ON THE GENERALIZATIONS AND PROPERTIES OF ABRAMOVICH–WICKSTEAD SPACES

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ABSTRACT

ON THE GENERALIZATIONS AND PROPERTIES OF ABRAMOVICH–WICKSTEAD SPACES

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In this thesis, we study two problems. The first one is to introduce the general version of Abramovich-Wickstead type space and investigate its order properties. In particular, we study the ideals, order bounded sets, disjointness properties, Dedekind completion and the norm properties of this Riesz space. We also define a new example of Riesz space-valued uniformly continuous functions, denoted by CD_0^r which generalizes the original Abramovich-Wickstead space. It is also shown that similar spaces CD_0 and CD_w introduced earlier by Alpay and Ercan are decomposable lattice-normed spaces.

The second one is related to analytic representations of different classes of dominated operators on these spaces. Our main theorems say that regular linear operators on CD_0^r or linear dominated operators on CD_0 may be represented as the sum of integration with respect to operator-valued measure and summation operation. In the case when the operator is order continuous or *bo*-continuous, then these representations reduce to summation parts.

Keywords: Riesz space, regular operator, lattice-normed space, dominated operator, vector measure with bounded variation

ÖZ

ABRAMOVICH–WICKSTEAD UZAYLARININ GENELLEŞTİRMELERİ VE ÖZELLİKLERİ ÜZERİNE

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Bu tezde iki problem ele alınmaktadır. Bunlardan ilki Abramovich-Wickstead türü uzayını genel versiyonunu tanıtmak ve onun sıralama özelliklerini araştırmaktır. Özel olarak bu Riesz uzayının ideallerini, sıra sınırlı kümelerini, ayrıklık özelliklerini, Dedekind tamlık ve norm özelliklerini çalışıyoruz. Aynı zamanda orjinal Abramovich-Wickstead uzayını genelleştiren yeni bir CD_0^r ile gösterilen Riesz uzayı değerli düzgün sürekli fonksiyon uzayını tanımlıyoruz. Ayrıca Alpay ve Ercan tarafından önceden tanıtılan benzer CD_0 ve CD_w uzaylarının gerçekte ayrıştırılabilir örgü-normlu uzaylar olduğu gösterilmektedir.

Ele alınan ikinci problem ise bu uzaylar üzerindeki baskın operatörlerin farklı sınıflarının analitik temsilleriyle ilgilidir. Temel teoremlerimiz CD_0^r üzerinde ki düzgün doğrusal operatörlerin veya CD_0 üzerinde ki baskın doğrusal operatörlerin operatör değerli ölçü integrali ve toplam operasyonunun toplamı olarak yazılabileceği ile ilgilidir. Operatörün sıra sürekli yada *bo*- sıra sürekli olması durumunda bu temsiller sadece toplamsal kısma indirgenmektedir.

Anahtar Kelimeler: Riesz uzayı, düzgün operatör, örgü-normlu uzay, baskın operatör, sınırlı varyasyonlu vektör ölçüsü

To my wife Elif

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CHAPTER 1

INTRODUCTION

1.1 State of the Art

In 1993, two peculiar new classes of *unital AM-spaces* $CD_0(K)$ and $CD_w(K)$, the elements of which are the sums of real-valued continuous functions and discrete functions on K, were introduced by Abramovich and Wickstead [33] for a quasi-Stonean space K without isolated points. They showed that neither class is almost Dedekind σ-complete, although $CD_w(K)$ has Cantor property. Finally, they identified the order continuous and sequentially order continuous duals of spaces in these classes. Further Alpay and Ercan [27] relaxed the condition on quasi-Stonean space K by taking it as a compact Hausdorff space without isolated points and introduced the spaces $CD_0(K,E)$ and $CD_w(K,E)$ for a normed space E. They investigated lattice-norm properties, the center and order continuous duals of these spaces under the assumption that E is a Banach lattice. From then Ercan [34] proved that $CD_0(K)$ and $C(K\times\{0,1\})$ are isometrically Riesz isomorphic spaces under a certain topology on $K\times\{0,1\}$. V. G. Troitsky [30] found a description of Ercan's topological space $K\times\{0,1\}$ as the *Alexandroff duplicate* of K and gave an elegant characterization of elements of $CD_0(K)$. T. Hoim and D. A. Robbins [29] introduced the space of sections $CD_0(K,\mathcal{X})$ of a *continuous Banach bundle* $\mathcal{X}$ over K and making use of Ercan's result proved that this space is linearly isometric to the space of all *continuous sections* of some continuous Banach bundle $\widetilde{\mathcal{X}}$ over the Alexandroff duplicate $\widetilde{K}$ of K. Some new properties of the space $CD_0(K,\mathcal{X})$ were investigated by A. E. Gutman and A. V. Koptev in [3], see also a survey paper [2].

These results naturally rise the following task: to introduce the general version of Abramovich–Wickstead type spaces, investigate its order properties, and find new examples of such spaces. As mentioned above, the works of both Abramovich–Wickstead and Alpay–Ercan contained some characterizations of order continuous duals of CD_0 and CD_w-type spaces. This naturally brought us the idea to investigate the linear operators in more general setting on these type of spaces.

The notion of a *dominated* or *majorized* operator was invented in the 1930s by L. V. Kantorovich. He introduced the fundamental notion of lattice-normed space by elements of a vector lattice and that of a linear operator between such spaces which is dominated by a positive linear or monotone sublinear operator. The idea of dominated operator can be stated as follows: if an operator under consideration is dominated by another operator, called a *dominant or majorant*, then the properties of the latter have a substantail influence on the properties of the former. Thus operators that have "nice" dominants must posses nice properties. In the succeeding years, many authors studied various particular cases of lattice-normed spaces and different classes of dominated operators, e.g., see [8, 11, 19, 31]. The general theory of dominated operators has been improved by A. G. Kusraev and his followers (A. E. Gutman, S. A. Malyugin, E. V. Kolesnikov, S. Z. Strizhevskii etc.). Different kinds (*bo*-continuous, disjointness preserving operators, integral operators in particular pseudointegral operators) and some analytical represantations of dominated operators were given in the book [4] by A. G. Kusraev. There exists an important relationship between lattice-normed spaces and continuous Banach bundles. A. G. Kusraev and V. Z. Strizhevskiĭ [7] proved that any lattice-normed space can be represented as the space of *almost global sections* of a suitable continuous Banach bundle. However, uniqueness of the bundle was not established and later A. E. Gutman found a class of uniqueness for The Kusraev–Strizhevskiĭ Representation Theorem, the class of *ample* (or *complete*) continuous Banach bundles. A detailed presentation of this theory can be found in [1, 4].

The spaces of vector-valued functions are often *br*-complete or *bo*-complete lattice-normed spaces and this peculiarity is important when studying the structure of the spaces or linear operators on them. As $CD_0(K,E)$ and $CD_w(K,E)$ spaces are lattice-normed spaces, this motivated us to use the technics in [4] to characterize some kinds of dominated operators on these spaces.

The modern *vector measure theory* includes two main lines of investigation. The first one, the study of measures with values in Banach or locally convex space, stems from classical works by S. Bohner, N. Dunford, I. M. Gelfand, and B. Pettis and together with a variety of interesting applications in geometry of Banach spaces and operator theory is covered by many monographs, see for example [12, 19, 20].

The second line dealing with vector lattice valued measures stems from L. V. Kantorovich, B. Z. Vulikh, and A. G. Pinsker [17], although such measures appeared implicitly earlier as *Boolean homomorphisms* or *spectral measures*. This study was concentrated mainly on measure extension problem and Riesz type representation theorems. In this respect two important results due to J. D. M. Wright should be mentioned:

1) A Dedekind σ-complete vector lattice possesses the measure extension property if and only if it is weakly (σ, ∞)-distributive [13];

2) The Riesz Representation Theorem for positive operators with values in a Dedekind complete vector lattice is true, but the representing measure is quasiregular and cannot be chosen regular [14].

J. D. M. Wright [15] obtained also the following characterization: Every quasiregular Borel measure on every compact space with values in a Dedekind complete vector lattice is regular if and only if this vector lattice is weakly (σ, ∞)-distributive. More details and further bibliography can be found in [4]. For a unified treatment of both lines see [5].

One of the main ideas of Dinculeanu's book [19] is domination of a linear operator on the space of vector-valued measurable functions by a measure. An operator $T : C(K, E) \to F$ is said to be dominated (or majored) if there exists a regular positive Borel measure ν such that

$$||T(f)|| \leq \int ||f(k)||\, d\nu(k), \quad for\ every\ f \in C(K, E).$$

We say that T is dominated by ν, or that ν dominates T. If T is dominated, then there exists a least positive regular measure dominating T. He ellobrated the space of dominated linear operators on $C(K, E)$ in details by using the regular Borel measures with bounded variation. Kusraev's definition and N. Dinculeanu's definition actually coincide for the dominated linear operators on $C(K, E)$. The systematic application of concept of domination leads to integral representation of broad classes of linear operators defined on spaces of measurable vector-valued functions.

Putting together the ideas of Abramovich–Wickstead type spaces, Kantorovich's dominated operators, and Dinculeanu's integral representations, we can state the following important problem.

1.2 Statement of the Problem

The aim of this work is to introduce and investigate new types of Abramovich–Wickstead spaces of vector-valued functions and obtain analytic representations of different classes of dominated operators on these spaces.

1.3 Review of Contents

Chapter 1 of this thesis presents the scope of the study as an introduction.
Chapter 2 contains some background related to theory of vector measures and lattice-normed spaces needed in this thesis.
Chapter 3 deals with the general version of Abramovich–Wickstead type spaces, denoted by $E \times_G F$ and investigating its order properties. In particular, we study the ideals, order bounded sets, disjointness properties, Dedekind completion and the norm properties of this Riesz space as well as we identify its center. We also define a new concrete example of Riesz space-valued uniformly continuous functions, denoted by CD_0^r which generalizes the original Abramovich–Wickstead space. It is also shown that similar spaces CD_0 and CD_w introduced earlier by Alpay and Ercan are actually decomposable lattice-normed spaces.
Chapter 4 is devoted to study the analytic represantation of different classes of dominated operators on CD_0^r and CD_0-type spaces. Our main representation theorems are that regular linear operators on CD_0^r or linear dominated operators on CD_0 may be constructed as the sum of integration with respect to operator-valued measure and summation operation (or integration with respect to discrete operator-valued measure). We have shown that if the operator is order continuous or *bo*-continuous, then these representations reduce to discrete parts.

More precisely, we can state the main results of this chapter as follows.
1. Let K be a non-empty set and F be a Dedekind complete vector lattice. Then we set

(1) $c_0(\mathbb{N}, E) = \{(e_n) \in E : \exists e \in E^+ \textit{ such that } e_n \in E(e)\ \forall n \textit{ and } \|e_n\|_e \to 0\}$,

(2) $l_1[K, L^r(E, F)]$ the space of operators $\alpha : K \to L^r(E, F)$ such that the infinite sum $\sum_{n=1}^{\infty} |\alpha(k_n)|(|e_n|)$ is an element of F for all $(k_n) \in K$ and $(e_n) \in c_0(\mathbb{N}, E)$.

As usual, $\sum_{n=1}^{\infty}|\alpha(k_n)|(|e_n|)$ is the supremum of the sums $\sum_{n=1}^{m}|\alpha(k_n)|(|e_n|)$. $l_1[K, L^r(E,F)]$ is a vector lattice under the pointwise operations. Then we have the following.

Theorem 1. *Let K be a compact Hausdorff space without isolated points and F be a Dedekind complete vector lattice. Then $L^r(CD_0^r(K,E),F)$ is lattice isomorphic to $qca(K, L^r(E,F)) \oplus l_1[K, L^r(E,F)]$ with the dual order on this direct sum defined by*

$$< \mu, \alpha > \geq 0 \Leftrightarrow \mu \geq 0 \text{ and } \alpha \geq 0 \text{ and } \mu(\{k\}) \geq \alpha(k)$$

for all $k \in K$, which if we identify α with a discrete measure on K, is precisely requiring that $\mu \geq \alpha \geq 0$.

2. Let K be a compact space and F be a Dedekind complete vector lattice. Then we set $l^1(K, L_n^r(E,F))$ the set of all maps $\beta = \beta(k)$ from K into $L_n^r(E,F)$ satisfying

(1) $\sup_{||f||_e \leq 1} \sum_k |\beta(k)|(|(f(k)|) \in F$ for each arbitrary but fixed $e \in E^+$ and $f \in CD_0^r(K,E)$,

(2) $\sum_k |\beta(k)|(f_\alpha(k)) \downarrow_\alpha 0$ whenever $f_\alpha \downarrow 0$.

Then we have the following.

Theorem 2. *Let K be a compact Hausdorff space without isolated points and F be a Dedekind complete vector lattice. Then $L_n^r(CD_0^r(K,E),F)$ is lattice isomorphic to $l^1(K, L_n^r(E,F))$.*

3. Let K be a non-empty set, E and F be two Banach spaces. Then we set $l_1(K, L(E,F))$ the set of mappings $\varphi : K \to L(E,F)$ such that the sum $\sum_{k\in K} ||\varphi(k)(f(k))|| < \infty$ for all $f \in c_0(K,E)$. Then we have the following.

Theorem 3. *Let K be a compact Hausdorff space without isolated points, E and F be two Banach spaces. Then $M(CD_0(K,E),F)$ is isometrically isomorphic to $rca(\mathfrak{B}, L(E,F)) \oplus l_1(K, L(E,F))$ where $rca(\mathfrak{B}, L(E,F))$ is the space of regular Borel measures $m : \mathfrak{B} \to L(E,F)$ with finite variation $|m|$.*

Theorem 4. *Let K be a compact Hausdorff space without isolated points, E and F be two Banach spaces. Then $M_n(CD_0(K,E),F)$ is isometrically isomorphic to $l^1(K, L(E,F))$.*

4. Let K be a compact Hausdorff space without isolated points, E and F be two Banach lattices with F Dedekind complete. Then we define $l^1(K, L_n^r(E,F))$ as the set of all maps $\varphi = \varphi(k)$ from K into $L_n^r(E,F)$ satisfying

$$\sum_k |\varphi(k)|(|f(k)|) \in F$$

where $f \in CD_0(K,E)$ and $\sum_k |\varphi(k)|(f_\alpha(k)) \downarrow_\alpha 0$ whenever $f_\alpha \downarrow 0$ in $CD_0(K,E)$. Then we have the following result.

Theorem 5. *Let K, E and F be as above definition. Then $L_n^r(CD_0(K,E),F)$ is isometrically lattice isomorphic to $l^1(K, L_n^r(E,F))$.*

5. Let K be a compact Hausdorff space without isolated points, E and F be two Banach lattices with F Dedekind complete. Then we define $l_w^1(K, L_n^r(E,F))$ as the set of all maps $\varphi = \varphi(k)$ from K into $L_n^r(E,F)$ satisfying

$$\sum_k |\varphi(k)|(|f(k)|) \in F$$

where $f \in CD_w(K,E)$ and $\sum_k |\varphi(k)|(f_\alpha(k)) \downarrow_\alpha 0$ whenever $f_\alpha \downarrow 0$ in $CD_w(K,E)$. Then we get the following result.

Theorem 6. *Let K, E and F be as above definition. Then $L_n^r(CD_w(K,E),F)$ is isometrically lattice isomorphic to $l_w^1(K, L_n^r(E,F))$.*

The main results presented above are new and original. These theorems and methods applied will be useful for further investigations of dominated operators on Abramovich–Wickstead type spaces of vector-valued functions.

1.4 Methods Applied

This work uses essentially the methods and technical tools from the following branches of modern analysis: Theory of vector lattices and positive operators, theory of dominated operators in lattice-normed spaces, and theory of vector measures (with values in Banach spaces and vector lattices). In particular, we use intensively the following concepts: order continuity, vector measures with bounded variation, norm order completeness and norm uniformly completeness of a lattice-normed space, decomposability of the majorant norm of a dominated operator, spaces with mixed norms, integration with respect to operator-valued measure, etc.

1.5 Publications and Reports

Some results of this thesis were published in the following two papers.

1) F. Polat, *Linear Operators on Abramovich-Wickstead type spaces*, Vladikavkaz Math. J.

(10), 46-55 (2008);

2) F. Polat, *Dominated Operators on Some Lattice-Normed spaces*, Proceedings of International Conference Order Analysis and Related Problems of Mathematical Modeling, Vladikavkaz, June 1-7, 2008.

Besides, some results of the thesis were delivered in the following seminars and symposium.

1) Joint Seminar on Analysis in the IAMI (Vladikavkaz, Russia, March 2008);

2) Seminar on Positivity and Its Applications (METU, Ankara, 13 June 2008);

3) International Symposium ”Positivity and Its Applications in Science and Economics” (Bolu, 17-19 September 2008).

CHAPTER 2

PRELIMINARIES

In this chapter, for the convenience of the reader, we present a general background needed in this thesis. For Riesz space theory, the reader can consult the book [10] by C. D. Aliprantis and O. Burkinshaw.

2.1 Vector Measures

In this section, we collect some necassary materials for this thesis. For more information about vector measures, we refer to [6, 12, 19].

1. Consider a nonempty set K and a σ-algebra $\mathcal{A}$ of the subsets of K. Let E be a Dedekind complete vector lattice. We shall call the mapping $\mu : \mathcal{A} \to E$ an (E-*valued*) measure if

(1) $\mu(\emptyset) = 0$,

(2) Whenever $\{A_n\}$ (n=1,2,...) is a sequence of pairwise disjoint elements of $\mathcal{A}$, then

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n) := o\text{-}\lim_{n} \sum_{k=1}^{n} \mu(A_k).$$

We say that a measure μ is *positive* and write $\mu \geq 0$ if $\mu(A) \geq 0$ for all $A \in \mathcal{A}$ and *bounded* if there exists $e \in E^+$ such that $|\mu(A)| \leq e$ for each $A \in \mathcal{A}$. We denote the set of all bounded E-valued measures on a σ-algebra $\mathcal{A}$ by $ca(K, \mathcal{A}, E)$. If $\mu, \nu \in ca(K, \mathcal{A}, E)$ and $t \in \mathbb{R}$, then we put by definition

(1) $(\mu + \nu)(A) := \mu(A) + \nu(A)$ $(A \in \mathcal{A})$;

(2) $(t\mu)(A) := t\mu(A)$ $(A \in \mathcal{A})$;

(3) $\mu \geq \nu \Leftrightarrow \mu - \nu \geq 0$.

One can prove that $ca(K, \mathcal{A}, E)$ is a Dedekind complete vector lattice. In particular, for every measure $\mu : \mathcal{A} \to E$, $\mu^+ := \mu \vee 0$ and $\mu^- := (-\mu)^+ = -(\mu \wedge 0)$ are the positive and negative parts respectively. It is easy to verify that

$$\mu^+(A) = \sup\{\mu(A') : A' \in \mathcal{A},\ A' \subset A\} \quad (A \in \mathcal{A}).$$

In the sequel, we shall consider special E-valued measures. Suppose that K is a compact topological space and $\mathcal{A}$ is the Borel σ-algebra. A positive measure $\mu : \mathcal{A} \to E$ is said to be *regular* if for every $A \in \mathcal{A}$ we have

$$\mu(A) = \inf\{\mu(U) : A \subset U,\ U \in Op(K)\}$$

where $Op(K)$ is the collection of all open subsets of K. If the latter condition is true only for closed $A \in \mathcal{A}$, then μ is called *quasiregular.* Finally, an arbitrary measure $\mu : \mathcal{A} \to E$ is said to be *regular* (*quasiregular*) if the positive measures μ^+ and μ^- are regular (quasiregular). Let $rca(K, E)$ and $qca(K, E)$ be the sets of E-valued Borel measures, regular and quasiregular respectively. It is seen from the definitions that $rca(K, E)$ and $qca(K, E)$ are vector sublattices in $ca(K, \mathcal{A}, E)$. Clearly, the supremum (infimum) of the increasing (decreasing) family of quasiregular measures bounded in $ca(A, \mathcal{A}, E)$ will also be quasiregular. The same holds for regular measures. Thus $qca(K, E)$ and $rca(K, E)$ are Dedekind complete vector lattices.

2. Now we will define the integral with respect to an arbitrary measure $\mu \in \mathrm{ca}(K, \mathcal{A}, E)$.

(1) Let us denote by $St(K, \mathcal{A})$ the set of all functions $\varphi : K \to \mathbb{R}$ of the form $\varphi = \sum_{k=1}^{n} a_k \chi_{A_k}$, where $A_1, \dots, A_n \in \mathcal{A}$, $a_1, \dots, a_n \in \mathbb{R}$, and χ_A is the characteristic function of a set A. Construct the operator $I_\mu : St(K, \mathcal{A}) \to E$ by putting

$$I_\mu\left(\sum_{k=1}^{n} a_k \chi_{A_k}\right) := \sum_{k=1}^{n} a_k \mu(A_k).$$

As it is seen I_μ is a linear operator; moreover, the normative inequality holds

$$|I_\mu(f)| \leq \|f\|_\infty |\mu|(K) \quad (f \in St(K, \mathcal{A})),$$

where $\|f\|_\infty := \sup_{\alpha \in \mathfrak{A}} |f(\alpha)|$. The subspace $St(K, \mathcal{A})$ is dense with respect to the norm in the space $l_\infty(K, \mathcal{A})$ of all bounded measurable functions. Therefore I_μ admits a unique linear extension (by continuity) to $l_\infty(K, \mathcal{A})$, with the above-mentioned normative inequality being preserved. In particular, if K is a compact space and $\mathcal{A}$ is the Borel σ-algebra, then $I_\mu(f)$ is defined for every continuous function $f \in C(K)$. Note also that $I_\mu \geq 0$ if and only if $\mu \geq 0$.

3. Now we give several results about analytical representation of linear operators which yields new formulas of subdifferentiation.

Suppose that for every $n \in \mathbb{N}$ a directed set $\mathrm{A}(n)$ is given. Take a sequence of decreasing nets $(e_{\alpha,n})_{\alpha\in\mathrm{A}(n)} \subset [0, e]$ in a Dedekind complete vector lattice E such that $\inf\{e_{\alpha,n} : \alpha \in \mathrm{A}(n)\} = 0$ for each $n \in \mathbb{N}$. If for any such sequence the equality

$$\inf_{\varphi\in\mathrm{A}} \sup_{n\in\mathbb{N}} e_{\varphi(n),n} = 0, \quad \mathrm{A} := \prod_{n\in\mathbb{N}} \mathrm{A}(n)$$

holds, then we call Dedekind complete vector lattice E as (σ, ∞)-distributive. For a Dedekind complete vector lattice of countable type (= with the countable chain condition) the property of (σ, ∞)-distributivity is equivalent to the *regularity* of the base. The latter means that the *diagonal principle* is fulfilled in the Boolean algebra $\mathcal{B}(E)$: if a double sequence $(b_{n,m})_{n,m\in\mathbb{N}}$ in $\mathcal{B}(E)$ is such that for every $n \in \mathbb{N}$ the sequence $(x_{n,m})_{m\in\mathbb{N}}$ decreases and o-converges to zero then there exists a strictly increasing sequence $(m(n))_{n\in\mathbb{N}}$ for which $o\text{-}\lim_{n\to\infty} x_{n,m(n)} = 0$.

The following theorem belongs to J. D. M. Wright [15].

Theorem 2.1 (Wright) *Let K be a compact topological space and let E be an arbitrary Dedekind complete vector lattice. The mapping $\mu \mapsto I_\mu$ implements a linear and lattice isomorphism of Dedekind complete vector lattices $qca(K, E)$ and $L^r(C(K), E)$.*

Theorem 2.2 *Let a Dedekind complete vector lattice E be (σ, ∞)-distributive. Then*

$$qca(K, E) = rca(K, E).$$

In addition, the mapping $\mu \mapsto I_\mu$ implements a linear and lattice isomorphism of Dedekind complete vector lattices $rca(K, E)$ and $L^r(C(K), E)$.

We omit the proofs of the Wright theorem and its improvements contained in Theorem 2.2, which demand considerations that are rather long and laborious in a technical sense.

2.2 Lattice–Normed Spaces

In this section, we give some definitions about lattice-normed spaces. We also collect some results concerning dominated operators which are related to lattice-normed spaces. For more details and proofs of theorems, the reader can consult the book [4] by A. G. Kusraev.

Let X be a vector space and E be a real vector lattice. A mapping $|.| : X \to E_+$ is called a *vector (E-valued) norm* if it satisfies the following axioms:

(1) $|x| = 0 \Leftrightarrow x = 0 \ (x \in X)$;

(2) $|\lambda x| = |\lambda||x| \ (\lambda \in \mathbb{R}, x \in X)$;

(3) $|x + y| \leq |x| + |y| \ (x, y \in X\ )$.

A vector norm is called *decomposable* or *Kantorovich norm* if

(4) for all $e_1, e_2 \in E_+$ and $x \in X$, from $|x| = e_1 + e_2$, it follows that there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $|x_k| = e_k$ ($k = 1, 2$).

A triple $(X, |.|, E)$ is a *lattice-normed space* if $|.|$ is an E-valued norm in the vector space X. The space E is called the *norm lattice* of X. If the vector norm is decomposable, then the space $(X, |.|, E)$ is called decomposable.

If $|x| \wedge |y| = 0$, then we call the elements $x, y \in X$ *disjoint* and write $x \perp y$. As in the case of a vector lattice, a set $M^{\perp} = \{x \in X : x \perp y \ for\ each\ y \in M\}$, with $\emptyset \neq M \subset X$ is called a *band* or a *component*.

Lemma 2.3 $[4, 2.1.2]$ *If the elements $x, y \in X$ are disjoint, then we have that* $|x+y| = |x|+|y|$.

Proof. Indeed, from the relations $|x| \wedge |y| = 0$ and $|x| \leq |x + y| + |y|$, we infer that

$$|x| \leq (|x + y| + |y|) \wedge |x| \leq |x + y| \wedge |x| \leq |x + y|.$$

Similarly, $|y| \leq |x + y|$; therefore, $|x| + |y| = |x| \vee |y| \leq |x + y|$. □

We now give the following important property of disjoint elements in the lattice-normed space $(X, |.|, E)$.

Lemma 2.4 [4, 2.1.3] *For every pair of disjoint elements* $e_1, e_2 \in E$*, then the decomposition* $x = x_1 + x_2$ *in* X *with* $|x_1| = e_1$ *and* $|x_2| = e_2$ *is unique.*

Proof. Assume $|x_1| = |y_1| = e_1$ and $|x_2| = |y_2| = e_2$, and $x = x_1 + x_2 = y_1 + y_2$. Then $x_1 - y_1 \perp x_2 - y_2$, since $|x_1 - y_1| \leq |x_1| + |y_1| = 2e_1$ and $|x_2 - y_2| \leq 2e_2$. By the previous lemma,

$$0 = |(x_1 - y_1) + (x_2 - y_2)| = |x_1 - y_1| + |x_2 - y_2|,$$

whence $x_1 = y_1$ and $x_2 = y_2$. □

Example 2.5 *In the definition above, if* $X = E$*, then the modulus of an element serves as its vector norm:* $|x| = x \vee (-x)$, $(x \in E)$*. Decomposability of this norm easily follows from the Riesz decomposition property holding in every vector lattice. If* $E = \mathbb{R}$*, then* X *is a normed space. We can use the conventional notation* $\|.\|$ *for the norm and omit references to the order structure of the norm lattice.*

Definition 2.6 [4, 2.1.5] *Let* $(X, |.|, E)$ *be a lattice-normed space.*

(1) *We say that a net* $(x_\alpha)_{\alpha \in A}$ *bo−converges to an element* $x \in X$ *and we write* $x = bo\text{-}\lim x_\alpha$ *if there exists a decreasing net* $(e_\gamma)_{\gamma \in \Gamma}$ *in* E *such that* $\inf_{\gamma \in \Gamma} e_\gamma = 0$ *and, for every* $\gamma \in \Gamma$*, there exists an index* $\alpha(\gamma) \in A$ *such that* $|x - x_\alpha| \leq e_\gamma$ *for all* $\alpha \geq \alpha(\gamma)$*.*

(2) *Let* $(x_\alpha)_{\alpha \in A}$ *be a net in* X*. Given an element* $e \in E^+$*, let the following condition be satisfied: for every* $\varepsilon > 0$*, there is an index* $\alpha(\varepsilon) \in A$ *such that* $|x - x_\alpha| \leq \varepsilon e$ *for all* $\alpha \geq \alpha(\varepsilon)$*. Then we say that* (x_α) *br-converges to* x *and write* $x = br\text{-}\lim x_\alpha$*.*

(3) *A net* (x_α) *is said to be bo−fundamental (br−fundamental) if the net* $(x_\alpha - x_\beta)_{(\alpha,\beta) \in A \times A}$ *bo−converges (br−converges) to zero.*

(4) $(X, |.|, E)$ *is called bo−complete (br−complete) if every bo-fundamental net (br−fundamental net) in it bo−converges (br−converges) to an element of the space.*

Recall that a normed (Banach) lattice is a vector lattice E that is simultenously a normed (Banach) space whose norm is monotone in the following sense: if $|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$ $(x, y \in E)$. If $(X, |.|, E)$ is a lattice-normed space with E a norm, then by definition, $|x| \in E$ for each $x \in X$ and we introduce a mixed norm in X by the formula

$$|||x||| = \|\, |x|\, \|, \quad (x \in X).$$

In this case, the normed space $(X, |||.|||)$ is called a space with *mixed norm*. Using the inequality $|\,|x| - |y|\,| \leq |x - y|$ and monotonicity of the norm in E, we have

$$\|\,|x| - |y|\,\| \leq |||x - y||| \quad (x, y \in X),$$

so that the vector norm $|.|$ is a norm continuous mapping from $(X, |||.|||)$ into E.

