OPEN BOOK DECOMPOSITIONS OF LINKS OF QUOTIENT SURFACE SINGULARITIES

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ABSTRACT

OPEN BOOK DECOMPOSITIONS OF LINKS OF QUOTIENT SURFACE SINGULARITIES

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In this thesis, we write explicitly the open book decompositions of links of quotient surface singularities that support the corresponding unique Milnor fillable contact structures. The page-genus of these Milnor open books are minimal among all Milnor open books supporting the corresponding unique Milnor fillable contact structures. That minimal page-genus is called Milnor genus. In this thesis we also investigate whether the Milnor genus is equal to the support genus for links of quotient surface singularities. We show that for many types of the quotient surface singularities the Milnor genus is equal to the support genus of the corresponding contact structure. For the remaining we are able to find an upper bound for the support genus which would be a step forward in understanding these contact structures.

Keywords: Quotient surface singularities, contact structures, open book decompositions, support genus

ÖZ

BÖLÜM YÜZEY TEKİLLİKLERİNİN DÜĞÜMLERİNİN AÇIK KİTAPLARI

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Bu tezde, bölüm yüzey tekilliklerinin düğümlerinin üzerindeki Milnor doldurulabilir kontakt yapıları destekleyen Milnor açık kitaplarını ayrıntılı olarak yazdık. Bu yazdığımız açık kitapların sayfaları, Milnor açık kitaplar arasında minimum sayfa-cinsine sahip olduğunu söyleyebiliyoruz. Bu minimum sayfa-cinsine Milnor cinsi denir. Bu tezde Milnor cinsinin, bölüm yüzey tekilliklerinin düğümleri için, destekleyen cinse eşit olup olmadığını da inceledik. Bölüm yüzey tekilliklerinin çoğu için Milnor cinsinin, destekleyen cinsine eşit olduğunu gösterdik. Diğerleri için ise, destekleyen cins icin birer üst sınır bulduk ki bunlar da ileride bu kontakt yapıları anlamada yardımcı birer adım olacaktır.

Anahtar Kelimeler: Bölüm yüzey tekillikleri, kontakt yapılar, açık kitaplar, destekleyen cins

To my family

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CHAPTER 1

INTRODUCTION

On a closed oriented 3-manifold, there is a one-to-one correspondence between contact structures (up to isotopy) and open book decompositions (up to positive stabilization/destabilization) by the result of Giroux, [12]. Using this correspondence, in their work [11], Etnyre and Özbağcı defined a new invariant, support genus. It is the minimal genus of a page of an open book supporting a fixed contact structure. There is not much information known about this new invariant.

One of the most important results about this invariant is proved in [9], although support genus of a contact structure is not defined at that time. The main theorem of [9] states that, if the symplectically fillable contact structure (M, ξ) is supported by an open book with planar (genus zero surface) pages, then the intersection form of its symplectic filling embeds in a negative definite diagonalizable quadratic form. By this result, Etnyre showed that the unique tight contact structure on Poincaré homology sphere cannot be supported by an open book with planar pages, hence its suport genus is at least one. Later, in their work [10], Etnyre and Özbağcı provide an open book, with pages genus one surface, supporting that unique tight contact structure. Therefore, it is the first and only known example of a contact structure with support genus one. In this thesis we tried to investigate the Milnor fillable contact structures on links of quotient surface singularities.

We describe open book decompositions of links of quotient surface singularities that support the corresponding unique Milnor fillable contact structures, using the construction given by Bhupal in his work [4]. For this purpose, in Chapter 2 first, we give definitions of contact structures and open book decompositions. Second, explain the surface singularities, and their links. We then describe the contact structures on these links. Next, we give the definitions of Milnor open books, Milnor fillable contact structures and state the theorems that we need to say that the Milnor open books of links of quoteint surface singularities that we construct supports the corresponding unique Milnor fillable contact structures.

In Chapter 3, we first explain the quotient surface singularities and then draw the minimal resolution graph of each type of quotient surface singlarity, which is investigated by Bhupal and Ono in detail in [5]. Next, we construct the Milnor open book decompositions, for the corresponding fundamental cycle, of the links of quotient surface singularities supporting the corresponding unique Milnor fillable contact structures. By the work of Bhupal and Altınok [1] and later by Némethi and Tosun [18], we conclude that the page-genus of these Milnor open books are minimal among all Milnor open books supporting the corresponding unique Milnor fillable contact structures. The question is whether are these give the support genus, i.e is it minimal over all open books? Most of these turn out to be planar open books, so we are able to conclude that, for these types, the support genus is zero. The pages of the open books of others are genus one surfaces, hence the support genus of the corresponding contact structure of each is atmost one. Moreover, in Section 3.3.2 we show that, for some of the types of quotient singularities this minimal page-genus is the support genus of the corresponding contact structure. Hence we have the following theorem according to the graphs given in Section 3.1:

Theorem 3.1 *The unique Milnor fillable contact structures on the links of quotient surface singularities have support genus zero for singularities of type*

- *cyclic,*
- *dihedral where* $b_r > 2$*,*
- *tetrahedral where* $b > 2$*,*
- *octahedral where* $b > 2$*,*
- *icosahedral where* $b > 2$

and have support genus one for

- *tetrahedral part (a) where* $b = 2$,
- *octahedral part (a) where* $b = 2$,
- *icosahedral part (a) and (b) where* $b = 2$.

For the remaining cases, the corresponding contact structures have support genus at most one.

CHAPTER 2

PRELIMINARIES

In this chapter we give some preliminary information. In Section 2.1 we define contact structures and contact manifolds. In Section 2.2 we give the definition of an open book decomposition and discuss the relation between contact structures and open book decompositions. In Section 2.3 we describe surface singularities and in Subsection 2.3.1 define the link of a surface singularity. In the last part of this section we define the contact structure on these links of surface singularities. In Section 2.4 we define a resolution of a singularity, its dual resolution graph, and give the relation between them. In the last section we define plumbed manifolds and explain the relation between a plumbed manifold and link of a surface singularity.

2.1 Contact structures

A **contact form** on a $2n+1$ manifold M is a 1-form α such that $\alpha \wedge (d\alpha)^n$ nowhere vanishes.

A **contact structure** on M is a hyperplane field $\xi \subset TM$ that can be written as the kernel of a contact form.

A contact manifold is $2n+1$ manifold equipped with a contact structure.

We will focus on contact 3 manifolds. We will assume that our contact structures are *cooriented* i.e. contact manifold (M, ξ) is oriented, the contact structure ξ is oriented and $\alpha \wedge (d\alpha)^n > 0$ (contact structure is positive).

2.2 Open Book Decompositions

An **open book decomposition** (or simply an **open book**) in a smooth (real) manifold M is a pair (B, θ) consisting of

- a proper submanifold B of codimension two in M with trivial normal bundle, so B admits neighbourhoods N diffeomorphic to $\mathbb{D}^2 \times B$ where $\mathbb{D}^2$ denotes the unit disc in $\mathbb{R}^2$ and B sits as $\{0\} \times B$ in $\mathbb{D}^2 \times B$,
- a locally trivial smooth fibration $\theta : M \setminus B \to \mathbb{S}^1$ such that there exists a neighbourhood N $(\mathbb{D}^2 \times B)$ of B in which θ is the normal angular coordinate, i.e., $\theta|_{N \setminus B} : N \setminus B \to \mathbb{S}^1$ is the composition of the projections

$$N \setminus B = \big(\mathbb{D}^2 \setminus \{0\}\big) \times B \to \mathbb{D}^2 \setminus \{0\} \to \mathbb{S}^1.$$

The submanifold B is called the **binding** of the open book and the compact surfaces $\theta^{-1}(t) \cup B$, for $t \in \mathbb{S}^1$, are called the **pages** of the open book.

Definition 2.1 *Two open books (B, θ) and (B', θ') in the manifolds M and M' respectively, are called isomorphic if there exists a diffeomorphism $\phi : (M, B) \to (M', B')$ which preserves the orientations and carries the fibres of θ to the fibres of θ'.*

There is a relation between contact structures and the open books given by Giroux [12].

Definition 2.2 *A contact structure ξ on M is supported by an open book (B, θ) of M (or (B, θ) supports ξ) if there is a contact 1-form α on M with $ker(\alpha) = \xi$ such that*

- *α induces a contact form on B (i.e. α is positive when restricted to B);*
- *$d\alpha$ induces a symplectic form on each fibre F of θ (i.e. $d\alpha$ is positive when restricted to the pages of (B, θ)).*

Theorem 2.3 ([12])

- *Every open book decomposition of a 3-manifold M supports a contact structure on M. Contact structures supported by the same open book decomposition are isotopic.*

- *Every contact structure on a 3-manifold M is supported by an open book decomposition of M. Any two open book decompositions supporting the same contact structure admit common positive stabilization.*

Using Giroux's theorem, in their work [11] Etnyre and Özbağcı defined an invariant for a fixed contact structure.

Definition 2.4 ([11]) *The support genus of a contact structure ξ on a 3-manifold M is the minimal genus of a page of an open book decomposition of M supporting ξ.*

2.3 Surface Singularities

Definition 2.5 *A surface singularity $(X, 0)$ is defined by:*

$$(X, 0) = (f_1 = \ldots = f_m = 0, 0) \subset (\mathbb{C}^N, 0),$$

where $f_i : (\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$ are germs of analytic functions with

$$r(p) = rank \left[\frac{\partial f_i}{\partial z_j}(p) \right]_{i=1\ldots,m;j=1,\ldots,N} = N - 2$$

for any generic (or smooth) point p of X. If $r(0) = N - 2$ then $(X, 0)$ is a smooth germ, i.e. analytically isomorphic to $(\mathbb{C}^2, 0)$. If $r(0) < N - 2$, but $r(p) = N - 2$ for all $p \in X \setminus 0$ then we say that $(X, 0)$ has an ***isolated singularity*** *at the origin.*

We now give the definition of a **link** of a surface singularity $(X, 0)$.

2.3.1 The Link of a Surface Singularity $(X, 0)$

Let $(X, 0)$ be a surface singularity, and let $(X, 0) \subset (\mathbb{C}^N, 0)$ be an arbitrary embedding. There exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, the $(2N-1)$-dimensional real sphere $S_\epsilon^{2N-1} = \left\{ z \in \mathbb{C}^N : |z| = \epsilon \right\}$ intersects $(X, 0)$ transversally (see [16]). For sufficiently small ϵ, the orientable 3-dimensional real manifold $M_X = X \cap S_\epsilon^{2N-1}$ is called the **link of** $(X, 0)$.

Examples:

1. $(X,0) = \left(\left\{x^2 + y^2 + z^n = 0\right\}, 0\right)$; the link of $(X,0) \subset (\mathbb{C}^3, 0)$ is $M_X = L(n, n-1)$, lens space.

2. $(X,0) = \left(\left\{x^2 + y^3 + z^5 = 0\right\}, 0\right)$; the link of $(X,0) \subset (\mathbb{C}^3, 0)$ is $M_X = PHS$, Poincaré Homology Sphere.

3. $(X,0) = \left(\left\{x^a + y^b + z^c = 0\right\}, 0\right)$ with $\gcd(a,b) = \gcd(b,c) = \gcd(c,a) = 1$; the link of $(X,0) \subset (\mathbb{C}^3, 0)$ is $M_X = \Sigma\,(a,b,c)$, integral homology sphere known as Seifert 3 manifold. Here $\gcd(a,b)$ is the greatest common divisor of a and b.

