

$$
\begin{equation*}
\dot{V}_{2}=-\mathbf{z}_{1}^{T} \mathbf{K}_{1} \mathbf{z}_{1}-\mathbf{z}_{2}^{T} \mathbf{K}_{2} \mathbf{z}_{2} \tag{3.19}
\end{equation*}
$$

If $\mathbf{K}_{2}$ is a symmetric and positive definite matrix, the closed loop system is globally stable.

An important point here is the time derivative of $\boldsymbol{\alpha}_{1}$. One should be careful here that direct differentiation of this variable through numerical algorithms embedded on DSP microcontroller board should be avoided (differentiation degrades the signal to noise ratio). So the differentiated value of $\boldsymbol{\alpha}_{1}$ should be obtained through the system state equation. This can be done by for example:

$$
\begin{equation*}
\dot{\boldsymbol{\alpha}}_{1}=\frac{\partial \boldsymbol{\alpha}_{1}}{\partial \mathbf{x}} \dot{\mathbf{x}} \tag{3.20}
\end{equation*}
$$

where $\dot{\mathbf{x}}$ is replaced by the right hand side of the state equation in (3.1). Lastly, the closed loop system equations are:

$$
\begin{align*}
& \dot{\mathbf{z}}_{1}=\mathbf{f}_{1}\left(\mathbf{x}_{1}\right) \mathbf{z}_{2}+\mathbf{K}_{1} \mathbf{z}_{1}  \tag{3.21}\\
& \dot{\mathbf{z}}_{2}=-\mathbf{P}_{2}^{-1} \mathbf{K}_{2} \mathbf{z}_{2}-\mathbf{P}_{2}^{-1} \mathbf{f}_{1}^{T}\left(\mathbf{x}_{1}\right) \mathbf{z}_{1}
\end{align*}
$$

The aim of the example given in this section is to present the main procedure of backstepping. Of course in the realistic applications, the selection of the Lyapunov functions and control inputs may be different. They could be changed according to the design requirements. In the next section basic theory behind the feedback linearization will be introduced. In the control laws of (3.10) and (3.18) the state variables $\mathbf{x}_{1}, \mathbf{x}_{2}$ and the errors $\mathbf{z}_{1}, \mathbf{z}_{2}$ are computed from the sensing system (i.e. gyroscopes and attitude determination circuitry).

### 3.3 The Basic Theory of Feedback Linearization

As it is understood from its name the method of feedback linearization is intended to obtain a linear system as a result of a nonlinear feedback. However, there exists no partition of the state like that of the back - stepping. In this section the basics of feedback linearization is presented for building the framework necessary for attitude control.

### 3.3.1 The Notion of the Vector Relative Degree

The vector relative degree is a measure between the input and output of a nonlinear system. Mathematically, it is the number of successive differentiations of the output where the plant input vector appears first time. Qualitatively, it is equivalent to the pole - zero difference in linear systems (more truly speaking the difference between the number of poles and zeros). In order to simplify the description, first of all single - input single - output (SISO) systems are considered. Consider the following nonlinear (SISO) system:

$$
\begin{align*}
& \dot{\mathbf{x}}=f(\mathbf{x})+g(\mathbf{x}) u  \tag{3.22}\\
& y=h(\mathbf{x})
\end{align*}
$$

where $\mathbf{x} \in \mathbb{R}^{n}$.

Definition 3.1: The nonlinear system of (3.22) is said to have relative degree of $r$ in the neighborhood of $\mathbf{x}_{0}$ if the following conditions are satisfied:

1. $L_{g} h(\mathbf{x})=L_{g} L_{f} h(\mathbf{x})=L_{g} L_{f}^{2} h(\mathbf{x})=L_{g} L_{f}^{r-2} h(\mathbf{x})=0$ identically in the neighborhood of $\mathbf{x}=\mathbf{x}_{0}$.
2. $L_{g} L_{f}^{r-1} h(\mathbf{x}) \neq 0$ at $\mathbf{x}=\mathbf{x}_{0}$.

The term $L_{f} h$ is the Lie derivative of the function $h(\mathbf{x})$ over the vector field $\mathbf{f}(\mathbf{x})$ which is mathematically defined by:

$$
\begin{align*}
& L_{f} h(\mathbf{x})=\left\{\frac{\partial}{\partial \mathbf{x}} h(\mathbf{x})\right\} \mathbf{f}(\mathbf{x})  \tag{3.23}\\
& L_{f}^{r} h(\mathbf{x})=\frac{\partial}{\partial \mathbf{x}}\left\{L_{f}^{r-1} h(\mathbf{x})\right\} \mathbf{f}(\mathbf{x})
\end{align*}
$$

One should note that the higher order derivatives are defined as recursive expressions. For cross Lie derivatives a similar identity could be written i.e:

$$
\begin{equation*}
L_{g} L_{f} h(\mathbf{x})=\frac{\partial}{\partial \mathbf{x}}\left\{L_{f} h(\mathbf{x})\right\} \mathbf{g}(\mathbf{x}) \tag{3.24}
\end{equation*}
$$

For multi - input and multi - output (MIMO) systems like the one shown below the relative degree is defined for each output as $r_{i}$ and the sum $r=r_{1}+r_{2}+\ldots+r_{m}$ is called as the total relative degree.

$$
\begin{align*}
& \dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})+\sum_{i=1}^{m} \mathbf{g}_{i}(\mathbf{x}) u_{i}, i=1 \ldots m \\
& \dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x}) \mathbf{u}, \mathbf{g}(\mathbf{x})=\left[\begin{array}{lll}
\mathbf{g}_{1}(\mathbf{x}) & \ldots & \mathbf{g}_{m}(\mathbf{x})
\end{array}\right], \mathbf{u}=\left[\begin{array}{lll}
u_{1} & \ldots & u_{m}
\end{array}\right]^{T}  \tag{3.25}\\
& y_{i}=h_{i}(\mathbf{x}), i=1 \ldots m, \mathbf{y}=\mathbf{h}(\mathbf{x}), \mathbf{h}=\left[\begin{array}{lll}
h_{1} & \ldots & h_{m}
\end{array}\right]^{T} \\
& \mathbf{x} \in \mathbb{R}^{n}
\end{align*}
$$

Definition 3.2: Let the nonlinear system of (3.25) has smallest relative degree of $r_{i}$ for output $y_{i}$ in the neighborhood of point $x_{0}$. Then one can write the following:

$$
\begin{equation*}
\frac{d^{r_{i}}}{d t^{r_{i}}} y_{i}=L_{f}^{i_{i}} h_{i}(\mathbf{x})+L_{g} L_{f}^{r_{i}-1} h_{i}(\mathbf{x}) \mathbf{u} \tag{3.26}
\end{equation*}
$$

In the above the term $L_{g} L_{f}^{i-1} h_{i}(\mathbf{x}) \in \mathbb{R}^{1 \times m}$ and for all outputs the following matrix is obtained:

$$
\mathbf{A}(\mathbf{x})=\left[\begin{array}{cccc}
L_{g_{1}} L_{f}^{i_{1}-1} h_{1}(\mathbf{x}) & L_{g_{2}} L_{f}^{h_{1}-1} h_{1}(\mathbf{x}) & \ldots & L_{g_{m}} L_{f}^{n_{1}-1} h_{1}(\mathbf{x})  \tag{3.27}\\
L_{g_{1}} L_{f}^{h_{2}-1} h_{2}(\mathbf{x}) & L_{g_{2}} L_{f}^{2_{2}-1} h_{2}(\mathbf{x}) & \ldots & L_{g_{m}} L_{f}^{r_{2}-1} h_{2}(\mathbf{x}) \\
\vdots & \vdots & \vdots & \vdots \\
L_{g_{1}} L_{f}^{m_{m}-1} h_{m}(\mathbf{x}) & L_{g_{2}} L_{f}^{r_{m}-1} h_{m}(\mathbf{x}) & \ldots & L_{g_{m}} L_{f}^{m_{m}-1} h_{m}(\mathbf{x})
\end{array}\right]
$$

This is called as decoupling matrix. Compiling those, one can make the definition for the MIMO vector relative degree in the neighborhood of a point $\mathbf{x}_{0}$ as:

1. Each output of the nonlinear system (3.25) has relative degree of $r_{i}$ according to Definition 3.1.
2. The decoupling matrix $\mathbf{A}(\mathbf{x})$ should be invertible in the neighborhood of $\mathbf{x}_{0}$.

Once the vector relative degree of the considered nonlinear system for each output is found, it will be easy to proceed to the linearization process. Here, the total relative degree $r$ is important. If $r=n$, the system is said to be of full relative degree, however if $r<n$ then the system is not full relative degree and has an internal nonlinear dynamics. If that is not stable, one can not say that the feedback linearization procedure is successfully completed.

### 3.3.2 Feedback Linearization Procedure

The feedback linearization procedure begins with the definition of the normal variables vector $\varphi_{i}$ as shown in the following:

$$
\boldsymbol{\varphi}_{i}=\left[\begin{array}{c}
h_{i}  \tag{3.28}\\
L_{f} h_{i} \\
\vdots \\
L_{f}^{r-1}
\end{array}\right]
$$

As one can understood from the notation each normal variable formed according to the representation above corresponds to each output. Each entry of a normal variable vector has the lie derivative of the relevant output variable $y_{i}=h_{i}$ up to the order $r_{i}-1$ where $r_{i}$ is the vector relative degree of the output $y_{i}$.

For all of the outputs of the nonlinear model given in (3.25) the vectors defined in (3.28) are compiled into a new vector denoted as $\boldsymbol{\varphi}=\left[\begin{array}{llll}\boldsymbol{\varphi}_{1}^{T} & \boldsymbol{\varphi}_{2}^{T} & \ldots & \boldsymbol{\varphi}_{m}^{T}\end{array}\right]^{T}$. For the sake of simplicity the representation in (3.28) is rewritten in a more formal way as shown below:

$$
\boldsymbol{\varphi}_{i}=\left[\begin{array}{c}
h_{i}  \tag{3.29}\\
L_{f} h_{i} \\
\vdots \\
L_{f}^{i-1}
\end{array}\right]=\left[\begin{array}{c}
\xi_{i}^{i} \\
\xi_{2}^{i} \\
\vdots \\
\xi_{i}^{i}
\end{array}\right]
$$

In order to proceed one should differentiate all of the elements of the above vector as presented in the following:

$$
\left.\begin{array}{l}
\frac{d}{d t} \xi_{1}^{i}=\frac{d}{d t} h_{i}=L_{f} h_{i}+\underbrace{L_{g_{1}} h_{i} u_{1}}_{0}+\ldots+\underbrace{L_{g_{m}} h_{i} u_{m}}_{0} \\
\frac{d}{d t} \xi_{2}^{i}=\frac{d}{d t} L_{f} h_{i}=L_{f}^{2} h_{i}+\underbrace{L_{g_{1}} L_{f} h_{i} u_{1}}_{0}+\ldots+\underbrace{L_{g_{m}} L_{f} h_{i} u_{m}}_{0} \\
\frac{d}{d t} \xi_{3}^{i}=\frac{d}{d t} L_{f}^{2} h_{i}=L_{f}^{3} h_{i}+\underbrace{L_{g_{1}} L_{f}^{2} h_{i} u_{1}}_{0}+\ldots+\underbrace{L_{g_{m}} L_{f}^{2} h_{i} u_{m}}_{0}
\end{array}\right\} \begin{gathered}
\text { The zeros in these equations are } \\
\text { due to the relative degree } \\
L_{g_{m}} L_{f}^{j} h_{i}=0, j=1, \ldots, r_{i}-2 \tag{3.30}
\end{gathered}
$$

Only the last equation in the above has components of input which is a natural result of the relative degree concept. If the last equation is rewritten for all outputs one has the following set of differential equations (they are also packed into matrix from):

$$
\begin{align*}
& \frac{d}{d t} \xi_{r_{1}}^{1}=\frac{d}{d t} L_{f}^{h_{1}-1} h_{1}=L_{f}^{\mathrm{r}_{1}} h_{i}+L_{g_{1}} L_{f}^{\mathrm{r}_{1}-1} h_{i} u_{1}+\ldots+L_{g_{m}} L_{f}^{\mathrm{r}_{1}-1} h_{i} u_{m} \\
& \left.\frac{d}{d t} \xi_{r_{2}}^{2}=\frac{d}{d t} L_{f}^{r_{2}^{2}-1} h_{2}=L_{f}^{r_{2}} h_{i}+L_{g_{1}} L_{f}^{r_{2}-1} h_{i} u_{1}+\ldots+L_{g_{m}} L_{f}^{r_{2}-1} h_{i} u_{m}\right\}= \\
& \begin{array}{l}
\vdots \\
\frac{d}{d t} \xi_{r_{m}}^{m}=\frac{d}{d t} L_{f}^{r_{m}-1} h_{m}=L_{f}^{r_{m}} h_{i}+L_{g_{1}} L_{f}^{r_{m}-1} h_{i} u_{1}+\ldots+L_{g_{m}} L_{f}^{r_{m}-1} h_{i} u_{m}
\end{array}  \tag{3.31}\\
& \frac{d}{d t}\left[\begin{array}{c}
L_{f}^{r_{1}-1} h_{1} \\
L_{f}^{r_{2}-1} h_{2} \\
\vdots \\
L_{f}^{r_{m}-1} h_{m}
\end{array}\right]=\left[\begin{array}{c}
L_{f}^{i_{1}} h_{1} \\
L_{f}^{L_{2}} h_{2} \\
\vdots \\
L_{f}^{r_{m}} h_{m}
\end{array}\right]+\left[\begin{array}{cccc}
L_{g_{1}} L_{f}^{\eta_{f}-1} h_{1} & L_{g_{2}} L_{f}^{r_{1}-1} h_{1} & \cdots & L_{g_{m}} L_{f}^{r_{1}-1} h_{1} \\
L_{g_{1}} L_{f}^{r_{2}-1} h_{2} & L_{g_{2}} L_{f}^{r_{2}-1} h_{2} & \cdots & L_{g_{m}} L_{f}^{r_{2}-1} h_{2} \\
\vdots & \vdots & \cdots & \vdots \\
L_{g_{1}} L_{f}^{r_{m}-1} h_{m} & L_{g_{2}} L_{f}^{m_{m}-1} h_{m} & \cdots & L_{g_{m}} L_{f}^{r_{m}-1} h_{m}
\end{array}\right]\left[\begin{array}{c}
u_{1} \\
u_{2} \\
\vdots \\
u_{m}
\end{array}\right]
\end{align*}
$$

So final form of the above equations with the desriptive symbols $(\mathbf{A}(\mathbf{x}), \mathbf{b}, \mathbf{v})$ written for easier presentation is given as:

$$
\begin{align*}
& {\left[\begin{array}{c}
\dot{\xi}_{1}^{i} \\
\dot{\xi}_{2}^{i} \\
\vdots \\
\xi_{r_{i}}^{i}
\end{array}\right]=\left[\begin{array}{c}
\xi_{2}^{i} \\
\xi_{3}^{i} \\
\vdots \\
L_{f}^{h_{i}} h_{i}+L_{g_{1}} L_{f}^{i_{2}-1} h_{i} u_{1}+\ldots+L_{g_{m}} L_{f}^{L_{f}^{i-1}} h_{i} u_{m}
\end{array}\right]} \tag{3.32}
\end{align*}
$$

If one applies the control input:

$$
\begin{equation*}
\mathbf{u}=\mathbf{A}^{-1}(\mathbf{x})[-\mathbf{b}+\mathbf{v}] \tag{3.33}
\end{equation*}
$$

(where $\mathbf{v}$ is an external control input like $\mathbf{v}=\left[\begin{array}{llll}v_{1} & v_{2} & \cdots & v_{n}\end{array}\right]^{T}$ ) to (3.32), the remaining is a controllable and observable linear system like shown in (3.34).

One can use the above resultant linear system to design a controller setting the vector $\mathbf{v}$ as system input. The result is a set of double integrator systems the number of which is equal to the order of the plant output. Here, any desired linear control method can be utilized for completing the feedback linearized controller.

The controller design process is easier if the resultant linear system corresponding to a single output $y_{i}$ is represented as shown below:

$$
\left[\begin{array}{c}
\dot{\xi}_{1}^{i}  \tag{3.35}\\
\dot{\xi}_{2}^{i} \\
\vdots \\
\dot{\xi}_{r_{i}}^{i}
\end{array}\right]=\left[\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \ldots \\
0 & 0 & 1 & 0 \ldots \\
0 \\
0 & 0 & 0 \ldots 0 & 0
\end{array}\right]\left[\begin{array}{c}
\xi_{1}^{i} \\
\xi_{2}^{i} \\
\vdots \\
\xi_{r_{i}}^{i}
\end{array}\right]+\left[\begin{array}{c}
0 \\
0 \\
\vdots \\
1
\end{array}\right] v_{i}
$$

The above is a controllable and observable linear time invariant system.

### 3.4 Input to State Stability

Input to state stability theories can provide useful tools in analyzing a closed loop system against external disturbances. In this section information related to input to state stability are presented.

### 3.4.1 Definition Concerning Input - To - State Stability

Here, some definitions about robust stability is presented. Those definitions are referenced throughout this work. The first definitions are about the class $K$ and class KL functions.

Definition 3.3: A function $\alpha:[0, a) \rightarrow[0, \infty)$ is said to be of class K if it is strictly increasing and $\alpha(0)=0$.

Definition 3.4: A continuous function $\beta:[0, a) \times[0, \infty) \rightarrow[0, \infty)$ is said to be of class KL if for each fixed $y$ the function $\beta(x, y)$ belongs to class K with respect to $x$ and for each fixed $x$ the function $\beta(x, y)$ is decreasing with $y$ and $\beta(x, y) \rightarrow 0, y \rightarrow \infty$.

Definition 3.5: Let $\mathbf{x}=\mathbf{0}$ is an equilibrium point of $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$ and let $V(\mathbf{x})$ is a continuously differentiable function from a domain $\mathbf{D}$ to $\mathbb{R}$ with the following properties:

$$
\begin{align*}
& V(0)=0 \& V(\mathbf{x})>0 \quad \forall \mathbf{x} \in[\mathbf{D}-\{\mathbf{0}\}]  \tag{3.36}\\
& \dot{V}(\mathbf{x})<0 \quad \forall \mathbf{x} \in[\mathbf{D}]
\end{align*}
$$

If the above condition is satisfied the point $\mathbf{x}=\mathbf{0}$ is a stable point.

Definition 3.6: Let $\mathbf{x}=\mathbf{0}$ be an equilibrium point for $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$. and let $V(\mathbf{x})$ is a continuously differentiable function from a domain $\mathbf{D}$ to $\mathbb{R}$ (containing the origin $\mathbf{x}=\mathbf{0}$ ) such that $\dot{V}(\mathbf{x})<0$ in $\mathbf{D}$ except $\mathbf{x}=\mathbf{0}$. Also assume that there exist a set $\mathbf{S} \in\{\mathbf{x} \in[\mathbf{D}] \mid \dot{V}(\mathbf{x})=0\}$ and suppose that no analytical solution stays identically in the set $\mathbf{S}$ other than $\mathbf{x}=\mathbf{0}$ then the origin is asymptotically stable.

Definition 3.7: Let $\mathbf{x}=\mathbf{0}$ be an equilibrium point for $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$. and let $V(\mathbf{x})$ is a continuously differentiable function from a domain $\mathbb{R}^{n}$ to $\mathbb{R}$ (containing the origin $\mathbf{x}=\mathbf{0}$ ) such that $\dot{V}(\mathbf{x})<0$ in $\mathbb{R}^{n}$ except $\mathbf{x}=\mathbf{0}$. Also assume that there exist a set $\mathbf{S} \in\left\{\mathbf{x} \in \mathbb{R}^{n} \mid \dot{V}(\mathbf{x})=0\right\}$ and suppose that no analytical solution stays identically in the set $\mathbf{S}$ other than $\mathbf{x}=\mathbf{0}$ then the origin is globally asymptotically stable.

Definition 3.8 [Khalil (1996)] The system $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}, \mathbf{w})$ is said to be input to state stable for bounded $\left\|\mathbf{x}\left(t_{0}\right)\right\|$ and $\|w(t)\|$ if the following is satisfied:

$$
\begin{equation*}
\|\mathbf{x}(t)\| \leq \beta\left(\left\|\mathbf{x}\left(t_{0}\right)\right\|, t-t_{0}\right)+\gamma\left(\sup _{t_{0} \leq \leq \leq t}\|w(\tau)\|\right) \tag{3.37}
\end{equation*}
$$

where $\beta$ is class KL and $\gamma$ is a class K function. There is also another definition of input - to - state stability [Sontag (1995)] which is given next.

Definition 3.9: [Sonntag (1995)] The system $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}, \mathbf{w})$ is said to be input to state stable if the following is satisfied:

$$
\begin{equation*}
\underbrace{\frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{w})}_{\dot{V}} \leq \underbrace{\beta(\|\mathbf{w}\|)-\alpha(\|\mathbf{x}\|)}_{\delta(\mathbf{w}, \mathbf{x})}, \mathbf{x} \in \mathbb{R}^{n}, \mathbf{w} \in \mathbb{R}^{m} \tag{3.38}
\end{equation*}
$$

$V(\mathbf{x})$ is a positive definite Lyapunov function (storage function), $\alpha$ and $\beta$ are class K functions. The above can also be thought as a dissipation inequality where $V(\mathbf{x})$ is the storage function and $\delta(\mathbf{w}, \mathbf{x})$ is the supply function. In fact the above definition has a relationship with the global asymptotic stability since when the input is zero (i.e. $\mathbf{w}=\mathbf{0})$ (3.38) becomes:

$$
\begin{equation*}
\underbrace{\frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, 0)}_{V} \leq \underbrace{-\alpha(\|\mathbf{x}\|)}_{\delta(\mathbf{x})} \tag{3.39}
\end{equation*}
$$

Since $\alpha$ is of class $K$, the above is a negative definite function so the above implies that the system $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}, \mathbf{0})$ is a globally asymptotically stable (GAS) system. So one can state that in order to discuss about the input to state stability property first of all the non - disturbed system (i.e. $\mathbf{w}=\mathbf{0}$ ) should be globally asymptotically stable.

The inequality in (3.38) states that the Lyapunov function to be used in the stability analysis of the system $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}, \mathbf{w})$ should satisfy the constraints given in that inequality in order to have stability for bounded inputs w. By that way along the trajectories of the system $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}, \mathbf{w})$ the derivative of the Lyapunov function $V(\mathbf{x})$ is bounded by the term $\beta(\|\mathbf{w}\|)-\alpha(\|\mathbf{x}\|)$. This property is called as the existence of the Lyapunov function in the dissipation form [Grüne (2004)].

# CHAPTER 4 <br> ATTITUDE CONTROL BY BACKSTEPPING 

### 4.1 Introduction

In this chapter, an attitude control system based on back - stepping is proposed. The purpose of this controller is to track the attitude of the satellite to the desired trajectory. Robustness issues are considered by reformulating the problem for input to state stability against the disturbance torques.

### 4.2 Attitude Error Formulations

In the attitude control research there are two approaches of attitude error representation. The first of which is the additive approach which has just the same definition as the classical control error $\left(\boldsymbol{\sigma}_{e}=\boldsymbol{\sigma}-\boldsymbol{\sigma}_{d}\right)$. The other one is derived from some cross product operations and called as multiplicative error definition (Appendix A presents some information on the derivation of attitude errors). That is:

$$
\begin{equation*}
\boldsymbol{\sigma}_{e}=\frac{\left[1-\boldsymbol{\sigma}_{d}^{T} \boldsymbol{\sigma}_{d}\right] \boldsymbol{\sigma}-\left[1-\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}\right] \boldsymbol{\sigma}_{d}+2 \mathbf{S}(\boldsymbol{\sigma}) \boldsymbol{\sigma}_{d}}{1+\left(\boldsymbol{\sigma}_{d}^{T} \boldsymbol{\sigma}_{d}\right)\left(\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}\right)+2 \boldsymbol{\sigma}_{d}^{T} \boldsymbol{\sigma}} \tag{4.1}
\end{equation*}
$$

In fact the correct error definition is the multiplicative version (above representation) since the attitude displacement is of the rotational type. In some research such as [Bar - Itzhack (1984), Schaub (1999)] additive attitude error definitions are used. The actual error definition shown in (4.1) brings computational complexity to application. Thus, in this research the additive error definition is preferred for controller design purposes however in simulations the actual error definition is applied.

### 4.3 The Back - Stepping Design Procedure

Before entering into the procedure in the detail it is convenient to rewrite the dynamic model and state the problem. the satellite's dynamic model was:

For the demanded torque generator:

$$
\begin{align*}
& \mathbf{x}=\left[\begin{array}{ll}
\boldsymbol{\sigma}^{T} & \left(\boldsymbol{\omega}_{i b}^{b}\right)^{T}
\end{array}\right]^{T} \\
& \mathbf{f}(\mathbf{x})=\left[\begin{array}{c}
\mathbf{G}(\boldsymbol{\sigma})\left\{\boldsymbol{\omega}_{i b}^{b}+\mathbf{c}_{2} \omega_{o}\right\} \\
\mathbf{J}^{-1}\left[-\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)\right]
\end{array}\right]  \tag{4.2}\\
& \mathbf{g}(\mathbf{x})=\left[\begin{array}{c}
\mathbf{0}_{3 \times 3} \\
-\mathbf{J}^{-1}
\end{array}\right], \mathbf{p}(\mathbf{x})=\left[\begin{array}{c}
\mathbf{0}_{3 \times 3} \\
\mathbf{J}^{-1}
\end{array}\right] \\
& \mathbf{u}=\boldsymbol{\tau}_{a}, \mathbf{w}=\boldsymbol{\tau}_{e}
\end{align*}
$$

For the reaction wheel velocity controller:

$$
\begin{align*}
& \mathbf{x}=\left[\begin{array}{ll}
\boldsymbol{\omega}_{s}^{T} & \boldsymbol{\tau}_{a}^{T}
\end{array}\right]^{T} \\
& \mathbf{f}(\mathbf{x})=\left[\begin{array}{c}
-\mathbf{J}^{-1}\left[-\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)\right]+\left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right) \boldsymbol{\tau}_{a} \\
-\mathbf{K}_{T} \mathbf{R}_{L} \mathbf{K}_{T}^{-1} \boldsymbol{\tau}_{a}+\mathbf{K}_{T} \mathbf{K}_{E} \boldsymbol{\omega}_{s}
\end{array}\right]  \tag{4.3}\\
& \mathbf{g}(\mathbf{x})=\left[\begin{array}{c}
\mathbf{0}_{3 \times 3} \\
\mathbf{K}_{T} \mathbf{L}
\end{array}\right], \mathbf{p}(\mathbf{x})=\left[\begin{array}{c}
-\mathbf{J}^{-1} \\
\mathbf{0}_{3 \times 3}
\end{array}\right] \\
& \mathbf{u}=\mathbf{U}_{a}, \mathbf{w}=\boldsymbol{\tau}_{e}
\end{align*}
$$

The problem is to force the attitude variable $\boldsymbol{\sigma}$ to track the desired trajectory provided by the variable $\boldsymbol{\sigma}_{d}$. Shortly this is a tracking controller problem.

The back - stepping approach starts simply by partitioning the satellite model state into sub - state vectors according to its characteristics. In each step, the corresponding sub - state vector is stabilized using Lyapunov theory of nonlinear control. The total number of steps is equal to the number of sub - states. In this work, the states are grouped as:

1. Attitude Vector (3 components)
2. Body angular velocities (3 components)
3. Reaction wheel velocity (3 components)
4. Reaction wheel torque (3 components)

So the back - stepping control design procedure for the satellite model of this research requires four steps. Since the attitude control structure involves a cascaded structure, the first two steps are included in the derivation of the demanded torque generator and the last two are included in the speed controller design. For each step, a new virtual state variable is defined and they constitute the states of the closed loop system.

### 4.3.1 Step - I (Demanded Torque Generator)

Since the problem is of tracking type, one has to define the first virtual state variable $\mathbf{z}_{1}$ as the attitude tracking error defined as shown below (based on additive error definition):

$$
\begin{equation*}
\mathbf{z}_{1}=\boldsymbol{\sigma}-\boldsymbol{\sigma}_{d} \tag{4.4}
\end{equation*}
$$

So it is not inconvenient to think that this goal of this step is to deal with the attitude error dynamics. Then one should take its time derivative for continuing ( $\left.\dot{\boldsymbol{\sigma}}_{d}=0\right)$ :

$$
\begin{equation*}
\dot{\mathbf{z}}_{1}=\dot{\boldsymbol{\sigma}}=\mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b} \tag{4.5}
\end{equation*}
$$

Although one is working on the first step, the procedure requires the second back stepping vector to complete the Lyapunov function. That forms a link between the first and second steps:

$$
\begin{equation*}
\boldsymbol{\omega}_{o b}^{b}=\boldsymbol{\alpha}_{1}+\mathbf{z}_{2} \tag{4.6}
\end{equation*}
$$

where $\boldsymbol{\alpha}_{1}$ is a virtual control input that is derived in the proceeding discussions and $\mathbf{z}_{2}$ is the second back - stepping state vector. Substituting (4.6) into (4.5) one can obtain:

$$
\begin{equation*}
\dot{\mathbf{z}}_{1}=\mathbf{G}(\boldsymbol{\sigma})\left(\boldsymbol{\alpha}_{1}+\mathbf{z}_{2}\right) \tag{4.7}
\end{equation*}
$$

Now one can select a quadratic Lyapunov function as:

$$
\begin{equation*}
V_{1}=\frac{1}{2} \mathbf{z}_{1}^{T} \mathbf{z}_{1} \tag{4.8}
\end{equation*}
$$

the time derivative of which is:

$$
\begin{equation*}
\dot{V}_{1}=\frac{1}{2} \dot{\mathbf{z}}_{1}^{T} \mathbf{z}_{1}+\frac{1}{2} \mathbf{z}_{1}^{T} \dot{\mathbf{z}}_{1} \tag{4.9}
\end{equation*}
$$

Substituting from (4.7) will yield:

$$
\begin{align*}
& \dot{V}_{1}=\frac{1}{2}\left[\left(\boldsymbol{\alpha}_{1}^{T}+\mathbf{z}_{2}^{T}\right) \mathbf{G}^{T}(\boldsymbol{\sigma})\right] \mathbf{z}_{1}+\frac{1}{2} \mathbf{z}_{1}^{T}\left[\mathbf{G}(\boldsymbol{\sigma})\left(\boldsymbol{\alpha}_{1}+\mathbf{z}_{2}\right)\right] \\
& =\frac{1}{2}\left[\boldsymbol{\alpha}_{1}^{T} \mathbf{G}^{T}(\boldsymbol{\sigma})\right] \mathbf{z}_{1}+\frac{1}{2} \mathbf{z}_{1}^{T}\left[\mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\alpha}_{1}\right]+\frac{1}{2} \mathbf{z}_{2}^{T} \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{z}_{1}+\frac{1}{2} \mathbf{z}_{1}^{T} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{z}_{2} \tag{4.10}
\end{align*}
$$

By selecting a virtual control input $\boldsymbol{\alpha}_{1}$ as:

$$
\begin{equation*}
\boldsymbol{\alpha}_{1}=-\mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1} \mathbf{z}_{1} \tag{4.11}
\end{equation*}
$$

where $\mathbf{K}_{1}$ is diagonal and positive definite matrix $\left(\mathbf{K}_{1}=k_{1} \mathbf{I}_{3 \times 3}\right.$, st $\left.k_{1}>0\right)$. The corresponding Lyapunov derivative $\dot{V}_{1}$ is obtained as:
$\dot{V}_{1}=-\frac{1}{2} \mathbf{z}_{1}^{T} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1} \mathbf{z}_{1}-\frac{1}{2} \mathbf{z}_{1}^{T} \mathbf{K}_{1} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{z}_{1}+\frac{1}{2} \mathbf{z}_{2}^{T} \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{z}_{1}+\frac{1}{2} \mathbf{z}_{1}^{T} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{z}_{2}$
$=-\frac{1}{2} \mathbf{z}_{1}^{T}\left\{\mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1}+\mathbf{K}_{1} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma})\right\} \mathbf{z}_{1}+\frac{1}{2} \mathbf{z}_{2}^{T} \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{z}_{1}+\frac{1}{2} \mathbf{z}_{1}^{T} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{z}_{2}$
In the above the quadratic term related to the variable $\mathbf{z}_{1}$ is negative definite (symmetric and positive definite inner matrix). So the goal of the first step is achieved. The remaining cross term i.e. $1 / 2\left\{\mathbf{z}_{2}^{T} \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{z}_{1}+\mathbf{z}_{1}^{T} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{z}_{2}\right\}$ is eliminated in the next step. An interesting property of the term $\mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma})$ is that the product is diagonal and in form of $s \mathbf{I}_{3 \times 3}$ where $s=\left(1+\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}\right)^{2} / 16$. This relationship can be found by direct manipulation.

