DYNAMIC COMPLEX HEDGING AND PORTFOLIO OPTIMIZATION IN ADDITIVE MARKETS

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ABSTRACT

DYNAMIC COMPLEX HEDGING AND PORTFOLIO OPTIMIZATION IN ADDITIVE MARKETS

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In this study, the geometric Additive market models are considered. In general, these market models are incomplete, that means: the perfect replication of derivatives, in the usual sense, is not possible. In this study, it is shown that the market can be completed by new artificial assets which are called “power-jump assets” based on the power-jump processes of the underlying Additive process. Then, the hedging portfolio for claims whose payoff function depends on the prices of the stock and the power-jump assets at maturity is derived. In addition to the previous completion strategy, it is also shown that, using a static hedging formula, the market can also be completed by considering portfolios with a continuum of call options with different strikes and the same maturity. What is more, the portfolio optimization problem is considered in the enlarged market. The optimization problem consists of choosing an optimal portfolio in such a way that the largest expected utility of the terminal wealth is obtained. For particular choices of the equivalent martingale measure, it is shown that the optimal portfolio consists only of bonds and stocks.

Keywords: Additive processes, Power-jump processes, Martingale Representation Property, Replicating Portfolio, Portfolio optimization.

ÖZ

ADDITIVE PİYASALARDA DİNAMİK KOMPLEKS RİSK MİNİMİZASYONU VE PORTFÖY OPTİMİZASYONU

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Bu çalışmada, geometrik Additive piyasa modelleri incelenmiştir. Genellikle, bu piyasa modelleri tam olmayıp, bu durum şu anlama gelmektedir: Türev ürünlerinin, bilinen haliyle, mükemmel bir şekilde riski minimize etmesi mümkün değildir. Bu çalışmada, piyasanın Additive süreçlerine bağlı kuvvet sıçrama süreçlerini içeren ve kuvvet sıçrama varlıkları olarak adlandırılan yapay varlıklarla tamamlanabileceği gösterilmiştir. Daha sonra alacak hakkına ait ödeme fonksiyonunun hisse senedi ve kuvvet varlıklarının vade sonu değerlerine bağlı riskten korunma portföyü ifade edilmiştir. Önceki tamamlama stratejisine ek olarak, dinamik risk minimizasyonu formülü kullanılarak, piyasanın aynı vadesonu ve farklı vadesonu fiyatına sahip satın alma hakkı veren sürekli opsiyonları içeren porföyleri göz önünde bulundurarak ta tamamlanabileceği gösterilmiştir. Ek olarak, genişletilmiş olan piyasada portföy optimizasyon problemi incelenmiştir. Problem; Optimal portföyün, nihai servete ait beklenen faydasının maksimum olarak belirlenmesini ifade etmektedir. Denk martengale ölçüsünün özel seçimlerinde, optimal porföyün sadece tahvil ve hisse senetlerini içerdiği gösterilmiştir.

Anahtar kelimeler: Toplama süreçleri, Üstel olarak sıçrama özelliği gösteren süreçler, Martingale Temsili Özelliği, Yineleme Portföyü, Potföy optimizasyonu.

To my family,

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CHAPTER 1

INTRODUCTION

In a complete market, any contingent claim can be valued on the basis of the unique equivalent martingale measure. This means that, any contingent claim can be replicated by an admissible self-financing portfolio. For instance, Black-Scholes model is a known complete market model, where the stock prices evolve according to a geometric Brownian motion. This model can be seen more detailed in Black and Scholes [3]. However, when the sources of randomness are more than the number of investment assets, the completeness vanishes.

In the real world, there are a lot of incomplete market models and specially most Additive market models are incomplete. In these type of markets, a general claim is not necessarily a stochastic integral of a stochastic process based in the model. This means that, the claim has an intrinsic risk. So, a risk minimising strategy must be used in these type of models.

There are some techniques to minimise risk. Mean-variance hedging strategy is one of them. This strategy can be seen in the study of Föllmer and Schweizer [14]. Quantile hedging is another risk minimising strategy that is studied in Föllmer and Leukert [15]. Main characteristic of this strategy is that, this strategy requires a large amount of initial capital. In our work, we prefer the previous strategy to minimize risk.

We still use superhedging strategy in incomplete market models, but the cost of these strategies in many cases are too high. For instance, superhedging cost of the call option is the price of the underlying asset in the call.

In this study, the geometric Additive market models are considered and these market models are based on additive processes. These market models are generally incomplete, so a contingent claim can not be replicated by a self-financing portfolio in the usual sense. In our study, we define some new artificial assets, so called "power-jump assets". Then, by using these new artificial assets, we complete the market. This type of a completion strategy for the Lévy case was done by Corcuera, Nualart and Schoutens [9]. We show that the enlarged market, where trading instruments are bond, stock and power jump assets, is complete.

In our study, it is also shown that the market can be completed by considering porfolios with a continuum of Europian call options with the same maturity and different strikes.

Other authors try to replicate complex derivatives by using liquid and non-redundant assets. For example, Balland [1] uses short-dated vanilla options, because of their liquid enough prices. Carr and Madan [5] uses self-decomposible laws at unit time and the associated self-similar additive processes. They show that the models based on these processes describe the option price surface equally well. Jacod and Protter [16] also try to complete incomplete markets. In their study, it is shown that the market can be completed by adding some new trading assets.

By giving the explicit hedging portfolios for claims whose payoff function depends on the prices of the stock and power jump processes at maturity, the portfolio optimization problem is considered. This problem includes choosing an optimal portfolio in such a way that the largest expected utility of the terminal wealth is obtained. In this thesis, a class of special utility functions, including HARA, logarithmic and exponential utilities, are considered. Then the optimal portfolio that maximizes the terminal expected utility is obtained by the martingale method. It is shown that for particular choices of the equivalent martingale measure, the optimal portfolio consists of only bonds and stocks.

The organization of this study is as follows. In chapter 2, basic definitions and theorems related to the Lévy processes and Additive processes are given. In chapter 3, geometric Additive market model and power-jump processes are given. In chapter 4, power-jump assets are considered and geometric Additive market is completed by these artificial assets. In chapter 4, the hedging portfolio whose payoff is a function of time, stock price and the new assets at maturity is given. In chapter 4, it is also shown that the market can be completed by considering portfolios with a continuum of call options with different strikes and the same maturity. In chapter 5, portfolio optimization problem in the enlarged Additive market is considered. Chapter 6, concludes the thesis.

CHAPTER 2

PRELIMINARIES

2.1 SOME DEFINITIONS AND EXAMPLES

Definitions and theorems in this part are mainly taken from [8], [22] and [23].

Assume that we are given in a complete, filtered probability space $(\Omega, \mathcal{F}_t, F, P)$ in which $F = (\mathcal{F}_t)_{0 \le t < \infty}$ satisfies the usual hypothesis, i.e. :

(i) $\mathcal{F}_0$ contains all the $P-$null sets $\mathcal{F}$.

(ii) $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u$, all t, $0 \le t < \infty$; that is, the filtration F is right continuous.

Definition 2.1.1 *A stochastic process* $X = \{X_t, t \ge 0\}$ *on* R^d *(d-dimensional Euclidean space) is stochastically continuous or continuous in probability if, for every* $t \ge 0$ *and* $\varepsilon > 0$, $\lim_{s \to t} P[|X_s - X_t| > \varepsilon] = 0.$

Definition 2.1.2 *A filtration or information flow on* $(\Omega, \mathcal{F}, P)$ *is an increasing family of* σ*-algebras* $(\mathcal{F}_t)_{t \in [0,T]}$: $\forall t \ge s \ge 0,\ \mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}.$

Definition 2.1.3 *A contingent claim is any stochastic variable* $X = \Phi(z)$*, where z is a stochastic variable drived by a stock price process.*

Definition 2.1.4 *A set is called Borel, if it can be constructed from open or closed sets by repeatedly taking countable unions and intersections.*

Definition 2.1.5 *A function* $f : [0,T] \to R^d$ *is said to be cádlág if it is right-continuous with left limits: for each* $t \in [0,T]$ *the limits*

$$f(t-) = \lim_{s \to t, s<t} f(s), \qquad f(t+) = \lim_{s \to t, s>t} f(s)$$

exist and $f(t) = f(t+)$.

Similarly, a function $f : [0,T] \to R^d$ *is said to be cádlád if it is left-continuous with right limits: for each* $t \in [0,T]$ *the limits*

$$f(t-) = \lim_{s \to t, s<t} f(s), \qquad f(t+) = \lim_{s \to t, s>t} f(s)$$

exist and $f(t) = f(t-)$.

Definition 2.1.6 *A probability measure* μ *on* R^d *is called infinitely divisible if, for any positive integer n,there is a probability measure* μ_n *on* R^d *such that* $\mu = \mu_n^{n*}$. *(* μ^{n*} *is the n-fold convolution of probability measure* μ *i.e.* $\mu^{n*} = \mu_* \mu_* \dots_* \mu$ *)*

Definition 2.1.7 *A real-valued, adapted process* $X = (X_t)_{0 \le t < \infty}$ *is called martingale (resp. Supermartingale, submartingale) with respect to the filtration* $F = \{\mathcal{F}_t, 0 \le t < \infty\}$ *if*

(i) $X_t = L^1(dP)$; *that is,* $E\{|X_t|\} < \infty$.
(ii) if $s \le t$, *then* $E\{X_t | \mathcal{F}_s\} = X_s$, *a.s. (resp.* $E\{X_t | \mathcal{F}_s\} \le X_s$, *resp.* $E\{X_t | \mathcal{F}_s\} \ge X_s$ *)*

Definition 2.1.8 *A family of random variables* $(U_\alpha)_{\alpha \in A}$ *is uniformly integrable if*

$$\lim_{n \to \infty} \sup_{\alpha} \int_{\{|U_\alpha| \ge n\}} |U_\alpha| dP = 0$$

Definition 2.1.9 *A random variable* T *is nonanticipating random time* ($(\mathcal{F}_t)$ *-stopping time*) *if*

$$\forall t \ge 0, \ \{T \le t\} \in \mathcal{F}_t.$$

Definition 2.1.10 *Let* $E \subset R^d$. *A radon measure on* (E, ε) *is a measure* μ *such that for every compact measurable set* $B \in \varepsilon$, $\mu(B) < \infty$.

Definition 2.1.11 *An adapted, càdlàg process X is a local martingale if there exists a sequence of increasing stopping times* T_n, *with* $\lim_{n \to \infty} T_n = \infty$ *a.s. such that* $X_{t \wedge T_n} \mathrm{I}_{\{T_n > 0\}}$ *is an uniformly integrable martingale for each n.*

Definition 2.1.12 *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space,* $E \subset R^d$ *and* μ *is a given (positive) Radan measure on* (E, ε). *A poisson random measure on E with intensity measure* μ *is an integer valued random measure:*

$$Q : \Omega \times \varepsilon \to N$$
$$(w, A) \to Q(w, A)$$

such that

1. *For (almost all)* $w \in \Omega$, $Q(w,.)$ *is an integer-valued Radon measure on E: for any bounded measurable* $A \subset E$, $Q(A) < \infty$ *is an integer valued random variable.*
2. *For each measurable* $A \subset E$, $Q(.,A) = Q(A)$ *is a Poisson random variable with parameter* $\mu(A)$:

$$\forall k \in N, \quad P(Q(A) = k) = e^{-\mu(A)} \frac{(\mu(A)^k)}{k!}.$$

3. *For disjoint measurable sets* $A_1,...,A_n \in \varepsilon$, *the variables* $Q(A_1),...,Q(A_n)$ *are independent.*

Definition 2.1.13 *A process H is said to be simple predictable if H has a representation*

$$H = H_0\, \mathrm{I}_{\{0\}}(t) + \sum_{i=1}^{n} H_i \mathrm{I}_{(T,T_{i+1}]}(t)$$

where $0 = T_1 \le ... \le T_{n+1} < \infty$ *is a finite sequence of stopping times,* $H_i \in \mathcal{F}_{T_{ii}}$ *with* $|H_i| < \infty$ *a.s.,* $0 \le i \le n$. *The collection of simple predictable processes is denoted S. We can topologize S by uniform convergence* $\{t, w\}$, *and we denote S endowed with this topology by* S_u.

Definition 2.1.14 *A process X is a total martingale if X is cádlág, adapted and* $\mathrm{I}_X : S \to L^0$ *is continuous. The linear mapping* $\mathrm{I}_X : S \to L^0$ *is defined for a given process X and for a given simple predictable process H as follows:*

$$I_X(H) = H_0 X_0 + \sum_{i=1}^{n} H_i (X_{T_{i+1}} - X_{T_i}).$$

Definition 2.1.15 *A process* $X = \{X_t, t \in [0,T]\}$ *is called a semimartingale if, for each* $t \in [0,\infty]$, X^t *is a total martingale. The notation* X^t *denotes the process* $(X_{s\wedge t})_{s\ge 0}$.

Corollary 2.1.1 *A semimartingale* $X = \{X_t, t \in [0,T]\}$ *admits the decomposition*

$$X_t = X_0 + A_t + M_t$$

where X_0 *is finite and* $\mathcal{F}_t$*-measurable,* $M = \{M_t, t \in [0,T]\}$ *is a local martingale with* $M_0 = 0$ *and A is a finite variation process with* $A_0 = 0$.

Definition 2.1.16 *Let X be semimartingale. The quadratic variation process of X, denoted* $[X,X] = ([X,X]_t)_{t\ge 0}$ *is defined by*

$$[X,X] = X^2 - \int X_- dX.$$

Definition 2.1.17 *An adapted process* $B = (B_t)_{0\le t<\infty}$ *taking values in* R^n *is called n-dimensional Brownian motion if*

(i) *for* $0 \le s < t < \infty$, $B_t - B_s$ *is independent of* $\mathcal{F}_s$.

(ii) *for* $0 \le s < t$, $B_t - B_s$ *is a Gaussian random variable with mean zero and variance matrix (t-s)C, for a given, non-random matrix C.*

Definition 2.1.18 *Let* $(\tau_i)_{i\geq 1}$ *be a sequence of independent exponential random variables with parameter* λ *and* $T_n = \sum_{i=1}^{n} \tau_i$. *The process* $\{N_t, t \geq 0\}$ *defined by*

$$N_t = \sum_{n\geq 1} \mathrm{I}_{t \geq T_n}$$

is called Poisson process with intensity λ.

Definition 2.1.19 *An adapted process* $X = (X_t)_{t\geq 0}$ *with* $X_0 = 0$ *a.s. is a Lévy process if*

(1) X has increments independent of the past; that is, $X_t - X_s$ *is independent of* $\mathcal{F}_s$, $0 \leq s < t < \infty$; *and,*

(2) X has stationary increments; that is, $X_t - X_s$ *has the same distribution as* X_{t-s}, $0 \leq s < t < \infty$; *and,*

(3) X_t *is continuous in probability; that is,* $\lim_{t\to s} X_t = X_s$, *where the limit is taken in probability.*

Definition 2.1.20 *Let* Λ *be a Borel set in R bounded away from* 0 *and let* $0 \notin \bar{\Lambda}$. *Define the random variables as following:*

$$T_\Lambda^1 = \inf\{t > 0 : \Delta X_t \in \Lambda\}$$

$$\vdots$$

$$T_\Lambda^{n+1} = \inf\{t > T_\Lambda^n : \Delta X_t \in \Lambda\}$$

Define

$$N_t^\Lambda = \sum_{0<s\leq t} \mathrm{I}_\Lambda(\Delta X_s) = \sum_{n=1}^{\infty} \mathrm{I}_{\{T_\Lambda^n \leq t\}}$$

Then the masure ν, *defined by*

$$\nu(\Lambda) = E\{N_1^\Lambda\} = E\{\sum_{0<s\leq 1} \mathrm{I}_\Lambda(\Delta X_s)\}$$

is called the Lévy measure of the Lévy process X.

Definition 2.1.21 *Let* $X = \{X_t, t \in [0,T]\}$ *be a stochastic process on* R^d *and consider the following properties:*

(1) For any choice of $m \geq 1$ *and* $0 \leq t_0 \leq t_1 \leq ... \leq t_m$, *the random variables* $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_m} - X_{t_{m-1}}$ *are independent.*

(2) $X_0 = 0$ *almost surely.*

(3) The process X_t is stochastically continuous.

(4) There is a $\pi_0 \in F$ with $P[\pi_0]=1$ such that for every $w \in \pi_0$, $X_t(w)$ is right continuous in $t \geq 0$ and has left limits in $t \geq 0$

If X satisfies the properties (1),(2) and (3), then it is called Additive in law. If the properties (1),(2),(3) and (4) are satisfied by X, then the process X is called Additive process.

Some known Additive processes can be seen in following examples:

Example 2.1.1 ***Brownian motion with time dependent volatility:***
Let $(W_t)_{t\geq 0}$ be a standart Brownian motion on R, $\sigma(t): R^+ \to R^+$ be a measurable function such that $\int_0^t \sigma^2(s)ds < \infty$ for all $t>0$ and $b(t): R^+ \to R$ be a continuous function. Then the process

$$X_t = b(t) + \int_0^t \sigma(s) dW_s$$

is an Additive process.

Example 2.1.2 ***Cox process with deterministic intensity:***
Let $\lambda(t): R^+ \to R^+$ be a positive measurable function such that $\Lambda(t) = \int_0^t \lambda(s)ds < \infty$ for all t and let Q be Poisson random measure on R^+ with intensity measure $\mu(A) = \int_A \lambda(s)ds$ for all $A \in \mathcal{B}(R^+)$. Then the process $(X_t)_{t\geq 0}$ defined path by path via $X_t(w) = \int_0^t Q(w, ds)$ is an Additive process.

Example 2.1.3 ***Time inhomogeneous jump-diffusion:***

Given positive functions $\sigma: R^+ \to R^+, \lambda: R^+ \to R^+$ as above and a sequence of independent random variables (Y_i) with distribution F the process defined by

$$X_t = \int_0^t \sigma(s) dW_s + \sum_{i=1}^{N_{\Lambda(t)}} Y_i$$

is an Additive process and $(N_t)_{t\geq 0}$ is a standart Poisson process.
Example 2.1.4 ***Lévy process with deterministic volatility:***

Extending (Example 2.1.1), we now consider Lévy processes with time dependent volatility. Consider a continuous function $\sigma(t): R^+ \to R^+$. *Let* $(L_t)_{t\geq 0}$ *be a Lévy process on* R. *Then*

$$X_t = \int_0^t \sigma(s) dL_s$$

is an Additive process.