2.3.2 Milnor Fillable Contact Structure on The Link of $(X,0)$

The link M_X carries a natural contact structure ξ_X given by $\xi_X = TM_X \cap JTM_X$ where J denotes the complex structure on $\mathbb{C}^N$. Suppose that f is a germ of a holomorphic function vanishing at 0 that defines also an isolated singularity at 0. Then $N(f) = f^{-1}(0) \cap M_X$, $\theta(f) = argf : M_X \setminus N(f) \to S^1$ defines an open book decomposition of M_X (see [7]). In [7], the following theorem is proved:

Theorem 2.6 ([7]) *Each Milnor open book decomposition of M_X supports the natural contact structure ξ_X on M_X.*

A contact structure ξ on a 3 manifold M is said to be **Milnor fillable** if (M, ξ) is contactomorphic to a link M_X with its natural contact structure ξ_X for some germ $(X,0) \subset (\mathbb{C}^N, 0)$ of a normal complex analytic surface singularity. In [7], the main theorem establishes the uniqueness property.

Theorem 2.7 ([7]) *Any Milnor fillable 3 manifold admits a unique Milnor fillable contact structure up to contactomorphism.*

2.4 The Resolution and The Resolution Graph of a Singularity

Theorem 2.8 *Let $(X,0)$ be a germ of a normal complex analytic space with an isolated singularity at the origin, such that $X^* = X \setminus 0$ is non-singular. Then there exists a non-singular complex surface $\widetilde{X}$ and a proper analytic map $\pi : \widetilde{X} \to X$, such that*

- $E = \pi^{-1}(0)$ *is a divisor in* $\widetilde{X}$*, i.e., a union of* 1*-dimensional compact curves in* $\widetilde{X}$*; and*

- $\pi|_{\pi^{-1}(X^*)} : \widetilde{X} \setminus E \to X^*$ *is a biholomorphic map.*

This theorem first stated by Jung and first completed proof was given by Hirzebruch (see [3] chapter III, for a proof). In the theorem above, the surface $\widetilde{X}$ is called a **resolution** of $(X, 0)$ and the map $\pi : \widetilde{X} \to X$ is called the **resolution map.** The divisor E is called the **exceptional divisor.**

Given a resolution $\widetilde{X}$, one can obtain new resolutions by performing blow-ups at points in E. Hence the resolution of $(X, 0)$ is not unique. Given a resolution, one can make blow-ups on it, if necessary, so that the exceptional divisor E is **good**, i.e.,

- each irreducible component E_i of E is non-singular; and

- E has normal crossings, i.e., E_i intersects E_j, $i \neq j$, transversally, in at most one point, and no three of them intersect, i.e., $E_i \cap E_j \cap E_k = \emptyset$ for distinct indices i, j, k.

Definition 2.9 *A resolution* $\pi : \widetilde{X} \to X$ *is* ***good*** *if its exceptional divisor is good.*

Definition 2.10 *A resolution* $\pi : X \to V$ *is* ***minimal*** *if given any other resolution* $\pi' : X' \to V$*, there is a proper analytic map* $p : X' \to X$ *such that* $\pi' = \pi \circ p$*.*

We have the following theorem, about the uniqueness of minimal resolutions (see [3] chapter III, 6.2).

Theorem 2.11 *Up to isomorphism, there exists a unique minimal resolution of* X*, and this is characterized by not containing non-singular rational curves with self-intersection* -1 *i.e., there is no rational irreducible exceptional divisor* E_i *with self intersection* $E_i^2 = -1$*.*

A minimal resolution is not necessarily good however we can make it good by performing blow-ups. There is a unique (up to isomorphism) *minimal good resolution.*

Consider an exceptional divisor E and $E = \cup_{i=1}^{r} E_i$ be its decomposition in non-singular irreducible components, meeting transversally and no three of them intersect, in a complex 2 manifold X. Note that E need not to be good. One can associate an $r \times r$ matrix $A = (E_{ij})$ to the divisor E, called the *intersection matrix of* E, as follows. On the diagonal of E, put the self-intersection numbers E_i^2, called the *weights* of these curves; and if a curve E_i meets E_j at E_{ij} points, put this number to the entry E_{ij} (hence to the entry E_{ji} also) of A.

An intersection matrix can be realized as a resolution graph of a (complex analytic) normal surface singularity if and only if it is negative definite by the following results of Mumford and Grauert.

Theorem 2.12 ([17]) *If E is the exceptional divisor of a resolution $\pi : X \to V$, where V is a normal surface, then the intersection matrix A is negative definite (and the weights of the irreducible components of the exceptional divisors E_i are all negative numbers.)*

Theorem 2.13 ([13]) *Conversely if the divisor E in X is such that the intersection matrix A is negative definite, then we can blow down E analytically; we get a normal complex surface V, in general with a singularity at the image 0 of E, and the projection $\pi : X \to V$ is a good resolution of $(V, 0)$ with exceptional divisor E.*

2.4.1 Dual Graph of The Resolution

One can associate a weighted graph $G = G(E)$ to a good exceptional divisor E in a complex 2 manifold X as follows. To each irreducible component E_i of E we associate a vertex v_i, and if the curves E_i and E_j meet, then we join the vertices v_i and v_j by an edge. Note that these vertices are joined with an edge, since E is a good divisor. Each vertex has two integers attached to it. Usually firstly it is written the genus g_i of the corresponding Riemann surface E_i and if $g_i = 0$, it is usually omitted. The second one is the weight of E_i. This weighted graph is called the **dual graph of the resolution** where E is the exceptional set of a good resolution of a normal singularity.

Dual graph of a resolution allows us to reconstruct the topology of the resolution, hence the topology of the link of the singularity also. For this purpose we will introduce the *plumbing* construction.

2.5 Plumbed Manifold $S(\Gamma)$

Let X be a real 2-dimensional oriented vector bundle over a Riemann surface Σ, and let $D(X)$ denote its $\mathbb{D}^2$ bundle $\pi : D(X) \to \Sigma$. The total space of $D(X)$, that is denoted by the same symbol, is a 4-dimensional smooth manifold with boundary the $\mathbb{S}^1$ bundle $S(X)$. Consider two such bundles X_i over Riemann surfaces Σ_i for $i = 1, 2$. In order to perform plumbing, we consider the corresponding disc bundles $\pi_i : D(X_i) \to \Sigma_i$. Choose small discs D_i in Σ_i, hence $\pi_i^{-1}(D_i)$ is a trivial bundle $D_i \times \mathbb{D}^2$. Identify these by sending each point $(x, y) \in \pi_1^{-1}(D_1) = D_1 \times \mathbb{D}^2$ to $(y, x) \in \pi_2^{-1}(D_2) = D_1 \times \mathbb{D}^2$. The result is a 4-dimensional oriented manifold with boundary and corners which can be smoothed off. Denote this 4 manifold by $P(X_1, X_2)$.

The boundary of this 4 manifold, denote it by $S(X_1, X_2)$ is obtained by plumbing the circle bundles $S(X_1)$ and $S(X_2)$. Remove small discs D_i from the base space Σ_i, and remove the inverse images of these discs from the bundle. Hence we have two 3 manifolds with boundary $\mathbb{S}^1 \times \mathbb{S}^1$. Identify these boundaries by gluing the meridians in one torus to the longitude in the other one. The resulting is a 3 manifold with corners which can be smoothed off.

A finite connected graph consisting of r vertices where each vertex A_i is decorated with a pair of integers (g_i, e_i), $g_i \geq 0$ is called a **plumbing graph**.

The dual graph of a good resolution of a normal singularity is a plumbing graph with negative definite intersection matrix. Hence we denote both of them with Γ.

Given a plumbing graph Γ we may perform plumbing according to the graph. For each vertex A_i, take a Riemann surface Σ_i of genus-g_i and an oriented $\mathbb{D}^2$ bundle $\pi : D(X_i) \to \Sigma_i$ with Euler class e_i. If there is an edge between A_i and A_j, plumb these bundles as explained above. If there is more than one edge connecting A_i to another vertices, choose pair wise disjoint small discs, as many as the edges, in the surface Σ_i, and do the plumbing by pairs. The result is a 4 manifold $P(\Gamma)$ with boundary $S(\Gamma)$. A manifold obtained this way is called a plumbed manifold, and it is referred either to the 4 manifold or to its boundary. (We will call plumbed manifold to its boundary.)

If we have a plumbing graph Γ with negative definite intersection form hence we have a good resolution $\pi : \widetilde{X} \to X$ of a normal surface singularity with dual graph Γ, then $\widetilde{X}$ is diffeomorphic to $P(\Gamma)$ and $\partial\widetilde{X} = L_X$ is diffeomorphic to the plumbed 3 manifold $S(\Gamma)$.

Definition 2.14 *Let $S(\Gamma)$ be a plumbed 3 manifold, with plumbing graph Γ with vertices $A_1, \ldots, A_r$. To any r-tuple $\underline{n} = (n_1, \ldots, n_r)$ of non-negative integers, one can associate a link $N(\underline{n})$ in $S(\Gamma)$ as follows. For each i, consider n_i generic fibre of the circle bundle $\pi_i : S(X_i) \to \Sigma_i$. Then take their union for $i = 1, \ldots, r$. The link $N(\underline{n})$, constructed as above is called a vertical link.*

Remark 2.15 *The vertical link is oriented by the orientations of the fibres.*

Definition 2.16 *Let $S(\Gamma)$ be a plumbed 3 manifold. An open book in $S(\Gamma)$ whose binding is a vertical link, pages are transverse to the fibres of the circle bundles $S(X_i) \to \Sigma_i$, and compatible with the orientations, is called horizontal.*

Given a good resolution of a normal surface singularity, $\pi : \widetilde{X} \to X$ and consider its negative definite intersection matrix $I(\Gamma)$ associated to its dual graph Γ having r vertices. For this $r \times r$ matrix, we can find r-tuples of integers $\underline{m} = (m_1, \ldots, m_r)$ and $\underline{n} = (n_1, \ldots, n_r)$ such that

$$I(\Gamma)\underline{m}^t = -\underline{n}^t, \tag{2.1}$$

where $m_i \geq 1$ and $n_i \geq 0$ for all i.

Remark 2.17 *By the work of M. Artin, for any r-tuple $\underline{n}$ of nonnegative integers satisfying Equality (2.1), there is a Milnor open book of the link M_X whose binding is equivalent to a link of type $N(\underline{n})$, by Artin (see [2])*

Proposition 2.18 ([8]) *Let M be a rational homology sphere and N be an oriented link in M. If N is the binding of an open book decomposition of M, then that open book is unique up to isotopy.*

In the next chapter, for the link M_X of an isolated surface singularity X, we will construct the Milnor open book which supports the unique Milnor fillable contact structure, as it is done in [4]. We will construct an open book satisfying Equality (2.1) with the smallest positive $\underline{m}$. We say that $\underline{m} = (m_1, \ldots, m_r) \leq \underline{m'} = (m'_1, \ldots, m'_r)$ if $m_i \leq m'_i$ for all i. That particular $\underline{m}$ gives the *fundamental cycle* for the resolution π. Hence by Remark 2.17, we can say that there is a Milnor open book of M_X, whose binding is equivalent to a link of type $N(\underline{n})$. Then by Proposition 2.18 the open book we constructed is isotopic to that Milnor open book. By Theorems 2.6 and 2.7 that open book supports the corresponding unique Milnor fillable contact structure. It is shown by Bhupal and Altınok in [1], later by Némethi and Tosun in [18] that that the page-genus of the Milnor open book we find for the fundamental cycle is the smallest through all other Milnor open books supporting that Milnor fillable contact structure.

CHAPTER 3

MILNOR OPEN BOOKS; SUPPORT GENUS PROBLEM

In Section 3.1 we give all types of Quotient Singularities and then write their dual resolution graphs as it is investigated in [5]. In Section 3.3.1 we construct explicitly Milnor open books, for the corresponding fundamental cycles, of the links of quotient surface singularities which are supported by the Milnor fillable contact structures. In the last section we show that for some of these types of singularities these open books give the support genus number for the corresponding contact structures.