### 4.3.2 Step - II (Demanded Torque Generator)

In the second step, the dynamics related to the second virtual state variable $\mathbf{z}_{2}$ is stabilized. To do that, the second back - stepping state variable is rewritten from the back - stepping transformation [Skjetne (2004)]:

$$
\begin{equation*}
\mathbf{z}_{2}=\boldsymbol{\omega}_{o b}^{b}-\boldsymbol{\alpha}_{1} \tag{4.13}
\end{equation*}
$$

From the relation between body angular velocity with respect to orbit plane and inertial frame (ECI) one can write:

$$
\begin{equation*}
\boldsymbol{\omega}_{o b}^{b}=\boldsymbol{\omega}_{i b}^{b}+\omega_{o} \mathbf{c}_{2} \tag{4.14}
\end{equation*}
$$

Combining (4.13) and (4.14) will yield:

$$
\begin{equation*}
\mathbf{z}_{2}=\boldsymbol{\omega}_{i b}^{b}+\omega_{o} \mathbf{c}_{2}-\boldsymbol{\alpha}_{1} \tag{4.15}
\end{equation*}
$$

Knowing that $\dot{\mathbf{c}}_{2}=\mathbf{S}\left(\mathbf{c}_{2}\right) \boldsymbol{\omega}_{o b}^{b}$ and differentiating the above leads to:

$$
\begin{equation*}
\dot{\mathbf{z}}_{2}=\dot{\boldsymbol{\omega}}_{i b}^{b}+\omega_{o} \mathbf{S}\left(\mathbf{c}_{2}\right) \boldsymbol{\omega}_{o b}^{b}-\dot{\boldsymbol{\alpha}}_{1} \tag{4.16}
\end{equation*}
$$

For the sake of simplicity it is convenient to multiply the above by the inertia like matrix $\mathbf{J}$ :

$$
\begin{equation*}
\mathbf{J} \dot{\mathbf{z}}_{2}=\mathbf{J} \dot{\boldsymbol{\omega}}_{i b}^{b}+\omega_{o} \mathbf{J S}\left(\mathbf{c}_{2}\right) \boldsymbol{\omega}_{o b}^{b}-\mathbf{J} \dot{\boldsymbol{\alpha}}_{1} \tag{4.17}
\end{equation*}
$$

And from (2.27):

$$
\begin{equation*}
\mathbf{J} \dot{\mathbf{z}}_{2}=-\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\boldsymbol{\tau}_{e}-\boldsymbol{\tau}_{a}+\omega_{o} \mathbf{J S}\left(\mathbf{c}_{2}\right) \boldsymbol{\omega}_{o b}^{b}-\mathbf{J} \dot{\boldsymbol{\alpha}}_{1} \tag{4.18}
\end{equation*}
$$

One can call $\boldsymbol{\tau}_{a}$ as $\boldsymbol{\tau}_{a r}$ in order to emphasize that the torque demand law derived in this section is a demanded (or requested) torque that is to be supplied to the reaction wheel velocity command generator.

$$
\begin{equation*}
\mathbf{J} \dot{\mathbf{z}}_{2}=-\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\boldsymbol{\tau}_{e}-\boldsymbol{\tau}_{a r}+\omega_{o} \mathbf{J S}\left(\mathbf{c}_{2}\right) \boldsymbol{\omega}_{o b}^{b}-\mathbf{J} \dot{\boldsymbol{\alpha}}_{1} \tag{4.19}
\end{equation*}
$$

Now one should construct the second part of the Lyapunov function as:

$$
\begin{equation*}
V_{2}=V_{1}+\frac{1}{2} \mathbf{z}_{2}^{T} \mathbf{J} \mathbf{z}_{2} \tag{4.20}
\end{equation*}
$$

the derivative of which is:

$$
\begin{equation*}
\dot{V}_{2}=\dot{V}_{1}+\frac{1}{2} \dot{\mathbf{z}}_{2}^{T} \mathbf{J} \mathbf{z}_{2}+\frac{1}{2} \mathbf{z}_{2}^{T} \mathbf{J} \dot{\mathbf{z}}_{2} \tag{4.21}
\end{equation*}
$$

Substitute $\dot{V}_{1}$ from Eq. (4.12) to obtain:

$$
\begin{align*}
& \dot{V}_{2}=-\frac{1}{2} \mathbf{z}_{1}^{T}\left\{\mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1}+\mathbf{K}_{1} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma})\right\} \mathbf{z}_{1}+ \\
& +\frac{1}{2} \mathbf{z}_{2}^{T} \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{z}_{1}+\frac{1}{2} \mathbf{z}_{1}^{T} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{z}_{2}+\frac{1}{2} \dot{\mathbf{z}}_{2}^{T} \mathbf{J} \mathbf{z}_{2}+\frac{1}{2} \mathbf{z}_{2}^{T} \mathbf{J} \dot{\mathbf{z}}_{2} \\
& =-\frac{1}{2} \mathbf{z}_{1}^{T}\left\{\mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1}+\mathbf{K}_{1} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma})\right\} \mathbf{z}_{1}+  \tag{4.22}\\
& +\frac{1}{2}\left[\dot{\mathbf{z}}_{2}^{T} \mathbf{J}+\mathbf{z}_{1}^{T} \mathbf{G}(\boldsymbol{\sigma})\right] \mathbf{z}_{2}+\frac{1}{2} \mathbf{z}_{2}^{T}\left[\mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{z}_{1}+\mathbf{J} \dot{\mathbf{z}}_{2}\right]
\end{align*}
$$

Substituting from (4.19) yields:

$$
\begin{align*}
& \dot{V}_{2}=-\frac{1}{2} \mathbf{z}_{1}^{T}\left\{\mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1}+\mathbf{K}_{1} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma})\right\} \mathbf{z}_{1}+ \\
& +\frac{1}{2}\left[-\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)^{T} \mathbf{S}^{T}\left(\boldsymbol{\omega}_{i b}^{b}\right)+\boldsymbol{\tau}_{e}^{T}-\boldsymbol{\tau}_{a r}^{T}-\mathbf{z}_{3}^{T}+\omega_{o}\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T} \mathbf{S}^{T}\left(\mathbf{c}_{2}\right) \mathbf{J}^{T}-\dot{\boldsymbol{\alpha}}_{1}^{T} \mathbf{J}^{T}+\mathbf{z}_{1}^{T} \mathbf{G}(\boldsymbol{\sigma})\right] \mathbf{z}_{2}+  \tag{4.23}\\
& +\frac{1}{2} \mathbf{z}_{2}^{T}\left[\mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{z}_{1}-\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\boldsymbol{\tau}_{e}-\boldsymbol{\tau}_{a r}-\mathbf{z}_{3}+\omega_{o} \mathbf{J S}\left(\mathbf{c}_{2}\right) \boldsymbol{\omega}_{o b}^{b}-\mathbf{J} \dot{\boldsymbol{\alpha}}_{1}\right]
\end{align*}
$$

In order to proceed one has to find a way to convert the terms related to the variable $\mathbf{z}_{2}$ to some negative definite quadratic functions so that the dynamics related to $\mathbf{z}_{1}$ and $\mathbf{z}_{2}$ become globally stable. The equations below illustrate $a$ possible manipulation for that purpose.

$$
\begin{align*}
& \dot{V}_{2}=-\frac{1}{2} \mathbf{z}_{1}^{T}\left\{\mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1}+\mathbf{K}_{1} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma})\right\} \mathbf{z}_{1}+ \\
& +\frac{1}{2} \underbrace{\left[-\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)^{T} \mathbf{S}^{T}\left(\boldsymbol{\omega}_{i b}^{b}\right)+\boldsymbol{\tau}_{e}^{T}-\boldsymbol{\tau}_{a r}^{T}+\omega_{o}\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T} \mathbf{S}^{T}\left(\mathbf{c}_{2}\right) \mathbf{J}^{T}-\dot{\boldsymbol{\alpha}}_{1}^{T} \mathbf{J}^{T}+\mathbf{z}_{1}^{T} \mathbf{G}(\boldsymbol{\sigma})\right]}_{-\boldsymbol{z}_{2}^{T} \mathbf{K}_{2}} \mathbf{z}_{2}+  \tag{4.24}\\
& +\frac{1}{2} \mathbf{z}_{2}^{T} \underbrace{\left[\mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{z}_{1}-\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\boldsymbol{\tau}_{e}-\boldsymbol{\tau}_{a r}+\omega_{o} \mathbf{J} \mathbf{S}\left(\mathbf{c}_{2}\right) \boldsymbol{\omega}_{o b}^{b}-\mathbf{J} \dot{\boldsymbol{\alpha}}_{1}\right]}_{-\mathbf{K}_{2} \mathbf{z}_{2}}
\end{align*}
$$

Note that, the first bracketed term is the transposed version of the second one. By replacing the bracketed terms with the corresponding gain terms ( $\mathbf{K}_{2}$ is symmetric and positive definite) as shown in the above (they are shown under the bracketed terms) one can obtain the torque demand law as shown in the following:

$$
\begin{align*}
& \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{z}_{1}-\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\boldsymbol{\tau}_{e}-\boldsymbol{\tau}_{a r}+\omega_{o} \mathbf{J S}\left(\mathbf{c}_{2}\right) \boldsymbol{\omega}_{o b}^{b}-\mathbf{J} \dot{\boldsymbol{\alpha}}_{1}=-\mathbf{K}_{2} \mathbf{z}_{2}  \tag{4.25}\\
& \boldsymbol{\tau}_{a r}=\mathbf{K}_{2} \mathbf{z}_{2}+\mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{z}_{1}-\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\omega_{o} \mathbf{J S}\left(\mathbf{c}_{2}\right) \boldsymbol{\omega}_{o b}^{b}-\mathbf{J} \dot{\boldsymbol{\alpha}}_{1}
\end{align*}
$$

The above torque demand law serves as the demanded torque to be supplied into the motor control unit. So it is more convenient to write the above law as:

$$
\begin{equation*}
\boldsymbol{\tau}_{a r}=\mathbf{K}_{2} \mathbf{z}_{2}+\mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{z}_{1}-\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\omega_{o} \mathbf{J S}\left(\mathbf{c}_{2}\right) \boldsymbol{\omega}_{o b}^{b}-\mathbf{J} \dot{\boldsymbol{\alpha}}_{1} \tag{4.26}
\end{equation*}
$$

where $\boldsymbol{\tau}_{a r}$ denoted the demanded (or the requested) torque from the reaction wheel control unit. That is designed in the third and fourth steps. The actual torque input $\boldsymbol{\tau}_{a}$ is drawn out of the DC motor. In (4.26) the reaction wheel velocity $\boldsymbol{\omega}_{s}$ is treated as an external signal and provided through gyroscopic measurement.

The control law in (4.26) yields $\dot{V}_{2}$ as:

$$
\begin{align*}
& \dot{V}_{2}=-\frac{1}{2} \mathbf{z}_{1}^{T}\left\{\mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1}+\mathbf{K}_{1} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma})\right\} \mathbf{z}_{1}-\underbrace{\frac{1}{2} \mathbf{z}_{2}^{T} \mathbf{K}_{2} \mathbf{z}_{2}-\frac{1}{2} \mathbf{z}_{2}^{T} \mathbf{K}_{2} \mathbf{z}_{2}}_{-\mathbf{-}_{2}^{T} \mathbf{K}_{2} \mathbf{z}_{2}}+  \tag{4.27}\\
& +\frac{1}{2} \boldsymbol{\tau}_{e}^{T} \mathbf{z}_{2}+\frac{1}{2} \mathbf{z}_{2}^{T} \boldsymbol{\tau}_{e}
\end{align*}
$$

Neglecting the disturbance torques $\boldsymbol{\tau}_{e}$ the above Lyapunov function provides a stable control law (torque command) generator.

In the above Lyapunov derivative equation there exist some terms including the disturbance torque vector $\boldsymbol{\tau}_{e}$. Obviously disturbing terms can deviate $\dot{V}_{i}$ from negative definiteness. All the comments about stability in the current and proceeding design sections assume that the disturbances torques are zero. Their effects on stability and performance will be investigated in Section 4.6. So without
the disturbances (4.27) is a negative definite Lyapunov derivative and the closed loop dynamics of $\mathbf{z}_{1} \& \mathbf{z}_{2}$ is globally stable.

Before going into the details of the third step, the reaction wheel velocity command generator should be presented. This is a differential equation taking the requested torque from (4.26) and produces the reaction wheel velocity after an integration process. That is:

$$
\begin{equation*}
\boldsymbol{\omega}_{s r}=\int_{0}^{t}\left\{\mathbf{J}^{-1} \mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right) \boldsymbol{\tau}_{a r}\right\} d t \tag{4.28}
\end{equation*}
$$

The variable $\boldsymbol{\omega}_{s r}$ is supplied as a reaction wheel velocity command to the third step.

### 4.3.3 Step - III (Speed Control)

In the current and proceeding step the motor control unit is designed. The motor control unit involves the torque and reaction wheel velocities as the state vectors. This step's purpose is to provide a control law for the torque tracking mechanism:

$$
\begin{equation*}
\mathbf{z}_{3}=\boldsymbol{\omega}_{s}-\boldsymbol{\omega}_{s r} \tag{4.29}
\end{equation*}
$$

As done in the first step, one should differentiate the above error definition in order to use in the formation of the control Lyapunov functions:

$$
\begin{equation*}
\dot{\mathbf{z}}_{3}=\dot{\boldsymbol{\omega}}_{s}-\dot{\boldsymbol{\omega}}_{s r} \tag{4.30}
\end{equation*}
$$

Substituting from the satellite model and using (4.28) yield (after some cancellations):

$$
\begin{equation*}
\dot{\mathbf{z}}_{3}=\left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right)\left\{\boldsymbol{\tau}_{a}-\boldsymbol{\tau}_{a r}\right\}-\mathbf{J}^{-1} \boldsymbol{\tau}_{e} \tag{4.31}
\end{equation*}
$$

One can multiply the above by $\mathbf{J}$ in order to make the proceeding analysis easier:

$$
\begin{equation*}
\mathbf{J}_{3}=\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right)\left\{\boldsymbol{\tau}_{a}-\boldsymbol{\tau}_{a r}\right\}-\boldsymbol{\tau}_{e} \tag{4.32}
\end{equation*}
$$

The Lyapunov function of this step can be defined as shown in the following:

$$
\begin{equation*}
V_{3}=V_{2}+\frac{1}{2} \mathbf{z}_{3}^{T} \mathbf{J z}_{3} \tag{4.33}
\end{equation*}
$$

and,

$$
\begin{equation*}
\dot{V}_{3}=\dot{V}_{2}+\frac{1}{2} \dot{\mathbf{z}}_{3}^{T} \mathbf{J} \mathbf{z}_{3}+\frac{1}{2} \mathbf{z}_{3}^{T} \mathbf{J} \dot{\mathbf{z}}_{3} \tag{4.34}
\end{equation*}
$$

Substituting from (4.32) enables one to obtain:

$$
\begin{align*}
& \dot{V}_{3}=\dot{V}_{2}+\frac{1}{2}\left\{\boldsymbol{\tau}_{a}^{T}\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right)^{T}-\boldsymbol{\tau}_{a r}^{T}\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right)^{T}-\boldsymbol{\tau}_{e}^{T}\right\} \mathbf{z}_{3}+  \tag{4.35}\\
& +\frac{1}{2} \mathbf{z}_{3}^{T}\left\{\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right) \boldsymbol{\tau}_{a}-\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right) \boldsymbol{\tau}_{a r}-\boldsymbol{\tau}_{e}\right\}
\end{align*}
$$

Before proceeding it is convenient here to define the fourth virtual state variable as:

$$
\begin{equation*}
\mathbf{z}_{4}=\boldsymbol{\tau}_{a}-\boldsymbol{\alpha}_{2} \tag{4.36}
\end{equation*}
$$

where $\boldsymbol{\alpha}_{2}$ is a virtual control variable (this forms an internal control loop and may be taught as a virtual torque command). The reaction wheel torque $\boldsymbol{\tau}_{a}$ is replaced by $\boldsymbol{\tau}_{a}=\mathbf{z}_{4}+\boldsymbol{\alpha}_{2}$ for linking the third and fourth steps. That is:

$$
\begin{align*}
& \dot{V}_{3}=\frac{1}{2}\left\{\mathbf{z}_{4}^{T}\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right)^{T}+\boldsymbol{\alpha}_{2}^{T}\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right)^{T}-\boldsymbol{\tau}_{a r}^{T}\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right)^{T}-\boldsymbol{\tau}_{e}^{T}\right\} \mathbf{z}_{3}+  \tag{4.37}\\
& +\frac{1}{2} \mathbf{z}_{3}^{T}\left\{\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right) \mathbf{z}_{4}+\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right) \boldsymbol{\alpha}_{2}-\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right) \boldsymbol{\tau}_{a r}-\boldsymbol{\tau}_{e}\right\}
\end{align*}
$$

Just like in the second step, the terms in the brackets above except $\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right) \mathbf{z}_{4}$ and $\boldsymbol{\tau}_{e}$ are changed by a term like $-\mathbf{K}_{3} \mathbf{z}_{3} \quad\left(\mathbf{K}_{3}\right.$ is positive definite and symmetric preferably in the form $\mathbf{K}_{3}=k_{3} \mathbf{I}_{3 \times 3}$ where $k_{3}>0$ ) to obtain a negative definite $\dot{V}_{3}$. So:

$$
\begin{equation*}
-\mathbf{K}_{3} \mathbf{z}_{3}=\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right) \boldsymbol{\alpha}_{2}-\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right) \boldsymbol{\tau}_{a r} \tag{4.38}
\end{equation*}
$$

So the virtual control input $\boldsymbol{\alpha}_{2}$ is obtained as shown below:

$$
\begin{equation*}
\boldsymbol{\alpha}_{2}=-\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right)^{-1} \mathbf{K}_{3} \mathbf{z}_{3}+\boldsymbol{\tau}_{a r} \tag{4.39}
\end{equation*}
$$

The resultant Lyapunov function derivative is:

$$
\begin{equation*}
\dot{V}_{3}=-\mathbf{z}_{3}^{T} \mathbf{K}_{3} \mathbf{z}_{3}-\frac{1}{2} \boldsymbol{\tau}_{e}^{T} \mathbf{z}_{3}-\frac{1}{2} \mathbf{z}_{3}^{T} \boldsymbol{\tau}_{e}+\frac{1}{2} \mathbf{z}_{4}^{T}\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right)^{T} \mathbf{z}_{3}+\frac{1}{2} \mathbf{z}_{3}^{T}\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right) \mathbf{z}_{4} \tag{4.40}
\end{equation*}
$$

### 4.3.4 Step - IV (Speed Control)

This step starts with the differentiation of the last additional state variable which is defined in (4.36). That is:

$$
\begin{equation*}
\dot{\mathbf{z}}_{4}=\dot{\boldsymbol{\tau}}_{a}-\dot{\boldsymbol{\alpha}}_{2} \tag{4.41}
\end{equation*}
$$

For the sake of simplicity one can multiply the above by $\mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{K}_{T}^{-1}$ and obtain the following:

$$
\begin{equation*}
\mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{K}_{T}^{-1} \dot{\mathbf{z}}_{4}=\mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{K}_{T}^{-1} \dot{\boldsymbol{\tau}}_{a}-\mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{K}_{T}^{-1} \dot{\boldsymbol{\alpha}}_{2} \tag{4.42}
\end{equation*}
$$

A Lyapunov function can be selected as:

$$
\begin{equation*}
V_{4}=V_{3}+\frac{1}{2} \mathbf{z}_{4}^{T} \mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{K}_{T}^{-1} \mathbf{z}_{4} \tag{4.43}
\end{equation*}
$$

and its time derivative is:

$$
\begin{equation*}
\dot{V}_{4}=\dot{V}_{3}+\frac{1}{2} \dot{\mathbf{z}}_{4}^{T} \mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{K}_{T}^{-1} \mathbf{z}_{4}+\frac{1}{2} \mathbf{z}_{4}^{T} \mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{K}_{T}^{-1} \dot{\mathbf{z}}_{4} \tag{4.44}
\end{equation*}
$$

Substituting from (2.35), (4.40) and (4.42) yields:

$$
\begin{align*}
& \dot{V}_{4}=-\mathbf{z}_{3}^{T} \mathbf{K}_{3} \mathbf{z}_{3}+\frac{1}{2} \mathbf{z}_{4}^{T}\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right)^{T} \mathbf{z}_{3}+\frac{1}{2} \mathbf{z}_{3}^{T}\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right) \mathbf{z}_{4}+ \\
& +\frac{1}{2}\left\{-\boldsymbol{\tau}_{a}^{T}+\boldsymbol{\omega}_{s}^{T} \mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{K}_{E}+\mathbf{U}_{a}^{T} \mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{L}-\dot{\boldsymbol{\alpha}}_{2}^{T} \mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{K}_{T}^{-1}\right\} \mathbf{z}_{4} \\
& +\frac{1}{2} \mathbf{z}_{4}^{T}\left\{-\boldsymbol{\tau}_{a}+\mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{K}_{E} \boldsymbol{\omega}_{s}+\mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{L} \mathbf{U}_{a}-\mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{K}_{T}^{-1} \dot{\boldsymbol{\alpha}}_{2}\right\}  \tag{4.45}\\
& =-\mathbf{z}_{3}^{T} \mathbf{K}_{3} \mathbf{z}_{3}+ \\
& +\frac{1}{2}\left\{\mathbf{z}_{3}^{T}\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right)-\boldsymbol{\tau}_{a}^{T}+\boldsymbol{\omega}_{s}^{T} \mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{K}_{E}+\mathbf{U}_{a}^{T} \mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{L}-\dot{\boldsymbol{\alpha}}_{2}^{T} \mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{K}_{T}^{-1}\right\} \mathbf{z}_{4} \\
& +\frac{1}{2} \mathbf{z}_{4}^{T}\left\{\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right)^{T} \mathbf{z}_{3}-\boldsymbol{\tau}_{a}+\mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{K}_{E} \boldsymbol{\omega}_{s}+\mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{L} \mathbf{U}_{a}-\mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{K}_{T}^{-1} \dot{\boldsymbol{\alpha}}_{2}\right\}
\end{align*}
$$

The terms inside the brackets can be replaced by $-\mathbf{K}_{4} \mathbf{z}_{4}$ (where $\mathbf{K}_{4}$ has the same properties as that of $\mathbf{K}_{3}$ again preferably in the form $\mathbf{K}_{4}=k_{4} \mathbf{I}_{3 \times 3}, k_{4}>0$ ) to make the function $\dot{V}_{4}$ negative definite. As a result:

$$
\begin{equation*}
-\mathbf{K}_{4} \mathbf{z}_{4}=\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right)^{T} \mathbf{z}_{3}-\boldsymbol{\tau}_{a}+\mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{K}_{E} \boldsymbol{\omega}_{s}+\mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{L} \mathbf{U}_{a}-\mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{K}_{T}^{-1} \dot{\boldsymbol{\alpha}}_{2} \tag{4.46}
\end{equation*}
$$

The solution for the motor input voltage is obtained from the above as:

$$
\begin{equation*}
\mathbf{U}_{a}=\mathbf{L}^{-1} \mathbf{R}_{L} \mathbf{K}_{T}^{-1}\left\{-\mathbf{K}_{4} \mathbf{z}_{4}-\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right)^{T} \mathbf{z}_{3}+\boldsymbol{\tau}_{a}-\mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{K}_{E} \boldsymbol{\omega}_{s}+\mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{K}_{T}^{-1} \dot{\boldsymbol{\alpha}}_{2}\right\} \tag{4.47}
\end{equation*}
$$

As a result the Lyapunov function corresponding to the reaction wheel speed controller is written as:

$$
\begin{equation*}
\dot{V}_{4}=-\mathbf{z}_{3}^{T} \mathbf{K}_{3} \mathbf{z}_{3}-\underbrace{\frac{1}{2} \mathbf{z}_{4}^{T} \mathbf{K}_{4} \mathbf{z}_{4}-\frac{1}{2} \mathbf{z}_{4}^{T} \mathbf{K}_{4} \mathbf{z}_{4}}_{-\mathbf{z}_{4}^{T} \mathbf{K}_{4} \mathbf{z}_{4}}-\frac{1}{2} \boldsymbol{\tau}_{e}^{T} \mathbf{z}_{3}-\frac{1}{2} \mathbf{z}_{3}^{T} \boldsymbol{\tau}_{e} \tag{4.48}
\end{equation*}
$$

which provides a stable speed control system (neglecting the disturbance torque inputs).

The third and fourth steps constitude a wheel speed control loop which is feeding information from the attitude estimator and gyroscopes attached on the satellite body and reaction wheels. The attitude and satellite body angular velocities
are assumed as external signals. The actuation torque is generated by holding the reaction wheel velocity at the necessary value produced by (4.28).

### 4.4 Application Issues

Considering the control laws generated up to this point, there are some important issues related to the implementation. The most important one is the computation of the derivative of virtual command $\boldsymbol{\alpha}_{1}$ namely $\left(\dot{\boldsymbol{\alpha}}_{1}\right)$. In the real time coding, direct differentiation of $\boldsymbol{\alpha}_{1}$ should be avoided because of the robustness issues. So to do that one can use the readily available attitude derivative in the kinematical equation. Using all that information the differentiation algorithm can be presented as follows:

$$
\begin{align*}
& \boldsymbol{\alpha}_{1}=-\mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1} \mathbf{z}_{1} \\
& \dot{\boldsymbol{\alpha}}_{1}=-\dot{\mathbf{G}}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1} \mathbf{z}_{1}-\mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1} \dot{\mathbf{z}}_{1} \tag{4.49}
\end{align*}
$$

The time derivative of the matrix $\dot{\mathbf{G}}(\boldsymbol{\sigma})$ can be computed by taking the derivative of the kinematics equation in (2.2) as shown below:

$$
\begin{align*}
& \dot{\mathbf{G}}(\boldsymbol{\sigma})=\frac{d}{d t}\left(\frac{1-\boldsymbol{\sigma}^{\mathrm{T}} \boldsymbol{\sigma}}{4} \mathbf{I}_{3 \times 3}+\frac{1}{2} \mathbf{S}(\boldsymbol{\sigma})+\frac{1}{2} \boldsymbol{\sigma} \boldsymbol{\sigma}^{\mathrm{T}}\right) \\
& =-\frac{\dot{\boldsymbol{\sigma}}^{\mathrm{T}} \boldsymbol{\sigma}+\boldsymbol{\sigma}^{\mathrm{T}} \dot{\boldsymbol{\sigma}}^{2}}{4} \mathbf{I}_{3 \times 3}+\frac{1}{2} \mathbf{S}(\dot{\boldsymbol{\sigma}})+\frac{1}{2}\left\{\dot{\boldsymbol{\sigma}} \boldsymbol{\sigma}^{\mathrm{T}}+\boldsymbol{\sigma} \dot{\boldsymbol{\sigma}}^{\mathrm{T}}\right\}  \tag{4.50}\\
& =-\frac{\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T} \mathbf{G}^{T}(\boldsymbol{\sigma}) \boldsymbol{\sigma}+\boldsymbol{\sigma}^{\mathrm{T}} \mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}}{4} \mathbf{I}_{3 \times 3}+\frac{1}{2} \mathbf{S}\left(\mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}\right)+\frac{1}{2}\left\{\mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b} \boldsymbol{\sigma}^{\mathrm{T}}+\boldsymbol{\sigma}\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T} \mathbf{G}^{T}(\boldsymbol{\sigma})\right\}
\end{align*}
$$

A similar case is the derivative of $\boldsymbol{\alpha}_{2}$ variable which was:

$$
\begin{equation*}
\boldsymbol{\alpha}_{2}=-\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right)^{-1} \mathbf{K}_{3} \mathbf{z}_{3}+\boldsymbol{\tau}_{a r} \tag{4.51}
\end{equation*}
$$

The time derivative of the above is:

$$
\begin{equation*}
\dot{\boldsymbol{\alpha}}_{2}=-\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right)^{-1} \mathbf{K}_{3} \dot{\mathbf{z}}_{3}+\dot{\boldsymbol{\tau}}_{a r} \tag{4.52}
\end{equation*}
$$

where $\dot{\mathbf{z}}_{3}$ is computed from (4.31) after deleting the disturbance torque term $\boldsymbol{\tau}_{e}$ and $\dot{\boldsymbol{\tau}}_{a r}$ is computed as:
$\dot{\boldsymbol{\tau}}_{a r}=\mathbf{K}_{2} \dot{\mathbf{z}}_{2}-\mathbf{S}\left(\dot{\boldsymbol{\omega}}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)-\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \dot{\boldsymbol{\omega}}_{i b}^{b}+\mathbf{I}_{s} \dot{\boldsymbol{\omega}}_{s}\right)+\omega_{o} \mathbf{J S}\left(\dot{\mathbf{c}}_{2}\right) \boldsymbol{\omega}_{i b}^{b}+\omega_{o} \mathbf{J S}\left(\mathbf{c}_{2}\right) \dot{\boldsymbol{\omega}}_{i b}^{b}-\mathbf{J} \ddot{\boldsymbol{\omega}}_{1}$

$$
\begin{align*}
& \ddot{\boldsymbol{\omega}}_{1}=-\ddot{\mathbf{G}}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1} \mathbf{z}_{1}-\dot{\mathbf{G}}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1} \mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}-k_{1} \frac{2\left(1+\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}\right)\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T} \mathbf{G}^{T}(\boldsymbol{\sigma}) \boldsymbol{\sigma}+\boldsymbol{\sigma}^{T} \mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}}{16} \mathbf{I}_{3 \times 3} \boldsymbol{\omega}_{o b}^{b}  \tag{4.54}\\
& -k_{1} \frac{\left(1+\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}\right)^{2}}{16} \dot{\boldsymbol{\omega}}_{o b}^{b} \\
& \ddot{\mathbf{G}}(\boldsymbol{\sigma})=-\frac{\left(\dot{\boldsymbol{\omega}}_{o b}^{b}\right)^{T} \mathbf{G}^{T}(\boldsymbol{\sigma}) \boldsymbol{\sigma}+\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T} \dot{\mathbf{G}}^{T}(\boldsymbol{\sigma}) \boldsymbol{\sigma}+2\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T} \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}+\boldsymbol{\sigma}^{T} \dot{\mathbf{G}}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}+\boldsymbol{\sigma}^{T} \mathbf{G}(\boldsymbol{\sigma}) \dot{\boldsymbol{\omega}}_{o b}^{b}}{4}+ \\
& +\frac{1}{2} \mathbf{S}\left\{\dot{\mathbf{G}}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}+\mathbf{G}(\boldsymbol{\sigma}) \dot{\boldsymbol{\omega}}_{o b}^{b}\right\}+ \\
& +\frac{1}{2} \mathbf{S}\left\{\dot{\mathbf{G}}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b} \boldsymbol{\sigma}^{T}+\mathbf{G}(\boldsymbol{\sigma}) \dot{\boldsymbol{\omega}}_{o b}^{b} \boldsymbol{\sigma}^{T}+2 \mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T} \mathbf{G}^{T}(\boldsymbol{\sigma})+\boldsymbol{\sigma}\left(\dot{\boldsymbol{\omega}}_{o b}^{b}\right)^{T} \mathbf{G}^{T}(\boldsymbol{\sigma})+\boldsymbol{\sigma}\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T} \dot{\mathbf{G}}^{T}(\boldsymbol{\sigma})\right\} \tag{4.55}
\end{align*}
$$

The term $\dot{\mathbf{z}}_{2}$ is computed from (4.16) and the other terms $\dot{\boldsymbol{\omega}}_{i b}^{b}, \dot{\boldsymbol{\omega}}_{o b}^{b}$ and $\dot{\boldsymbol{\omega}}_{s}$ are computed using the nonlinear model of the satellite (excluding the disturbance torques).