Example 2.1.5 ***Time changed Lévy process:***
Let $(L_t)_{t\geq 0}$ *be a Lévy process on* R^d *and let* $\nu(t): R^+ \to R^+$ *be a continuous increasing function such that* $\nu(0) = 0$. *Then the process* $(X_t)_{t\geq 0}$ *defined path by path via*

$$X_t(w) = L_{\nu(t)}(w)$$

is an Additive process.

Definition 2.1.22 *A strategy* Φ *is called self-financing if the following equation is satisfied for all* $n \in \{0,1,...,N-1\}$

$$\Phi_n S_n = \Phi_{n+1} S_n .$$

Definition 2.1.23 *For a given filtration* $\mathcal{F}: \mathcal{F}_0, \mathcal{F}_1, ..., \mathcal{F}_N$ *an adapted sequence* $(H_n)_{0<n\leq N}$ *of random variables is predictable if, for all* $n \geq 1$, H_n *is* $\mathcal{F}_{n-1}$ *measurable.*

Definition 2.1.24 *An adapted, càdlàg process X is a potential if it is a non-negative supermartingale such that* $\lim_{t\to\infty} E\{X_t\} = 0$.

Definition 2.1.25 *Let* $(\Omega, \mathcal{A}, P)$ *be a probability space. A probability measure* $\tilde{P}$ *on* $(\Omega, \mathcal{A})$ *is absolutely continuous relative to* P, *if*

$$\forall A \in \mathcal{A},\ P(A) = 0 \Rightarrow \tilde{P}(A) = 0.$$

Definition 2.1.26 *A stochastic process* X_t *is called previsible if it satisfies following two properties:*

(i) $X_0 = 0$,

(ii) $X_t^2 - [X, X]_t$ *is uniformly integrable martingale.*

Definition 2.1.27 *Suppose that* τ *is a partition of* $[0, \infty]$ *and that* $X_{t_i} \in L^1$, *each* $t_i \in \tau$. *Define*

$$C(X, \tau) := \sum_{i=0}^{n} \left| E\{X_{t_i} - X_{t_{i+1}} \middle| \mathcal{F}_{t_i}\} \right|$$

The variation of X along τ *is defined to be*

$$Var_{\tau}(X) = E\{C(X,\tau)\},$$

The variation of X is defined to be

$$Var(X) = \sup_{\tau} Var_{\tau}(X),$$

where the supremum is taken over all such partitions.

Definition 2.1.28 *Let* $A = (A_t)$ *be a cádlág process. Then, A is called a finite variation process (FV) if almost all of the paths of A of the paths of A are of finite variation on each compact interval of* R_+.

Definition 2.1.29 *Let* $\{X_t, t \geq 0\}$ *be a stochastic process on* R^d. *It is called self-similar if, for any a>0 there is b>0 such that*

$$\{X_{at} : t \geq 0\} \overset{d}{=} \{bX_t : t \geq 0\}$$

Definition 2.1.30 *Let* p *be a probability measure on* R^d. *It is called selfdecomposable, or of class L, if for any b>1,there is a probability measure* ρ_b *on* R^d *such that*

$$\hat{\mu}(z) = \hat{\mu}(b^{-1}z)\, \hat{\rho}_b(z).$$

Definition 2.1.31 *A function* $g : R \to [0,\infty)$ *is called submultiplicative if there exists a constant* $c > 0$ *such that* $g(x+y) \leq cg(x)g(y)$ *for all* $x, y \in R.$

Definition 2.1.32 *A semimartingale X is called quadratic pure jump if* $[X,X]^C = 0.$

2.2 SOME THEOREMS

Theorem 2.2.1 (Lévy-Khintchine formula) *Let X be Lévy process with Lévy measure ν. Then*

$$E\{e^{iuX_t}\} = e^{-t\Psi(u)},$$

where

$$\Psi(u) = \frac{\sigma^2}{2}u^2 - i\alpha u + \int_{\{|x|\geq 1\}} (1 - e^{iux})\nu(dx) + \int_{\{|x|<1\}} (1 - e^{iux} + iux)\nu(dx).$$

Moreover given ν, σ^2, α, the corresponding Lévy process is unique in distribution.

Theorem 2.2.2 (Lévy Decomposition Theorem) *Let $X = \{X_t, t \in [0,T]\}$ be a Lévy process. Then X has a decomposition*

$$\begin{aligned} X_t &= B_t + \int_{\{|x|<1\}} x(N_t(.,dx) - t\nu(dx)) \\ &\quad + tE\{X_1 - \int_{\{|x|\geq 1\}} xN_1(.,dx)\} + \int_{\{|x|\geq 1\}} xN_t(.,dx) \\ &= B_t + \int_{\{|x|<1\}} x(N_t(.,dx) - t\nu(dx)) + \alpha t + \sum_{0<s\leq t} \Delta X_s \mathrm{I}_{\{|\Delta X_s|\geq 1\}}, \end{aligned}$$

where B is a Brownian motion; for any set Λ, $0 \notin \bar{\Lambda}$, $N_t^{\Lambda} = \int_{\Lambda} N_t(.,dx)$ is a Poissson process independent of B; N_t^{Λ} is independent of N_t^{Γ} if Λ and Γ are disjoint; N_t^{Λ} has parameter $\nu(\Lambda)$; and $\nu(dx)$ is a measure on $R \setminus \{0\}$ such that $\int \min(1, x^2)\nu(dx) < \infty$.

Theorem 2.2.3 *Let $X = \{X_t, t \in [0,T]\}$ be a Lévy process with triplet (λ_t, D_t, ν_t).*

(i) If $\nu(R) < \infty$, then almost all paths of X have a finite number of jumps on every compact interval. In that case, the Lévy process has finite activity.

(ii) If $\nu(R) = \infty$, then almost all paths of X have an infinite number of jumps on every compact interval. In that case, the Lévy process has infinite activity.

Theorem 2.2.4 *Let X be a Lévy process and let $\mathcal{G}_t = \mathcal{F}_t^0 \vee N$ where $(\mathcal{F}_t^0)_{0\leq t<\infty}$ is the natural filtration of X, and $\mathcal{N}$ are the P-null sets of $\mathcal{F}$. Then $(\mathcal{G}_t)_{0\leq t<\infty}$ is right continuous.*

Theorem 2.2.5 *Let Λ be Borel with $0 \notin \bar{\Lambda}$. Let ν be the Lévy measure of X, and let $f\mathrm{I}_{\Lambda} \in L^2(d\nu)$. Then*

$$E\left\{\int_{\Lambda} f(x)N_t(.,dx)\right\} = t\int_{\Lambda} f(x)\nu(dx)$$

and also $E\{(\int_{\Lambda} f(x)N_t(.,dx) - t\int_{\Lambda} f(x)\nu(dx))^2\} = t\int_{\Lambda} f(x)^2\nu(dx).$

Theorem 2.2.6 (Doob Decomposition) *A potential* $(X_n)_{n\in N}$ *has a decomposition* $X_n = M_n - A_n$, *where* $A_{n+1} > A_n$ *a.s.* $A_0 = 0$, $A_n \in \mathcal{F}_{n-1}$, *and* $M_n = E\{A_\infty | \mathcal{F}_n\}$. *Such a decomposition is unique.*

Theorem 2.2.7 (Girsanov theorem) *Let* $(\theta_t)_{0\le t\le T}$ *be an adapted process satisfying* $\int_0^T \theta_s{}^2 ds < \infty$ *a.s. and such that the process* $(L_t)_{0\le t\le T}$ *defined by*

$$L_t = \exp\left(-\int_0^t \theta(s)dB_s - \frac{1}{2}\int_0^t \theta_s{}^2 ds\right)$$

is a martingale. Then, under the probability $P^{(L)}$ *with density* L_T *relative to* P, *the process* $(W_t)_{0\le t\le T}$ *defined by* $W_t = B_t + \int_0^t \theta_s ds$ *is a standart Brownian motion.*

Theorem 2.2.8 (Itô's formula for continuous semimartingales) *Let X be a continuous semimartingale and let f be a* C^2 *real function. Then f(X) is again a semimartingale and the following formula holds:*

$$f(X_t) - f(X_0) = \int_{0_+}^t f'(X_s)dX_s + \frac{1}{2}\int_{0_+}^t f''(X_s)d[X,X]_s.$$

Theorem 2.2.9 (Itô's formula for semimartingales that have jumps) *Let X be a semimartingale and let f be a* C^2 *real function. Then* $f(X)$ *is again a semimartingale and the following formula holds:*

$$f(X_t) - f(X_0) = \int_{0_+}^t f'(X_{s_-})dX_s + \frac{1}{2}\int_{0_+}^t f''(X_{s_-})d[X,X]_s^C$$
$$+ \sum_{0<s\le t}\{f(X_s) - f(X_{s_-}) - f'(X_{s_-})\Delta X_s\}.$$

Theorem 2.2.10 (Lévy-Itô Decomposition) *Let* $X = \{X_t, t\ge 0\}$ *be an additive process on* R^d *defined on a probability space* $(\Omega,,\mathcal{F},P)$ *with system of generating triplets* $\{(\lambda_t, D_t, \nu_t)\}$ *and define the measure* $\tilde{\nu}$ *on H (exponent of the*

non-trivial broad-sense self similar process X) by $\tilde{\nu}((0,t)\times B)$ *for* $B \in \mathcal{B}(R^d)$. *Using* Ω_0 *from definition 2.1.21, define, for* $B \in \mathcal{B}(H)$,

$$Q(B,w) = \begin{cases} \#\{\mathrm{s} : (\mathrm{s}, \mathrm{X_s(w)} - \mathrm{X_{s_-}(w)}) \in \mathrm{B}\} & \text{for } w \in \Omega_0 \\ 0 & \text{for } w \notin \Omega_0 \end{cases}$$

Then the following holds:

(i) $\{Q(B): B \in \mathcal{B}(H)\}$ is a Poisson random measure on H with intensity measure $\tilde{\nu}$.

(ii) There is $\Omega_1 \in \mathcal{F}$ *with* $P[\Omega_1]=1$ *such that, for any* $w \in \Omega_1$

$$X_t^1(w) := \lim_{\varepsilon \downarrow 0} \int_{(0,t]\times D(\varepsilon,1]} \{x(Q(d(s,x),w) - x\tilde{\nu}(d(s,x))\}$$

$$+ \int_{(0,t]\times D(1,\infty)} xQ(d(s,x),w).$$

is defined for all $t \in [0,\infty)$ *and the the convergence is uniform in t on any bounded interval. The process* $\{X_t^1\}$ *is an additive process on* R^d *with* $\{(0,0,\nu_t)\}$ *as the system of generating triplets.*

(iii) Define $X_t^2(w) := X_t(w) - X_t^1(w)$ *for* $w \in \Omega_1$

There is $\Omega_2 \in \mathcal{F}$ *with* $P[\Omega_2]=1$ *such that, for any* $w \in \Omega_2$, $X_t^2(w)$ *is continuous in* t. *The process* $\{X_t^2\}$ *is an additive process on* R^d *with* $\{(\lambda_t, D_t, 0)\}$ *as the system of generating triplets.*

(iv) The two processes $\{X_t^1\}$ *and* $\{X_t^2\}$ *are independent.*

Theorem 2.2.11 *Let* φ *be a characteristic function, and let F be the corresponding distribution function. Then* φ *is analytic if and only if the following conditions hold:*

(i) F has moments α_k *of all orders k.*

(ii) There exists a positive number γ *such that* $|\alpha_k| \le k!\gamma^k$ *for all* $k \ge 1$.

Theorem 2.2.12 *Let* $\beta \in R$, *and let* Π *be a Lévy measure. Then*

$$E(e^{\beta X_t}) < \infty \text{ for all } t \ge 0 \text{ if and only if } \int_{|x|\ge 1} e^{\beta x}\Pi(dx) < \infty.$$

Theorem 2.2.13 *Suppose that g is measurable, submultiplicative and bounded on compacts and let* Π *be a Lévy measure. Then*

$$\int_{|x|\ge 1} g(x)\Pi(dx) < \infty \text{ if and only if } E(g(X_t)) < \infty \text{ for all } t > 0.$$

CHAPTER 3

THE GEOMETRIC ADDITIVE MARKET MODEL AND THE POWER JUMP PROCESSES

3.1 THE GEOMETRIC ADDITIVE MARKET MODEL

Consider a market model where the stock price process $S = \{S_t, t \in [0,T]\}$ is a geometric additive process and satisfies the stochastic differential equation

$$\frac{dS_t}{S_{t_-}} = dX_t, \qquad S_0 > 0 \tag{3.1}$$

where $X = \{X_t, t \in [0,T]\}$ is an additive process.

In our market model, we will consider that we have a riskless asset or bond

$$B_t = \exp\left(\int_0^t r_s ds\right) \tag{3.2}$$

where r_t is deterministic spot interest rate.

We know the theory of integration and stochastic differential equations for semimartingales. In our study, this theory is considered for Additive processes.
We consider that our additive process is defined on a complete, filtered-probability space $(\Omega, \mathcal{F}_t, F, P)$.
The filtration F is the natural filtration generated by the stock price process S completed with P null sets $\mathcal{N}$ ie:
$F = \{\mathcal{F}_t, t \in [0,T]\} \cup \mathcal{N}$ where $\mathcal{F}_t = \sigma\{S_s, 0 \le s \le t\}$.
X_t has a cádlág modification. That means that the process X_t is right continuous with left limits.
X_t has an infinitely divisible distribution for all t. It's distribution is determined by generating triplets (λ_t, D_t, ν_t).
In above, λ_t is called as location parameter of X_t which is a continuous function.
D_t is called as Gaussian covariance. D_t is nonnegative and increasing continuous function.

ν_t is called as Lévy measure. ν_t is an increasing(in t) positive measure on R such that $\nu_t(\{0\}) = 0$ and $\nu_s(C) \to \nu_t(C)$ as $s \to t$ for all measurable sets $C \subset \{x, |x| \geq \delta\}$ for some $\delta > 0$ and

$$\int_R (1 \wedge x^2)\ \nu_t(dx) < \infty \tag{3.3}$$

for all $t \in [0, T]$.

The occurence of the generating triplets for Lévy and Additive processes is granted by the following theorem in Sato [23]:

Theorem 3.1.1 *If φ is an infinitely divisible distribution on R^d, then*

$$\hat{\varphi}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{R^d} (e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle \mathrm{I}_D(x)\nu(dx) \right] (z \in R^d) \tag{3.4}$$

where A is a symmetric, nonnegative-definite $d \times d$ matrix; ν is a measure on R^d satisfying

$$\nu(\{0\}) = 0 \ \text{and} \ \int_{R^d} (|x|^2 \wedge 1)\nu(dx) < \infty \ \text{and} \ \nu \in R^d$$

The representation of $\hat{\varphi}(z)$ in (i) by A, ν and γ are unique.

Conversely, if A is a symmetric, nonnegative definite $d \times d$ matrix; ν is a measure satisfying properties given above and $\gamma \in R^d$, then there exists an infinitely divisible distribution φ whose characteristic function is given by (3.4).

Proof of this theorem can be seen in Sato [23].

Definition 3.1.1 *We call (γ, A, ν), in the above theorem, generating triplet of φ.*
A is called Gaussian covariance matrix and ν is called Levy measure of φ and γ is called location parameter.

By Theorem (2.2.10), Additive processes can be written as following decomposition:

$$X_t = X_t^1 + X_t^2. \tag{3.5}$$

The process $X^2 = \{X_t^2, t \in [0,T]\}$ is the continuous part of X_t and the process $X^1 = \{X_t^1, t \in [0,T]\}$ is defined as:

$$X_t^1 := \lim_{\epsilon \downarrow 0} \int_{\{s\in(0,t],\epsilon<|x|<1\}} x(Q(ds,dx) - \tilde{\mu}(ds,dx)) + \int_{\{s\in(0,t],|x|\geq 1\}} xQ(ds,dx), \tag{3.6}$$

where $Q(ds,dx)$ is a Poisson Random measure on $[0,T]\times(R\setminus\{0\})$ with intensity measure $\tilde{\mu}(d(s,x))$. ($\tilde{\mu}(d(s,x))$ is defined by $\tilde{\mu}((0,T]\times C) := \mu(C)$ for all measurable $C \subset \mathcal{B}(R)$).

The process $X^2 = \{X_t^2, t\in[0,T]\}$ is defined as:

$$X_t^2 := X_t - X_t^1. \tag{3.7}$$

Here, the processes X_t^1 and X_t^2 are also additive processes. Generating triplets for these additive processes are respectively $(0,0,\nu_t)$ and $(\lambda_t, D_t, 0)$.
We know the theory of stochastic integration for semimartingales and all Additive processes are not semimartingale.
So, in our study we work only with Additive processes which are semimartingale. By doing this, we are working with a subclass of additive processes (the set of natural additive processes as they are called in Sato [24]). By the definition given in Sato [24] we can say that an additive process is natural, if it's location parameter has bounded variation.
By Theorem 2.6 in Sato [24], an Additive process is natural if and only if it has a factoring. And, an Additive process is semimartingale if and only if it is natural.
Thus, to make an additive process natural, we must define a factoring. A factoring is defined as follows: A factoring is a pair $(\{c_t, t\in[0,T]\},\rho)$, where ρ is a continuous (atomless) finite measure on [0,T] and c_t are family of infinitely divisible distributions such that characteristic function of X_t is equal to $\exp\int_0^t \log(\hat{c}_s(u)\rho(du)$

($\hat{c}_t$ corresponds the characteristic function of c_t.)

Denote the generating triplets of c_t as $(\eta_t, \chi_t^2, \sigma_t)$. By the theorem 2.6 and lemma 2.7 in Sato [24] , we can write these generating triplets as following:

$$\lambda_t = \int_0^t \eta_s \rho(ds), \tag{3.8}$$

$$D_t = \int_0^t \chi_s^2 \rho(ds), \tag{3.9}$$

$$\nu_t(C) = \int_0^t \sigma_s(C)\rho(ds), \quad \forall C \in \mathcal{B}(R) \tag{3.10}$$

By the above definitions, it is clear that $\int_0^T \left(\int_R (1 \wedge x^2)\sigma_t(dx) \right)\rho(ds) < \infty$ for all $t \in [0,T]$.