3.1 Quotient Singularities

We will consider germs of quotient singularities $(\mathbb{C}^2/G, 0)$, where G is a finite subgroup of $GL(2,\mathbb{C})$. A group $G \subset GL(2,\mathbb{C})$ is called *small* if it contains no reflection elements. By an important result of D. Prill (see [19]) every such quotient singularity is isomorphic to the quotient of $\mathbb{C}^2$ by a small subgroup of $GL(2,\mathbb{C})$. The small subgroups G_1 is conjugate to G_2 if and only if $(\mathbb{C}^2/G_1, 0)$ is analytically isomorphic to $(\mathbb{C}^2/G_2, 0)$. Therefore to classify quotient singularities $(\mathbb{C}^2/G, 0)$, it suffices to consider the small subgroups $G \subset GL(2,\mathbb{C})$ up to conjugation. In order to classify such groups $G \subset GL(2,\mathbb{C})$, it suffices to consider the finite subgroups of $SO(3)$ up to conjugation (see [5] for details). So G is conjugate to either a cyclic subgroup, dihedral subgroup, tetrahedral subgroup, octahedral subgroup or icosahedral subgroup.

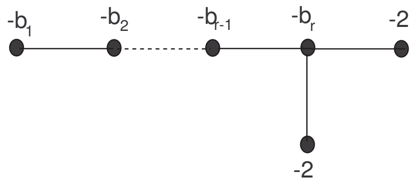
Quotient surface singularities can be divided into five families ([5]). Their minimal resolution graphs are given below.

1. **Cyclic Quotient singularities**: $A_{n,q}$, where $0 < q < n$ and $\gcd(n,q) = 1$. The minimal resolution of $A_{n,q}$ is given by where b_i are defined by

$$\frac{n}{q} = [b_1, b_2, \ldots, b_r] = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \frac{1}{b_r}}},$$

with $b_i \geq 2$ for all i.

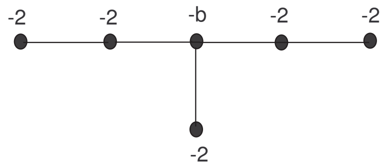
2. **Dihedral singularities**:



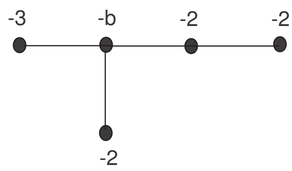
where $b_i \geq 2$ for all i.

3. **Tetrahedral singularities**:

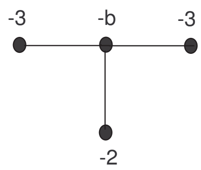
(a)



(b)



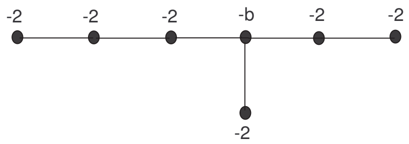
(c)



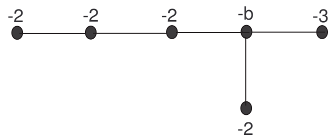
where $b \geq 2$ for all b.

4. **Octahedral singularities**:

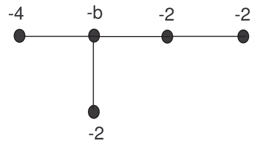
(a)



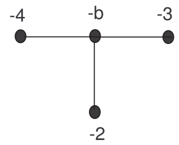
(b)



(c)



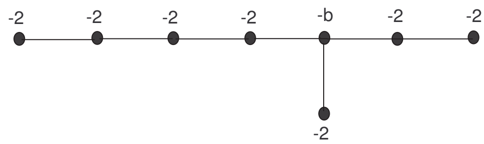
(d)



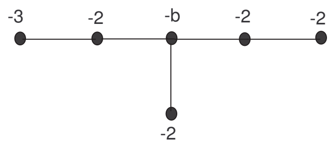
where $b \geq 2$ for all b.

5. **Icosahedral singularities**:

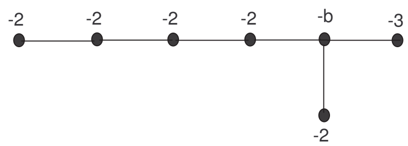
(a)



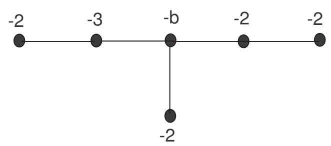
(b)



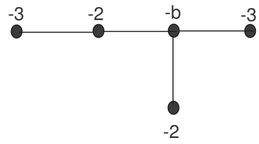
(c)



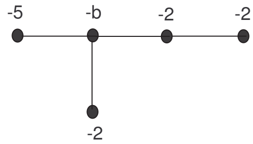
(d)



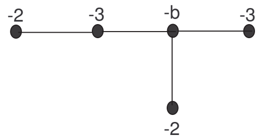
(e)



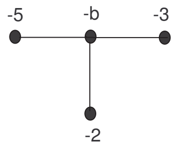
(f)



(g)



(h)



where $b \geq 2$ for all b.

3.2 Main Theorem

According to the dual graphs given above, we state the following theorem:

Theorem 3.1 *The unique Milnor fillable contact structures on the links of quotient surface singularities have support genus zero for singularities of type*

- *cyclic,*
- *dihedral where* $b_r > 2$*,*
- *tetrahedral where* $b > 2$*,*
- *octahedral where* $b > 2$*,*
- *icosahedral where* $b > 2$

and have support genus one for

- *tetrahedral part (a) where* $b = 2$*,*

- *octahedral part (a) where $b = 2$,*

- *icosahedral part (a) and (b) where $b = 2$.*

For the remaining cases, the corresponding contact structures have support genus at most one.

3.3 Proof of The Main Theorem

The minimal resolution graphs that occur for quotient surface singularities are given in Section 3.1. From these minimal resolution graphs we write the minimal page-genus Milnor open book decompositions of the links of the corresponding quotient surface singularity according to the construction given in [4].

3.3.1 Milnor Open Books on Links of Quotient Surface Singularities

1. **Cyclic quotient singularities**:

   Intersection matrix $I(\Gamma)$ for the Cyclic Quotient singularity is

$$\begin{bmatrix} -b_1 & 1 & 0 & \cdots & 0 \\ 1 & -b_2 & 1 & 0 & \cdots \\ & \ddots & \ddots & \ddots & \\ \cdots & 0 & 1 & -b_{r-1} & 1 \\ 0 & \cdots & 0 & 1 & -b_r \end{bmatrix}$$

   We construct the Milnor open book which supports the unique Milnor fillable contact structure, as it is done in [4]. We will construct an open book satisfying Equality (2.1) with the smallest positive $\underline{m}$.

   Consider r-tuple of integers $\underline{m} = (1, 1, \ldots, 1)$ which gives the fundamental cycle of the resolution.

$$\begin{bmatrix} -b_1 & 1 & 0 & \cdots & 0 \\ 1 & -b_2 & 1 & 0 & \cdots \\ & \ddots & \ddots & \ddots & \\ \cdots & 0 & 1 & -b_{r-1} & 1 \\ 0 & \cdots & 0 & 1 & -b_r \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} b_1 - 1 \\ b_2 - 2 \\ \vdots \\ b_{r-1} - 2 \\ b_r - 1 \end{bmatrix}$$

Page Σ of the open book associated to $\underline{m}$ consists of the following pieces. A collection of surfaces F_i, for $i = 1, \ldots, r$; an annulus U_t^i, for $i = 1, \ldots, r$, $t = 1, \ldots, n_i$, for each binding component of the open book; and a collection of annuli $U_l^{i,j}$, $l = 1, \ldots, \gcd(m_i, m_j)$ for each pair (i, j) with $1 \leq i < j \leq r$ such that $(A_i, A_j) \in \mathcal{E}$, where $\mathcal{E}$ denotes the set of edges of the graph Γ.

First we find F_i for $i = 1, \ldots, r$. For the vertex A_i of valency v_i, F_i is the m_i-cover of the sphere with $v_i + n_i$ boundary components. Let $g(F)$ denotes the genus of surface F. If $n_i > 0$, then F_i is connected and the genus of F_i can be calculated;

$$2 - 2g(F_i) - \sum_{(A_i, A_j) \in \mathcal{E}} \gcd(m_i, m_j) - n_i = m_i(2 - v_i - n_i) \tag{3.1}$$

$$genus(F_i) = 1 + \frac{(v_i + n_i - 2)\, m_i - \sum_{(A_i, A_j) \in \mathcal{E}} \gcd(m_i, m_j) - n_i}{2}. \tag{3.2}$$

If $n_i = 0$ then F_i has $d_i = \gcd\left(\{m_i\} \cup \{m_j |\, (A_i, A_j) \in \mathcal{E}\}\right)$ components and the genus of these components F_i^s, $s = 1, \ldots, d_i$ can be calculated similarly.

$$genus(F_i^s) = 1 + \frac{(v_i - 2)\, m_i/d_i - \sum_{(A_i, A_j) \in \mathcal{E}} \gcd(m_i, m_j)/d_i}{2}. \tag{3.3}$$

From Equations (3.2) and (3.3), we find the genus of these surfaces. Observe that in the Cyclic case, $m_i = 1$ for all i so, these two equations becomes the same in both cases ($n_i > 0$ and $n_i = 0$). Hence we find that $genus(F_i) = 0$ for all i. Number of boundary components of F_i is $n_i + \sum_{(A_i, A_j) \in \mathcal{E}} \gcd(m_i, m_j)$. Therefore each F_i is a sphere with b_i boundary components.

Next, we glue the annuli to the F_i's. We have only one annulus $U_1^{i,j}$ for each edge of the graph (connecting the surfaces F_i and F_i). There are n_i annuli U_t^i which are not used to plumbed the surfaces, and hence will give the binding

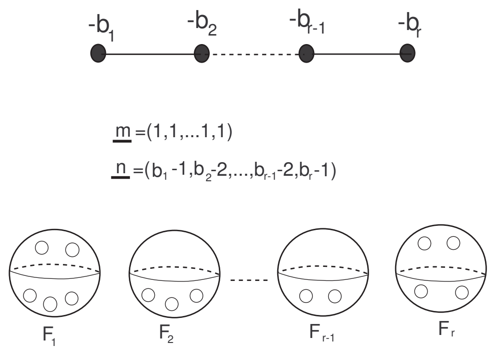


Figure 3.1: F_i is a sphere with b_i boundary components

components of the open book. As seen from Figure 3.2, page Σ is a sphere with N boundary components, where $N = b_1 + b_2 + \cdots + b_r - 2(r-1)$.

In order to find the monodromy ϕ, we only consider $\phi|_{U_t^i}$ and $\phi|_{U_l^{i,j}}$. Then we will take their composition. From [4], we know that $\phi|_{U_t^i}^{m_i} = t_{\delta_i}$ for $i = 1, \ldots, N$, where δ_i are the core circles of U_t^i and hence they are the boundary components of the page Σ. We now find the monodromy restricted to each annulus $U_l^{i,j}$. We know that $\phi|_{U_l^{i,j}}^{m_i m_j / \gcd(m_i, m_j)} = t_{c_i}$, where c_i is the core of the annulus $U_l^{i,j}$ (see Figure 3.2). Hence, in this case we have
$\phi|_{U_j^1} = t_{\delta_j^1}$ for $j = 1, \ldots, b_1 - 1$, $\phi|_{U_j^i} = t_{\delta_j^i}$ for $i = 2, \ldots, r-1$ and $j = 1, \ldots, b_i - 2$, $\phi|_{U_j^r} = t_{\delta_j^r}$ for $j = 1, \ldots, b_r - 1$, $\phi|_{U_1^{i,i+1}} = t_{c_i}$ for $i = 1, \ldots, r-1$. By composing them all, the monodromy is

$$\phi = \left(t_{\delta_1^1} \ldots t_{\delta_{b_1-1}^1}\right) \ldots \left(t_{\delta_1^i} \ldots t_{\delta_{b_i-2}^i}\right) \ldots \left(t_{\delta_1^r} \ldots t_{\delta_{b_r-1}^r}\right) \left(t_{c_1} \ldots t_{c_{r-1}}\right),$$

where $i = 2, \ldots, r-1$.