A *Banach space with mixed norm* is a pair (X, E) in which E is a Banach lattice and X is a $br-$complete lattice-normed space with E-valued norm.

The following proposition justifies the definition.

Proposition 2.7 [4, 7.1.2] *Let E be a Banach lattice. Then $(X, |||.|||)$ is a Banach space if and only if the lattice-normed space $(X, |.|, E)$ is complete with respect to relative uniform convergence.*

Proof. $(\Leftarrow)$ Take a Cauchy sequence $(x_n) \subset X$. Without loss of generality, we may assume that $|||x_{n+1} - x_n||| \leq \frac{1}{n^3}$, $n \in \mathbb{N}$. Assign

$$e_n = |x_1| + \sum_{k=1}^{n} k|x_{k+1} - x_k|, \quad n \in \mathbb{N}.$$

Then we may estimate

$$\|e_{n+l} - e_n\| = \left\| \sum_{k=n+1}^{n+l} k|x_{k+1} - x_k| \right\| \leq \sum_{k=n+1}^{n+l} k|||x_{k+1} - x_k||| \leq \sum_{k=n+1}^{n+l} \frac{1}{k^2} \to 0$$

when $k, l \to \infty$. Thus the sequence (e_n) is a Cauchy sequence and it has a limit $e = \lim_{n\to\infty} e_n$. Since $e_{n+k} \geq e_n$, $(n, k \in \mathbb{N})$, we have $e = \sup e_n$. If $n \geq m$, then

$$m|x_{n+l} - x_n| \leq \sum_{k=n+1}^{n+l} k|x_{k+1} - x_k| \leq e_{n+l} - e_n \leq e;$$

consequently, $|x_{n+l} - x_n| \leq (\frac{1}{m})e$. This means that the sequence (x_n) is br-fundamental and so that the limit $x := br - \lim_{n\to\infty} x_n$ exists. It is clear that $\lim_{n\to\infty} |||x - x_n||| = 0$.

$(\Rightarrow)$ Suppose that a sequence $(x_n) \in X$ is br-fundamental; i.e., $|x_n - x_m| \leq \lambda_k e$ $(m, n, k \in \mathbb{N}$ *and* $m, n \geq k)$ where $0 \leq e \in E$ and $\lim_{k\to\infty} \lambda_k = 0$. Then

$$|||x_n - x_m||| \leq \lambda_k \|e\| \to 0 \text{ } as \text{ } k \to \infty$$

Therefore the limit $x := \lim_{n\to\infty} x_n$ exists. By continuity of the vector norm, we have

$$|x - x_n| \leq \lambda_k e \quad (n \geq k),$$

therefore, $x = br$-$\lim x_n$. □

Definition 2.8 $[4, 4.1.1]$ *Let $(X, |.|, E)$ and $(Y, |.|, F)$ be two lattice-normed spaces. Then a linear operator $T : X \rightarrow Y$ is said to be dominated or majorized if there exists a positive linear operator $S : E \rightarrow F$ such that*

$$|Tx| \leq S(|x|) \quad (x \in X).$$

Remark 2.9 *Let F be a Dedekind complete vector lattice and $maj(T)$ be the set of all dominants of T. It is clear that $maj(T)$ is a convex set in the Dedekind complete vector lattice $\tilde{L}(E, F)$. If there is a least element in $maj(T)$ with respect to the order induced from $\tilde{L}(E, F)$, then it is called the least or exact dominant of T and denoted by $|T|$. Consequently $|T|$ is a positive linear operator from E to F. Moreover $|T| = \inf maj(T) \in maj(T)$ and the inequality*

$$|Tx| \leq |T|(|x|) \quad (x \in X)$$

holds. The set of all dominated operators from X to Y is denoted by $M(X, Y)$. Thus,

$$T \in M(X, Y) \Leftrightarrow \; maj(T) \neq \emptyset.$$

The following theorem gives the sufficient condition for a linear dominated operator to have an exact dominant.

Theorem 2.10 $[4, 4.1.2]$ *Let $(X, |.|, E)$ and $(Y, |.|, F)$ be two lattice normed spaces with X decomposable and F Dedekind complete. Then every dominated operator $T : X \rightarrow Y$ has exact dominant $|T|$.*

Theorem 2.11 $[4, 4.2.7]$ *Let $(X, |.|, E)$ and $(Y, |.|, F)$ be two decomposable lattice-normed spaces with Y bo-complete. Then for each dominated operator $T : X \rightarrow Y$ and each representation $|T| = S_1 + S_2$ where $0 \leq S_1, S_2 : E \rightarrow F$, there exist dominated operators $T_1, T_2 : X \rightarrow Y$ such that $T = T_1 + T_2$ and $|T_k| = S_k$ $(k = 1, 2)$. If the operators S_1 and S_2 are disjoint, then there exists a unique pair of operators T_1 and T_1 satisfying the condition under consideration.*

Definition 2.12 $[4, 4.3.1]$ *Let $T \in M(X, Y)$. Then T is called norm order continuous or bo-continuous if for every net $(x_\alpha) \subset X$, from the equality bo-$\lim_\alpha x_\alpha = 0$ it follows that bo-$\lim_\alpha T(x_\alpha) = 0$. That is to say, it follows from $|x_\alpha| \rightarrow^o 0$ in E that $|T(x_\alpha)| \rightarrow^o 0$ in F where E and F are norm lattices of X and Y respectively. The set of all bo-continuous operators $T \in M(X, Y)$ will be denoted by $M_n(X, Y)$.*

We now give the following characterization about *bo*-order continuous dominated operators.

Theorem 2.13 [4, 4.3.4] *Let X be a decomposable lattice-normed space and F be a Dedekind complete vector lattice. Then a dominated operator $T : X \to Y$ is bo-order continuous if and only if its least dominant $|T|$ is order continuous.*

CHAPTER 3

SOME GENERALIZATIONS OF ABRAMOVICH – WICKSTEAD SPACES

Recall that a topological space is called basically disconnected if the closure of any F_σ open set is open. A compact Hausdorff space that is basically disconnected is called *quasi-Stonean space*. For a quasi-Stonean space K without isolated points, the following function spaces were introduced by Y. A. Abramovich and A. W. Wickstead in [33].

$$l_w^\infty(K) = \{f : f \ is\ real\ valued,\ bounded\ and\ \{k : f(k) \neq 0\}\ is\ countable\};$$

$$c_0(K) = \{f : f\ is\ real\ valued\ and\ \{k : |f(k)| > \varepsilon\}\ is\ finite\ for\ each\ \varepsilon > 0\}.$$

These spaces were used to define $CD_0(K) = C(K) \oplus c_0(K)$ and $CD_w(K) = C(K) \oplus l_w^\infty(K)$ where $C(K)$ is the space of real valued continuous functions on K. Both of the spaces $CD_0(K)$ and $CD_w(K)$ are Banach lattices under the pointwise order and supremum norm. These types of spaces can be called Abramovich–Wickstead spaces, or shortly AW-spaces as in [35]. Further Alpay and Ercan [28] showed that for a compact Hausdorff space K without isolated points, $CD_0(K)$ is isometrically isomorphic to $C(K) \otimes c_0(K)$ where the order on $C(K) \otimes c_0(K)$ is defined as follows:

$$0 \leq (f, g) \Longleftrightarrow 0 \leq f(k)\ and\ 0 \leq f(k) + g(k)\ for\ each\ k \in K.$$

They also proved that $|(f, g)| = (|f|, |f + g| - |f|)$ for each $f \in C(K)$ and $g \in c_0(K)$. This motivated us to define a new Riesz space product of two Riesz spaces under the similar order above. In this chapter, we investigate order properties of this new Riesz space. In particular, we construct some concrete examples of this new Riesz space.

3.1 The Representation of $E \times_G F$–space

For ordered vector subspaces E and F of an ordered vector space G, we consider coordinate-wise algebraic operations on $E \times F$, that is,

$$(x, y) + (x', y') = (x + x', y + y') \text{ and } \lambda(x, y) = (\lambda x, \lambda y)$$

for each $(x, y) \in E \times F$ and for each $\lambda \in \mathbb{R}$.

Definition 3.1 *Let E and F be ordered vector subspaces of an ordered vector space G. We define an order on $E \times F$ as follows:*

$$0 \leq (x, y) \in E \times F \Leftrightarrow 0 \leq x \textit{ and } 0 \leq x + y \textit{ in } G.$$

$E \times_G F$ denotes the space $E \times F$ equipped with this order.

Now we give another definition which will be useful in the next theorem. For the details see [26].

Definition 3.2 *Let G be a Riesz space, and let X and Y be vector subspaces of G. We call X an order ideal with respect to Y if $|x + y| - |y| \in X$ for all $x \in X$, $y \in Y$.*

Theorem 3.3 *Let X be a vector subspace of a Riesz space G. Then X is a Riesz subspace of G if and only if it is an ideal with respect to $\{0\}$, and X is an order ideal if and only if it is an ideal with respect to G.*

Theorem 3.4 *Let E and F be ordered vector subspaces of a Riesz space G. Then $E \times_G F$ is a Riesz space if E is a Riesz subspace and F is an order ideal with respect to E. Moreover,*

$$|(x, y)| = (|x|, |x + y| - |x|),$$

for all $(x, y) \in E \times_G F$. In particular, if F is an ideal in G, then $G \times_G F$, $G \times_G G$ and $0 \times_G F$ are Riesz spaces.

Proof. Since $(0, 0) \leq (x, y)$ and $(x, y) \leq (0, 0)$ imply that $0 \leq x$, $0 \leq x + y$, $0 \leq -x$ and $0 \leq -x - y$, we have $x = 0$ and $x + y = 0$ so $y = 0$, then $E \times_G F$ is an ordered vector space.

Let $(x, y) \in E \times_G F$. Since F is an order ideal with respect to E, we have $|x+y| - |x| \in F$. Also $0 \leq |x| - x$, $0 \leq |x+y| - |x| - y + |x| - x$ and $0 \leq |x| + x$, $0 \leq |x+y| - |x| + y + |x| + x$, imply that $(x, y) \leq (|x|, |x+y| - |x|)$ and $(-x, -y) \leq (|x|, |x+y| - |x|)$, i.e, $(|x|, |x+y| - |x|)$ is an upper bound for $\{(x, y), -(x, y)\}$. Suppose that (g, p) is another upper bound of $\{(x, y), -(x, y)\}$. Then,

$$0 \leq g - x,\ 0 \leq g + p - x - y,\ 0 \leq x + g \text{ and } 0 \leq x + g + p + y.$$

This shows that $(|x|, |x + y| - |x|) \leq (g, p)$ and so $|\ (x, y)\ | = (|\ x\ |, |\ x + y\ | - |\ x\ |)$. In particular, since G is an ideal in itself, $G \times_G G$ is a Riesz space. Also by the previous theorem, F is an ideal with respect to $\{0\}$ and F is an order ideal with respect to G, so $0 \times_G F$ and $G \times_G F$ are Riesz spaces. □

Example 3.5 *(1) Let E and F be Riesz spaces. Then $G = E \times F$ is a Riesz space under the pointwise order such that $|(x, y)| = (|x|, |y|)$. Let $E_0 = \{(x, 0) : x \in E\}$ and $F_0 = \{(0, y) : y \in F\}$. Then E_0 and F_0 are Riesz subspaces of G. Since*

$$|(x, 0) + (0, y)| - |(x, 0)| = |(x, y)| - (|x|, 0) = (|x|, |y|) - (|x|, 0) = (0, |y|) \in F_0,$$

F_0 is an order ideal with respect to E_0. By previous theorem, $E_0 \times_G F_0$ is a Riesz space.

(2) Let G be a Riesz space, E be a Riesz subspace of G and F be an ordered subspace of G. If F is an ideal in E, then F is an order ideal with respect to E by the first theorem. So $E \times_G F$ is a Riesz space by the previous theorem.

(3) Suppose that a Riesz subspace F is an order ideal with respect to a Riesz subspace E in a Riesz space G. Let E' be a Riesz subspace of E. Let $x \in E'$ and $y \in F$. Then $|x + y| - |x| \in F$ as $E' \subset E$. So F is an order ideal with respect to all Riesz subspaces of E.

(4) Suppose that a Riesz subspace F is an order ideal with respect to Riesz subspaces E_α for each α in a Riesz space G. Then F is an ideal with respect to $\bigcap E_\alpha$. To see this, let $x \in F$ and $y \in \bigcap E_\alpha$, then $|x + y| - |y| \in F$ as $y \in E_\alpha$ for each α.

(5) Let G, G' be Riesz spaces and $T : G \to G'$ be a lattice homomorphism. Then if E and F are Riesz subspaces of G for which F is an order ideal with respect to E. Then we have that the Riesz subspace $T(F)$ of G' is an order ideal with respect to the Riesz subspace $T(E)$ of G'. Similarly, if F' is a Riesz subspace of G' which is an order ideal with respect to a Riesz subspace E' of G', then the Riesz subspace $T^{-1}(F')$ of G is an order ideal with respect to the Riesz subspace $T^{-1}(E')$ of G.

(6) Let E and F_α be Riesz subspaces of a Riesz space G for each α. Suppose F_α is an order ideal with respect to E for each α. Then $\bigcup F_\alpha$ is a Riesz subspace of G which is an order ideal with respect to E.

(7) Let G be a Riesz space, E be a Riesz subspace of G and F be an ordered subspace of G. Suppose that F is an order ideal with respect to E. Then, the ideal $I(F)$ generated by F in G is also an ideal with respect to E. That is to say, $|x + y| - |x| \in I(F)$ for each $x \in E$ and $y \in I(F)$. Suppose $y \in I(F)$, then there exists $f \in F$ such that $|y| \leq |f|$. But then $|\,|x + y| - |x|\,| \leq |x + y - x| = |y| \leq |f|$. This implies $|x + y| - |x| \in I(F)$. Thus $E \times_G I(F)$ is a Riesz space.

(8) Let E, F and G be as in (7). Suppose F is an order ideal in G. If F is an order ideal with respect to E, then F is an order ideal with respect to the ideal $I(E)$ generated by E in G. We have to show that for every $x \in I(E)$ and $y \in F$, $|x+y|-|x| \in F$. But $|\,|x+y|-|x|\,| \leq |x+y-x| = |y|$. Since $y \in F$ and F is an order ideal in G, we have $|x + y| - |x| \in F$. Thus $I(E) \times_G F$ is a Riesz space.

(9) Suppose that F is an order ideal with respect to E. Then $I(F)$ is an order ideal with respect to $I(E)$. That is to say, $|x + y| - |x| \in I(F)$ for all $x \in I(E)$ and $y \in I(F)$. But $|\,|x + y| - |x|\,| \leq |x + y - x| = |y|$ and $|y| \leq |f|$ for some $f \in F$. So $|x + y| - |x| \in I(F)$. Thus $I(E) \times_G I(F)$ is a Riesz space.

(10) Suppose that F is an ideal in E. Then $I(F)$ is an ideal in $I(E)$. Clearly, $F \subset E$ implies $I(F) \subset I(E)$. Assume that $|x| \leq |y|$ for all $y \in I(F)$ and $x \in I(E)$. Since $y \in I(F)$, there exists some $f \in F$ such that $|y| \leq |f|$. This implies that $x \in I(F)$. So $I(F)$ is an order ideal in $I(E)$. Thus $I(E) \times_G I(F)$ is a Riesz space.

(11) Suppose that F is an ideal with respect to E in a normed Riesz space G. Let $x \in E$ and $y \in \overline{F}$. Then there exists a sequence $(y_n) \subset F$ such that $y_n \to y$. As the lattice operations are continuous, $|x + y_n| - |x| \to |x + y| - |x|$. As $|x + y_n| - |x| \in F$ for each $n \in \mathbb{N}$, $|x + y| - |x| \in \overline{F}$. So $E \times_G \overline{F}$ is a Riesz space. As a corollary to this, we obtain that if G is a Banach lattice and if F is an order ideal with respect to E, then the completion F is alsa an ideal with respect to E.

(12) Suppose that F is an ideal with respect to E in a normed Riesz space G. Suppose F is closed. Let $x \in \overline{E}$ and $y \in F$. Then there exists a sequence $(x_n) \subset E$ such that $x_n \to x$. As the

lattice operations are continuous, we have that $|x_n+y|-|x_n| \to |x+y|-|x|$. *As* $|x_n+y|-|x_n| \in F$ *for each* $n \in \mathbb{N}$, $|x+y|-|x| \in F$. *So* $\overline{E} \times_G F$ *is a Riesz space.*

(13) Let T *be a lattice homomorphism on the Riesz space* E, *then* $T^{-1}(\{0\})$ *is an order ideal with respect to* $T(E)$.

Recall that a sequence (x_n) in a Riesz space L is said to be order convergent to an element x of L (denoted by $x_n \to^o x$) if there exists a sequence $y_n \downarrow 0$ such that $|x_n - x| \leq y_n$.

We can characterize order convergent sequences in $E \times_G F$ space as follows.

Proposition 3.6 *If* $x_n \to^o x$ *in* E *and* $y_n \to^o y$ *in* F, *then* $(x_n, y_n) \to^o 0$ *in* $E \times_G F$.

Proof. Let $x_n \to^o x$ in E and $y_n \to^o y$ in F. Then there exist sequences (p_n) in E and (r_n) in F such that $|x_n - x| \leq p_n \downarrow 0$ and $|y_n - y| \leq r_n \downarrow 0$. So $|(x_n, y_n) - (x, y)| = |(x_n - x, y_n - y| = (|x_n - x|, |y_n - y + x_n - x| - |x_n - x|) \leq (|x_n - x|, |y_n - y|) \leq (p_n, r_n)$. But $p_n \downarrow 0$ and $r_n \downarrow 0$ imply $(p_n, r_n) \downarrow 0$ in $E \times_G F$ and this completes the proof. □

3.2 Ideals and Central Operators in $E \times_G F$–space

In this section, we deal with the ideals and central operators of $E \times_G F$-spaces. First we turn our attention to characterize principal ideals of E and G by using the principal ideals of $E \times_G F$-spaces. Next proposition contains this characterization.

Proposition 3.7 *If* (x, y) *belongs to the principal ideal generated by* (x_0, y_0) *in* $E \times_G F$, *then* x *belongs to the principal ideal generated by* x_0 *in* E *and* $x + y$ *belongs to the principal ideal generated by* $x_0 + y_0$ *in* G.

Proof. It is enough to consider only positive elements. Let (x_0, y_0) be a positive element of $E \times_G F$ and $I_{(x_0,y_0)}$ be the order ideal generated by (x_0, y_0) in $E \times_G F$. Let (x, y) be a positive element of $I_{(x_0,y_0)}$. Then there exists $\lambda > 0$ such that $0 \leq (x, y) \leq \lambda(x_0, y_0)$ in $E \times_G F$. By definition of the order in $E \times_G F$, this yields $0 \leq x \leq \lambda x_0$ and $0 \leq x + y \leq \lambda(x_0 + y_0)$, and this completes the proof. □

It is a natural task to consider projections of $E \times_G F$ onto E and investigate its properties. Next proposition is dealt with this.

Proposition 3.8 *Let E be a Riesz subspace of G and F be an order ideal with respect to E. Then 1) $(x_\alpha, y_\alpha) \downarrow (0,0)$ in $E \times_G F$ implies $x_\alpha \downarrow 0$ in E and $x_\alpha + y_\alpha \downarrow 0$ in G;*

2) The projection map $P : E \times_G F \longrightarrow E$ defined as $P(x,y) = x$ is an order continuous lattice homomorphism.

Proof. 1)Let $(x_\alpha, y_\alpha) \downarrow (0,0)$ in $E \times_G F$. Then $0 < (u,v) \leq (x_\alpha, y_\alpha)$ implies that $u \leq 0$ and $u + v \leq 0$ so that $x_\alpha \downarrow 0$ in E and $x_\alpha + y_\alpha \downarrow 0$ in G.

2) Since $P(|(x,y)|) = P(|x|, |x+y| - |x|) = |x| = |P(x,y)|$, P is a lattice homomorphism. The fact that P is order continuous follows from (1). $\square$

Example 3.9 (1) *For each ideal I in E, $P^{-1}(I)$ is an ideal in $E \times_G F$. That is to say the set $\{(x,y) : x \in I, y \in F\}$ is an ideal of $E \times_G F$.*

(2) *$I \times_G F$ and $I_{x_0} \times_G F$ are ideals of $E \times_G F$ where I is the ideal in E and I_{x_0} is a principal ideal generated by x_0 in E.*

(3) *Let J be an order ideal of the Riesz space F, then $(0, J) = \{(0, x) : x \in J\}$ is an order ideal of $E \times_G F$. If I is an order ideal of E with $I \cap F = \{0\}$, then $(I, 0)$ is an order ideal in $E \times_G F$. Thus we have that $(I, 0) + (0, J) = (I, J)$ is an order ideal of $E \times_G F$.*

(4) *Identifying E with $E_0 = \{(x,0) : x \in E\}$, we see, in fact, that P is a projection of $E \times_G F$ onto E. In fact if E_1 is a subspace of E for which there exists a projection P_1 from E onto E_1, then the subspace E_1 is the image (onto) of the projection $P_1 o P : E \times_G F \longrightarrow E_1$. Suppose now E_1 is a Riesz subspace of E. If F is an order ideal with respect to E, then automatically an order ideal with respect to E_1. That is to say, $|x+y| - |x| \in F$ for each $x \in E_1$. Thus for each Riesz subspace E_1 of E, $E_1 \times_G F$ is a Riesz subspace of $E \times_G F$. Since P is a Riesz homomorphism, for each Riesz subspace H of $E \times_G F$, $P(H)$ is a Riesz subspace of E*

(5) *If E is not a uniformly complete Riesz space, then $E \times_G F$ is not a uniformly complete Riesz space, see* [32, Thm.59.3].

For the following definition, see [32].

Definition 3.10 *The Riesz homomorphism π of L into M is called a Riesz σ-homomorphism if π preserves countable suprema, i.e., if $f = \sup f_n (n = 1, 2, ...)$ in L then $\pi f = \sup \pi f_n$ holds in M.*

A sufficient condition for Riesz homomorphism π to be Riesz σ – homomorphism is that $f_n \downarrow 0$ in L^+ implies $\pi f_n \downarrow 0$ in M^+. Evidently, the kernel of a Riesz σ-homomorphism is a σ-ideal in L.

The projection map $P : E \times_G F \longrightarrow E$ defined as $P(x, y) = x$ is a Riesz σ-homomorphism since $(x_n, y_n) \downarrow (0, 0)$ in $E \times_G F$ implies $x_n \downarrow 0$ in E. We immediately have the following results by using $[32, Thm.18.11]$.

Proposition 3.11 (1) *$KerP = \{0\} \times_G F$ is a σ-ideal in $E \times_G F$.*

(2) *For any σ-ideal N of E, $P^{-1}(N) = N \times_G F$ is a σ-ideal in $E \times_G F$.*

(3) *For any ideal A in $E \times_G F$, $P(A)$ is an ideal in E.*

Definition 3.12 *The Riesz homomorphism π of L into M is called a normal Riesz homomorphism if π preserves arbitrary suprema, i.e., if it follows from $f = \sup f_\alpha$ (where α runs through an arbitrary index set) in L that $\pi f = \sup \pi f_\alpha$ holds in M.*

A sufficient condition for the Riesz homomorphism π to be a normal Riesz homomorphism is that $0 \leq f_\alpha \uparrow f$ in L^+ implies $0 \leq \pi f_\alpha \uparrow \pi f$ in M^+ (or equivalently, $f_\alpha \downarrow 0$ in L^+ implies $\pi f_\alpha \downarrow 0$ in M^+). Evidently, the kernel of a normal Riesz homomorphism is a band in L.

The projection map $P : E \times_G F \longrightarrow E$ defined as $P(x, y) = x$ is a normal Riesz homomorphism since $(x_\alpha, y_\alpha) \downarrow (0, 0)$ in $E \times_G F$ implies $x_\alpha \downarrow 0$ in E. We immediately have the following results by using $[32, Thm.18.12]$.

Proposition 3.13 (1) *$KerP = \{0\} \times_G F$ is a band in $E \times_G F$.*

(2) *For any band N of E, $P^{-1}(N) = N \times_G F$ is a band in $E \times_G F$.*

(3) *For any projection band H in E, $H \times_G F$ is a projection band in $E \times_G F$.*

(4) *If f is an atom in E, then the band generated by f, B_f, is a projection band in E. So $B_f \times_G F$ is a projection band in $E \times_G F$.*

The following theorem is useful to characterize the disjoint complement of kernel of projection map P, $\{0\} \times_G F$ and Dedekind completeness of $E \times_G F$. For the details of the following theorem, see [$32, Thm.66.3$].

Theorem 3.14 *If L is Dedekind complete and π is a normal Riesz homomorphism from L onto M with kernel K_π, then M and disjoint complement $(K_\pi)^\perp$ are Riesz isomorphic, and so M is Dedekind complete.*

By using the previous theorem, we immediately have the following corollary.

Corollary 3.15 (1) $(K_P)^\perp = (\{0\} \times_G F)^\perp \cong E$

(2) *If E is not Dedekind complete, then $E \times_G F$ is not Dedekind complete.*

For further discussion, it will be convenient to introduce a property for the Riesz space L which is intermediate between the principal projection property that is to say every principal band in L is a projection band and the Archimedean property. For the details see [32].

Definition 3.16 *The Riesz space L is said to have sufficiently many projections if every nonzero band contains a nonzero projection band.*

Proposition 3.17 *If E has sufficiently many projections, then $E \times_G F$ has sufficiently many projections.*

Proof. Suppose that E has sufficiently many projections. Let B be a band in $E \times_G F$. Then $P(B)$ is a band in E. So $P(B)$ contains a nonzero projection band in E. Then $P^{-1}(P(B)) \subset B$ and $P^{-1}(P(B))$ is a projection band of $E \times_G F$. □

We now consider operators on E, G and on $E \times_G F$. For an operator T on G, T may not map E into E and F into F. One way to get around this, we may restrict our attention to central operators and assume that E and F are ideals of G. Since central operators map ideals to ideals, each central operator of the Riesz space G will give rise to an operator of $E \times_G F$ into itself. Another way is to assume that F is an order ideal in E. Then F is an order ideal with respect to E. Therefore we ensure that $E \times_G F$ is a Riesz space. Then if $T : E \to E$ is a central operator on E, the relation

$$\overline{T}(x, y) = (Tx, Ty)$$

gives rise to a central operator $\overline{T}$ on $E \times_G F$. Suppose $(x, y) \geq 0$ in $E \times_G F$, then $x \geq 0$ and $x + y \geq 0$ in G. Then we have that $-\lambda x \leq Tx \leq \lambda x$ and $-\lambda(x + y) \leq T(x + y) \leq \lambda(x + y)$. Thus $-\lambda(x, y) \leq (Tx, Ty) \leq \lambda(x, y)$. So the operator $\overline{T} : E \times_G F \longrightarrow E \times_G F$ is a central operator. Observe also that $\overline{T}$ is a positive operator whenever T is a positive operator and $\|\overline{T}\|_0 \leq \|T\|_0$ where $\|.\|_0$ denotes the order unit norm of $Z(E)$ and $Z(E \times_G F)$. So we get the following result.