The elements of $(\eta_t, \chi_t^2, \sigma_t)$ are called the local characteristics of X_t. We consider natural additive processes with these local characteristics.

Generalized version of Lévy -Itô Formula gives us:

$$X_t = \int_0^t \chi_s dW_s + J_t \tag{3.11}$$

where $W = \{W_t, t \in [0,T]\}$ is a standart Brownian motion and $J = \{J_t, t \in [0,T]\}$ is a jump process independent of W. And the process $J = \{J_t, t \in [0,T]\}$ satisfies

$$J_t = \int_{\{s \in (0,t], |x|<1\}} x(Q(ds,dx) - \sigma_s(dx)ds) + \int_{\{s \in (0,t], |x| \geq 1\}} x(Q(ds,dx)) + \int_0^t \eta(s)ds. \tag{3.12}$$

In the above equation $Q(ds,dx)$ is a Poisson Random measure on $[0,T] \times R \setminus \{0\}$ with intensity measure $\sigma_t(dx)dt$. (We consider that $\rho(dt) = dt$ is the Lebesque measure.)

Decomposition (3.11) gives us the process $X = \{X_t, t \in [0,T]\}$ is a semimartingale with a quadratic variation

$$[X,X]_t = \int_0^t \chi_s^2 ds + \sum_{s \in (0,t]} \left|\Delta X_s\right|^2. \tag{3.13}$$

Suppose that the family of Lévy measures $\{\sigma_t\}_{t \in (0,T]}$ satisfies the following property, for some δ >0 and $\tau > 0$,

$$\sup_{t \in (0,T]} \int_{(-\delta,\delta)^C} \exp(\tau|x|)\sigma_t(dx) < \infty \tag{3.14}$$

By the theorems 2.2.11 and 2.2.12, we can write that $E(e^{\tau|x|}) < \infty$.

By Taylor series expansion, we can write $\exp\tau(|x|) = \tau|x| + \frac{\tau^2|x^2|}{2!} + \ldots + \frac{\tau^k|x|^k}{k!} + \ldots + \ldots$

Since, the previous series is converging, each term in the sum must be finite. So, the integral of each term in the sum must be finite.

Therefore, we obtain $\int_{-\infty}^{\infty} |x|^j \sigma_t(dx) < \infty$ for all $j \geq 2$ and all $t \in [0,T]$. Thus, all moments of X_t and J_t exist and we can define following functions:

$$m_j(t) := \int_{-\infty}^{\infty} |x|^j \sigma_t(dx), \; j \geq 2, \tag{3.15}$$

$$M_j(t) := \int_0^t m_j(s)ds, \; j \geq 2. \tag{3.16}$$

Moreover, the Doob decomposition of J in terms of a martingale part and a predictable process of finite variation, is given by

$$J_t = N_t + \int_0^t a_s ds$$

where

$$N_t := \int_0^t \int_{-\infty}^{+\infty} xM(ds,dx)$$

is a martingale and $M(dt,dx) := Q(dt,dx) - \sigma_t(dx)dt$ is the compensated Poisson random measure on $[0,T] \times (R \setminus \{0\})$.

Now, if we use the Itô formula for cádlág semimartingales, then we will find the solution of stochastic differential equation (3.1) as follows:

Take $f(S_t) = \log(S_t)$

It is clear that $dS_t = S_{t\_} dX_{t\_} = S_{t\_} \, d\left(\int_0^t \chi_s dW_s + J_t \right)$.

By Itô's Formula, we can write:

$$f(S_t) = f(S_0) + \int_0^t f'(S_{s\_})dS_{s\_} + \frac{1}{2}\int_0^t f''(S_{s\_})d[S,S]_s^C$$
$$+ \sum_{0<s\leq t} \{ f(S_s) - f(S_{s\_}) - f'(S_{s\_})\Delta S_s \}.$$

We also have $S_s - S_{s\_} = \Delta S_s = S_{s\_} \Delta J_s$.

$$\Rightarrow: S_s = S_{s\_}(1 + \Delta J_s).$$

Thus,

$$\log(S_t) = \log(S_0) + \int_0^t \frac{1}{S_{s_-}} S_{s_-} d\left(\int_0^t \chi_s dW_s + J_t\right) + \frac{1}{2}\int_0^t -\frac{1}{S_{s_-}^2} S_{s_-}^2 \chi_s^2 ds$$

$$+ \sum_{0<s\le t} (\log(1+\Delta X_s) - \frac{1}{S_{s_-}} S_{s_-} \Delta X_s)$$

$$\log(S_t) = \log(S_0) + \left(X_t - \frac{1}{2}\int_0^t \chi_s^2 ds\right) + \sum_{0<s\le t} (\log(1+\Delta X_s) - (\Delta X_s))$$

Finally, if we use property of Logarithm function then we will find the solution (3.1) as

$$S_t = S_0 \exp\left(X_t - \frac{1}{2}\int_0^t \chi_s^2 ds\right) \prod_{0<s\le t} (1+\Delta X_s)\exp(-\Delta X_s) \ . \qquad (3.17)$$

By the decomposition of J_t, this decomposition can be written as follows:

$$S_t = S_0 \exp\left(\int_0^t \chi_s dW_s + N_t + \int_0^t \left(a_s - \frac{\chi_s^2}{2}\right) ds\right) \prod_{0<s\le t} (1+\Delta J_s)\exp(-\Delta J_s)$$

with dynamics

$$dS_t = \chi_t S_{t_-} dW_t + S_{t_-} \int_{-\infty}^{+\infty} x M(dt,dx) + S_{t_-} a_t dt.$$

In order to ensure that $S_t > 0$ for all $t \ge 0$ a.s., we require that $\Delta X_t > -1$ for all t. Thus,we shall assume that the family of Lévy measures $\{\sigma_t\}_{t\in[0,T]}$ is supported on (-1,+∞).

3.2 THE POWER JUMP PROCESSES

Now, let us define "The Power Jump Processes" as follows:

$$X_t^{(1)} := X_t,$$

$$X_t^{(j)} := \sum_{0<s\le t} (\Delta X_s)^j, \;\; j \ge 2 \;\text{ where }\; \Delta X_s = X_s - X_{s_-} \tag{3.18}$$

and

$$J_t^{(1)} := J_t,$$

$$J_t^{(j)} := \sum_{0<s\le t} (\Delta J_s)^j, \; j \ge 2\,. \tag{3.19}$$

Definition 3.2.1 *The process $J^{(j)}$ as defined above are called j-th power jump process.*

By definition, it is clear that $X_t^{(j)} = J_t^{(j)}$.

These processes have jumps at the same point as the original additive process, but the size of the jump of $X_t^{(j)}$ is equal to the size of the original jumps to the power j.

Let us find expectation of Power Jump Processes by recalling (3.15) and (3.16):

$$E[X_t^{(1)}] = \int_0^t \eta_s ds + \int_0^t \int_{\{|x|\ge 1\}} x\sigma_s(dx)ds := \int_0^t m_1(s)ds := M_1(t),$$

$$E[X_t^{(j)}] = E\left[\sum_{0<s\le t} (\Delta X_s)^j\right] = \int_0^t \int_{-\infty}^{\infty} x^j \sigma_s(dx)ds := \int_0^t m_j(s)\,ds := M_j(t), \quad j \ge 2$$

Now, let us define compensated Power Jump Processes:

$$Z_t^{(j)} := X_t^{(j)} - E[X_t^{(j)}] = X_t^{(j)} - M_j(t), \;\; j \ge 1. \tag{3.20}$$

By the definition, compensated Power Jump Processes are martingales. These processes are called Teugel Martingales.

From the Orthonormalization procedure described in (Balland. P [1] and Nualart. D and Schoutens W. [9] , we can define the following sequence of strongly orthonormal martingales $\{Y^{(j)},\ j \geq 1\}$:

$$Y_t^{(j)} := \int_0^t b_{j,j}(s)dZ_s^{(j)} + \int_0^t b_{j,j-1}(s)dZ_s^{(j-1)} + \ldots + \int_0^t b_{j,1}(s)dZ_s^{(1)},\ j \geq 1 \qquad (3.21)$$

Here, $b_{j,i}$ are the coefficients of the orthonormalization of the following polynomials with time dependent coefficients,

$$\{\mathrm{I}_{\{s<t\}}, \mathrm{I}_{\{s<t\}}x,\ \mathrm{I}_{\{s<t\}}x^2, \ldots, \mathrm{I}_{\{s<t\}}x^{j-1}\},$$

with respect to the measure $\psi_s(dx)ds = (x^2\sigma_s(dx) + \chi_s^2\delta(dx))ds$ defined in $[0,T]\times R$.
Here, the orthogonalization is considered with respect to the following inner product:

$$\langle m,n \rangle = \int_0^T \int_{-\infty}^{\infty} m_s(x)n_s(x)(x^2\sigma_s(dx) + \chi_s^2\delta(dx))ds$$

Here, the coefficients $m_t(x)$ and $n_t(x)$ are real polynomials with time dependent coefficients and δ is dirac delta.
By the Martingale Representation Property(MRP), we can say that any square integrable Q-martingale can be represented as an orthogonal sum of stochastic integrals with respect to the orthonormalized power jump processes $\{Y^{(j)}, j \geq 1\}$.
This means that, any square integrable martingale $M = \{M_t, t \in [0,T]\}$ submits the representation:

$$M_t = M_0 + \sum_{j=1}^{\infty} \int_0^t h_s^{(j)} dY_s^{(j)}. \qquad (3.22)$$

In the above equation $h_s^{(j)},\ j \geq 1$ are predictable and $E\left[\int_0^t \sum_{j=1}^{\infty} \left|h_s^{(j)}\right|^2 ds\right] < \infty.$

CHAPTER 4

MARKET COMPLETENESS AND THE HEDGING PORTFOLIOS

4.1 MARKET COMPLETENESS

Assume that there exists at least one equivalent martingale measure Q such that X remains a natural additive process under Q. If Q satisfies previous property, then it is called structure preserving. Now, let us see the theorem 3.2 in Chan [7]. This theorem is given for Lévy market models, but we can apply the results in these theorems for our geometric Additive market model.

Theorem 4.1.1 *Let* $\tilde{P}$ *be a measure which is absolutely continuous with respect to P on* $\mathcal{F}_t$. *Then*

$$\frac{d\tilde{P}}{dP}\Big|\mathcal{F}_T = Z_T,$$

where the process $Z = \{Z_t, t \in [0,T]\}$ *is defined as follows:*

$$Z_t := \exp\left[\int_0^t G_s\,dB_s - \frac{1}{2}\int_0^t G_s^2\,ds + \int_0^t\int_R h(s,x)M(ds,dx) - \int_{[0,t)\times R}[H(s,x)-1-h(s,x)]\nu(dx)ds\right]$$
$$\times \prod_{0<s\le t} H(s,\Delta X_s)\exp(-h(s,\Delta X_s))$$

.

Here G_t *and* $H(t,x)$ *are previsible and Borel previsable processes respectively and* $M(ds,dx) = Q(ds,dx) - ds\,\nu(dx)$, *where Q is a poisson random measure.*
$H \ge 0$, $H(t,0) = 1$ *for all* $t \ge 0$ *and* $h(t,x)$ *is another Borel previsible process such that*

$$\int_R \big[H(t,x) - 1 - h(t,x)\big]\nu(dx) < \infty.$$

Moreover, under $\tilde{P}$, *the process*

$$\tilde{B}_t = B_t - \int_0^t G_s\,ds$$

is a Brownian motion and the process X is a quadratic pure jump process with compensator measure by $\tilde{\nu}(dt,dx) = dt\,\tilde{\nu}_t(dx)$ *where* $\tilde{\nu}_t(dx) = H(t,x)\nu(dx)$ *and previsable part is given by*

$$\tilde{\alpha}_t = E_{\tilde{P}}[X_t] = \alpha t + \int_0^t \int_R x(H(s,x)-1)\nu(dx)ds \quad \left(\alpha = E\left(X_1 - \int_{\{|x|\geq 1\}} x\,\nu(dx)\right)\right).$$

Suppose that $E\left[\int_0^t G_s^2 ds\right] < \infty$ *and* $H \geq 0$ *where* $H(t,0)=1$ *for all* $t \geq 0$. $h(t,x)$ *is Borel previsible process such that*

$$\int_R |H(t,x) - 1 - h(t,x)|\nu(dx) < \infty.$$

Proposition 4.1.1 *Let* $F(t,x)$ *and* $f(t,x)$ *be Borel functions satisfying the following assumptions:*

(a) $F(t,x) > 0$ *for all x in the support of the Lévy measure* σ_t *and all* $t \in [0,T]$ *and assume that there exist constants* $\alpha, \beta > 0$ *such that* $F(t,x) \geq \alpha > 0$ *for all* $x \in (-\beta, \beta)$ *and* $t \in [0.T]$,

(b) $\int_0^T \int_R |F(t,x) - 1 - f(t,x)|\sigma_t(dx)dt < \infty,$

(c) $\int_0^T \int_{-\infty}^{+\infty} |f(t,x)|^2 \sigma_t(dx)dt < \infty,$

(d) *There is an* $\varepsilon > 0$ *such that*

$$\int_0^T \int_{-\varepsilon}^{+\varepsilon} |F(t,x) - 1|^2 \sigma_t(dx)dt < \infty.$$

Then, the process $U = \{U_t, t \in [0,T]\}$ *is defined by*

$$U_t = \exp\left(\int_0^t \int_{-\infty}^{+\infty} f(s,x)M(ds,dx) - \int_0^t \int_{-\infty}^{+\infty} (F(s,x) - 1 - f(s,x))\sigma_s(dx)ds\right)$$
$$\times \prod_{0<s\leq t} F(s,\Delta J_s)\exp(-f(s,\Delta J_s))$$

and does not depend on f and it is a local martingale.

Theorem 4.1.2 *Let* $X = (X_t, 0 \leq t \leq T\}$ *be a non-homogeneous Lévy process(Additive process) with local characteristics* $(\eta_t, \chi_t^2, \sigma_t)$ *under some probability measure* P. *Then,* *if exists a probability measure* Q *equivalent to P, such that X is also a* Q *non-homogeneous Lévy process with local characteristics* $(\tilde{\eta}_t, \tilde{\chi}^2, \tilde{\sigma}_t)$, *then the followings are hold:*

(i) $\tilde{\sigma}_t(dx) = H(t,x)\sigma_t(dx)$ *for some Borel function* $H : R^+ \times R \to R^+$,
for some Borel function

(ii) $\tilde{\eta}_t = \eta_t + \int_{-\infty}^{+\infty} x\mathrm{I}_{\{|x|<1\}}(H(t,x)-1)\sigma_t(dx) + G_t\chi_t^2$, $\quad G_t : R^+ \to R^+$ *such that* $\int_0^t \chi_s^2 G_s^2 ds < \infty$,

(iii) $\tilde{\chi}_t = \chi_t$,

(iv) *The density process* $\varsigma_t := \dfrac{d\mathrm{Q}_t}{dP_t}$ *is given by*

$$\exp\left[\int_0^t \chi_s G_s dW_s - \frac{1}{2}\int_0^t \chi_s^2 G_s^2 ds\right]$$

$$\times \exp\left[\lim_{\varepsilon \to 0}\left(\int_0^t\left(\int_{|x|>\tau} \log H(s,x) Q((0,s],dx) - \int_{|x|>\tau}(H(s,x)-1)\sigma_s(dx)\right)ds\right)\right]$$

with $E_p[\varsigma_t] = 1$, *for every* $t \in [0,T]$.

The above theorems imply following results:

$$\tilde{W} = \{\tilde{W}_t, 0 \le t \le T\} \text{ with } \tilde{W}_t = W_t - \int_0^t \chi_s G_s ds \tag{4.1}$$

is a Brownian motion under Q.

$$J_t = \tilde{N}_t + \int_0^t \left(a_s + \int_{-\infty}^{+\infty} x(H(s,x)-1)\sigma_s(dx)\right)ds \tag{4.2}$$

where $\tilde{N} = \{\tilde{N}_t, t \in [0,T]\}$ is a Q-martingale and

$$\tilde{N}_t = N_t - \int_0^t \int_{-\infty}^{+\infty} x(H(s,x)-1)\sigma_s(dx)ds,$$

and $\quad \tilde{M}(dt,dx) = Q(dt,dx) - \tilde{\sigma}_t(dx)dt$

$$= M(dt,dx) + (H(s,x)-1)\sigma_s(dx). \tag{4.3}$$

The discounted price process $\tilde{S}$ can be written as:

$$\tilde{S}_t = S_0 \exp\left(\int_0^t \chi_s d\tilde{W}_s + \tilde{N}_t + \int_0^t \left(a_s - r_s + \chi_s^2 G_s - \frac{\chi_s^2}{2}\right) ds\right)$$

$$\times \exp\left(\int_0^t \int_{-\infty}^{+\infty} x(H(s,x)-1)\sigma_s(dx) ds\right) \prod_{0<s\le t} (1+\Delta \tilde{N}_s) \exp(-\Delta \tilde{N}_s).$$

By Proposition 4.1.1, the process

$$\exp\left(\int_0^t \chi_s d\tilde{W}_s + \tilde{N}_t - \int_0^t \frac{\chi_s^2}{2} ds\right) \prod_{0<s\le t} (1+\Delta \tilde{N}_s) \exp(-\Delta \tilde{N}_s)$$

is a Martingale. Thus, a necessary and sufficient condition for $\tilde{S}$ to be a Q-martingale is the existence of G_t and $H(t,x)$, for which the process ς_t is a positive martingale such that

$$\chi_t^2 G_t + a_t - r_t + \int_{-\infty}^{+\infty} x(H(s,x)-1)\sigma_s(dx) = 0 \tag{4.4}$$

Thus, by (4.1), (4.2) and (4.4), we have:

$$X_t = \int_0^t \chi_s d\tilde{W}_s + \tilde{N}_t + \int_0^t r_s ds.$$

Therefore $X_t - \int_0^t r_s ds$ is a martingale. Also, the dynamics of $\tilde{S}$ under Q is given by:

$$\tilde{S}_t = S_0 \exp\left(\int_0^t \chi_s d\tilde{W}_s + \tilde{N}_t - \frac{1}{2}\int_0^t \chi_s^2 ds\right) \prod_{0<s\le t} (1+\Delta \tilde{N}_s) \exp(-\Delta \tilde{N}_s). \tag{4.5}$$

Here the new generalized Lévy measure is $\tilde{\sigma}_t(dx) = H(t,x)\sigma_t(dx)$.