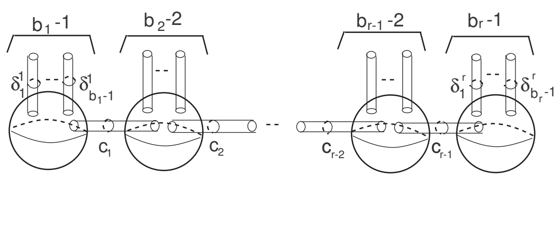


Figure 3.2: Page Σ for Cyclic Quotient Singularities

2. **Dihedral singularities**:

Intersection matrix $I(\Gamma)$ for the Dihedral singularity is

$$
\begin{bmatrix}
-b_1 & 1 & 0 & \cdots & & & \\
1 & -b_2 & 1 & 0 & \cdots & & \\
 & \ddots & \ddots & \ddots & & & \\
\cdots & 0 & 1 & -b_{r-1} & 1 & 0 & 0 \\
 & \cdots & 0 & 1 & -b_r & 1 & 1 \\
 & & \cdots & 0 & 1 & -2 & 0 \\
 & & \cdots & 0 & 1 & 0 & -2
\end{bmatrix}
$$

We investigate Dihedral singularities in three cases.

- Case 1: $b_r > 2$

$$
\begin{bmatrix}
-b_1 & 1 & 0 & \cdots & & & \\
1 & -b_2 & 1 & 0 & \cdots & & \\
 & \ddots & \ddots & \ddots & & & \\
\cdots & 0 & 1 & -b_{r-1} & 1 & 0 & 0 \\
 & \cdots & 0 & 1 & -b_r & 1 & 1 \\
 & & \cdots & 0 & 1 & -2 & 0 \\
 & & \cdots & 0 & 1 & 0 & -2
\end{bmatrix}
\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
= -
\begin{bmatrix} b_1 - 1 \\ b_2 - 2 \\ \vdots \\ b_{r-1} - 2 \\ b_r - 3 \\ 1 \\ 1 \end{bmatrix}
$$

In this case the fundamental cycle is $\underline{m} = (1, \ldots, 1)$. Then it is clear from the construction that we have a planar open book with $N = b_1 + b_2 + \cdots + b_r - 2(r-1)$ boundary components (see Figure 3.3).

Now we write the monodromy restricted to each annulus, then composing them all, we find the monodromy of the open book decomposition. $\phi|_{U_j^1} = t_{\delta_j^1}$ for $j = 1, \ldots, b_1 - 1$, $\phi|_{U_j^i} = t_{\delta_j^i}$ for $i = 2, \ldots, r-1$ and $j = 1, \ldots, b_i - 2$, $\phi|_{U_j^r} = t_{\delta_j^r}$ for $j = 1, \ldots, b_r - 3$, $\phi|_{U_1^{r+1}} = t_{\delta_1^{r+1}}$ and $\phi|_{U_1^{r+2}} = t_{\delta_1^{r+2}}$, $\phi|_{U_1^{i,i+1}} = t_{c_i}$ for $i = 1, \ldots, r-1$, $\phi|_{U_1^{r,r+1}} = t_{\delta_1^{r+1}}$ and $\phi|_{U_1^{r,r+2}} = t_{\delta_1^{r+2}}$. Hence the total monodromy ϕ is

$$
\phi = \left(t_{\delta_1^1} \ldots t_{\delta_{b_1-1}^1}\right) \ldots \left(t_{\delta_1^i} \ldots t_{\delta_{b_i-2}^i}\right) \ldots \left(t_{\delta_1^r} \ldots t_{\delta_{b_r-3}^r}\right) \left(t_{\delta_1^{r+1}}\right)^2 \left(t_{\delta_1^{r+2}}\right)^2 \left(t_{c_1} \ldots t_{c_{r-1}}\right),
$$

where $i = 2, \ldots, r-1$.

- Case 2: $b_r = 2$ and $b_{r-1} > 2$

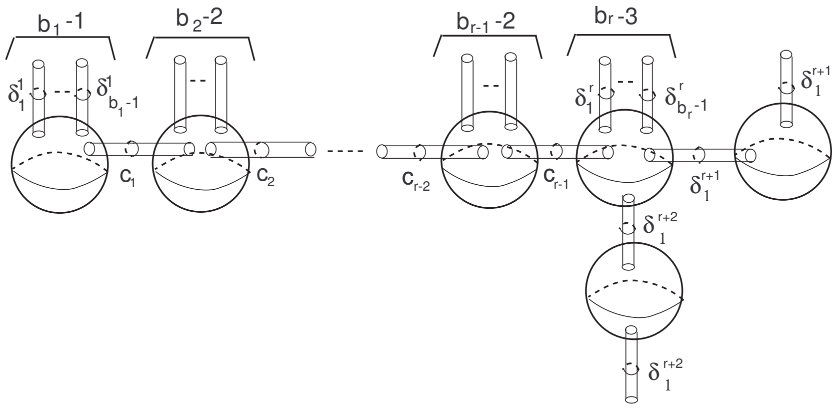


Figure 3.3: Page Σ for Dihedral Singularities Case 1

$$\begin{bmatrix} -b_1 & 1 & 0 & \cdots & & & \\ 1 & -b_2 & 1 & 0 & \cdots & & \\ & \ddots & \ddots & \ddots & & & \\ \cdots & 0 & 1 & -b_{r-1} & 1 & 0 & 0 \\ & \cdots & 0 & 1 & -2 & 1 & 1 \\ & & \cdots & 0 & 1 & -2 & 0 \\ & & \cdots & 0 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} b_1 - 1 \\ b_2 - 2 \\ \vdots \\ b_{r-1} - 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The surface F_r is a torus with a puncture and, all the other surfaces F_i where $i \neq r$ is a sphere with punctures. There are annuli U_t^i for joining the surfaces F_i and F_j, and $U_l^{i,j}$ that will be the binding components, see Figure 3.4. At the end we have a genus-1 surface with $N = b_1 + \cdots + b_{r-1} - 2r + 3$ boundary components as a page.

We now write the monodromy restricted to each annulus, then composing them all, we find the monodromy of the open book decomposition.
$\phi|_{U_j^1} = t_{\delta_j^1}$ for $j = 1, \ldots, b_1 - 1$, $\phi|_{U_j^i} = t_{\delta_j^i}$ for $i = 2, \ldots, r-2$ and $j = 1, \ldots, b_i - 2$, $\phi|_{U_j^{r-1}} = t_{\delta_j^{r-1}}$ for $j = 1, \ldots, b_{r-1} - 3$ and $\left(\phi|_{U_1^r}\right)^2 = t_{\delta_1^r}$, $\phi|_{U_1^{i,i+1}} = t_{c_i}$ for $i = 1, \ldots, r-2$ and $\left(\phi|_{U_1^{r-1,r}}\right)^2 = t_{c_{r-1}}$.
Combining these, we have

$$\phi = \left(t_{\delta_1^1} \ldots t_{\delta_{b_1-1}^1}\right) \ldots \left(t_{\delta_1^i} \ldots t_{\delta_{b_i-2}^i}\right) \ldots \left(t_{\delta_1^{r-1}} \ldots t_{\delta_{b_{r-1}-3}^{r-1}}\right) \left(t_{c_1} \ldots t_{c_{r-2}}\right) \left(t_{c_{r-1}} t_{\delta_1^r}\right)^{1/2};$$

Note that the surface F_r with the annuli $U_1^{r,r+1}$ and U_1^r is a torus with two boundary compenets (namely c_{r-1} and δ_1^r). We use Relation (A.7)

$$(t_{\alpha_1} t_{\alpha_2} t_\beta)^4 = t_{c_{r-1}} t_{\delta_1^r};$$

Using this relation, the total monodromy ϕ is

$$\phi = (t_{\alpha_1} t_{\alpha_2} t_\beta)^2 \left(t_{\delta_1^1} \dots t_{\delta_{b_1-1}^1} \right) \dots \left(t_{\delta_1^i} \dots t_{\delta_{b_i-2}^i} \right) \dots \left(t_{\delta_1^{r-1}} \dots t_{\delta_{b_{r-1}-3}^{r-1}} \right) \left(t_{c_1} \dots t_{c_{r-2}} \right)$$

where $i = 2, \dots, r-2$.

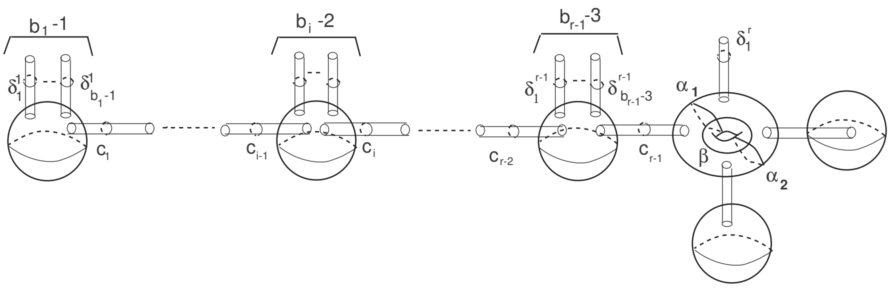


Figure 3.4: Page Σ for Dihedral Singularities Case 2

- Case 3: $b_r = b_{r-1} = \ldots = b_i = 2$ and (if applicaple) $b_{i-1} > 2$

$$\begin{bmatrix}
-b_1 & 1 & 0 & \cdots & & & & & & \\
1 & -b_2 & 1 & 0 & \cdots & & & & & \\
 & \ddots & \ddots & \ddots & & & & & & \\
\cdots & 0 & 1 & -b_{i-1} & 1 & 0 & \cdots & & & \\
 & \cdots & 0 & 1 & -2 & 1 & 0 & \cdots & & \\
 & & & & \ddots & \ddots & \ddots & & & \\
 & & & \cdots & 0 & 1 & -2 & 1 & 0 & 0 \\
 & & & & \cdots & 0 & 1 & -2 & 1 & 1 \\
 & & & & & \cdots & 0 & 1 & -2 & 0 \\
 & & & & & \cdots & 0 & 1 & 0 & -2
\end{bmatrix}
\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 2 \\ \vdots \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}
= -
\begin{bmatrix} b_1 - 1 \\ b_2 - 2 \\ \vdots \\ b_{i-1} - 3 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

As it can be seen from Figure 3.5, each F_i is a sphere with boundary components c_{i-1} and δ_1^i. However by gluing each annulus, the page is a genus-1 surface with $N = b_1 + \cdots + b_{i-1} - 2i + 3$ boundary components. $\phi|_{U_j^1} = t_{\delta_j^1}$ for $j = 1, \dots, b_1 - 1$, $\phi|_{U_j^k} = t_{\delta_j^k}$ for $j = 2, \dots, i-2$ and

$j = 1,\dots,b_k-2$, $\phi|_{U_j^{i-1}} = t_{\delta_j^{i-1}}$ for $j = 1,\dots,b_{i-1}-3$ and $\left(\phi|_{U_1^i}\right)^2 = t_{\delta_1^i}$ $\phi|_{U_1^{j,j+1}} = t_{c_j}$ for $j = 1,\dots,i-2$ and $\left(\phi|_{U_1^{i-1,i}}\right)^2 = t_{c_{i-1}}$, $\left(\phi|_{U_1^{j,j+1}}\right)^2 = t_\alpha$ and $\left(\phi|_{U_2^{j,j+1}}\right)^2 = t_\alpha$ for $j = i,\dots,r-1$.
Combining these we have

$$\phi = \left(t_{\delta_1^1}\dots t_{\delta_{b_1-1}^1}\right)\dots\left(t_{\delta_j^k}\dots t_{\delta_{b_k-2}^k}\right)\dots\left(t_{\delta_1^{i-1}}\dots t_{\delta_{b_{i-1}-3}^{i-1}}\right)\left(t_{c_1}\dots t_{c_{i-2}}\right)\left(t_{c_{i-1}}t_{\delta_1^i}\right)^{1/2}.$$

We can use Relation (A.7) for the torus with two boundary components (namely c_{i-1} and δ_1^i). Hence monodromy of the open book is

$$\phi = \left(t_{\delta_1^1}\dots t_{\delta_{b_1-1}^1}\right)\dots\left(t_{\delta_j^k}\dots t_{\delta_{b_k-2}^k}\right)\dots\left(t_{\delta_1^{i-1}}\dots t_{\delta_{b_{i-1}-3}^{i-1}}\right)\left(t_{c_1}\dots t_{c_{i-2}}\right)\left(t_{\alpha_1}t_{\alpha_2}t_\beta\right)^2$$

where $k = 2,\dots,i-2$.