Proposition 3.18 *Let E be a Riesz subspace of G and F be an order ideal in E. If $T : E \to E$ is a a positive central operator on E, the relation*

$$\overline{T}(x, y) = (Tx, Ty)$$

gives rise to a central operator $\overline{T}$ on $E \times_G F$ such that $\|\overline{T}\|_0 \leq \|T\|_0$ where $\|.\|_0$ denotes the order unit norm of $Z(E)$ and $Z(E \times_G F)$.

One is temped to conjecture that each central operator on $E \times_G F$ gives rise to a central operator on E and on F. The problem here is the following; an operator $\overline{T}$ on $E \times_G F$ may not map $E \cong \{(x, 0) : x \in E\}$ into itself. Similarly, it may not map $F \cong \{(0, y) : y \in F\}$ into itself. One way to approach this is to assume that E and F are order ideals of the big space G. But then: suppose that E is an ideal in G, then $|(x, y)| \leq |(x, 0)|$ imply that $(|x|, |x + y| - |x|) \leq (|x|, 0)$. This gives us $|x + y| \leq |x|$. So $x + y \in E$. Then $y \in E \Rightarrow F \subset E$, even if we assume that $F \cap E = \{0\}$. But, clearly if $E_0 = \{(x, 0) : x \in E\}$ and $F_0 = \{(0, y) : x \in F\}$ are ideals of $E \times_G F$, then a central operator $\overline{T} : E \times_G F \longrightarrow E \times_G F$ defines maps $T_0 : E \longrightarrow E$ and $T_1 : F \longrightarrow F$ as $T_0(x, 0) = \overline{T}(x, 0)$, $T_1(0, y) = \overline{T}(0, y)$ such that $-\lambda(x, 0) \leq T_0(x, 0) \leq \lambda(x, 0)$ and $-\lambda(0, y) \leq T_1(0, y) \leq \lambda(0, y)$. These show that $T_0 \in Z(E)$ and $T_1 \in Z(F)$. So we get the following result.

Proposition 3.19 *if $E_0 = \{(x, 0) : x \in E\}$ and $F_0 = \{(0, y) : y \in F\}$ are ideals of $E \times_G F$, then a central operator $\overline{T} : E \times_G F \longrightarrow E \times_G F$ defines maps $T_0 : E \longrightarrow E$ and $T_1 : F \longrightarrow F$ as $T_0(x, 0) = \overline{T}(x, 0)$, $T_1(0, y) = \overline{T}(0, y)$ such that $T_0 \in Z(E)$ and $T_1 \in Z(F)$.*

3.3 Order Bounded Sets and Dedekind Completion of $E \times_G F$–space

In this section, we investigate order bounded sets and Dedekind completeness properties of $E \times_G F$–spaces. In particular, we give two examples of concrete $E \times_G F$–spaces such that one has Cantor property and the other one does not have Cantor property.

As the following proposition shows, we can characterize order bounded subsets of E and F by using order bounded subsets of $E \times_G F$.

Proposition 3.20 *Let $B = \{(x, y) : x \in E, y \in F\}$ be an order bounded subset of $E \times_G F$. If F is an order ideal with respect to E, then $\{x : (x, y) \in B\}$ is an order bounded subset of E and $\{y : (x, y) \in B\}$ is an order bounded subset of G. If F is an ideal in E, then $\{x : (x, y) \in B\}$ and $\{y : (x, y) \in B\}$ are order bounded subsets of E.*

Proof. Let $B = \{(x, y) : x \in E, y \in F\}$ be an order bounded subset of $E \times_G F$. Let $0 \leq (a, b)$ in $E \times_G F$ be such that $|(x, y)| \leq (a, b)$ for all $(x, y) \in B$. That is to say $(|x|, |x + y| - |x|) \leq (a, b)$. Then $|x| \leq a$ for all $(x, y) \in B$ and $\{x : (x, y) \in B\}$ is an order bounded subset of E. On the other hand, $|x + y| \leq a + b$ for all $(x, y) \in B$. Thus $|\,|y| - |x|\,| \leq |x + y| \leq a + b$ yields $|y| \leq (a + b) + |x| \leq 2a + b$ for all $(x, y) \in B$. If F is an ideal of E, then $\{y : (x, y) \in B\}$ is an order bounded subset of E. If F is an order ideal with respect to E, then $\{y : (x, y) \in B\}$ is an order bounded subset of G. □

Proposition 3.21 *If $E \times_G F$ is a Dedekind complete Riesz space, then E and F are Dedekind complete Riesz spaces in their own rights.*

Proof. Suppose that $E \times_G F$ is Dedekind complete Riesz space. Let $0 \leq x_\alpha \uparrow\leq x$ in E. Let $0 \leq y \in F$ be arbitrary. Then $0 \leq (x_\alpha, y) \uparrow\leq (x, y)$ in $E \times_G F$. Let (z_1, z_2) be the supremum of (x_α, y) in $E \times_G F$. Suppose $0 \leq x_\alpha \uparrow\leq z_1$ and if $0 \leq x_\alpha \leq z$ for all α, then $(x_\alpha, y) \leq (z, y)$ in $E \times_G F$ and we have $(z_1, z_2) \leq (z, y)$ which yields $z_1 \leq z$. Thus z_1 is the supremum of (x_α) in E. Similarly, let $0 \leq y_\alpha \uparrow\leq y$ in F. We choose $x \in E^+$ and consider $0 \leq (x, y_\alpha)$ in $E \times_G F$. Then $0 \leq (x, y_\alpha) \uparrow\leq (x, y)$ in $E \times_G F$. Let (z_1, z_2) be the supremum of (x, y_α) in $E \times_G F$. It follows that $x = z_1$ and z_2 is the supremum of (y_α) in F. □

Proposition 3.22 *Let E be a Dedekind complete Riesz subspace of G and F be a band in E, then $E \times_G F$ is Dedekind complete Riesz space.*

Proof. Let $(x_\alpha, y_\alpha) \in E \times_G F$ be such that $0 \leq (x_\alpha, y_\alpha) \uparrow\leq (x, y)$ in $E \times_G F$. Then $0 \leq x_\alpha \uparrow\leq x$ in E and as E is Dedekind complete, (x_α) has a supremum in E, say B_1. On the other hand, $y_\alpha \leq 2x + y$ in E. Then the supremum of (y_α) exists in E and as F is assumed to be a band in E, this supremum, say B_2, belongs to F. Then $B_1 + B_2$ is an upper bound for $(x_\alpha + y_\alpha)$ in E and if z is the supremum of $(x_\alpha + y_\alpha)$ in E, we have $0 \leq z \leq B_1 + B_2$. Suppose $0 \leq z < B_1 + B_2$. Then by Riesz Decomposition Property, $z = u + v$ where $0 \leq u < B_1, 0 \leq v < B_2$. But then there exists α_0 and x_{α_0} with $u < x_{\alpha_0}$ and y_{α_0} with $v < y_{\alpha_0}$. Thus $u + v = z < x_{\alpha_0} + y_{\alpha_0}$ which is a contradiction. Hence $z = B_1 + B_2$ and (B_1, B_2) is the supremum (x_α, y_α) in $E \times_G F$. So $E \times_G F$ is Dedekind complete. □

Remark 3.23 *If E is Dedekind complete and F is an ideal of E, then $F^\delta \subset E$ and F^δ is an ideal of E where F^δ is the Dedekind completion of F. Since both F^δ and E are Riesz spaces, it is enough to consider positive elements only. Let $0 \leq y \leq x \in F^\delta$ with $y \in E$. Since $x \in F^\delta$, there exists $0 \leq x_\alpha \uparrow x$ with $x_\alpha \in F$ for all α. Then $0 \leq x_\alpha \wedge y \uparrow x \wedge y = y$. As $0 \leq x_\alpha \wedge y \leq x_\alpha$ for each α, $x_\alpha \wedge y \in F$ and $y \in F^\delta$.*

Thus if E is Dedekind complete and if F is an ideal of E, then $E \times_G F^\delta$ has a Riesz space structure from the previous remark. In this case the Dedekind completion of $E \times_G F$ is $E \times_G F^\delta$ as the next proposition shows.

Proposition 3.24 *If E is Dedekind complete and F is an ideal of E, then $E \times_G F^\delta$ is the Dedekind completion of $E \times_G F$.*

Proof. We already know that $E \times_G F^\delta$ is Dedekind complete and contains $E \times_G F$. We now show that $E \times_G F$ is a majorizing order dense Riesz subspace of $E \times_G F^\delta$. Let $(x, y) \in E \times_G F^\delta$. Choose $y_0 \in F$ with $y \leq y_0$, then $(x, y) \leq (x, y_0) \in E \times_G F$. This shows that $E \times_G F$ is a majorizing Riesz subspace of $E \times_G F^\delta$.

Let $0 < (x, y)$ in $E \times_G F^\delta$. Then $0 < x$ and $0 < x + y$. Then we have that $0, -y < x \Rightarrow y^- < x$. Hence $0 < x - y^- \leq x$. Since $y \in F^\delta$, by the order denseness of F in F^δ, there exists $0 < f \leq y^+$. So $0 < (x - y^-, f) \leq (x, y)$ where $(x - y^-, f) \in E \times_G F$. This shows that $E \times_G F$ is an order dense Riesz subspace of $E \times_G F^\delta$. □

Similar considerations above proposition will yield the following result.

Proposition 3.25 *If F is an ideal of E, then the Dedekind completion of $E \times_G F$ is $E^\delta \times_G F^\delta$.*

Proof. We have seen earlier that $E^\delta \times_G F^\delta$ is Dedekind complete and has a Riesz space structure as F^δ is an order ideal of E^δ. It is also easy to see that $E \times_G F$ is majorizing in $E^\delta \times_G F^\delta$. Thus the only problem is to show that $E \times_G F$ is order dense in $E^\delta \times_G F^\delta$. To see this Let $0 < (x, y)$ in $E^\delta \times_G F^\delta$. Then $0 < x$ and $0 < x + y$. So $0, -y < x \Rightarrow y^- < x$. Hence $0 < x - y^- \in E^\delta$. As E is order dense in E^δ, there exists $x_1 \in E$ such that $0 < x_1 \leq x - y^-$. Also there exists $f \in F$ such that $0 < f \leq y^+$ as F is order dense in F^δ. But then $x_1 + f \leq x - y^- + f \leq x - y^- + f + y^+ = x + y$. Then we get $0 < (x_1, f) \leq (x, y)$ where $(x_1, f) \in E \times_G F$. This completes the proof. □

Definition 3.26 *An Archmedean Riesz space L is said to have Cantor property (or σ- interpolation property) if for any sequences (x_n) and (y_m) in L such that $x_n \leq y_m$ for each $n, m \in \mathbb{N}$, there exists an element $g \in L$ such that $x_n \leq g \leq y_m$ for each $n, m \in \mathbb{N}$.*

As the following examples show that the space $E \times_G F$ has Cantor property in some cases. The first example contains a concrete $E \times_G F$–space having Cantor property. For the details of examples, see [33].

Example 3.27 *Let Q be a quasi-Stonean space with no isolated points.*
Let G=$B(Q)$= the space of all bounded real-valued functions on Q, $E = C(Q)$ and $F = l_\infty^w(Q)$. Let $x_n, z_m \in E \times_G F$ with $x_n = (a_n, b_n) \leq z_m = (a'_m, b'_m)$ for all $m, n \in \mathbb{N}$. It follows that $a_n \leq a'_m$ for all $m, n \in \mathbb{N}$. As E has certainly the Cantor property, there is $g \in E$ with $a_n \leq g \leq a'_m$ for each $n, m \in \mathbb{N}$. Then the set

$$C = \bigcup_{n \in \mathbb{N}} supp(b_n) \cup \bigcup_{n \in \mathbb{N}} supp(b'_n)$$

is countable since it is the union of two countable sets. For each $q \in C$ choose any $p(q) \in \mathbb{R}$ with $a_n(q) + b_n(q) \leq a'_m(q) + b'_m(q)$. Define $d \in l_\infty^w(Q)$ by $d(q) = p(q) - g(q)$ if $q \in C$ and $d(q) = 0$ if $q \notin C$. Then clearly, we have that $(a_n, b_n) \leq (g, d) \leq (a'_m, b'_m)$ on $E \times_G F$ and $(g, d) \in E \times_G F$. So $E \times_G F$ has Cantor property.

Now we give an example of $E \times_G F$–space which has no Cantor property.

Example 3.28 *Let Q be a quasi-Stonean space with no isolated points. Let G=$B(Q)$= the space of all bounded real-valued functions on Q, $E = C(Q)$ and $F = c_0(Q)$. To prove that $E \times_G F$ is not a Cantor space, we may find disjoint sets $T = \{t_1, t_2, ...\}$ and $U = \{u_1, u_2, ...\}$ with p in the closure of both $\{t_n, t_{n+1}, ...\}$ and $\{u_n, u_{n+1}, ...\}$ for each $n \in \mathbb{N}$. Define $b_n \in c_0(Q)$ to be the function with value -1 on $\{t_1, t_2, ..., t_n\}$ and otherwise 0. Similarly let $d_n \in c_0(Q)$ be the function with value 1 on $\{u_1, u_2, ..., u_n\}$ and otherwise 0. Then we have $(1_Q, b_n) \geq (0, d_m)$ for all $m, n \in \mathbb{N}$. If there were $f \in C(Q)$ and $c \in c_0(Q)$ such that $(1_Q, b_n) \geq (f, c) \geq (0, d_n)$ for each $n \in \mathbb{N}$, then $1_Q + b_n \geq f + c \geq d_n$ for each n. As $\{t_1, t_2, ..., t_n\} \subset \{t_1, t_2, ..., t_{n+k}\}$ and $\{u_1, u_2, ..., u_n\} \subset \{u_1, u_2, ..., u_{n+k}\}$ for each $k \in \mathbb{N}$, then we certainly have $f + c = 0$ on T and 1 on U. Since $c \in c_0(Q), c(t_n) \to 0$ and $c(u_n) \to 0$ as $n \to \infty$. Thus*

$$0 = \lim(f(t_n) + c(t_n)) = \lim f(t_n) = f(p) = \lim f(u_n) = \lim(f(u_n) + c(u_n)) = 1$$

and this contradiction shows that $E \times_G F$ has no Cantor property.

3.4 The Norm Properties of $E \times_G F$–space

In this section, we investigate some properties of $E \times_G F$-spaces such as Dunford-Pettis property, weakly sequentially continuity of lattice operations and Schur property. For these purposes, we need to make $E \times_G F$-space a normed Riesz space(Banach lattice). Consider the norm $\|(x, y)\|_{E \times_G F} = max\{\|x\|, \|x + y\|\}$ on $E \times_G F$. With respect to this norm, we have the following.

Theorem 3.29 *Let G be a normed Riesz space. We consider the following norm on $E \times_G F$:*

$$\|(x, y)\|_{E \times_G F} = max\{\|x\|, \|x + y\|\}.$$

Then we have the following:

(1) *$E \times_G F$ is a normed Riesz space.*

(2) *(e, f) is an order unit for $E \times_G F$ whenever e and f are order units for E and F respectively.*

(3) *$E \times_G F$ is a Banach lattice whenever E and F are Banach lattices in their own rights.*

(4) *$E \times_G F$ is an AM-space whenever E and F are Banach lattices in their own rights and G is an AM-space.*

(5) *$E \times_G F$ is an AL-space whenever E and F are Banach lattices in their own rights and G is an AL-space.*

(6) *If G has an order continous norm, then $E \times_G F$ has an order continuous norm.*

Proof.

(1) It is enough to show that the norm defined above is a lattice norm. Let $|(x, y)| \leq |(x', y')|$. Then from Theorem 3.4, we have

$$(|x|, |x + y| - |x|) \leq (|x'|, |x' + y'| - |x'|).$$

This gives us $|x| \leq |x'|$ and $|x + y| \leq |x' + y'|$. Since G is a normed Riesz space, we have $||x|| \leq ||x'||$ and $||x + y|| \leq ||x' + y'||$. So we have $||(x, y)||_{E\times_G F} \leq ||(x', y')||_{E\times_G F}$.

(2) We will show that (e, f) is an order unit for $E \times_G F$. Let $(x, y) \in E \times_G F$. Then there exist $0 \leq \lambda, \alpha \in \mathbb{R}$ such that $|x| \leq \lambda e$ and $|y| \leq \alpha f$. So $|x| \leq max\{\lambda, \alpha\}e$ and $|x + y| \leq \lambda e + \alpha f \leq max\{\lambda, \alpha\}(e + f)$. This gives us that $|(x, y)| \leq max\{\lambda, \alpha\}(e, f)$. So (e, f) is an order unit for $E \times_G F$.

(3) Let $(x_n, y_n)_n$ be a Cauchy sequence in $E \times_G F$. Then (x_n) is a Cauchy sequence in E as

$$||x_n - x_m|| \leq ||(x_n, y_n) - (x_m, y_m)||_{E\times_G F}.$$

Then there exist $x \in E$ such that $x_n \to x$. The equality

$$||y_n - y_m|| = ||(0, y_n) - (0, y_m)||_{E\times_G F},$$

and the inequality

$$\begin{aligned} ||(0, y_n) - (0, y_m)||_{E\times_G F} &= ||(0, y_n) - (x_n, 0) + (x_n, 0) + (x_m, 0) - (x_m, 0) - (0, y_m)||_{E\times_G F} \\ &\leq ||(x_n, y_n) - (x_m, y_m)||_{E\times_G F} + ||x_n - x_m|| \end{aligned}$$

show that (y_n) is a Cauchy sequence in F. Let $y_n \to y \in F$. We also claim that $(x_n, y_n) \to (x, y)$ in $E \times_G F$. Let $\varepsilon > 0$ be given. As $x_n \to x$ and $y_n \to y$, there exist n_0

and n_1 in $\mathbb{N}$ such that $\|x_n - x\| < \varepsilon$ for each $n \geq n_0$ and $\|y_n - y\| < \varepsilon$ for each $n \geq n_1$. Let $N = \max\{n_0, n_1\}$. As

$$\begin{aligned}
\|(x_n, y_n) - (x, y)\|_{E\times_G F} &= \|(x_n, 0) + (0, y_n) - (x, 0) - (0, y)\|_{E\times_G F} \\
&\leq \|(x_n - x, 0)\|_{E\times_G F} + \|(0, y_n - y)\|_{E\times_G F} \\
&= \|x_n - x\| + \|y_n - y\|,
\end{aligned}$$

we have that $\|(x_n, y_n) - (x, y)\|_{E\times_G F} < 2\varepsilon$ for each $n \geq N$. Therefore we proved the claim. So $E \times_G F$ is a Banach lattice.

(4) Since $E \times_G F$ is a Banach lattice by (3), it is enough to show that the norm defined above is an M-norm, that is if $(x, y) \wedge (x', y') = 0$, then

$$\|(x, y) + (x', y')\| = max\{\|(x, y)\|, \|(x', y')\|\}$$

for each $(x, y), (x', y') \in (E \times_G F)^+$. Let $(x, y) \wedge (x', y') = 0$. Then $x \wedge x' = 0$ and $(x + y) \wedge (x' + y') = 0$ in G. Therefore, we have $\|x + x'\| = max\{\|x\| + \|x'\|\}$ and $\|x + y + x' + y'\| = max\{\|x + y\|, \|x' + y'\|\}$ as G is an AM-space. These give us $\|(x, y) + (x', y')\|_{E\times_G F} = max\{\|(x, y)\|_{E\times_G F}, \|(x', y')\|_{E\times_G F}\}$. This completes the proof.

(5) It is enough to show that the norm defined above is an L-norm, i.e., if $(x, y) \wedge (x', y') = 0$, then

$$\|(x, y) + (x', y')\|_{E\times_G F} = \|(x, y)\|_{E\times_G F} + \|(x', y')\|_{E\times_G F}$$

for each $(x, y), (x', y') \in (E \times_G F)^+$. Let $(x, y) \wedge (x', y') = 0$. Then, we have $x \wedge x' = 0$ and $(x + y) \wedge (x' + y') = 0$ in G. Then we have $\|x + x'\| = \|x\| + \|x'\|$ and $\|x + y + x' + y'\| = \|x + y\| + \|x' + y'\|$ as G is an AL-space. These give us

$$\|(x, y) + (x', y')\|_{E\times_G F} = \|(x, y)\|_{E\times_G F} + \|(x', y')\|_{E\times_G F}.$$

This completes the proof.

(6) Let $(x_\alpha, y_\alpha) \downarrow 0$ in $E \times_G F$. This implies that $x_\alpha \downarrow 0$ and $x_\alpha + y_\alpha \downarrow 0$ in G. As G has an order continuous norm, then for each $\varepsilon > 0$, there exist some α_0 and α_1 such that $\|x_\alpha\| < \frac{1}{2}\varepsilon$ for each $\alpha \geq \alpha_0$ and $\|x_\alpha + y_\alpha\| < \frac{1}{2}\varepsilon$ for each $\alpha \geq \alpha_1$. Taking $\alpha_3 = \max\{\alpha_0, \alpha_1\}$, we have that

$$\|(x_\alpha, y_\alpha)\|_{E\times_G F} = max\{\|x_\alpha\|, \|x_\alpha + y_\alpha\|\} < \varepsilon$$

for each $\alpha \geq \alpha_3$.

$\square$

Example 3.30 *(1) Let c be the space of all convergent sequences, c_0 be the space of all sequences converging to zero and l_∞ be the space of all bounded sequences. Consider $c \times_{l_\infty} c_0$. We see that l_∞ having order continuous norm is indispensable.*

(2) Consider $c_0 \times_{l_\infty} l_1$ where l_1 is the space of all absolutely summable sequences. We know that l_1 is an order ideal in c_0. So $c_0 \times_{l_\infty} l_1$ is a Banach lattice having order continuous norm. But l_∞ does not have order continuous norm. So the converse of part (6) in the previous theorem is not true in general.

Consider the norm $||(x, y)||_\infty = \max\{||x||, ||y||\}$ on $E \times_G F$. Actually, two norms $||.||_\infty$ and $||.||_{E\times_G F}$ are equivalent as the next proposition shows.

Proposition 3.31 *Let G be a normed Riesz space, E and F be normed Riesz subspaces of G such that F is an order ideal with respect to E. Two norms $||(x, y)||_{E\times_G F}$ and $||(x, y)||_\infty$ on $E \times_G F$ are equivalent. In particular, the projections $P : E \times_G F \to E$ and $Q : E \times_G F \to F$ are continuous.*

Proof. Let $\{(x_n, y_n)\}$ be sequence in $E \times_G F$ which converges to (x, y) with respect to the norm $||.||_{E\times_G F}$ in $E \times_G F$. Then given $\varepsilon > 0$, there exists $N(\varepsilon)$ such that $||(x_n, y_n) - (x, y)||_{E\times_G F} < \varepsilon$ if $n \geq N(\varepsilon)$. Thus $||(x_n - x, y_n - y)||_{E\times_G F} < \varepsilon$ if $n \geq N(\varepsilon)$. So we get

$$\max\{||x - x_n||, ||y - y_n + x - x_n||\} < \varepsilon$$

if $n \geq N(\varepsilon)$. In particular, this shows that $||y - y_n + x - x_n|| < \varepsilon$ if $n \geq N(\varepsilon)$. Thus we have

$$||y - y_n|| = ||y - y_n + x - x_n - (x - x_n)|| \leq ||y - y_n + x - x_n|| + ||x - x_n|| \leq 2\varepsilon$$

if $n \geq N(\varepsilon)$. Thus if $(x_n, y_n) \to (x, y)$ with respect to $||.||_{E\times_G F}$ in $E \times_G F$, then $x_n \to x$ in E and $y_n \to y$ in F. Thus both of the projections $P : E \times_G F \to E$ and $Q : E \times_G F \to F$ are continuous. In particular, we have that $||x|| \leq K||(x, y)||_{E\times_G F}$ and $||y|| \leq K||(x, y)||_{E\times_G F}$. This implies that $||(x, y)||_\infty = \max\{||x||, ||y||\} \leq K||(x, y)||_{E\times_G F}$. On the other hand,

$$\begin{aligned} \max\{||x||, ||x + y||\} &\leq \max\{||x||, ||x|| + ||y||\} \\ &\leq \max\{||x||, 2\max\{||x||, ||y||\}\} = 2\max\{||x||, ||y||\}. \end{aligned}$$

Therefore, $\|(x,y)\|_{E\times_G F} \leq 2\|(x,y)\|_\infty$. Thus $\|(x,y)\|_{E\times_G F}$ is equivalent to $\|(x,y)\|_\infty$. $\square$

From now on, we assume that E and F are Banach sublattices of a Banach lattice G.

Without assuming that F is an order ideal in E, we have a map

$$T : G' \to (E \times_G F)'$$

defined by

$$Tf(x,y) = f(x+y) \text{ for each } f \in G' \text{ and } (x,y) \text{ in } E \times_G F.$$

As

$$|Tf(x,y)| = |f(x+y)| \leq \|f\|_{G'}\|x+y\| \leq \|f\|_{G'}\|(x,y)\|_{E\times_G F},$$

Tf defines a continuous linear functional on $E\times_G F$ and $\|Tf\|_{(E\times_G F)'} \leq \|f\|_{G'}$. If we restrict T to E', then $\|Tf\| \geq |Tf(x,0)| = |f(x)|$. Taking the supremum over $x \in E$ such that $\|x\| \leq 1$, we get $\|f\|_{E'} \leq \|Tf\|_{(E\times_G F)'}$. Hence T is an isometry when it is restricted to E'. In what follows, we assume that F is an order ideal in E and use the fact that $E' = (E \times_G F)'$. In this case, we have the following.

Proposition 3.32 *Let F be an order ideal in E. If E has Dunford-Pettis Property, shortly DPP, i.e., for all $(x_n) \subset E$, $x_n \to 0$ weakly and $(f_n) \subset E'$, $f_n \to 0$ weakly, then $\lim_{n\to\infty} f_n(x_n) = 0$, then $E \times_G F$ has DPP.*

Proof. Suppose that E has DPP. Let (x_n, y_n) be a sequence in $E \times_G F$ such that $(x_n, y_n) \to 0$ weakly in $E \times_G F$. Let $(f_n) \subset (E \times_G F)'$ such that $f_n \to 0$ weakly. Since the projections $P : E \times_G F \to E$ and $Q : E \times_G F \to F$ are norm continuous, they are weakly continuous, we have $x_n \to 0$ weakly in E and $y_n \to 0$ weakly in F. Thus $x_n + y_n \to 0$ weakly in E. Since E has DPP, then

$$\lim_{n\to\infty} f_n(x_n, y_n) = \lim_{n\to\infty} f_n(x_n + y_n) = 0.$$

$\square$

If F were not an order ideal in E, then the result would be as follows: if G has DPP, then $E \times_G F$ has DPP for each F.