Under Q, the discounted stock price process $\tilde{S} = \{\tilde{S}_t = S_t / B_t, 0 \le t \le T\}$ and the process $\tilde{X} = \{X_t - \int_0^t r_s ds, 0 \le t \le T\}$ are both Q-martingales. It is clear that $\Delta X_t = \Delta \tilde{X}_t$ and $X_t^{(j)} = \tilde{X}_t^{(j)}$ for $j \ge 2$.

We have that for $j \ge 2$, $\tilde{m}_j = \int_R \tilde{x}^j \, \tilde{\sigma}_t(dx)$ where $\{\tilde{\sigma}_t\}_{t\in[0,T]}$ is the family of Lévy measures of X (and $\tilde{X}$) under Q.

Let us now define the new artificial assets for completing the market: *the Power jump assets*

$$\bar{Z}^{(j)} = \{\bar{Z}_t^{(j)}, t \in [0,T]\} \text{ given by } \bar{Z}_t^{(j)} := B_t Z_t^{(j)},\ j \geq 2. \tag{4.6}$$

And, we can also introduce the othonormalized power-jump assets:

$$\bar{Y}^{(j)} = \{\bar{Y}_t^{(j)}, t \in [0,T]\} \text{ given by } \bar{Y}_t^{(j)} := B_t\, Y_t^{(j)}, j \geq 2. \tag{4.7}$$

By definition, the processes $\{\bar{Z}^{(j)}, j \geq 1\}$ and $\{\bar{Y}^{(j)}, j \geq 1\}$ are both Q-martingales.

An attainable contingent claim is a non-negative random variable $X \in L^2(\Omega,,\mathcal{F}_t,\mathrm{Q})$ in $L^2(\mathrm{Q})$, if there exists a self-financing portfolio whose values, at time *T*, converges in $L^2(\mathrm{Q})$ to X. In our market, a portfolio $\Psi = \{\Psi^m, m \geq 1\}$ is a sequence of finite-dimensional predictable processes.

$$\{\Psi_t^m = (\omega_t^m, \alpha_t^m, \alpha_t^{(2),m}, \dots, \alpha_t^{(c_m),m}), 0 \leq t \leq T, m \geq 2\}$$

Here, ω_t^m corresponds the number of bonds at time t, α_t^m corresponds the number of stocks at time t, and $\alpha_t^{(j),m}$ is the *j*th-power jump assets $Y^{(j)}$ and c_m is an integer depends on *m*. This portfolio $\Psi = \{\Psi^m, m \geq 1\}$ is self financing if Ψ^m is self financing for each finite *m*.

Theorem 4.1.3 *Consider that our market model is* M_{Q} *,where traded assets are a bond with price process is given by (3.2) a stock with dynamics given by (3.1) and the Power jump assets* $\{\bar{Z}^{(j)}, j \geq 2\}$. *If the additive process X satisfies (3.14) and if there is at least one equivalent martingale measure* Q *that is sructure preserving, then the market* M_{Q} *is complete in the sense that any square integrable contingent claim* $C \in L^2(\Omega, \mathcal{F}_t, \mathrm{Q})$ *can be replicated in* L^2.

Proof:

Let us take a square integrable contingent claim C with maturity T. Let

$$M_t = E_{\mathrm{Q}}\left[\exp\left(-\int_0^T r_s ds\right) C \middle| \mathcal{F}_t\right].$$

By using Martingale Representation Property, we know that

$$M_t = M_0 + \int_0^t h_s^{(1)} d\tilde{X}_s + \sum_{j=2}^{\infty} \int_0^t h_s^{(j)} dY_s^{(j)}.$$

Now, let us define the following process as:

$$M_t^N := M_0 + \int_0^t h_s^{(1)} d\tilde{X}_s + \sum_{j=2}^{N} \int_0^t h_s^{(j)} dY_s^{(j)}.$$

Then we have that $\lim_{N\to\infty} M_t^N = M_t$ in $L^2(\mathrm{Q})$. Now, let us define sequence of portfolios as follows:

$$\Psi^N := \{\Psi_t^N = (\omega_t^N, \alpha_t, \alpha_t^{(2)}, \alpha_t^{(3)}, \ldots, \alpha_t^{(N)}),\ t \geq 0, N \geq 2\} \text{ given by}$$

$$\omega_t^N = M_{t\_}^N - \alpha_t S_{t\_} \exp\left(-\int_0^t r_s ds\right) - \exp\left(-\int_0^t r_s ds\right) \sum_{j=2}^{N} \alpha_t^{(j)} \bar{Y}_t^{(j)},$$

$$\alpha_t = \exp\left(\int_0^t r_s ds\right) h_t^{(1)} S_{t\_}^{-1},$$

$$\alpha_t^{(j)} = h_t^{(j)}. \qquad j = 2,3,\ldots,N.$$

In the above , ω_t^N corresponds to the number of bonds at time t, α_t corresponds to the number of stocks at time t and $\alpha_t^{(j)}$ corresponds to the number of power jump assets $\bar{Y}_t^{(j)}$. Firstly, we must show that the sequence of portfolios $\Psi^N := \{\Psi_t^N, t \in [0,T]\}$ is self financing that replicates C. If we look at the values V_t^N at time t, then we will find

$$V_t^N = \omega_t^N \exp\left(\int_0^t r_s ds\right) + \alpha_t S_t + \sum_{j=2}^{N} \alpha_t^{(j)} \bar{Y}_t^{(j)} = \exp\left(\int_0^t r_s ds\right) M_t^N.$$

Thus, we can say that the sequence of portfolios $\{\Psi^N, N \geq 2\}$ is replicating the claim C. Denote the Gain process by:

$$D_t^N = r_t \int_0^t \omega_s^N e^{r_s} ds + \int_0^t \alpha_s dS_s + \sum_{j=2}^{N} \int_0^t \alpha_s^{(j)} d\bar{Y}_s^{(j)}.$$

If the portfolio is self financing then it must satisfy the following equation:

$$D_t^N + M_0 = B_t M_t^N.$$

If we substitute ω_t^N, α_t and $\alpha_t^{(j)}$ in D_t^N, then we will obtain:

$$D_t^N = r_t \int_0^t M_{s_-}^N e^{r_s} ds - r_t \int_0^t e^{r_s} h_s^{(1)} ds - r_t \sum_{j=2}^N \int_0^t h_s^{(j)} e^{r_s} \bar{Y}_{s_-}^{(j)} ds$$
$$+ \int_0^t e^{r_s} h_s^{(1)} S_{s_-}^{-1} dS_s + \sum_{j=2}^N \int_0^t h_s^{(j)} d\bar{Y}_s^{(j)} .$$

Integrating by parts gives us:

$$r_t \int_0^t M_s^N e^{r_s} ds = B_t M_t^N - M_0 - \int_0^t h_s^{(1)} e^{r_s} d\tilde{X}_s - \sum_{j=2}^N \int_0^t h_s^{(j)} e^{r_s} dY_s^{(j)}.$$

By using the above equation, and by using the equations $\bar{Y}_t^{(j)} = B_t Y_t^{(j)}$ and $dS_t = S_{t_-}(rdt + d\tilde{X}_t)$, we will obtain the following result:

$$D_t^N = B_t M_t^N - M_0 - \int_0^t h_s^{(1)} e^{r_s} d\tilde{X}_s - \sum_{j=2}^N \int_0^t h_s^{(j)} e^{r_s} dY_s^{(j)}$$
$$- r_t \int_0^t e^{r_s} h_s^{(1)} ds - r_t \sum_{j=2}^N \int_0^t h_s^{(j)} e^{r_s} \bar{Y}_{s_-}^{(j)} ds$$
$$+ \int_0^t e^{r_s} h_s^{(1)} S_{s_-}^{-1} dS_s + \sum_{j=2}^N \int_0^t h_s^{(j)} d\bar{Y}_s^{(j)}$$

$$= B_t M_t^N - M_0 - r_t \int_0^t h_s^{(1)} e^{r_s} ds - \int_0^t h_s^{(1)} e^{r_s} d\tilde{X}_s + \int_0^t h_s^{(1)} e^{r_s} S_{s_-}^{-1} dS_s$$

$$= B_t M_t^N - M_0, \qquad \text{Q.E.D. .}$$

4.2 HEDGING PORTFOLIOS

Lemma 4.2.1 *Consider a real function* $h(s,x,y)$ *on* $R^+ \times R^+ \times R^+$ *which is infinitely differentiable in the y-variable and satisfies* $h(s,x,0)=0$ *and* $\frac{\partial h}{\partial y}(s,x,0)=0.$

Set

$$a_j(s,x) = \frac{1}{j!}\frac{\partial^j h}{\partial y^j}(s,x,0)$$

and assume that

$$\sup_{x<K, s\le s_0} \sum_{j=2}^{\infty} \left| a_j(s,x) \right| R^j < \infty.$$

For all $K, R > 0$*,* $s_0 = 0.$ *Then we have*

$$\sum_{t<s\le T} h(s, S_{s\_}, \Delta J_s) = \sum_{j=2}^{\infty} \int_t^T \frac{1}{j!}\frac{\partial^j}{\partial y^j} h(s, S_{s\_}, 0) dZ_s^{(j)} + \int_t^T \int_{-\infty}^{\infty} h(s, S_{s\_}, y) \tilde{\sigma}(dy) ds.$$

Proof:

The function $h(s,x,y)$ can be expanded as

$$h(s,x,y) = \sum_{j=2}^{\infty} a_j(s,x) y^j$$

Then, we have

$$\begin{aligned}
\sum_{t<s\le T} h(s, S_{s_}, \Delta J_s) &= \sum_{t<s\le T} \sum_{j=2}^{\infty} a_j(s, S_{s_})(\Delta J_s)^j. \\
&= \sum_{j=2}^{\infty} \int_t^T a_j(s, S_{s_}) dJ_s^{(j)} \\
&= \sum_{j=2}^{\infty} \int_t^T a_j(s, S_{s_}) dZ_s^{(j)} + \sum_{j=2}^{\infty} \int_t^T a_j(s, S_{s_}) m_j ds.
\end{aligned}$$

Here, $m_j = \int_{-\infty}^{\infty} y^j \tilde{\sigma}(dy)$. Hence, we have

$$\begin{aligned}
\sum_{0<s\le T} h(s, S_{s_}, \Delta J_s) &= \sum_{j=2}^{\infty} \int_t^T a_j(s, S_{s_}) dZ_s^{(j)} + \int_t^T \int_{-\infty}^{\infty} \sum_{j=2}^{\infty} a_j(s, S_{s_}) y^j \tilde{\sigma}(dy) ds. \\
&= \sum_{j=2}^{\infty} \int_t^T a_j(s, S_{s_}) dZ_s^{(j)} + \int_t^T \int_{-\infty}^{\infty} h(s, S_{s_}, y) \tilde{\sigma}(dy) ds. \quad \text{Q.E.D. .}
\end{aligned}$$

Now, let us compute the hedging portfolio that replicates a contingent claim C whose payoff is a function of the value, at maturity, of the stock price S, of an absolutely continuous process $V^1=\{V_t^1, 0 \le t \le T\}$ and of a jump process $V^2=\{V_t^2, 0 \le t \le T\}$ satisfy the following definitions:

$$V_t^1 := \int_0^t f(S_s)ds, \tag{4.8}$$

$$V_t^2 := \int_0^t \int_{-\infty}^{\infty} u(x)\tilde{M}(ds,dx). \tag{4.9}$$

In the above $f(x)$ is a continuous function and u is a smooth function satisying the property $u(0) = u'(0) = 0$ and $\int_0^t \int_{-\infty}^{\infty} |u(x)|\tilde{\sigma}_s(dx)ds < \infty$ and

$\tilde{M}(ds,dx) = Q(ds,dx) - \tilde{\sigma}_s(dx)ds$ is the compensated Poisson Random measure. The payoff is a function of S_T, V_T^1 and V_T^2 that is of the form $p(S_T, V_T^1, V_T^2)$. By the independence of $\frac{S_T}{S_t}$ and $V_T^2 - V_t^2$ wrt $\mathcal{F}_t$, we obtain the price function of the contingent claim C, at time t as follows:

$$\begin{aligned}
&\exp\left(-\int_t^T r_s ds\right) E_Q\left(p(S_T, V_T^1, V_T^2)\middle|\mathcal{F}_t\right) \\
&= \exp\left(-\int_t^T r_s ds\right) E_Q\left[p\left(\frac{S_T}{S_t}S_t, \int_t^T f\left(\frac{S_s}{S_t}S_t\right)ds + V_t^1, V_T^2 - V_t^2 + V_t^2\right)\middle|\mathcal{F}_t\right] \\
&= E_Q\left[p\left(\frac{S_T}{S_t}x_1, \left(\int_t^T f\left(\frac{S_s}{S_t}x_1\right)ds + x_2, V_T^2 - V_t^2 + x_3\right)\right)\right]_{x_1=S_t, x_2=V_t^1, x_3=V_t^2} \\
&:= F(t, S_t, V_t^1, V_t^2).
\end{aligned}$$

In the above equation, the price function $F(t,x)$ must satisfy a partial differential integral equation (PIDE).

Our notation that we will use in the PIDE is $x := (x_1, x_2, x_3)$, $D_0 := \frac{\partial}{\partial t}$, $D_i := \frac{\partial}{\partial x_i}$ and $D_1^j := \frac{\partial^j}{\partial x_1^j}$.

The price function $F(t,x)$ is a solution of the PIDE:

$$\begin{aligned} &D_0F(t,x)+f(x_1)D_2F(t,x)+r_tx_1D_1F(t,x)+\frac{1}{2}\chi_t^2x_1^2D_1^2F(t,x) \\ &-D_3F(t,x)\int_R u(y)\tilde{\sigma}_t(dy)+DF(t,x)=r_tF(t,x) \end{aligned} \tag{4.10}$$

$$F(T,x)=p(x). \tag{4.11}$$

where

$$DF(t,x):=\int_{-\infty}^{\infty}(F(t,x_1(1+y),x_2,x_3+u(y))-F(t,x)-x_1yD_1F(t,x)\tilde{\sigma}_t(dy)$$

We guarantee the entity of the equtions by the following proof:

Proof:

The discounted process $\exp\left(-\int_0^t r_s ds\right)F(t,T_t)$ is a Q-martingale by assumption. ($T_t=(S_t,V_t^1,V_t^2)$).

Therefore, we have following equation $\exp\left(-\int_0^t r_s ds\right)F(t,T_t)=F(0,T_0)+M_t+C_t$

where M_t is a local martingale and C_t is a finite variation process
Itô's formula for semimartingales gives us:

$$\begin{aligned} \exp\left(-\int_0^t r_s ds\right)F(t,T_t) &= F(0,T_0)+\int_0^t(-r_se^{-r_s}F(s,T_{s\_})+e^{-r_s}D_0F(s,T_{s\_}))ds \\ &+\int_0^t e^{-r_s}D_1F(s,T_{s\_})dS_s+\int_0^t e^{-r_s}l(S_{s\_}D_2F(s,T_{s\_})ds \\ &-\int_R u(y)\tilde{\sigma}(dy)\tilde{\sigma}(dy)\int_0^t e^{-r_s}D_3F(s,T_{s\_})ds \\ &+\frac{\chi^2}{2}\int_0^t e^{-r_s}S_{s\_}^2D_1^2F(s,T_{s\_})ds \\ &+\sum_{0<s\le t}e^{-r_s}[F(s,T_{s\_})-F(s,T_{s\_})-D_1F(s,T_{s\_})\Delta S_s] \end{aligned}$$

It is clear that $S_s = S_{s\_}(1+\Delta J_s)$ and $V_s=V_{s\_}+u(\Delta J_s)$

Applying Lemma (4.2.1) to h, we will find:

$$\sum_{0<s\le t} e^{-r_s}[F(s,T_s)-F(s,T_{s\_})-D_1F(s,T_{s\_})\Delta S_{s\_}]$$
$$=\sum_{j=2}^{\infty}\int_0^t \frac{e^{-r_s}}{j!}\frac{\partial^j}{\partial y^j}h(s,T_{s\_},0)dT_s^{(j)}+\int_0^t\int_{-\infty}^{\infty} e^{-r_s}h(s,T_{s\_},y)\tilde{\sigma}(dy)ds$$

($h(t,x,y):=F(t,x_1(1+y),x_2,x_3+u(y)),F(t,x)-x_1yD_1F(t,x)$, $x=(x_1,x_2,x_3)$, $t\in[0,T]$, $y\in R$)

Thus, we have:

$$\exp\left(-\int_0^t r_s ds\right)F(t,T_t)=F(0,T_0)+\int_0^t\Big(e^{-r_s}(r_sF(s,T_{s\_}))+D_0F(s,T_{s\_})$$
$$+\frac{\chi^2}{2}S_{s\_}^2D_1^2F(s,T_{s\_})-f(S_{s\_})-D_2F(s,T_{s\_})$$
$$-D_3F(S_{s\_})\int_r u(y)\tilde{\sigma}(dy)+\int_{-\infty}^{+\infty}h(s,T_{s\_},y)\tilde{\sigma}(dy))ds$$
$$+\sum_{j=2}^{\infty}\frac{e^{-r_s}}{j!}\frac{\partial}{\partial y^j}h(s,T_{s\_},0)dT_s^{(j)}.$$

Under Q, dynamics of S_t is given by:

$$dS_t=\chi_tS_{t\_}d\tilde{W}_t+S_{t\_}d\tilde{N}_t+r_tS_{t\_}dt$$

where $\tilde{N}_t=\int_0^t\int_R x\tilde{M}(ds,dx).$

($\tilde{M}(ds,dx):=Q(ds,dx)-\tilde{\sigma}(dx)ds$ is compensated Poisson random measure on $[0,T]\times R/\{0\}$. Hence, we have that $\tilde{W}$ is a Q brownian motion and $\tilde{N}$ is a Q martingale.