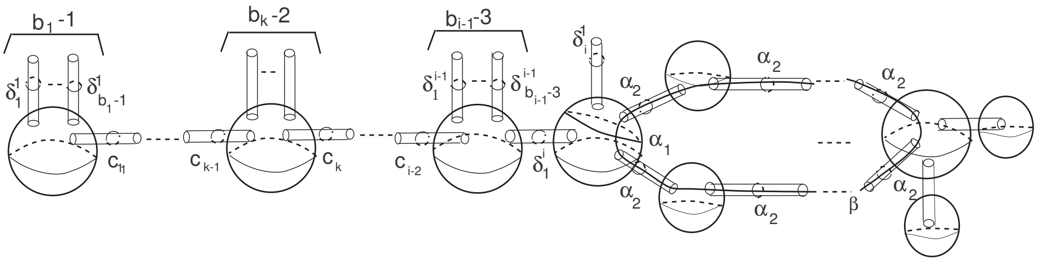


Figure 3.5: Page Σ for Dihedral Singularities Case 3

3. **Tetrahedral singularities**:

   (a) Tetrahedral singularity of type (a) has an intersection matrix $I(\Gamma)$

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix}$$

   We investigate this type of singularity in two cases.

   - Case 1: if $b = 2$

This is the singularity of type E_6 and it was investigated in [4]. Consider the fundamental cycle $\underline{m} = (1,2,3,2,1,2)$. Then from Equation (2.1) we have $\underline{n} = (0,0,0,0,0,1)$.

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \\ 2 \end{bmatrix} = - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Page Σ of the open book associated to $\underline{m}$ is a torus with one boundary component, built up as union of the central torus with three punctures and the annuli. In order to find the monodromy, we now write the monodromy restricted to those surfaces.

$\left(\phi|_{U_1^{3,6}}\right)^6 = t_\delta$ and $\left(\phi|_{U_1^6}\right)^2 = t_\delta$.

$\phi = (t_\delta)^{\frac{1}{6}+\frac{1}{2}} = (t_\delta)^{\frac{2}{3}}$.

Page Σ is a torus with one boundary component (namely δ) so we can use Relation (A.9)

$$(t_\alpha t_\beta)^6 = t_\delta.$$

The next theorem was proved by C. Bonatti and L. Paris (c.f. [6], Theorem 3.6). It will be useful for us in writing the roots of elements in the mapping class group $MCG(\Sigma)$ of a torus with boundary Σ. Here $MCG(\Sigma)$ is defined to be the group of isotopy classes of self-diffeomorphisms of Σ. Diffeomorphisms and isotopies of Σ are assumed to be the identity on the boundary of Σ.

Theorem 3.2 *If Σ is a torus with non-empty boundary components, then each element f in $MCG(\Sigma)$ has at most one m-root up to conjugation for all $m \geq 1$.*

Using Relation (A.9) and Theorem 3.2, monodromy ϕ of the open book is

$$\phi = (t_\delta)^{\frac{2}{3}} = (t_\alpha t_\beta)^4.$$

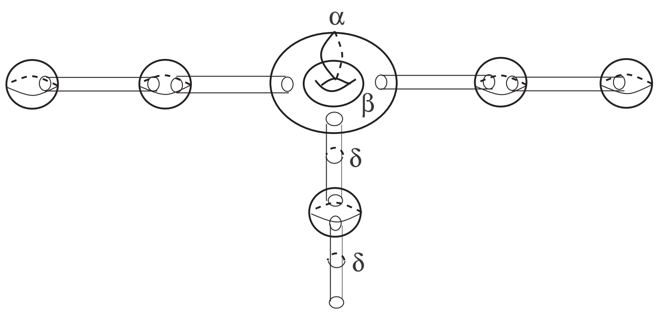


Figure 3.6: Page Σ for Tetrahedral Singularities of type (a) Case 1.

- Case 2: if $b > 2$

  Similarly we write page Σ of the open book associated to the fundamental cycle $\underline{m} = (1, 1, 1, 1, 1, 1)$.

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 0 \\ b-3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

  As it can be seen from Figure 3.7, Σ is a sphere with b boundary components, built up as union of the central sphere with b boundary components, five spheres with two boundary components and $b + 5$ annuli. To find the monodromy, we look at the monodromy restricted to each annulus.
  $\phi|_{U_1^{1,2}} = t_{\delta_1^1}$, $\phi|_{U_1^{2,3}} = t_{\delta_1^1}$, $\phi|_{U_1^{3,4}} = t_{\delta_1^5}$, $\phi|_{U_1^{4,5}} = t_{\delta_1^5}$, $\phi|_{U_1^{3,6}} = t_{\delta_1^6}$.
  $\phi|_{U_1^1} = t_{\delta_1^1}$, $\phi|_{U_i^3} = t_{\delta_i^3}$ for $i = 1, \ldots, b-3$, $\phi|_{U_1^5} = t_{\delta_1^5}$, $\phi|_{U_1^6} = t_{\delta_1^6}$.
  Hence, the monodromy ϕ of the open book is

$$\phi = \left(t_{\delta_1^1}\right)^3 \left(t_{\delta_1^3} \ldots t_{\delta_{b-3}^3}\right) \left(t_{\delta_1^5}\right)^3 \left(t_{\delta_1^6}\right)^3.$$

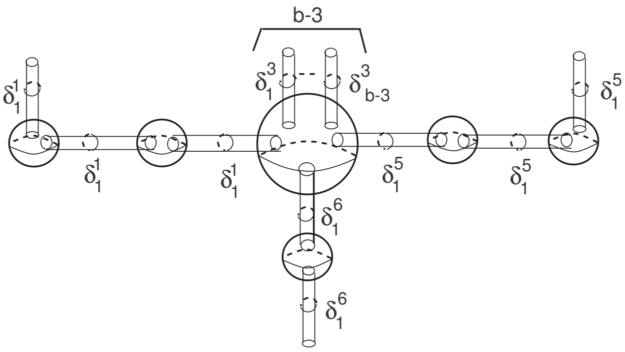


Figure 3.7: Page Σ for Tetrahedral Singularities of type (a) Case 2.

(b) Intersection matrix $I(\Gamma)$ for Tetrahedral singularity of type (b) is

$$\begin{bmatrix} -3 & 1 & 0 & 0 & 0 \\ 1 & -b & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & -2 \end{bmatrix}$$

- Case 1: if $b = 2$

  The fundamental cycle for this resolution is $\underline{m} = (1, 2, 2, 1, 1)$.

$$\begin{bmatrix} -3 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

  As it can be seen from Figure 3.8, the fiber Σ of the open book associated to $\underline{m}$ is a sphere with b boundary components, built up as union of the central sphere with four boundary components, a sphere with four boundary components, two spheres with one boundary component, a sphere with two boundary components, and seven annuli, two of which, gives the binding components. As can be seen from Figure 3.8, after gluing these alltogether we end up with Σ being a genus-1 surface with two boundary components (namely δ_1 and δ_2).

  $\phi|_{U_1^1} = t_{\delta_1}$, $\left(\phi|_{U_1^{1,2}}\right)^2 = t_{\delta_1}$, $\phi|_{U_1^3} = t_{\delta_2}$, $\left(\phi|_{U_1^{2,3}}\right)^2 = t_{\alpha_1}$ and,

$\left(\phi|_{U_2^{2,3}}\right)^2 = t_{\alpha_2}.$
Hence, the monodromy ϕ of the open book is

$$\phi = t_{\delta_1} \left(t_{\delta_1} t_{\delta_2} t_{\alpha_1} t_{\alpha_2}\right)^{1/2}.$$

By using Relation (A.8) and Theorem 3.2

$$\phi = t_{\delta_1} \left(t_{\alpha_1} t_{\alpha_2} t_\beta t_{\alpha_1} t_{\alpha_2} t_\beta t_{\alpha_2}\right).$$

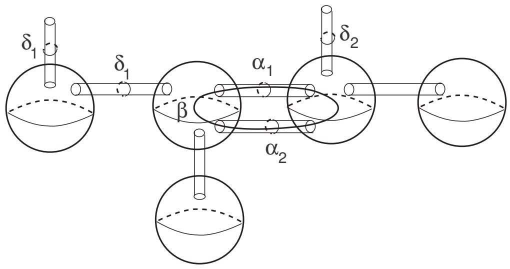


Figure 3.8: Page Σ for Tetrahedral Singularities of type (b) Case 1.

- Case 2: if $b > 2$

  The fundamental cycle for this case is $\underline{m} = (1, 1, 1, 1, 1)$.

$$\begin{bmatrix} -3 & 1 & 0 & 0 & 0 \\ 1 & -b & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 2 \\ b-3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

As it can be seen from Figure 3.9,page Σ of the open book associated to $\underline{m}$ is a sphere with $b+1$ boundary components, built up as union of the central sphere with b boundary components, and four other spheres corresponding to the other vertices, and annuli. As it can be seen from Figure 3.9 after gluing these we end up with Σ being a sphere with $b + 1$ boundary components. To find the monodromy, we look at the monodromy restricted to each annulus.
$\phi|_{U_1^1} = t_{\delta_1^1}$, $\phi|_{U_2^1} = t_{\delta_2^1}$, $\phi|_{U_i^2} = t_{\delta_i^2}$ for $i = 1, \ldots, b-3$, $\phi|_{U_1^4} = t_{\delta_1^4}$,

$\phi|_{U_1^5} = t_{\delta_1^5}$. $\phi|_{U_1^{1,2}} = t_{c_1}$, $\phi|_{U_1^{2,3}} = t_{\delta_1^4}$, $\phi|_{U_1^{2,5}} = t_{\delta_1^5}$, $\phi|_{U_1^{3,4}} = t_{\delta_1^4}$. Hence the monodromy ϕ of the open book is

$$\phi = t_{\delta_1^1} t_{\delta_2^1} \left(t_{\delta_1^2} \dots t_{\delta_{b-3}^2} \right) \left(t_{\delta_1^4} \right)^3 \left(t_{\delta_1^5} \right)^2 t_{c_1}.$$

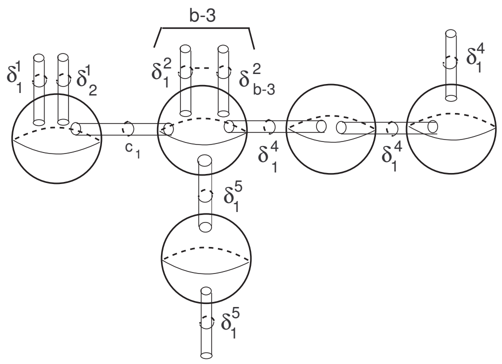


Figure 3.9: Page Σ for Tetrahedral Singularities of type (b) Case 2.