Definition 3.33 *Lattice operations in a Banach lattice are said to be weakly sequentially continuous if $x_n \to 0$ weakly $\Rightarrow |x_n| \to 0$ weakly.*

Proposition 3.34 *Let F be an order ideal in E. If E has weakly sequentially lattice operations, then $E \times_G F$ has weakly sequentially lattice operations.*

Proof. Suppose that $(x_n, y_n) \to 0$ weakly. Since the canonical projections are weakly continuous, we have $x_n \to 0$ weakly in E and $y_n \to 0$ weakly in F then $x_n + y_n \to 0$ weakly in E. Thus $|(x_n, y_n)| = (|x_n|, |x_n + y_n| - |x_n|)$ and $f_n \in E' = (E \times_G F)'$, we have $f(|x_n|, |x_n + y_n| - |x_n|) = f(|x_n + y_n|)$ we see that $f(|x_n + y_n|) \to 0$ since E has weakly sequentially continuous lattice operations. □

If F were not an order ideal in E, then the result would be as follows: if G has weakly sequentially continuous lattice operations, then $E \times_G F$ has weakly sequentially continuous lattice operations for each F.

Proposition 3.35 *Let F be an order ideal in E. If E has Schur property, i.e., $x_n \to 0$ weakly in $E \Rightarrow \|x_n\| \to 0$, then $E \times_G F$ has Schur property.*

Proof. Suppose $(x_n, y_n) \to 0$ weakly in $E \times_G F$. Then using projections again, we have $x_n \to 0$ weakly in E and $y_n \to 0$ weakly in F so that $x_n + y_n \to 0$ weakly in E. Firstly, from $x_n \to 0$ weakly $\Rightarrow \|x_n\| \to 0$ and from $x_n + y_n \to 0$ weakly $\Rightarrow \|x_n + y_n\| \to 0$. Thus $\|(x_n, y_n)\|_{E \times_G F} = max\{\|x_n\|, \|x_n + y_n\|\} \to 0$. □

If F were not an order ideal in E, then the result would be as follows: if G has Schur property, then $E \times_G F$ has Schur property for each F.

From now on we assume that F is an order ideal with respect to E.

Definition 3.36 *A Banach lattice L is said to be a KB (Kantorovich-Banach)–space whenever every increasing norm bounded sequence of L^+ is norm convergent.*

When is $E \times_G F$ a KB-space? Now we will try to find an answer for this question. One of them is below.

Proposition 3.37 *If G is a KB-space, E and F are Banach lattices, then $E \times_G F$ is a KB-space.*

Proof. Let $0 \leq (x_n, y_n) \uparrow$ be a norm bounded sequence in $E \times_G F$. Then $0 \leq x_n \uparrow$ is norm bounded in E and $0 \leq x_n + y_n \uparrow$ is norm bounded in G. As G is a KB-space, (x_n) and $(x_n + y_n)$

are norm convergent in G, say $x_n \to x$ and $x_n + y_n \to z$. This gives us $x_n \to x$ and $y_n \to z - x$. As E and F are Banach lattices, $(x, z - x)$ belongs to $E \times_G F$. So $(x_n, y_n) \to (x, z - x)$ in $E \times_G F$. This completes the proof. $\square$

Proposition 3.38 *Let E and F be Banach lattices. If $E \times_G F$ is a KB-space, then E and F are KB-spaces.*

Proof. Let $0 \leq x_n \uparrow$ be a norm bounded sequence in E. Then $0 \leq (x_n, 0) \uparrow$ is a norm bounded sequence in $E \times_G F$. This implies $(x_n, 0)$ is norm convergent in $E \times_G F$. But we have $\|x_n\| = \|(x_n, 0)\|_{E \times_G F}$. This shows that (x_n) is norm convergent in E. So E is a KB-space. Similarly we can show that F is a KB space. $\square$

From the proposition above, we immediately have the following corollary.

Corollary 3.39 *If E or F is not a KB space, then $E \times_G F$ is not a KB space.*

Example 3.40 *Consider $c \times_{l_\infty} c_0$. As c is not a KB-space, so $c \times_{l_\infty} c_0$ is not a KB-space from the previous corollary.*

There is a connection between being a KB-space and containing c_0 for a Banach lattice L. If c_0 is not (lattice) embeddable in L, then L is a KB-space. For the details see $[10, Thm. 14.13]$.

Example 3.41 *It is well-known that c_0 is always lattice embeddable in an infinite dimensional AM-space. Thus if G is a Banach lattice and if one consider the principal ideal I_x generated by $x \in G^+$, then from the previous theorem $\overline{I_x} \times_G F$ is not a KB space for any F which is an order ideal with respect to I_x since I_x is an AM space which contains c_0.*

Recall that a closed vector subspace Y of a Banach space X is said to be *complemented* whenever there exists another closed vector subspace Z of X such that $X = Y \oplus Z$. Also recall that a Banach space Y is said to embed complementably into another Banach space X whenever there exists an embedding $T : Y \to X$ so that $T(Y)$ is complemented in X.

Regarding embeddings of Banach spaces into KB-spaces, we have the following remarkable result of W. B. Johnson and L. Tzafriri.

Theorem 3.42 *If a Banach space X embeds complementably into a Banach lattice and c_0 does not embed in X, then X also embeds complementably in a KB–space.*

Remark 3.43 *(1) Let G be a KB–space. Then c_0 is not embeddable in $E \times_G F$ for every Banach lattices E and F, otherwise $E \times_G F$ is not a KB–space. Also in this case, c_0 is not lattice embeddable in E and F as E and F are lattice embeddable in $E \times_G F$.*

(2) if E is a complemented sublattice of G and c_0 is not embeddable in E, then by considering a KB space G' containing E (as given in the previous theorem), $E \times_{G'} F$ is a KB–space for every order ideal F with respect to E.

(3) Let G be a Banach lattice containing c_0. Let F be an ideal with respect to c_0. Then we have that $c_0 \times_G F$ is not a Grothendieck space. This is because the operator $P_c : c_0 \times_G F \to c_0$ is a lattice homomorphism. On the other hand it is well-known that there does not exist any surjective linear operator T from a Grothendieck space to c_0. In particular, we see that $c_0 \times_{l_\infty} l_1$ is not a Grothendieck space.

In Banach lattices, the norm topology and the relatively uniform topology (ru–topology) always coincide. This may not be true in normed Riesz spaces.

Proposition 3.44 *Let G be a normed Riesz space in which the norm and ru–topology coincide. Then $E \times_G F$ has the same property for each Riesz subspaces E and F in which F is an order ideal with respect to E.*

Proof. It suffices to show that every norm convergent sequence has a ru-convergent subsequence. Let (x_n, y_n) be a norm convergent sequence in $E \times_G F$. Then (x_n) and $(x_n + y_n)$ are both norm convergent sequences in G. Let (x_{n_k}) be a ru-convergent subsequence of (x_n). Let $(x_{n_l} + y_{n_l})$ be a ru-convergent subsequence of the norm convergent sequence $(x_{n_k} + y_{n_k})$. Since (x_{n_l}, y_{n_l}) is a ru-convergent subsequence of (x_n, y_n), we see that ru-uniform topology and norm topology coincide in $E \times_G F$. □

Corollary 3.45 *Let E, F and G be as above. Then $\overline{F_{ru}}$ is an ideal with respect to E for every F which is an ideal with respect to E.*

3.5 Disjointness Properties of $E \times_G F$–space

In Banach lattice theory, disjoint sequences play an important role to characterize many properties of given Banach lattice such as laterally σ-completeness properties, weakly compact order intervals, lower p-estimate property. In this section, we will characterize some properties of $E \times_G F$–spaces by using their disjoint sequences. We start with the following proposition which gives a relation between the disjoint sequences of $E \times_G F$–spaces and the spaces E and G.

Proposition 3.46 *Let E be a Riesz subspace of a Riesz space G and F be an ordered vector subspace of G such that F is an order ideal with respect to E. Then $(x, y) \perp (x', y')$ in $E \times_G F$ $\Leftrightarrow$ $x \perp x'$ in E and $x + y \perp x' + y'$ in G.*

Proof. Let $(x, y) \perp (x', y')$ in $E \times_G F$. Then $|(x, y) + (x', y')| = |(x, y) - (x', y')| \Leftrightarrow |(x + x', y + y')| = |(x - x', y - y')| \Leftrightarrow (|x + x'|, |x + y + x' + y'| - |x + x'|) = (|x - x'|, |x + y - x' - y'| - |x - x'|)$. So we get $|x + x'| = |x - x'|$ and $|x + y + x' + y'| = |x + y - x' - y'|$. Therefore $x \perp x'$ in E and $x + y \perp x' + y'$ in G. □

Proposition 3.47 *Let E be an order dense ideal in G and F be an order ideal in E. Then the ideal $I(E \times_G F)$ generated by $E \times_G F$ is an order dense ideal in $G \times_G G$.*

Proof. Let $(x, y) \in (E \times_G F)^{\perp}$. Then $(x, y) \perp (e, f)$ for each $(e, f) \in E \times_G F$. Then $x \perp e$ and $x + y \perp e + f$. As E is an order dense ideal in G, we get $x = 0$. So $y \perp e + f$ for each $e \in E$ and $f \in F$. Taking $f = 0$, we get $y \perp e$. Again as E is an order dense ideal in G, we get $y = 0$. Let $I(E \times_G F)$ be the ideal generated by $E \times_G F$ in $G \times_G G$. Then $E \times_G F \subset I(E \times_G F) \Rightarrow I(E \times_G F)^{\perp} = 0$. □

Proposition 3.48 *If F is an order ideal in E, then $(E \times_G F)^{\perp}$ is a subset of $F^{\perp}$.*

Proof. Let $(x, y) \in (E \times_G F)^{\perp}$. Then $x \perp e$ and $x + y \perp e + f$ for each $e \in E$ and $f \in F$. This implies that $x \in E^{\perp}$ and $x + y \in E^{\perp}$. We get $x \in E^{\perp}$ and $y = x + y - x \in E^{\perp}$. As $E^{\perp} \subset F^{\perp}$, we get $x \in F^{\perp}$, $y \in F^{\perp}$. □

We now give the following definition. For the details see [9].

Definition 3.49 *A Riesz space L is said to be laterally σ-complete, if the supremum of every disjoint sequence of L^+ exists in L.*

Proposition 3.50 *If E is a laterally σ-complete Riesz subspace of G and F is a band of E, then $E \times_G F$ is a laterally σ-complete Riesz space.*

Proof. Let $0 \leq (x_n, y_n)$ be a disjoint sequence in $E \times_G F$. Then we have $(x_n, y_n) \perp (x_m, y_m) = 0$ for each $n \neq m$. This implies that $x_n \perp x_m$ and also $x_n + y_n \perp x_m + y_m$. As E is laterally σ-complete, $\sup\{x_n\}$ and $sup\{x_n + y_n\}$ exist in E. Let $\sup\{x_n\} = x$ and $\sup\{x_n + y_n\} = z$. As $y_n \leq x_n + y_n \leq z$ and F is a band in E, $\sup\{y_n\}$ exists in F, let $\sup\{y_n\} = y$. We claim that $z = x + y$. Assume that $0 \leq z < x + y$. Then by the Riesz Decomposition Property, there exist x_1 and x_2 in E such that $0 \leq x_1 < x$ and $0 \leq x_2 < y$ such that $x_1 + x_2 = z$. Then there exists an n_0 such that $x_1 < x_{n_0}$ and $x_2 < y_{n_0}$. Thus $x_1 + x_2 = z < x_{n_0} + y_{n_0}$, which is a contradiction. Hence $z = x + y$ and (x, y) is the supremum of (x_n, y_n) in $E \times_G F$. □

We now state a theorem which will be useful in the next proposition. For the details see [9].

Theorem 3.51 *Let $T : L \to F$ be a positive operator between two Archimedean Riesz spaces. If L is laterally σ-complete, then the operator T is σ-order continuous.*

Remark 3.52 *Let E be an Archimedean Riesz subspace of G and F be an ideal in E. Let $0 \leq n(x, y) \leq (x', y')$ for each $n \in \mathbb{N}$ and $(x, y), (x', y') \in E \times_G F$. Then $0 \leq nx \leq x'$ and $0 \leq n(x + y) \leq x' + y'$ in E. As E is Archimedean, we get $x = 0$ and $x + y = 0$. Hence we get $x = 0$ and $y = 0$. So $E \times_G F$ is Archimedean*

Proposition 3.53 *Let E be an Archimedean laterally σ-complete Riesz subspace of G and F be a band in E. Then the projection map, $P : E \times_G F \to E$ defined by $P(x, y) = x$ is σ-order continuous.*

Proof. As E is an Archimedean laterally σ-complete Riesz space and F is a band in E, $E \times_G F$ is an Archimedean laterally σ-complete Riesz space. Also the projection map P is positive. Therefore P is σ-order continuous by the previous theorem. □

Now we give a characterization of super Dedekind completeness of $E \times_G F$ by using disjointeness property of this space. For this purpose, we give the following well-known definitions.

Definition 3.54 *The Riesz space L is said to have countable sup property (CSP) if every disjoint net (f_α) such that $f_\alpha \leq f$ for some $f \in L$ is countable.*

Definition 3.55 *A Riesz space L is called super Dedekind complete if L is Dedekind complete and has countable sup property.*

Proposition 3.56 *If E has CSP, then $E \times_G F$ has CSP.*

Proof. Suppose that E has CSP and let (f_α, g_α) be a disjoint net in $E \times_G F$ such that $(f_\alpha, g_\alpha) \leq (f, g)$. This implies that (f_α) is a disjoint net in E and $f_\alpha \leq f$. As E has CSP, we get that (f_α) is countable and so is (f_α, g_α). □

Using the proposition above, we immediately have the following corollary.

Corollary 3.57 *Let E be a Dedekind complete Riesz space which has CSP and F be a band in E, then $E \times_G F$ is super Dedekind complete.*

Proposition 3.58 *Let F be an order ideal in E. Assume that $T : E \to E$ is a positive orthomorphism on E. Then the relation $\overline{T}(x, y) = (Tx, Ty)$ gives rise to an orthomorphism on $E \times_G F$.*

Proof. Suppose $(x, y) \geq 0$ in $E \times_G F$. Then $x \geq 0$ and $x + y \geq 0$ in G. By the positiveness of T, we get $0 \leq T(x)$ and $0 \leq T(x + y) = T(x) + T(y)$. So $\overline{T}$ is a positive operator. Therefore $\overline{T}$ is an order bounded operator. Now assume that $(x, y) \perp (x', y')$ in $E \times_G F$. This implies that $x \perp x'$ and $x + y \perp x' + y'$ in E. As T is an orthomorphism on E, we have $x \perp T(x')$ and $x + y \perp T(x' + y')$ $\Rightarrow (x, y) \perp \overline{T}(x', y')$. □

Proposition 3.59 *Let F be an order ideal with respect to E. If e is a weak order unit of E and $e + f$ is a weak order unit of $E + F$, then (e, f) is a weak order unit of $E \times_G F$.*

Proof. Let $e > 0$ be a weak order unit in E and $e + f > 0$ be a weak order unit in $E + F$. Then $(e, f) > 0$ in $E \times_G F$. Let $(e, f) \perp (x, y)$ in $E \times_G F$. This implies that $(e, f) \wedge |(x, y)| = 0$. Hence $(e, f) \wedge (|x|, |x + y| - |x|) = 0$. Therefore $e \wedge |x| = 0$ and $(e + f) \wedge |x + y| = 0$. As e and $e + f$ are the weak order units of E and $E + F$ respectively, we get $x = 0$ and $x + y = 0$. So $(x, y) = (0, 0)$. □

Proposition 3.60 *Let F be an order ideal in E. If e is a weak order unit of E and f is a weak order unit of F, then (e, f) is a weak order unit of $E \times_G F$.*

Proof. Let $e > 0$ be a weak order unit in E and $f > 0$ be a weak order unit in F. Then $(e, f) > 0$ in $E \times_G F$. Let $(e, f) \perp (x, y)$ in $E \times_G F$. This implies that $(e, f) \wedge |(x, y)| = 0$. Hence $(e, f) \wedge (|x|, |x + y| - |x|) = 0$. Therefore $e \wedge |x| = 0$ and $(e + f) \wedge |x + y| = 0$. As e is a weak order unit of E, we get $x = 0$. Then $(e + f) \wedge |y| = 0$. But $0 < f \leq (e + f)$. This implies that $f \wedge |y| \leq (e + f) \wedge |y| = 0$. This yields that $f \wedge |y| = 0$. As f is a weak order unit of F, we get $y = 0$. □

Definition 3.61 *A Riesz space L is said to have a finite or countable order basis if there exists a sequence $(v_n) \subset L^+$ such that if $f \in L$ and $|f| \wedge v_n = 0$ for each $n \in \mathbb{N}$ imply that $f = 0$.*

By using the definition above and disjointness property of $E \times_G F$, we may characterize the finite or countable order basis of $E \times_G F$ as in the next proposition.

Proposition 3.62 *Let E be a Riesz subspace of G and F be an order ideal in E. If E has a countable order basis, then $E \times_G F$ has a countable order basis.*

Proof. Assume that E has a countable order basis. Then there exists a sequence $(v_n) \subset E^+$ such that $|f| \wedge v_n = 0$ for each $n \in \mathbb{N}$. Then the sequence $(v_n, 0)$ in $E \times_G F$ is positive. Let $|(f, g)| \wedge (v_n, 0) = (0, 0)$ for each $n \in \mathbb{N}$. Then $|f| \wedge v_n = 0$ and $|f + g| \wedge v_n = 0$ for each $n \in \mathbb{N}$. As E has a countable order basis and $f, f + g \in E$, we have that $f = 0$ and $f + g = 0$ $\Rightarrow (f, g) = (0, 0)$. □

Definition 3.63 *The element p in a Riesz space L is called a component of an element $e > 0$ in L whenever p and $e - p$ are disjoint, i.e., whenever $p \perp (e - p)$.*

The above definition is justified by the following lemma. For the details of lemma, see $[32, Thm. 38.2]$.

Lemma 3.64 *Any component p of $e > 0$ satisfies $0 \leq p \leq e$.*

Proposition 3.65 *If (e, f) is a component of (x, y) in $E \times_G F$, then e is a component of x in E and $e + f$ is a component of $x + y$ in G.*

Proof. Let $(0, 0) \leq (e, f)$ be a component of (x, y) in $E \times_G F$. This implies that $(e, f) \wedge ((x, y) - (e, f)) = 0$. Then $(e, f) \wedge (x - e, y - f) = 0$. This gives $e \wedge (x - e) = 0$ and $(e + f) \wedge (x + y - (e + f)) = 0$. □

For a positive element x in a Banach lattice L, the order interval $[0, x]$ is *weakly compact* if every disjoint sequence of $[0, x]$ is norm convergent to zero. By using this characterization of weakly compact order intervals, we have the following.

Proposition 3.66 *If G is a Banach lattice having weakly compact order intervals, then $E \times_G F$ has weakly compact order intervals.*

Proof. Let (x_n, y_n) be a disjoint sequence of $[0, (e, f)]$ in $E \times_G F$. Then (x_n) is a disjoint sequence in E such that $(x_n) \subset [0, e]$ and $(x_n + y_n)$ is a disjoint sequence in G such that $(x_n + y_n) \subset [0, e + f]$. As G has weakly compact order intervals, we have that $\|x_n\| \to 0$ and $\|x_n + y_n\| \to 0$. This implies that $\|(x_n, y_n)\|_{E \times_G F} = \max\{\|x_n\|, \|x_n + y_n\|\} \to 0$. □

Proposition 3.67 *Suppose that G is a Banach lattice with the property that $Sol(A)$, solid hull of every norm bounded subset A is relatively weakly compact. Then norm bounded subsets of $E \times_G F$ has the same property for each Banach lattices E and F where F is an order ideal with respect to E.*

Proof. Let A be a norm bounded subset of $E \times_G F$. It is enough to show that every norm bounded disjoint sequence in $Sol(A)$ is norm convergent to zero. Let (x_n, y_n) be a norm bounded disjoint sequence in $Sol(A)$. Then there exists some $M > 0$ such that $\|(x_n, y_n)\| \leq M$. Then (x_n) and $(x_n + y_n)$ are norm bounded disjoint sequences in G. Then $\|x_n\| \to 0$ and $\|x_n + y_n\| \to 0$ imply that $\|(x_n, y_n)\|_{E \times_G F} = \max\{\|x_n\|, \|x_n + y_n\|\} \to 0$ in $E \times_G F$. This completes the proof. □

Let us recall that the norm of a normed Riesz space satisfies a *lower p-estimate* if and only if $(\|x_n\|) \in l_p$ for every disjoint order bounded sequence $(x_n) \in E^+$.

Proposition 3.68 *Let G be a normed Riesz space satisfying a lower p-estimate for some p. Then $E \times_G F$ satisfies a lower p-estimate for each $E \subset G$ and F an ideal with respect to E.*

Proof. Let (x_n, y_n) be an order bounded disjoint sequence in $(E \times_G F)^+$. Then (x_n) and $(x_n + y_n)$ are order bounded disjoint sequences in G^+. Then $(\|x_n\|)$ and $(\|x_n + y_n\|)$ are in l_p. Hence $\|(x_n, y_n)\|_{E \times_G F} = \max\{\|x_n\|, \|x_n + y_n\|\}$ also belongs to l_p and $E \times_G F$ satisfies a lower p-estimate. □

It is well-known that every L-weakly compact subset A of a normed Riesz space L is relatively weakly compact.

Proposition 3.69 *Let G be a Banach lattice with the property that every relatively weakly compact subset is L-weakly compact. Then $E \times_G F$ has the same property for each Riesz subspaces E and F of G such that F is an order ideal with respect to E.*

Proof. Let A be a relatively weakly compact subset of $E \times_G F$. By Theorem 3.6.8 in ([23]), it suffices to show that each disjoint weakly null sequence (x_n, y_n) in A is norm convergent to zero. It is easily seen that (x_n) and $(x_n + y_n)$ are weakly null disjoint sequences in G. Thus $\|x_n\|$ and $\|x_n + y_n\|$ are convergent to zero in G and therefore in $E \times_G F$. □

3.6 A New Type of Abramovich – Wickstead Spaces

In this section we introduce a new type of Abramovich–Wickstead spaces. We show that this space is a Riesz space under pointwise order. We start with the following definition which contains the building blocks of this space.

Definition 3.70 *For a compact space K and a relatively uniformly complete vector lattice E, we set*

(1) *$C(K, E(e))$ the space of all mappings from K into $E(e)$ which are continuous in the sense of the norm $\|.\|_e$ where $E(e)$ denotes the ideal generated by $e \in E^+$ and*

$$\|u\|_e := \inf\{\lambda > 0 : |u| \le \lambda e\} \quad (u \in E(e)).$$

Then, we set

$$C_r(K,E) := \bigcup \{C(K,E(e)) : e \in E^+\}$$

and call the elements of this set r*-continuous or uniformly continuous functions on* K.

It is clear that $C_r(K,E)$ *is contained in* $l_\infty(K,E)$, *the space of order bounded functions from* K *into* E, *since in* $E(e)$ *norm boundedness coincides with order boundedness. Moreover,* $C_r(K,E)$ *is a vector sublattice in* $l_\infty(K,E)$.

(2) $c_0(K,E(e))$ *the space of all mappings* d *from* K *into* $E(e)$ *such that for all* $\varepsilon > 0$, *the set* $\{k \in K : \|d(k)\|_e \geq \varepsilon\}$ *is finite. Then we set*

$$c_0^r(K,E) := \bigcup \{c_0(K,E(e)) : e \in E^+\}.$$

It is clear that $c_0^r(K,E)$ *is contained in* $l_\infty(K,E)$. *Moreover,* $c_0^r(K,E)$ *is a vector sublattice in* $l_\infty(K,E)$.

Now we give the following theorem which will be useful in the sequel.

Theorem 3.71 *Let* K *be a compact Hausdorff space. For any* $f \in C_r(K,E)$ *and* $\varepsilon > 0$ *there exist* $e \in E^+$ *and finite collections* $\varphi_1,\ldots,\varphi_n \in C(K)$ *and* $e_1,\ldots,e_n \in E$ *such that*

$$\sup_{\alpha\in K}\left| f(\alpha) - \sum_{k=1}^{n} \varphi_k(\alpha)e_k \right| \leq \varepsilon e.$$

Proof. By the assumption, $f \in C(K,E(e))$ for some $e \in E^+$. According to the Kakutani and Kreĭn Theorem, $E(e)$ is linearly isometric and lattice isomorphic to $C(Q)$ for some compact Hausdorff space Q. Therefore one can assume that $f \in C(K,C(Q))$. However, the spaces $C(K,C(Q))$ and $C(K\times Q)$ are isomorphic as Banach lattices. It remains to note that, according to the Stone-Weierstrass Theorem, the subspace of the functions $(\alpha,q) \mapsto \sum_{k=1}^{n} \varphi_k(\alpha)e_k(q)$, where $\varphi_1,\ldots,\varphi_n \in C(K)$ and $e_1,\ldots,e_n \in C(Q)$, is dense in $C(K\times Q)$. □

Definition 3.72 *Let* K *be a compact Hausdorff space without isolated points and* E *be a relatively uniformly complete vector lattice. We denote by* $CD_0^r(K,E)$ *the set of* E*-valued functions on* K *each of which is the sum of two* E*-valued functions* f *and* d, *where* $f \in C_r(K,E)$ *and* $d \in c_0^r(K,E)$.

For a finite subset S of K and $e \in E$, $\chi_S \otimes e$ is in $CD_0^r(K,E)$. It is easy to see that $CD_0^r(K,E)$ is an ordered vector space under the pointwise order.

We now give the following important lemma which will be used in this section.

Lemma 3.73 *Let K be a compact space and E be a relatively uniformly complete vector lattice. If $f \in C_r(K,E) \cap c_0^r(K,E)$, then there exists an element $e \in E^+$ such that the function $f \in C(K,E(e)) \cap c_0(K,E(e))$.*

Proof. Let $f \in C_r(K,E) \cap c_0^r(K,E)$. By assumption, there exist e_1 and $e_2 \in E^+$ such that $f \in C(K,E(e_1))$ and $f \in c_0(K,E(e_2))$. Let $e = e_1 \vee e_2$. Then clearly, $E(e_1)$ and $E(e_2) \subset E(e)$. We will show that $f \in C(K,E(e)) \cap c_0(K,E(e))$. Let $f \in C(K,E(e_1))$. As $E(e_1) \subset E(e)$, we have

$$\{\lambda > 0 : |f(x)| \le \lambda e_1,\ x \in K\} \subset \{\beta > 0 : |f(x)| \le \beta e,\ x \in K\}.$$

If we take the infimum of these sets, we get

$$\inf\{\beta > 0 : |f(x)| \le \beta e,\ x \in K\} \le \inf\{\alpha > 0 : |f(x)| \le \alpha e_1,\ x \in K\},$$

hence $\|f(x)\|_e \le \|f(x)\|_{e_1}$ and $f \in C(K,E(e))$.

Let $f \in c_0(K,E(e_2))$. As $E(e_2) \subset E(e)$, we have similary $\|f(x)\|_e \le \|f(x)\|_{e_2}$. Fix $\varepsilon > 0$. It follows that

$$\{x \in K : \varepsilon \le \|f(x)\|_e\} \subset \{x \in K : \varepsilon \le \|f(x)\|_{e_2}\},$$

hence $\{x \in K : \varepsilon \le \|f(x)\|_e\}$ is finite. Therefore $f \in c_0(K,E(e)$ and this completes the proof. $\square$

Lemma 3.74 *Let K be a compact Hausdorff space without isolated points and E be a relatively complete vector lattice. Then, $C_r(K,E) \cap c_0^r(K,E) = \{0\}$.*

Proof. Suppose the contrary; let $0 \neq f \in C_r(K,E) \cap c_0^r(K,E)$. Let us assume $f(x) \neq 0$. By using the previous lemma, there exists some $e \in E^+$ such that $f \in C(K,E(e)) \cap c_0(K,E(e))$. Then there exists a neighborhood V of x such that for $y \in V$ we have $\|f(y)\|_e > \|f(x)\|_e/2$. But since x is not isolated, V is uncountable, which is a contradiction since $f \in c_0(K,E(e))$. $\square$

It now follows that the decomposition of an element of $CD_0^r(K,E)$-space into a sum of an r-continuous function and one with finite support is unique. So $CD_0^r(K,E)$ deserves to be called an Abramovich–Wickstead space.