Another applicational issue is the reference trajectory generation. It is useful to provide a reference trajectory generator which has adjustable features. Especially the transient part of the reference trajectory should not be too fast in order not to load the actuator too much. So a trajectory smoothing function at the reference input of the attitude controller (demanded torque generator) is recommended. A useful example for this purpose is the standard second order system function. In Laplace domain this can be shown as:

$$
\begin{equation*}
G_{s m}(s)=\frac{\omega_{n}^{2}}{s^{2}+2 \zeta \omega_{n} s+\omega_{n}^{2}}=\frac{\theta_{\text {rraj }}(s)}{\theta_{r e f}(s)} \tag{4.56}
\end{equation*}
$$

where $\theta_{r e f}(s)$ is the reference attitude function, $\theta_{r r a j}(s)$ is the reference trajectory that is to be supplied to the demanded torque generator law, $\omega_{n}$ is the natural frequency and $\zeta$ is the damping ratio of the transfer function .To have a smooth trajectory $\zeta$ can be selected as unity. For practical purposes, the reference attitude can be provided as a step function, namely $\theta_{\text {ref }}(t)=\theta_{0} u(t)$. The state space representation of (4.56) is:

$$
\frac{d}{d t}\left[\begin{array}{c}
\theta_{r v a j}(t)  \tag{4.57}\\
\hat{\theta}_{r r a j}(t)
\end{array}\right]=\left[\begin{array}{cc}
0 & 1 \\
-\omega_{n}^{2} & -2 \zeta \omega_{n}
\end{array}\right]\left[\begin{array}{c}
\theta_{m p a}(t) \\
\hat{\theta}_{r a j}(t)
\end{array}\right]+\left[\begin{array}{c}
0 \\
\omega_{n}^{2}
\end{array}\right] \theta_{r r f}(t)
$$

So $\theta_{\text {rraj }}(t), \dot{\theta}_{\text {traj }}(t)$ and $\ddot{\theta}_{t r a j}(t)$ can easily be computed from the above well known state equation if necessary.

### 4.5 Final Structure of the Control

The final form of the back - stepping control can be expressed as shown in (4.58). It should be stressed again that the derivatives of the state variables $\boldsymbol{\omega}_{i b}^{b}$ (or $\boldsymbol{\omega}_{o b}^{b}$ ) found in those control laws should not be differentiated numerically instead they should be continued from the dynamical model equations presented in (4.2) and (4.3).

Demanded Torque Generator:
$\boldsymbol{\tau}_{a r}=\mathbf{K}_{2} \mathbf{z}_{2}+\mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{z}_{1}-\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\omega_{o} \mathbf{J S}\left(\mathbf{c}_{2}\right) \boldsymbol{\omega}_{o b}^{b}-\mathbf{J} \dot{\boldsymbol{\alpha}}_{1}$
$\boldsymbol{\omega}_{o b}^{b}=\boldsymbol{\omega}_{i b}^{b}+\omega_{o} \mathbf{c}_{2}$
$\dot{\boldsymbol{\omega}}_{o b}^{b}=\dot{\boldsymbol{\omega}}_{i b}^{b}+\omega_{o} \dot{\mathbf{c}}_{2}$
$\dot{\mathbf{c}}_{2}=\mathbf{S}\left(\mathbf{c}_{2}\right) \boldsymbol{\omega}_{o b}^{b}$
$\boldsymbol{\alpha}_{1}=-\mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1} \mathbf{z}_{1}$
$\mathbf{z}_{1}=\boldsymbol{\sigma}-\boldsymbol{\sigma}_{d}$
$\dot{\mathbf{z}}_{1}=\dot{\boldsymbol{\sigma}}=\mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}$
$\dot{\boldsymbol{\alpha}}_{1}=-\dot{\mathbf{G}}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1} \mathbf{z}_{1}-\mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1} \dot{\mathbf{z}}_{1}$
$\dot{\mathbf{G}}(\boldsymbol{\sigma})=-\frac{\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T} \mathbf{G}^{T}(\boldsymbol{\sigma}) \boldsymbol{\sigma}+\boldsymbol{\sigma}^{\mathrm{T}} \mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}}{4} \mathbf{I}_{3 \times 3}+\frac{1}{2} \mathbf{S}\left(\mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}\right)+\frac{1}{2}\left\{\mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b} \boldsymbol{\sigma}^{\mathrm{T}}+\boldsymbol{\sigma}\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T} \mathbf{G}^{T}(\boldsymbol{\sigma})\right\}$
Reaction Wheel Velocity Controller:
$\mathbf{U}_{a}=\mathbf{L}^{-1} \mathbf{R}_{L} \mathbf{K}_{T}^{-1}\left\{-\mathbf{K}_{4} \mathbf{z}_{4}-\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right)^{T} \mathbf{z}_{3}+\boldsymbol{\tau}_{a}-\mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{K}_{E} \boldsymbol{\omega}_{s}+\mathbf{K}_{T} \mathbf{R}_{L}^{-1} \mathbf{K}_{T}^{-1} \dot{\boldsymbol{\alpha}}_{2}\right\}$
$\boldsymbol{\alpha}_{2}=-\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right)^{-1} \mathbf{K}_{3} \mathbf{z}_{3}+\boldsymbol{\tau}_{a r}$
$\dot{\boldsymbol{\alpha}}_{2}=-\left(\mathbf{J I}_{s}^{-1}+\mathbf{I}_{3 \times 3}\right)^{-1} \mathbf{K}_{3} \dot{\mathbf{z}}_{3}+\dot{\boldsymbol{\tau}}_{a r}$
$\dot{\boldsymbol{\tau}}_{a r}=\mathbf{K}_{2} \dot{\mathbf{z}}_{2}-\mathbf{S}\left(\dot{\boldsymbol{\omega}}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)-\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \dot{\boldsymbol{\omega}}_{i b}^{b}+\mathbf{I}_{s} \dot{\boldsymbol{\omega}}_{s}\right)+\omega_{o} \mathbf{J S}\left(\dot{\mathbf{c}}_{2}\right) \boldsymbol{\omega}_{i b}^{b}+\omega_{o} \mathbf{J S}\left(\mathbf{c}_{2}\right) \dot{\boldsymbol{\omega}}_{i b}^{b}-\mathbf{J} \ddot{\boldsymbol{\alpha}}_{1}$
$\ddot{\boldsymbol{\alpha}}_{1}=-\ddot{\mathbf{G}}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1} \mathbf{z}_{1}-\dot{\mathbf{G}}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1} \mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}-k_{1} \frac{2\left(1+\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}\right)\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T} \mathbf{G}^{T}(\boldsymbol{\sigma}) \boldsymbol{\sigma}+\boldsymbol{\sigma}^{T} \mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}}{16} \mathbf{I}_{3 \times 3} \boldsymbol{\omega}_{o b}^{b}$
$-k_{1} \frac{\left(1+\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}\right)^{2}}{16} \dot{\boldsymbol{\omega}}_{o b}^{b}$
$\ddot{\mathbf{G}}(\boldsymbol{\sigma})=-\frac{\left(\dot{\boldsymbol{\omega}}_{o b}^{b}\right)^{T} \mathbf{G}^{T}(\boldsymbol{\sigma}) \boldsymbol{\sigma}+\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T} \dot{\mathbf{G}}^{T}(\boldsymbol{\sigma}) \boldsymbol{\sigma}+2\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T} \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}+\boldsymbol{\sigma}^{T} \dot{\mathbf{G}}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}+\boldsymbol{\sigma}^{T} \mathbf{G}(\boldsymbol{\sigma}) \dot{\boldsymbol{\omega}}_{o b}^{b}}{4}+$
$+\frac{1}{2} \mathbf{S}\left\{\dot{\mathbf{G}}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}+\mathbf{G}(\boldsymbol{\sigma}) \dot{\boldsymbol{\omega}}_{o b}^{b}\right\}+$
$+\frac{1}{2} \mathbf{S}\left\{\dot{\mathbf{G}}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b} \boldsymbol{\sigma}^{T}+\mathbf{G}(\boldsymbol{\sigma}) \dot{\boldsymbol{\omega}}_{o b}^{b} \boldsymbol{\sigma}^{T}+2 \mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T} \mathbf{G}^{T}(\boldsymbol{\sigma})+\boldsymbol{\sigma}\left(\dot{\boldsymbol{\omega}}_{o b}^{b}\right)^{T} \mathbf{G}^{T}(\boldsymbol{\sigma})+\boldsymbol{\sigma}\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T} \dot{\mathbf{G}}^{T}(\boldsymbol{\sigma})\right\}$
(4.58)

### 4.6 Stability Analysis

In this section, the back - stepping attitude controller is analyzed for input - to - state stability. The main purpose is to investigate the stability of the back - stepping controller against the external disturbance torques. For that purpose the definitions in 3.4.1 is utilized. The analysis is performed separately for the attitude (first and second steps) and the wheel speed (third and fourth steps) controllers. So it is convenient to write the Lyapunov functions $\dot{V}_{2}$ and $\dot{V}_{4}$ as:

$$
\begin{align*}
& \dot{L}_{1}=\dot{V}_{2}=-\frac{1}{2} \mathbf{z}_{1}^{T}\left\{\mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1}+\mathbf{K}_{1} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma})\right\} \mathbf{z}_{1}-\underbrace{\frac{1}{2} \mathbf{z}_{2}^{T} \mathbf{K}_{2} \mathbf{z}_{2}-\frac{1}{2} \mathbf{z}_{2}^{T} \mathbf{K}_{2} \mathbf{z}_{2}}_{-\mathbf{z}_{2}^{T} \mathbf{K}_{2} \mathbf{z}_{2}}+ \\
&+\frac{1}{2} \boldsymbol{\tau}_{e}^{T} \mathbf{z}_{2}+\frac{1}{2} \mathbf{z}_{2}^{T} \boldsymbol{\tau}_{e}  \tag{4.59}\\
&=-\frac{1}{2} \mathbf{z}_{1}^{T}\left\{\mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1}+\mathbf{K}_{1} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma})\right\} \mathbf{z}_{1}-\mathbf{z}_{2}^{T} \mathbf{K}_{2} \mathbf{z}_{2}+\mathbf{z}_{2}^{T} \boldsymbol{\tau}_{e} \\
& \dot{L}_{2}=\dot{V}_{4}=-\mathbf{z}_{3}^{T} \mathbf{K}_{3} \mathbf{z}_{3}-\underbrace{-\frac{1}{2} \mathbf{Z}_{4}^{T} \mathbf{K}_{4} \mathbf{z}_{4}-\frac{1}{2} \mathbf{z}_{4}^{T} \mathbf{K}_{4} \mathbf{z}_{4}}_{-\mathbf{z}_{4}^{T}}-\frac{1}{2} \boldsymbol{\tau}_{e}^{T} \mathbf{z}_{3}-\frac{1}{2} \mathbf{z}_{3}^{T} \boldsymbol{\tau}_{e}  \tag{4.60}\\
&=-\mathbf{z}_{3}^{T} \mathbf{K}_{3} \mathbf{z}_{3}-\mathbf{z}_{4}^{T} \mathbf{K}_{4} \mathbf{z}_{4}-\mathbf{z}_{3}^{T} \boldsymbol{\tau}_{e}
\end{align*}
$$

### 4.6.1 Analysis of the Demanded Torque Generator

In order to proceed one should remember the following property of vector norms:

$$
\begin{equation*}
(\mathbf{x}-\mathbf{y})^{T}(\mathbf{x}-\mathbf{y})=\mathbf{x}^{T} \mathbf{x}-\mathbf{x}^{T} \mathbf{y}-\mathbf{y}^{T} \mathbf{x}+\mathbf{y}^{T} \mathbf{y}=\mathbf{x}^{T} \mathbf{x}-2 \mathbf{x}^{T} \mathbf{y}+\mathbf{y}^{T} \mathbf{y} \geq 0 \tag{4.61}
\end{equation*}
$$

Since $\mathbf{x}^{T} \mathbf{x}=\|\mathbf{x}\|^{2}$, one can say that:

$$
\begin{align*}
& \|\mathbf{x}\|^{2}-2 \mathbf{x}^{T} \mathbf{y}+\|\mathbf{y}\|^{2} \geq 0 \\
& \rightarrow\|\mathbf{x}\|^{2}+\|\mathbf{y}\|^{2} \geq 2 \mathbf{x}^{T} \mathbf{y} \\
& \rightarrow \frac{1}{2}\left\{\|\mathbf{x}\|^{2}+\|\mathbf{y}\|^{2}\right\} \geq \mathbf{x}^{T} \mathbf{y}  \tag{4.62}\\
& \rightarrow\left\{\|\mathbf{x}\|^{2}+\|\mathbf{y}\|^{2}\right\} \geq \mathbf{x}^{T} \mathbf{y}
\end{align*}
$$

Using the above property, the equation (4.59) can be rewritten as an inequality:

$$
\begin{equation*}
\dot{L} \leq-\frac{1}{2} \mathbf{z}_{1}^{T}\left\{\mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1}+\mathbf{K}_{1} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma})\right\} \mathbf{z}_{1}-\mathbf{z}_{2}^{T} \mathbf{K}_{2} \mathbf{z}_{2}+\left\|\mathbf{z}_{2}^{T}\right\|^{2}+\left\|\boldsymbol{\tau}_{e}\right\|^{2} \tag{4.63}
\end{equation*}
$$

It is obvious that $\left\|\mathbf{z}_{2}^{T}\right\|^{2}=\left\|\mathbf{z}_{2}\right\|^{2}$ so the above is rewritten as:

$$
\begin{equation*}
\dot{L} \leq-\frac{1}{2} \mathbf{z}_{1}^{T}\left\{\mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1}+\mathbf{K}_{1} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma})\right\} \mathbf{z}_{1}-\mathbf{z}_{2}^{T} \mathbf{K}_{2} \mathbf{z}_{2}+\left\|\mathbf{z}_{2}\right\|^{2}+\left\|\boldsymbol{\tau}_{e}\right\|^{2} \tag{4.64}
\end{equation*}
$$

A very important property of the matrix eigenspace is:

$$
\begin{equation*}
\lambda_{\text {min }}(\mathbf{A}) \mathbf{x}^{T} \mathbf{x} \leq \mathbf{x}^{T} \mathbf{A} \leq \lambda_{\text {max }}(\mathbf{A}) \mathbf{x}^{T} \mathbf{x} \tag{4.65}
\end{equation*}
$$

Applying the above property to (4.64) results in:
$\dot{L} \leq-\frac{1}{2} \mathbf{z}_{1}^{T} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1} \mathbf{z}_{1}-\frac{1}{2} \mathbf{z}_{1}^{T} \mathbf{K}_{1} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{z}_{1}-\left\|\mathbf{z}_{2}\right\|^{2} \lambda_{\text {min }}\left(\mathbf{K}_{2}\right)+\left\{\left\|\boldsymbol{\tau}_{\boldsymbol{c}}\right\|^{2}+\left\|\mathbf{z}_{2}\right\|^{2}\right\}$
and
$\dot{L} \leq-\frac{1}{2}\left\|\mathbf{z}_{1}\right\|^{2} \lambda_{\text {min }}\left(\mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1}\right)-\frac{1}{2}\left\|\mathbf{z}_{1}\right\|^{2} \lambda_{\text {min }}\left(\mathbf{K}_{1} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma})\right)-\left\{\lambda_{\text {min }}\left(\mathbf{K}_{2}\right)-1\right\}\left\|\mathbf{z}_{2}\right\|^{2}+\left\|\boldsymbol{\tau}_{\boldsymbol{e}}\right\|^{2}$
The last inequality can be written in a simplified form considering $\mathbf{z}=\left[\begin{array}{ll}\mathbf{z}_{1}^{T} & \mathbf{z}_{2}^{T}\end{array}\right]^{T}$ as presented in the following:

$$
\begin{equation*}
\dot{L} \leq-M_{1}\left(\| \| \|^{2}\right)+\left\|\boldsymbol{\tau}_{e}\right\|^{2} \tag{4.68}
\end{equation*}
$$

where $M_{1}$ is a constant and can be selected as shown below (this can be inferred from (4.67) easily):

$$
\begin{equation*}
M=\min \left[\frac{1}{2}\left\{\lambda_{\text {min }}\left(\mathbf{K}_{1} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma})\right)+\lambda_{\text {min }}\left(\mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1}\right)\right\},\left\{\lambda_{\text {min }}\left(\mathbf{K}_{2}\right)-1\right\}\right] \tag{4.69}
\end{equation*}
$$

By the virtue of [Khalii (1996)], the system is said to be input - to - state stable if $M_{1}>0$. Because of that in addition to the positive definiteness of $\mathbf{K}_{1,2}$ 's the values of $\lambda_{\text {min }}\left(\mathbf{K}_{2}\right)$ should be greater than unity. The product $\mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma})$ is a diagonal matrix and written as:

$$
\begin{equation*}
\mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma})=\mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{G}(\boldsymbol{\sigma})=\frac{\left(1+\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}\right)^{2}}{16} \mathbf{I}_{3 \times 3} \tag{4.70}
\end{equation*}
$$

The above is an interesting result of the matrix $\mathbf{G}(\boldsymbol{\sigma})$. The above is infact a diagonal matrix with equal entries and is monotonically increasing from $1 / 16 @(\boldsymbol{\sigma}=0)$ to infinity. So one can easily deduce that the minimum value of $\mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma})$ is equal to $1 / 16$. With the selection of $\mathbf{K}_{1}=k_{1} \mathbf{I}_{3 \times 3}$ the following identity can be written:

$$
\begin{equation*}
\frac{1}{2}\left\{\lambda_{\text {min }}\left(\mathbf{K}_{1} \mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma})\right)+\lambda_{\text {min }}\left(\mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma}) \mathbf{K}_{1}\right)\right\}=0.0625 k_{1} \tag{4.71}
\end{equation*}
$$

If one desires to have the value of $M_{1}$ to be greater than unity, an example selection can be $\left(k_{1}=20, k_{2}=2.5\right)$. In this case the value of is $M_{1}=\min \{0.0625 \cdot 20,2.5-1\}=1.25$. Other details concerning the application are given in Chapter 6.

### 4.6.2 Analysis of the Speed Controller

A similar operation on two vectors $\mathbf{x}$ and $\mathbf{y}$ like that of (4.61) yields the following result:

$$
\begin{align*}
& (\mathbf{x}+\mathbf{y})^{T}(\mathbf{x}+\mathbf{y})=\mathbf{x}^{T} \mathbf{x}+\mathbf{x}^{T} \mathbf{y}+\mathbf{y}^{T} \mathbf{x}+\mathbf{y}^{T} \mathbf{y}=\mathbf{x}^{T} \mathbf{x}+2 \mathbf{x}^{T} \mathbf{y}+\mathbf{y}^{T} \mathbf{y} \geq 0 \\
& \|\mathbf{x}\|^{2}+2 \mathbf{x}^{T} \mathbf{y}+\|\mathbf{y}\|^{2} \geq 0 \\
& \rightarrow\|\mathbf{x}\|^{2}+\|\mathbf{y}\|^{2} \geq-2 \mathbf{x}^{T} \mathbf{y}  \tag{4.72}\\
& \rightarrow \frac{1}{2}\left\{\|\mathbf{x}\|^{2}+\|\mathbf{y}\|^{2}\right\} \geq-\mathbf{x}^{T} \mathbf{y} \\
& \rightarrow\left\{\|\mathbf{x}\|^{2}+\|\mathbf{y}\|^{2}\right\} \geq-\mathbf{x}^{T} \mathbf{y}
\end{align*}
$$

Using the above rule, the Lyapunov function derivative on (4.60) can be rewritten as:

$$
\begin{equation*}
\dot{L}_{2} \leq-\mathbf{z}_{3}^{T} \mathbf{K}_{3} \mathbf{z}_{3}-\mathbf{z}_{4}^{T} \mathbf{K}_{4} \mathbf{z}_{4}+\left\|\mathbf{z}_{3}\right\|^{2}+\left\|\boldsymbol{\tau}_{e}\right\|^{2} \tag{4.73}
\end{equation*}
$$

The eigenvalue inequality rule in (4.65) enables one to modify the above inequality as:

$$
\begin{align*}
& \dot{L}_{2} \leq-\mathbf{z}_{3}^{T} \mathbf{K}_{3} \mathbf{z}_{3}-\mathbf{z}_{4}^{T} \mathbf{K}_{4} \mathbf{z}_{4}+\left\|\mathbf{z}_{3}\right\|^{2}+\left\|\boldsymbol{\tau}_{e}\right\|^{2} \\
& \dot{L}_{2} \leq-\lambda_{\text {min }}\left(\mathbf{K}_{3}\right)\left\|\mathbf{z}_{3}\right\|^{2}-\lambda_{\text {min }}\left(\mathbf{K}_{4}\right)\left\|\mathbf{z}_{4}\right\|^{2}+\left\|\mathbf{z}_{3}\right\|^{2}+\left\|\boldsymbol{\tau}_{\boldsymbol{t}}\right\|^{2}  \tag{4.74}\\
& \dot{L}_{2} \leq-\left\{\lambda_{\text {min }}\left(\mathbf{K}_{3}\right)-1\right\}\left\|\mathbf{z}_{3}\right\|^{2}-\lambda_{\text {min }}\left(\mathbf{K}_{4}\right)\left\|\mathbf{z}_{4}\right\|^{2}+\left\|\boldsymbol{\tau}_{e}\right\|^{2}
\end{align*}
$$

The last inequality can be written in the following form:

$$
\begin{equation*}
\dot{L} \leq-M_{2}\left(\|\boldsymbol{z}\|^{2}\right)+\left\|\boldsymbol{\tau}_{\varepsilon}\right\|^{2} \tag{4.75}
\end{equation*}
$$

with the acceptance that $\mathbf{z}=\left[\begin{array}{ll}\mathbf{z}_{3}^{T} & \mathbf{z}_{4}^{T}\end{array}\right]^{T}$. The value of the variable $M_{2}$ is:

$$
\begin{equation*}
M_{2}=\min \left[\left\{\lambda_{\text {min }}\left(\mathbf{K}_{3}\right)-1\right\}, \lambda_{\text {min }}\left(\mathbf{K}_{4}\right)\right] \tag{4.76}
\end{equation*}
$$

$M_{2}$ should be positive in order to have a stable closed loop against the disturbance torques. The selection criteria of this case is simpler than that of the demanded torque generator. For $\mathbf{K}_{3,4}=k_{3,4} \mathbf{I}_{3 \times 3}$, one should satisfy $k_{3}>1$ and $k_{4}>0$ in order that $M_{2}>0$. In the case of $M_{2}>1$, the condition can be $k_{3}>2$ and $k_{4}>1$. For the details of application one can refer to Chapter 6.

### 4.7 The Stability of Wheel Velocity Command Generator:

As it is stated in the Section 4.3.2 the required actuation torque is generated by integrating the noise free part of the reaction wheel dynamics. The reaction wheel dynamical equation shown below:

$$
\begin{equation*}
\dot{\boldsymbol{\omega}}_{s}=\mathbf{J}^{-1} \mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right) \boldsymbol{\tau}_{a r} \tag{4.77}
\end{equation*}
$$

Is not a stable system as can be understood from its linearized version shown below:

$$
\begin{equation*}
\frac{\partial \dot{\boldsymbol{\omega}}_{s}}{\partial \boldsymbol{\omega}_{s}}=\mathbf{J}^{-1} \mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right) \mathbf{I}_{s} \tag{4.78}
\end{equation*}
$$

Since the skew symmetric matrix $\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)$ is structurally singular the above should have at least one eigenvalue at the origin. That will be the same for the product $\mathbf{J}^{-1} \mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right) \mathbf{I}_{s}$. So the linearized system should be considered as unstable in the vicinity of the linearization point. Structural singularity means that the linearized system is unstable in each linearization point. So the nonlinear system in (4.77) should be considered as unstable. In fact this is normal since the reaction wheels are free running mechanical structures that spins at a certain velocity according to the torque supplied by the brushless dc motors. The reaction wheel velocity command generator is an artificial system that operates as a part of the control computer software to produce the necessary reaction wheel velocity according to the requested torque from the demanded torque generator. So it is convenient to put a velocity limiter to the reference input of the reaction wheel speed controller.

## CHAPTER 5 <br> ATTITUDE CONTROL WITH FEEDBACK LINEARIZATION

### 5.1 Introduction

In this chapter, the second method of attitude control in this work which is the input output feedback linearization (IOFL) is presented. In this case there exists no partition of the model state vector into sub - states due to the non - recursive nature of feedback linearization method. Both the demanded torque generator and the reaction wheel velocity controllers are designed by IOFL method.