(c) Intersection matrix $I(\Gamma)$ for Tetrahedral singularity of type (c) is

$$\begin{bmatrix} -3 & 1 & 0 & 0 \\ 1 & -b & 1 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix}$$

- Case 1: if $b = 2$

  The fundamental cycle for this resolution is $\underline{m} = (1, 2, 1, 1)$.

$$\begin{bmatrix} -3 & 1 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

  Page Σ of the open book associated to $\underline{m}$ is a torus with three boundary components, built up as union of the central torus with four boundary components, two spheres with two boundary components, a sphere with one boundary component, and six annuli. After gluing these alltogether, we end up with Σ being a torus with three boundary components (see Figure 3.10). The monodromy restricted to each annulus is

$\phi|_{U_1^1} = t_{\delta_1}$, $\left(\phi|_{U_1^{1,2}}\right)^2 = t_{\delta_1}$, $\left(\phi|_{U_1^2}\right)^2 = t_{\delta_2}$, $\left(\phi|_{U_1^{2,3}}\right)^2 = t_{\delta_3}$, $\phi|_{U_1^3} = t_{\delta_3}$.
The monodromy ϕ of the open book is

$$\phi = t_{\delta_1} t_{\delta_3} \left(t_{\delta_1} t_{\delta_2} t_{\delta_3}\right)^{1/2},$$

using 3-holed torus relation (A.2) and Theorem 3.2, we can write the monodromy as

$$\phi = t_{\delta_1} t_{\delta_3} \left(t_{\alpha_1} t_{\alpha_3} t_{\beta} t_{\alpha_2} t_{\alpha_3} t_{\beta}\right)$$

where $\beta, \alpha_1, \alpha_2, \alpha_3$.

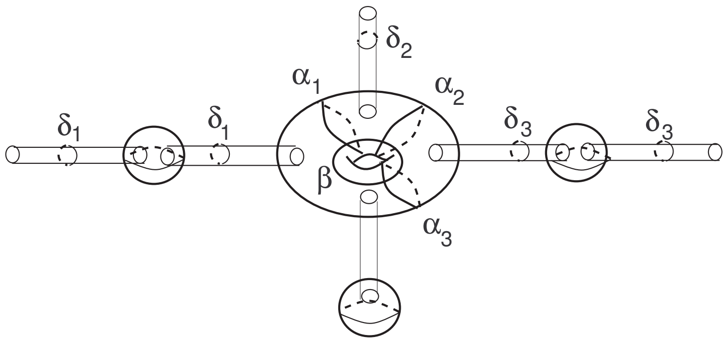


Figure 3.10: Page Σ for Tetrahedral Singularities of type (c) Case 1.

- Case 2: if $b > 2$

$$\begin{bmatrix} -3 & 1 & 0 & 0 \\ 1 & -b & 1 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 2 \\ b-3 \\ 2 \\ 1 \end{bmatrix}$$

Page Σ of the open book associated to the fundamental cycle $\underline{m} = (1,1,1,1)$ is a sphere with $b+2$ boundary components, built up as union of the central sphere with b boundary components, and three other spheres corresponding to the other vertices, and the annuli. As it can be seen from Figure 3.11 after gluing these we end up with Σ being a sphere with $b+2$ boundary components. Now we write the monodromy restricted to each annulus, to get the total monodromy of the open book.

$\phi|_{U_1^1} = t_{\delta_1^1}$, $\phi|_{U_2^1} = t_{\delta_2^1}$, $\phi|_{U_i^2} = t_{\delta_i^2}$ for $i = 1, \ldots, b-3$, $\phi|_{U_1^3} = t_{\delta_1^3}$, $\phi|_{U_2^3} = t_{\delta_2^3}$, $\phi|_{U_1^4} = t_{\delta_1^4}$, $\phi|_{U_1^{1,2}} = t_{c_1}$, $\phi|_{U_1^{2,3}} = t_{c_2}$, $\phi|_{U_1^{2,4}} = t_{\delta_1^4}$.
Hence, the monodromy ϕ of the open book is

$$\phi = t_{\delta_1^1} t_{\delta_2^1} \left(t_{\delta_1^2} \ldots t_{\delta_{b-3}^2} \right) t_{\delta_1^3} t_{\delta_2^3} \left(t_{\delta_1^4} \right)^2 t_{c_1} t_{c_2}.$$

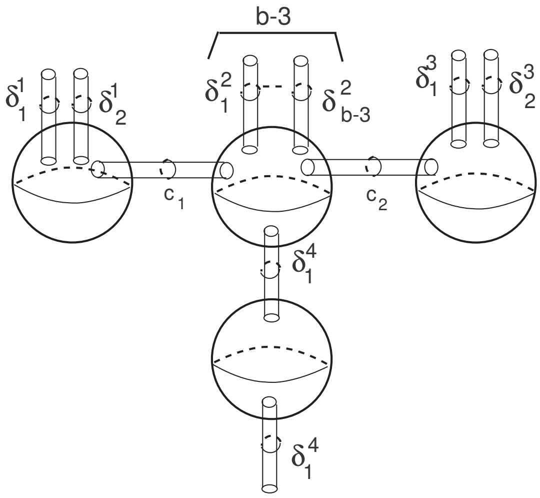


Figure 3.11: Page Σ for Tetrahedral Singularities of type (c) Case 2.

4. **Octahedral singularities**:

(a) This quotient surface singularity's intersection matrix $I(\Gamma)$ is

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -b & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix}$$

- Case 1: if $b = 2$

  This is the singularity of type E_7. Since this is a simple surface singularity, which is investigated by [4]. Consider the fundamental cycle

$\underline{m} = (1,2,3,4,3,2,2)$. Then we have $\underline{n} = (0,0,0,0,0,1,0)$ by Equation (2.1).

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 3 \\ 2 \\ 2 \end{bmatrix} = - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

We can now write page Σ of the open book associated to $\underline{m}$ which is a torus with one boundary component, built up as union of the central torus with four boundary components and the annuli. The monodromy restricted to each annulus can be written as
$\left(\phi|_{U_1^{4,5}}\right)^{12} = t_\delta$, $\left(\phi|_{U_1^{5,6}}\right)^{6} = t_\delta$, $\left(\phi|_{U_1^{6}}\right)^{2} = t_\delta$.
Hence $\phi = (t_\delta)^{1/12+1/6+1/2} = (t_\delta)^{3/4}$. Using Relation (A.9) and Theorem 3.2, we can get the monodromy

$$\begin{aligned} t_\delta &= (t_\alpha t_\beta)^6 \\ (t_\delta)^3 &= (t_\alpha t_\beta)^{18} \\ &= \left(t_\beta\,(t_\alpha t_\beta)^4\right)^4 \\ \phi &= t_\beta\,(t_\alpha t_\beta)^4\,. \end{aligned}$$

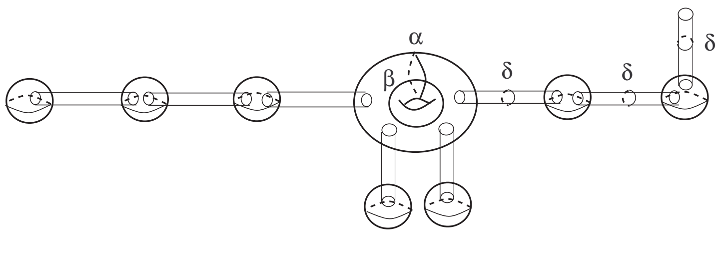


Figure 3.12: Page Σ for Octahedral Singularities of type (a) Case 1.

- Case 2: if $b > 2$

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -b & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 0 \\ 0 \\ b-3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Page Σ of the open book associated to the fundamental cycle $\underline{m} = (1,1,1,1,1,1,1)$ is a sphere with b boundary components, built up as union of the central sphere with b boundary components, six spheres with two boundary components and $b+6$ annuli (see Figure 3.13). We now find the monodromy restricted to each annulus.

$\phi|_{U_1^1} = t_{\delta_1^1}$, $\phi|_{U_1^{1,2}} = t_{\delta_1^1}$, $\phi|_{U_1^{2,3}} = t_{\delta_1^1}$, $\phi|_{U_1^{3,4}} = t_{\delta_1^1}$, $\phi|_{U_i^4} = t_{\delta_i^4}$ for $i = 1, \ldots, b-3$, $\phi|_{U_1^6} = t_{\delta_1^6}$, $\phi|_{U_1^{4,5}} = t_{\delta_1^6}$, $\phi|_{U_1^{5,6}} = t_{\delta_1^6}$, $\phi|_{U_1^7} = t_{\delta_1^7}$, $\phi|_{U_1^{4,7}} = t_{\delta_1^7}$.

Hence the monodromy of the open book is

$$\phi = \left(t_{\delta_1^1}\right)^4 \left(t_{\delta_1^4} \ldots t_{\delta_{b-3}^4}\right) \left(t_{\delta_1^6}\right)^3 \left(t_{\delta_1^7}\right)^2.$$

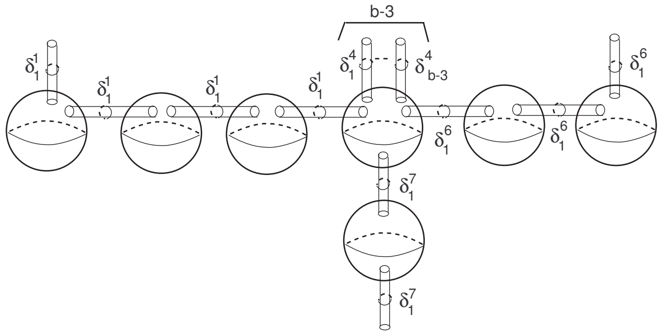


Figure 3.13: Page Σ for Octahedral Singularities of type (a) Case 2.

(b) Intersection matrix $I(\Gamma)$ for Octahedral singularity of type (b) is

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -b & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 \end{bmatrix}$$

- Case 1: if $b = 2$

  The fundamental cycle for this resolution is $\underline{m} = (1, 2, 2, 2, 1, 1)$.

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

  Page Σ of the open book associated to $\underline{m}$ is a torus with two boundary components (see Figure 3.14). The monodromy restricted to each annulus is
  $\left(\phi|_{U_1^2}\right)^2 = t_{\delta_1}$, $\left(\phi|_{U_1^{2,3}}\right)^2 = t_{\alpha_1}$, $\left(\phi|_{U_1^{3,4}}\right)^2 = t_{\alpha_1}$, $\left(\phi|_{U_2^{2,3}}\right)^2 = t_{\alpha_2}$, $\left(\phi|_{U_2^{3,4}}\right)^2 = t_{\alpha_2}$, $\phi|_{U_1^5} = t_{\delta_2}$, $\left(\phi|_{U_1^{4,5}}\right)^2 = t_{\delta_2}$,
  Hence the monodromy ϕ of the open book is

$$\phi = (t_{\delta_1})^{1/2} (t_{\delta_2})^{3/2} t_{\alpha_1} t_{\alpha_2},$$

  which can be written as

$$\phi = t_{\delta_2} t_{\alpha_1} t_{\alpha_2} (t_{\alpha_1} t_{\alpha_2} t_{\beta})^2$$

  by using Relation (A.7) and Theorem 3.2.