Lemma 3.75 *Let K be a compact Hausdorff space without isolated points and E be a relatively uniformly complete vector lattice. Let $p \in CD_0^r(K,E)$. Then $p^+ = \sup(p,0)$ exists in $CD_0^r(K,E)$*

Proof. Let $p \in CD_0^r(K,E)$. Let $r(k) = f^+(k) + [-f^-(k) + h(k)] \vee (-f^+(k))$ for each $k \in K$ where f and h are continuous and discrete parts of p respectively. Let $s(k) = (-f^-(k)+h(k)) \vee (-f^+(k))$. Let $\varepsilon > 0$ be given. Then there exists some $e \in E^+$ and $n_0 \in \mathbb{N}$ such that

$$\{k \in K : \varepsilon \leq \|s(k)\|_e\} \subset \{k \in K : \frac{1}{n_0} \leq \|h(k)\|_e\}.$$

Indeed, if this were not true, then for some sequence (k_n) in K, we would have $\varepsilon \leq \|s(k_n)\|_e$ while $\|h(k_n)\|_e < \frac{1}{n}$ for all $n \in \mathbb{N}$. By compactness of K, we can find a subnet (k_α) of (k_n) that converges to some $k_0 \in K$. As $\|h(k_\alpha)\|_e \to 0$ in $E(e)$, we have that

$$\varepsilon \leq \|s(k_\alpha)\|_e = \|(-f^-(k_\alpha) + h(k_\alpha)) \vee (-f^+(k_\alpha))\|_e \to \| - f^-(k_0) \vee (-f^+(k_0))\|_e = 0$$

which is a contradiction. Hence $r \in CD_0^r(K,E)$ whenever $p \in CD_0^r(K,E)$. On the other hand,

$$r(k) = f^+(k) + [-f^-(k) + h(k)] \vee (-f^+(k)) = [f^+(k) - f^-(k) + h(k)] \vee 0 = (p(k))^+$$

for each $k \in K$. So r is indeed p^+. Therefore, continuous part of r is f^+, where $f^+(k) = (f(k))^+$ by uniqueness of decomposition. □

We summarize what we have from the previous lemma as follows:

Proposition 3.76 *Let K be a compact Hausdorff space without isolated points and E be a relatively uniforly complete vector lattice. Then $CD_0^r(K,E)$ is a vector lattice under the pointwise ordering : $0 \leq p \in CD_0^r(K,E) \Leftrightarrow 0 \leq p(k)$ in E for all $k \in K$.*

Just like real-valued function space $CD_0(K)$ in [33], suprema and infima are easy to identify in $CD_0^r(K,E)$. We shall write $h_\gamma \uparrow h$ if the net h_γ is increasing and $sup(h_\gamma) = h$.

Proposition 3.77 *Let K be a compact Hausdorff space without isolated points and E be a relatively uniformly complete vector lattice. If $h_\gamma \uparrow h$ in $CD_0^r(K,E)$, then $h_\gamma(k) \uparrow h(k)$ in E for all $k \in K$.*

Proof. Let k_0 be an arbitrary but fixed point of K. Then $h(k_0)$ is an upper bound of $\{h_\gamma(k_0) : \gamma \in \Gamma\}$ in $E(e)$ for some $e \in E^+$. Let v be another upper bound for $\{h_\gamma(k_0) : \gamma \in \Gamma\}$. If $v \wedge h(k_0) = h(k_0)$, then the proof is obvious. On the other hand, if $v \wedge h(k_0) < h(k_0)$, then we can find some $0 < e_1 \in E(e)$ such that $v \wedge h(k_0) + e_1 \leq h(k_0)$. Then $h - \chi_{k_0} \otimes e_1$ is an upper bound in $CD_0^r(K, E)$ for the family $\{h_\gamma : \gamma \in \Gamma\}$, contradicting the definition of h. $\square$

From the proposition above, we conclude that order convergence in $CD_0^r(K, E)$ is pointwise.

3.7 Abramovich–Wickstead Spaces as Lattice–Normed Spaces

In this section, we show that Abramovich–Wickstead type spaces are actually decomposable *br*-complete lattice-normed spaces. For the sake of convenience, we give the following definitions which were given in [27].

Definition 3.78 *For a non-empty set K and a normed space E, we define $c_0(K, E)$ as the space of E-valued functions, f on K such that for each $\varepsilon > 0$ the set $\{k \in K : \|f(k)\| > \varepsilon\}$ is finite and $l_w^\infty(K, E)$ as the space of all bounded E-valued functions, d on K such that the set $\{k \in K : \|d(k)\| \neq 0\}$ is countable.*

Proposition 3.79 *Let K be a topological space and E be a normed space. Then $C(K, E)$ which is the space of E-valued continuous functions on K, $c_0(K, E)$ and $l_w^\infty(K, E)$ are decomposable lattice-normed spaces with norm lattices $C(K)$, $c_0(K)$ and $l_w^\infty(K)$ respectively.*

Proof. Let $f \in C(K, E)$. Define its vector norm by $|f|(k) = \|f(k)\|$ $(k \in K)$. Then $|f| \in C(K)$, since $f : K \to E$ and $\|.\| : E \to \mathbb{R}^+$ are continuous. Therefore the norm lattice of $C(K, E)$ is $C(K)$. $|.|$ is also a decomposable norm. Indeed, assume that $|f| = h_1 + h_2$ for some $h_1, h_2 \in C(K)^+$. Define a vector-valued function $f_1 : K \to E$ such that $f_1(k) = \frac{f(k)}{\|f(k)\|} h_1(k)$ when $f(k) \neq 0$ and $f_1(k) = 0$ when $f(k) = 0$. Let $k_\alpha \to k$ in K. Let $f(k) \neq 0$, then

$$f_1(k_\alpha) = \frac{f(k_\alpha)}{\|f(k_\alpha)\|} h_1(k_\alpha) \to \frac{f(k)}{\|f(k)\|} h_1(k) = f_1(k).$$

Let $f(k) = 0$. Suppose the contrary that $f_1(k_\alpha) \nrightarrow 0$. Then there exist some $\varepsilon > 0$ and a subnet (k_β) of (k_α) such that

$$\varepsilon < \|f_1(k_\beta)\| = h_1(k_\beta) \leq \|f(k_\beta)\| \to 0,$$

which is a contradiction. Therefore $f_1 \in C(K,E)$ and $f_2 := f - f_1 \in C(K,E)$. Moreover, $|f_k| = h_k$ $(k = 1,2)$, since

$$|f_1|(k) = \|f_1(k)\| = \|\frac{f(k)}{\|f(k)\|}h_1(k)\| = |h_1(k)| = h_1(k)$$

and

$$\begin{aligned}
|f_2|(k) = |f - f_1|(k) &= \|f(k) - f_1(k)\| = \|f(k) - \frac{f(k)}{\|f(k)\|}h_1(k)\| \\
&= \frac{\|\, f(k)\|f(k)\| - f(k)h_1(k)\|}{\|f(k)\|} \\
&= \frac{\|f(k)(h_1(k) + h_2(k)) - f(k)h_1(k)\|}{\|f(k)\|} \\
&= \frac{\|f(k)h_2(k))\|}{\|f(k)\|} = h_2(k).
\end{aligned}$$

Let now $f \in c_0(K,E)$. Define its vector norm $|f|(k) = \|f(k)\|$. Then we have $|f| \in c_0(K)$, since for each $\varepsilon > 0$, the set

$$\{k \in K : |f|(k) > \varepsilon\} = \{k \in K : \|f(k)\| > \varepsilon\}$$

is finite. $|.|$ is also a decomposable norm. Indeed, assume that $|f| = h_1 + h_2$ for some $h_1, h_2 \in c_0(K)^+$. Define a vector-valued function $f_1 : K \to E$ such that $f_1(k) = \frac{f(k)}{\|f(k)\|}h_1(k)$ when $f(k) \neq 0$ and $f_1(k) = 0$ when $f(k) = 0$. If $f(k) \neq 0$, then the set

$$\{k \in K : \|f_1(k)\| > \varepsilon\} = \{k \in K : h_1(k) > \varepsilon\}$$

is finite as $h_1 \in c_0(K)^+$ so that $f_1 \in c_0(K,E)$. If $f(k) = 0$, then $f_1(k) = 0$ so that clearly $f_1 \in c_0(K,E)$. Therefore $f_1 \in c_0(K,E)$ and $f_2 := f - f_1 \in c_0(K,E)$. Moreover, $|f_k| = h_k$ $(k = 1,2)$.

The third assertion can be proved similarly. □

The following definition was given in [27].

Definition 3.80 *Let K be a compact Hausdorff space without isolated points and E be a normed space. We define $CD_0(K,E)$ as the set of E-valued functions on K such that each of which is the sum of two E-valued functions f and d, where $f \in C(K,E)$ and $d \in c_0(K,E)$. Similarly, we define $CD_w(K,E)$ as the set of all E-valued functions on K each of which is the sum of two E-valued functions f and d, where $f \in C(K,E)$ and $d \in l_w^\infty(K,E)$.*

For a finite subset S of K and $e \in E$, the vector–valued function $\chi_S \otimes e$ is in $CD_0(K, E)$ where $\chi_S \otimes e(k) = e$ if $k \in S$ and 0 otherwise. On the other hand, for a countable subset S of K and $e \in E$, $\chi_S \otimes e$ is in $CD_w(K, E)$.

Lemma 3.81 *Let K be a compact Hausdorff space without isolated points and E be a normed space. Then $C(K, E) \cap l_w^\infty(K, E) = \{0\}$.*

Proof. Suppose the contrary; let $0 \neq f \in C(K, E) \cap l_w^\infty(K, E)$. Assume that $f(x) \neq 0$. Then there exists a neighborhood V of x such that for $y \in V$ we have $\|f(y)\| > \|f(x)\|/2$. But since x is not isolated, V is uncountable, which is a contradiction since $f \in l_w^\infty(K, E)$. □

It now follows that the decomposition of an element of $CD_0(K, E)$ or of $CD_w(K, E)$ into a sum of a continuous function and one with finite (or countable) support is unique.

Lemma 3.82 *Let K be a compact Hausdorff space without isolated points and E be a normed space. Then $CD_0(K, E)$ and $CD_w(K, E)$ are decomposable lattice-normed spaces with norm lattices $CD_0(K)$ and $CD_w(K)$ respectively.*

Proof. Let $f \in CD_0(K, E)$. Therefore $f = f_1 + f_2$ ($f_1 \in C(K, E)$, $f_2 \in c_0(K, E)$). We define its vector norm as $|f|(k) = |f_1 + f_2|(k) = \|f_1(k) + f_2(k)\|$ for each $k \in K$, since $|f_1| \in C(K)$ and $|f_2| \in c_0(K)$ are disjoint (Proposition 3.79) so that $|f| = |f_1 + f_2| = |f_1| + |f_2| \in C(K) \oplus c_0(K) = CD_0(K)$ (Lemma 2.3). $|.|$ is also a decomposable norm. Indeed, assume that $|f| = h_1 + h_2$ for some $h_1, h_2 \in CD_0(K)^+$. Then $|f_1 + f_2| = |f_1| + |f_2| = h_1 + h_2 = h_1' + h_1'' + h_2' + h_2''$ where $h_1', h_2' \in C(K)$ and $h_1'', h_2'' \in c_0(K)$. Then $|f_1| = h_1' + h_2'$ and $|f_2| = h_1'' + h_2''$. As $C(K, E)$ and $c_0(K, E)$ are decomposable lattice-normed spaces (Proposition 3.79), there exist $f_k' \in C(K, E)$ such that $|f_k'| = h_k'$ ($k = 1, 2$) and $f_1 = f_1' + f_2'$ and there exist $f_k'' \in c_0(K, E)$ such that $|f_k''| = h_k''$ ($k = 1, 2$) and $f_2 = f_1'' + f_2''$. Therefore $|f_k' + f_k''| = |f_k'| + |f_k''| = h_k' + h_k'' = h_k$ ($k = 1, 2$) and also we have $f = f_1 + f_2 = f_1' + f_2' + f_1'' + f_2''$.

The second assertion can be proved similarly. □

The following observation about the norm in $CD_0(K, E)$ and $CD_w(K, E)$ was given in [27]. For the sake of convenience, we give its proof.

Lemma 3.83 *Let K be a compact Hausdorff space without isolated points and E be a normed space. Then for each function $f \in C(K, E)$, we have*

$$\|f\| = \inf_{d \in c_0(K,E)} \|f + d\| = \inf_{d \in l_w^\infty(K,E)} \|f + d\|$$

where $\|f + d\| = \sup_{k \in K} \|f(k) + d(k)\|$.

Proof. As constantly zero function belongs to $c_0(K, E)$, we have $\inf_{d \in c_0(K,E)} \|f + d\| \le \|f\|$. Suppose the contrary, $\inf_{d \in c_0(K,E)} \|f + d\| < \|f\|$. Then for some $\varepsilon > 0$ and $d \in c_0(K, E)$, we would have $\|f + d\| + \varepsilon \le \|f\|$. As the mapping $k \mapsto \|f(k)\|$ is a continuous function from K to $\mathbb{R}$, we have $\|f + d\| + \varepsilon \le \|f(k_0)\|$ for some $k_0 \in K$. But the last inequality is not only true for $k_0 \in K$ but also for an open neighborhood U of k_0, i.e. $\|f(k) + d(k)\| + \varepsilon \le \|f(k)\|$ for all $k \in U$. As $d \in c_0(K, E)$ and U is uncountable, there exists some $k_1 \in U$ such that $d(k_1) = 0$. But then $\|f(k_1)\| + \varepsilon \le \|f(k_1)\|$ which is a contradiction.

The second equality can be proved similarly. □

Lemma 3.84 *Let K be a non-empty set and E be a normed space. Then $c_0(K, E)$ and $l_w^\infty(K, E)$ are closed subspaces of $B(K, E)$ the space of bounded E-valued functions on K.*

Proof. Suppose that a sequence of functions $(f_n) \in c_0(K, E)$ converges in supremum norm to $f \in B(K, E)$. Fix $\varepsilon > 0$, then $\|f_n - f\| < \frac{\varepsilon}{2}$ for some n. It follows that

$$\{k \in K : \|f(k)\| > \varepsilon\} \subset \{k \in K : \|f_n(k)\| > \frac{\varepsilon}{2}\},$$

hence $\{k \in K : \|f(k)\| > \varepsilon\}$ is finite.

The second assertion can be proved similarly. □

The following theorem which will be used in the sequel was given in [27].

Theorem 3.85 *Let K be a compact Hausdorff space without isolated points and E be a Banach space. Then $CD_0(K, E)$ and $CD_w(K, E)$ are Banach spaces under the supremum norm.*

Proof. Let (h_n) be a Cauchy sequence in $CD_0(K, E)$. Suppose $h_n = f_n + d_n$ where $f_n \in C(K, E)$ and $d_n \in c_0(K, E)$. Then from Lemma 3.83, $\|f_n - f_m\| \le \|h_n - h_m\|$ so that (f_n) is a Cauchy sequence in $C(K,E)$. Norm completeness of $C(K,E)$ yields a function $f \in C(K,E)$ such that $f_n \to f$ in supremum norm. The inequality

$$\|d_n - d_m\| = \|d_n + f_n - f_n + f_m - f_m - d_m\| \leq \|h_n - h_m\| + \|f_n - f_m\|$$

implies that (d_n) forms a Cauchy sequence in $c_0(K,E)$. It follows from Lemma 3.84 that $d_n \to d \in c_0(K,E)$. Therefore $h_n = f_n + d_n \to f + d \in CD_0(K,E)$, hence $CD_0(K,E)$ is a Banach space. The second assertion can be proved similarly as $l^{\infty}{}_w(K,E)$ is a closed subspace of $B(K,E)$. □

Remark 3.86 *Let us denote the supremum norm of $CD_0(K)$ by $\|.\|_\infty$. From [33], we know that $CD_0(K)$ is a Banach lattice so that its norm is monotone in the following sense: if*

$$|x| \leq |y| \Rightarrow \|x\|_\infty \leq \|y\|_\infty \ (x, y \in CD_0(K)).$$

Since $CD_0(K,E)$ is a lattice-normed space with norm lattice $CD_0(K)$, then $|f| \in CD_0(K)$ for each $f \in CD_0(K,E)$. Then

$$\|f\| = \sup_{k \in K} \|f(k)\| = \sup_{k \in K} |f|(k) = \| \, |f| \, \|_\infty,$$

so that $\|.\|$ is a mixed norm in $CD_0(K,E)$ which is introduced in the second chapter of this thesis. Hence $CD_0(K,E)$ is a Banach space with mixed norm $\|.\|$ from the previous theorem. In view of the inequality $| \ |f| - |g| \ | \leq |f - g|$ and monotonicity of the norm in $CD_0(K)$, we have

$$\| \, |f| - |g| \, \|_\infty \leq \|f - g\| \quad (f, g \in CD_0(K,E)),$$

so that the vector norm $|.|$ is a norm continuous function from $(CD_0(K,E), \|.\|)$ into $CD_0(K)$.

The same considerations yield that $CD_w(K,E)$ is a Banach space with mixed norm $\|.\|$ and the vector norm $|.|$ of $CD_w(K,E)$ is a norm continuous function from $(CD_w(K,E), \|.\|)$ into $(CD_w(K), \|.\|_\infty)$.

Actually the following lemma is a direct consequence of the Proposition 2.7. Nevertheless we give its proof for the sake of convenience.

Lemma 3.87 *Let K be a compact Hausdorff space without isolated points and E be a Banach space. Then $CD_0(K,E)$ and $CD_w(K,E)$ are br-complete lattice-normed spaces with $CD_0(K)$ and $CD_w(K)$-valued norms respectively.*

Proof. Suppose that a sequence $(f_n) \in CD_0(K,E)$ is br-fundamental; that is to say, $|f_n - f_m| \leq \lambda_k g$ $(m,n,k \in \mathbb{N}$ *and* $m,n \geq k)$ where $0 \leq g \in CD_0(K)$ and $\lim_{k\to\infty} \lambda_k = 0$. Then

$$||f_n - f_m|| \leq \lambda_k ||g||_\infty \to 0 \; as \; k \to \infty$$

Therefore the limit $f := \lim_{n\to\infty} f_n$ exists. By continuity of the vector norm, we have

$$|f - f_n| \leq \lambda_k g \quad (n \geq k),$$

therefore, $f = br\text{-}\lim f_n$.

The second assertion can be proved similarly. □

3.8 Aleksandrov duplicates and $CD_0(K)$–spaces

In this section, we define a concrete $E \times_G F$ space. We show that this space can be represented as the space of real-valued continuous functions on the Aleksandrov duplicate. As a corollary, we obtain the main result of [18].

Throughout this section, Σ denotes a compact Hausdorff topology on K and Γ denotes a locally compact Hausdorff topology on a non-empty subset A of K such that the identity map $i : (A,\Gamma) \to (A,\Sigma)$ is continuous. These spaces are denoted by K_Σ and A_Γ respectively. As usual, the Banach lattice of real-valued K_Σ-continuous functions on K equipped with sup norm and pointwise ordering is denoted by $C(K_\Sigma)$. $B(K)$ denotes the space of real-valued bounded functions on K. We denote the set **{**$d \in B(K) : d(k) = 0$ **for all** $k \notin A$**,** d **is** Γ**-continuous on** A **such that** $\forall\, \varepsilon > 0$, **there exists a compact set** M **in** A **with** $|d(k)| < \varepsilon$ **for each** $k \in A \setminus M$ **}** by $C_0(A_\Gamma)$ which is equipped with supremum norm and pointwise ordering.

Lemma 3.88 *$C_0(A_\Gamma)$ is a closed subspace of $B(K)$.*

Proof. Suppose that a sequence of functions (f_n) in $C_0(A_\Gamma)$ converges in supremum norm to $f \in B(K)$. Fix $\varepsilon > 0$, then $||f_n - f|| < \varepsilon/2$ for some n. It follows that

$$\{k \in A : |f(k)| \geq \varepsilon\} \subset \{k \in A : |f_n(k)| \geq \varepsilon/2\}.$$

As $\{k \in A : |f_n(k)| \geq \varepsilon/2\}$ is compact and the set $\{k \in A : |f(k)| \geq \varepsilon\}$ is closed so that $\{k \in A : |f(k)| \geq \varepsilon\}$ is compact. □

It follows that $C_0(A_\Gamma)$ equipped with sup norm and pointwise ordering is a Banach lattice.

$C(K_\Sigma) \times_{B(K)} C_0(A_\Gamma)$ is a special example of an $E \times_G F$ as seen by the following lemma.

Lemma 3.89 *$C(K_\Sigma) \times_{B(K)} C_0(A_\Gamma)$ equipped with coordinatewise algebraic operations is a Banach lattice with respect to the order*

$$0 \leq (f, d) \Leftrightarrow 0 \leq f(k) \textit{ for all } k \in K \textit{ and } 0 \leq f(a) + d(a) \textit{ for all } a \in A$$

and the norm

$$\|(f, d)\|=\max\{\|f\|, \|f + d\|\} \textit{ where } \|.\| \textit{ is the supremum norm.}$$

Proof. it is enough to show that $C_0(A_\Gamma)$ is an order ideal with respect to $C(K_\Sigma)$. That is to say $|f+d|-|f| \in C_0(A_\Gamma)$ for each $f \in C(K_\Sigma)$ and $d \in C_0(A_\Gamma)$. Clearly, $|f+d|-|f|$ is Γ-continuous on A. Also for each $\varepsilon > 0$,

$$\{a \in A : |f(a) + d(a)| - |f(a)| < \varepsilon\} \subset \{a \in A : |d(a)| < \varepsilon\}.$$

Then $C_0(A_\Gamma)$ is an order ideal with respect to $C(K_\Sigma)$. So $C(K_\Sigma) \times_{B(K)} C_0(A_\Gamma)$ is a Riesz space by Theorem 3.4. Moreover,

$$|(f, d)| = (|f|, |f + d| - |f|).$$

As $C(K_\Sigma)$ and $C_0(A_\Gamma)$ are Banach lattices, $C(K_\Sigma) \times_{B(K)} C_0(A_\Gamma)$ is a Banach lattice by Theorem 3.29. This completes the proof. □

From now on, $C(K_\Sigma) \times C_0(A_\Gamma)$ denotes $C(K_\Sigma) \times_{B(K)} C_0(A_\Gamma)$. If K_Σ has no isolated points and $A = K$ and Γ is discrete, then $C(K_\Sigma) \cap C_0(K_\Gamma) = \{0\}$, and $CD_0(K_\Sigma) = C(K_\Sigma) \oplus C_0(K_\Gamma)$ is a Banach lattice under pointwise ordering and supremum norm. Moreover, it is easy to see that $CD_0(K_\Sigma)$ and $C(K_\Sigma) \times C_0(K_\Gamma)$ are isometrically Riesz isomorphic spaces.

Let $K \times \{0\} \cup A \times \{1\}$ be topologized by the open base $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, where

$$\mathcal{A}_1 = \{H \times \{1\} : \text{H is } \Gamma\text{-open in A}\}$$

and

$$\mathcal{A}_2 = \{G \times \{0, 1\} \setminus M \times \{1\} : \text{G is } \Sigma\text{-open, M is } \Gamma\text{-compact in A}\}.$$

When Γ is discrete, this space is called **Aleksandrov duplicate** and denoted by $D(K, A)$, see [21]. For the sake of convenience, we denote $K \times \{0\} \cup A \times \{1\}$ under the topology defined above by $D(K_\Sigma \otimes A_\Gamma)$. In the case Γ is discrete topology and $A = K$, we denote this space by $A(K)$. The space $A(K)$ has been constructed by R. Engelking in [24] and it is generalized for arbitrary locally compact Hausdorff space in [25]. It is known that $A(K)$ is a compact Hausdorff space, see [24] and [16]. The space $A([0,1])$ (where $[0,1]$ is topologized under the usual metric) has been constructed by P. S. Aleksandrov and P. S. Uryson in [22] as an example of a compact Hausdorff space containing a discrete dense subspace.

We now give the following proposition which will be useful in the proof of main result of this section.

Proposition 3.90 (i) *K_Σ and the subspace $K \times \{0\}$ of $D(K_\Sigma \otimes A_\Gamma)$ are homeomorphic.*

(ii) *If $k_\alpha \to k$ in A_Γ, then $(k_\alpha, 1) \to (k, 1)$ in $D(K_\Sigma \otimes A_\Gamma)$.*

Proof.

(i) It is enough to show that the map $g : K_\Sigma \to K \times \{0\}$ defined by $g(k) = (k, 0)$ is a homeomorphism. Let (k_α) be a net in K_Σ such that $k_\alpha \to k$ in K_Σ. Let assume that $U = G \times \{0,1\} \setminus M \times \{1\}$ is a neighborhood of $(k, 0)$ in $D(K_\Sigma \otimes A_\Gamma)$ as above. Then $(k, 0) \in G \times \{0\}$. So $k \in G$. As $k_\alpha \to k$, there exists an α_0 such that $k_\alpha \in G$ for each $\alpha \geq \alpha_0$. Then $(k_\alpha, 0) \in U$ for each $\alpha \geq \alpha_0$. So $(k_\alpha, 0) \to (k, 0)$ in $D(K_\Sigma \otimes A_\Gamma)$. Conversely, assume that $(k_\alpha, 0) \to (k, 0)$ in $D(K_\Sigma \otimes A_\Gamma)$. Let G be a Σ-open neighborhood of k. Then $G \times \{0,1\} \setminus M \times \{1\}$ where M is Γ-compact is a neighborhood of $(k, 0)$ in $D(K_\Sigma \otimes A_\Gamma)$. Then there exists an α_0 such that $(k_\alpha, 0) \in G \times \{0,1\} \setminus M \times \{1\}$ for each $\alpha \geq \alpha_0$. So $k_\alpha \in G$ for each $\alpha \geq \alpha_0$. This implies that $k_\alpha \to k$ in K_Σ. So g is a homeomorphism. This completes the proof.

(ii) Let $k_\alpha \to k$ in A_Γ. Let U be a neighborhood of $(k, 1)$ in $D(K_\Sigma \otimes A_\Gamma)$. Then either $U = H \times \{1\}$ where H is Γ-open in A or $U = G \times \{0,1\} \setminus M \times \{1\}$ where G is Σ-open in K and M is Γ-compact in A.

If $U = H \times \{1\}$, then $k \in H$. So there exists an α_0 such that $k_\alpha \in H$ for each $\alpha \geq \alpha_0$. Then $(k_\alpha, 1) \in H \times \{1\}$ for each $\alpha \geq \alpha_0$.

If $U = G \times \{0,1\} \setminus M \times \{1\}$, then $k \in G$ but $k \notin M$. As the identity map $i : (A, \Gamma) \to (A, \Sigma)$ is continuous, $k_\alpha \to k$ in (A, Σ). So there exists an α_0 such that $k_\alpha \in G$ for each $\alpha \geq \alpha_0$. Then $(k_\alpha, 1) \in G \times \{0,1\} \setminus M \times \{1\}$ as M is Γ-compact and $k \notin M$.

In both of two cases, $(k_\alpha, 1) \to (k, 1)$ in $D(K_\Sigma \otimes A_\Gamma)$. This completes the proof.

$\square$

Further, we will often identify K_Σ with $K \times \{0\}$.