### 5.2 Preparing for IOFL Attitude Control

Since the feedback linearization requires the usage of affine nonlinear models it is convenient to use the nonlinear model in (2.39) and (2.40). The model which is required for the development of the demanded torque generator is repeated here for convenience:

$$
\begin{align*}
& \mathbf{x}=\left[\begin{array}{ll}
\boldsymbol{\sigma}^{T} & \left(\boldsymbol{\omega}_{o b}^{b}\right)^{T}
\end{array}\right]^{T} \\
& \mathbf{f}(\mathbf{x})=\left[\begin{array}{c}
\mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b} \\
\mathbf{J}^{-1}\left[-\mathbf{S}\left(\boldsymbol{\omega}_{o b}^{b}-\mathbf{c}_{2} \omega_{o}\right)\left(\mathbf{I}\left[\boldsymbol{\omega}_{o b}^{b}-\mathbf{c}_{2} \omega_{o}\right]+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)\right]-\mathbf{S}\left(\boldsymbol{\omega}_{o b}^{b}\right) \mathbf{c}_{2} \omega_{o}
\end{array}\right] \\
& \mathbf{g}(\mathbf{x})=\left[\begin{array}{c}
\mathbf{0}_{3 \times 3} \\
-\mathbf{J}^{-1}
\end{array}\right], \mathbf{p}(\mathbf{x})=\left[\begin{array}{c}
\mathbf{0}_{3 \times 3} \\
\mathbf{J}^{-1}
\end{array}\right]  \tag{5.1}\\
& \mathbf{u}=\boldsymbol{\tau}_{a}, \mathbf{w}=\boldsymbol{\tau}_{e} \\
& \dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}) \mathbf{x}+\mathbf{g}(\mathbf{x}) \mathbf{u}+\mathbf{p}(\mathbf{x}) \mathbf{w}
\end{align*}
$$

The procedure starts with the attitude dynamics in (5.1) (to be used in the development of the demanded torque generator). The first step of the input output feedback linearization is to compute the total relative degree of the nonlinear system in (5.1). As the output of the system is the attitude vector (i.e. $\mathbf{h}(\mathbf{x})=\boldsymbol{\sigma})$ then

$$
\begin{align*}
& L_{\mathbf{g}_{j}} h_{1}=\frac{\partial \sigma_{1}}{\partial \mathbf{x}} \mathbf{g}_{j}=\left[\begin{array}{llllll}
1 & 0 & 0 & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{c}
\mathbf{0}_{3 \times 3} \\
-\mathbf{J}^{-1}
\end{array}\right]=0 \\
& L_{\mathbf{g}_{j}} h_{1}=\frac{\partial \sigma_{2}}{\partial \mathbf{x}} \mathbf{g}_{j}=\left[\begin{array}{llllll}
0 & 1 & 0 & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{c}
\mathbf{0}_{3 \times 3} \\
-\mathbf{J}^{-1}
\end{array}\right]=0  \tag{5.2}\\
& L_{\mathbf{g}_{j}} h_{1}=\frac{\partial \sigma_{3}}{\partial \mathbf{x}} \mathbf{g}_{j}=\left[\begin{array}{llllll}
0 & 0 & 1 & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{c}
\mathbf{0}_{3 \times 3} \\
-\mathbf{J}^{-1}
\end{array}\right]=0
\end{align*}
$$

So the relative degree should be at least one for each output ( $h_{i}=\sigma_{i}$ ). Now processing further:

$$
\left.\left.\begin{array}{l}
L_{\mathrm{g}_{j}} L_{\mathrm{f}} \sigma_{1}=\frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial \sigma_{1}}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x})\right) \mathbf{g}_{j}(\mathbf{x})=\frac{\partial}{\partial \mathbf{x}}\left(\left[\begin{array}{lllll}
1 & 0 & 0 & 0 & 0
\end{array} 0\right.\right.
\end{array}\right] \mathbf{f}(\mathbf{x})\right) \mathbf{g}_{j}(\mathbf{x}), ~\left(\frac{\partial}{\mathbf{g}_{j}} L_{\mathrm{f}} \sigma_{2}=\frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial \sigma_{2}}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x})\right) \mathbf{g}_{j}(\mathbf{x})=\frac{\partial}{\partial \mathbf{x}}\left(\left[\begin{array}{llllll}
0 & 1 & 0 & 0 & 0 & 0
\end{array}\right] \mathbf{f}(\mathbf{x})\right) \mathbf{g}_{j}(\mathbf{x}) .\right.
$$

Compiling the above into a single equation yields:
$L_{\mathbf{g}_{j}} L_{\mathrm{f}} \boldsymbol{\sigma}=\frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial \sigma_{1}}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x})\right) \mathbf{g}_{j}(\mathbf{x})=\frac{\partial}{\partial \mathbf{x}}(\left[\begin{array}{cccccc}1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0\end{array}\right] \underbrace{\mathbf{f}(\mathbf{x})}_{\mathbf{G}(\boldsymbol{\sigma}) \mathbf{\omega}_{o b}^{b}}) \mathbf{g}_{j}(\mathbf{x})$
And,
$L_{\mathrm{g}} L_{\mathrm{f}} \boldsymbol{\sigma}=\frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial \sigma_{1}}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x})\right)=\frac{\partial}{\partial \mathbf{x}}\left(\mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}\right) \mathbf{g}(\mathbf{x})=\frac{\partial}{\partial \mathbf{x}}\left(\mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}\right)\left[\begin{array}{c}\mathbf{0}_{3 \times 3} \\ -\mathbf{J}^{-1}\end{array}\right]$
$=\left[\begin{array}{cccc}\times & \times & \times & \\ \times & \times & \times & \mathbf{G}(\boldsymbol{\sigma}) \\ \times & \times & \times & \end{array}\right]\left[\begin{array}{c}\mathbf{0}_{3 \times 3} \\ -\mathbf{J}^{-1}\end{array}\right] \neq \mathbf{0}$
The symbol $\times$ represents don't care entries where their contents have no meaning since they are multiplied by zero. The matrix $\mathbf{G}(\boldsymbol{\sigma})$ is always invertible so the above identity will never be zero. In addition to that one should also check the determinant of the matrix:

$$
\begin{equation*}
\mathbf{A}(\mathbf{x})=-\mathbf{G}(\boldsymbol{\sigma}) \mathbf{J}^{-1} \tag{5.6}
\end{equation*}
$$

The above expression is obtained simply by using (3.27) and the results from (5.3), (5.4) and (5.5). Since $\mathbf{G}(\boldsymbol{\sigma})$ and $\mathbf{J}^{-1}$ are fully invertible matrices $\mathbf{A}(\mathbf{x})$ is said to be invertible. So, it can be concluded that the relative degree of the nonlinear satellite model of $(5.1)$ is equal to two for each output. So the total relative degree is equal to six. So the nonlinear system in (5.1) can be completely linearized.

The normal variables are selected for each output as defined in (3.28) and combined into the vector $\varphi$ (as defined in section 3.3.2) as shown below:

$$
\begin{align*}
& \boldsymbol{\varphi}_{1}=\left[\begin{array}{c}
h_{1} \\
L_{f} h_{1}
\end{array}\right]=\left[\begin{array}{c}
\sigma_{1} \\
L_{f} \sigma_{1}
\end{array}\right], \boldsymbol{\varphi}_{2}=\left[\begin{array}{c}
h_{2} \\
L_{f} h_{2}
\end{array}\right]=\left[\begin{array}{c}
\sigma_{2} \\
L_{f} \sigma_{2}
\end{array}\right], \boldsymbol{\varphi}_{3}=\left[\begin{array}{c}
h_{3} \\
L_{f} h_{3}
\end{array}\right]=\left[\begin{array}{c}
\sigma_{3} \\
L_{f} \sigma_{3}
\end{array}\right] \\
& \boldsymbol{\varphi}=\left[\begin{array}{c}
\phi_{1} \\
\phi_{2} \\
\phi_{3} \\
\phi_{4} \\
\phi_{5} \\
\phi_{6}
\end{array}\right]=\left[\begin{array}{c}
\sigma_{1} \\
L_{f} \sigma_{1} \\
\sigma_{2} \\
L_{f} \sigma_{2} \\
\sigma_{3} \\
L_{f} \sigma_{3}
\end{array}\right]=\left[\begin{array}{l}
\xi_{1}^{1} \\
\xi_{2}^{1} \\
\xi_{1}^{2} \\
\xi_{2}^{2} \\
\xi_{1}^{3} \\
\xi_{2}^{3}
\end{array}\right] \tag{5.7}
\end{align*}
$$

The Jacobian of the above vector function can be computed after some elementary row operations:

$$
\mathbf{Q}=\frac{\partial \varphi}{\partial \mathbf{x}}=\left[\begin{array}{cccc}
1 & 0 & 0 &  \tag{5.8}\\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & \\
\times & \times & \times & \\
\times & \times & \times & \mathbf{G}(\boldsymbol{\sigma}) \\
\times & \times & \times &
\end{array}\right]
$$

As expected, the above matrix is invertible which is the basic requirement of a diffeomorphism.

### 5.3 Construction of the IOFL Demanded Torque Generator

In this section the feedback control laws are generated using the outcome of the previous section. As it is stated in 3.3.2, the feedback linearization controller is implemented by using the following feedback:

$$
\begin{align*}
& \boldsymbol{\tau}_{a}=\mathbf{A}^{-1}(\mathbf{x})[-\mathbf{b}+\mathbf{v}] \\
& \mathbf{A}^{-1}(\mathbf{x})=-\mathbf{J G}^{-1}(\boldsymbol{\sigma})  \tag{5.9}\\
& \mathbf{b}=\left[\begin{array}{l}
L_{f}^{2} \sigma_{1} \\
L_{f}^{2} \sigma_{2} \\
L_{f}^{2} \sigma_{3}
\end{array}\right]=\left[\begin{array}{l}
\frac{\partial}{\partial \mathbf{x}}\left\{\frac{\partial \sigma_{1}}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x})\right\} \mathbf{f}(\mathbf{x}) \\
\left.\frac{\partial}{\partial \mathbf{x}}\left\{\frac{\partial \sigma_{2}}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x})\right\} \mathbf{f}(\mathbf{x})\right]=\frac{\partial}{\partial \mathbf{x}}\left\{\mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}\right\} \mathbf{f}(\mathbf{x}) \\
\frac{\partial}{\partial \mathbf{x}}\left\{\frac{\partial \sigma_{3}}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x})\right\} \mathbf{f}(\mathbf{x})
\end{array}\right]
\end{align*}
$$

The Jacobian matrices associated with $\mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}$ can be analytically computed by getting use of the derivations in [Crassidis (1996)]. First of all, the Jacobian in (5.9) should be expanded as shown in the following:

$$
\begin{equation*}
\frac{\partial}{\partial \mathbf{x}}\left\{\mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}\right\}=\left[\frac{\partial}{\partial \boldsymbol{\sigma}}\left\{\mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}\right\} \quad \frac{\partial}{\partial \boldsymbol{\omega}_{o b}^{b}}\left\{\mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}\right\}\right] \tag{5.10}
\end{equation*}
$$

The components of the above are computed as done in below:

$$
\begin{align*}
& \frac{\partial}{\partial \boldsymbol{\sigma}}\left\{\mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}\right\}=\frac{1}{2}\left[\boldsymbol{\sigma}\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T}-\left(\boldsymbol{\omega}_{o b}^{b}\right) \boldsymbol{\sigma}^{T}-\mathbf{S}\left(\boldsymbol{\omega}_{o b}^{b}\right)+\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T} \boldsymbol{\sigma} \mathbf{I}_{3 \times 3}\right] \\
& \frac{\partial}{\partial \boldsymbol{\omega}_{o b}^{b}}\left\{\mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}\right\}=\mathbf{G}(\boldsymbol{\sigma}) \tag{5.11}
\end{align*}
$$

So the vector b can be obtained as:

$$
\begin{align*}
& \mathbf{b}=\left[\frac{1}{2}\left[\boldsymbol{\sigma}\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T}-\left(\boldsymbol{\omega}_{o b}^{b}\right) \boldsymbol{\sigma}^{T}-\mathbf{S}\left(\boldsymbol{\omega}_{o b}^{b}\right)+\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T} \boldsymbol{\sigma} \mathbf{I}_{3 \times 3}\right] \mathbf{G}(\boldsymbol{\sigma}) \quad \mathbf{0}_{3 \times 3}\right] \mathbf{f}(\mathbf{x}) \\
& \mathbf{b}=\frac{1}{2}\left[\boldsymbol{\sigma}\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T}-\left(\boldsymbol{\omega}_{o b}^{b}\right) \boldsymbol{\sigma}^{T}-\mathbf{S}\left(\boldsymbol{\omega}_{o b}^{b}\right)+\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T} \boldsymbol{\sigma} \mathbf{I}_{3 \times 3}\right] \mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}+  \tag{5.12}\\
& +\mathbf{G}(\boldsymbol{\sigma})\left\{\mathbf{J}^{-1}\left[-\mathbf{S}\left(\boldsymbol{\omega}_{o b}^{b}-\mathbf{c}_{2} \omega_{o}\right)\left(\mathbf{I}\left[\boldsymbol{\omega}_{o b}^{b}-\mathbf{c}_{2} \omega_{o}\right]+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)\right]-\mathbf{S}\left(\boldsymbol{\omega}_{o b}^{b}\right) \mathbf{c}_{2} \omega_{o}\right\}
\end{align*}
$$

Then the whole linearizing control law is:

$$
\begin{align*}
& \boldsymbol{\tau}_{a}=\frac{1}{2} \mathbf{J G}^{-1}(\boldsymbol{\sigma})\left[\boldsymbol{\sigma}\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T}-\left(\boldsymbol{\omega}_{o b}^{b}\right) \boldsymbol{\sigma}^{T}-\mathbf{S}\left(\boldsymbol{\omega}_{o b}^{b}\right)+\left(\boldsymbol{\omega}_{o b}^{b}\right)^{T} \boldsymbol{\sigma} \mathbf{I}_{3 \times 3}\right] \mathbf{G}(\boldsymbol{\sigma}) \boldsymbol{\omega}_{o b}^{b}+  \tag{5.13}\\
& +\left[-\mathbf{S}\left(\boldsymbol{\omega}_{o b}^{b}-\mathbf{c}_{2} \omega_{o}\right)\left(\mathbf{I}\left[\boldsymbol{\omega}_{o b}^{b}-\mathbf{c}_{2} \omega_{o}\right]+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)\right]-\mathbf{J S}\left(\boldsymbol{\omega}_{o b}^{b}\right) \mathbf{c}_{2} \omega_{o}-\mathbf{J G} \mathbf{G}^{-1}(\boldsymbol{\sigma}) \mathbf{v}
\end{align*}
$$

The vector $\mathbf{v}$ is an artificial control input like:

$$
\mathbf{v}=\left[\begin{array}{l}
v_{1}  \tag{5.14}\\
v_{2} \\
v_{3}
\end{array}\right]
$$

The application of the linearizing feedback control law presented in (5.13) yields a controllable and observable linear system in terms of the normal variables in (5.7). Since, the relative degree of the satellite model for each output is same and equal to two, the product consists of three second order linear state equations (incorporating the normal variables) corresponding to each relevant plant (satellite) output. In fact, each linear system is a double integrator. Mathematical expressions for those systems is:

$$
\frac{d}{d t}\left[\begin{array}{l}
\xi_{1}^{i}  \tag{5.15}\\
\xi_{2}^{i}
\end{array}\right]=\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right]\left[\begin{array}{l}
\xi_{1}^{i} \\
\xi_{2}^{i}
\end{array}\right]+\left[\begin{array}{l}
0 \\
1
\end{array}\right] v_{i}
$$

For the sake of simplicity it is convenient to represent the state variables of the above system as $\xi_{1}^{i}=\alpha_{i} \& \xi_{2}^{i}=\beta_{i}$. By that way one can get rid of the confusing terms. The whole linearized system is:

$$
\frac{d}{d t}\left[\begin{array}{l}
\xi_{1}^{1}  \tag{5.16}\\
\xi_{2}^{1} \\
\xi_{1}^{2} \\
\xi_{2}^{2} \\
\xi_{1}^{3} \\
\xi_{2}^{3}
\end{array}\right]=\left[\begin{array}{llllll}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{l}
\xi_{1}^{1} \\
\xi_{2}^{1} \\
\xi_{1}^{2} \\
\xi_{2}^{2} \\
\xi_{1}^{3} \\
\xi_{2}^{3}
\end{array}\right]+\left[\begin{array}{lll}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{array}\right] \underbrace{\left[\begin{array}{l}
v_{1} \\
v_{2} \\
v_{3}
\end{array}\right]}_{\mathrm{v}}
$$

Now that the resultant linear systems are available, one can proceed with the generation of the final control law on the input $\mathbf{v}$. As it is stated before, the linear systems corresponding to each output $\left(\sigma_{i}\right)$ are identical so a design for one output is adequate. One can start the designs by defining a tracking error on the states:

$$
\begin{align*}
& e_{\alpha_{i}}=\alpha_{i}-\alpha_{i}^{d} \\
& e_{\beta_{i}}=\beta_{i}-\beta_{i}^{d} \tag{5.17}
\end{align*}
$$

In the above, the symbols $\alpha_{i}, \alpha_{i}^{d}, \beta_{i}$ and $\beta_{i}^{d}$ are:

$$
\begin{align*}
& \xi_{1}^{i}=\alpha_{i}=\sigma_{i} \& \xi_{2}^{i}=\beta_{i}=\dot{\sigma}_{i} \\
& \alpha_{i}^{d}=\sigma_{i}^{d}  \tag{5.18}\\
& \beta_{i}^{d}=\dot{\sigma}_{i}^{d}
\end{align*}
$$

In this research, it is assumed that the desired input is constant (in other words it is adequately smooth such that the first and second derivatives can be assumed as zero). So the elements of $(5.18)$ can be rewritten as:

$$
\begin{align*}
& \xi_{1}^{i}=\alpha_{i}=\sigma_{i} \& \xi_{2}^{i}=\beta_{i}=\dot{\sigma}_{i} \\
& \alpha_{i}^{d}=\sigma_{i}^{d} \\
& \beta_{i}^{d}=\dot{\sigma}_{i}^{d}=0  \tag{5.19}\\
& \dot{\alpha}_{i}^{d}=\dot{\sigma}_{i}^{d}=0 \\
& e_{\beta_{i}}=\beta_{i}
\end{align*}
$$

If one differentiates (5.17) the result is obtained as:

$$
\begin{align*}
& \dot{e}_{\alpha_{i}}=\alpha_{i}-\dot{\alpha}_{i}^{d}=\dot{\alpha}_{i} \\
& \dot{e}_{\beta_{i}}=\beta_{i}-\beta_{i}^{d}=\dot{\beta}_{i} \tag{5.20}
\end{align*}
$$

Knowing that (5.15) can be rewritten as shown below:

$$
\frac{d}{d t}\left[\begin{array}{l}
\alpha_{i}  \tag{5.21}\\
\beta_{i}
\end{array}\right]=\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right]\left[\begin{array}{l}
\alpha_{i} \\
\beta_{i}
\end{array}\right]+\left[\begin{array}{l}
0 \\
1
\end{array}\right] v_{i}
$$

Since the above plant contains an integrator there is no need to include an artificial integrator into the linear controller. So one can propose a controller of the following form:

$$
\begin{align*}
& v_{i}=-k_{i}^{1}\left(\alpha_{i}-\sigma_{i}^{d}\right)-k_{i}^{2}\left(\dot{\sigma}_{i}\right) \\
& v_{i}=-k_{i}^{1}\left(\alpha_{i}-\alpha_{i}^{d}\right)-k_{i}^{2}\left(\beta_{i}\right) \\
& v_{i}=-k_{i}^{1} \alpha_{i}-k_{i}^{2} \beta_{i}+k_{i}^{1} \alpha_{i}^{d}  \tag{5.22}\\
& v_{i}=-\left[\begin{array}{ll}
k_{i}^{1} & k_{i}^{2}
\end{array}\right]\left[\begin{array}{c}
\alpha_{i} \\
\beta_{i}
\end{array}\right]+k_{i}^{1} \alpha_{i}^{d}
\end{align*}
$$

The above control law converts (5.21) into the following:

$$
\frac{d}{d t}\left[\begin{array}{c}
\alpha_{i}  \tag{5.23}\\
\beta_{i}
\end{array}\right]=\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right]\left[\begin{array}{c}
\alpha_{i} \\
\beta_{i}
\end{array}\right]-\left[\begin{array}{l}
0 \\
1
\end{array}\right]\left[\begin{array}{ll}
k_{i}^{1} & k_{i}^{2}
\end{array}\right]\left[\begin{array}{c}
\alpha_{i} \\
\beta_{i}
\end{array}\right]+\left[\begin{array}{l}
0 \\
1
\end{array}\right] k_{i}^{1} \alpha_{i}^{d}
$$

and,

$$
\frac{d}{d t}\left[\begin{array}{l}
\alpha_{i}  \tag{5.24}\\
\beta_{i}
\end{array}\right]=\left[\begin{array}{cc}
0 & 1 \\
-k_{i}^{1} & -k_{i}^{2}
\end{array}\right]\left[\begin{array}{c}
\alpha_{i} \\
\beta_{i}
\end{array}\right]+\left[\begin{array}{l}
0 \\
1
\end{array}\right] k_{i}^{1} \alpha_{i}^{d}
$$

Considering (5.19) and (5.20) the error state equation can be rewritten as:

$$
\begin{align*}
& \frac{d}{d t}\left[\begin{array}{l}
e_{\alpha_{i}} \\
e_{\beta_{i}}
\end{array}\right]=\left[\begin{array}{cc}
0 & 1 \\
-k_{i}^{1} & -k_{i}^{2}
\end{array}\right]\left[\begin{array}{c}
\alpha_{i} \\
\beta_{i}
\end{array}\right]+\left[\begin{array}{c}
0 \\
k_{i}^{1}
\end{array}\right] \alpha_{i}^{d} \\
& \frac{d}{d t}\left[\begin{array}{l}
e_{\alpha_{i}} \\
e_{\beta_{i}}
\end{array}\right]=\left[\begin{array}{cc}
0 & 1 \\
-k_{i}^{1} & -k_{i}^{2}
\end{array}\right]\left[\begin{array}{c}
e_{\alpha_{i}}+\alpha_{i}^{d} \\
e_{\beta_{i}}
\end{array}\right]+\left[\begin{array}{c}
0 \\
k_{i}^{1}
\end{array}\right] \alpha_{i}^{d} \tag{5.25}
\end{align*}
$$

So as a result the error dynamics is obtained as:

$$
\begin{align*}
& \frac{d}{d t}\left[\begin{array}{l}
e_{\alpha_{i}} \\
e_{\beta_{i}}
\end{array}\right]=\left[\begin{array}{cc}
0 & 1 \\
-k_{i}^{1} & -k_{i}^{2}
\end{array}\right]\left[\begin{array}{l}
e_{\alpha_{i}} \\
e_{\beta_{i}}
\end{array}\right] \\
& e_{\alpha_{i}}=\sigma_{i}-\sigma_{i}^{d}  \tag{5.26}\\
& e_{\beta_{i}}=\dot{\sigma}_{i}
\end{align*}
$$

The first element of the state vector $e_{\alpha_{i}}$ represents the attitude tracking error and the second one $e_{\beta_{i}}$ is the rate of change of the attitude (since the derivative of the desired attitude is assumed as zero). For a practical application the above proposal is adequate. The parameters $k_{i}^{1}$ and $k_{i}^{2}$ can be tuned by any desired linear control techniques such as linear quadratic regulator (LQR).

### 5.4 The Reaction Wheel Velocity Command Generator

The reaction wheel velocity command generation is exactly the same as that of the back - stepping case presented in Section 4.3.2.

### 5.5 The Wheel Speed Control System

The reaction wheel speed control system uses the following differential equations for modeling the reaction wheel and actuator dynamics:

$$
\begin{align*}
& \dot{\boldsymbol{\omega}}_{s}=-\mathbf{J}^{-1}\left[-\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\boldsymbol{\tau}_{e}\right]+\left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right) \boldsymbol{\tau}_{a}  \tag{5.27}\\
& \dot{\boldsymbol{\tau}}_{a}=-\mathbf{K}_{T} \mathbf{R}_{L} \mathbf{K}_{T}^{-1} \boldsymbol{\tau}_{a}+\mathbf{K}_{T} \mathbf{K}_{E} \boldsymbol{\omega}_{s}+\mathbf{K}_{T} \mathbf{L} \mathbf{U}_{a}
\end{align*}
$$

The output for the above system is the reaction wheel velocity $\left(\boldsymbol{\omega}_{s}\right)$. The first step in this section is the computation of the vector relative degree for each component of the output vector $\boldsymbol{\omega}_{s}$. As it can be easily deduced from (5.27) that when one takes the derivative of $\boldsymbol{\omega}_{s}$ twice the input voltage vector $\mathbf{U}_{a}$ is obtained first time. So the vector relative degree for each of the outputs are equal to two. The total relative degree of the plant is equal to six. The control law generation procedure is almost similar to that of the demanded torque generation law.

The necessary procedure starts with the reformation of the plant state equations as nonlinear affine form:

$$
\begin{align*}
& \mathbf{x}=\left[\begin{array}{ll}
\boldsymbol{\omega}_{s}^{T} & \boldsymbol{\tau}_{a}^{T}
\end{array}\right]^{T} \\
& \mathbf{f}(\mathbf{x})=\left[\begin{array}{c}
\mathbf{J}^{-1} \mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right) \boldsymbol{\tau}_{a} \\
\\
-\mathbf{K}_{T} \mathbf{R}_{L} \mathbf{K}_{T}^{-1} \boldsymbol{\tau}_{a}+\mathbf{K}_{T} \mathbf{K}_{E} \boldsymbol{\omega}_{s}
\end{array}\right] \\
& \mathbf{g}(\mathbf{x})=\left[\begin{array}{c}
\mathbf{0}_{3 \times 3} \\
\mathbf{K}_{T} \mathbf{L}
\end{array}\right], \mathbf{p}(\mathbf{x})=\left[\begin{array}{c}
-\mathbf{J}^{-1} \\
\mathbf{0}_{3 \times 3}
\end{array}\right]  \tag{5.28}\\
& \mathbf{u}=\mathbf{U}_{a}, \mathbf{w}=\boldsymbol{\tau}_{e} \\
& \dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}) \mathbf{x}+\mathbf{g}(\mathbf{x}) \mathbf{u}+\mathbf{p}(\mathbf{x}) \mathbf{w}
\end{align*}
$$

The decoupling matrix $\mathbf{A}(\mathbf{x})$ for this case can be computed as shown below:

$$
\begin{align*}
& \frac{\partial \boldsymbol{\omega}_{s}}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x})=\left[\begin{array}{llllll}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{c}
\mathbf{J}^{-1} \mathbf{S}\left(\begin{array}{c}
\left.\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right) \boldsymbol{\tau}_{a} \\
-\mathbf{K}_{T} \mathbf{R}_{L} \mathbf{K}_{T}^{-1} \boldsymbol{\tau}_{a}+\mathbf{K}_{T} \mathbf{K}_{E} \boldsymbol{\omega}_{s}
\end{array}\right] \\
=\mathbf{J}^{-1} \mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right) \boldsymbol{\tau}_{a} \\
\frac{\partial}{\partial \mathbf{x}}\left\{\frac{\partial \boldsymbol{\omega}_{s}}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x})\right\} \mathbf{g}(\mathbf{x})=\frac{\partial}{\partial \mathbf{x}}\left[\mathbf{J}^{-1} \mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right) \boldsymbol{\tau}_{a}\right]\left[\begin{array}{c}
\mathbf{0}_{3 \times 3} \\
\mathbf{K}_{T} \mathbf{L}
\end{array}\right] \\
=\left[\begin{array}{l}
\times \times \times \\
\times \times \times
\end{array}\right]\left[\begin{array}{c}
\mathbf{0}_{3 \times 3} \\
\times \times \mathbf{K}_{T} \mathbf{L}
\end{array}\right]=\left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right) \mathbf{K}_{T} \mathbf{L}=\mathbf{A}(\mathbf{x}) \\
\times \times \times
\end{array}\right.
\end{align*}
$$

The second step is the computation of the $\mathbf{b}(\mathbf{x})$ vector which is shown below:

$$
\begin{align*}
& \mathbf{b}(\mathbf{x})=\frac{\partial}{\partial \mathbf{x}}\left\{\frac{\partial \boldsymbol{\omega}_{s}}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x})\right\} \mathbf{f}(\mathbf{x})= \\
& \frac{\partial}{\partial \mathbf{x}}\left[\mathbf{J}^{-1} \mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right) \boldsymbol{\tau}_{a}\right]\left[\begin{array}{c}
\mathbf{J}^{-1} \mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right) \boldsymbol{\tau}_{a} \\
-\mathbf{K}_{T} \mathbf{R}_{L} \mathbf{K}_{T}^{-1} \boldsymbol{\tau}_{a}+\mathbf{K}_{T} \mathbf{K}_{E} \boldsymbol{\omega}_{s}
\end{array}\right] \\
& =\left[\begin{array}{ll}
\mathbf{J}^{-1} \mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right) \mathbf{I}_{s} & \left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right)
\end{array}\right]\left[\begin{array}{c}
\mathbf{J}^{-1} \mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right) \boldsymbol{\tau}_{a} \\
-\mathbf{K}_{T} \mathbf{R}_{L} \mathbf{K}_{T}^{-1} \boldsymbol{\tau}_{a}+\mathbf{K}_{T} \mathbf{K}_{E} \boldsymbol{\omega}_{s}
\end{array}\right]  \tag{5.30}\\
& =\mathbf{J}^{-1} \mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right) \mathbf{I}_{s} \mathbf{J}^{-1} \mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)+\mathbf{J}^{-1} \mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right) \mathbf{I}_{s}\left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right) \boldsymbol{\tau}_{a}+ \\
& +\left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right)\left\{-\mathbf{K}_{T} \mathbf{R}_{L} \mathbf{K}_{T}^{-1} \boldsymbol{\tau}_{a}+\mathbf{K}_{T} \mathbf{K}_{E} \boldsymbol{\omega}_{s}\right\}
\end{align*}
$$

As a result the linearizing feedback control law on $\mathbf{U}_{a}$ is obtained as:

$$
\begin{align*}
& \mathbf{U}_{a}=\mathbf{L}^{-1} \mathbf{K}_{T}^{-1}\left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right)^{-1}\left\{-\mathbf{J}^{-1} \mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right) \mathbf{I}_{s} \mathbf{J}^{-1} \mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right)\left(\mathbf{I} \boldsymbol{\omega}_{i b}^{b}+\mathbf{I}_{s} \boldsymbol{\omega}_{s}\right)-\mathbf{J}^{-1} \mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right) \mathbf{I}_{s}\left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right) \boldsymbol{\tau}_{a}\right\}  \tag{5.31}\\
& +\left\{\mathbf{L}^{-1} \mathbf{R}_{L} \mathbf{K}_{T}^{-1} \boldsymbol{\tau}_{a}-\mathbf{L}^{-1} \mathbf{K}_{E} \boldsymbol{\omega}_{s}\right\}+\mathbf{L}^{-1} \mathbf{K}_{T}^{-1}\left(\mathbf{I}_{s}^{-1}+\mathbf{J}^{-1}\right)^{-1} \mathbf{v}
\end{align*}
$$

The final linear system is a double integrator as expected and shown as:

$$
\frac{d}{d t}\left[\begin{array}{c}
\omega_{s}^{i}  \tag{5.32}\\
\dot{\omega}_{s}^{i}
\end{array}\right]=\left[\begin{array}{cc}
0 & 1 \\
0 & 0
\end{array}\right]\left[\begin{array}{c}
\omega_{s}^{i} \\
\dot{\omega}_{s}^{i}
\end{array}\right]+\left[\begin{array}{l}
0 \\
1
\end{array}\right] v_{i}
$$

 an external control input whose law is designed by the linear control part of this procedure. The linear controller design will exactly be the same as that of the demanded torque generator if one makes the substitution $\alpha_{i}=\omega_{s}^{i}$ and $\beta_{i}=\dot{\omega}_{s}^{j}$. The control input $v_{i}$ is obtained as:

$$
\begin{equation*}
v_{i}=-k_{i}^{1}\left(\omega_{s}^{i}-\omega_{s r}^{i}\right)-k_{i}^{2}\left(\dot{\omega}_{s}^{i}\right) \tag{5.33}
\end{equation*}
$$

where $\omega_{s r}^{i}$ is the $i^{\text {th }}$ component of the desired reaction wheel velocity command that is generated by (4.28). The controller gains $k_{i}^{1}$ and $k_{i}^{2}$ are computed by the process described in Section 5.3. The variable $\dot{\omega}_{s}$ is produced directly from the reaction wheel velocity dynamics (excluding the torque $\boldsymbol{\tau}_{e}$ ) in (5.27). Following the derivation presented in the Section 5.3 with the designation of $\alpha_{i}=\omega_{s}^{i}$ and $\beta_{i}=\dot{\omega}_{s}^{s}$ and using the assumptions in (5.19) the closed loop actuator dynamics is obtained as shown below:

$$
\begin{align*}
& \frac{d}{d t}\left[\begin{array}{l}
e_{\alpha_{i}} \\
e_{\beta_{1}}
\end{array}\right]=\left[\begin{array}{cc}
0 & 1 \\
-k_{i}^{1} & -k_{i}^{2}
\end{array}\right]\left[\begin{array}{l}
e_{\alpha_{i}} \\
e_{\beta_{1}}
\end{array}\right] \\
& e_{\alpha_{i}}=\omega_{s}^{i}-\omega_{s}^{i}  \tag{5.34}\\
& e_{\beta_{i}}=\dot{\omega}_{s}^{i}
\end{align*}
$$

The first element of the state vector $e_{\alpha_{i}}$ represents the reaction wheel velocity tracking error and the second one $e_{\beta_{i}}$ is the rate of change of the reaction wheel velocity (since the derivative of the velocity command is assumed as zero). For a practical application the above proposal is adequate. The parameters $k_{i}^{1}$ and $k_{i}^{2}$ can be tuned by any desired linear control techniques such as linear quadratic regulator (LQR).