Therefore, we have:

$$\int_0^t e^{-r_s}D_1F(s,T_{s\_})dS_s=\chi_t\int_0^t e^{-r_s}S_{s\_}D_1F(s,T_{s\_})d\tilde{W}_s+\int_0^t e^{-r_s}S_{s\_}D_1F(s,T_{s\_})d\tilde{N}_s$$
$$+r_t\int_0^t e^{-r_s}S_{s\_}D_1F(s,T_{s\_})ds.$$

Thus; $\exp\left(-\int_0^t r_s ds\right) F(t,T_t) = F(0,T_0) + M_t + C_t$,

$$M_t := \sum_{j=2}^{\infty}\int_0^t \frac{e^{-r_s}}{j!}\frac{\partial}{\partial y^j} h(s,T_{s\_},0)dT_s^{(j)} + \chi_t \int_0^t e^{-r_s} S_{s\_} D_1F(s,T_{s\_})d\tilde{W}_s$$
$$+\int_0^t e^{-r_s} S_{s\_} D_1F(s,T_{s\_})d\tilde{N}_s,$$

$$C_t := \int_0^t e^{-r_s}(-r_sF(s,T_{s\_}) + D_0F(s,T_{s\_}) + \frac{\chi^2}{2} S_{s\_}^2 D_1^2F(s,T_{s\_}) + f(S_{s\_})D_2F(s,T_{s\_})$$
$$-D_3F(s,T_{s\_})\int_R u(y)\tilde{\sigma}(dy) + r_sS_{s\_} D_1F(s,T_{s\_}) + \int_R h(s,T_{s\_},y)\tilde{\sigma}(dy)ds.$$

The process $C = \{C_t, t \in [0,T]\}$ is predictable, finite variation process and the process $M = \{M_t, t\in[0,T]\}$ is Q-local martingale.

The series $\sum_{j=2}^{\infty}\int_0^t \left(\frac{e^{-r_s}}{j!}\right)\left(\frac{\partial^j}{\partial y^j}\right) h(s,T_{s\_},0)dT_s^{(j)}$ converges in L^1 (Q), and the processes $\int_0^t \left(\frac{e^{-r_s}}{j!}\right)\left(\frac{\partial^j}{\partial y^j}\right) h(s,T_{s\_},0)dT_s^{(j)}$ are martingales.

The condition $C_t \equiv 0$ gives

$$-r_tF(s,T_{s\_}) + D_0F(s,T_{s\_}) + \frac{\chi^2}{2}\ S_{s\_}^2\ D_1^2F(t,T_{s\_}) + f(S_{s\_})\ D_2F(s,T_{s\_})$$
$$-D_3F(s,T_{s\_})\ \int_R u(y)\ \tilde{\sigma}(y)\ +\ r_t\ S_{s\_}\ \ D_1F(s,T_{s\_})\ +\ \int_R h(s,T_{s\_},y)\ \tilde{\sigma}(y) = 0$$

By the above equation, we can say that the pricing function satisfies the (PIDE)

$$\Phi F(t,y) + DF(t,y) = 0, \quad \text{Q.E.D.}\ .$$

For simplification, we will use the notation $V_t := (V_t^1, V_t^2)$ in the next theorem.

Theorem 4.2.1 *Let* $Y \in [\Omega, \mathcal{F}_t, \mathrm{Q}]$ *be a contingent claim with payoff* $Y = p(S_T, V_T)$ *and a price function F of class* $C^{1,\infty,2,\infty}$. *Let us define the function* $h:[0,T]\times R^3 \to R$ *given by:*

$$h(t,x,y) := F(t, x_1(1+y), x_2, x_3 + u(y)) - F(t,x) - x_1 y D_1F(t,x). \quad (4.12)$$

Assume that $h(t,x,y)$ *is analytic in y for all* $x \in R^3$, $t \in [0,T]$. *So, we have Taylor series representation of h as following:*

$$h(t,x,y) = \sum_{j=2}^{\infty} \frac{1}{j!} \frac{\partial^j}{\partial y^j}(t,x,0)\ y^j, \tag{4.13}$$

for all $y \in R$.
Then Y has a self–financing portfolio, that is given by, at time t:

$$\text{number of bonds} = \omega_t = B_t^{-1}\,(F(t,S_{t\_}V_{t\_}) - S_{t\_}D_1F(t,S_{t\_},V_{t\_})), \tag{4.14}$$

$$\text{number of stocks} = \alpha_t = D_1F(t,S_{t\_},V_{t\_}), \tag{4.15}$$

$$\text{number of power jump assets} = \alpha_t^{(j)} = \frac{\frac{\partial^j}{\partial y^j} h(t,S_{t\_},V_{t\_},0)}{j!}, \qquad j=2,3,\ldots . \tag{4.16}$$

Proof:

Applying Itô's Formula to $F(t,S_t,V_t)$ gives us:

$$\begin{aligned}
F(t,S_t,V_t) &= F(0,S_0,V_0) + \int_0^t D_0F(s,S_{s_},V_{s_})ds + \int_0^t D_1F(s,S_{s_},V_{s_})dS_s \\
&\quad + \int_0^t D_2F(s,S_{s_},V_{s_})dV_s^1 + \int_0^t D_3F(s,S_{s_},V_{s_})dV_s^2 \\
&\quad + \frac{1}{2}\int_0^t \chi_s^2 S_{s_}^2 D_1^2F(s,S_{s_},V_{s_})ds \\
&\quad + \sum_{0<s\le t} F(s,S_s,V_S) - F(s,S_{s_},V_{s_}) - D_1F(s,S_{s_},V_{s_})\Delta S_s \\
&= F(0,S_0,V_0) + \int_0^t D_0F(s,S_{s_},V_{s_})ds \\
&\quad + \int_0^t D_1F(s,S_{s_},V_{s_})dS_s + \int_0^t f(S_{s_})D_2F(s,S_{s_},V_{s_})ds \\
&\quad + \int_R u(y)\tilde{\sigma}(y)\int_0^t D_3F(s,S_{s_},V_{s_})ds + \frac{1}{2}\chi_t^2 S_{t_}^2 \int_0^t D_1^2F(s,S_{s_},V_{s_})ds \\
&\quad + \sum_{0<s\le t} F(s,S_s,V_S) - F(s,S_{s_},V_{s_}) - D_1F(s,S_{s_},V_{s_})\Delta S_s
\end{aligned}$$

$$= F(0,S_0,V_0)$$

$$+\int_0^t \begin{bmatrix} D_0F(s,S_{s_-},V_{s_-}) + \frac{1}{2}\chi_s^2 S_{s_-}^2 D_1^2 F(s,S_{s_-},V_{s_-}) + \\ f(S_{s_-})D_2F(s,S_{s_-},V_{s_-}) - D_3F(s,S_{s_-},V_{s_-})\int_R u(y)\tilde{\sigma}(y) \end{bmatrix} ds + \int_0^t D_1F(s,S_{s_-},V_{s_-})dS_s$$

$$+\sum_{0<s\le t} F(s,S_s,V_S) - F(s,S_{s_-},V_{s_-}) - D_1F(s,S_{s_-},V_{s_-})\Delta S_s.$$

Clearly $\Delta S_s = S_{s_-}\Delta J_s$. Therefore;

$$\sum_{0<s\le t} F(s,S_s,V_S) - F(s,S_{s_-},V_{s_-}) - D_1F(s,S_{s_-},V_{s_-})\Delta S_s = h(s,S_{s_-},V_{s_-},\Delta J_s).$$

Take the following PIDE:

$$D_0F(t,x) + f(x)D_2(F(t,x) + r_tD_2F(t,x) + D_3F(t,x)\int_R u(y)\tilde{\sigma}_t(dy) + DF(t,x) = r_tF(t,x),$$

$$F(T,x) = p(x).$$

We will find following:

$$F(t,S_t,V_t) = F(0,S_0,V_0) + \int_0^t D_1F(s,S_{s_-},V_{s_-})dS_s$$

$$+\int_0^t \frac{F(s,S_{s_-},V_{s_-}) - S_{s_-}D_1F(s,S_{s_-},V_{s_-})}{B_s}dB_s$$

$$+\sum_{0<s\le t} h(s,S_{s_-},V_{s_-},\Delta J_s) - \int_0^t\int_R h(s,S_{s_-},V_{s_-},y)\tilde{\sigma}_t(dy)ds$$

$$+$$

$$= F(0,S_0,V_0) + \int_0^t D_1F(s,S_{s_-},V_{s_-})dS_s$$

$$+\int_0^t \frac{F(s,S_{s_-},V_{s_-}) - S_{s_-}D_1F(s,S_{s_-},V_{s_-})}{B_s}dB_s + \sum_{j=2}^{\infty}\int_0^t \frac{\left(\frac{\partial^j}{\partial z^j}\right)h(s,T_{s_-},0)}{j!B_s}d\bar{Z}_s^{(j)},$$

Q.E.D. .

Proposition 4.2.1 *Let us denote* $\tilde{F}^{(m)}(t,S_t)$ *discounted price function of the contingent claims* $Y_{(m)}$ *with payoffs* $Y_{(m)} = S_T^m$ *where* $m \geq 2$ *and set* $\tilde{F}^{(1)}(t,S_t) := \tilde{S}_t$*. Then, we can write the following representation formula:*

$$\tilde{F}^{(m)}(t,S_t) = \tilde{F}^{(m)}(0,S_0) + \sum_{j=1}^{m} \int_0^t \frac{F^{(m)}(s,S_{s_-})}{B_s} \binom{m}{j} dZ_s^{(j)}, \qquad m \geq 1, \quad (4.17)$$

And, its inverse representation

$$dZ_t^{(j)} = (-1)^j \, B_t \left(\sum_{m=1}^{j} \binom{j}{m} (-1)^m \frac{1}{F^{(m)}(t,S_{t_-})} d\tilde{F}^{(m)}(t,S_t) \right), \quad j \geq 1. \quad (4.18)$$

Proof:

Since $\left(\dfrac{S_T}{S_t}\right)^m$ is independent with respect to $\mathcal{F}_t$, we have the price function of the derivatives $Y_{(m)}$, as follows:

$$\begin{aligned} F^{(m)}(t,S_t) &= \exp\left(-\int_t^T r_s ds\right) E_Q\left(S_T^m \middle| \mathcal{F}_t\right) \\ &= \exp\left(-\int_t^T r_s ds\right) S_t^m \, E_Q\left(\left(\frac{S_T}{S_t}\right)^m\right) \\ &= \Pi^{(m)}(t,T) S_t^m. \end{aligned}$$

Here $\Pi^{(m)}(t,T)$ is a deterministic function. If we first apply Itô lemma and then if we apply theorem 4.2.1 to $F(t,x) = \Pi^{(m)}(t,T)\, x^m$, then we will find:

$$\begin{aligned} F^{(m)}(t,S_t) = F^{(m)}(0,S_0) &+ \int_0^t D_0 F^{(m)}(s,S_{s_-}) ds + \int_0^t D_1 F^{(m)}(s,S_{s_-}) dS_{s_-} \\ &+ \frac{1}{2}\chi_t^2 S_{t_-}^2 \int_0^t D_1^2 F^{(m)}(s,S_{s_-}) ds \\ &+ \sum_{0<s\leq t} F^{(m)}(s,S_s) - F^{(m)}(s,S_{s_-}) - S_{s_-} D_1^{(m)} F(s,S_{s_-}). \end{aligned}$$

Setting $h(t,S_{t_-},V_{t_-},0) = \sum_{0<s\leq t} F^{(m)}(s,S_s) - F^{(m)}(s,S_{s_-}) - S_{s_-} D_1^{(m)}(s,S_{s_-})$ and applying Lemma 4.2.1 will give us following:

$$F^{(m)}(t,S_t)=F^{(m)}(0,S_0)+\int_0^t \frac{r_s B_s F^{(m)}(s,S_{s_-})-mr_s B_s F^{(m)}(s,S_{s_-})-\sum_{j=2}^{m}\binom{m}{j}\bar{Z}_t^{(j)}}{B_s}ds$$

$$+\int_0^t \frac{mF^{(m)}(s,S_{s_-})}{S_{s_-}}dS_s+\frac{1}{B_t}\sum_{j=2}^{\infty}\binom{m}{j}d\bar{Z}_t^{(j)}.$$

Therefore, we have

$$dF^{(m)}(t,S_t)=r_t F^{(m)}(t,S_{t_-})\left(1-m-\frac{1}{B_t}\sum_{j=2}^{m}\binom{m}{j}\bar{Z}_t^{(j)}\right)dt+\frac{mF^{(m)}(t,S_{t_-})}{S_{t_-}}dS_t$$

$$+\frac{1}{B_t}\sum_{j=2}^{m}\binom{m}{j}d\bar{Z}_t^{(j)}.$$

We have $dS_t = B_t\, d\tilde{S}_t + r_t\, B_t\, \tilde{S}_{t_-}\, dt$. Here $\tilde{S}=\left(\tilde{S}=\left\{\tilde{S}_t := \exp\left(-\int_0^t r_s ds\right)S_t, t\in[0,T]\right\}\right)$

and

$$d\tilde{F}^{(m)}(t,S_t)=\frac{1}{B_t}dF^{(m)}(t,S_t)-\frac{r_t}{B_t}F^{(m)}(t,S_t)dt$$

and

$$d\bar{Z}_t^{(j)}=B_t dZ_t^{(j)}+r_t B_t Z_t^{(j)}dt.$$

If we use these equations and the identity $Z_t^{(1)} = X_t - \int_0^t r_s ds = \int_0^t \frac{B_s}{S_{s_-}}d\tilde{S}_s$, then we will find:

$$d\tilde{F}^{(m)}(t,S_t)=\frac{r_t}{B_t}F^{(m)}(t,S_{t\_})dt-m\frac{r_t}{B_t}F^{(m)}(t,S_{t\_})dt-\frac{r_t}{B_t^2}\sum_{j=2}^{m}\binom{m}{j}\bar{Z}_t^{(j)}dt$$

$$+\frac{mF^{(m)}(t,S_{t\_})}{S_{t\_}}d\tilde{S}_t+\frac{mr_tF^{(m)}(t,S_{t\_})}{S_{t\_}}\tilde{S}_{t\_}\,dt+\frac{1}{B_t}\sum_{j=2}^{m}\binom{m}{j}d\bar{Z}_t^{(j)}$$

$$+\frac{1}{B_t}\sum_{j=2}^{m}\binom{m}{j}r_tZ_t^{(j)}dt-\frac{r_t}{B_t}F^{(m)}(t,S_t)dt$$

$$=\frac{mF^{(m)}(t,S_{t\_})}{S_{t\_}}d\tilde{S}_t+\frac{F^{(m)}(t,S_{t\_})}{B_t}\sum_{j=2}^{m}\binom{m}{j}d\bar{Z}_t^{(j)}$$

$$=\sum_{j=1}^{m}\frac{F^{(m)}(t,S_{t\_})}{B_t}\binom{m}{j}d\bar{Z}_t^{(j)} \qquad \text{Q.E.D..} \qquad (4.19)$$

We now introduce the hedging formula for contingent claims in terms of call options with the same maturity and different strikes. In order to persue this goal, let $p(x)$ be a real function of class C^2 in $(0,\infty)$ and let $\tilde{C}_t(K):=\frac{1}{B_T}E_{\mathrm{Q}}[(S_T-K)_+|\mathcal{F}_t]$ be the discounted price function of a call option with maturity T and strike K. $C_t(K)$ denotes one call option with strike K and maturity T.

By (Carr. P. And Madan. D. [6]) we can write this twice differentiable payoff as follows:

$$p(S_T)=p(K)+p'(K)(S_T-K)^+ +\int_K^\infty p''(K)(S_T-K)^+\,dK. \qquad (4.20)$$

The first term can be interpreted as the payoff from a static position in $p(K)$ pure discound bond. The second term can be interpreted as the payoff from $p'(K)$ calls struck at K. The third term arises from a static position in $p''(K)dK$ calls at all strikes greater than K.

Proof:
If we take the Taylor expansion of $p(S_T)$ at K with the first error remainder term, then we have following equation:

$$p(S_T)=p(K)+p'(K)(S_T-K)^+ +E_1(K).$$

$$p(S_T)=p(K)+p'(K)(S_T-K)^+ +\int_K^\infty p''(K)(S_T-K)^+\,dK. \qquad \text{Q.E.D. .}$$

Discounting and taking conditional expectation of each side of the above equation for $0\le K<\infty$ will give us following:

$$E_Q\left[B_T^{-1} p(S_T) \middle| \mathcal{F}_t\right] = B_T^{-1} p(0) + p'(0)\tilde{S}_t + \int_0^\infty p''(K)\tilde{C}_t(K)dK. \tag{4.21}$$

Theorem 4.2.2 *Let U be a contingent claim with payoff $U = p(S_T, V_T)$ and a price function $F(t, S_t, V_t)$ such that $F(t,x) \in C^{1,\infty,2\infty}$ and*

$h(t,x,y) := F(t, x_1(1+y), x_2, x_3 + u(y)) - F(t,x) - x_1 y D_1 F(t,x)$ is analytic in y for all $x \in R^3$ and $t \in [0,T]$. Set

$$R(t,K) := \sum_{m=2}^{\infty} \frac{\frac{\partial^{m-1}}{\partial y^{m-1}} h(t, S_{t_-}, V_{t_-}, -1)}{(m-2)!\Pi^{(m)}(t,T)} \left(\frac{K}{S_{t_-}}\right)^{m-2}. \tag{4.22}$$

Assume that

$$\int_0^\infty \sum_{m=2}^{\infty} \frac{\left|\frac{\partial^{m-1}}{\partial y^{m-1}} h(t, S_{t_-}, V_{t_-}, -1)\right|}{(m-2)!\Pi^{(m)}(t,T)} \left(\frac{K}{S_{t_-}}\right)^{m-2} C_s(K)dK < \infty. \tag{4.23}$$

Then, we have the following representation:

$$\sum_{j=2}^{\infty} \int_0^t \alpha_s^{(j)} dZ_s^{(j)} = \int_0^t \int_0^\infty \frac{B_s}{S_{s_-}^2} R(s,K) d\tilde{C}_s(K)dK - \int_0^t \frac{B_s h(s, S_{s_-}, V_{s_-}, -1)}{S_{s_-}} d\tilde{S}_s, \tag{4.24}$$

Hedging portfolio, in terms of bonds, stocks and call options, is given by

$$\omega_t = B_t^{-1}[F(t, S_{t_-}, V_{t_-}) - S_{t_-} D_1 F(t, S_{t_-}, V_{t_-})]$$
$$+ B_t^{-1}\left[h(t, S_{t_-}, V_{t_-}, -1) - \int_0^\infty \frac{R(t,K)}{S_{t_-}^2} C_{t_-}(K)dK\right],$$
$$\alpha_t = D_1 F(t, S_{t_-}, V_{t_-}) - \frac{h(t, S_{t_-}, V_{t_-}, -1)}{S_{t_-}},$$
$$\alpha_t^{(K)} = \frac{R(t,K)}{S_{t_-}^2}.$$

where $\alpha_t^{(K)}$ is the number of call options in the hedging portfolio, at time t, with strike K.