Lemma 3.91 *$D(K_\Sigma \otimes A_\Gamma)$ is compact. In particular $K \times \{0\}$ is a closed subspace of $D(K_\Sigma \otimes A_\Gamma)$.*

Proof. Consider an open cover $\{O_i\}_{i \in I}$ of $D(K_\Sigma \otimes A_\Gamma)$. By replacing each set in the cover by a union of basic open neighborhoods of all points in the set, we can assume that the cover is formed by basic open neighborhoods. Hence the cover is of the form

$$\{H_\alpha\}_{\alpha \in I} \cup \{G_\gamma \times \{0, 1\} \setminus M_\gamma \times \{1\}\}_{\gamma \in \Omega}$$

where H_α is a Γ-open set in A and G_γ is a Σ-open set and M_γ is a Γ-compact set in A. It is easy to see that $\{G_\gamma \times \{0\}\}_{\gamma \in \Omega}$ is an open cover of $K \times \{0\}$, so that there is a finite subcover $G_{\gamma_1} \times \{0\}, ..., G_{\gamma_n} \times \{0\}$. But then

$$G_{\gamma_1} \times \{0, 1\} \setminus M_{\gamma_1} \times \{1\} \cup ... \cup G_{\gamma_n} \times \{0, 1\} \setminus M_{\gamma_n} \times \{1\}$$

only misses finitely many Γ-compact sets $M_{\gamma_1} \times \{1\}, ..., M_{\gamma_n} \times \{1\}$.
As M_{γ_j} (j=1,2,...n) is compact in A , then we have $M_{\gamma_j} \times \{1\} \subset \cup H_\alpha \times \{1\}$. Therefore we have $M_{\gamma_j} \times \{1\} \subset \cup_{p=1}^{n} H_{p^j} \times \{1\}$. Hence if we add the corresponding open sets from the cover then we obtain a finite cover of the entire $D(K_\Sigma \otimes A_\Gamma)$. $\square$

Note that $D(K_\Sigma \otimes A_\Gamma)$ is not a Hausdorff space as we can not seperate the points $(k, 0)$ and $(k, 1)$ for each $k \in K$.

Definition which is similar to the following was introduced in [34].

Definition 3.92 *Let $((k_\alpha, r_\alpha))$ be a net in $K \times \{0\} \cup A \times \{1\}$. We say that the net $((k_\alpha, r_\alpha))$ converges to (k, r) in $K \times \{0\} \cup A \times \{1\}$ (in notation $(k_\alpha, r_\alpha) \longrightarrow (k, r)$) if*

$$f(k_\alpha) + r_\alpha d(k_\alpha) \longrightarrow f(k) + rd(k)$$

for each $f \in C(K_\Sigma)$ and $d \in C_0(A_\Gamma)$. $D(K \odot A)$ denotes $K \times \{0\} \cup A \times \{1\}$ equipped with this convergence.

The proof of the following theorem is a simple consequence of the above definition.

Theorem 3.93 *Under the convergence in the previous definition $D(K \odot A)$ is a Hausdorff topological space (not necessarily $\Sigma \cap A \subset \Gamma$).*

In [34], it was proved that $D(K \odot K)$ is a compact Hausdorff space under the convergence given in the above definition if K_Σ has no isolated point and K_Γ is discrete. For certain Banach lattices, some representations of it have been constructed in [34] with the topology induced by this.

Mimicking the proof of Theorem 3 in [18], we can identify $C(K_\Sigma) \times C_0(A_\Gamma)$ with the space of real-valued continuous functions on $D(K_\Sigma \otimes A_\Gamma)$ as follows.

Theorem 3.94 *$C(D(K_\Sigma \otimes A_\Gamma))$ and $C(K_\Sigma) \times C_0(A_\Gamma)$ are isometrically Riesz isomorphic spaces.*

Proof. Let us assume that $f : K \times \{0\} \cup A \times \{1\} \to \mathbb{R}$ is a map. We claim that $f \in C(D(K_\Sigma \otimes A_\Gamma))$ if and only if

(i) the map $h : K \to \mathbb{R}$ defined by $h(k) = f(k, 0)$ is Σ-continuous, and

(ii) the map $d : A \to \mathbb{R}$ defined by $d(k) = f(k, 1) - f(k, 0)$ belongs to the space $C_0(A_\Gamma)$.

Indeed, suppose that (i) and (ii) are satisfied. Then the map $k \to f(k, 1)$ is $\Gamma-$ continuous on A. Because it is the sum of $d(k)$ and $h(k)$. The first one is Γ-continuous on A by (ii), the second one is $\Sigma-$ continuous by (i), and so is Γ-continuous on A as $\Sigma \cap A \subset \Gamma$. It follows that f is continuous at each point of $A \times \{1\}$.

Let $k \in K$. Let us show that f is continuous at $(k, 0)$. Let $\varepsilon > 0$ be given. Then

$$H = \{k \in A : |f(k, 1) - f(k, 0)| \geq \varepsilon/2\}$$

is Γ- compact by (ii). Further, by (i) there is a Σ-open set G containing k such that we have $|f(k, 0) - f(l, 0)| < \varepsilon/2$ for each $l \in G$.

Set $U = (G \times \{0, 1\}) \backslash H \times \{1\}$. Then we have that U is a neighborhood of $(k, 0)$ in $D(K_\Sigma \otimes A_\Gamma)$. Further, if $(l, i) \in U$, then:

either $i = 0$, then $|f(k, 0) - f(l, 0)| < \varepsilon/2 < \varepsilon$, or $i = 1$, then $l \notin H$ and hence

$$|f(l,1) - f(k,0)| \leq |f(l,1) - f(l,0)| + |f(l,0) - f(k,0)| < \varepsilon.$$

Conversely, suppose that f is continuous. Then clearly (i) holds. Further, the map $k \to f(k,1)$ on A is Γ-continuous, and so is $d(k)$. It remains to show that the set

$$H = \{k \in A : |d(k)| = |f(k,1) - f(k,0)| \geq \varepsilon\}$$

is Γ- compact for each $\varepsilon > 0$.

Suppose that for some $\varepsilon > 0$, H is not Γ-compact. Now, by compactness of (K, Σ) there is $k \in K$ such that $cl_\Gamma(G \cap A) \cap H$ is not Γ-compact for any Σ- neighbourhood G of k (otherwise H would be covered by finitely many Γ- compact subsets and hence itself would be Γ-compact).

Let $U = (G \times \{0,1\}) \setminus M \times \{1\}$ be a basic open set in $D(K_\Sigma \otimes A_\Gamma)$ containing $(k,0)$ such that for each $(l,i) \in U$ we have $|f(l,i) - f(k,0)| < \varepsilon/2$. As $cl_\Gamma(G \cap A) \cap H$ is not Γ-compact, there is $l \in H \cap (G \cap A \setminus M)$. Then both (l,i) and $(l,0)$ belong to U, hence $|f(l,1) - f(l,0)| < \varepsilon$. However, $|f(l,1) - f(l,0)| \geq \varepsilon$ (as $l \in H$), a contradiction. This proves the claim.

Define the map $\pi : C(K_\Sigma) \times C_0(A_\Gamma) \to C(D(K_\Sigma \otimes A_\Gamma))$ by $\pi(f,d)(k,r) = f(k)$ for each $k \in K \setminus A$ and $\pi(f,d)(k,r) = f(k) + d(k)$ for each $k \in A$. It is clear that π is a bipostive, one-to-one linear operator. Let $f \in C(D(K_\Sigma \otimes A_\Gamma)$ be given. Define $h : K \to \mathbb{R}$ by $h(k) = f(k,0)$ and $d : A \to \mathbb{R}$ by $d(k) = f(k,1) - f(k,0)$. Then from the above observation $(h,d) \in C(K_\Sigma) \times C_0(A_\Gamma)$ and $\pi(h,d) = f$. So π is also onto. It is also clear that $\|\pi(f,d)\| = \|(f,d)\|$. $\square$

As a corollary to the previous theorem, we obtain the main result of [18].

Corollary 3.95 *$C(D(K_\Sigma \otimes K_\Gamma))$ and $C(K_\Sigma) \times C_0(K_\Gamma)$ are isometrically Riesz isomorphic spaces.*

The following theorem is a consequence of previous theorem.

Theorem 3.96 *$D(K_\Sigma \otimes A_\Gamma)$ and $D(K \odot A)$ are homeomorphic spaces.*

Proof. It is enough to show that the identity map, $i : D(K_\Sigma \otimes A_\Gamma) \to D(K \odot A)$ is a homeomorphism. Let $(k_\alpha, r_\alpha) \to (k,r)$ in $D(K_\Sigma \otimes A_\Gamma)$. Then we have that $F(k_\alpha, r_\alpha) \to F(k,r)$ for each $F \in C(D(K_\Sigma \otimes A_\Gamma))$. From the previous theorem,

$$F(k_\alpha, r_\alpha) = \pi(f,d)(k_\alpha, r_\alpha) \to F(k,r) = \pi(f,d)(k,r)$$

for each $f \in C(K_\Sigma)$ and $d \in C_0(A_\Gamma)$.

This implies that $f(k_\alpha) + r_\alpha d(k_\alpha) \to f(k) + rd(k)$. So we get $(k_\alpha, r_\alpha) \to (k, r)$ in $D(K \odot A)$. This completes the proof. □

If K_Σ has no isolated points, $A = K$ and Γ is discrete, then we get the following result of [34].

Corollary 3.97 *If K_Σ has no isolated points, then the spaces $CD_0(K)$ and $C(A(K))$ are isometrically Riesz isomorphic spaces.*

CHAPTER 4

LINEAR OPERATORS ON ABRAMOVICH – WICKSTEAD SPACES

4.1 Linear Operators on Generalized Abramovich – Wickstead Spaces

In this section, we characterize some types of linear operators on generalized Abramovich–Wickstead spaces such as order continuous, weakly compact, M–weakly compact, L–weakly compact and absolutely summing operators. Throughout this section, we assume that E and F are Riesz subspaces of a Riesz space G such that F is an order ideal with respect E. Let $T : G \to X$ be a linear operator where G and X are normed Riesz spaces. Consider the operator $\overline{T} : E \times_G F \to X$ such that $\overline{T}(e, f) = Te + Tf = T(e + f)$. Then we have the following.

Proposition 4.1 *Let G and X be normed Riesz spaces. If $T : G \to X$ is a linear operator, then T induces a linear map $\overline{T} : E \times_G F \to X$ such that $\overline{T}(e, f) = Te + Tf = T(e + f)$. Moreover, we have*

(1) If T is a positive operator, then $\overline{T}$ is positive.

(2) If T is a Riesz homomorphism, then $\overline{T}$ is a Riesz homomorphism.

(3) If T is a continuous operator, then $\overline{T}$ is a continuous operator with respect to $\|.\|_{E\times_G F}$.

Proof. The linearity of induced map directly follows from the linearity of T.

(1) Suppose that T is positive and let $(0, 0) \leq (e, f)$ in $E \times_G F$. Then $0 \leq e$ and $0 \leq e + f$ in G. So $\overline{T}(e, f) = T(e + f)$ is positive.

(2) Suppose that T is a Riesz homomorphism. Then

$$\overline{T}|(e, f)| = \overline{T}(|e|, |e + f| - |e|) = T(|e + f|) = |T(e + f)| = |\overline{T}(e, f)|,$$

hence $\overline{T}$ is a Riesz homomorphism.

(3) Suppose that T is a continuous operator. Let $\|(x_n, y_n)\| \to 0$ in $E \times_G F$ with respect to the norm $\|(x, y)\|_{E\times_G F} = \max\{\|x\|, \|x + y\|\}$. Then $\|x_n\| \to 0$ and $\|x_n + y_n\| \to 0$ in G. Then $\|\overline{T}(x_n, y_n)\| = \|T(x_n + y_n)\| \to 0$ in X, since T is a continuous operator. □

Concerning the order continuity of the induced map $\overline{T}$, we have the following characterization.

Proposition 4.2 *Let G and X be Riesz spaces with X Dedekind complete. If T is an order continuous operator, then $\overline{T}$ is an order continuous operator.*

Proof. Let $(e_\alpha, f_\alpha) \downarrow (0, 0)$ in $E \times_G F$. This implies that $e_\alpha \downarrow 0$ and $e_\alpha + f_\alpha \downarrow 0$ in G. As T is order continuous,

$$\inf |\overline{T}(e_\alpha, f_\alpha)| = \inf |T(e_\alpha + f_\alpha)| = 0,$$

hence $\overline{T}$ is an order continuous operator. □

Definition 4.3 *The Riesz space L is said to have order continuity property if every lattice homomorphism on L is order continuous.*

Proposition 4.4 *Let G be a Riesz space with order continuity property and E be a majorizing Riesz subspace of G. Then $E \times_G F$ has order continuity property for every order ideal F with respect to E.*

Proof. Suppose that G is a Riesz space which has order continuity property, X is a Dedekind complete Riesz space and E is a majorizing Riesz subspace of G. Let $T : E \times_G F \to X$ be a lattice homomorphism. Using the lattice embedding $J : E \to E \times_G F$, then the map $\pi : E \to E \times_G F \to X$ is a lattice homomorphism. Then we may extend π to G such that the extended map, $\overline{\pi}$, is a lattice homomorphism. As G has order continuity property, then the map T is order continuous by previous proposition. □

Proposition 4.5 *Let E be a Riesz subspace of G and X be a Riesz space and $\pi : E \to X$ be a lattice homomorphism. If there is a lattice homomorphism extension of π to G, then there is a lattice homomorphism extension of π to $E \times_G F$.*

Proof. Let $\Phi_{E\times_G F}(\pi)$ and $\Phi_G(\pi)$ denote the set of all possible lattice homomorphism extensions of π to $E \times_G F$ and G respectively. We will show that $\Phi_{E\times_G F}(\pi) \neq \emptyset$. Let $\overline{\pi} \in \Phi_G(\pi)$. Then define a map $\pi_0 : E \times_G F \to X$ by $\pi_0(e, f) = \overline{\pi}(e + f)$. As $\overline{\pi}$ is a lattice homomorphism, so is π_0. Therefore $\pi_0 \in \Phi_{E\times_G F}(\pi) \neq \emptyset$. □

Let G be a Banach lattice and X be a Banach space. By using the linear map $T : G \to X$, we may characterize many properties of the induced map $\overline{T} : E \times_G F \to X$ defined by $\overline{T}(e, f) = T(e + f)$ where E and F are Banach lattices in G such that F is an order ideal with respect to E. We consider the norm

$$\|(x, y)\| = \max\{\|x\|, \|x + y\|$$

on $E \times_G F$. The details are in the following proposition.

Proposition 4.6 (1) *Assume that $T : G \to X$ is a compact operator. Then $\overline{T} : E \times_G F \to X$ is a compact operator. If the given operator T factors through a Banach space Z with continuous factors, then so does the operator $\overline{T}$.*

(2) *If $T : G \to X$ is a weakly compact operator, then $\overline{T} : E \times_G F \to X$ is a weakly compact operator.*

(3) *Let $T : G \to X$ be a continuous operator. If T is an order weakly compact operator, then $\overline{T}$ is an order weakly compact operator.*

(4) *Let $T : G \to X$ be a continuous operator. If T is an M-weakly compact operator, then $\overline{T}$ is an M-weakly compact operator.*

(5) *Let $T : G \to X$ be a continuous operator. If T is an L-weakly compact operator, then $\overline{T}$ is an L-weakly compact operator.*

(6) *If $T : G \to X$ is a Dunford-Pettis operator, then $\overline{T}$ is a Dunford-Pettis operator.*

(7) *Let $T : G \to X$ be a continuous operator. If T is a cone absolutely summing operator, i.e., $(\|Tx_n\|) \in l_1$ for every order bounded disjoint sequence $(x_n) \in G^+$, then $\overline{T}$ is a cone absolutely summing operator.*

Proof.

(1) Let (x_n, y_n) be a norm bounded sequence in $E \times_G F$. Then there exists $M > 0$ such that $\|(x_n, y_n)\| \leq M$ for each $n \in \mathbb{N}$. This implies that $\max\{\|x_n\|, \|x_n + y_n\|\} \leq M$. This gives us that (x_n) and $(x_n + y_n)$ are norm bounded sequences in G. As T is a compact operator, $T(x_n + y_n)$ has a convergent subsequence $T(x_{n_k} + y_{n_k})$ in X. But $\overline{T}(x_{n_k}, y_{n_k}) = T(x_{n_k} + y_{n_k})$. So $\overline{T}(x_n, y_n)$ has a convergent subsequence in X. Therefore $\overline{T}$ is a compact operator. Now assume that T factors through a Banach space Z with continuous factors, i.e., there exist continous operators $R : G \to Z$ and $S : Z \to X$ such that $T = SR$. Let $\overline{R} : E \times_G F \to Z$ be defined as $\overline{R}(e, f) = R(e + f)$. As R is a continuous operator, so is $\overline{R}$. Then we have that $S\overline{R}(e, f) = SR(e + f) = T(e + f) = \overline{T}(e, f)$.

(2) Let (x_n, y_n) be a norm bounded sequence in $E \times_G F$. Then there exists $M > 0$ such that $\|(x_n, y_n)\| \leq M$ for each $n \in \mathbb{N}$. This implies that $\max\{\|x_n\|, \|x_n + y_n\|\} \leq M$. This gives us that (x_n) and $(x_n + y_n)$ are norm bounded sequences in G. As T is a weakly compact operator, $T(x_n + y_n)$ has a weakly convergent subsequence $T(x_{n_k} + y_{n_k})$ in X. But we have $\overline{T}(x_{n_k}, y_{n_k}) = T(x_{n_k} + y_{n_k})$. So $\overline{T}(x_n, y_n)$ has a weakly convergent subsequence in X. Therefore $\overline{T}$ is a weakly compact operator.

(3) Let $T : G \to X$ be a continuous operator. Then the induced map $\overline{T} : E \times_G F \to X$ is a continuous operator. Let (x_n, y_n) be an order bounded disjoint sequence of $E \times_G F$. Then there exists a positive element (x, y) in $E \times_G F$ such that $|(x_n, y_n)| \leq (x, y)$. This implies that $|x_n| \leq x$ and $|x_n + y_n| \leq (x + y)$. As (x_n, y_n) is a disjoint sequence in $E \times_G F$, we have that (x_n) is an order bounded disjoint sequence in E and $(x_n + y_n)$ is an order bounded disjoint sequence in G. As T is an order weakly compact operator, we have that $\lim \|\overline{T}(x_n, y_n)\| = \lim \|T(x_n + y_n)\| = 0$. Thus $\overline{T}$ is an order weakly compact operator.

(4) Let $T : G \to X$ be a continuous operator. Then $\overline{T} : E \times_G F \to X$ is a continuous operator. Assume that T is an M-weakly compact operator. Let (x_n, y_n) be a norm bounded disjoint sequence of $E \times_G F$. Then $(x_n + y_n)$ is a norm bounded disjoint sequence of G. We have that $\lim \|\overline{T}(x_n, y_n)\| = \lim \|T(x_n + y_n)\| = 0$, since T is an M-weakly compact operator. So $\overline{T}$ is an M-weakly compact operator.

(5) Let $T : G \to X$ be a continuous operator. Then the induced map defined above, $\overline{T} : E \times_G F \to X$ is a continuous operator. Assume that T is an L-weakly compact operator. Let (x_n) be a disjoint sequence in the solid hull of $\overline{T}(U_{E\times_G F})$, $Sol(\overline{T}(U_{E\times_G F}))$, where $U_{E\times_G F}$ denotes the closed unit ball of $E \times_G F$. As $\overline{T}(U_{E\times_G F}) \subset T(U_G)$ where U_G

denotes the closed unit ball of G, we have that $Sol(\overline{T}(U_{E\times_G F}) \subset Sol(T(U_G))$. So (x_n) is a disjoint sequence in $Sol(T(U_G))$. Then $\lim \|x_n\| = 0$, since T is an L-weakly compact operator. So $\overline{T}$ is an L-weakly compact operator.

(6) Let $T : G \to X$ be a Dunford-Pettis operator. Let $(x_n, y_n) \to 0$ be a weakly convergent sequence in $E \times_G F$. As the projections $P : E \times_G F \to E$ and $Q : E \times_G F \to F$ are norm continuous, they are weakly continous, we have that $x_n \to 0$ weakly in E and $y_n \to 0$ weakly in F. Then $x_n + y_n \to 0$ weakly in G. Then

$$\lim \|\overline{T}(x_n, y_n)\| = \lim \|T(x_n + y_n)\| = 0,$$

since T is a Dunford-Pettis operator. So $\overline{T}$ is a Dunford-Pettis operator.

(7) Let $T : G \to X$ be a cone absolutely summing operator. Let (x_n, y_n) be an order bounded disjoint sequence in $(E \times_G F)^+$. Then (x_n) and $(x_n + y_n)$ are order bounded disjoint sequences in G^+. We get that $(\|\overline{T}(x_n, y_n)\|) \in l_1$, since $(\|\overline{T}(x_n, y_n)\|) = (\|T(x_n + y_n)\|)$. So $\overline{T}$ is cone absolutely summing.

□

4.2 Linear Operators on A New Type of Abramovich–Wickstead Spaces

Throughout this section, unless stated otherwise, E will denote a *relatively uniformly complete vector lattice* and for a vector valued function f, $\chi_k \otimes f$ will denote the function which takes $f(k)$ at k and 0 otherwise. In this section we give two characterizations about the regular and order continuous regular operators from $CD_0^r(K, E)$ into a Dedekind complete vector lattice F. The symbols L^r and L_n^r will denote the space of regular and order continuous regular operators respectively.

We start with the following lemma which will be used in the sequel.

Lemma 4.7 *Let K be a compact Hausdorff space and F be a Dedekind complete vector lattice. Then for every positive linear operator $T : C_r(K, E) \to F$ there exists a positive operator $T' : C(K) \to L^r(E, F)$ such that*

$$T(\varphi \otimes e) = T'(\varphi)e \ \text{for all } \varphi \in C(K) \text{ and } e \in E.$$

The correspondence $T \to T'$ is linear and one-to-one.

Proof. Let $T : C_r(K, E) \to F$ be a positive linear operator. For each $\varphi \in C(K)$ and $e \in E$, the function $\varphi \otimes e$ defined by $\varphi \otimes e(k) = \varphi(k)e$ belongs to $C_r(K, E)$; we put

$$T(\varphi \otimes e) = T'(\varphi)e \ for\ all\ \varphi \in C(K)\ and\ e \in E.$$

For fixed $\varphi \in C(K)$, the mapping $T'(\varphi) : e \mapsto T'(\varphi)e$ of E into F is evidently linear. Moreover, if $0 \leq e \in E$ and $0 \leq \varphi \in C(K)$, then $T'(\varphi)e = T(\varphi \otimes e) \geq 0$, therefore $T'(\varphi) \in L_+(E, F)$. Thus, the mapping $T' : \varphi \to T'(\varphi)$ of $C(K)$ into $L^r(E, F)$ is linear and positive.

It is easy to verify that the mapping $T \mapsto T'$ is linear. In order to prove that this mapping is one-to-one, let $S : C_r(K, E) \to F$ be a positive linear operator such that

$$S(\varphi \otimes e) = T'(\varphi)e,\ \ for\ \varphi \in C(K)\ and\ e \in E.$$

Let $f \in C_r(K, E)$. Then by Theorem 3.71, there exists a sequence (f_n) of the form $\sum \varphi_i \otimes e_i$ (finite sum) with $\varphi_i \in C(K)$ and $e_i \in E$ converging relatively uniformly to f. Then we have $T(f_n) = S(f_n)$ for every n. On the other hand T and S are relatively uniformly continuous on $C_r(K, E)$, therefore

$$T(f) = \lim_{n\to\infty} T(f_n) = \lim_{n\to\infty} S(f_n) = S(f),$$

consequently $T = S$. $\square$

Theorem 4.8 *Let K be a compact space F be a Dedekind complete vector lattice. Then there exists a lattice isomorphism $T' \leftrightarrow \mu$ between the set of regular operators $T' : C(K) \to L^r(E, F)$ and the set of countably additive quasiregular Borel measures $\mu : K \to L^r(E, F)$ given by the equality*

$$T'(f) = \int f\, d\mu, \quad for\ every\ f \in C(K).$$

Proof. Proof directly follows from Theorem 2.1, since $L^r(E, F)$ is a Dedekind complete vector lattice. $\square$

Let F be a Dedekind complete vector lattice and $\mu \in qca(K, L^r(E, F))$. Then the integral $I_\mu : C(K) \to L^r(E, F)$ can be extended to $C_r(K, E)$. We can identify the algebraic tensor product $C(K) \otimes E$ with a subspace in $C_r(K, E)$, assigning the mapping $\alpha \mapsto \sum_{k=1}^{n} \varphi_k(\alpha)e_k$

($\alpha \in K$) to $\sum_{k=1}^{n} \varphi_k \otimes e_k$, where $e_k \in E$ and $\varphi_k \in C(K)$. Define I_μ on $C(K) \otimes E$ by the formula

$$I_\mu\left(\sum_{k=1}^{n} \varphi_k \otimes e_k\right) := \sum_{k=1}^{n} \int_K \varphi_k \, d\mu e_k.$$

If $f \in C_r(K, E)$, then using Theorem 3.71, there exist $e \in E^+$ and a sequence $(f_n) \subset C(K) \otimes E$ such that

$$\sup_{\alpha \in K} |f(\alpha) - f_n(\alpha)| \le \frac{1}{n} e.$$

Put by definition

$$\int_K f \, d\mu := I_\mu(f) := o\text{-}\lim I_\mu(f_n).$$

The soundedness of the above definitions easily follows from the following lemma.

Lemma 4.9 *Let K be a compact Hausdorff space and F be a Dedekind complete vector lattice. Then for every positive linear operator $T' : C(K) \to L^r(E, F)$, there exists a unique positive linear operator $T : C_r(K, E) \to F$ such that*

$$T'(\varphi)e = T(\varphi \otimes e) \ for\ every\ \varphi \in C(K)\ and\ e \in E.$$

Proof. Let $T' : C(K) \to L^r(E, F)$ be a positive operator. Define an operator $\overline{T} : C(K) \otimes E \to F$ by setting

$$\overline{T}\left(\sum_{k=1}^{n} \varphi_k \otimes e_k\right) := \sum_{k=1}^{n} T'(\varphi_k)(e_k) \quad (\varphi_k \in C(K)\ and\ e_k \in E).$$

As T' is linear and positive, $\overline{T}$ is a linear and positive operator.

Let $f \in C_r(K, E)$. From Theorem 3.71, there exists a sequence $(f_n) \in C(K) \otimes E$ of the form $\sum \varphi_i \otimes e_i$ (finite sum) with $\varphi_i \in C(K)$ and $e_i \in E$ converging relatively uniformly to f. Then (f_n) is a relatively uniformly Cauchy sequence. Since $\overline{T}$ is linear and positive, we have

$$|\overline{T}(f_n) - \overline{T}(f_m)| = |\overline{T}(f_n - f_m)| \le \overline{T}|f_n - f_m|.$$

This shows that $(\overline{T}(f_n))$ is a relatively uniformly Cauchy sequence in F because (f_n) converges. As F is Dedekind complete, then $(\overline{T}(f_n))$ converges relatively uniformly to an element of F. Let (f_n) and (f'_n) be two sequences in $C(K) \otimes E$ such that $f_n \to f$ and $f'_n \to f$ relatively uniformly. Then from the inequality,

$$|f_n - f'_n| \le |f_n - f| + |f'_n - f|,$$

we get that $\lim_n \overline{T}(f_n) = \lim_n \overline{T}(f'_n)$. Hence we define a map $T : C_r(K, E) \to F$ by setting

$$T(f) = \lim_n \overline{T}(f_n).$$

The mapping T defined above coincides with $\overline{T}$ on $C(K) \otimes E$ since for each $f \in C_r(K, E)$ we can set $f_n = f$ for each n. This implies that $T'(\varphi)e = T(\varphi \otimes e)$ for every $\varphi \in C(K)$ and $e \in E$. The linearity and positivity of T come from the linearity and positivity of T'.