### 5.6 The Linear Quadratic Regulator

The linear quadratic regulator (LQR) is a simple method of optimization based controller design technique where a quadratic performance index is minimized by the algorithm. General form of the performance index is:

$$
\begin{equation*}
J=\int_{0}^{\infty}\left(\mathbf{x}^{T} \mathbf{Q} \mathbf{x}+\mathbf{u}^{T} \mathbf{R u}\right) d t \tag{5.35}
\end{equation*}
$$

for a linear system of the form:

$$
\begin{align*}
& \dot{\mathbf{x}}=\mathbf{A x}+\mathbf{B u} \\
& \mathbf{x} \in \mathbb{R}^{n}  \tag{5.36}\\
& \mathbf{u} \in \mathbb{R}^{m}
\end{align*}
$$

where the feedback input is $\mathbf{u}=-\mathbf{K x}$. The control gain $\mathbf{K} \in \mathbb{R}^{m \times n}$ is $\mathbf{K}=\mathbf{R}^{-1} \mathbf{B}^{T} \mathbf{P}$ and $\mathbf{P} \in \mathbb{R}^{n \times n}$ is a symmetric and positive definite matrix the solution of which is found from the Riccati equation presented below:

$$
\begin{equation*}
\mathbf{A}^{T} \mathbf{P}+\mathbf{P} \mathbf{A}-\mathbf{P} \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^{T} \mathbf{P}=-\mathbf{Q} \tag{5.37}
\end{equation*}
$$

In all above $\mathbf{Q} \in \mathbb{R}^{n \times n}$ and $\mathbf{R} \in \mathbb{R}^{m \times m}$ are also symmetric and positive definite matrices. The closed loop system is obtained as $\dot{\mathbf{x}}=(\mathbf{A}-\mathbf{B K}) \mathbf{x}$. For the attitude control problem presented in this chapter, the elements of the controller design process are:

$$
\begin{align*}
& \dot{\mathbf{x}}=\mathbf{A} \mathbf{x}+\mathbf{B u} \\
& \mathbf{x}=\left[\begin{array}{l}
\alpha_{i} \\
\beta_{i}
\end{array}\right] \\
& \mathbf{u}=u=v_{i}  \tag{5.38}\\
& \mathbf{A}=\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right], \mathbf{B}=\left[\begin{array}{l}
0 \\
1
\end{array}\right] \\
& \mathbf{Q}=\left[\begin{array}{ll}
q_{a} & q_{s} \\
q_{s} & q_{b}
\end{array}\right], \mathbf{R}=R
\end{align*}
$$

The performance of the IOFL based controller is investigated through the simulations presented in the next chapter.

## CHAPTER 6 <br> NUMERICAL APPLICATIONS AND SIMULATIONS

### 6.1 Motivation

In this chapter, the theoretical and applicational information presented in Chapter 4 and Chapter 5 is simulated for a realistic satellite application. The selected model belongs to BILSAT - I satellite designed by cooperation between Surrey Satellite Technology Limited (University of Surrey - UK) and Space Technologies Research Institute of Turkish Scientific and Technological Research Council (TUBITAK). This satellite was launched from Russia in 2003 and has been involved in experimental observation processes performed by the institute.

### 6.2 The Parameters of BILSAT - I

The BILSAT - I is a microsatellite whose photograph is shown in Figure 6-1. Its parameters are presented in Table 6-1.


Figure 6-1 A photograph of BILSAT satellite before launch

| Moment of Inertia: $\mathbf{I}^{n o m}=\left[\begin{array}{ccc} I_{x x} & -I_{x y} & -I_{x z} \\ -I_{x y} & I_{y y} & -I_{y z} \\ -I_{x z} & -I_{y z} & I_{z z} \end{array}\right]$ | $\begin{aligned} & I_{x x}=9.8194 \mathrm{~kg} \cdot \mathrm{~m}^{2}, I_{y y}=9.7030 \mathrm{~kg} \cdot \mathrm{~m}^{2}, \\ & I_{z z}=9.7309 \mathrm{~kg} \cdot \mathrm{~m}^{2} \\ & I_{x y}=0.0721 \mathrm{~kg} \cdot \mathrm{~m}^{2}, I_{x z}=0.2893 \mathrm{~kg} \cdot \mathrm{~m}^{2}, \\ & I_{y z}=0.1011 \mathrm{~kg} \cdot \mathrm{~m}^{2} \end{aligned}$ |
| :---: | :---: |
| Mass of the Satellite: | $m=120 \mathrm{~kg}$ |
| Satellite Orbital Velocity: | $\omega_{o}=0.0010831 \mathrm{rad} / \mathrm{sec}$ |
| The Full Orbit Time: | $t_{o}=5801.2 \mathrm{sec}$ |
| Moment of Inertia of the Wheels: $\mathbf{I}_{s}=\left[\begin{array}{ccc} I_{s_{1}} & 0 & 0 \\ 0 & I_{s_{2}} & 0 \\ 0 & 0 & I_{s_{3}} \end{array}\right]$ | $I_{s_{1}}=I_{s_{2}}=I_{s_{3}}=0.008 \mathrm{~kg} \cdot \mathrm{~m}^{2}$ |
| Total Inertia Matrix: | $\mathbf{J}=\left[\begin{array}{lll}9.8114 & -0.0721 & -0.2893 \\ -0.0721 & 9.695 & -0.1011 \\ -0.2892 & -0.1011 & 9.7229\end{array}\right]$ |
| Maximum Torque available from the Wheels: | $\left\|\tau_{\text {max }}\right\|=0.02 \mathrm{~N} \cdot \mathrm{~m}$ |
| Maximum Wheel Speed: | $\omega_{s}^{\max }=5000 \mathrm{rpm}$ |
| Motor Armature Resistance: | $0.696 \Omega$ |
| Motor Armature Inductance: | $528.8 \mu \mathrm{H}$ |
| Motor Torque Constant: | $0.038 \mathrm{Nm} / \mathrm{A}$ |
| Motor Back - EMF Constant: | $0.038 \mathrm{~V} / \mathrm{rad} / \mathrm{s}$ |
| Motor Friction Constant: | $1.604 \cdot 10^{-05} \mathrm{Nm} / \mathrm{rad} / \mathrm{s}$ |

Table 6-1 Parameters of BILSAT

### 6.3 Implementation of Back - stepping

The implementation of back - stepping requires the selection of proper control gains (i.e. the four matrix gains $\mathbf{K}_{1}, \mathbf{K}_{2}, \mathbf{K}_{3}, \mathbf{K}_{4}$ ). A larger gain selection may lead to larger torque demand which is not desirable. Another restriction on control gain selection comes from the input to state stability condition derived in (4.68) and (4.69) as the coefficient of $\|\mathbf{z}\|^{2}$ should be negative to obtain input to state stability in the dissipative sense. In this research, the control gain matrices are in diagonal form as shown in the following:
$\mathbf{K}_{1}=\left[\begin{array}{ccc}K_{11} & 0 & 0 \\ 0 & K_{12} & 0 \\ 0 & 0 & K_{13}\end{array}\right], \mathbf{K}_{2}=\left[\begin{array}{ccc}K_{21} & 0 & 0 \\ 0 & K_{22} & 0 \\ 0 & 0 & K_{23}\end{array}\right], \mathbf{K}_{3}=\left[\begin{array}{ccc}K_{31} & 0 & 0 \\ 0 & K_{32} & 0 \\ 0 & 0 & K_{33}\end{array}\right], \mathbf{K}_{4}=\left[\begin{array}{ccc}K_{41} & 0 & 0 \\ 0 & K_{42} & 0 \\ 0 & 0 & K_{43}\end{array}\right]$
\&
$K_{11}=K_{12}=K_{13}=k_{1}$
$K_{21}=K_{22}=K_{23}=k_{2}$
$K_{31}=K_{32}=K_{33}=k_{3}$
$K_{41}=K_{42}=K_{43}=k_{4}$

In order to complete the design procedure one should select the control gains according to the input to state stability restrictions presented in Section 4.6. Two such control gains are presented in the table given below:

Table 6-2 The control gain sets for back - stepping simulations

| $k_{1}$ | $k_{2}$ | $k_{3}$ | $k_{4}$ | $M_{1}, M_{2}$ |
| :---: | :---: | :---: | :---: | :---: |
| 20 | 2.5 | 2.5 | 1.5 | $1.25 / 1.5$ |
| 40 | 3.6 | 3.6 | 2.5 | $2.5 / 2.5$ |
| 10 | 1.7 | 1.7 | 0.7 | $0.625 / 0.7$ |
| 5 | 11 | 1.4 | 0.4 | $0.3125 / 0.4$ |
| 160 | 21 | 2.5 | 10 | $10 / 10$ |
| 320 | 1.5 | $20 / 1.5$ |  |  |

Table 6-3 Control gains for back - stepping attitude control simulation

In this chapter, the simulation results concerning the second case (which is shaded in the table above) is presented in the proceeding sections.

### 6.4 Simulation Outline

The simulation outline describes the initial and final conditions of the nonlinear satellite attitude simulation. The simulation is composed of single - run and a multi run phases. In a single - run simulation a target attitude is defined and the satellite is forced to track towards the given target. There are no parametric uncertainties existent in the simulation environment. The purpose is solely to verify the control laws derived in this work. In contrast, the multi - run simulations involve parametric uncertainties in the satellite inertias (including the reaction wheel inertias) and motor electrical parameters. Simulations are repeated several times and the results are superimposed on the same figures in order to analyze the performance deviations of the attitude controller during a realistic operation. In this research, a number of 100 consecutive simulations are planned for the multi - run phase.

The natural frequency of the smoothing system in (4.56) is selected as $\omega_{n}=0.02 \mathrm{rad} / \mathrm{sec}$ in a critically damped condition $(\zeta=1)$. By that way the settling time of the desired trajectory is about 380 seconds (about $7 \%$ of the full orbit time).

In the multi - run simulations, it is mentioned that there are parametric uncertainties in the satellite and actuator models which have to be taken into account. The unknown portion in inertia can be modeled by using a uniformly distributed random variable $\delta_{i i}$ in the interval $\{-0.10,0.10\}$ (Doruk, [2005]). The uncertainty representation is:

$$
\begin{equation*}
\mathbf{I}^{\text {actual }}=\mathbf{I}^{\text {nom }}(1+\Delta \mathbf{I}) \tag{6.2}
\end{equation*}
$$

where,

$$
\Delta \mathbf{I}=\left[\begin{array}{ccc}
\delta I_{x x} & -\delta I_{x y} & -\delta I_{x z}  \tag{6.3}\\
-\delta I_{x y} & \delta I_{y y} & -\delta I_{y z} \\
-\delta I_{x z} & -\delta I_{y z} & \delta I_{z z}
\end{array}\right]
$$

The uncertainties in the motor parameters (armature resistance and inductance, torque and back emf constants) are modeled in a similar fashion. For those, same level of uncertainty is assumed and is existent on the nominal value of the parameter in the following form:

$$
\begin{equation*}
P_{m}=P_{m}^{n o m}(1+\delta P) \tag{6.4}
\end{equation*}
$$

where $P_{m}$ and $P_{m}^{n o m}$ are the actual and nominal value of the interested motor parameter. $\delta P$ represents the unknown percentage of the uncertainty (accepted in multiplicative form).

The initial and final conditions of the example simulations are:
The Initial Conditions: $\phi_{0}=0^{\circ}, \theta_{0}=0^{\circ}, \psi_{0}=0^{\circ}$
Target (final) orientation of the satellites: $\phi_{\text {ref }}=20^{\circ}, \theta_{\text {ref }}=40^{\circ}, \psi_{\text {ref }}=60^{\circ}$

### 6.4.1 Results of the Single Run Simulations

In this section the results of a single run simulation for the second case in Table $6-3$ is presented starting from Figure 6-2. The results of the other cases are not presented since thay have produced almost similar results. The dynamic model in (2.27) and (2.35) are simulated in closed loop. The controllers used in this simulation are given in (4.58).




Figure 6-2 Desired attitude trajectory in degrees (Single Run)


Figure 6-3 Desired trajectory represented in MRP (Single Run)


Figure 6-4 Attitude tracking error represented in MRP (Single Run)


Figure 6-5 Reaction wheel torque requirement (Single Run)


Figure 6-6 Angular velocity of the satellite body (Single Run)


Figure 6-7 Reaction wheel velocity command (Single Run)


Figure 6-8 Reaction wheel velocity error (Single Run)


Figure 6-9 Reaction wheel velocity represented in revolutions per minute (Single Run)

The single - run simulation results shows an applicable characteristic in uncertainty free environment. The important variables in this case are the attitude errors, torque requirements and reaction wheel velocities. According to the results, the torque requirement and wheel velocitiy are far below the maximum value that can be exerted by the reaction wheel motor. The attitude tracking errors are converging to zero as expected. Shortly one can say that the simulation results successfully verified that the controller is working as expected.

Nevertheless, those results may only be useful for controller verification. In a real case, one should take care of the parametrical uncertainties existent on both the satellite body and the motor armature parameters.

### 6.4.2 Results of Multi - Run Simulations

In this section the results of the multi - run simulation corresponding to the second case of Table 6-3 are presented. Since the uncertainties can occur in several configurations (there are more than 15 parameters in the satellite model), the simulation should be repeated several times to obtain reliable information of robustness against parametric uncertainties. In each simulation, a different level of
deviation is added on the nominal value of the considered parameter (by using the uncertainty models in (6.2) and (6.3).

The desired attitude trajectory in terms of MRP and Euler angles are not drawn again since they have already been plotted in the single - run simulations section.


Figure 6-10 Attitude error variation in MRP (Multi Run)


Figure 6-11 Attitude error variation in Euler angles (Multi Run)


Figure 6-12 Torque demand variation (Multi Run)


Figure 6-13 Reaction wheel velocity demand variation (Multi Run)


Figure 6-14 Reaction wheel velocity error (Multi Run)




Figure 6-15 Reaction wheel velocity in revolutions per minute (Multi Run)

According to the multi - run simulation results the uncertainties existent in the satellite parameters and the actuator does not affect the stability of the proposed back - stepping controller. The constant disturbance torque does not have any destabilizing effects on the controller's performance.

### 6.5 Simulation of Feedback Linearization

In this section, a numerical application on attitude control by input output feedback linearization (IOFL) is presented. The dynamic model in (2.27) and (2.35) are simulated in closed loop. The controllers used in this simulation are given in (5.13), (5.22), (5.31) and (5.33). The LQR for this case are presented as shown below:

For the demanded torque generator $q_{a, b}$ is replaced by $q_{1,2}$ and $q_{s}=0$.
For the demanded torque generator $q_{a, b}$ is replaced by $q_{3,4}$ and $q_{s}=0$.

The $q_{i}{ }^{\prime} s$ used here have no relationship with the quaternion representation.
In the first trial the input weighting coefficient $R$ is taken as unity. The coefficients of Hata! Başvuru kaynağı bulunamadı. assumes that $R$ is always equal to unity. However, as it is also indicated in the table given below, some of the cases become unstable during the simulations. In addition to that in multi - run simulation with the same conditions as that of the back - stepping case none of the cases can handle the uncertainties in the actuator motor parameters. Because of that, some additional trials have been performed with different $R$ values.

Table 6-4 Gains for Feedback Linearization Case (with $R=1$ )

| CASE | $q_{1}$ | $q_{2}$ | $q_{3}$ | $q_{4}$ | RESULT |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 0.001 | $1 \cdot 10^{-8}$ | 0.1 | $1 \cdot 10^{-8}$ | OK |
| 2 | 0.1 | $1 \cdot 10^{-8}$ | 0.1 | $1 \cdot 10^{-8}$ | UNSTABLE |
| 3 | 1 | $1 \cdot 10^{-8}$ | 0.1 | $1 \cdot 10^{-8}$ | UNSTABLE |
| 4 | 0.001 | $1 \cdot 10^{-8}$ | 1 | $1 \cdot 10^{-8}$ | OK |
| 5 | 0.001 | $1 \cdot 10^{-8}$ | 5 | $1 \cdot 10^{-8}$ | OK |
| 6 | 0.1 | $1 \cdot 10^{-8}$ | 5 | $1 \cdot 10^{-8}$ | OK |
| 7 | 1 | $1 \cdot 10^{-8}$ | 5 | $1 \cdot 10^{-8}$ | UNSTABLE |
| 8 | 1 | $1 \cdot 10^{-8}$ | 10 | $1 \cdot 10^{-8}$ | UNSTABLE |
| 9 | 1 | $1 \cdot 10^{-8}$ | 10 | $1 \cdot 10^{-8}$ | OK |

In some of the trials the motor uncertainties can be handled, one such case uses the following controller gains:

$$
\begin{align*}
& q_{1}=100 \\
& q_{2}=10^{-8}  \tag{6.5}\\
& q_{s}=0 \\
& R_{1}=10^{8}
\end{align*}
$$

for the demanded torque generator whereas for the reaction wheel speed controller:

$$
\begin{align*}
q_{3} & =10^{5} \\
q_{4} & =10^{-8}  \tag{6.6}\\
q_{s} & =0 \\
R_{2} & =1
\end{align*}
$$

In the next section the results of the simulations concerning the controller configurations given above are given. The resultant control gains for (6.5) and (6.6) are $K_{1}=\left[\begin{array}{ll}0.001 & 0.044721\end{array}\right]$ and $K_{2}=\left[\begin{array}{ll}316.23 & 25.149\end{array}\right]$ respectively.

### 6.5.1 Single Run Simulations for IOFL based Attitude Control

In this section the single run simulation results concerning the attitude controller based on the input output feedback linearization approach. The desired attitude trajectories are same as that of the back - stepping based simulation.


Figure 6-16 Attitude error in MRP (Single Run)




Figure 6-17 Attitude error in Euler angles (Single Run)


Figure 6-18 Torque demand (Single Run)


Figure 6-19 Reaction wheel velocity demand (Single Run)


Figure 6-20 Reaction wheel velocity error (Single Run)


Figure 6-21 Reaction wheel velocity in revolutions per minute (Single Run)

The results showed that free of parametric uncertainties and with the properly selected quadratic performance indices (refer to Hata! Başvuru kaynağı bulunamadı.) the IOFL based approach can provide an applicable configuration however one should perform a more realistic simulation including parametric uncertainties in satellite inertia and motor parameters. This is performed in the next section.

### 6.5.2 Multi Run Simulations for IOFL based Attitude Control

In this section the results of the multi - run simulations for IOFL control case corresponding to the conditions given in (6.5) and (6.6).




Figure 6-22 Attitude error in MRP (Multi Run)


Figure 6-23 Attitude error in Euler angles (Multi Run)


Figure 6-24 Torque demand (Multi Run)


Figure 6-25 Reaction wheel velocity demand (Multi Run)




Figure 6-26 Reaction wheel velocity error (Multi Run)




Figure 6-27 Reaction wheel velocity in revolutions per minute (Multi Run)

The multi - run simulations show that the IOFL controllers are robust against the satellite body and reaction wheel inertia uncertainties provided that the quadratic performance index coefficients $q_{1}, q_{3}, R_{1}$ and $R_{2}$ are selected properly. The motor uncertainties can also be handled by the last configuration. However, this configuration requires comparably larger gains for the reaction wheel speed control system. In the following two figures the results of the last simulation repeated with an elongated time span (to 320000 seconds) are given. The purpose of this is to show whether the reaction wheel velocity and its tracking error converges or not.


Figure 6-28 Reaction wheel velocity demand (Multi Run - time span 320000 seconds)


Figure 6-29 Reaction wheel velocity error (Multi Run - time span 320000 seconds)

As one can understand from the last presentation the reaction wheel velocity and its tracking error converges to values around zero after a very long time about 90000 seconds (about 25 hours). From the normal simulation curves it is undertstood that the controllers are operating properly but the reaction wheel operates longer than that of the back - stepping based controller. Because of that, the energy demands of the proposed IOFL based controller may be larger than that of the back - stepping based controller.

### 6.6 Long Range Simulation

The purpose of long range simulations is to demonstrate the behavior of the attitude controllers when the target attitude is leading to a rotation where the Modified Rodriguez Parameters are approaching to the singular ranges. The MRP are singular at $\pm 360^{\circ}$ of rotations which means that the MRP have infinite range of values i.e. $-\infty<\boldsymbol{\sigma}<\infty$. The test is performed by using the initial and final values of attitude (in terms of Modified Rodriguez Parameters) in Table 6-5. The corresponding angle of rotation is found from the formula

Table 6-5 The long range test information

| Case Nr. | Initial Attitude | Final Attitude | Angle of rotation <br> about the axis of <br> rotation |
| :---: | :---: | :---: | :---: |
| 1 | $\boldsymbol{\sigma}_{0}=\left[\begin{array}{lll}-1 & -1 & -1\end{array}\right]$ | $\boldsymbol{\sigma}_{f}=\left[\begin{array}{ll}1 & 1 \\ 1\end{array}\right]$ | $-180^{\circ} \rightarrow 180^{\circ}$ |
| 2 | $\boldsymbol{\sigma}_{0}=\left[\begin{array}{lll}-2 & -2 & -2\end{array}\right]$ | $\boldsymbol{\sigma}_{f}=\left[\begin{array}{lll}2 & 2 & 2\end{array}\right]$ | $-253.74^{\circ} \rightarrow 253.74^{\circ}$ |
| 3 | $\boldsymbol{\sigma}_{0}=\left[\begin{array}{lll}-4 & -4 & -4\end{array}\right]$ | $\boldsymbol{\sigma}_{f}=\left[\begin{array}{lll}4 & 4 & 4\end{array}\right]$ | $-303.86^{\circ} \rightarrow 303.86^{\circ}$ |

The simulation results are presented in the form of figures just like that of the single run simulations presented in this chapter. The long range simulations are performed using the back - stepping controllers.

### 6.6.1 Results for the Case 1





Figure 6-30 Desired attitude trajectory for the long range simulations Case 1


Figure 6-31 Attitude tracking error for the long range simulations Case 1


Figure 6-32 Torque demand for the long range simulations Case 1


Figure 6-33 Reaction wheel velocity command for the long range simulations Case 1


Figure 6-34 Reaction wheel velocity error for the long range simulations Case 1

### 6.6.2 Results of the Case 2:





Figure 6-35 Desired attitude trajectory for the long range simulations Case 2


Figure 6-36 Attitude tracking error for the long range simulations Case 2


Figure 6-37 Torque demand for the long range simulations Case 2




Figure 6-38 Reaction wheel velocity command for the long range simulations Case 2


Figure 6-39 Reaction wheel velocity for the long range simulations Case 2

### 6.6.3 Results of the Case 3:





Figure 6-40 Desired attitude trajectory for the long range simulations Case 3


Figure 6-41 Attitude tracking error for the long range simulations Case 3


Figure 6-42 Torque demand for the long range simulations Case 3


Figure 6-43 Reaction wheel command for the long range simulations Case 3


Figure 6-44 Reaction wheel velocity error for the long range simulations Case 3

### 6.6.4 Comments

According to the results one can conclude that as the total angle of rotation increases the system produces unwanted responses such as the impulsive figures in the torque demand. This occurs especially in the third case where the total angle of rotation is the highest. This justifies the fact that as the angle of rotation increases the MRP approaches to singularity and the system has to consume more energy. In addition to that the numerical operations used in the simulation frameworks produces ill conditioned states due to the growing values of attitude and related system variables. This result also validates that numerical inversion and differentiation (not used here but will worsen everything further) should be avoided as much as possible for realistic application.

## CHAPTER 7 CONCLUSION \& FUTURE WORK

### 7.1 Summary

In this work, some approaches intended for satellite attitude control are proposed. The attitude is modeled by the Modified Rodriguez Parameters (MRP) instead of quaternion since it is a minimal representation and provides fully invertible kinematics. Two controllers are proposed which are based on integrator back stepping and input output feedback linearization (IOFL). The method of back stepping is a recursive approach which provides the derivation of the control laws after the selection of a Lyapunov function whereas in many conventional and modern control approaches the Lyapunov function is used for analysis only. Thus the stability of the controller is verified in the design step already. Another reason for the selection of the current methodology is due to the affinity of the satellite model where back - stepping is one of the most applicable approaches for those type of systems. Both the attitude dynamics and the reaction wheel dynamics are full relative degree and proper systems where the inversion of portions of the model is easier than many other plants. The back - stepping approach combined with the MRP provides a new contribution to the literature since there exists no similar approach except the one based on quaternion [Kristiansen (2005)]. In addition, the control approach of this research is also different in the way that it also incorporates the dynamics of the actuation element which is the reaction wheel and its driver motor. As a result, a cascaded controller framework is developed. That is also a contribution to the attitude control literature. Another contribution is the proposed analytical stability analysis approach that is based on Lyapunov theory. It is applied on the back - stepping based controller in order to analyze the stability of the controller against the disturbance torques coming from space. A comparison is made with the IOFL based attitude control in order to understand the differences of the two control approaches in performance. Both of the controller proposals are examined using single (free of uncertainties) and multi - run (parametric uncertainties involved) simulations.

During the development phase, it is assumed that the satellite is operating in the normal stabilization mode (i.e. the detumbling phase is completed). The satellite
is actuated by three reaction wheels mounted on the satellite body and they are driven by three brushless dc motors. The study is divided into two sections which involves the design of demanded torque generator and speed controllers respectively. The demanded torque generator produces a torque reference which is required to track the satellite to the desired trajectory and it is used for generation of a reaction wheel velocity command to be used by the speed controller. The speed controller tracks the given velocity command and exerts a torque on the satellite body. The integrator back - stepping stabilizes the satellite attitude recursively. This nature enables one to consider each state vector (attitude, body angular rates, torque and reaction wheel angular velocity vectors - having three components each) separately. The back - stepping procedure has two steps for the demanded torque generator and two steps for the speed controller. Each individual step defines a new artificial state vector which constitutes the state vectors of the closed loop system. The first and third ones are the attitude and reaction wheel velocity errors. Initial stability analysis is made through the Lyapunov functions selected in each of the design steps. The procedure assumes no external disturbances acting on the satellite body for the design phase. In this context, the closed loop is globally stable in the sense of Lyapunov. During design and analysis, multiplicative properties of the MRP kinematical matrix $\mathbf{G}(\boldsymbol{\sigma})$ brought a very useful advantage since the product $\mathbf{G}(\boldsymbol{\sigma}) \mathbf{G}^{T}(\boldsymbol{\sigma})$ is exactly a diagonal and positive definite matrix. Stability analysis becomes quite easier by getting use of that property. It is also used in the stability analysis against the disturbance torques since norm based inequalities are used in the analysis. This is an important advantage of using MRP kinematics over other attitude representations.

The second method that is preferred for the attitude control approach is the input output feedback linearization (IOFL). Like back - stepping this methodology has also nonlinearity elimination characteristics and it results in a double integrator linear system. Linear quadratic theory is applied to the remaining linear systems and the design is completed. The same approach is used both for the demanded torque generator and the speed controller. The stability against the disturbance torques are made through the multi - run simulations. This is performed by providing constant disturbances to the external torque inputs of the dynamic model during the simulations.

### 7.2 Simulations

For both of the control approaches two types of simulations are planned. Those are the single and multi - run simulations where the former one results a single plot of the necessary variables and aims at checking whether the generated control law has been performing as expected. No uncertainities are involved in this type of simulation. The latter one incorporates parametric uncertainties in the inertia and motor armature parameters. Since the uncertainties are provided as random deviations from the nominal parameter values, it is required that the simulation to be repeated several times. In this work, the determined number of runs is equal to 100 . By this way a healthier information could be collected concerning the deviations from the ideal operational characteristics.

For the back - stepping based approach six sets of control gain parameters are selected according to the rules derived from the stability analysis (against the disturbance torques) presented in Chapter 4. Both the single and multi run simulations show that the back - stepping based approach is working successfully. It is robust against both the satellite body and motor armature parameter inertias as well as the constant disturbance torque provided during the simulation.

For the feedback linearization case, at first nine sets of linear quadratic performance indices with unity input weight are investigated. The basic result, is that the reaction wheel velocity controller should considerably be faster than the demanded torque generator in order to have stable operation. In the multi - run simulations the IOFL based controller is not successful in presence of motor armature parametric uncertainties. When the control gains of the reaction wheel velocity controller increased further this problem is solved. However, the required level of the reaction wheel velocity controller gains are quite large. This may not be a problematic case for a realistic operation but generally it is not desired.

In stable back - stepping and IOFL simulations, the level of reaction wheel torques and angular velocities are far below $0.02 \mathrm{~N} \cdot \mathrm{~m}$ and 5000 rpm respectively which are the maximum possible values that can be provided by the reaction wheel motors.

According to the obtained results, the IOFL based attitude controller designed in this research is not as robust as the back - stepping based approach. The design is very sensitive to the parametric uncertainties existent on the motor model. A very small deviation may lead to oscillations and instabilities in the closed loop. This issue is existent even if there is no disturbance torques and uncertainties in
the inertial parameters of the satellite body. Therefore, it is not safe to use such a controller in realistic applications.

### 7.3 Future Work

1. In the IOFL controller design the linear quadratic regulator (LQR) theory is used to stabilize the double integrator linear system obtained after feedback. Different control approaches can be explored and this may result in a more satisfactory controller.
2. The theoretical robustness analysis approach used in the analysis of back stepping based attitude controller is based on dissipative input to state stability (ISS) criterion [Sonntag (1995), Khalil (1996)]. It is not applied to the IOFL based approach because it did not give any suitable results. Different approaches may be helpful in the analysis of the IOFL based approach in obtaining meaningful results.
3. Another alternative may be to use back - stepping in the demanded torque generator and IOFL in the reaction wheel velocity controller sections. That could decrease the computational complexity of the overall application. However, the design should be verified by simulations in order to find whether the controller is stable for all control gain.
4. An adaptive design may solve the ambiguities in the control design approaches (both for back - stepping and IOFL based approaches) such as the sensitivities to parametric uncertainties. Those types of the controllers generally does not require the model to be explicitly known. So these controllers may be more robust than their deterministic versions.
5. In the back - stepping approach the Control Lyapunov Functions are selected as quadratic functions which makes the computational process easier. However, the Lyapunov function can be selected in any form provided that it is a positive definite function. So other studies can be performed by using different Lyapunov functions and forming new control laws. A comparative study could be helpful for further development and serves as a guidance for the attitude controller developers. Another approach concerning this issue is the usage of angular momentum terms as states and forming the Lyapunov function in terms of angular momentum terms as done in [Hall (1998)].
6. The simulation environment in this research is based on the MATLAB Simulink which consumes too much computational resources which limits the noise modeling capacity. A solution to this issue can be iteration of the the overall system equations by an algorithm written in C language. By that way different noise models such as the fuel slosh can also be incorporated into simulation environment without exhausting the computational resources. Also the simulation runs much faster than the Simulink based versions.