Proof:

From (4.18), we can obtain the value of the hedging portfolio in the first n discounted power-jump assets $Z_t^{(j)}$, $2 \le j \le n$. It is given by

$$\sum_{j=2}^{n} \alpha_t^{(j)} dZ_t^{(j)} = B_t \sum_{j=2}^{n} \alpha_t^{(j)} (-1)^j \left(\sum_{m=1}^{j} \binom{j}{m} (-1)^m \frac{1}{F^{(m)}(t, S_{t\_})} d\tilde{F}^{(m)}(t, S_t) \right).$$

Therefore,

$$\sum_{j=2}^{n} \alpha_t^{(j)} dZ_t^{(j)} = B_t \sum_{m=1}^{n} \frac{(-1)^m}{F^{(m)}(t, S_{t\_})} \left(\sum_{j=m\vee 2}^{n} \binom{j}{m} (-1)^j \alpha_t^{(j)} \right) d\tilde{F}^{(m)}(t, S_t).$$

We know that

$$\alpha_t^{(j)} = \frac{\frac{\partial^j}{\partial y^j} h(t, S_{t\_}, V_{t\_}, 0)}{j!}.$$

So, we have

$$\sum_{j=2}^{n} \alpha_t^{(j)} dZ_t^{(j)}$$

$$= B_t \sum_{m=1}^{n} \frac{(-1)^m}{F^{(m)}(t, S_{t\_})} \left(\sum_{j=m\vee 2}^{n} \binom{j}{m} (-1)^j \frac{\frac{\partial^j}{\partial y^j} h(t, S_{t\_}, V_{t\_}, 0)}{j!} \right) d\tilde{F}^{(m)}(t, S_{t\_}).$$

Assumption (4.13) implies that $\sum_{j=2}^{\infty} \alpha_t^{(j)} dZ_t^{(j)}$ converges for every $w \in \Omega$. Hence,

$$\sum_{j=2}^{\infty} \alpha_t^{(j)} dZ_t^{(j)}$$

$$= \lim_{n\to\infty} B_t \left[\sum_{m=1}^{n} \frac{1}{m! F^{(m)}(t, S_{t\_})} \left(\sum_{j=m\vee 2}^{n} (-1)^{j-m} \frac{\frac{\partial^j}{\partial y^j} h(t, S_{t\_}, V_{t\_}, 0)}{(j-m)!} \right) d\tilde{F}^{(m)}(t, S_t) \right]. \quad (4.25)$$

If we now consider the representation formula (4.21) with $p(x) = x^m$, then we have

$$d\tilde{F}(t,S_t)=\int_0^\infty m(m-1)K^{m-2}d\tilde{C}_t(K)dK. \qquad (4.26)$$

Thus,

$$\sum_{j=2}^\infty \alpha_t^{(j)}dZ_t^{(j)}=\int_0^\infty \frac{B_t}{S_{t_-}^2}R(t,K)d\tilde{C}_t(K)dK-\frac{B_t}{S_{t_-}}h(t,S_{t_-},V_{t_-},-1)d\tilde{S}_t. \qquad (4.27)$$

It is obvious that the series

$$R(t,K):=\sum_{m=2}^\infty \frac{\frac{\partial^{m-1}}{\partial y^{m-1}}h(t,S_{t_-},V_{t_-},-1)}{(m-2)!\Pi^{(m)}(t,T)}\left(\frac{K}{S_{t_-}}\right)^{m-2}$$

is absolutely convergent for each t and each K. By the definition of $\Pi^{(m)}(t,T)$ we can write

$$\left|\Pi^{(2)}(t,T)\right|\le\left|\Pi^{(k)}(t,T)\right| \text{ for all } 0\le t\le T.$$

We have also that h is analytic in y. Using this fact gives us

$$\sum_{m=2}^\infty \left|\frac{\frac{\partial^{m-1}}{\partial y^{m-1}}h(t,S_{t_-},V_{t_-},-1)}{(m-2)!\Pi^{(m)}(t,T)}\right|\left(\frac{K}{S_{t_-}}\right)^{m-2}$$

$$\le\left|\frac{1}{\Pi^{(2)}(t,T)}\right|\sum_{m=2}^\infty \left|\frac{\frac{\partial^{m-1}}{\partial y^{m-1}}h(t,S_{t_-},V_{t_-},-1)}{(m-2)!}\right|\left|\frac{K}{S_{t_-}}\right|^{m-2}<\infty$$

for all $t\in[0,T]$. By assumption (4.23) we can apply Fubini's theorem to (4.27). So, we obtain

$$\sum_{j=2}^\infty \int_0^t \alpha_s^{(j)}dZ_s^{(j)}=\int_0^t\int_0^\infty \frac{B_s}{S_{s_-}^2}R(s,K)d\tilde{C}_s(K)dK-\int_0^t \frac{B_s h(s,S_{s_-},V_{s_-},-1)}{S_{s_-}}d\tilde{S}_s. \qquad (4.28)$$

By the above equation we can obtain number of bonds, number of stocks and number of call options in the hedging portfolio as follows:

$$\omega_t = B_t^{-1}[F(t,S_{t_-},V_{t_-}) - S_{t_-} D_1 F(t,S_{t_-},V_{t_-})]$$
$$+ B_t^{-1}\left[h(t,S_{t_-},V_{t_-},-1) - \int_0^\infty \frac{R(t,K)}{S_{t_-}^2} C_{t_-}(K)dK \right],$$
$$\alpha_t = D_1 F(t,S_{t_-},V_{t_-}) - \frac{h(t,S_{t_-},V_{t_-},-1)}{S_{t_-}}$$
$$\alpha_t^{(K)} = \frac{R(t,K)}{S_{t_-}^2}$$

In the above, ω_t corresponds number of bonds at time t, α_t corresponds number of stocks at time t and $\alpha_t^{(j)}$ corresponds number of call options with the same maturity T and different strikes K.

It is pointed out that, replacing (4.26) in (4.18) gives us

$$dZ_t^{(j)} = (-1)^j B_t \left(\sum_{m=1}^{j} \binom{j}{m} (-1)^m \frac{1}{F^{(m)}(t,S_{t_-})} \int_0^\infty m(m-1)K^{m-2} d\,\tilde{C}_t(K)dK \right)$$

This gives us the replication formula for the power jump assets in terms of call options with the same maturity and with a continuum of strikes. (4.24) gives a dynamic hedging portfolio in terms of call options and of the discounted stock, that is equivalent to the hedging portfolio in terms of power-jump assets.

Remark 4.2.1 *Now, let us investigate the relationship between the usual exponential Lévy model and the geometric Lévy model(stochastic exponential model). Assume that our stock price process* $S = \{S_t, t \in [0,T]\}$ *is given by the equation*

$$S_t = S_0 e^{\tilde{x}_t}, \qquad S_0 > 0 \tag{4.29}$$

where $\tilde{X} = \{\tilde{X}_t, t \in [0,T]\}$ *is a Lévy process. We can say that the process can be modelled as a stochastic exponential of a Lévy process, that is defined as the solution of the linear stochastic differential equation (3.1) and denoted by* $S_t = S_0\,\Omega(X_t)$*,where* $X = \{X_t, t \in [0,T]\}$ *is a Lévy process related to* $\tilde{X}$.

Then, the following properties can be written:

(1) *If* $\tilde{X}$ *is a Lévy process with characteristic triplet* $(\tilde{\eta}, \tilde{\chi}^2, \tilde{\sigma})$ *then the usual exponential* $e^{\tilde{x}_t}$ *is of the form* $\Omega(X_t)$ *for some Lévy process X with characteristic triplet given by* (η, χ^2, σ), *where*

$$\eta = \tilde{\eta} + \frac{\tilde{\chi}^2}{2} + \int (\mathrm{I}_{\{|e^x - 1| \le 1\}}(e^x - 1) - x\mathrm{I}_{\{|x| \le 1\}})\tilde{\sigma}(dx),$$
$$\chi^2 = \tilde{\chi}^2,$$
$$\tilde{\sigma}(U) = \int_R \mathrm{I}_U (e^x - 1)\tilde{\sigma}(dx), U \in \mathcal{B}(R)$$

(2) *If X is a Lévy process with characteristic triplet given by* (η , χ^2, σ) *then the stochastic exponential* $\Omega(X_t)$ *is of the form* $e^{\bar{x}_t}$, *for some Lévy process* $\tilde{X}$ *with characteristic triplet given by* $(\tilde{\eta}, \tilde{\chi}^2, \tilde{\sigma})$, *where*

$$\tilde{\eta} = \eta - \frac{\tilde{X}^2}{2} + \int \mathrm{I}_{\{|\log(1+x)| \le 1\}}(\log(1+x) - x\mathrm{I}_{\{|x| \le 1\}})\sigma(dx),$$
$$\tilde{\chi}^2 = \chi^2,$$
$$\tilde{\sigma}(D) = \int_R \mathrm{I}_D (\log(1+x)\sigma(dx), \ D \in \mathcal{B}(\mathrm{R})$$

By (3.18) and (4.29)

$$\tilde{S}_t = \tilde{S}_{t\_} \exp(\Delta \tilde{J}_t),$$
$$S_t = S_{t\_} \exp(1 + \Delta J_t),$$

where $\tilde{J}$ *is the jump part of* $\tilde{X}$ *and J is the jump part of* X.

If the stock dynamics is defined by (4.29) then we can define the hedging portfolio as following. Here the price function $F(t,x)$ *is the solution of the following PIDE:*

$$D_0F(t,x) + f(x_1)D_2F(t,x) + r_t x_1 D_1F(t,x) + \frac{1}{2}\chi_t^2 x_1^2 D_1^2 F(t,x)$$
$$- D_3F(t,x)\int_R u(y)\tilde{\sigma}_t(dy) + DF(t,x) = r_t F(t,x)$$

$$F(T,x)=p(x),$$

where

$$DF(t,x):=\int_{-\infty}^{\infty} h(t,x,y)\tilde{\sigma}_t(dy)$$

and

$$h(t,x,y):=F(t,x_1e^y,x_2,x_3+u(y))-F(t,x)-x_1(e^y-1)\ -x_1D_1F(t,x_1,x_2,x_3),$$

Here, the contingent claim has a payoff that depends only on the stock price at maturity. So we have that $h(t,x,y)=F(t,xe^y)-F(t,x)-x(e^y-1)D_1F(t,x)$ *and*

$$h(t,x,0)=0,$$
$$\frac{\partial}{\partial y}h(t,x,y)=0,$$
$$\frac{\partial^n}{\partial y^n}h(t,x,0)=\sum_{m_1,m_2,\dots,m_n,\,2\le m\le n}\frac{n!}{m_1!m_2!\dots m_n!(2!)^{m_1}(3!)^{m_2}\dots(n!)^{m_n}}D_1^{(m)}F(t,x)x^m.$$

Here, the sum is over all partitions of n,that is, over all n-tuples $(m_1,m_2,\dots,m_n)$ *such that*

$$1m_1+2m_2+3m_3+\dots+nm_n=n,$$

We have the notation of $m:=m_1+m_2+\dots+m_n$,
And, the hedging portfolio is given by

$$\alpha_t^{(j)}=\frac{\displaystyle\sum_{m_1,m_2,\dots,m_n,2\le m\le j}\frac{n!}{m_1!m_2!\dots m_n!(2!)^{m_2}(3!)^{m_3}\dots(n!)^{m_n}}D_1^{(m)}F(t,S_{t\_})S_{t\_}^m}{j!},\quad j=2,3,\dots,$$

Until now we supposed that the contingent claims with a price function $F(t,x)$ satisfying the analytic assumptions in Theorem 4.2.2. However, these regularity conditions are strong and we would like to obtain hedging formulas for more general contingent claims. For getting such formula, we will consider the discounted orthonormalized power jump processes $\{Y^{(j)},j\ge 2\}$ Recall the orthonormalization coefficients from the orthonormalization procedure and consider the orthonormal real polynomial $p_t^{(j)}(y),\ j\ge 1$ with these time dependent coefficients.

Lemma 4.2.2 *Let* $f:[0,T]\times R^4\to R$ *be a measurable function such that*

$$E\left[\int_0^t\int_R |f(s,x,y)|^2 \tilde{\sigma}_s(dy)ds\right] < \infty.$$

Then we have

$$\sum_{0<s\le t} f(s,S_{s\_},V_{s\_},\Delta J_s) = \sum_{j=1}^{\infty}\int_0^t \langle f(s,x,.), p_s^{(j)}(.)\rangle_{L^2(\tilde{\sigma}_s)} dY_s^{(j)}$$
$$+\int_0^t\int_R f(s,x,y)\tilde{\sigma}_s(dy)ds.$$

Theorem 4.2.3 *Let Y be a contingent claim with payoff* $Y = p(S_T, V_T)$ *and a price function* $F(t,S_t,V_t)$ *such that* $F(t,x)$ *is of class* $C^{1,2,2,2}$ *in* $[0,T]\times R^3$.

Consider the function

$$h(t,x,y) := F(t,x_1(1+y),x_2,x_3+u(y)) - F(t,x) - x_1 y D_1 F(t,x) \qquad (4.30)$$

and suppose that

$$E\left[\int_0^t\int_R |h(s,x,y)|^2 \tilde{\sigma}_s(dy)ds\right] < \infty.$$

Set

$$N^{(k)}(s,K) := \sum_{j=2}^{k}\sum_{m=2}^{j}\frac{(-1)^{j-m}\alpha_s^{(j,k)}\binom{j}{m}m(m-1)}{\Pi^{(m)}(s,T)}\left(\frac{K}{S_{s\_}}\right)^{m-2},$$

where $\alpha_s^{(j)} := \int_R h(s,S_{s\_},V_{s\_},y)p_s^{(j)}(y)\tilde{\sigma}_s(dy)$ *and* $\alpha_s^{(j,k)} := \sum_{j=i}^{k} b_{j,i}(s)\alpha_s^{(j)}$.

Then we have the representation formula

$$\sum_{j=1}^{\infty}\int_0^t \alpha_s^{(j)} dY_s^{(j)}$$
$$= \lim_{k\to\infty}\left[\int_0^{\infty}\int_0^t \frac{B_s N^{(k)}(s,K)}{S_{s\_}^2} d\tilde{C}_s(K)dK - \sum_{j=1}^{k}\int_0^t \frac{j(-1)^j \alpha_s^{(j)} B_s}{S_{s\_}} d\tilde{S}_s\right].$$

In addition to this, the hedging portfolio in terms of bonds, stocks and call options is given by

$$\omega_t = \frac{1}{B_t}[F(t, S_{t_-}, V_{t_-}) - S_{t_-} D_1 F(t, S_{t_-}, V_{t_-})]$$

$$+ \frac{1}{B_t} \lim_{k \to \infty} \left[\sum_{j=1}^{k} \frac{j(-1)^j \alpha_s^{(j,k)}}{S_{s_-}} - \int_0^{\infty} \frac{N^{(k)}(t, K)}{S_{t_-}^2} C_{t_-}(K) dK \right],$$

$$\alpha_t = D_1 F(t, S_{t_-}, V_{t_-}) - \lim_{k \to \infty} \sum_{j=1}^{k} \int_0^t \frac{j(-1)^j \alpha_t^{(j,k)}}{S_{t_-}},$$

$$\alpha_t^{(K)} = \lim_{k \to \infty} \frac{N^{(k)}(t, K)}{S_{t_-}^2}.$$

Proof:

Applying Ito's formula to $F(t, S_t, V_t)$ gives

$$F(t, S_t, V_t) = F(t, S_0, V_0) + \int_0^t F(s, S_{s_-}, V_{s_-}) dS_s$$

$$+ \int_0^t \left(D_0 F(s, S_{s_-}, V_{s_-}) \frac{1}{2} \chi_s^2 S_{s_-}^2 D_1^2 F(s, S_{s_-}, V_{s_-}) + f(S_{s_-}) D_2 F(s, .S_{s_-}, V_{s_-}) \right) ds$$

$$+ \sum_{0 < s \le t} [F(s, S_s, V_{s_-}) - F(s, S_{s_-}, V_{s_-}) - D_1 F(s, S_{s_-}, V_{s_-}) \Delta S_s].$$

It is clear that , $\Delta S_s = S_{s_-} \Delta J_s$. Thus;

$$\sum_{0 < s \le t} \left(F(s, S_s, V_{s_-}) - F(s, S_{s_-}, V_{s_-}) - S_{s_-} \Delta J_s D_2 F(s, S_{s_-}, V_{s_-}) \right)$$

$$= h(s, S_{s_-}, V_{s_-}, \Delta J_s).$$

Using the Partial Differential equations (4.10) and (4.11) gives the following:

$$F(t,S_t,V_t) = F(t,S_0,V_0) + \int_0^t D_1 F(t,S_{s_-},V_{s_-})dS_s$$
$$+ \int_0^t \frac{(F(s,S_{s_-},V_{s_-}) - S_{s_-} D_1 F(s,S_{s_-},V_{s_-}))}{B_s} dB_s$$
$$+ \sum_{0<s\le t} h(s,S_{s_-},V_{s_-},\Delta J_s) - \int_0^t \int_R h(s,S_{s_-},V_{s_-},y)\tilde{\sigma}_s(dy)ds. \quad (4.31)$$

This gives us the representation of the hedging portfolio in terms of bonds and stocks.