- Case 2: if $b > 2$

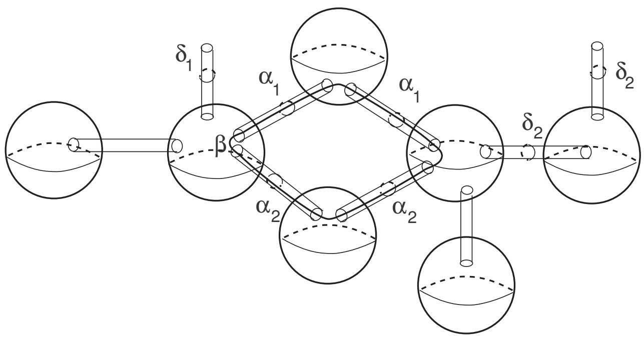


Figure 3.14: Page Σ for Octahedral Singularities of type (b) Case 1.

The fundamental cycle for this case is $\underline{m} = (1,1,1,1,1,1)$.

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -b & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 0 \\ 0 \\ b-3 \\ 2 \\ 1 \end{bmatrix}$$

As it can be seen in Figure 3.15, page Σ of the open book associated to $\underline{m}$ is a sphere with $b+1$ boundary components, built up as union of the central sphere with b boundary components, and five other spheres corresponding to the other vertices, and annuli. The monodromy restricted to each annulus is

$\phi|_{U_1^1} = t_{\delta_1^1}$, $\phi|_{U_1^{1,2}} = t_{\delta_1^1}$, $\phi|_{U_1^{2,3}} = t_{\delta_1^1}$, $\phi|_{U_1^{3,4}} = t_{\delta_1^1}$, $\phi|_{U_i^4} = t_{\delta_i^4}$ for $i = 1,\ldots,b-3$, $\phi|_{U_1^{4,5}} = t_{c_1}$, $\phi|_{U_1^5} = t_{\delta_1^5}$, $\phi|_{U_2^5} = t_{\delta_2^5}$, $\phi|_{U_1^{4,6}} = t_{\delta_1^6}$, $\phi|_{U_1^6} = t_{\delta_1^6}$.

Hence the monodromy of the open book is

$$\phi = \left(t_{\delta_1^1}\right)^4 \left(t_{\delta_1^4} \ldots t_{\delta_{b-3}^4}\right) t_{\delta_1^5} t_{\delta_2^5} \left(t_{\delta_1^6}\right)^2 t_{c_1}.$$

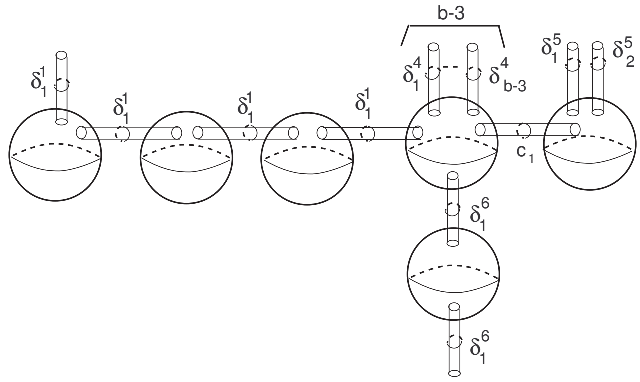


Figure 3.15: Page Σ for Octahedral Singularities of type (b) Case 2.

(c) Intersection matrix $I(\Gamma)$ for Octahedral singularity of type (c) is

$$\begin{bmatrix} -4 & 1 & 0 & 0 & 0 \\ 1 & -b & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & -2 \end{bmatrix}$$

- Case 1: if $b = 2$

  The fundamental cycle for this resolution is $\underline{m} = (1, 2, 2, 1, 1)$.

$$\begin{bmatrix} -4 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

  Page Σ of the open book associated to $\underline{m}$ is a torus with three boundary components, built up as union of five spheres and eight annuli. As it is clear from from 3.16 after gluing these we end up with Σ being a torus with three boundary components. The monodromy restricted to each annulus is

  $\phi|_{U_1^1} = t_{\delta_1}$, $\phi|_{U_2^1} = t_{\delta_2}$, $\left(\phi|_{U_1^{1,2}}\right)^2 = t_{c_1}$, $\left(\phi|_{U_1^{2,3}}\right)^2 = t_{\alpha_1}$, $\left(\phi|_{U_2^{2,3}}\right)^2 = t_{\alpha_2}$, $\left(\phi|_{U_1^3}\right)^2 = t_{\delta_3}$.

  The monodromy of the open book is

$$\phi = t_{\delta_1} t_{\delta_2} \left(t_{c_1} t_{\delta_3} t_{\alpha_1} t_{\alpha_2}\right)^{1/2}.$$

Since with c_1, δ_3 we have a torus with 2 boundary components and hence by using (A.8) and Theorem 3.2

$$\phi = t_{\delta_1} t_{\delta_2} \left(t_{\alpha_1} t_{\alpha_2} t_\beta t_{\alpha_1} t_{\alpha_2} t_\beta t_{\alpha_2} \right).$$

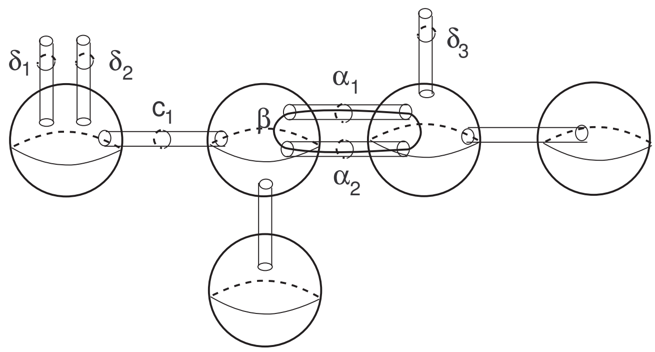


Figure 3.16: Page Σ for Octahedral Singularities of type (c) Case 1.

- Case 2: if $b > 2$

Fundamental cycle is given by $\underline{m} = (1,1,1,1,1)$.

$$\begin{bmatrix} -4 & 1 & 0 & 0 & 0 \\ 1 & -b & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 3 \\ b-3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Page Σ of the open book associated to $\underline{m}$ is a sphere with $b+2$ boundary components, built up as union of the central sphere with b boundary components, and four other spheres corresponding to the other vertices, and annuli. As it can be seen from Figure 3.17 after gluing these we end up with Σ being a sphere with $b+2$ boundary components. We find the monodromy restricted to each annulus.

$\phi|_{U_1^1} = t_{\delta_1^1}$, $\phi|_{U_2^1} = t_{\delta_2^1}$, $\phi|_{U_3^1} = t_{\delta_3^1}$, $\phi|_{U_i^2} = t_{\delta_i^2}$ for $i = 1, \ldots, b-3$, $\phi|_{U_1^4} = t_{\delta_1^4}$, $\phi|_{U_1^5} = t_{\delta_1^5}$, $\phi|_{U_1^{1,2}} = t_{c_1}$, $\phi|_{U_1^{2,3}} = t_{\delta_1^4}$, $\phi|_{U_1^{3,4}} = t_{\delta_1^4}$, $\phi|_{U_1^{2,5}} = t_{\delta_1^5}$.

Hence the monodromy ϕ of the open book is

$$\phi = t_{\delta_1^1} t_{\delta_2^1} t_{\delta_3^1} \left(t_{\delta_1^2} \ldots t_{\delta_{b-3}^2} \right) \left(t_{\delta_1^4} \right)^3 \left(t_{\delta_1^5} \right)^2 t_{c_1}.$$

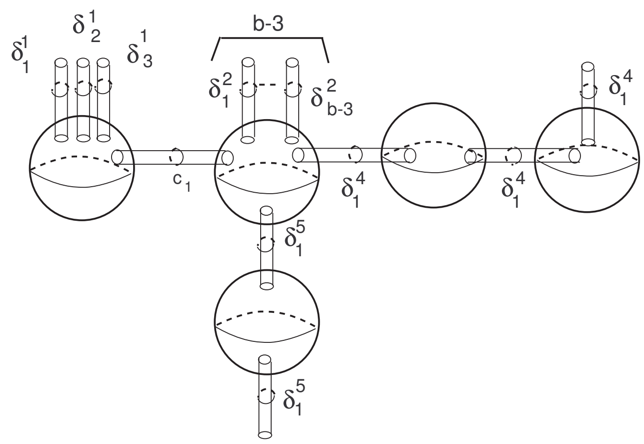


Figure 3.17: Page Σ for Octahedral Singularities of type (c) Case 2.

(d) Intersection matrix $I(\Gamma)$ for Octahedral singularity of type (d) is

$$\begin{bmatrix} -4 & 1 & 0 & 0 \\ 1 & -b & 1 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix}$$

- Case 1: if $b = 2$

  The fundamental cycle for this type of resolution is $\underline{m} = (1, 2, 1, 1)$.

$$\begin{bmatrix} -4 & 1 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

  Page Σ of the open book associated to $\underline{m}$ is built up as union of the central torus with four boundary components, and three other spheres corresponding to the other vertices, and annuli. As it can be seen from Figure 3.18 after gluing these we end up with Σ, a torus with four boundary components. We now find the monodromy restricted to each annulus.

  $\phi|_{U_1^1} = t_{\delta_1}$, $\phi|_{U_2^1} = t_{\delta_2}$, $\left(\phi|_{U_1^2}\right)^2 = t_{\delta_3}$, $\phi|_{U_1^3} = t_{\delta_4}$, $\left(\phi|_{U_1^{1,2}}\right)^2 = t_{c_1}$, $\left(\phi|_{U_1^{2,3}}\right)^2 = t_{\delta_4}$.

Hence the monodromy ϕ of the open book is

$$\phi = t_{\delta_1} t_{\delta_2} t_{\delta_4} \left(t_{c_1} t_{\delta_3} t_{\delta_4}\right)^{1/2}$$

which we can write equivalently

$$\phi = t_{\delta_1} t_{\delta_2} t_{\delta_4} \left(t_{\alpha_1} t_{\alpha_3} t_{\beta} t_{\alpha_2} t_{\alpha_3} t_{\beta}\right).$$

Since we can use 3-holed torus Relation (A.2) for the torus with boundary components c_1, δ_3 and δ_4 and Theorem 3.2.

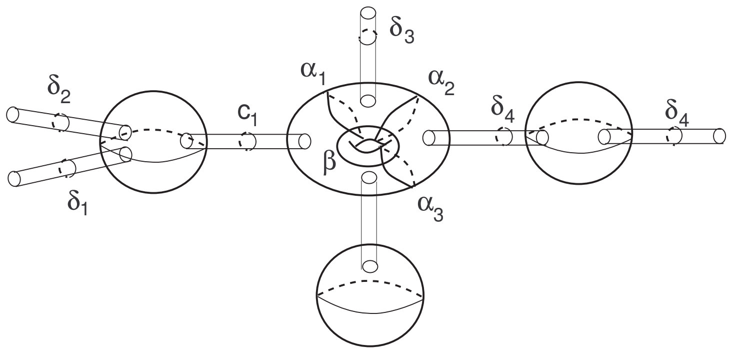


Figure 3.18: Page Σ for Octahedral Singularities of type d Case 1.

- Case 2: if $b > 2$

  The fundamental cycle for this case is $\underline{m} = (1, 1, 1, 1)$.

$$\begin{bmatrix} -4 & 1 & 0 & 0 \\ 1 & -b & 1 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 3 \\ b-3 \\ 2 \\ 1 \end{bmatrix}$$

  Page Σ of the open book associated to $\underline{m}$ is a sphere with $b+3$ boundary components, built up as union of the central sphere with b boundary components, and three other spheres corresponding to the other vertices, and annuli. As it can be seen from Figure 3.19 after gluing these we end up with Σ being a sphere with $b+3$ boundary components. The monodromy restricted to each annulus is

  $\phi|_{U_1^1} = t_{\delta_1^1}$, $\phi|_{U_2^1} = t_{\delta_2^1}$, $\phi|_{U_3^1} = t_{\delta_3^1}$, $\phi|_{U_i^2} = t_{\delta_i^2}$ for $i = 1, \ldots, b-3$, $\phi|_{U_1^3} = t_{\delta_1^3}$, $\phi|_{U_2^3} = t_{\delta_2^3}$, $\phi|_{U_1^4} = t_{\delta_1^4}$, $\phi|_{U_1^{1,2}} = t_{c_1}$, $\phi|_{U_1^{2,3}} = t_{c_2}$, $\phi|_{U_1^{2,4}} = t_{\delta_1^4}$.