## BIBLIOGRAPHY

1. Akella M., "Rigid body attitude tracking without angular velocity feedback", Systems \& Control Letters, Vol 42, 2001, pp 321-326
2. Akella M.R., Halbert J.T., Kotamraju G.R., "Rigid body attitude control with inclinometer and low-cost gyro measurements", Systems \& Control Letters, Vol 49, 2003, pp 151-159
3. Al Gohary A., "New optimal control laws for attitude of a rigid body motion without angular velocity measurements", Chaos, Solitons \& Fractals, Volume 25, Issue 3, August 2005, pp 557-571
4. Al Gohary A., "New control laws for stabilization of a rigid body motion using rotors system", Mechanics Research Communications, Volume 33, Issue 6, November-December 2006, pp 818-829
5. Antonsen J., "Attitude control of a micro satellite with the use of reaction control thrusters", M.Sc Thesis, Hogskolen I Narvik, 2004
6. Bajodah, A.H., Hodges D.H., "Linear Parametrization of Nonlinear Spacecraft Control by Pseudoinversion of the Control Jacobian," Proceedings of the AIAA Guidance, Navigation and Control Conference, San Francisco, California, Aug. 15-18, 2005
7. Bang H.C., Choi H.D., "Attitude Control of a Bias Momentum Satellite Using Moment of Inertia", IEEE Transactions On Aerospace And Electronic Systems Vol. 38, No. 1 January 2002, pp. 243-250
8. Bang H.C., Oh C.S., "Attitude Maneuver Control of Flexible Spacecraft by Observer-based Tracking Control", KSME International Journal, Vol. 18, No. 1, 2004, pp. 122-131
9. Bang H.C., Lee J.S, and Eun Y.J., "Nonlinear Attitude Control for a Rigid Spacecraft by Feedback Linearization", KSME International Journal, Vol, 18, No. 2, 2004, pp. 203-210
10. Bang H., Ha C-K., Kim J.H., "Flexible spacecraft attitude maneuver by application of sliding mode control", Acta Astronautica, Volume 57, Issue 11, December 2005, pp 841-850
11. Bar-Itzhack I.Y., Oshman Y., "Attitude Determination from Vector Observations: Quaternion Estimation", IEEE Transactions On Aerospace And Electronic Systems Vol. AES-21, NO. 1 January 1985, pp 128-136
12. Battilotti S., "Global output regulation and disturbance attenuation with global stability via measurement feedback for a class of nonlinear systems", IEEE Trans. Automat. Control Vol 41 (Issue 3) (1996) pp 315-327
13. Blindheim F.R., "Attitude Control of a micro satellite: Three-axis attitude control of the ESEO using reaction thrusters", M.Sc Thesis, Hogskolen I Narvik, 2004
14. Bondhus A.K., Pettersen K.Y., Gravdahl J.T., "Leader/Follower synchronization of satellite attitude without angular velocity measurements", Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference 2005 Seville, Spain, December 12-15, 2005, pp 7270 - pp 7277
15. Caccavale F., Villani L., "Output feedback control for attitude tracking", Systems \& Control Letters, Volume 38, Issue 2, 4 October 1999, pp 91-98
16. Charbonnel C.V., Duc G., Le Ballois S., "Low order robust attitude control of an earth observation satellite", Control Engineering Practice, No 7, 1999, pp 493-506
17. Chen C.C., Chien T.L., Wei C.L., "Application of feedback linearization to tracking and almost disturbance decoupling control of the ARIMA ball and beam system", Journal of Optimization Theory and Applications Vol 121, No 2, 2004, pp 279 - 300
18. Chen C.C., Lin Y.F., "Application of feedback linearization to the tracking and almost disturbance decoupling control of multi - input multi - output nonlinear systems", IEE Proceedings on Control Theory \& Applications, Vol 153, No 3, 2006, pp 331 - 341
19. Chiappa C., Bodineau G., Boulade S., Beugnon C., " $\mu-\mathrm{Mu})$-lteration Technique : Application to attitude control of satellite with large flexible appendages", Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference 2005, Seville, Spain, pp 641 646
20. Childs D., Tapley B., Fowler W., "Suboptimal attitude control of a spin stabilized axisymmetric spacecraft", IEEE Transactions on Automatic Control, Vol AC-14, Issue 6, 1969, pp 736-740
21. Costic B.T., Dawson D.M., de Queiroz M.S., Kapila V., "A quaternion based adaptive attitude tracking controller without the velocity measurements", 39th IEEE Conference on Decision and Control Australia, 2000, pp 2424-2429
22. Crassidis J.L., Markley F.L., "Sliding Mode Control Using Modified Rodrigues Parameters", AlAA Journal of Guidance, Control \& Dynamics, Vol 19, No 6, 1996, pp 1381-1383
23. Crassidis J.L., and Markley F.L., "Nonlinear Predictive Control of Spacecraft", AIAA Journal of Guidance, Control, and Dynamics, Vol. 20, No. 6, Nov.-Dec. 1997, pp. 1096-1103
24. Crassidis J.L., Vadali S.R., Markley F.L., "Optimal Variable-Structure Control Tracking Of Spacecraft Maneuvers", Proceedings of the Flight Mechanics Symposium, NASA-Goddard Space Flight Center, Greenbelt, MD, May 1999, pp. 201-214
25. Crassidis J.L., Vadali S.R., Markley F.L., "Optimal Variable-Structure Control Tracking of Spacecraft Maneuvers", AIAA Journal of Guidance, Control and Dynamics, Vol. 23, No. 3, May-June 2000, pp. 564-566
26. Crouch P.E., "Spacecraft attitude control and stabilization: Application of geometric control theory to rigid body models", IEEE Transactions on Automatic Control, Vol AC - 29, Issue 4, 1984, pp 321-331
27. Diebel J., "Representing Attitude: Euler angles, unit quaternion and rotation vectors", Technical Report, Stanford University, 2006
28. Doruk R.O., Kocaoğlan E., "A robustness analysis procedure for realistic missiles", Turkish Journal of Electrical Engineering and Computer Sciences, Vol 13, Issue 2, 2005, pp 241-260
29. Dracopoulos D.C., Jones A.J., "Neuro-Genetic Adaptive Attitude Control", Neural Computing \& Applications, Vol 2 (Issue 4), 1994, pp 183-204
30. Dunn S., Edelmann R., "Minimum power spacecraft attitude control laws for small constant disturbance torques", IEEE Transaction on Automatic Control, Vol 13, Issue 6, 1968, pp 691-694
31. Dwyer T.A.W., Fadali M.S., Chen N., "Single step optimization of feedback decoupled spacecraft attitude manuevers", Proceedings of the $24^{\text {th }}$ IEEE Conference on Decision and Control, Florida, 1985, pp 669-671
32. Fauske K.M., "NCUBE Attitude Control", NTNU DEC Engineering Report, 2002
33. Fearnsides J.J., Chu S.C., "A minimum fuel mode for steady state attitude control of an each orbiting spacecraft", IEEE Transactions on Automatic Control, Vol 115, Issue 2, 1970, pp 362-365
34. Fossen T.I., Berge A., "Nonlinear vectorial back - stepping design for global exponential tracking of marine vessels in the presence of actuator dynamics", Proceedings of the IEEE Conference on Decision and Control (CDC - 97), San Diego pp 4237 - 4242
35. Froelich R., Patapoff H., "Reaction wheel attitude control for space vehicles", IRE Transactions on Automatic Control, Vol 4, Issue 3, 1959, pp 139-149
36. Godhavn J.M., Egeland O., "Attitude Control of an Underactuated Satellite", Proceedings of the $34^{\text {th }}$ Conference on Decision \& Control New Orleans, LA December 1995, pp 3986-3988
37. Greval M.S., Shiva M., "Application of Kalman Filtering to Gyroless Attitude Determination and Control System for Environmental Satellites", Proceedings of the $34^{\text {th }}$ Conference on Decision \& Control New Orleans, LA - December 1995, pp 1544-1552
38. Guan P., Liu X.J., Lara-Rosano F., Chen J.B., "Adaptive Fuzzy Attitude Control of Satellite based on Linearization", Proceeding of the 2004 American Control Conference Boston, Massachusetts, 2004, pp 1091-1096
39. Guan P., Liu X.J., Liu J.Z., "Flexible satellite attitude control via sliding mode technique", 44th IEEE Conference on Decision and Control and the European Control Conference, Sevilla, Spain, 2006, pp 1258-1263
40. Hall C.D., Tsiotras P., Shen H., "Tracking Rigid Body Motion Using Thrusters And Momentum Wheels", AIAA conference paper, Ref No: AIAA 98-4471, 1998
41. Hall, C., Tsiotras, P., Shen, H., "Tracking Rigid Body Motion Using Thrusters and Momentum Wheels", Journal of the Astronautical Sciences, Vol. 50, No. 3, 2003, pp 311-323
42. Hartman P., "A Lemma in the Theory of Structural Stability of Differential Equations", Proceedings of the American Mathematical Society, Vol. 11, No. 4, 1960, pp 610-620
43. Harvey C.A., "Limit cycle bounds of a satellite attitude control systems", IEEE Transactions on Automatic Control, Vol 17, Issue 4, 1972, pp 526-528
44. Hegrenaes O., Gravdahl J.T., Tondel P., "Satellite attitude control using explicit model predictive control", Automatica Vol 41, Issue 12, 2005, pp 2107 - 2114
45. Huang J., Lin C.F., Hadaegh F.Y., Chiang R.Y., "Spacecraft Attitude Control Using Nonlinear Servomechanism Theory", IEEE Conference paper, TM-6-2, 1994, pp 941-946
46. Hull R.A., Ham C., Johnson R.W., "Systematic Design of Attitude Control Systems for a Satellite in a Circular Orbit with Guaranteed Performance and Stability", Reference No: SSC00-VIII-5, Session: VIII (Advanced Subsystems \& Components 1), 2000
47. Isidori A., "Nonlinear Control Systems: An Introduction", 2nd Edition, Springer Verlag, 1989
48. Isley W.C., Endres D.L., "ATS - 6 Spacecraft attitude precision pointing and slewing adaptive control experiment", IEEE Transactions on Aerospace and Electronic Systems, Vol AES-11, Issue 6, 1975, pp 1158-1164
49. Iyer A., Singh S.N., "Minimal realizations from MFDs and attitude control of spinning satellite using gyrotorquers", Proceedings of the $28^{\text {th }}$ IEEE Conference on Decision and Control, Los Angeles, 1987, pp 1269-1274
50. Iyer A., Singh S.N., "Nonlinear Adaptive Attitude Control Of Satellite Using Gyrotorquers", Proceedings of the 29th Conference on Decision and Control Honolulu, Hawaii December 1990, pp 3357-3361
51. Jan Y.W., Chiou J.C., "Attitude control system for ROCSAT-3 microsatellite: a conceptual design", Acta Astronautica, Volume 56, Issue 4, February 2005, pp 439-452
52. Joshi S.M., "Damping enhancement and attitude control of large structures", 19th IEEE Conference on Decision and Control including Symposium on Adaptive Processes, Vol 19, Part 1, 1980, pp 804-806
53. Joshi S.M., Kelkar A.G., Wen J.T.Y., "Robust Attitude Stabilization of Spacecraft Using Nonlinear Quaternion Feedback", IEEE Transactions On Automatic Control, Vol. 40, No. Io, October 1995, pp 1800-1803
54. Kaplan C., "Leo Satellites: Attitude Determination And Control Components ; Some Linear Attitude Control Techniques", M.Sc Thesis, METU 2006
55. Karataş S., "Leo Satellites: Dynamic Modelling, Simulations And Some Nonlinear Attitude Control Techniques", M.Sc Thesis, METU 2006
56. Khalil H.K., "Nonlinear Systems, Second Edition", Prentice Hall, 1996
57. Kim J., Kim J., Crassidis J.L., "Disturbance accommodating sliding mode controller for spacecraft attitude maneuvers", American Astronautics Society Conference Paper, Ref. No: AAS98-310, 1998, pp 141-153
58. Kim J., and Crassidis J.L., "A Comparative Study of Sliding Mode Control and Time-Optimal Control", Proceedings of the AIAA/AAS Astrodynamics Specialist Conference, Boston, MA, Aug. 1998, AIAA Paper \#98-4473
59. Kim J.-R., and Crassidis J.L., "Model-Error Control Synthesis Using Approximate Receding-Horizon Control Laws", AIAA Guidance, Navigation, and Control Conference, Montreal, CA, Aug. 2001, AlAA Paper \#01-4220
60. Kim J.-R., and Crassidis J.L., "Robust Spacecraft Attitude Control Using ModelError Control Synthesis", AIAA Guidance, Navigation, and Control Conference, Monterey, CA, Aug. 2002, AlAA Paper \#02-4576
61. Kim J.-R., and Crassidis J.L., "Spacecraft Attitude Control Using Approximate Receding-Horizon Model-Error Control Synthesis", AlAA Journal of Guidance, Control, and Dynamics, Vol. 29, No. 5, Sept.-Oct. 2006, pp. 1023-1031
62. Kim K-S., Kim Y., "Robust backstepping control for slew maneuver using nonlinear tracking function", Control Systems Technology, IEEE Transactions on, Volume 11, Issue 6, 2003, pp 822-829
63. Kristiansen R., "Attitude Control Of a Microsatellite", M.Sc Thesis, NTNU, 2000
64. Kristiansen R., Nicklasson PJ., "Satellite attitude control by quaternion based backstepping", American Control Conference, 2005, W-B11-4, pp 907-912
65. Kristic M., Tsiotras P., "Inverse Optimal Stabilization of a Rigid Spacecraft", IEEE Transactions On Automatic Control, Vol. 44, No. 5, May 1999, pp 1042-1048
66. Krogstad T.R., "Attitude Control of Satellites in Clusters", M.Sc Thesis, NTNU, 2005
67. Lai L-C., Yang C-C., Wu C-J., "Time-optimal maneuvering control of a rigid spacecraft", Acta Astronautica, Volume 60, Issues 10-11, May-June 2007, pp 791-800
68. Lappas V.J., Steyn W.H., Underwood C.I., "Attitude control for small satellites using control moment gyros", Acta Astronautica, Volume 51, Issues 1-9, JulyNovember 2002, pp 101-111
69. Larson V., Likins P., Marsh E., "Optimal estimation and attitude control of a solar electric propulsion spacecraft", IEEE Transactions on Aerospace and Electronic Systems, Vol AES-13, Issue 1, 1977, pp 35-47
70. Li C., Ma G., Song B., "Spacecraft attitude determination and control based on nonlinear $\mathrm{H}_{\infty}$ control", IEEE Conference on $1^{\text {st }}$ International Symposium on systems and Control in Aeronautics and Astronautics ISSCAA, 2006, pp 395 399
71. Lin J.Y., Mingori D.L., "Linear Quadratic Gaussian Synthesis with Reduced Parameter Sensitivity," International Journal of Control, Vol. 56,No. 4. pp. 901922, 1992.
72. Lin Z., Saberi A., Teel A.R., "The almost disturbance decoupling problem with internal stability for linear systems subject to input saturation state feedback case", Automatica, Volume 32, Issue 4, April 1996, pp 619-624
73. Long M.R., "Spacecraft Attitude Tracking Control, Master Thesis", Virginia Tech University, 1999
74. Lovera M., Astolfi A., "Spacecraft attitude control using magnetic actuators", Automatica, No. 40, 2004, pp 1405-1414
75. Luo W., Chu Y-C., Ling K-V., "Inverse optimal adaptive control for attitude tracking of spacecraft", Automatic Control, IEEE Transactions on Volume 50, Issue 11, 2005, pp 1639-1654
76. Makovec K.L., "A Nonlinear Magnetic Controller for Three-Axis Stability of Nanosatellites", MSc Thesis, Virginia Tech, 2001
77. Marino R., Tomei P., "Nonlinear output feedback tracking with almost disturbance decoupling", IEEE Transactions on Automatic Control, Vol 44, No 1, 1999, pp 18-28
78. Markley F.L., "Attitude error representations for Kalman filtering", Journal of Guidance, Control \& Dynamics, Vol 26 No 2, 2003, pp 311-317
79. McDuffie J.H., Shtessel Y.B., "A decoupled sliding mode controller and observer for satellite attitude control", Americal Control Conference, New Mexico, 1997, pp 564-565
80. Mobasser S., Weisenberg D., "A sun gate for Galileo spacecraft attitude control", IEEE Aerospace Applications Conference, 1990, pp 137-146
81. Monaco S., Stronelli S., "A nonlinear attitude control law for a satellite with flexible appandages", Proceedings of the $24^{\text {th }}$ IEEE Conference on Decision and Control, Florida, 1985, pp 1654-1659
82. Mothlag R.B., Mothlag M.R., "Adaptive robust attitude tracking control of spacecraft", IEEE Conference on Control Applications, Toronto, Canada, 2005, MC4-4, pp 498-503
83. Murugesan S., "Application to Al to real time intelligent attitude control of spacecraft", IEEE International Symposium on Intelligent Control, 1989, pp 287 - 292
84. Myung H., Bang H., Oh C.H., Tahk M-J., "Nonlinear Predictive Attitude Control Of Spacecraft Under External Disturbance", IFAC 15th Triennial World Congress, Barcelona, Spain, 2002
85. Nam S.K., Kim K.K., "Fuzzy control based on $H_{\infty}$ output feedback for attitude stabilization of flexible satellite", IEEE Fuzzy Systems Conference, 2001, pp 159 162
86. Nicosia S., Tomei P., "Nonlinear observer and output feedback attitude control of spacecraft", IEEE Trans. Of Aerospc and Electr. Syst., Vol 28 No 4, 1992, pp 970-977
87. Overby J.E., "Attitude control for the Norwegian student satellite NCube", M.Sc Thesis, NTNU, 2004
88. Pande K.C., Venkatachalam R, "Semipassive pitch attitude control of satellites by solar radiation pressure", IEEE Transactions on Aerospace and Electronic Systems, Vol AES-15, Issue 2, 1979, pp 194-198
89. Park Y., Tahk M-P., "Robust attitude stabilization of a spacecraft using minimal kinematic parameters", IEEE International Conference on Robotics and Automation, Korea, 2001, pp 1621-1626
90. Park Y., "Robust and optimal attitude stabilization of spacecraft with external disturbances", Aerospace Science and Technology, Volume 9, Issue 3, April 2005, pp 253-259
91. Parrish D.K., Ridgely D.B., "Attitude control of a satellite using the SDRE method", American Control Conference, New Mexico, 1997, pp 942 - 946
92. Porcelli G., Connoly A., "Optimal attitude control of spinning space body - A graphical approach", IEEE Transactions on Automatic Control, Vol AC-12, Issue 3, 1967, pp 241-249
93. Prieto D., Bona B, "Orbit and Attitude Control for the European Satellite GOCE", IEEE Network, Sensing and Control Conference, 2005, pp 728-733
94. Ruud C., "Coordinated control of satellites in formation", M.Sc Thesis, Hogskolen I Narvik, 2005
95. Sadati N., Meghdari A., Tehrani N.D., "Optimal Tracking Neuro-Controller in Satellite Attitude Control", IEEE ICIT Conference Bangkok, THAILAND, 2002, pp 54-59
96. Schaub H., Junkins J.L., Robinett R.D., "Globally Stable Feedback Laws for Near-Minimum-Fuel and Near-Minimum-Time Pointing Maneuvers for a Landmark-Tracking Spacecraft", Journal of the Astronautical Sciences Vol. 44, No. 4, Oct.-Dec. 1996, pp 443-466
97. Schaub H., Akella M., Junkins J.L., "Adaptive Control of Nonlinear Attitude Motions Realizing Linear Closed-Loop Dynamics." In American Control Conference, San Diego, June 2-6, 1999, Paper No. TA02-6, pp 1563-1567
98. Schaub H., "Rigid body kinematics package for Mathematica", Wolfram Research Inc., 1999
99. Sepahban A.E., Podraza G., "Digital implementation of time - optimal attitude control", IEEE Transactions on Automatic Control, Vol AC-9, Issue 2, 1964, pp 164-174
100. Sharma R., Tewari A., "Optimal nonlinear tracking of spacecraft attitude maneuvers", IEEE Transactions of Control Systems Technology, Vol 12, No 5, 2004, pp 677-682
101. Shen H., and Tsiotras P., "Time-Optimal Control of Axi-Symmetric Rigid Spacecraft with Two Controls "AlAA Journal of Guidance, Control, and Dynamics, Vol. 22, No.5, pp. 682-694, 1999
102. Show L., Juang J., Yang C., "Nonlinear $\mathrm{H}_{\infty}$ robust control for satellite large angle attitude maneuver", American Control Conference, 2001, pp 1357 1362
103. Showman R.D., "Simplified processing of star tracker commands for satellite attitude control", IEEE Transactions on Automatic Control, Vol AC-12, Issue 4, 1967, pp 353-359
104. Shuster M.D., "A survey of attitude representations", The Journal of Astronautical Sciences, Vol 41, No 4, 1993, pp 439-517
105. Silani E., Lovera M., "Predictive attitude control techniques for satellites with magnetic actuators", IFAC 15th Triennial World Congress, Barcelona, Spain, 2002,
106. Singh S.N., Arajuo A.D., "Asymptotic reproducibility in nonlinear systems and attitude control of gyrostat", IEEE Transactions on Aerospace and Electronic Systems, Vol AES-20, Issue 2, 1984, pp 94-103
107. Singh S.N., "Model reference adaptive attitude control of spacecraft using reaction wheels", Proceedings of the $28^{\text {th }}$ IEEE Conference on Decision and Control, Greece, 1986, pp 1514-1519
108. Singh S.N., "Robust nonlinear attitude control of flexible spacecraft", IEEE Transactions on Aerospace and Electronic Systems, Vol AES-23, Issue 3, 1987, pp 380-387
109. Singh SN., "Nonlinear adaptive attitude control of spacecraft", IEEE Transactions on Aerospace and Electronic Systems, Vol AES-23, Issue 3, 1987, pp 371-379
110. Singh S.N., Iyer A., "Nonlinear decoupling sliding mode control and attitude control of spacecraft", IEEE Transactions on Aerospace and Electronic Systems, Vol AES - 25, Issue 5, 1989, pp 621-633
111. Singh S.N., Yim W., "Nonlinear adaptive backstepping design for spacecraft attitude control using solar radiation pressure", Proceedings of The 41st IEEE Conference on Decision and Control, Las Vegas, Nevada, USA, 2002, pp 1239-1244
112. Skelton C.E., Hall C.D., "Mixed Control Moment Gyro and Momentum Wheel Attitude Control Strategies", American Astronautics Society Conference Paper, Ref. No: AAS03-558, 2003
113. Skjetne R., Fossen T.I., "On Integral Control in Backstepping: Analysis of Different Techniques", Proceeding of the 2004 American Control Conference Boston, Massachusetts, 2004, pp 1899-1904
114. Sontag E.D., "On the input-to-state stability property", European J. Control, Vol 1, 1995, pp 24-36
115. Tafazoli S., Khorasani K., "Attitude recovery of flexible spacecraft using nonlinear control", IEEE Conference Proceeding, 2004, pp 585-588
116. Tafazoli S., "On attitude recovery of spacecraft using nonlinear control", PhD Thesis, Concordia University, Canada, 2005
117. Tandale M.D., Valasek J., "Adaptive Dynamic Inversion Control with Actuator Saturation Constraints Applied to Tracking Spacecraft Maneuvers", Dynamics and Control of Systems and Structures in Space (DCSSS), 6th Conference July 2004 Riomaggiore, Italy,
118. Tienel J., Sanner R.M., "A Coupled Nonlinear Spacecraft Attitude Controller and Observer With an Unknown Constant Gyro Bias and Gyro Noise", IEEE Transactions On Automatic Control, Vol. 48, No. 11, November 2003, pp 2011 - 2014
119. Tongchet S., Kuntanapreeda S., "A fuzzy neural bang bang controller for satellite attitude control", Journal of KMITNB, Vol 11, No 4, 2001,
120. Topland M.P., "Nonlinear Attitude Control of the Micro-Satellite ESEO", M.Sc Thesis, NTNU, 2004
121. Tsai H., Lin J., Chiang T., Cheng C., "Optimal satellite attitude control system design by combination of eigenstructure assignment and LEQG/LTR methods", IEEE $5^{\text {th }}$ World Congress on Intelligent Control and Automation, 2004, pp 407-412
122. Tsiotras P., Longuski J.M., "Spin axis stabilization of symmetric spacecraft with two control torques", Systems and Control Letters, Vol 23, 1994, pp 395-402
123. Tsiotras P., "New Control Laws for the Attitude Stabilization of Rigid Bodies", 13th IFAC Symposium on Automatic Control in Aerospace, Palo Alto, California, Sept. 12-16, pp 316-321, 1994
124. Tsiotras P., Corless M., Longuski JM., "A Novel Approach to the Attitude Control of Axi - Symmetric Spacecraft", Automatica, Vol 31, No 8, 1995, pp 1099-1112
125. Tsiotras P., "Stabilization and Optimality Results for the Attitude Control Problem," AIAA Journal of Guidance, Control, and Dynamics, Vol. 19, No. 4, pp. 772-779, 1996
126. Tsiotras P., "Stabilization and Optimality Results for the Attitude Control Problem", Journal of Guidance Control and Dynamics Vol 19, No 4, 1996, pp 772-779
127. Tsiotras, P., "Further Passivity Results for the Attitude Control Problem," IEEE Transactions on Automatic Control, Vol. 43, No. 11, pp. 1597-1600, 1998
128. Tsiotras P., Luo J., "Control of underactuated spacecraft with bounded inputs", Automatica Vol 36, 2000, pp 1153-1169
129. Tsiotras, P., Shen, H. and Hall, C., "Satellite Attitude Control and Power Tracking with Momentum Wheels," AIAA Journal of Guidance, Control, and Dynamics, Vol. 24, No. 1, pp. 23-34, 2001
130. Turner J.A., "An open source extensible spacecraft simulation and modeling environment framework", MSC Thesis, Aerospace Engineering, Virginia Polytechnic Institute, 2003
131. Walchko K., Mason P.C., "Development Of A Fuzzy Sliding Mode Controller For Satellite Attitude Control", GSRP Final Report, UFL, 2000
132. Walchko K., "Robust Nonlinear Attitude Control With Disturbance Compensation", PhD Dissertation, University of Florida, 2003
133. Wang P., Shtessel, YB., Wang Y-Q, "Satellite attitude control using only magnetorquers", System Theory, Proceedings of the Thirtieth Southeastern Symposium on, 1998, pp 500-504
134. Wang Q., Yuan J., Zhu Z., "The Application of Error Quaternion and PID Control Method in Earth Observation Satellite's Attitude Control System", ${ }^{\text {st }}$ International Symposium on systems and Control in Aeronautics and Astronautics ISSCAA, 2006, pp 128-131
135. Weiss H., "Quaternion based rate/attitude tracking system with application to gimbal attitude control", Proceedings of ICCON - 89, IEEE International Conference on Control and Applications, 1989, pp 488-494
136. Wen J.T.Y., Kreutz-Delgado K., "The attitude control problem," IEEE Transactions on Automatic Control, vol. 36, no. 10, pp. 1148-1162, 1991
137. Wiener T.F., "Theoretical analysis of gimballess inertial reference equipment using delta - modulated instruments", D.Sc Theis, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, 1962
138. Willems J.C., "Almost invariant subspace: an approach to high gainfeedback design - Part I: Almost controlled invariant subspaces", IEEE Trans. Autom. Control., AC-26, Issue 1, 1981, pp 235-252
139. Wisniewski R., "Satellite Attitude Control Using Only Electromagnetic Actuation", M.Sc Thesis, Aalborg University of Denmark, 1996
140. Wisniewski R., Stoustrup J., "Periodic H2 Synthesis for Spacecraft Attitude Control with Magnetometers", Journal of Guidance, Control, and Dynamics, vol. 27, no. 5, pp. 874-881, 2004
141. Won C-H., "Comparative study of various control methods for attitude control of a LEO satellite", Aerospace Science and Technology, No 5, 1999, pp 323 333
142. Won C-H., "Satellite Structure Attitude Control with Parameter Robust RiskSensitive Control Synthesis", Proceeding of the 2004 American Control Conference Boston, Massachusetts June 30-July 2, 2004, pp 3538-3543
143. Wong H., de Queiroz MS., Kapila V., "Adaptive tracking control using synthesized velocity from attitude measurements", Automatica, Volume 37, Issue 6, June 2001, pp 947-953
144. Wu F-X., Zhang W.J., "Robust control of the aircraft attitude", I-SAIRAS, June 12-22, 2001,Canada
145. Xi B., Yi G., Shen B., "Attitude Tracking Control Using Recursive Passivation Technique", IEEE Conference on $1^{\text {st }}$ International Symposium on systems and Control in Aeronautics and Astronautics ISSCAA, 2006, pp 404-407
146. Yamashita T., Ogura N., Kurii T., Hashimoto T., "Improved satellite attitude control using a disturbance compensator", Acta - Astronautica, Vol 55, 2004, pp 15-25
147. Yoon H., Tsiotras P. "Spacecraft Adaptive Attitude Control And Power Tracking With Single-Gimballed Variable Speed Control Moment Gyroscopes, AIAA Journal of Guidance, Control, and Dynamics, Vol. 25, No. 6, pp. 10811090, 2002
148. Zeng Y., "Robust dynamic feedback and variable structure adaptive control of rigid and flexible spacecraft", MSc Thesis, University of Nevada Las Vegas, 1997

## APPENDIX A

## A SUMMARY OF ATTITUDE REPRESENTATIONS

In this appendix, a summary of the general attitude representations are presented. Some of those attitude representations are used throughout this research. Information presented here is a combined summary of [Shuster (1993), Wertz (1985)]. The discussion starts with the review of vector properties.