In addition to this, note that

$$M_t = \sum_{0<s\le t} h(s,S_{s_-},V_{s_-},\Delta J_s) - \int_0^t \int_R h(s,S_{s_-},V_{s_-},y)\tilde{\sigma}_s(dy)ds$$
$$= \sum_{0<s\le t} h(s,S_{s_-},V_{s_-},\Delta J_s) - E\left(\sum_{0<s\le t} h(s,S_{s_-},V_{s_-},\Delta J_s) \right)$$

is a square integrable martingale and, by Lemma 4.2.2, we have that

$$M_t = \sum_{j=1}^{\infty} \int_0^t \alpha_s^{(j)} dY_s^{(j)}$$

where

$$\alpha_s^{(j)} = \int_R f(s,S_{s_-},V_{s_-},y) p_s^{(j)}(y)\tilde{\sigma}_s(dy) \text{ and } E\left[\int_0^t \sum_{j=1}^{\infty} \left|\alpha_s^{(j)}\right|^2 ds \right] < \infty.$$

By (3.21), we have

$$\alpha_s^{(j)} dY_s^{(j)} = \alpha_s^{(j)} \sum_{i=1}^{j} b_{j,i}(s) dZ_s^{(j)}.$$

Generaly, we can write

$$\sum_{j=1}^{\infty} \int_0^t \alpha_s^{(j)} dY_s^{(j)} = \lim_k \sum_{j=1}^{k} \int_0^t \alpha_s^{(j)} dY_s^{(j)}$$
$$= \lim_k \sum_{i=1}^{k} \int_0^t \left(\sum_{j=i}^{k} b_{j,i} \alpha_s^{(j)} \right) dZ_s^{(j)} = \lim_k \sum_{i=1}^{k} \int_0^t \alpha_s^{(i,k)} dZ_s^{(i)},$$

where, $\alpha_s^{(i,k)} := \sum_{j=i}^{k} b_{j,i}(s) \alpha_s^{(j)}$.

Recalling (4.18) gives us:

$$\sum_{j=i}^{\infty}\int_0^t \alpha_s^{(j)} dY_s^{(j)} = \lim_k \sum_{j=i}^{k}\int_0^t \alpha_s^{(j)} dZ_s^{(j)}$$

$$= \lim_k \sum_{j=i}^{k}\int_0^t B_s \alpha_s^{(j,k)} (-1)^j \sum_{m=1}^{j} \binom{j}{m} (-1)^m \frac{1}{F^{(k)}(s, S_{s_-})} d\tilde{F}^{(k)}(s, S_s)$$

$$= \lim_k \sum_{m=i}^{k} (-1)^m \int_0^t \frac{B_s}{F^{(m)}(s, S_{s_-})} \sum_{j=m}^{k} \alpha_s^{(j,k)} (-1)^j \binom{j}{m} d\tilde{F}^{(m)}(s, S_s).$$

The representation (4.26) yields

$$\sum_{j=1}^{\infty}\int_0^t \alpha_s^{(j)} dY_s^{(j)} =$$

$$\lim_k \left[\sum_{j=2}^{k} (-1)^j \int_0^{\infty}\int_0^t B_s \alpha_s^{(j,k)} \sum_{m=2}^{j} (-1)^m \binom{j}{m} m(m-1) \frac{K^{m-2}}{\Pi^{(m)}(s,T) S_{s_-}^m} d\tilde{C}_s(K) dK \right]$$

$$- \lim_k \left[\sum_{j=1}^{k} \int_0^t \frac{j(-1)^j \alpha_s^{(j,k)} B_s}{S_{s_-}} d\tilde{S}_s \right].$$

Combining with the representation (4.31), we obtain the hedging portfolio.

Remark 4.2.3 *Take a call option struck at K_* in an additive market with bond price process B_t. Then, its price function is given by*

$$F(t, S_t) = \frac{B_t}{B_T} E_Q\left[(S_T - S_t)_+ \middle| \mathcal{F}_t\right]$$

$$= \frac{B_t}{B_T} S_t E_Q\left[\left(\frac{S_T}{S_t} - \frac{K_*}{x}\right)_+\right]\Bigg|_{x=S_t}$$

$$= \frac{B_t}{B_T} S_t \Phi(t,x)\Big|_{x=S_t}$$

where $\Phi(t,x) := E_Q\left[\left(\frac{S_t}{S_T} - \frac{K_*}{x}\right)\right]$. *Price function* $F(t,x) = \frac{B_t}{B_T} S_t \Phi(t,x)$ *must satisfy the partial differential equations (4.10) and (4.11).Thus,*

$$\frac{\partial}{\partial t}\Phi(t,x) - r_t x \frac{\partial}{\partial x}\Phi(t,x) + \frac{\chi_t^2}{2} x^2 \frac{\partial^2}{\partial x^2}\Phi(t,x)$$

$$\int_{-1}^{\infty} \left((1+y) \left(\Phi\left(t, \frac{x}{1+y}\right) - \Phi(t,x) \right) + yx \frac{\partial}{\partial x}\Phi(t,x) \right) \tilde{\sigma}_t(dy) = 0$$

$$\Phi(T,x)=(x-K_*)_+$$

Suppose that $F(t,X)=e^{-r_{T-t}(T-t)}x\Phi(t,x)$ *is analytic in x for all* $x>0$ *and* $t\in[0,T]$. *By theorem (4.2.2) ,the portfolio in the power jump assets* $Z^{(j)}$, $j\geq 2$ *can be represented by*

$$\sum_{j=2}^{\infty}\int_0^t \alpha_s^{(j)}dZ_s^{(j)}=\int\frac{B_s}{S_{s_-}^2}R(s,K)d\tilde{C}_s(K)dK-\int_0^t\frac{B_sh(s,S_{s_-},V_{s_-},-1)}{S_{s_-}}d\tilde{S}_s.$$

CHAPTER 5

PORTFOLIO OPTIMIZATION

In this section, the portfolio optimization problem in the complete Additive Market is considered. This problem consists of choosing an optimal portfolio in such a way that the largest expected utility of the terminal wealth is obtained.

In this section, a class of utility functions, including HARA, logarithmic and exponential utilities, are considered. Then, the optimal portfolio that maximizes the terminal expected utility is obtained by the martingale method. Then, the optimal wealth is found and the hedging portfolio replicating this wealth is obtained respectively.

In this section, it is shown that for particular choices of the equivalent martingale measure in our market, the optimal portfolio only consists of bonds and stocks.

5.1 THE OPTIMAL WEALTH AND THE OPTIMAL PORTFOLIO

Definition 5.1.1 *A utility function is a map* $U(x) := R \to R \cup \{-\infty\}$*, which is strictly increasing and continuous on* $\{U > -\infty\}$*, of class* C^{∞} *and strictly concave on the interior of* $\{U > -\infty\}$*, and such that marginal utility tends to zero when wealth tends to infinity, i.e.,*

$$U^{'}(\infty) := \lim_{x \to \infty} U'(x) = 0.$$

Let us denote the interior of $\{U > -\infty\}$ *by* $dom(U)$*. We will consider only the two following cases:*

Case 5.1.1 $dom(U) = (0, \infty)$ and U satisfies

$$U'(0) := \lim_{x \to 0^{+}} U'(x) = \infty.$$

Case 5.1.2 $dom(U) = R$ and U satisfies

$$U'(-\infty) := \lim_{x \to -\infty} U'(x) = \infty.$$

The HARA utility functions $U(x) = \dfrac{x^{1-p}}{1-p}$ for $p \in R_{+} \backslash \{0,1\}$ and the logarithm utility $U(x) = \log(x)$ are examples of case 5.1.1 and the exponential utility function $U(x) = -\dfrac{1}{a} e^{-ax}$ is example of case 5.1.2.

Let us fix a structure preserving martingale Q. Our aim is to solve the optimal investment problem in the non-homogeneous Lévy market by using the so called "Martingale method."

We consider that, an initial wealth w_0 and an utility function U, that we want to find the optimal terminal wealth W_T, that is, the value of W_T that maximizes $E_p(U(W_T))$ and can be replicated by a portfolio with initial value w_0.
We have the information that, under an equivalent measure Q, which is structure preserving, any random variable $W_T \in L^2(\Omega, \mathcal{F}_t, \mathrm{Q})$ can be replicated and $w_0 = E_{\mathrm{Q}}\left[\frac{W_T}{B_T}\right]$.
Thus, we will consider the optimization problem

$$\max\left\{E_p(U(W_T)) : E_{\mathrm{Q}}\left(\frac{W_T}{B_T}\right) = w_0\right\}.$$

The corresponding Lagrangian is

$$E_P(U(W_T)) - \lambda E_{\mathrm{Q}}\left(\frac{W_T}{B_T} - w_0\right) = E_P\left(U(W_T) - \lambda_T\left(\frac{d\mathrm{Q}_T}{dP_T}\frac{W_T}{B_T} - w_0\right)\right). \tag{5.1}$$

Definition 5.1.2 W_T *is called the optimal terminal wealth if it is a solution to the optimization problem* (5.1),

The optimal terminal wealth is given by the following equation:

$$W_T = (U')^{-1}\left(\frac{\lambda_T}{B_T}\varsigma_T\right)$$

where λ_T *is the solution of the equation*

$$E_{\mathrm{Q}}\left[\frac{1}{B_T}(U')^{-1}\left(\frac{\lambda_T}{B_T}\frac{d\mathrm{Q}_T}{dP_T}\right)\right] = w_0. \tag{5.2}$$

Remark 5.1.1 *In order to hedge the optimal terminal wealth* W_T, *we need to know its price process:*

$$E_{\mathrm{Q}}\left[\frac{B_t}{B_T}W_T|\mathcal{F}_T\right] = E_{\mathrm{Q}}\left[\frac{B_t}{B_T}(U')^{-1}\left(\frac{\lambda_T}{B_T}\varsigma_T|\mathcal{F}_T\right)\right].$$

and this depends on the utility function considered. Assume that the utility function satisfies $(U')^{-1}(xy) = b_1(x)(U')^{-1}(y) + b_2(x)$, *for any* $x, y \in R$, *for certain* C^∞ *functions* $b_1(x)$ *and* $b_2(x)$.
Then the price function W_T *verifies:*

$$E_{\mathrm{Q}}\left[\frac{B_t}{B_T}W_T\middle|\mathcal{F}_t\right]=E_{\mathrm{Q}}\left[\frac{B_t}{B_T}(U')^{-1}\left(\frac{\lambda_T}{B_T}\frac{d\mathrm{Q}_T}{dP_T}\right)\middle|\mathcal{F}_t\right]$$

$$=E_{\mathrm{Q}}\left[b_1\left(\frac{\lambda_{T,t}}{B_{T,t}}\frac{d\mathrm{Q}_{T,t}}{dP_{T,t}}\right)\right](U')^{-1}\left(\frac{\lambda_t}{B_t}\frac{d\mathrm{Q}_t}{dP_t}\right)+E_Q\left[b_2\left(\frac{\lambda_{T,t}}{B_{T,t}}\frac{d\mathrm{Q}_{T,t}}{dP_{T,t}}\right)\right]$$

$$=\pi(t,T)W_t+\omega(t,T) \tag{5.3}$$

So we have $E_{\mathrm{Q}}\left[\frac{B_t}{B_T}W_T\middle|\mathcal{F}_t\right]=\pi(t,T)W_t+\omega(t,T)$, *for certain deterministic functions* $\pi(t,T)$ *and* $\omega(t,T)$ *(In the above* $\frac{d\mathrm{Q}_{T,t}}{dP_{T,t}}=\frac{d\mathrm{Q}_T/dP_T}{d\mathrm{Q}_t/dP_t}$ *)*

with

$$W_t=(U')^{-1}\left(\frac{\lambda_t}{B_t}\varsigma_t\right)$$

where

$$E_{\mathrm{Q}}\left[\frac{1}{B_t}(U')^{-1}\left(\frac{\lambda_t}{B_t}\varsigma_t\right)\right]=\omega_0.$$

Lemma 5.1.1 $(U')^{-1}(xy)=b_1(x)(U')^{-1}(y)+b_2(x)$, *for any* $x,y\in(0,\infty)$ *if and only if* $U'(x)/U''(x)=mx+n$, *for any* $x\in dom(U)$, *and some* $m,n\in R$.

Proof:

Suppose first that $(U')^{-1}(xy)=b_1(x)(U')^{-1}(y)+b_2(x)$. Write $I(x)=(U')^{-1}(x)$. Then, by differentiating with respect to x, we have

$$yI'(xy)=b_1'(x)I(y)+b_2'(x)$$

Thus, by taking $y=I^{-1}(u)$ and $x=1$, we obtain

$$I^{-1}(u)I'(I^{-1}(u))=\frac{U'(u)}{U''(u)}$$

$$=b_1'(1)u+b_2'(1)$$

Suppose that $U'(x)/U''(x)=mx+n$. By integrating the differential equation we have that

$$U(x)=B_1\log(x-n)+B_2 \qquad \text{if } m\neq 1,$$

$$U(x) = \frac{B_1}{m(1+1/m)}(mx+n)^{1+1/m} + B_2 \qquad \text{if} \quad m \notin \{-1,0\},$$

$$U(x) = B_1 n e^{x/n} + B_2 \qquad \text{if} \quad m = 0.$$

where B_1 and B_2 are integration constants. Hence, $(U')^{-1}(y) = ky^m - n/m$, if $m \neq 0$ and $(U')^{-1}(y) = n \log y + k$ if $m = 0$, where k is constant.

Lemma 5.1.2 *Consider a utility function U such that* $\frac{U'(x)}{U''(x)} = mx + n,$ *for any* $x \in dom(U)$ *and for some m,n*$\in \mathbb{R}$*. Then;*

$$(U')^{-1}(y) = \begin{cases} ky^m - \frac{n}{m} & \text{if} \quad m \neq 0 \\ n \log y + k & \text{if} \quad m = 0 \end{cases} \tag{5.4}$$

The proof of this lemma has been already done in the above.

In the above ς_t has the following representation:

$$\begin{aligned} \varsigma_t &= \frac{dQ_t}{dP_t} \\ &= \exp\left(\int_0^t \chi_s G_s dW_s - \frac{1}{2}\int_0^t \chi_s^2 G_s^2 ds + \int_0^t \int_{-\infty}^{+\infty} \log H(s,x) M(ds,dx) \right) \\ &\quad / \exp\left(\int_0^t \int_{-\infty}^{+\infty} (H(s,x) - 1 - \log H(s,x)) \sigma_s(dx) ds \right) \end{aligned} \tag{5.5}$$

where G and H verifying following assumptions:

(i) $\int_0^T \int_{-\infty}^{+\infty} \left|\log H(s,x)\right|^2 \sigma_s(dx) ds < \infty,$

(ii) $\int_0^T \int_{-\infty}^{+\infty} \left|H(s,x) - 1 - \log H(s,x)\right| \sigma_s(dx) ds < \infty,$

(iii) $\chi_t^2 G_t + a_t + r_t + \int_{-\infty}^{+\infty} x(H(t,x) - 1)\sigma_t(dx) = 0.$

We can express this density process in terms of the Q*-Brownian motion* $\tilde{W}$ *and the compensated random measure* $\tilde{M}(dt,dx)$*. From (4.1) and (4.3), we have*

$$\varsigma_t = \frac{dQ_t}{dP_t}$$

$$= \exp\left(\int_0^t \chi_s G_s dW_s - \frac{1}{2}\int_0^t \chi_s^2 G_s^2 ds + \int_0^t \int_{-\infty}^{+\infty} \log H(s,x) M(ds,dx) \right)$$

$$/\exp\left(\int_0^t \int_{-\infty}^{+\infty} (H(s,x) - 1 - \log H(s,x))\sigma_s(dx)ds \right)$$

Theorem 5.1.1 *Assume that the utility function* U *satisfies* $\frac{U'(x)}{U''(x)} = mx + n$, *for any* $x \in dom(U)$ *and for some m,n*$\in R$. *Assume also that* Q *is a structure preserving equivalent martingale measure, the associated function H satisfies the following assumptions:*

(i) $\int_0^T \int_{-\infty}^{+\infty} \left|\log H(s,x)\right|^2 \sigma_s(dx)ds < \infty,$

(ii) $\int_0^T \int_{-\infty}^{+\infty} \left|H(s,x) - 1 - \log H(s,x)\right| \sigma_s(dx)ds < \infty,$

(iii) $\sup_{t\in[0,T.]} \int_{(-\varepsilon,\varepsilon)^C} \exp(\lambda|x|)\tilde{\sigma}_t(dx) = \sup_{t\in[0,T]} \int_{(-\varepsilon,\varepsilon)} \exp(\lambda|x|)H(t,x)\sigma_t(dx).$

In addition to the assumptions in the above $H(t,0) = 1$ *and* $\left.\frac{\partial}{\partial y} H(t,y)\right|_{y=0} = G_t$ *for all*

$t \in [0,T]$

(a) If

$$H(t,x) = \begin{cases} (1 + mG_t x)^{\frac{1}{m}}, & \text{if } m \neq 0, \\ H(t,x) = \exp(G_t x), & \text{if } m = 0, \end{cases} \tag{5.5}$$

then the optimal portfolio consists only of bonds and stocks, and the number of shares is given by

$$\phi_t^1 = \frac{\pi(t,T)(mW_{t_-} + n)G_t}{S_{t_-}}. \tag{5.6}$$

(b) If $m \neq 0$ *and* $(H(t,x))^m$ *is an analytic function in the x variable, and does not have the form (5.5) then, in general, the optimal portfolio will consist of bonds, stocks and power-jump assets. The number of shares of these assets is given by*

$$\phi_t^{(j)} = \frac{\pi(t,T)}{j!B_t}\left(W_{t_-} + \frac{m}{n}\right)\frac{\partial^j}{\partial y^j}(H(t,y))^m\bigg|_{y=0}, \quad j = 2,3,...., \tag{5.7}$$

(c) If $m = 0$ *and* $\log H(t,x)$ *is an analytic function in the x variable, and does not have the form (5.5) then, in general, the optimal portfolio will consist of bonds, stocks and power-jump assets. The number of shares of these assets is given by*

$$\phi_t^{(j)} = \frac{n\pi(t,T)}{j!B_t}\left(W_{t_-} + \frac{m}{n}\right)\frac{\partial^j}{\partial y^j}(\log(H(t,y))^m\bigg|_{y=0}, \quad j = 2,3,..., \tag{5.8}$$

Proof:

Let us define

$$R_t := \int_0^t \chi_s G_s d\tilde{W}_s,$$

$$K_t := \int_0^t \int_{-\infty}^{+\infty} \log H(s,x)\tilde{M}(ds,dx).$$

which are Q-martingales. The discounted wealth price process, given by

$$\hat{W}_t := \frac{1}{B_t} E_Q\left[\frac{B_t}{B_T} W_T \middle| \mathcal{F}_T\right] = \frac{\pi(t,T)}{B_t} W_t + \frac{\omega(t,T)}{B_t},$$

is also a martingale and can also be written in terms of the processes R_t and K_t.