For uniqueness, let T'' be another extension of $\overline{T}$. We want to prove that if $T(a) = T''(a)$ for every $a \in C(K) \otimes E$, then $T = T''$. For each $f \in C_r(K, E)$, there exists a sequence $f_n \in C(K) \otimes E$ by density such that $f_n \to f$ relatively uniformly. We have $T(f_n) = T''(f_n)$ and so

$$T(f) = \lim_n T(f_n) = \lim_n T''(f_n) = T''(f).$$

This completes the proof. □

Theorem 4.10 *Let K be a compact Hausdorff space and F be a Dedekind complete vector lattice. Then there exists a lattice isomorphism $T \leftrightarrow T'$ between the space of regular operators $T : C_r(K, E) \to F$ and the space of regular operators $T' : C(K) \to L^r(E, F)$ given by the equality*

$$T(\varphi \otimes e) = T'(\varphi)e, \ \ for\ \varphi \in C(K) \ and\ e \in E.$$

If T and T' are in correspondence, then there exists a common countably additive quasiregular Borel measure $\mu : K \to L^r(E, F)$ such that

$$T(f) = \int f \, d\mu, \quad for\ f \in C_r(K, E),$$

and

$$T'(\varphi) = \int \varphi \, d\mu, \quad for\ \varphi \in C(K).$$

Proof. Let $T : C_r(K, E) \to F$ be a regular operator. Let $T' : C(K) \to L^r(E, F)$ be the regular operator corresponding to T (Lemma 4.7) by the equality

$$T(\varphi \otimes x) = T'(\varphi)x, \ \ for\ \varphi \in C(K) \ and\ x \in E.$$

We know that the correspondence is linear and one-to-one. We have

$$T(\varphi \otimes x) = T'(\varphi)x = \int \varphi \, d\mu \ x$$

for every $x \in E$, therefore

$$T(\varphi \otimes x) = \int \varphi \otimes x \, d\mu, \quad for\ every\ \varphi \in C(K).$$

Conversely, let $T' : C(K) \to L^r(E, F)$ be a regular operator, and let $\mu : K \to L^r(E, F)$ be the countably additive quasiregular measure corresponding to T'. If we put

$$T(f) = \int f \, d\mu, \quad f \in C_r(K, E),$$

then $T : C_r(K, E) \to F$ is a regular operator and we have

$$T(\varphi \otimes x) = T'(\varphi)x, \ \ for\ \varphi \in C(K)\ and\ x \in E.$$

$\square$

Now we give the following definition which will be useful in the sequel.

Definition 4.11 *Let K be a non-empty set and F be a Dedekind complete vector lattice. Then we set*

(1) $c_0(\mathbb{N}, E) = \{(e_n) \in E : \exists e \in E^+ \ such\ that\ e_n \in E(e)\ \forall n\ and\ \|e_n\|_e \to 0\}$,

(2) $l_1[K, L^r(E, F)]$ *the space of operators* $\alpha : K \to L^r(E, F)$ *such that the infinite sum* $\sum_{n=1}^{\infty} |\alpha(k_n)|(|e_n|) \in F$ *for all* $(k_n) \in K$ *and* $(e_n) \in c_0(\mathbb{N}, E)$.

As usual, $\sum_{n=1}^{\infty} |\alpha(k_n)|(|e_n|)$ *is the supremum of the sums* $\sum_{n=1}^{m} |\alpha(k_n)|(|e_n|)$. $l_1[K, L^r(E, F)]$ *is a vector lattice under the pointwise operations.*

Theorem 4.12 *Let K and F be as above. Then $L^r(c_0^r(K, E), F)$ is lattice isomorphic to $l_1[K, L^r(E, F)]$.*

Proof. Let $\phi : L^r(c_0^r(K, E), F) \to l_1[K, L^r(E, F)]$ be defined by $\phi(G)(k)(e) = G(\chi_k \otimes e)$ for each $G \in L^r(c_0^r(K, E), F)$, $k \in K$ and $e \in E$. It is easy to verify that ϕ is a linear mapping. Then $\phi(G)(k)$ is an order bounded operator from E into F as $\phi(G^+)(k)$ and $\phi(G^-)(k)$ are order bounded for each G. Thus $\phi(G)$ is a map from K into $L^r(E, F)$.

Let us recall that $\phi(G)$ should also satisfy $\sum_{n=1}^{\infty} |\phi(G)(k_n)|(|e_n|) \in F$ for all sequences $(k_n) \in K$ and $(e_n) \in c_0(\mathbb{N}, E)$.

Let $G \in L^r(c_0^r(K,E),F)$. Then we have

$$\begin{aligned}\sum_{n=1}^{m} |\phi(G)(k_n)|(|e_n|) &\leq \sum_{n=1}^{m} |G|(\chi_{k_n} \otimes |e_n|) \\ &= |G|\left(\sum_{n=1}^{m} \chi_{k_n} \otimes |e_n|\right) \leq |G|\left(\sum_{n=1}^{\infty} \chi_{k_n} \otimes |e_n|\right) \in F,\end{aligned}$$

therefore

$$\sum_{n=1}^{\infty} |\phi(G)(k_n)|(|e_n|) = \sup_m \sum_{n=1}^{m} |G|(\chi_{k_n} \otimes |e_n|) \in F.$$

Thus the map $\phi(G)$ we have defined belongs to $l_1[K, L^r(E,F)]$.

We now show that ϕ is bipositive. Clearly $\phi(G)(k)(e) \geq 0$ whenever $e \geq 0$ and $G \geq 0$, i.e. $\phi(G)(k)$ is positive for all $G \geq 0$. Suppose that $\phi(G) \geq 0$ for some $G \in L^r(c_0^r(K,E),F)$, and take $0 \leq f \in c_0^r(K,E)$. As $\sum_{k\in S} \chi_k \otimes f \uparrow_S f$ in $c_0^r(K,E)$, we have $\sum_{k\in S} G(\chi_k \otimes f) \to G(f)$. By definition $G(\chi_k \otimes f) = \phi(G)(k)(f(k)) \geq 0$ and thus $G(f) \geq 0$ for each $0 \leq f \in c_0^r(K,E)$, i.e. $G \geq 0$.

Let now $\phi(G) = 0$ for some $G \in L^r(c_0^r(K,E),F)$. Then $G(\chi_k \otimes f) = 0$ for each $k \in K$ and $0 \leq f \in c_0^r(K,E)$. As $\sum_{k\in S} \chi_k \otimes f \uparrow_S f$ in $c_0^r(K,E)$, we have $0 = \sum_{k\in S} G(\chi_k \otimes f) \to G(f)$ or $G(f) = 0$. The fact that $c_0^r(K,E)$ is vector lattice leads to $G = 0$.

To show that ϕ is surjective, let $0 \leq \alpha \in l_1[K, L^r(E,F)]$. Let $f \in c_0^r(K,E)$. Then there exists an at most countable subset (k_n) of K such that $f(k) = 0$ for all $k \neq k_n$ and there exists some $e \in E^+$ such that $f(k_n) \in E(e)$ for each n and $\|f(k_n)\|_e \to 0$. Hence we can define

$$G(f) = \sum_{n\in\mathbb{N}} \alpha(k_n)(f(k_n)),$$

which certainly belongs to F as $f(k_n) \in c_0(\mathbb{N}, E)$. We now verify that $\phi(G) = \alpha$. Let $0 \leq e \in E$, then

$$\phi(G)(k_0)(e) = G(\chi_{k_0} \otimes e) = \sum_{n\in\mathbb{N}} \alpha(k_n)(\chi_{k_0} \otimes e(k_n)) = \alpha(k_0)(e).$$

Since $e \in E$ is arbitrary, we conclude that $\phi(G)(k_0) = \alpha(k_0)$ and k_0 is arbitrary, we have $\phi(G) = \alpha$. Since $l_1[K, L^r(E,F)]$ is a vector lattice, the proof of surjectivity of ϕ is now complete. □

Now we are in a position to give one of the main result of this section as follows:

Theorem 4.13 *Let K be a compact Hausdorff space without isolated points and F be a Dedekind complete vector lattice. Then we have that $L^r(CD_0^r(K,E),F)$ is lattice isomorphic* *to* $qca(K, L^r(E,F) \oplus l_1[K, L^r(E,F)]$ *with the dual order on this direct sum defined by*

$$< \mu, \alpha > \geq 0 \Leftrightarrow \mu \geq 0 \text{ and } \alpha \geq 0 \text{ and } \mu(\{k\}) \geq \alpha(k)$$

for all $k \in K$*, which if we identify* α *with a discrete measure on* K*, is precisely requiring that* $\mu \geq \alpha \geq 0$.

Proof. Let $T \in L^r(CD_0^r(K,E), F)$. Then certainly T splits into two regular operators T_1 and T_2 where $T_1 : C_r(K,E) \to F$ and $T_2 : c_0^r(K,E) \to F$. Then by Theorem 4.10 there exists an element $\mu \in qca(K, L^r(E,F)$ such that T_1 can be identified with μ . On the other hand, by Theorem 4.12 there exists a $\alpha \in l_1[K, L^r(E,F)]$ such that T_2 can be identified with α. We thus have a map from $L^r(CD_0^r(K,E), F)$ into $qca(K, L^r(E,F) \oplus l_1[K, L^r(E,F)]$.

Now suppose that $\mu \in qca(K, L^r(E,F))$ and $\alpha \in l_1[K, L^r(E,F)]$. We can certainly define an operator $T \in L^r(CD_0^r(K,E), F)$ by

$$T(f) = \int f_1 \, d\mu + \sum_{k \in K} \alpha(k)(f_2(k)),$$

for $f = f_1 + f_2 \in C_r(K,E) \oplus c_0^r(K,E)$. The map from $qca(K, L^r(E,F)) \oplus l_1[K, L^r(E,F)]$ into $L^r(CD_0^r(K,E), F)$ is easily seen to be lattice isomorphism by Theorem 4.10 and Theorem 4.12. □

Now we give the following definition which will be used in our final result.

Definition 4.14 *Let* K *be a compact space and* F *be a Dedekind complete vector lattice. Then we set* $l^1(K, L_n^r(E,F))$ *the set of all maps* $\beta = \beta(k)$ *from* K *into* $L_n^r(E,F)$ *satisfying*

(1) $\sup_{||f||_e \leq 1} \sum_k |\beta(k)|(|(f(k)|) \in F$ *for each arbitrary but fixed* $e \in E^+$ *and* $f \in CD_0^r(K,E)$ *where* $||f||_e = \sup_{k \in K} ||f(k)||_e$.

(2) $\sum_k |\beta(k)|(f_\alpha(k)) \downarrow_\alpha 0$ *in* F *whenever* $f_\alpha \downarrow 0$ *in* $CD_0^r(K,E)$.

As usual, $\sum_k |\beta(k)|(|(f(k)|)$ *is the supremum of the sums* $\sum_S |\beta(k)|(|f(k)|)$ *where* S *is a finite subset of* K.

$l^1(K, L_n^r(E,F))$ *is a vector lattice under pointwise operations.*

We close this section with a result about order continuous operators on $CD_0^r(K,E)$-spaces.

Theorem 4.15 *Let K be a compact Hausdorff space without isolated points and F be a Dedekind complete vector lattice. Then $L_n^r(CD_0^r(K,E),F)$ is isomorphic to $l^1(K, L_n^r(E,F))$.*

Proof. Define $\phi : L_n^r(CD_0^r(K,E),F) \to l^1(K, L_n^r(E,F))$ via $\phi(G)(k)(e) = G(\chi_k \otimes e)$ for each $G \in L_n^r(CD_0^r(K,E),F)$, $k \in K$ and $e \in E$. It is easy to see that ϕ is linear. Then $\phi(G)(k)$ is order bounded, since $\phi(G^+)(k)$ and $\phi(G^-)(k)$ are order bounded F-valued operators for each G on $CD_0^r(K,E)$. If $e_\alpha \downarrow 0$ in E, then $\chi_k \otimes e_\alpha \downarrow 0$ in $CD_0^r(K,E)$ for each $k \in K$. So $\phi(G)(k)(e_\alpha) = G(\chi_k \otimes e_\alpha)$ is order convergent to 0 so that $\phi(G)(k) \in L_n^r(E,F)$ for each $G \in L_n^r(CD_0^r(K,E),F)$. Thus $\phi(G)$ is a map from K into $L_n^r(E,F)$.

Now we will show that $\phi(G)$ is an element of $l^1(K, L_n^r(E,F))$. Let S be a finite subset of K and $G \in L_n^r(CD_0^r(K,E),F)$. Then

$$\begin{aligned}
\sum_{k\in S} |\phi(G)(k)|(|f(k)|) &= \sum_{k\in S} |\phi(G^+ - G^-)(k)|(|f(k)|) \\
&\leq \sum_{k\in S} \phi(G^+)(k)(|f(k)|) + \sum_{k\in S} \phi(G^-)(k)(|f(k)|) \\
&= \sum_{k\in S} G^+(\chi_k \otimes |f|) + \sum_{k\in S} G^-(\chi_k \otimes |f|) \\
&= G^+\left(\sum_{k\in S} \chi_k \otimes |f|\right) + G^-\left(\sum_{k\in S} \chi_k \otimes |f|\right)
\end{aligned}$$

for each $f \in CD_0^r(K,E)$. Hence we get

$$\sum_{k\in S} |\phi(G)(k)|(|f(k)|) \leq G^+(|f|) + G^-(|f|) = |G|(|f|)$$

as $\sum_{k\in S} \chi_k \otimes |f| \uparrow_S |f|$, G^+ and G^- are order continuous. Let e be an arbitrary but fixed element of E^+. Then

$$\sup_{\|f\|_e \leq 1} \sum_k |\phi(G)(k)|(|f(k)|) \leq \sup_{\|f\|_e \leq 1} |G|(|f|) \leq |G|(e) \in F,$$

as $|f| \leq \|f\|_e e$.

So far we have shown that $\phi(G)$ satisfies the first condition of Definition 4.14. We also have to show that

$$\sum_k |\phi(G)(k)|(f_\alpha(k)) \downarrow_\alpha 0 \textit{ in } F$$

for each $f_\alpha \in CD_0^r(K,E)$ such that $f_\alpha \downarrow 0$. It is enough to show this for positive elements in $L_n^r(CD_0^r(K,E),F)$. Let $0 \leq G \in L_n^r(CD_0^r(K,E),F)$ and $f_\alpha \downarrow 0$ in $CD_0^r(K,E)$. For a fixed α, we have $\sum_{k\in S} \chi_k \otimes f_\alpha \uparrow_S \ f_\alpha$. As G is order continuous and positive,

$$G\left(\sum_{k\in S} \chi_k \otimes f_\alpha\right) = \sum_{k\in S} G(\chi_k \otimes f_\alpha) \uparrow G(f_\alpha).$$

Thus

$$\sum_{k\in K} |\phi(G)(k)|(f_\alpha(k)) = \sum_{k\in K} \phi(G)(k)(f_\alpha(k)) = \sum_{k\in K} G(\chi_k \otimes f_\alpha) = G(f_\alpha) \downarrow 0.$$

Hence the map $\phi(G)$ we have defined belongs to $l^1(K, L_n^r(E, F))$.

We now show that it is bipositive. Certainly $\phi(G)(k)(e) \geq 0$ whenever $e \geq 0$ and $G \geq 0$, i.e., $\phi(G)(k)$ is positive for all $G \geq 0$. Now assume that $\phi(G) \geq 0$ for some $G \in L_n^r(CD_0^r(K, E), F)$ and take $0 \leq f \in CD_0^r(K, E)$. We have $\sum_{k\in S} G(\chi_k \otimes f) \to G(f)$, since $\sum_{k\in S} \chi_k \otimes f \uparrow_S \ f$ in $CD_0^r(K, E)$. As $G(\chi_k \otimes f) = \phi(G)(k)(f) \geq 0$, $G(f) \geq 0$ for each $0 \leq f \in CD_0^r(K, E)$, i.e., $G \geq 0$. We now show that ϕ is one-to-one. Let $\phi(G) = 0$ for some $G \in L_n^r(CD_0^r(K, E), F)$. Then $G(\chi_k \otimes f) = 0$ for each $k \in K$ and $0 \leq f \in CD_0^r(K, E)$. As G is order continuous and $\sum_{k\in S} \chi_k \otimes f \uparrow_S \ f$, this gives that $0 = \sum_{k\in S} G(\chi_k \otimes f) \to G(f)$ or $G(f) = 0$. As $CD_0^r(K, E)$ is a vector lattice, we get $G = 0$.

To show that ϕ is surjective, take an arbitrary $0 \leq \alpha \in l^1(K, L_n^r(E, F))$ and let us define $G : CD_0^r(K, E)_+ \to F$ by $G(f) = \sum_{k\in K} \alpha(k)(f(k))$. G is additive on $CD_0^r(K, E)$ and so $G(f) = G(f^+) - G(f^-)$ extends G to $CD_0^r(K, E)$. We now verify that $\phi(G) = \alpha$. If $0 \leq e \in E$, then

$$\phi(G)(k_0)(e) = G(\chi_{k_0} \otimes e) = \sum_{k\in K} \alpha(k)(\chi_{k_0} \otimes e)(k) = \alpha(k_0)e.$$

Since $e \in E$ is arbitrary, we conclude that $\phi(G)(k_0) = \alpha(k_0)$ and k_0 is arbitrary, we have $\phi(G) = \alpha$. $\square$

4.3 Linear Dominated Operators on Abramovich–Wickstead Spaces

4.3.1 Dominated Operators on $C(K, E)$–spaces

Let K be a locally compact Hausdorff space and E be a Banach space. Recall that the support of a function $f : K \to E$ is the closure of the set $\{k : f(k) \neq 0\}$. The space of the continuous functions with compact support is denoted by $\mathcal{K}(K, E)$. If K is compact, then we have $\mathcal{K}(K, E) = C(K, E)$, where $C(K, E)$ is the space of continuous functions $f : K \to E$. N. Dinculeanu [19] has given some integral representations of dominated operators on $\mathcal{K}(K, E)$.

In this section, We adapt some results of N. Dinculeanu about the integral representations of linear operators on $\mathcal{K}(K,E)$ to the integral representations of linear dominated operators on $C(K,E)$. The proofs of the theorems follow directly the lines in the Dinculeanu's proofs. Nevertheless, we give their proofs for the sake of convenience.

The regular Borel measures with finite variation will play an important role in this section. We begin with the definition of variation of a vector measure and give some properties of it.

Definition 4.16 *Let $\mathfrak{B}$ be a σ-algebra of subsets of a set X and $E = (E, \|.\|)$ be a Banach space. Then*

(1) *the function $m : \mathfrak{B} \to E$ is called a* **vector measure** *(or* **E-valued measure***) if $m(\emptyset) = 0$ and $m\left(\bigcup_k A_k\right) = \sum_k m(A_k)$ for any sequence (A_k) of pairwise disjoint sets from $\mathfrak{B}$,*

(2) *the function $|m| : \mathfrak{B} \to \mathbb{R}^+ \cup \{+\infty\}$ defined by the following formula :*

$$|m|(A) = \sup\{\sum_{n=1}^{j} \|m(A_n)\| : A_n \in \mathfrak{B}, \bigcup_{n=1}^{j} A_n = A,\ A_k \cap A_p = \emptyset\ \forall k \neq p\}$$

is called the variation of m. $|m|$ is additive and monotone. It is also a measure. m is called a vector measure of **finite variation** *if $|m|$ is finite, i.e, $|m|(X) < +\infty$. It is easy to see that m is σ-finite or finite E-valued measure if and only if $|m|$ is σ-finite or finite.*

We now give the following well-known fact which will be useful in the sequel.

Theorem 4.17 *Let K be a compact Hausdorff space and E be a Banach space. Then the space of the functions $\sum \varphi_i \otimes x_i$ (finite sum) with $\varphi_i \in C^+(K)$ and $x_i \in E$ is (uniformly) dense in $C(K,E)$.*

Proof. Let $f \in C(K,E)$ and let $\varepsilon > 0$. For every $k \in K$, there exists an open neighborhood V of k such that if $k' \in V$, then $\|f(k) - f(k')\| < \frac{\varepsilon}{2}$. Then

$$\|f(k') - f(k'')\| < \varepsilon,\ if\ k', k'' \in V.$$

Since K is compact, we can find a finite family (V_i) of open sets covering K, such that for each i we have

$$\|f(k') - f(k'')\| < \varepsilon,\ if\ k', k'' \in V_i.$$

Let (φ_i) be a continuous partition of the unity subordinated to the family (V_i), i.e. $0 \le \varphi_i \le 1$, the support of φ_i is contained in V_i and $\sum \varphi_i(k) = 1$ for $k \in K$. For each i take a point $k_i \in V_i$ and put $x_i = f(k_i)$. Then for each $k \in K$,

$$\|\sum \varphi_i(k)x_i - f(k)\| \le \sum \varphi_i(k)\|f(k_i) - f(k)\| \le \varepsilon$$

so that $\|\sum \varphi_i \otimes x_i - f\| \le \varepsilon$. □

By using the previous theorem, we have the following result.

Theorem 4.18 *Let K be a compact space, E and F be Banach spaces. Then for every linear and continuous operator $T : C(K,E) \to F$, there exists a linear and continuous operator $T' : C(K) \to L(E,F)$ such that*

$$T(\varphi \otimes x) = T'(\varphi)x, \ for\ \varphi \in C(K)\ and\ x \in E.$$

The correspondence $T \mapsto T'$ is linear and one-to-one.

Proof. For every $\varphi \in C(K)$ and $x \in E$, we have $\varphi \otimes x \in C(K,E)$; we put

$$T(\varphi \otimes x) = T'(\varphi)x.$$

For fixed $\varphi \in C(K)$, the mapping $T'(\varphi) : x \mapsto T'(\varphi)x$ of E into F is linear and continuous:

$$\|T'(\varphi)x\|_F = \|T(\varphi \otimes x)\|_F \le \|T\|\,\|\varphi\|\,\|x\|,$$

therefore $\|T'(\varphi)\|_{L(E,F)} \le \|T\|\,\|\varphi\|$ and $T'(\varphi) \in L(E,F)$.

The mapping $T' : \varphi \to T'(\varphi)$ of $C(K)$ into $L(E,F)$ is linear.

It is easy to verify that the mapping $T \mapsto T'$ is linear. In order to prove that this mapping is one-to-one, let $S : C(K,E) \to F$ be a linear and continuous operator such that

$$S(\varphi \otimes x) = T'(\varphi)x, \ for\ \varphi \in C(K)\ and\ x \in E.$$

Let $f \in C(K,E)$. Then by Theorem 4.17, there exists a sequence (f_n) of the form $\sum \varphi_i \otimes x_i$ (finite sum) with $\varphi_i \in C(K)$ and $x_i \in E$ converging uniformly to f. Then $T(f_n) = S(f_n)$ for every n. On the other hand T and S are continuous on $C(K,E)$, therefore

$$T(f) = \lim_{n\to\infty} T(f_n) = \lim_{n\to\infty} S(f_n) = S(f),$$

consequently $T = S$. □

Remark 4.19 *There are linear and continuous mappings $T' : C(K) \to L(E, F)$ which do not correspond to any linear mapping $T : C(K, E) \to F$. However, if T' is dominated (see Definition 4.20), in particular $F = \mathbb{C}$ and T is continuous, then T' can be obtained from a dominated linear mapping $T : C(K, E) \to F$ (Theorem 4.25).*

We now give the following definition which is given for the linear operators
$T : C(K, E) \to F$ by N. Dinculeanu [19].

Definition 4.20 *A linear mapping $T : C(K, E) \to F$ is said to be dominated (or majored) if there exists a regular positive Borel measure ν such that*

$$||T(f)||_F \leq \int ||f(k)||_E \, d\nu(k), \quad for\ every\ f \in C(K, E).$$

We say that T is dominated by ν, or that ν dominates T. If T is dominated, then there exists a least positive regular measure dominating T. We shall denote the least regular measure dominating T by μ_T.

Proposition 4.21 *Let K be a compact Hausdorff space, E and F be two Banach spaces. If the linear operator $T : C(K, E) \to F$ is dominated in the sense of Definition 2.8, then there exists a regular Borel measure m such that*

$$||T(f)||_F \leq \int ||f(k)||_E \, dm(k) \quad (f \in C(K, E)).$$

Proof. From Definition 2.8, the dominant of T is a positive linear functional $S : C(K) \to \mathbb{R}$ such that

$$||T(f)||_F \leq S(|f|) \quad (f \in C(K, E)).$$

By Riesz Representation Theorem, there exists a bounded regular Borel measure m such that

$$S(|f|) = \int |f|(k) \, dm(k) = \int ||f(k)||_E \, dm(k).$$

□

It follows from the previous proposition that Definition 2.8 and Definition 4.20 coincide for the dominated linear operators $T : C(K, E) \to F$.

Let μ be a positive measure defined on a σ-algebra $\mathfrak{B}$ of subsets of K and E be a Banach space. Let p be a real number such that $0 < p < \infty$.

We denote by $\mathfrak{L}^p_E(K,\mu)$ the set of all μ-measurable functions $f : K \to E$ such that $|f|^p$ is μ-integrable. For every function $f \in \mathfrak{L}^p_E(K,\mu)$, we put

$$N_p(f,\mu) = \left(\int \|f(k)\|^p_E \, d\mu(k)\right)^{\frac{1}{p}}.$$

If a function f belongs to $\mathfrak{L}^p_E(K,\mu)$ we say that f has the p-th power of μ-integrable. We shall write also $\mathfrak{L}^p_E(\mu)$ or $\mathfrak{L}^p_E$ instead of $\mathfrak{L}^p_E(K,\mu)$, and $N_p(f)$ or $\|f\|_p$ instead of $N_p(f,\mu)$. If $E = \mathbb{R}$ we shall write $\mathfrak{L}^p(\mu)$ instead of $\mathfrak{L}^p_{\mathbb{R}}(\mu)$. The set $\mathfrak{L}^p_E(\mu)$ is a vector space. In case $p = 1$, the space $\mathfrak{L}^1_E$ defined here equals to the space of the μ-integrable functions that is to say f is μ-measurable and $\|f(k)\|_E \in \mathfrak{L}^1(\mu)$.

Proposition 4.22 *Let K be a compact Hausdorff space, E be a Banach space and μ be a positive regular Borel measure. Then the space $C(K,E)$ of continuous functions $f : K \to E$ is dense in $\mathfrak{L}^p_E(\mu)$ for $1 \leq p < \infty$.*

Proof. See [Proposition 33 in section 5 of [19]]. $\square$

We now give the following fact which will be used in obtaining one of the main results in this section.

Proposition 4.23 *Let K be a compact space, E and F be two Banach spaces, m and n be two regular Borel measures with finite variations, with values in $L(E,F)$. Then we have $m = n$ if and only if*

$$\int f(k)\, dm(k) = \int f(k)\, dn(k), \quad for\ every\ f \in C(K,E).$$

Proof. If $m = n$, then evidently $\int f(k)\, dm(k) = \int f(k)\, dn(k)$, for every $f \in C(K,E)$. Conversely, suppose that

$$\int f(k)\, dm(k) = \int f(k)\, dn(k), \quad for\ every\ f \in C(K,E).$$

We have $|m| \leq |m| + |n|$ and $|n| \leq |m| + |n|$, therefore

$$\mathfrak{L}^1_E(|m| + |n|) \subset \mathfrak{L}^1_E(|m|) \cap \mathfrak{L}^1_E(|n|).$$

We have also

$$\left\|\int f(k)\, dm(k)\right\|_F \leq \int \|f(k)\|_E \, d|m|(k) \leq \int \|f(k)\|_E \, d(|m| + |n|)(k),$$

$$\left\| \int f(k)\, dn(k) \right\|_F \leq \int \|f(k)\|_E \, d|n|(k) \leq \int \|f(k)\|_E \, d(|m| + |n|)(k),$$

therefore the linear mappings $f \to \int f \, dm$ and $f \to \int f \, dn$ of $\mathfrak{L}^1_E(|m| + |n|)$ into F are continuous. On the other hand, $|m|$ and $|n|$ are regular since m and n are regular (Proposition 22 of section 15 in [19]), therefore $|m| + |n|$ is regular, hence $C(K, E)$ is dense in $\mathfrak{L}^1_E(|m| + |n|)$ (Proposition 4.22). Since these two mappings coincide on the dense subspace $C(K, E)$, they coincide on the whole space $\mathfrak{L}^1_E(|m| + |n|)$. In particular, if $A \in \mathfrak{B}$ and $x \in E$, then $\chi_A x \in \mathfrak{L}^1_E(|m| + |n|)$, therefore

$$\int \chi_A(k) x \, dm(k) = \int \chi_A(k) x \, dn(k),$$

hence

$$m(A)x = n(A)x.$$

It follows that $m(A) = n(A)$ for every $A \in \mathfrak{B}$, therefore $m = n$. □

Theorem 4.24 *Let K be a compact space, E and F be Banach spaces. Then, there exists an isomorphism $T \leftrightarrow m$ between the set of linear dominated operators $T : C(K, E) \to F$ and the set of regular Borel measures $m : \mathfrak{B} \to L(E, F)$ with finite variation $|m|$, given by the equality*

$$T(f) = \int f(k) \, dm(k), \quad for\ every\ f \in C(K, E).$$

Moreover, if T and m are in correspondence, we have $\mu_T = |m|$.