## A. 1 Properties of vectors in 3 - D space:

## A.1.1 Properties of scalar products

1. $\mathbf{u} \cdot \mathbf{v}=\mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot \mathbf{v} \geq 0$
3. $\mathbf{v} \cdot \mathbf{v}=0$ only if $\mathbf{v}=0$
4. $\mathbf{u} \cdot(a \mathbf{v})=a(\mathbf{u} \cdot \mathbf{v})$
5. $(\mathbf{u}+\mathbf{v}) \cdot \mathbf{w}=\mathbf{u} \cdot \mathbf{w}+\mathbf{v} \cdot \mathbf{w}$
6. $|v|=\sqrt{v \bullet v}$

## A.1.2 Properties of cross products

$$
\begin{align*}
& \mathbf{u} \times \mathbf{v}=-\mathbf{v} \times \mathbf{u} \\
& (a \mathbf{u}) \times \mathbf{v}=a(\mathbf{u} \times \mathbf{v})  \tag{A.2}\\
& (\mathbf{u}+\mathbf{v}) \times \mathbf{w}=\mathbf{u} \times \mathbf{w}+\mathbf{v} \times \mathbf{w}
\end{align*}
$$

## A.1.3 Right Handed Orthonormal Bases

A three dimensional vector is generally rewritten as:

$$
\begin{equation*}
\mathbf{v}=v_{1} \mathbf{e}_{1}+v_{2} \mathbf{e}_{2}+v_{3} \mathbf{e}_{3} \tag{A.3}
\end{equation*}
$$

where $\left(\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}\right)$ are linearly independent basis vectors. These basis vectors are classified as orthonormal if the following is valid:

$$
\begin{equation*}
\mathbf{e}_{i} \cdot \mathbf{e}_{j}=\delta_{i j} \tag{A.4}
\end{equation*}
$$

where $\delta_{i j}$ is Kronecker delta which is:

$$
\delta_{i j}=\left\{\begin{array}{l}
1, \text { if } i=j  \tag{A.5}\\
0, \text { otherw } .
\end{array}\right.
$$

The elements of the triple $\left(\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}\right)$ will be of right handed type if $\mathbf{e}_{i} \cdot\left(\mathbf{e}_{j} \times \mathbf{e}_{k}\right)=\epsilon_{i j k}$ where $\epsilon_{i j k}$ is called as Levi Civita symbol and it has the following property:

$$
\begin{align*}
& \epsilon_{123}=\epsilon_{231}=\epsilon_{312}=1 \\
& \epsilon_{132}=\epsilon_{213}=\epsilon_{321}=-1 \tag{A.6}
\end{align*}
$$

all other elements are vanished. This symbol has also the property given below:

$$
\begin{align*}
& \sum_{k=1}^{3} \epsilon_{i j k} \epsilon_{m n k}=\delta_{i n} \delta_{j m}-\delta_{i n} \delta_{j m} \\
& \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{i j k} \epsilon_{m j k}=2 \delta_{i m} \tag{A.7}
\end{align*}
$$

One can represent two vectors $\mathbf{u}$ and $\mathbf{v}$ in terms of $\left(\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}\right)$ as shown in the below:

$$
\begin{align*}
\mathbf{u} & =u_{1} \mathbf{e}_{1}+u_{2} \mathbf{e}_{2}+u_{3} \mathbf{e}_{3}  \tag{A.8}\\
\mathbf{v} & =v_{1} \mathbf{e}_{1}+v_{2} \mathbf{e}_{2}+v_{3} \mathbf{e}_{3}
\end{align*}
$$

In matrix form:

$$
\mathbf{e}_{1}=\left[\begin{array}{l}
1  \tag{A.9}\\
0 \\
0
\end{array}\right], \mathbf{e}_{2}=\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right], \mathbf{e}_{3}=\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]
$$

and,

$$
\mathbf{u}=\left[\begin{array}{l}
u_{1}  \tag{A.10}\\
u_{2} \\
u_{3}
\end{array}\right], \mathbf{v}=\left[\begin{array}{l}
v_{1} \\
v_{2} \\
v_{3}
\end{array}\right]
$$

A general dot product is rewritten as:

$$
\begin{equation*}
\mathbf{u} \cdot \mathbf{v}=\sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{i j} u_{i} v_{j}=\sum_{k=1}^{3} u_{k} v_{k} \tag{A.11}
\end{equation*}
$$

And a simple cross product is expanded as:

$$
\begin{equation*}
\mathbf{e}_{i} \times \mathbf{e}_{j}=\sum_{k=1}^{3} \epsilon_{i j k} \mathbf{e}_{k} \tag{A.12}
\end{equation*}
$$

For the general case:

$$
\begin{align*}
& \mathbf{e}_{i} \times \mathbf{e}_{j}=\sum_{k=1}^{3} \epsilon_{i j k} \mathbf{e}_{k} \\
& \mathbf{u} \times \mathbf{v}=\sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{i j k} u_{i} v_{j} \mathbf{e}_{k} \tag{A.13}
\end{align*}
$$

So in the matrix form the cross product of the vectors $\mathbf{u}$ and $\mathbf{v}$ are written as:

$$
\begin{equation*}
\mathbf{u} \times \mathbf{v}=\mathbf{S}(\mathbf{u}) \mathbf{v} \tag{A.14}
\end{equation*}
$$

where $\mathbf{S}(\mathbf{u})$ is the skew symmetric matrix former and expressed as shown in the following:

$$
\mathbf{v}=\left[\begin{array}{l}
v_{1}  \tag{A.15}\\
v_{2} \\
v_{3}
\end{array}\right], \mathbf{S}(\mathbf{v})=\left[\begin{array}{ccc}
0 & -v_{3} & v_{2} \\
v_{3} & 0 & -v_{1} \\
-v_{2} & v_{1} & 0
\end{array}\right]
$$

Some other properties concerning vector products and skew symmetric matrices are:

$$
\begin{align*}
& \text { 1. } \mathbf{S}^{T}(\mathbf{u})=-\mathbf{S}(\mathbf{u}) \\
& \text { 2. } \mathbf{S}(\mathbf{u}) \mathbf{v}=-\mathbf{S}(\mathbf{v}) \mathbf{u} \\
& \text { 3. } \mathbf{S}(\mathbf{u}) \mathbf{u}=0 \\
& \text { 4. } \mathbf{S}(\mathbf{u}) \mathbf{S}(\mathbf{v})=-(\mathbf{u} \cdot \mathbf{v}) \mathbf{I}_{3 \times 3}+\mathbf{v u}^{T}  \tag{A.16}\\
& \text { 5. } \mathbf{S}^{3}(\mathbf{u})=-|\mathbf{u}|^{2} \mathbf{S}(\mathbf{u}) \\
& \text { 6. } \mathbf{S}(\mathbf{u}) \mathbf{S}(\mathbf{v})-\mathbf{S}(\mathbf{v}) \mathbf{S}(\mathbf{u})=\mathbf{v} \mathbf{u}^{T}-\mathbf{u} \mathbf{v}^{T}=\mathbf{S}(\mathbf{u} \times \mathbf{v}) \\
& \text { 7. } \mathbf{u v}{ }^{T} \mathbf{S}(\mathbf{w})+\mathbf{S}(\mathbf{w}) \mathbf{v u}^{T}=-\mathbf{S}(\mathbf{u} \times[\mathbf{v} \times \mathbf{w}])
\end{align*}
$$

The Grossmann and Jacobi identities are:

$$
\begin{align*}
& \mathbf{a} \times(\mathbf{b} \times \mathbf{c})=(\mathbf{a} \cdot \mathbf{c}) \mathbf{b}-(\mathbf{a} \cdot \mathbf{b}) \mathbf{c}  \tag{A.17}\\
& \mathbf{a} \times(\mathbf{b} \times \mathbf{c})+\mathbf{b} \times(\mathbf{c} \times \mathbf{a})+\mathbf{c} \times(\mathbf{a} \times \mathbf{b})=0
\end{align*}
$$

## A.1.4 Orthogonal transformations:

A vector $\mathbf{x}$ can be represented in terms of different orthonormal bases. If one defines two spaces $E^{a}$ and $E^{b}$ spanned by the bases $\left(\mathbf{e}_{1}^{a}, \mathbf{e}_{2}^{a}, \mathbf{e}_{3}^{a}\right)$ and $\left(\mathbf{e}_{1}^{b}, \mathbf{e}_{2}^{b}, \mathbf{e}_{3}^{b}\right)$ respectively ( $\mathbf{e}_{i}$ 's are all unit vectors). If the vector $\mathbf{x}$ can be defined as an element of the space $E^{a}$ then it is expressed as:

$$
\begin{equation*}
\mathbf{x}=x_{1}^{a} \mathbf{e}_{1}^{a}+x_{2}^{a} \mathbf{e}_{2}^{a}+x_{3}^{a} \mathbf{e}_{3}^{a} \tag{A.18}
\end{equation*}
$$

Similarly for the space $E^{b}$ :

$$
\begin{equation*}
\mathbf{x}=x_{1}^{b} \mathbf{e}_{1}^{b}+x_{2}^{b} \mathbf{e}_{2}^{b}+x_{3}^{b} \mathbf{e}_{3}^{b} \tag{A.19}
\end{equation*}
$$

The basis vectors can be written in terms of each other as:

$$
\begin{align*}
\mathbf{e}_{i}^{b} & =\sum_{j=1}^{3} C_{i j} \mathbf{e}_{j}^{a}  \tag{A.20}\\
\mathbf{e}_{i}^{a} & =\sum_{j=1}^{3} C_{i j}^{\prime} \mathbf{e}_{j}^{b}
\end{align*}
$$

where:

$$
\begin{align*}
C_{i j} & =\mathbf{e}_{i}^{b} \cdot \mathbf{e}_{j}^{a}  \tag{A.21}\\
C_{i j}^{\prime} & =\mathbf{e}_{i}^{a} \cdot \mathbf{e}_{j}^{b}
\end{align*}
$$

The direct result of the above facts is $\left[C_{i j}\right]=\left[C_{i j}^{\prime}\right]^{T}$ or $\mathbf{C}^{\prime}=\mathbf{C}^{T}$. In geometrical point of view the elements $C_{i j}$ and $C_{i j}^{\prime}$ are the cosines of the angles between two sets of axes. Thus, they are also called as direction cosines in kinematics. Similarly, $\mathbf{C}$
and $\mathbf{C}^{\prime}$ are the direction cosine matrices. Since it is known that the components of a vector can be extracted by scalar (dot) multiplication with the basis vectors (i.e $x_{i}^{a}=\mathbf{e}_{i}^{a} \cdot \mathbf{x}$ and $\left.x_{i}^{b}=\mathbf{e}_{i}^{b} \cdot \mathbf{x}\right)$ one can also write:

$$
\begin{align*}
& x_{i}^{b}=\sum_{j=1}^{3} C_{i j} x_{j}^{a}  \tag{A.22}\\
& x_{i}^{a}=\sum_{j=1}^{3} C_{i j}^{\prime} x_{j}^{b}
\end{align*}
$$

If the vectors $\mathbf{x}^{a}$ and $\mathbf{x}^{b}$ are expressed as $\mathbf{x}^{a}=\left[\begin{array}{lll}x_{1}^{a} & x_{2}^{a} & x_{3}^{a}\end{array}\right]^{T}$ and $\mathbf{x}^{b}=\left[\begin{array}{lll}x_{1}^{b} & x_{2}^{b} & x_{3}^{b}\end{array}\right]^{T}$ then the following can be written:

$$
\begin{align*}
& \mathbf{x}^{b}=\mathbf{C} \mathbf{x}^{a}  \tag{A.23}\\
& \mathbf{x}^{a}=\mathbf{C}^{T} \mathbf{x}^{b}
\end{align*}
$$

The direction cosine matrix has a property that is $\mathbf{C}^{T} \mathbf{C}=\mathbf{I}_{3 \times 3}$ or $\mathbf{C C}^{T}=\mathbf{I}_{3 \times 3}$. This also means that $\mathbf{C}^{-1}=\mathbf{C}^{T}$ and $\left|\mathbf{C}^{T} \mathbf{C}\right|=1$. So the determinant of the matrix $\mathbf{C}$ is either 1 or -1 . If $|\mathbf{C}|=1$, it will represent a rotation matrix. In this case one can denote it by $\mathbf{R}(\mathbf{R}=\mathbf{C})$. In the case of $|\mathbf{C}|=-1$, it is an improper direction cosine matrix and does not have any meaning in kinematics.

## A. 2 Rotation Matrices and Composition

The rotation matrix is important in representing transformation from one position to another in the three dimensional space. Generally rotations involve more than one axis (axis of rotation) so the concept of rotational composition should be discussed. Composition can be explained as the combination of two or more successive rotations around two or more different axes.

## A.2.1 Introduction to the rotational composition:

Consider a rotational motion depicted by the diagram shown below:


Figure 1 Diagram depicting two successive rotations

According to Figure 1, $\mathbf{y}=\mathbf{P x}, \mathbf{z}=\mathbf{R y}$ and $\mathbf{z}=\mathbf{R P x}$. The composition of successive set of rotations is also a rotation. There are some important properties of rotation matrices related with dot and cross products which are given below:

$$
\begin{align*}
& (\mathbf{R u}) \cdot(\mathbf{R v})=\mathbf{u} \cdot \mathbf{v} \\
& (\mathbf{R u}) \times(\mathbf{R v})=\mathbf{R}(\mathbf{u} \times \mathbf{v}) \tag{A.24}
\end{align*}
$$

## A.2.2 Rotation matrices and angular rotation

In this section rotation matrices around specific axes are presented. Consider a rotation around the axis represented by the unit vector $\mathbf{e}_{3}$ (or more commonly $\mathbf{z}$ ) which is illustrated in Figure 2.


Figure 2 Rotation around $e_{3}$ axis

So the resultant rotation in terms of trigonometric functions is:

$$
\begin{align*}
& e_{1}^{b}=e_{1}^{a} \cos \gamma+e_{2}^{a} \sin \gamma \\
& e_{2}^{b}=-e_{1}^{a} \sin \gamma+e_{2}^{a} \cos \gamma  \tag{A.25}\\
& e_{3}^{b}=e_{3}^{a}
\end{align*}
$$

So in matrix form the rotation around $\mathbf{e}_{3}=\left[\begin{array}{lll}0 & 0 & 1\end{array}\right]^{T}$ is represented in matrix form as shown in below:

$$
\mathbf{R}\left(\mathbf{e}_{3}, \gamma\right)=\left[\begin{array}{ccc}
\cos \gamma & \sin \gamma & 0  \tag{A.26}\\
-\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{array}\right]
$$

If two vectors are represented as done in (A.18) (in terms of the bases of the vector space $E^{a}$ and $E^{b}$ ), the relationship between $\mathbf{x}^{b}$ and $\mathbf{x}^{a}$ for a rotation around the $\mathbf{e}_{3}=\left[\begin{array}{lll}0 & 0 & 1\end{array}\right]^{T}$ unit vector can be written as:

$$
\begin{equation*}
\mathbf{x}^{b}=\mathbf{R}\left(\mathbf{e}_{3}, \gamma\right) \mathbf{x}^{a} \tag{A.27}
\end{equation*}
$$

Similarly the rotations around the axes represented by $\mathbf{e}_{1}=\left[\begin{array}{lll}1 & 0 & 0\end{array}\right]^{T}$ and $\mathbf{e}_{2}=\left[\begin{array}{lll}0 & 1 & 0\end{array}\right]^{T}:$

$$
\begin{align*}
& \mathbf{R}\left(\mathbf{e}_{1}, \gamma\right)=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \gamma & \sin \gamma \\
0 & -\sin \gamma & \cos \gamma
\end{array}\right]  \tag{A.28}\\
& \mathbf{R}\left(\mathbf{e}_{2}, \gamma\right)=\left[\begin{array}{ccc}
\cos \gamma & 0 & -\sin \gamma \\
0 & 1 & 0 \\
\sin \gamma & 0 & \cos \gamma
\end{array}\right]
\end{align*}
$$

By doing some manipulations one can write the followings:

$$
\begin{align*}
& \mathbf{R}\left(\mathbf{e}_{3}, \gamma\right) \mathbf{e}_{1}=\cos \gamma \mathbf{e}_{1}-\sin \gamma \mathbf{e}_{2}=\cos \gamma \mathbf{e}_{1}-\sin \gamma \mathbf{e}_{3} \times \mathbf{e}_{1} \\
& \mathbf{R}\left(\mathbf{e}_{3}, \gamma\right) \mathbf{e}_{2}=\cos \gamma \mathbf{e}_{2}+\sin \gamma \mathbf{e}_{1}=\cos \gamma \mathbf{e}_{2}-\sin \gamma \mathbf{e}_{3} \times \mathbf{e}_{2}  \tag{A.29}\\
& \mathbf{R}\left(\mathbf{e}_{3}, \gamma\right) \mathbf{e}_{3}=\mathbf{e}_{3}
\end{align*}
$$

So by analogy the followings can also be written:

$$
\begin{align*}
& \mathbf{R}(\mathbf{n}, \gamma) \mathbf{v}_{\perp}=\cos \gamma \mathbf{v}_{\perp}-\sin \gamma \mathbf{n} \times \mathbf{v}_{\perp}=\cos \gamma \mathbf{v}_{\perp}-\sin \gamma \mathbf{S}(\mathbf{n}) \mathbf{v}_{\perp}  \tag{A.30}\\
& \mathbf{R}(\mathbf{n}, \gamma) \mathbf{n}=\mathbf{n}
\end{align*}
$$

where $\mathbf{v}_{\perp}$ is orthogonal projection of $\mathbf{v}$ on plane perpendicular to $\mathbf{n}$. So a general rotation of angle $\gamma$ around $\mathbf{n}$ is represented by:

$$
\begin{equation*}
\mathbf{R}(\mathbf{n}, \gamma) \mathbf{v}=\mathbf{v}_{\perp}+\cos \gamma \mathbf{v}_{\perp}-\sin \gamma \mathbf{S}(\mathbf{n}) \mathbf{v}_{\perp} \tag{A.31}
\end{equation*}
$$

Since $\mathbf{v}_{\perp}=\mathbf{n n}^{T} \mathbf{v}$ or $\mathbf{v}_{\perp}=\mathbf{S}^{2}(\mathbf{n}) \mathbf{v}$ the above equation can be rewritten as:

$$
\begin{equation*}
\mathbf{R}(\mathbf{n}, \gamma) \mathbf{v}=\mathbf{n n}^{T} \mathbf{v}-\cos \gamma \mathbf{S}^{2}(\mathbf{n}) \mathbf{v}-\sin \gamma \mathbf{S}(\mathbf{n}) \mathbf{v} \tag{A.32}
\end{equation*}
$$

A further manipulation leads to:

$$
\begin{align*}
& \mathbf{R}(\mathbf{n}, \gamma)=\mathbf{I}_{3 \times 3}-\sin \gamma \mathbf{S}(\mathbf{n})+(1-\cos \gamma) \mathbf{S}^{2}(\mathbf{n}) \\
& \mathbf{R}(\mathbf{n}, \gamma)=\left[\begin{array}{lll}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{array}\right] \tag{A.33}
\end{align*}
$$

If one demands the rotational angle $\gamma$ from the rotation matrix $\mathbf{R}(\mathbf{n}, \gamma)$ the following can be obtained:

$$
\begin{equation*}
\cos \gamma=\frac{1}{2}(\operatorname{tr} \mathbf{R}-1) \tag{A.34}
\end{equation*}
$$

And the rotational axis vector $\mathbf{n}$ is obtained back as:

$$
\mathbf{n}=\frac{1}{2 \sin \gamma}\left[\begin{array}{l}
R_{23}-R_{32}  \tag{A.35}\\
R_{31}-R_{13} \\
R_{12}-R_{21}
\end{array}\right]
$$

The equation in (A.33) is also called as Euler's formula.

## A. 3 Euler Angles

Consider the illustration shown in Figure 3. This figure depicts the following rotational sequence:

1. An angle of $\psi$ around the axis based on the unit vector $\mathbf{e}_{3}$
2. An angle of $\theta$ around the axis based on the unit vector $\mathbf{e}_{2}$
3. An angle of $\phi$ around the axis based on the unit vector $\mathbf{e}_{1}$


Figure 3 Euler angle rotation sequence

$$
E \longrightarrow R\left(e_{3}, \psi\right) \longrightarrow E^{\prime \prime \prime}-R\left(e_{2}, \theta\right) \longrightarrow E^{\prime \prime}-R\left(e_{1}, \phi\right) \longrightarrow E^{\prime}
$$

Figure 4 Another illustration for Figure 3

The rotational composition corresponding to Figure 3 is:

$$
\begin{align*}
& \mathbf{R}\left(\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}, \phi, \theta, \psi\right)=\mathbf{R}\left(\mathbf{e}_{1}, \phi\right) \mathbf{R}\left(\mathbf{e}_{2}, \theta\right) \mathbf{R}\left(\mathbf{e}_{3}, \psi\right) \\
& =\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{array}\right]\left[\begin{array}{ccc}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{array}\right]\left[\begin{array}{ccc}
\cos \psi & \sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{array}\right]  \tag{A.36}\\
& =\left[\begin{array}{ccc}
\cos \theta \cos \psi & \cos \theta \sin \psi & -\sin \theta \\
\sin \phi \sin \theta \cos \psi-\cos \phi \sin \psi & \sin \phi \sin \theta \sin \psi-\cos \phi \cos \psi & \sin \phi \cos \theta \\
\cos \phi \sin \theta \cos \psi+\sin \phi \sin \psi & \cos \phi \sin \theta \sin \psi-\sin \phi \cos \psi & \cos \phi \cos \theta
\end{array}\right]
\end{align*}
$$

In kinematics this approach is also called as roll - pitch - yaw sequence. This is a preferred angular representation in aerospace applications. There are also other representations which are obtained by changing the rotation sequence. The generic rotation matrix is formed by:

$$
\begin{equation*}
\mathbf{R}\left(\mathbf{e}_{i}, \mathbf{e}_{j}, \mathbf{e}_{k}, \phi, \theta, \psi\right)=\mathbf{R}\left(\mathbf{e}_{i}, \phi\right) \mathbf{R}\left(\mathbf{e}_{j}, \theta\right) \mathbf{R}\left(\mathbf{e}_{k}, \psi\right) \tag{A.37}
\end{equation*}
$$

However, there is a restriction that two consecutive rotation sequence should not be equal i.e. $i \neq j$ and $j \neq k$. Because of that, there exist 12 different representations that include six symmetric and six asymmetric representations. The relationships that enable one to obtain the triple $(\phi, \theta, \psi)$ from the rotation matrix in (A.36) are shown below:

$$
\begin{align*}
& \phi=\tan ^{-1}\left\{\frac{R_{23}}{R_{33}}\right\} \\
& \theta=-\sin ^{-1}\left\{R_{13}\right\}  \tag{A.38}\\
& \psi=\tan ^{-1}\left\{\frac{R_{12}}{R_{11}}\right\}
\end{align*}
$$

Some more discussion can be found in [Shuster (1993), Diebel (2006)]. The Euler angle representation is not a commonly used attitude representation in the aerospace literature due to the computational burdens that is brought by them. The most common issue is the singularities associated with the Euler angle representations. In order to represent the attitude completely one should employ at least two sets of Euler angles. This increases the computational burdens.

## A. 4 Quaternion Representation

In order to resolve the singularities in the Euler angle representation, the quaternion approach is proposed. It is a virtual representation, which can be
represented as $\mathbf{q}=\varepsilon_{1} \mathbf{i}+\varepsilon_{2} \mathbf{j}+\varepsilon_{3} \mathbf{k}+\eta$ (like a complex number). It does not have to have a unit magnitude (i.e. $\|\mathbf{q}\|=\sqrt{\varepsilon_{1}^{2}+\varepsilon_{2}^{2}+\varepsilon_{3}^{2}+\eta^{2}}=1$ ) but in order to define rotations with the quaternion one has to represent it in the normalized form which can be performed by:

$$
\begin{equation*}
\widehat{\mathbf{q}}=\frac{\mathbf{q}}{\|\mathbf{q}\|} \tag{A.39}
\end{equation*}
$$

In attitude control problems the quaternion is often represented as a four dimensional vector $\mathbf{q}=[\varepsilon, \eta]=\left[\begin{array}{llll}\varepsilon_{1} & \varepsilon_{2} & \varepsilon_{3} & \eta\end{array}\right]^{T}$ with its elements:

$$
\begin{align*}
& \eta=\cos \frac{\gamma}{2} \\
& \boldsymbol{\varepsilon}=\sin \frac{\gamma}{2} \mathbf{n}=\left[\begin{array}{l}
\varepsilon_{1} \\
\varepsilon_{2} \\
\varepsilon_{3}
\end{array}\right] \tag{A.40}
\end{align*}
$$

where $\gamma$ is the angle of rotation and $\mathbf{n}$ is the unit vector representing the axis of rotation. As it is understood from (A.40) the "rotation quaternion" is a unit norm element (for example a vector) in four dimensional space. The rotation matrix in terms of the quaternion parameters can be obtained by doing some manipulations on the Euler's rotation formula in (A.33) which is rewritten here for convenience.

$$
\begin{equation*}
\mathbf{R}(\mathbf{n}, \gamma)=(\cos \gamma) \mathbf{I}_{3 \times 3}+(1-\cos \gamma) \mathbf{m n}^{T}-\sin \gamma \mathbf{S}(\mathbf{n}) \tag{A.41}
\end{equation*}
$$

Using (A.40) and after doing some manipulations (especially trigonometric half angle relationships) the quaternion rotation matrix is found as shown below:

$$
\begin{gather*}
\mathbf{R}(\mathbf{q})=\left(q_{0}^{2}-\mathbf{q}^{T} \mathbf{q}\right) \mathbf{I}_{3 \times 3}+2 \mathbf{q} \mathbf{q}^{T}-2 q_{0} \mathbf{S}(\mathbf{q})  \tag{A.42}\\
\mathbf{R}(\mathbf{q})=\left[\begin{array}{ccr}
\eta^{2}+\varepsilon_{1}^{2}-\varepsilon_{2}^{2}-\varepsilon_{3}^{2} & 2 \varepsilon_{1} \varepsilon_{1}+2 \eta \varepsilon_{3} & 2 \varepsilon_{1} \varepsilon_{3}-2 \eta \varepsilon_{2} \\
2 \varepsilon_{1} \varepsilon_{2}-2 \eta \varepsilon_{3} & \eta^{2}-\varepsilon_{1}^{2}+\varepsilon_{2}^{2}-\varepsilon_{3}^{2} & 2 \varepsilon_{2} \varepsilon_{3}+2 \eta \varepsilon_{1} \\
2 \varepsilon_{1} \varepsilon_{3}+2 \eta \varepsilon_{2} & 2 \varepsilon_{2} \varepsilon_{3}-2 \eta \varepsilon_{1} & \eta^{2}-\varepsilon_{1}^{2}-\varepsilon_{2}^{2}+\varepsilon_{3}^{2}
\end{array}\right] \tag{A.43}
\end{gather*}
$$

Due to the unity of the norm of the rotation quaternion, one can also write the following:

$$
\begin{align*}
& t r \mathbf{R}=3 \eta^{2}-\varepsilon_{1}^{2}-\varepsilon_{2}^{2}-\varepsilon_{3}^{2} \\
& 1+t r \mathbf{R}=4 \eta^{2}  \tag{A.44}\\
& \eta= \pm \frac{1}{4} \sqrt{1+t r \mathbf{R}}
\end{align*}
$$

The components of $\varepsilon$ can be computed from (A.43) as:

$$
\begin{align*}
& \varepsilon_{1}=\frac{1}{4 \eta}\left(R_{23}-R_{32}\right) \\
& \varepsilon_{2}=\frac{1}{4 \eta}\left(R_{31}-R_{13}\right)  \tag{A.45}\\
& \varepsilon_{3}=\frac{1}{4 \eta}\left(R_{12}-R_{21}\right)
\end{align*}
$$

The above conversion is valid only for $\eta \neq 0$. If that is not the case then there will be numerical accuracy problems as $\eta \rightarrow 0$. To solve this problem, there are other sets of equations that do the same back computation. Those are:

$$
\begin{align*}
& \varepsilon_{1}= \pm \frac{1}{2} \sqrt{1+R_{11}-R_{22}-R_{33}} \\
& \varepsilon_{2}=\frac{1}{4 \varepsilon_{1}}\left(R_{12}+R_{21}\right) \\
& \varepsilon_{3}=\frac{1}{4 \varepsilon_{1}}\left(R_{13}+R_{31}\right)  \tag{A.46}\\
& \eta=\frac{1}{4 \varepsilon_{1}}\left(R_{23}-R_{32}\right) \\
& \varepsilon_{3}= \pm \frac{1}{2} \sqrt{1-R_{11}-R_{22}+R_{33}} \\
& \varepsilon_{1}=\frac{1}{4 \varepsilon_{3}}\left(R_{31}+R_{13}\right) \\
& \varepsilon_{2}=\frac{1}{4 \varepsilon_{3}}\left(R_{32}+R_{23}\right)  \tag{A.47}\\
& \eta=\frac{1}{4 \varepsilon_{3}}\left(R_{12}-R_{21}\right) \\
& \varepsilon_{2}= \pm \frac{1}{2} \sqrt{1-R_{11}+R_{22}-R_{33}} \\
& \varepsilon_{1}=\frac{1}{4 \varepsilon_{2}}\left(R_{21}+R_{12}\right)  \tag{A.48}\\
& \varepsilon_{3}=\frac{1}{4 \varepsilon_{2}}\left(R_{23}+R_{32}\right) \\
& \eta=\frac{1}{4 \varepsilon_{2}}\left(R_{31}-R_{13}\right)
\end{align*}
$$

The best accuracy is obtained if one of the computational approaches of (A.44) to (A.48) which has the largest argument inside the square root is preferred. The computational software package MATLAB uses the same approach.