$$\hat{W}_t = f(t, R_t, K_t),$$

where

$$f(t,x,y) := \frac{\pi(t,T)}{B_t}(U')^{-1}\left(\frac{\lambda_t}{B_t}\exp(g(t,x,y))\right) + \frac{\omega(t,T)}{B_t}$$

and

$$g(t,x,y) := x + y + \frac{1}{2}\int_0^t \chi_s^2 G_s^2 ds + \int_0^t \int_{-\infty}^{+\infty} (H(s,x)\log H(s,x) + 1 - H(s,x))\sigma_s(dx)ds.$$

In this proof, we will consider the two cases in (5.4).
First, consider $m \neq 0$. Then we have:

$$(U')^{-1}\left(\frac{\lambda_t}{B_t}\exp(g(t,x,y)\right) = \left(\frac{\lambda_t}{B_t}\right)^m \exp(mg(t,x,y)) - \frac{n}{m}.$$

Let us now apply the Itô's Formula for semimartingales to $\hat{W}_t = f(t, R_t, K_t)$. Then we will find following result:

$$d\hat{W}_t = \frac{\partial f}{\partial t}(t, R_{t_-}, K_{t_-})dt + \frac{\partial f}{\partial x}(t, R_{t_-}, K_{t_-})dR_t + \frac{\partial f}{\partial y}(t, R_{t_-}, K_{t_-})dK_t$$
$$+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, R_{t_-}, K_{t_-})d[R_t, R_t]_t^C + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(t, R_{t_-}, K_{t_-})d[K_t, K_t]_t^C$$
$$+\frac{\partial^2 f}{\partial x \partial y}(t, R_{t_-}, K_{t_-})d[R_t, K_t]_t^C$$
$$+\left[\hat{W}_t - \hat{W}_{t_-} - \frac{\partial f}{\partial x}(t, R_{t_-}, K_{t_-})\Delta R_t - \frac{\partial f}{\partial y}(t, R_{t_-}, K_{t_-})\Delta K_t\right].$$

By definition, R_t is a continuous process (It is an Itô integral) and K_t is a jump process and we have that

$$\Delta R_t = 0,$$
$$\Delta K_t = \log H(t, \Delta J_t),$$
$$d[R_t, R_t]_t^C = (\chi_t G_t)^2 dt,$$
$$d[K_t, K_t]_t^C = 0,$$

Hence,

$$d\hat{W}_t = A_t dt + \left(m\left(\hat{w}_{t_-} - \frac{\omega(t,T)}{B_t} \right) + n\frac{\pi(t,T)}{B_t} \right)\chi_t G_t d\hat{W}_t$$
$$+\left(\hat{W}_{t_-} - \frac{\omega(t,T)}{B_t} + \frac{n\pi(t,T)}{mB_t} \right)\int_{-\infty}^{+\infty}((H(t,x))^m - 1)\tilde{M}(dt, dx).$$

where A_t is the finite variation part of the process $\hat{W}$. It is obvious that the Itô integral and the integral with respect to the compensated random measure are Q-martingales and in order to $\hat{W}$ be a martingale, the finite variation part A_t, must be zero. Thus;

$$d\hat{W}_t = \left(\frac{\pi(t,T)(mW_{t_-} + n)}{B_t} \right)\chi_t G_t d\tilde{W}_t$$
$$+\left(\frac{\pi(t,T)\left(W_{t_-} + \frac{n}{m} \right)}{B_t} \right)\int_{-\infty}^{+\infty}((H(t,x))^m - 1)\tilde{M}(dt, dx)$$

By (4.5) we can write following equation:

$$d\tilde{S}_t = \chi_t \tilde{S}_{t\_} d\tilde{W}_t + \tilde{S}_{t\_} \int_{-\infty}^{+\infty} x \tilde{M}(dt,dx).$$

This Q-dynamics give:

$$d\hat{W}_t = \left(\frac{\pi(t,T)(mW_{t\_}+n)}{B_t}\right) G_t \frac{d\tilde{S}_t}{\tilde{S}_t}$$
$$+\int_{-\infty}^{+\infty}\left[\frac{\pi(t,T)}{B_t}\left(\left(W_{t\_}+\frac{n}{m}\right)((H(t,x))^m-1)-G_t(mW_{t\_}+n)x\right)\right]\tilde{M}(dt,dx).$$

Since $\tilde{S}_t = B_t^{-1} S_t$, we can write the following:

$$d\hat{W}_t = \frac{\pi(t,T)(mW_{t\_}+n)G_t}{S_t} d\tilde{S}_t$$
$$+\int_{-\infty}^{+\infty}\left[\frac{\pi(t,T)}{B_t}(mW_{t\_}+n)\left(\frac{(H(t,x))^m-1}{m}-G_t x\right)\right]\tilde{M}(dt,dx).$$

In order to ensure that the optimal portfolio will consists only of bonds and stocks, the jump part of of $d\hat{W}_t$ must be zero and, hence,

$$H(t,x) = (1+mG_t x)^{\frac{1}{m}}$$

By (4.4) we know that the function G_t must satisfy

$$\chi_t^2 G_t + a_t - r_t + \int_{-\infty}^{+\infty} x\left((1+mG_t x)^{\frac{1}{m}} - 1\right)\sigma_t(dx) = 0.$$

In addition to the previous equation, the wealth invested in stocks at time t is then given by

$$\phi_t^1 = \frac{G_t \pi(t,T)(mW_{t\_}+n)}{S_{t\_}}.$$

Generally, if the function $H(t,x)$ does not satisfy (5.5) then the structure preserving martingale measure, that characterizes the market, is such that the optimal portfolio includes bonds, stocks and derivatives which can be expressed by the power jump assets. Since $H(t,x)$ is analytic in $x,$ we can expand the integrant function

$$\frac{(H(t,x))^m - 1}{m} - G_t x$$

in terms of powers of x, with $H(t,0)=1$ and $\frac{\partial}{\partial x}H(t,x)\Big|_{x=0} = G_t$. Then

$$\frac{(H(t,x))^m - 1}{m} - G_t x = \frac{1}{a}\sum_{j=2}^{\infty} \left.\frac{\frac{\partial}{\partial y^j}(H(t,y))^m}{j!}\right|_{y=0} x^j$$

and the number of power-jump assets in the optimal portfolio is given by

$$\phi_t^j = \frac{\pi(t,T)}{j! B_t}\left(W_{t\_} + \frac{n}{m}\right)\frac{\partial^j}{\partial y^j}((H(t,y))^m\Big|_{y=0}, \quad j = 2,3,...,$$

In the second case of (5.4), when $m = 0,$ we have

$$(U')^{-1}\left(\frac{\lambda_t}{B_t}\exp(g(t,x,y))\right) = n\left(\log\left(\frac{\lambda_t}{B_t}\right) + g(t,x,y)\right) + k.$$

Applying to Itô Formula for semimartingales to $\hat{W}_t = f(t, R_t, K_t)$ gives us following:

$$d\hat{W}_t = A_t dt + \frac{n\pi(t,T)\chi_t G_t}{B_t} d\tilde{W}_t$$
$$+\left(\frac{n\pi(t,T)}{B_t}\right)\int_{-\infty}^{+\infty}\log(H(s,x))\tilde{M}(dt,dx),$$

where A_t is a finite variation process. Similarly to the case $m \neq 0,$ the finite variation part A_t, must be zero and

$$d\hat{W}_t = \frac{n\pi(t,T)G_t}{B_t}\left(\frac{d\tilde{S}_t}{\tilde{S}_t}\right)$$
$$+\left(\frac{n\pi(t,T)}{B_t}\right)\int_{-\infty}^{+\infty}(\log(H(s,x)) - G_t x)\tilde{M}(dt,dx).$$

Thus,

$$d\hat{W}_t = \frac{n\pi(t,T)G_t}{S_t} d\tilde{S}_t$$
$$+\left(\frac{n\pi(t,T)}{B_t}\right)\int_{-\infty}^{+\infty}(\log(H(t,x)) - G_t x)\tilde{M}(dt,dx).$$

In order to ensure that the optimal portfolio will consist only of bonds and stocks, we requare that

$$H(t,x) = \exp(G_t x),$$

where, by (4.4), the function G_t satisfies

$$\chi_t^2 G_t + a_t - r_t + \int_{-\infty}^{+\infty} x(\exp(G_t x) - 1)\sigma_t(dx) = 0.$$

The wealth invested in stocks is given by:

$$\phi_t^1 = \frac{n\pi(t,T)G_t}{S_{t_-}}.$$

If $H(t,x)$ does not satisfy (5.5) then the optimal portfolio includes also power jump assets. Let us expand the integrant function

$$\log(H(t,x)) - G_t x$$

In terms of powers of x, considering that $H(t,0) = 1$ and $\left.\frac{\partial}{\partial x} H(t,x)\right|_{x=0} = G_t$. Since $\log H(t,x)$ is an anlytic function in the x variable, we have

$$\log(H(t,x)) - G_t x = \sum_{j=2}^{\infty} \left.\frac{\frac{\partial}{\partial y^j}(\log(H(t,y)))}{j!}\right|_{y=0} x^j,$$

and the number of the power-jump assets in the optimal portfolio is given by,

$$\phi_t^j = \frac{b\pi(t,T)}{j! B_t} \left.\frac{\partial^j}{\partial y^j}(\log(H(t,y)))\right|_{y=0}, j = 2,3,..., \quad \text{Q.E.D.}\ .$$

5.2 APPLICATION

Example 5.2.1 *Consider the logarithm utility* $U(x) = \log x$. *Then*

$$U'(x) = (U')^{-1}(x) = x^{-1}$$

and $\dfrac{U'(x)}{U''(x)} = mx + n$ *with* $m = -1$ *and* $n = 0$. *By solving*

$$E_{\mathrm{Q}}\left(\frac{1}{B_t}(U')^{-1}\left(\frac{\lambda_T}{B_T}\frac{d\mathrm{Q}_T}{dP_T}\right)\right) = w_0.$$

we have

$$W_T = w_0 B_T \frac{dP_T}{d\mathrm{Q}_T}.$$

Thus, we obtain $E_{\mathrm{Q}}\left[\dfrac{B_t}{B_T}W_T\middle|\mathcal{F}_t\right] = w_0 B_t E_{\mathrm{Q}}\left[\dfrac{dP_T}{d\mathrm{Q}_T}|\mathcal{F}_t\right] = w_0 B_t \dfrac{dP_t}{d\mathrm{Q}_t} = W_t.$

Therefore, $\pi(t,T) = 1$ *and* $\omega(t,T) = 0$. *Applying theorem (5.1.1), the optimal portfolio consists only of bonds and stocks if*

$$H(t,x) = \frac{1}{1 - G_t x}, \tag{5.9}$$

and $G_t x$ *satisfies*

$$\chi_t^2 G_t + \eta_t - r_t + \int_0^t \int_{-\infty}^{+\infty} x\left((1 - G_t x)^{-1} - 1\right)\sigma_s(dx) = 0.$$

The fraction of wealth invested in stocks, at time t, *is given by*

$$\frac{\alpha_t S_{t_-}}{W_{t_-}} = -G_t.$$

If the function H satisfies the assumptions in Theorem 5.1.1 but is not of the form (5.9) then, in general, the optimal portfolio also includes power-jump assets. The number of these assets is given by

$$\phi_t^j = \frac{W_{t\_}}{j! B_t} \frac{\partial}{\partial y^j} (H(t,y))^{-1} \Bigg|_{y=0}, j = 2,3,...,$$

Example 5.2.2 *Consider the HARA utilities* $U(x) = \dfrac{x^{1-p}}{1-p}$ *with* $p \in R_+ \setminus \{0,1\}$. *Then*

$$(U')^{-1}(x) = x^{-\frac{1}{p}}$$

and $\dfrac{U'(x)}{U''(x)} = mx + n$ *with* $m = -\dfrac{1}{p}$ *and* $n = 0.$

Solving $E_{\mathrm{Q}}\left(\dfrac{1}{B_t}(U')^{-1}\left(\dfrac{\lambda_T}{B_T}\dfrac{d\mathrm{Q}_T}{dP_T}\right)\right) = w_0$ *gives us*

$$W_T = w_0 B_T \frac{(dP_T / d\mathrm{Q}_T)^{1/p}}{E_{\mathrm{Q}}((dP_T / d\mathrm{Q}_T)^{1/p}}.$$

By using these equation, we obtain

$$\begin{aligned}
E_Q\left[\frac{B_t}{B_T} W_T \middle| \mathcal{F}_t\right] &= w_0 B_t \frac{E_{\mathrm{Q}}[(dP_T / d\mathrm{Q}_T\ )^{1/p} | \mathcal{F}_t]}{E_{\mathrm{Q}}((dP_T / d\mathrm{Q}_T)^{1/p})} \\
&= w_0 B_t \frac{E_{\mathrm{Q}}\left((dP_{T,t} / d\mathrm{Q}_{T,t})^{1/p} (dP_t / d\mathrm{Q}_t)^{1/p} \middle| \mathcal{F}_t\right)}{E_{\mathrm{Q}}\left((dP_{T,t} / d\mathrm{Q}_{T,t})^{1/p} (dP_t / d\mathrm{Q}_t)^{1/p}\right)} \\
&= w_0 B_t \frac{(dP_t / d\mathrm{Q}_t)^{1/p}}{E_{\mathrm{Q}}\left((dP_t / d\mathrm{Q}_t)^{1/p}\right)} \\
&= W_t
\end{aligned}$$

Therefore $\pi(t,T) = 1$ *and* $\omega(t,T) = 0$. *Applying theorem (5.1.1), the optimal portfolio consists of only of bonds and stocks if*

$$H(t,x) = \frac{1}{\left(1 - \dfrac{G_t}{p} x\right)^p}$$

and G_t *satisfies*

$$\chi_t^2 G_t + \eta_t - r_t + \int_0^t \int_{-\infty}^{+\infty} x\left(\left(1-\frac{G_t}{p}x\right)^{-p} - 1\right)\sigma_s(dx) = 0.$$

The fraction of wealth invested in stocks, at time t, *is given by*

$$\frac{\alpha_t S_{t_-}}{W_{t_-}} = -\frac{G_t}{p}.$$

If the equivalent martingale measure is such that the optimal portfolio cannot be hedged by bonds and stocks, then the number of shares of the power-jump assets is given by

$$\phi_t^j = \frac{W_{t_-}}{j! B_t} \frac{\partial}{\partial y^j} (H(t,y))^{-1/p} \Bigg|_{y=0}, \; j = 2,3,...,$$

Example 5.2.3 *Consider exponential utility function is given by*

$$U(x) = -\frac{1}{\beta} e^{-\beta x}$$

with $\beta \in (0,\infty)$ *. Then,* $U'(x) = e^{-\beta x}$, $(U')^{-1}(x) = -\frac{1}{m}\log x$ *and* $\frac{U'(x)}{U''(x)} = -\frac{1}{\beta}.$

Hence, we are in the case $m = 0$ *and* $n = -\frac{1}{\beta}$ *of (5.4). We have that*

$$E_{\mathrm{Q}}\left[\frac{B_t}{B_T} W_T | \mathcal{F}_t\right] = w_0 B_t + \frac{B_t}{\beta B_T}\left(E_{\mathrm{Q}}\left[\log \frac{dP_T}{d\mathrm{Q}_T} | \mathcal{F}_t\right] - E_{\mathrm{Q}}\left[\log \frac{dP_T}{d\mathrm{Q}_T}\right]\right)$$

$$= w_0 B_t + \frac{B_t}{\beta B_T}\left(E_{\mathrm{Q}}\left[\log \frac{dP_{T,t}}{d\mathrm{Q}_{T,t}} | \mathcal{F}_t\right] - E_Q\left[\log \frac{dP_{T,t}}{d\mathrm{Q}_{T,t}}\right]\right)$$

$$+ \log \frac{dP_t}{d\mathrm{Q}_t} - E_Q\left(\log \frac{dP_t}{d\mathrm{Q}_t}\right)$$

$$= w_0 B_t + \frac{B_t}{\beta B_T}\left(\log \frac{dP_t}{d\mathrm{Q}_t} - E_Q\left(\log \frac{dP_t}{d\mathrm{Q}_t}\right)\right)$$

$$= \frac{B_t}{B_T} W_t + w_0 B_t \left(1 - \frac{B_t}{B_T}\right).$$

Thus, in this case $\pi(t,T)=\dfrac{B_t}{B_T}$ *and* $\omega(t,T)=w_0B_t(1-B_t/B_T)$. *By (5.4), the optimal portfolio includes only bonds and stocks if* $H(y)=\exp\left(\dfrac{G}{k}y\right)$.

If this is the case, then G_t *satisfies the equation*

$$\chi_t^2 G_t+\eta_t-r_t+\int_0^t\int_{-\infty}^{+\infty}x\left(\exp\left(\frac{G}{k}y\right)-1\right)\sigma_s(dx)=0$$

and

$$\phi_t^1=-\frac{\pi(t,T)G_t}{\beta S_{t_-}}.$$

If the optimal portfolio includes power-jump assets, then the number of shares is given by

$$\phi_t^{(j)}=-\frac{B_t}{j!\beta}\frac{\partial^j}{\partial y^j}\log H(y)\bigg|_{y=0},\ j=2,3,...,$$

CHAPTER 6

CONCLUSION

In this study, the general geometric Additive market models are considered. These market models are generally incomplete, this means that, the perfect replication of derivatives, in the usual sense, is not possible. It is offered that the geometric Additive market should be enlarged by so called “power-jump assets” based on power-jump processes of the underlying Additive process. By using Martingale Representation Property for Additive processes, it is shown that the enlarged market is complete. After doing this, the hedging portfolios for claims whose payoff function depends on the prices of the stock and the power-jump assets at maturity are derived. In addition to the previous completion strategy, it is offered that the market should also be completed by considering portfolios with a continuum of call options with the same maturity and different strikes. What is more, the portfolio optimization problem is considered in the enlarged market. The optimization problem includes choosing an optimal portfolio in such a way that the largest expected utility of the terminal wealth is obtained. In our study, a class of special utility functions, including the HARA, logarithmic and exponential utilities are considered. Then, optimal portfolio that maximizes the terminal expected utility is obtained by the martingale method. It is shown that for particular choices of the equivalent martingale measure in the market, optimal portfolio consists of only of bonds and stocks. This includes the solution to the problem of utility maximization in the real market, consisting only of the bond and the stock.

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