Proof. Let first $m : \mathfrak{B} \to L(E, F)$ be a regular Borel measure with finite variation $|m|$ and consider the space $\mathfrak{M}_E(\mathfrak{B})$ of the totally $\mathfrak{B}$-measurable functions such that $f : K \to E$. Then $C(K, E) \subset \mathfrak{M}_E(\mathfrak{B})$ so that

$$T(f) = \int f(k) \, dm(k), \quad for\ f \in C(K, E).$$

From this equality, we deduce that

$$\|T(f)\|_F \leq \int \|f(k)\|_E \, d|m|(k), \quad for\ f \in C(K, E),$$

therefore T is dominated by the regular measure $|m|$. The correspondence $m \leftrightarrow T$ is linear: from Proposition 4.23, we deduce that this correspondence is one-to-one. It remains to prove that every linear dominated operator T corresponds to some countably additive regular Borel measure m with finite variation $|m|$ and $\mu_T = |m|$. Let $T : C(K, E) \to F$ be a linear dominated operator. Let ν be a positive regular Borel measure such that

$$\|T(f)\|_F \leq \int \|f(k)\|_E \, d\nu(k), \quad for\ f \in C(K, E).$$

From this inequality we deduce that T is continuous on $C(K,E)$ for the topology of $\mathfrak{L}^1_E(\nu)$. Since ν is regular and $C(K,E)$ is dense in $\mathfrak{L}^1_E(\nu)$, the linear operator T can be extended uniquely to a continuous linear mapping of $\mathfrak{L}^1_E(\nu)$ into F denoted also by T, and we still have

$$\|T(f)\|_F \leq \int \|f(k)\|\, d\nu(k), \quad for\ f \in \mathfrak{L}^1_E(\nu).$$

For every set $A \in \mathfrak{B}$, and for every $x \in E$ we have $\chi_A x \in \mathfrak{L}^1_E(\nu)$. Put

$$m(A)x = T(\chi_A x).$$

The mapping $m(A) : x \mapsto m(A)x$ of E into F is linear and continuous:

$$\|m(A)x\|_F = \|T(\chi_A x)\|_F \leq \int \|\chi_A(k)x\|_E\, d\nu(k) = \|x\|_E \nu(A),$$

therefore $m(A) \in L(E,F)$ and

$$\|m(A)\|_{L(E,F)} \leq \nu(A).$$

It is easy to verify that the set function $m : \mathfrak{B} \to L(E,F)$ is additive. From the last inequality we deduce that m is regular, countably additive and with finite variation $|m|$ and $|m| \leq \nu$. We now show that

$$T(f) = \int f(k)\, dm(k), \quad f \in \mathfrak{L}^1_E(\nu).$$

For every step function $f = \sum \chi_{A_i} x_i$, we have

$$T(f) = T(\sum \chi_{A_i} x_i) = \sum T(\chi_{A_i} x_i) = \sum m(A_i)x_i := \int f(k)\, dm(k).$$

Since $|m| \leq \nu$, we have $\mathfrak{L}^1_E(\nu) \subset \mathfrak{L}^1_E(|m|)$, therefore if $f \in \mathfrak{L}^1_E(\nu)$ we have

$$\left\| \int f(k)\, dm(k) \right\|_F \leq \int \|f(k)\|_E\, d|m|(k) \leq \int \|f(k)\|_E\, d\nu(k),$$

whence the mapping $f \to \int f\, dm$ of $\mathfrak{L}^1_E(\nu)$ into F is continuous. Since the two linear mappings $\int f\, dm$ and T of $\mathfrak{L}^1_E(\nu)$ into F are continuous and coincide on the set of step functions which is dense, we deduce that they coincide on the whole space:

$$T(f) = \int f(k)\, dm(k), \quad f \in \mathfrak{L}^1_E(\nu).$$

This equality is valid, in particular, for $f \in C(K,E)$. Since ν is an arbitrary regular Borel measure dominating T and $|m| \leq \nu$, we deduce that $|m|$ is the smallest regular measure dominating T, i.e. $\mu_T = |m|$. □

Theorem 4.25 *There exists an isomorphism $T \leftrightarrow T'$ between the space of linear dominated operators $T : C(K, E) \to F$ and the space of linear dominated operators $T' : C(K) \to L(E, F)$ given by the equality*

$$T(\varphi \otimes x) = T'(\varphi)x, \; for \; \varphi \in C(K) \; and \; x \in E.$$

If T and T' are in correspondence, then there exists a regular measure $m : \mathfrak{B} \to L(E, F)$ with finite variation $|m|$ such that

$$T(f) = \int f(k) \, dm(k), \quad for \; f \in C(K, E),$$

and

$$T'(\varphi) = \int \varphi(k) \, dm(k), \quad for \; \varphi \in C(K),$$

and we have

$$\mu_T = \mu_{T'} = |m|.$$

Proof. Let $T : C(K, E) \to F$ be a linear dominated operator and $m : \mathfrak{B} \to L(E, F)$ be the regular measure with finite variation $|m|$ such that

$$T(f) = \int f(k) \, dm, \quad for \; every \; f \in C(K, E).$$

Let $T' : C(K) \to L(E, F)$ be the linear mapping corresponding to T by the equality (Theorem 4.18)

$$T(\varphi \otimes x) = T'(\varphi)x, \; for \; \varphi \in C(K) \; and \; x \in E.$$

We know that the correspondence is linear and one-to-one. Hence

$$T'(\varphi)x = T(\varphi \otimes x) = \int (\varphi \otimes x)(k) \, dm = \left(\int \varphi(k) \, dm(k) \right) x$$

for every $x \in E$, therefore

$$T'(\varphi) = \int \varphi(k) \, dm(k), \quad for \; every \; \varphi \in C(K).$$

From this equality we deduce that T' is dominated and that

$$\mu_T = \mu_{T'} = |m|.$$

Conversely, let $T' : C(K) \to X = L(E, F)$ be a linear dominated mapping, and let $m : \mathfrak{B} \to L(\mathbb{R}, X) = X$ be the regular measure corresponding to T' by Theorem 4.24. If we put

$$T(f) = \int f(k) \, dm(k), \quad f \in C(K, E),$$

then $T : C(K,E) \to F$ is a dominated linear mapping and we have

$$T(\varphi \otimes x) = T'(\varphi)x, \;\; for\ \varphi \in C(K)\ and\ x \in E.$$

□

4.3.2 Dominated Operators on $c_0(K,E)$–spaces

In this section we give some characterizations about dominated operators on $c_0(K,E)$-spaces. This section will play an important role in characterizing dominated operators on $CD_0(K,E)$ spaces. Throughout this section, the word *isometry* means vector norm preserving linear bijective operator. We now give the following fact which will be useful in the sequel.

Lemma 4.26 *Let K be a non-empty set and E be a Banach space. Then the space of the functions $\sum \varphi_i \otimes x_i$ (finite sum) with $\varphi_i \in c_0(K)$ and $x_i \in E$ is (uniformly) dense in $c_0(K,E)$.*

Proof. Let $f \in c_0(K,E)$ and $\varepsilon > 0$. Then there exists an at most countable subset $(k_n) \in K$ such that $f(k) = 0$ for all $k \neq k_n$ and there exists some $n_0 \in \mathbb{N}$ such that $\|f(k_n)\| < \varepsilon$ for each $n \geq n_0$. Let $f_m(k_n) = f(k_n)$ for each $1 \leq n \leq m$ and 0 otherwise. Then for each $k \in K$,

$$\|f_m(k_n) - f(k)\| = \|f(k_{n+1})\| < \varepsilon \quad for\ each\ n \geq N = n_0 - 1.$$

This completes the proof since $f_m(k_n) = \sum_{n=1}^{m} \chi_{k_n} \otimes f(k_n)$. □

Definition 4.27 *Let K be a non-empty set, E and F be two Banach spaces. We define $l_1(K, L(E,F))$ as the set of mappings $\varphi : K \to L(E,F)$ such that*

$$\sum_{k \in K} \|\varphi(k)(f(k))\|_F < \infty, \;\; for\ all\ f \in c_0(K,E).$$

As usual, $\sum_{k\in K} \|\varphi(k)(f(k))\|_F$ is the supremum of all the sums
$\sum_S \|\varphi(k)(f(k))\|_F$ where S is a finite subset of K .

$l_1(K, L(E,F))$ is a lattice-normed space with norm lattice $l_1(K)$ (the set of real-valued absolutely summable functions on K).

Now we give the following result which will be used in the next section.

Theorem 4.28 *Let K be a non-empty set and E and F be two Banach spaces. Then we have that $M(c_0(K,E),F)$ is isometrically isomorphic to $l_1(K,L(E,F))$.*

Proof. Let $\phi : M(c_0(K,E),F) \to l_1(K,L(E,F))$ be defined by $\phi(G)(k)(e) = G(\chi_k \otimes e)$ for each $G \in M(c_0(K,E),F)$, $k \in K$ and $e \in E$. Then $\phi(G)(k)$ is clearly a linear operator from E into F. Since $c_0(K,E)$ is decomposable (Proposition 3.79) and $\mathbb{R}$ (the norm lattice of F) is Dedekind complete, the least dominant $|G|$ exists (Theorem 2.10) and is a linear continuous functional on $c_0(K)$. We can identify $|G|$ with a function $\alpha \in l_1(K) = l_1(K,\mathbb{R})$ in the sense that

$$|G|(g) = \sum_{k\in K} \alpha(k)g(k) \ (g \in c_0(K)).$$

Taking this observation into account we deduce that

$$\|\phi(G)(k)(e)\|_F = \|G(\chi_k \otimes e\|_F \leq |G|(|\chi_k \otimes e|) = \sum_{k\in K} |\alpha(k)| \, \|e\|_E,$$

therefore $\|\phi(G)(k)\|_{L(E,F)} \leq \sum_{k\in K} |\alpha(k)| < \infty$ and $\phi(G)(k) \in L(E,F)$. Thus $\phi(G)$ is a map from K into $L(E,F)$.

We now show that $\sum_{k\in K} \|\phi(G)(k)(f(k))\|_F < \infty$ for each $f \in c_0(K,E)$.

For any finite subset S of K, we have

$$\begin{aligned}
&\sum_{k\in S} \|\phi(G)(k)(f(k))\|_F = \sum_{k\in S} \|G(\chi_k \otimes f)\|_F \\
&\leq \sum_{k\in S} |G|(|\chi_k \otimes f|) = |G|\left(\sum_{k\in S} \|f(k)\|_E \chi_k\right) \\
&\leq |G|(|f|),
\end{aligned}$$

therefore

$$\sum_{k\in K} \|\phi(G)(k)(f(k))\|_F \leq |G|(|f|) < \infty.$$

Hence $\phi(G)$ satisfies the restriction to be an element of $l_1(K,L(E,F))$. Let $0 \leq g \in c_0(K)$ such that $|f| \leq g$ for $f \in c_0(K,E)$, then we have from the previous inequality that

$$\begin{aligned}
\sup_{|f|\leq g} \sum_{k\in K} \|\phi(G)(k)(f(k))\|_F &= \sum_{k\in K} \|\phi(G)(k)\|_{L(E,F)} \leq \sup_{|f|\leq g} |G|(|f|) \\
&\leq |G|(g) = \sum_{k\in K} \alpha(k)g(k),
\end{aligned}$$

therefore $\phi(G)$ is dominated and $|\phi(G)| \le \alpha = |G|$.

It is easy to verify that ϕ is a linear map. Assume that $\phi(G) = 0$ for some $G \in M(c_0(K, E), F)$. Then $G(\chi_k \otimes e) = 0$ for each $k \in K$ and $e \in E$. This yields

$$G(\sum_{i=1}^{n} \chi_{k_i} \otimes e_i) = \sum_{i=1}^{n} G(\chi_{k_i} \otimes e_i) = 0.$$

Since $\sum_{i=1}^{n} \chi_{k_i} \otimes e_i$ is dense in $c_0(K, E)$ (Lemma 4.26), then (by continuity) $G(f) = 0$ for each $f \in c_0(K, E)$. This implies that $G = 0$. Thus ϕ is one-to-one.

To show that ϕ is surjective and an isometry, let $\alpha \in l_1(K, L(E, F))$ and define

$$G(f) = \sum_{k \in K} \alpha(k)(f(k)), \quad f \in c_0(K, E)$$

which certainly converges and it is clear that G is linear. We now verify that $\phi(G) = \alpha$. Let $e \in E$, then

$$\phi(G)(k_0)(e) = G(\chi_{k_0} \otimes e) = \sum_{k \in K} \alpha(k)(\chi_{k_0} \otimes e(k)) = \alpha(k_0)(e),$$

Since $e \in E$ is arbitrary, we conclude that $\phi(G)(k_0) = \alpha(k_0)$ since k_0 is arbitrary, we have $\phi(G) = \alpha$. Also

$$\begin{aligned} \|G(f)\| &= \|\sum_{k \in K} \alpha(k)(f(k))\|_F \le \sum_{k \in K} \|\alpha(k)(f(k))\|_F \\ &\le \sum_{k \in K} \|\alpha(k)\|_{L(E,F)} \, \|f(k)\|_E = \sum_{k \in K} |\alpha|(k) \, |f|(k), \end{aligned}$$

therefore G is dominated and $|G| \le |\alpha| = |\phi(G)|$. □

4.3.3 Dominated Operators on $CD_0(K, E)$–spaces

In this section, we give two characterizations about dominated and *bo*-continuous dominated operators on $CD_0(K, E)$. The following theorem contains the first characterization.

Theorem 4.29 *Let K be a compact Hausdorff space without isolated points, E and F be two Banach spaces. Then we have that $M(CD_0(K, E), F)$ is isometrically isomorphic to $rca(\mathfrak{B}, L(E, F)) \oplus l_1(K, L(E, F))$ where $rca(\mathfrak{B}, L(E, F))$ is the space of regular Borel measures $m : \mathfrak{B} \to L(E, F)$ with finite variation $|m|$.*

Proof. Let $T \in M(CD_0(K,E),F)$. Then certainly T splits into two dominated linear operators $T_1 : C(K,E) \to F$ and $T_2 : c_0(K,E) \to F$. By Theorem 4.24 and Theorem 4.28 , there exist some $m \in rca(\mathfrak{B}, L(E,F))$ and $\alpha \in l_1(K, L(E,F))$ such that T_1 and T_2 can be identified with m and α respectively. We thus have a mapping of $M(CD_0(K,E),F)$ into $rca(\mathfrak{B}, L(E,F)) \oplus l_1(K, L(E,F))$.

Now suppose that $m \in rca(\mathfrak{B}, L(E,F))$ and $\alpha \in l_1(K, L(E,F))$. We can certainly define a map ϕ on $M(CD_0(K,E),F)$ by

$$\phi(f) = \int f_1(k)\, dm(k) + \sum_{k \in K} \alpha(k)(f_2(k)),$$

for $f = f_1 + f_2 \in C(K,E) \oplus c_0(K,E)$. It follows from Theorem 4.24 and Theorem 4.28 that the map from $rca(\mathfrak{B}, L(E,F)) \oplus l_1(K, L(E,F))$ into $M(CD_0(K,E),F)$ is an isometric isomorphism. $\square$

Now we give a characterization about *bo*-continuous operators on $CD_0(K,E)$.

Theorem 4.30 *Let K be a compact Hausdorff space without isolated points, E and F be two Banach spaces. Then $M_n(CD_0(K,E),F)$ is isometrically isomorphic to $l^1(K, L(E,F))$.*

Proof. Let $G \in M_n(CD_0(K,E),F)$. Then $|G|$ is a positive order continuous linear functional on $CD_0(K)$ (Theorem 2.13). Also from Theorem 6.1 in [33], we know that $|G|$ can be identified with an element $\varphi \in l_1(K)$ so that by using Theorem 2.11 integral part in Theorem 4.29 vanishes. Thus $M_n(CD_0(K,E),F)$ is isometrically isomorphic to $l^1(K, L(E,F))$ again by Theorem 4.29. $\square$

4.4 Order Continuous Operators on Abramovich–Wickstead Spaces

Throughout this section the symbol L_n^r denotes the space of regular order continuous operators and $\chi_k \otimes f$ denotes the vector valued function which takes the value $f(k)$ at k and 0 otherwise. The following definitions and theorems were given in [27].

Definition 4.31 *Let K be a compact Hausdorff space without isolated points and E be a Banach lattice. Then the set of all maps $\beta = \beta(k)$ from K into $E_n^\sim$ satisfying*

$$\sup_{\|f\| \le 1} \sum_k |\beta(k)|(|f(k)|) < \infty$$

where $f \in CD_0(K,E)$ and $\sum_k |\beta(k)|(f_\alpha(k)) \downarrow_\alpha 0$ whenever $f_\alpha \downarrow 0$ will be denoted by $D_0(K,E_n^\sim)$. As usual, $\sum_k |\beta(k)|(|f(k)|)$ is the supremum of the sums $\sum_S |\beta(k)|(|f(k)|)$ where $S \subset K$ and is finite. $D_0(K,E_n^\sim)$ is a normed Riesz space under pointwise operations and supremum norm.

Theorem 4.32 *Let K and E be as above. Then $CD_0(K,E)_n^\sim$ and $D_0(K,E_n^\sim)$ are isometrically lattice isomorphic spaces.*

Definition 4.33 *Let K be a compact Hausdorff space without isolated points and E be a Banach lattice. Then the set of all maps $\beta = \beta(k)$ from K into $E_n^\sim$ satisfying*

$$\sup_{\|f\|\leq 1} \sum_k |\beta(k)|(|f(k)|) < \infty$$

where $f \in CD_w(K,E)$ and $\sum_k |\beta(k)|(f_\alpha(k)) \downarrow_\alpha 0$ whenever $f_\alpha \downarrow 0$ will be denoted by $D_w(K,E_n^\sim)$. As usual, $\sum_k |\beta(k)|(|f(k)|)$ is the supremum of the sums $\sum_S |\beta(k)|(|f(k)|)$ where S is a finite subset of K. $D_w(K,E_n^\sim)$ is a normed Riesz space under pointwise operations and supremum norm.

Theorem 4.34 *Let K and E be as above. Then $CD_w(K,E)_n^\sim$ and $D_w(K,E_n^\sim)$ are isometrically lattice isomorphic spaces.*

In this section, we give a generalization of Theorem 4.32 and Theorem 4.34 in two directions. In the first direction we replace $CD_0(K,E)_n^\sim$ (or $CD_w(K,E)_n^\sim$) by $L_n^r(CD_0(K,E),F)$ (or $L_n^r(CD_w(K,E),F)$) where E and F are Banach lattices with F Dedekind complete. We take F as a Dedekind complete Banach lattice to ensure that $L_n^r(CD_0(K,E),F)$ (or $L_n^r(CD_w(K,E),F)$) is a Dedekind complete Banach lattice under the regular norm $\|.\|_r$. In the second direction, we replace $E_n^\sim$ by $L_n^r(E,F)$. We now give the following definition which is a modification of Definition 4.31.

Definition 4.35 *Let K be a compact Hausdorff space without isolated points, E and F be two Banach lattices with F Dedekind complete. Then we define $l^1(K,L_n^r(E,F))$ as the set of all maps $\varphi = \varphi(k)$ from K into $L_n^r(E,F)$ satisfying*

$$\sum_k |\varphi(k)|(|f(k)|) \in F$$

where $f \in CD_0(K,E)$ and $\sum_k |\varphi(k)|(f_\alpha(k)) \downarrow_\alpha 0$ in F whenever $f_\alpha \downarrow 0$ in $CD_0(K,E)$.

As usual, $\sum_k |\varphi(k)|(|f(k)|)$ is the supremum of the sums $\sum_S |\varphi(k)|(|f(k)|)$ where $S \subset K$ and is finite.

$l^1(K, L_n^r(E, F))$ is a Banach lattice under pointwise operations and supremum norm.

We now give the following theorem which is the main result of this section.

Theorem 4.36 *Let K, E and F be as above definition. Then $L_n^r(CD_0(K, E), F)$ is isometrically lattice isomorphic to $l^1(K, L_n^r(E, F))$.*

Proof. Let us define a map $\phi : L_n^r(CD_0(K, E), F) \to l^1(K, L_n^r(E, F))$ at $e \in E$ by the formula $\phi(G)(k)(e) = G(\chi_k \otimes e)$ for each $G \in L_n^r(CD_0(K, E), F)$ and $k \in K$. It is clear that ϕ is a linear map. Using the linearity and the fact that $\phi(G^+)(k)$ and $\phi(G^-)(k)$ are order bounded F-valued operators for each G on $CD_0(K, E)$, $\phi(G)(k)$ is order bounded.

Moreover, if $e_\alpha \downarrow 0$ in E, then $\chi_k \otimes e_\alpha \downarrow 0$ in $CD_0(K, E)$ for each $k \in K$. Using the order continuity of G, we have that $G(\chi_k \otimes e)$ is order convergent to 0 so that $\phi(G)(k) \in L_n^r(E, F)$ for each $G \in L_n^r(CD_0(K, E), F)$. We thus have a map $\phi(G)$ from K into $L_n^r(E, F)$.

Now we will show that

$$\sum_k |\varphi(k)|(|f(k)|) \in F, \ \ f \in CD_0(K, E).$$

Let S be a finite subset of K and $G \in L_n^r(CD_0(K, E), F)$. Then

$$\begin{aligned} \sum_{k \in S} |\phi(G)(k)|(|f(k)|) &= \sum_{k \in S} |\phi(G^+ - G^-)(k)|(|f(k)|) \\ &\leq \sum_{k \in S} \phi(G^+)(k)(|(f(k)|) + \sum_{k \in S} \phi(G^-)(k)(|f(k)|) \\ &= \sum_{k \in S} G^+(\chi_k \otimes |f|) + \sum_{k \in S} G^-(\chi_k \otimes |f|) \\ &= G^+\left(\sum_{k \in S} \chi_k \otimes |f|\right) + G^-\left(\sum_{k \in S} \chi_k \otimes |f|\right) \end{aligned}$$

for each $f \in CD_0(K, E)$. But we know that $\sum_{k \in S} \chi_k \otimes |f| \uparrow_S |f|$, since G^+ and G^- are order continuous, we obtain

$$\sum_{k \in S} |\phi(G)(k)|(|f(k)|) \leq G^+(|f|) + G^-(|f|) = |G|(|f|),$$

so that

$$\sum_{k \in K} |\phi(G)(k)|(|f(k)|) \in F,$$

since F is Dedekind complete. We also have to show that

$$\sum_{k} |\phi(G)(k)|(f_\alpha(k)) \downarrow_\alpha 0 \textit{ in } F$$

for each $f_\alpha \in CD_0(K, E)$ such that $f_\alpha \downarrow 0$. It is enough to show this for positive elements in $L_n^r(CD_0(K, E), F)$. Let $0 \leq G \in L_n^r(CD_0(K, E), F)$ and $f_\alpha \downarrow 0$ in $CD_0(K, E)$. For a fixed α, we have $\sum_{k \in S} \chi_k \otimes f_\alpha \uparrow_S f_\alpha$. As G is order continuous and positive,

$$G\left(\sum_{k \in S} \chi_k \otimes f_\alpha\right) = \sum_{k \in S} G(\chi_k \otimes f_\alpha) \uparrow G(f_\alpha),$$

so that

$$\begin{aligned} \sum_{k \in K} |\phi(G)(k)|(f_\alpha(k)) &= \sum_{k \in K} \phi(G)(k)(f_\alpha(k)) \\ &= \sum_{k \in K} G(\chi_k \otimes f_\alpha) = G(f_\alpha) \downarrow 0. \end{aligned}$$

Hence the map $\phi(G)$ is an element of $l^1(K, L_n^r(E, F))$.

We now show that it is bipositive. Certainly $\phi(G) \geq 0$ whenever $G \geq 0$. Now assume that $\phi(G) \geq 0$ for some $G \in L_n^r(CD_0(K, E), F)$ and let us take $0 \leq f \in CD_0(K, E)$. We have $\sum_{k \in S} G(\chi_k \otimes f) \to G(f)$, since $\sum_{k \in S} \chi_k \otimes f \uparrow_S f$ in $CD_0(K, E)$. As $G(\chi_k \otimes f) = \phi(G)(k)(f) \geq 0$ and thus $G(f) \geq 0$ for each $0 \leq f \in CD_0(K, E)$, i.e., $G \geq 0$.

To show that ϕ is one-to-one, let $\phi(G) = 0$ for some $G \in L_n^r(CD_0(K, E), F)$. Then $G(\chi_k \otimes f) = 0$ for each $k \in K$ and $0 \leq f \in CD_0(K, E)$. As G is order continuous and $\sum_{k \in S} \chi_k \otimes f \uparrow_S f$, this gives that $0 = \sum_{k \in S} G(\chi_k \otimes f) \to G(f)$ or $G(f) = 0$. As $CD_0(K, E)$ is a vector lattice, we get $G = 0$.

To show that ϕ is surjective, take an arbitrary $0 \leq \alpha \in l^1(K, L_n^r(E, F))$ and let us define $G : CD_0(K, E)_+ \to F_+$ by $G(f) = \sum_{k \in K} \alpha(k)(f(k))$. As G is additive on $CD_0(K, E)$ and so $G(f) = G(f^+) - G(f^-)$ extends G to $CD_0(K, E)$. We now verify that $\phi(G) = \alpha$. If $0 \leq e \in E$, then

$$\phi(G)(k_0)(e) = G(\chi_{k_0} \otimes e) = \sum_{k \in K} \alpha(k)(\chi_{k_0} \otimes e)(k) = \alpha(k_0)e.$$

Since $e \in E$ is arbitrary, we conclude that $\phi(G)(k_0) = \alpha(k_0)$ and k_0 is arbitrary, we have $\phi(G) = \alpha$.

Finally we show that ϕ is an isometry. Let $G \in L_n^r(CD_0(K,E),F)$ and $f \in CD_0(K,E)$. Then

$$\begin{aligned}
\|G\|_r &= \sup_{\|f\|\leq 1} \| \, |G|(f) \, \| = \sup_{\|f\|\leq 1} \| \, |G|(|f|) \, \| = \sup_{\|f\|\leq 1} \| \, |G| \left(\sum_{k\in K} \chi_k \otimes |f| \right) \, \| \\
&= \sup_{\|f\|\leq 1} \| \sum_{k\in K} |G|(\chi_k \otimes |f|) \, \| = \|\phi(|G|)\| = \|\phi(G)\|_r .
\end{aligned}$$

This completes the proof. □

Definition 4.37 *Let K be a compact Hausdorff space without isolated points, E and F be two Banach lattices with F Dedekind complete. Then we define $l_w^1(K, L_n^r(E,F))$ as the set of all maps $\varphi = \varphi(k)$ from K into $L_n^r(E,F)$ satisfying*

$$\sum_k |\varphi(k)|(|f(k)|) \in F$$

where $f \in CD_w(K,E)$ and $\sum_k |\varphi(k)|(f_\alpha(k)) \downarrow_\alpha 0$ in F whenever $f_\alpha \downarrow 0$ in $CD_w(K,E)$.

$l_w^1(K, L_n^r(E,F))$ is a Banach lattice under pointwise operations and supremum norm. The following theorem is similar to Theorem 4.36 so we omit its proof.

Theorem 4.38 *Let K, E and F be as above definition. Then $L_n^r(CD_w(K,E),F)$ is isometrically lattice isomorphic to $l_w^1(K, L_n^r(E,F))$.*

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