## A.4.1 Conversion between quaternion and Euler angles

The conversion relationships from Euler angles to rotation quaternion can be obtained by applying the following procedure. First of all the rotation quaternion vector is rewritten for representing a general rotation:

$$
\left[\begin{array}{l}
\eta  \tag{A.49}\\
\varepsilon
\end{array}\right]=\left[\begin{array}{c}
\cos \frac{\gamma}{2} \\
0 \\
0 \\
0
\end{array}\right]+\sin \frac{\gamma}{2}\left[\begin{array}{l}
0 \\
1 \\
0 \\
0
\end{array}\right]+\sin \frac{\gamma}{2}\left[\begin{array}{l}
{\left[\begin{array}{l}
0 \\
0 \\
1 \\
1 \\
0
\end{array}\right]}
\end{array}+\sin \frac{\gamma}{2}\left[\begin{array}{l}
{\left[\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
1
\end{array}\right]}
\end{array}=\left[\begin{array}{c}
\cos \frac{\gamma}{2} \\
\sin \frac{\gamma}{2} \\
\sin \frac{\gamma}{2} \\
\sin \frac{\gamma}{2}
\end{array}\right]=\left[\begin{array}{c}
\cos \frac{\gamma}{2} \\
\sin \frac{\gamma}{2} \mathbf{n}
\end{array}\right]\right.\right.
$$

Since the compositions of rotations are also rotations, the following can be written for the rotation quaternion:

$$
\mathbf{R}\left(\left[\begin{array}{c}
\mu  \tag{A.50}\\
\boldsymbol{\mu}
\end{array}\right]\right) \mathbf{R}\left(\left[\begin{array}{c}
\eta \\
\varepsilon
\end{array}\right]\right)=\mathbf{R}\left(\left[\begin{array}{c}
\mu \\
\boldsymbol{\mu}
\end{array}\right] \otimes\left[\begin{array}{l}
\eta \\
\varepsilon
\end{array}\right]\right)
$$

The sign $\otimes$ denotes the quaternion product which is defined as:

$$
\left[\begin{array}{l}
\mu  \tag{A.51}\\
\boldsymbol{\mu}
\end{array}\right] \otimes\left[\begin{array}{l}
\eta \\
\boldsymbol{\varepsilon}
\end{array}\right]=\left[\begin{array}{c}
\mu \eta-\boldsymbol{\mu} \cdot \varepsilon \\
\mu \boldsymbol{\varepsilon}+\eta \boldsymbol{\mu}-\boldsymbol{\mu} \times \boldsymbol{\varepsilon}
\end{array}\right]
$$

So one can thought this conversion as first computing the rotation matrix in terms of the Euler angles and then finding the argument of the resultant matrix in terms of the quaternion. This is done through a triple quaternion product that is described as shown:

$$
\left[\begin{array}{c}
\mu  \tag{A.52}\\
\boldsymbol{\mu}
\end{array}\right]=\left[\begin{array}{c}
\cos \frac{\phi}{2} \\
\sin \frac{\phi}{2} \\
0 \\
0
\end{array}\right],\left[\begin{array}{l}
\eta \\
\varepsilon
\end{array}\right]=\left[\begin{array}{c}
\cos \frac{\theta}{2} \\
0 \\
\sin \frac{\theta}{2} \\
0
\end{array}\right],\left[\begin{array}{l}
\chi \\
\chi
\end{array}\right]=\left[\begin{array}{c}
\cos \frac{\psi}{2} \\
0 \\
0 \\
\sin \frac{\psi}{2}
\end{array}\right]
$$

In the above $[\mu, \mu]^{T}$ denotes a rotation of angle $\phi$ around the axis $\mathbf{e}_{1},[\eta, \varepsilon]^{T}$ denotes a rotation of angle $\theta$ around the axis $\mathbf{e}_{2}$ and $[\chi, \chi]^{T}$ denotes a rotation of angle $\psi$ around the axis $\mathbf{e}_{3}$. By using the associative property of quaternion product:

$$
\left[\begin{array}{l}
\varepsilon  \tag{A.53}\\
\varepsilon
\end{array}\right]_{\text {final }}=\left[\begin{array}{l}
\mu \\
\mu
\end{array}\right] \otimes\left[\begin{array}{l}
\eta \\
\varepsilon
\end{array}\right] \otimes\left[\begin{array}{l}
\chi \\
\chi
\end{array}\right]=\left[\begin{array}{l}
\mu \\
\mu
\end{array}\right] \otimes\left\{\left[\begin{array}{l}
\eta \\
\varepsilon
\end{array}\right] \otimes\left[\begin{array}{l}
\chi \\
\chi
\end{array}\right]\right\}=\left\{\left[\begin{array}{c}
\mu \\
\mu
\end{array}\right] \otimes\left[\begin{array}{l}
\eta \\
\varepsilon
\end{array}\right]\right\} \otimes\left[\begin{array}{c}
\chi \\
\chi
\end{array}\right]
$$

Substituting (A.52) into (A.53) yields:

$$
\left[\begin{array}{l}
\eta  \tag{A.54}\\
\varepsilon_{1} \\
\varepsilon_{2} \\
\varepsilon_{3}
\end{array}\right]=\left[\begin{array}{l}
\cos \frac{\phi}{2} \cos \frac{\theta}{2} \cos \frac{\psi}{2}+\sin \frac{\phi}{2} \sin \frac{\theta}{2} \sin \frac{\psi}{2} \\
\sin \frac{\phi}{2} \cos \frac{\theta}{2} \cos \frac{\psi}{2}-\cos \frac{\phi}{2} \sin \frac{\theta}{2} \sin \frac{\psi}{2} \\
\cos \frac{\phi}{2} \sin \frac{\theta}{2} \cos \frac{\psi}{2}+\sin \frac{\phi}{2} \cos \frac{\theta}{2} \sin \frac{\psi}{2} \\
\sin \frac{\psi}{2} \cos \frac{\phi}{2} \cos \frac{\theta}{2}-\sin \frac{\phi}{2} \sin \frac{\theta}{2} \cos \frac{\psi}{2}
\end{array}\right]
$$

For the reverse operation i.e. conversion from quaternion to Euler angle can be obtained directly from the rotation matrices written in terms of both parameters. So one can easily have the following relations:

$$
\begin{align*}
& \phi=\tan ^{-1}\left\{\frac{R_{23}}{R_{33}}\right\}=\tan ^{-1}\left\{\frac{2 \varepsilon_{2} \varepsilon_{3}+2 \eta \varepsilon_{1}}{\eta^{2}-\varepsilon_{1}^{2}-\varepsilon_{2}^{2}+\varepsilon_{3}^{2}}\right\} \\
& \theta=-\sin ^{-1}\left\{R_{13}\right\}=-\sin ^{-1}\left\{2 \varepsilon_{1} \varepsilon_{3}-2 \eta \varepsilon_{2}\right\}  \tag{A.55}\\
& \psi=\tan ^{-1}\left\{\frac{R_{12}}{R_{11}}\right\}=\tan ^{-1}\left\{\frac{2 \varepsilon_{1} \varepsilon_{2}+2 \eta \varepsilon_{3}}{\eta^{2}+\varepsilon_{1}^{2}-\varepsilon_{2}^{2}-\varepsilon_{3}^{2}}\right\}
\end{align*}
$$

## A. 5 The Gibbs Vector

The Gibbs (or Rodriguez) vector is a derivative of quaternion representation that is defined by:

$$
\boldsymbol{\rho}=\left[\begin{array}{l}
\rho_{1}  \tag{A.56}\\
\rho_{2} \\
\rho_{3}
\end{array}\right]=\tan \frac{\gamma}{2} \mathbf{n}=\frac{\sin \frac{\gamma}{2}}{\cos \frac{\gamma}{2}} \mathbf{n}=\frac{\boldsymbol{\varepsilon}}{\eta}
$$

## A.5.1 Conversion between quaternion and the Gibbs Vector

Since the Gibbs vector is related to quaternion by simple trigonometric identities, basic trigonometric function conversion routines can be utilized for obtaining the attitude conversion rules between quaternion and Gibbs vector. The rightmost term in (A.56) is the formula of conversion from quaternion to Gibbs vector. The reverse operation is simply obtained from trigonometric identities as:

$$
\left[\begin{array}{l}
\eta  \tag{A.57}\\
\boldsymbol{\varepsilon}
\end{array}\right]= \pm \frac{1}{\sqrt{1+\boldsymbol{\rho}^{T} \boldsymbol{\rho}}}\left[\begin{array}{l}
1 \\
\boldsymbol{\rho}
\end{array}\right]
$$

## A.5.2 Rotation matrix in terms of Gibbs vector

Rotation matrix in terms of Gibbs vector is easily obtained by using (A.57) and (A.42) as shown below:

$$
\begin{gather*}
\mathbf{R}(\boldsymbol{\rho})=\frac{1}{1+\boldsymbol{\rho}^{T} \mathbf{\rho}}\left\{\left[1-\boldsymbol{\rho}^{T} \boldsymbol{\rho}\right] \mathbf{I}_{3 \times 3}+2 \boldsymbol{\rho} \boldsymbol{\rho}^{T}-2 \mathbf{S}(\boldsymbol{\rho})\right\}  \tag{A.58}\\
\mathbf{R}(\boldsymbol{\rho})=\frac{1}{1+\boldsymbol{\rho}^{T} \boldsymbol{\rho}}\left[\begin{array}{ccc}
1+\rho_{1}^{2}-\rho_{2}^{2}-\rho_{3}^{2} & 2 \rho_{1} \rho_{2}+2 \rho_{3} & 2 \rho_{1} \rho_{3}-2 \rho_{2} \\
2 \rho_{1} \rho_{2}-2 \rho_{3} & 1-\rho_{1}^{2}+\rho_{2}^{2}-\rho_{3}^{2} & 2 \rho_{2} \rho_{3}+2 \rho_{1} \\
2 \rho_{1} \rho_{3}+2 \rho_{2} & 2 \rho_{2} \rho_{3}-2 \rho_{1} & 1-\rho_{1}^{2}-\rho_{2}^{2}+\rho_{3}^{2}
\end{array}\right] \tag{A.59}
\end{gather*}
$$

Back transformation from the rotation matrix to the Gibbs vector is simply performed by:

$$
\boldsymbol{\rho}=\frac{1}{1+\operatorname{tr}(\mathbf{R})}\left[\begin{array}{l}
R_{23}-R_{32}  \tag{A.60}\\
R_{31}-R_{13} \\
R_{12}-R_{21}
\end{array}\right]
$$

The composition rule for the Gibbs vector can be derived from the quaternion composition rule (A.51) as:

$$
\begin{equation*}
\boldsymbol{\rho}^{\prime \prime}=\frac{\boldsymbol{\rho}^{\prime}+\boldsymbol{\rho}-\boldsymbol{\rho}^{\prime} \times \boldsymbol{\rho}}{1-\boldsymbol{\rho}^{\prime} \boldsymbol{\rho}} \tag{A.61}
\end{equation*}
$$

A diagram showing the composition rule above is given in Figure 5.

$$
E \longrightarrow R(\rho) \longrightarrow E^{\prime} \longrightarrow R\left(\rho^{\prime}\right) \longrightarrow E^{\prime \prime}
$$

Figure 5 Gibbs vector composition

## A. 6 Cayley - Klein parameters

The Cayley - Klein parameters are derived from the quaternion vector by forming the Cayley - Klein matrix as shown in the following:

$$
\mathbf{Q}\left(\left[\begin{array}{l}
\eta  \tag{A.62}\\
\varepsilon
\end{array}\right]\right)=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]=\left[\begin{array}{ll}
\eta-j \varepsilon_{3} & \varepsilon_{2}-j \varepsilon_{1} \\
\varepsilon_{2}+j \varepsilon_{1} & \eta+j \varepsilon_{3}
\end{array}\right]
$$

where $j$ is the "complex number" or $\sqrt{-1}$. A property of the above matrix is:

$$
\begin{align*}
& \mathbf{Q}^{{ }^{* T} \mathbf{Q}=\mathbf{I}_{2 \times 2}}  \tag{A.63}\\
& \mathbf{Q}^{-1}=\mathbf{Q}^{* T}
\end{align*}
$$

The determinant of $\mathbf{Q}$ is equal to $\boldsymbol{\varepsilon}^{T} \boldsymbol{\varepsilon}+\eta^{2}$ which is unity. Another obvious property is related with composition that is:

$$
\mathbf{Q}\left(\left[\begin{array}{c}
\mu  \tag{A.64}\\
\boldsymbol{\mu}
\end{array}\right]\right) \mathbf{Q}\left(\left[\begin{array}{l}
\eta \\
\boldsymbol{\varepsilon}
\end{array}\right]\right)=\mathbf{Q}\left(\left[\begin{array}{c}
\mu \\
\boldsymbol{\mu}
\end{array}\right] \otimes\left[\begin{array}{l}
\eta \\
\boldsymbol{\varepsilon}
\end{array}\right]\right)
$$

Vectors are represented as matrices in Cayley - Klein representation. For example a vector $\mathbf{v}=\left[\begin{array}{lll}v_{1} & v_{2} & v_{3}\end{array}\right]^{T}$ is expressed in matrix form as:

$$
\mathbf{V}=\left[\begin{array}{cc}
v_{3} & v_{1}-j v_{2}  \tag{A.65}\\
v_{1}+j v_{2} & -v_{3}
\end{array}\right]
$$

where $\operatorname{det}(\mathbf{V})=-\left(v_{1}^{2}+v_{2}^{2}+v_{3}^{2}\right)=-\mathbf{v}^{2}$. The matrix $\mathbf{Q}$ can be used in deriving rotation matrices as given below:

$$
\begin{equation*}
\mathbf{V}_{l}=\mathbf{Q} \mathbf{V}_{b} \mathbf{Q}^{* T} \tag{A.66}
\end{equation*}
$$

Where the subscripts $b$ and $i$ denote the body and inertial frames respectively. In this transformation:

1. Hermitian property
2. Trace
3. Determinant
4. Orthogonality

Properties are all preserved. So the transformation is equivalent to quaternion rotational transformation.

## A. 7 Modified Rodriguez Parameters (MRP)

This representation of attitude is a minimal representation first appeared in [Wiener (1962)]. It is represented by the following trigonometric identity:

$$
\begin{equation*}
\boldsymbol{\sigma}=\mathbf{n} \tan \frac{\gamma}{4}=\frac{\sin \frac{\gamma}{4}}{\cos \frac{\gamma}{4}} \mathbf{n} \tag{A.67}
\end{equation*}
$$

Multiplying the denominator and numerator of the above by $2 \cos \frac{\gamma}{4}$ and using the double angle formulations in trigonometry one can obtain the equations shown below:

$$
\begin{equation*}
\boldsymbol{\sigma}=\frac{2 \cos \frac{\gamma}{4} \sin \frac{\gamma}{4}}{2 \cos \frac{\gamma}{4} \cos \frac{\gamma}{4}}=\frac{\sin \frac{\gamma}{2}}{2 \cos ^{2} \frac{\gamma}{4}}=\frac{\sin \frac{\gamma}{2}}{1+\cos \frac{\gamma}{2}}=\frac{\varepsilon}{1+\eta} \tag{A.68}
\end{equation*}
$$

The above is obviously the approach of conversion from quaternion to MRP vector. The reverse operation can be performed easily by noting that:

$$
\begin{equation*}
\eta=\cos \frac{\gamma}{2}=\cos ^{2} \frac{\gamma}{4}-1=\frac{2}{\sec ^{2} \frac{\gamma}{4}}-1=\frac{2-\sec ^{2} \frac{\gamma}{4}}{\sec ^{2} \frac{\gamma}{4}}=\frac{1-\tan ^{2} \frac{\gamma}{4}}{1+\tan ^{2} \frac{\gamma}{4}}=\frac{1-\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}}{1+\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}} \tag{A.69}
\end{equation*}
$$

From (A.68) $\varepsilon=(1+\eta) \boldsymbol{\sigma}$. Thus:

$$
\begin{equation*}
\boldsymbol{\varepsilon}=\boldsymbol{\sigma}+\boldsymbol{\sigma} \eta=\boldsymbol{\sigma}+\frac{1-\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}}{1+\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}} \boldsymbol{\sigma}=\frac{2 \boldsymbol{\sigma}}{1+\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}} \tag{A.70}
\end{equation*}
$$

As it can be easily understood from (A.67) the MRP representation, have a singularity at rotations of $\pm 2 \pi$ radians. Because of that, the representation can be modified for rotations of $\gamma-2 \pi$ radians. That is shown below:

$$
\begin{aligned}
& \tan (\alpha-\beta)=\frac{\sin (\alpha-\beta)}{\cos (\alpha-\beta)}=\frac{\sin \alpha \cos \beta-\cos \alpha \sin \beta}{\cos \alpha \cos \beta+\cos \alpha \cos \beta} \\
& \tan \left(\frac{\gamma-2 \pi}{4}\right)=\tan \left(\frac{\gamma}{4}-\frac{\pi}{2}\right)=\frac{\sin \frac{\gamma}{4} \cos \frac{\pi}{2}-\cos \frac{\gamma}{4} \sin \frac{\pi}{2}}{\cos \frac{\gamma}{4} \cos \frac{\pi}{2}+\sin \frac{\gamma}{4} \sin \frac{\pi}{2}}=-\frac{\cos \frac{\gamma}{4}}{\sin \frac{\gamma}{4}}=-\cot \frac{\gamma}{4} \quad \text { (A.71) } \\
& -\cot \frac{\gamma}{4}=-\frac{1}{\tan \frac{\gamma}{4}}=-\frac{\tan \frac{\gamma}{4}}{\tan ^{2} \frac{\gamma}{4}}
\end{aligned}
$$

Then:

$$
\begin{align*}
& \boldsymbol{\sigma}^{\prime}=\tan \left(\frac{\gamma-2 \pi}{4}\right) \mathbf{n}=-\frac{\cos \frac{\gamma}{4}}{\sin \frac{\gamma}{4}} \mathbf{n}=-\cot \frac{\gamma}{4} \mathbf{n} \\
& -\cot \frac{\gamma}{4} \mathbf{n}=-\frac{1}{\tan \frac{\gamma}{4}} \mathbf{n}=-\frac{\tan \frac{\gamma}{4}}{\tan ^{2} \frac{\gamma}{4}} \mathbf{n}=-\frac{\boldsymbol{\sigma}}{\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}}  \tag{A.72}\\
& =-\frac{\cos \frac{\gamma}{4}}{\sin \frac{\gamma}{4}} \mathbf{n}=-\frac{\sin \frac{\gamma}{2}}{1-\cos \frac{\gamma}{2}} \mathbf{n}=\frac{-\boldsymbol{\varepsilon}}{1-\eta}
\end{align*}
$$

As it is obviously seen from the above computation $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^{\prime}$ corresponds to $\left[\begin{array}{ll}\eta & \boldsymbol{\varepsilon}^{T}\end{array}\right]^{T}$ and $-\left[\begin{array}{ll}\eta & \boldsymbol{\varepsilon}^{T}\end{array}\right]^{T}$ respectively. This means the attitudes represented by $\boldsymbol{\sigma}$ and $\sigma^{\prime}$ are equivalent. In order to resolve the singularity in MRP representation one can use the following switching mechanism:

$$
\boldsymbol{\sigma}^{\text {processed }}=\left\{\begin{array}{cc}
\boldsymbol{\sigma} & \text { if } \boldsymbol{\sigma}^{T} \boldsymbol{\sigma}<1  \tag{A.73}\\
\frac{-\boldsymbol{\sigma}}{\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}} & \text { otherwise }
\end{array}\right.
$$

As a result of the rule, the angle of rotation is limited in $-\pi \leq \gamma \leq \pi$.

## A.7.1 Rotation matrix

The rotation matrix in terms of the Modified Rodriguez Parameters are derived from the quaternion rotation matrix in (A.42). The matrix defined in (A.42) is modified by using the properties given in (A.16). One of those properties is $\mathbf{S}^{2}(\mathbf{u})=-(\mathbf{u} \cdot \mathbf{u}) \mathbf{I}_{3 \times 3}+\mathbf{u u}^{T}$ and by utilizing this property, the standard quaternion rotation matrix is rewritten as:

$$
\begin{align*}
& \mathbf{R}(\varepsilon)=\left(\eta^{2}-\boldsymbol{\varepsilon}^{T} \varepsilon\right) \mathbf{I}_{3 \times 3}+2 \boldsymbol{\varepsilon} \varepsilon^{T}-2 \eta \mathbf{S}(\varepsilon)  \tag{A.74}\\
& =\mathbf{I}_{3 \times 3}+2 \mathbf{S}^{2}(\varepsilon)-2 \eta \mathbf{S}(\varepsilon)
\end{align*}
$$

And by direct substitution of (A.69) and (A.70) the rotation matrix in terms of MRP's are obtained as shown below:

$$
\begin{equation*}
\mathbf{R}(\boldsymbol{\sigma})=\mathbf{I}_{3 \times 3}+\frac{8 \mathbf{S}^{2}(\boldsymbol{\sigma})}{\left(1+\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}\right)^{2}}-\frac{4\left(1-\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}\right) \mathbf{S}(\boldsymbol{\sigma})}{\left(1+\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}\right)^{2}} \tag{A.75}
\end{equation*}
$$

## A.7.2 MRP Composition Rule

The composition rule for the Modified Rodriguez Parameters can be derived from the quaternion composition rules like the Gibbs vector. Before proceeding, it will be convenient to write the following:

$$
\begin{align*}
& \mu=\frac{1-\boldsymbol{\sigma}_{x}^{T} \boldsymbol{\sigma}_{x}}{1+\boldsymbol{\sigma}_{x}^{T} \boldsymbol{\sigma}_{x}} \\
& \boldsymbol{\mu}=\frac{2 \boldsymbol{\sigma}_{x}}{1+\boldsymbol{\sigma}_{x}^{T} \boldsymbol{\sigma}_{x}} \\
& \eta=\frac{1-\boldsymbol{\sigma}_{y}^{T} \boldsymbol{\sigma}_{y}}{1+\boldsymbol{\sigma}_{y}^{T} \boldsymbol{\sigma}_{y}}  \tag{A.76}\\
& \boldsymbol{\varepsilon}=\frac{2 \boldsymbol{\sigma}_{y}}{1+\boldsymbol{\sigma}_{y}^{T} \boldsymbol{\sigma}_{y}}
\end{align*}
$$

Applying the quaternion composition rule in (A.50) yields:

$$
\left[\begin{array}{c}
\chi \\
\chi
\end{array}\right]=\left[\begin{array}{l}
\mu \\
\boldsymbol{\mu}
\end{array}\right] \otimes\left[\begin{array}{l}
\eta \\
\boldsymbol{\varepsilon}
\end{array}\right]=\left[\begin{array}{c}
\frac{1-\boldsymbol{\sigma}_{x}^{T} \boldsymbol{\sigma}_{x}}{1+\boldsymbol{\sigma}_{x}^{T} \boldsymbol{\sigma}_{x}} \frac{1-\boldsymbol{\sigma}_{y}^{T} \boldsymbol{\sigma}_{y}}{1+\boldsymbol{\sigma}_{y}^{T} \boldsymbol{\sigma}_{y}}-\frac{4 \boldsymbol{\sigma}_{x}^{T} \boldsymbol{\sigma}_{y}}{1+\boldsymbol{\sigma}_{x}^{T} \boldsymbol{\sigma}_{x}} \\
\frac{2\left(1-\boldsymbol{\sigma}_{x}{ }^{T} \boldsymbol{\sigma}_{x}\right) \boldsymbol{\sigma}_{y}}{\left(1+\boldsymbol{\sigma}_{x}^{T} \boldsymbol{\sigma}_{x}\right)\left(1+\boldsymbol{\sigma}_{y}{ }^{T} \boldsymbol{\sigma}_{y}\right)}+\frac{2\left(1-\boldsymbol{\sigma}_{y}{ }^{T} \boldsymbol{\sigma}_{y}\right) \boldsymbol{\sigma}_{x}}{\left(1+\boldsymbol{\sigma}_{x}^{T} \boldsymbol{\sigma}_{x}\right)\left(1+\boldsymbol{\sigma}_{y}{ }^{T} \boldsymbol{\sigma}_{y}\right)}-\frac{4 \boldsymbol{\sigma}_{x} \times \boldsymbol{\sigma}_{y}}{\left(1+\boldsymbol{\sigma}_{x}^{T} \boldsymbol{\sigma}_{x}\right)\left(1+\boldsymbol{\sigma}_{y}{ }^{T} \boldsymbol{\sigma}_{y}\right)}
\end{array}\right]
$$

(A.77)

Finally conversion back into MRP yields the following:

$$
\begin{align*}
& \boldsymbol{\sigma}_{z}=\frac{\chi}{1+\chi} \\
& \boldsymbol{\sigma}_{z}=\frac{\left(1-\boldsymbol{\sigma}_{x}^{T} \boldsymbol{\sigma}_{x}\right) \boldsymbol{\sigma}_{y}+\left(1-\boldsymbol{\sigma}_{y}{ }^{T} \boldsymbol{\sigma}_{y}\right) \boldsymbol{\sigma}_{x}-2 \boldsymbol{\sigma}_{x} \times \boldsymbol{\sigma}_{y}}{1+\boldsymbol{\sigma}_{x}{ }^{T} \boldsymbol{\sigma}_{x} \boldsymbol{\sigma}_{y}{ }^{T} \boldsymbol{\sigma}_{y}+2 \boldsymbol{\sigma}_{x}{ }^{T} \boldsymbol{\sigma}_{y}} \tag{A.78}
\end{align*}
$$

The composition rules are used for deriving the attitude tracking errors in the simulations.

## A. 8 Kinematical Differential Equation

In this section, the differential kinematics that is used for attitude motion is presented. Some of the equations can be derived from the axis - angular differential equation shown below:

$$
\begin{align*}
\dot{\gamma} & =\mathbf{n} \cdot \boldsymbol{\omega} \\
\dot{\mathbf{n}} & =\frac{1}{2}\left[\mathbf{n} \times \boldsymbol{\omega}-\cot \left(\frac{\gamma}{2}\right) \mathbf{n} \times(\mathbf{n} \times \boldsymbol{\omega})\right] \tag{A.79}
\end{align*}
$$

## A.8.1 The Kinematical Differential Equation for Rotation Matrix

The kinematical equation for the rotation matrices is an important model for the derivation of the kinematical differential equations for various attitude
representations. The derivation of kinematics equation for the rotation matrices can be started with the following definition:

$$
\begin{equation*}
\mathbf{R}(t+\Delta t)=\boldsymbol{\Phi}(t+\Delta t) \mathbf{R}(t) \tag{A.80}
\end{equation*}
$$

One should note that the above definition is based on the composition rule for attitude definitions. Another important here is that the matrix $\boldsymbol{\Phi}(t+\Delta t)$ should also be a matrix and for small values of $\Delta t$ the value $\boldsymbol{\Phi}(t)$ should also be sufficiently small. One can express $\boldsymbol{\Phi}(t+\Delta t)$ as shown in the following:

$$
\begin{equation*}
\boldsymbol{\Phi}(t+\Delta t)=\mathbf{I}_{3 \times 3}-\mathbf{S}(\Delta \chi(t))+\mathbf{O}\left(|\Delta \chi(t)|^{2}\right) \tag{A.81}
\end{equation*}
$$

where $\Delta \chi(t)$ is a positional three dimensional vector having the following property:

$$
\begin{equation*}
\lim _{\Delta x \rightarrow 0}|\Delta x(t)| \rightarrow 0 \tag{A.82}
\end{equation*}
$$

and $\mathbf{O}\left(|\Delta \chi(t)|^{2}\right)$ defines the higher order terms. Using the standard definition of the derivative:

$$
\begin{align*}
& \lim _{\Delta \Delta 0} \frac{\mathbf{R}(t+\Delta t)-\mathbf{R}(t)}{\Delta t}=\frac{1}{\Delta t}\left\{-\mathbf{S}(\Delta \chi(t)) \mathbf{R}(t)+\mathbf{O}\left(|\Delta \chi(t)|^{2}\right)\right\} \\
& \mathbf{R}(t)=-\mathbf{S}(\boldsymbol{\omega}(t)) \mathbf{R}(t)  \tag{A.83}\\
& \boldsymbol{\omega}(t)=\lim _{\Delta t \rightarrow 0} \frac{\Delta \chi(t)}{\Delta t}
\end{align*}
$$

For the case of spinning space bodies, the velocity term $\boldsymbol{\omega}(t)$ represents the body coordinate frame angular velocity.

## A.8.2 Quaternion

The kinematics of the attitude quaternion is derived in a way similar to the rotation matrix differential equation and presented in [Shuster (1993)]. So only the results are given here:

$$
\begin{align*}
& \dot{\eta}=-\frac{1}{2} \varepsilon^{T} \boldsymbol{\omega}  \tag{A.84}\\
& \dot{\varepsilon}=\frac{1}{2}\left[\eta I_{3 \times 3}+\mathbf{S}(\varepsilon)\right] \boldsymbol{\omega}
\end{align*}
$$

## A.8.3 Gibbs Vector

The kinematics of the Gibbs vector follows from that of the quaternion and expressed as:

$$
\begin{equation*}
\dot{\boldsymbol{\rho}}=\frac{1}{2}\{\boldsymbol{\omega}-\boldsymbol{\omega} \times \boldsymbol{\rho}+(\boldsymbol{\omega} \cdot \boldsymbol{\rho}) \boldsymbol{\rho}\} \tag{A.85}
\end{equation*}
$$

## A.8.4 Cayley Klein Parameters

Since the Cayley - Klein parameters are thought of a matrix representation for quaternion the kinematics is also a matrix differential equation. So one can write:

$$
\begin{align*}
& \dot{\mathbf{H}}=\frac{1}{2} \boldsymbol{\Gamma}(\boldsymbol{\omega}) \mathbf{H} \\
& \boldsymbol{\Gamma}(\boldsymbol{\omega})=\left[\begin{array}{cc}
\omega_{z} & \omega_{x}-j \omega_{y} \\
\omega_{x}+j \omega_{y} & -\omega_{z}
\end{array}\right], \quad \boldsymbol{\omega}=\left[\begin{array}{c}
\omega_{x} \\
\omega_{y} \\
\omega_{z}
\end{array}\right] \tag{A.86}
\end{align*}
$$

As easily understood, the matrix $\boldsymbol{\Gamma}$ Hermitian symmetric $\left(\boldsymbol{\Gamma}^{* T}=\boldsymbol{\Gamma}\right)$.

## A.8.5 Modified Rodriguez Parameters (MRP)

Like that of the Gibbs vector the kinematical differential equation of the MRP vector follows from the quaternion and written as:

$$
\begin{equation*}
\dot{\boldsymbol{\sigma}}=\frac{1}{4}\left\{\left[1-\boldsymbol{\sigma}^{T} \boldsymbol{\sigma}\right] \boldsymbol{\omega}-2 \boldsymbol{\omega} \times \boldsymbol{\sigma}+2(\boldsymbol{\omega} \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma}\right\} \tag{A.87}
\end{equation*}
$$

For further information and details of the derivations, the following references could be read [Shuster (1993), Diebel (2006), Wiener (1962)].

## A.8.6 Euler Angles

The Euler Angle kinematics vary according to the selected Euler angle set from the twelve different rotation sequences. In this appendix, the kinematics related used in the MATLAB Aerospace Toolbox. The relationship between the body fixed angular velocity vector and the time derivatives of the Euler angles are given by the transformation:

$$
\begin{align*}
& \boldsymbol{\omega}_{o b}^{b}=\left[\begin{array}{l}
p \\
q \\
r
\end{array}\right]=\left[\begin{array}{l}
\dot{\phi} \\
0 \\
0
\end{array}\right]+\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{array}\right]\left[\begin{array}{c}
0 \\
\dot{\theta} \\
0
\end{array}\right]+\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{array}\right]\left[\begin{array}{ccc}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{array}\right]\left[\begin{array}{c}
0 \\
0 \\
\dot{\psi}
\end{array}\right]=\boldsymbol{\Psi}\left[\begin{array}{c}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{array}\right] \\
& {\left[\begin{array}{c}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{array}\right]=\boldsymbol{\Psi}^{-1} \boldsymbol{\omega}_{o b}^{b}=\left[\begin{array}{ccc}
1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\
0 & \cos \phi & -\sin \phi \\
0 & \frac{\sin \phi}{\cos \theta} & \frac{\cos \phi}{\cos \theta}
\end{array}\right]\left[\begin{array}{l}
p \\
q \\
r
\end{array}\right]} \tag{A.88}
\end{align*}
$$

## VITA