Hence the monodromy ϕ of the open book is

$$\phi = t_{\delta_1^1} t_{\delta_2^1} t_{\delta_3^1} \left(t_{\delta_1^2} \dots t_{\delta_{b-3}^2} \right) t_{\delta_1^3} t_{\delta_2^3} \left(t_{\delta_1^4} \right)^2 t_{c_1} t_{c_2}.$$

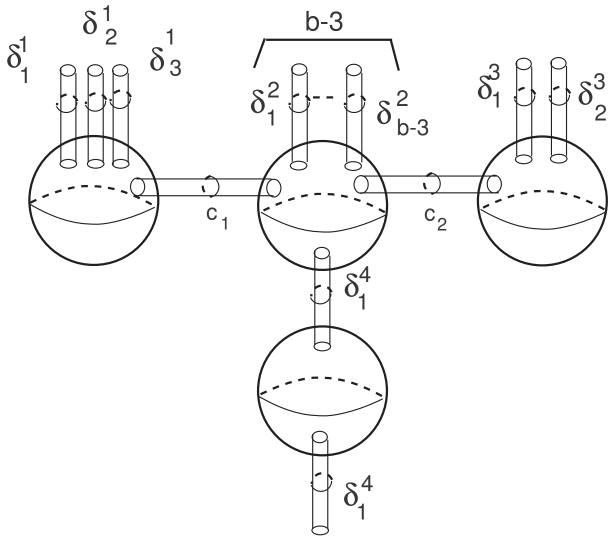


Figure 3.19: Page Σ for Octahedral Singularities of type d Case 2.

5. **Icosahedral singularities**:

(a) Intersection matrix $I(\Gamma)$ for Icosahedral singularity of type (a) is

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -b & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix}$$

- Case 1: if $b = 2$

  This is the singularity of type E_8. Since this is a simple surface singularity, which is investigated by [4]. The fundamental cycle for this type is given by $\underline{m} = (2,3,4,5,6,4,2,3)$, then $\underline{n} = (1,0,0,0,0,0,0,0)$ from Equation (2.1).

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 4 \\ 2 \\ 3 \end{bmatrix} = - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We can now write page Σ of the open book associated to $\underline{m}$ which is a torus with one boundary component, built up as union of the central torus with six boundary components and the annuli (see Figure 3.20). We find the monodromy restricted to each annulus.
$\left(\phi|_{U_1^1}\right)^2 = t_\delta$, $\left(\phi|_{U_1^{1,2}}\right)^6 = t_\delta$, $\left(\phi|_{U_1^{2,3}}\right)^{12} = t_\delta$, $\left(\phi|_{U_1^{3,4}}\right)^{20} = t_\delta$, $\left(\phi|_{U_1^{4,5}}\right)^{30} = t_\delta$.
Hence $\phi = t_\delta^{1/2+1/6+1/12+1/20+1/30} = t_\delta^{5/6}$ ϕ is 5/6 of a full turn around δ. Using Relation (A.9) and Theorem 3.2, the monodromy ϕ is

$$\phi = t_\delta^{5/6} = (t_\alpha t_\beta)^5 .$$

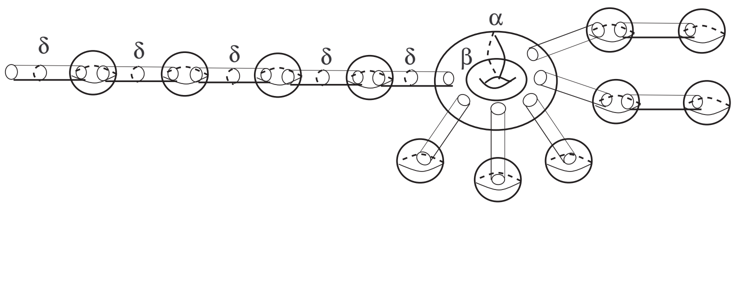


Figure 3.20: Page Σ for Icosahedral Singularities of type (a) Case 1.

- Case 2: if $b > 2$

  Similarly we find page Σ of the open book associated to $\underline{m}$. For this purpose, we find fundamental cycle $\underline{m}$.

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -b & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ b-3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Page Σ of the open book associated to $\underline{m}$ is a sphere with b boundary components, built up as union of the central sphere with b boundary components, seven spheres with two boundary components and $b+7$ annuli (see Figure 3.21). The monodromy restricted to those annuli is $\phi|_{U_1^1} = t_{\delta_1^1}$, $\phi|_{U_1^{1,2}} = t_{\delta_1^1}$, $\phi|_{U_1^{2,3}} = t_{\delta_1^1}$, $\phi|_{U_1^{3,4}} = t_{\delta_1^1}$, $\phi|_{U_1^{4,5}} = t_{\delta_1^1}$, $\phi|_{U_i^5} = t_{\delta_i^5}$ for $i = 1, \ldots, b-3$, $\phi|_{U_1^7} = t_{\delta_1^7}$, $\phi|_{U_1^{5,6}} = t_{\delta_1^7}$, $\phi|_{U_1^{6,7}} = t_{\delta_1^7}$, $\phi|_{U_1^8} = t_{\delta_1^8}$, $\phi|_{U_1^{5,8}} = t_{\delta_1^8}$.
Hence the monodromy of the open book is

$$\phi = \left(t_{\delta_1^1}\right)^5 \left(t_{\delta_1^5} \ldots t_{\delta_{b-3}^5}\right) \left(t_{\delta_1^7}\right)^3 \left(t_{\delta_1^8}\right)^2.$$

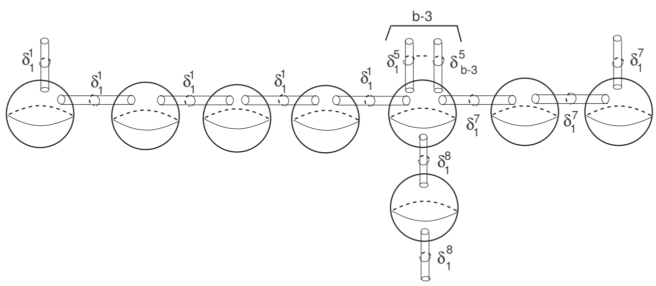


Figure 3.21: Page Σ for Icosahedral Singularities of type (a) Case 2.

(b) Intersection matrix $I(\Gamma)$ for Icosahedral singularity of type (b) is

$$\begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix}$$

- Case 1: if $b = 2$

  For this case, the fundamental cycle is $\underline{m} = (1, 2, 3, 2, 1, 2)$.

$$\begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \\ 2 \end{bmatrix} = - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

  We can see from Figure 3.22 that, page Σ of the open book associated to $\underline{m}$ is a torus with two boundary components, built up as union of the central torus with three boundary components, and five spheres with two boundary components, and seven annuli. The monodromy restricted to each annulus is
  $\phi|_{U_1^1} = t_{\delta_1}$, $\left(\phi|_{U_1^{1,2}}\right)^2 = t_{\delta_1}$, $\left(\phi|_{U_1^{2,3}}\right)^6 = t_{\delta_1}$, $\left(\phi|_{U_1^6}\right)^2 = t_{\delta_2}$, $\left(\phi|_{U_1^{3,6}}\right)^6 = t_{\delta_2}$.
  Hence the monodromy ϕ of the open book is

$$\phi = t_{\delta_1} \left(t_{\delta_1} t_{\delta_2}\right)^{2/3},$$

  then using the 2-holed torus Relation (A.6) and Theorem 3.2, we can write the monodromy as

$$\phi = t_{\delta_1} \left(t_{\alpha_1} (t_{\alpha_2})^2 t_{\beta}\right)^2.$$

- Case 2: if $b > 2$

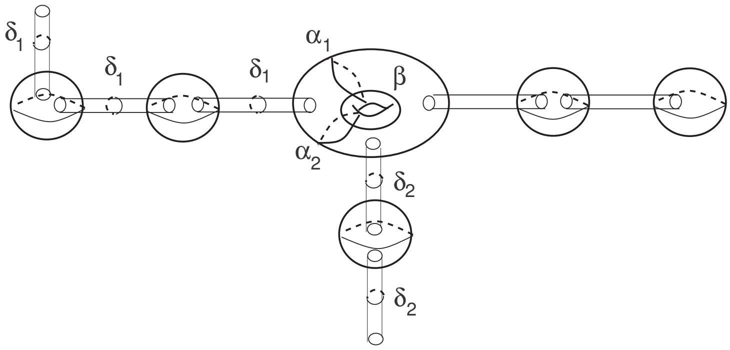


Figure 3.22: Page Σ for Icosahedral Singularities of type (b) Case 1.

The fundamental cycle is $\underline{m} = (1,1,1,1,1,1)$.

$$\begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 2 \\ 0 \\ b-3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

As it can be seen in Figure 3.23, page Σ of the open book associated to $\underline{m}$ is a sphere with $b+1$ boundary components, built up as union of the central sphere with b boundary components, and five other spheres corresponding to the other vertices, and annuli. In order to find the total monodromy of the open book, we look at the monodromy restricted to the annulus.

$\phi|_{U_1^1} = t_{\delta_1^1}$, $\phi|_{U_2^1} = t_{\delta_2^1}$, $\phi|_{U_i^3} = t_{\delta_i^3}$ for $i = 1, \ldots, b-3$, $\phi|_{U_1^5} = t_{\delta_1^5}$, $\phi|_{U_1^6} = t_{\delta_1^6}$. $\phi|_{U_1^{1,2}} = t_{c_1}$, $\phi|_{U_1^{2,3}} = t_{c_1}$, $\phi|_{U_1^{3,4}} = t_{\delta_1^5}$, $\phi|_{U_1^{4,5}} = t_{\delta_1^5}$, $\phi|_{U_1^{3,6}} = t_{\delta_1^6}$.

Hence the monodromy ϕ of the open book is

$$\phi = t_{\delta_1^1} t_{\delta_2^1} \left(t_{\delta_1^3} \ldots t_{\delta_{b-3}^3} \right) \left(t_{\delta_1^5} \right)^3 \left(t_{\delta_1^6} \right)^2 (t_{c_1})^2 .$$

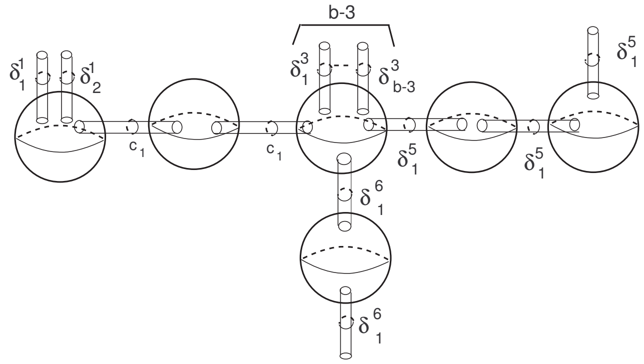


Figure 3.23: Page Σ for Icosahedral Singularities of type (b) Case 2.

(c) Intersection matrix $I(\Gamma)$ for Icosahedral singularity of type (c) is

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -b & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 \end{bmatrix}$$

- Case 1: if $b = 2$

  The fundamental cycle is $\underline{m} = (1, 2, 2, 2, 2, 1, 1)$.

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

  Page Σ of the open book associated to $\underline{m}$ is a torus with two boundary components (see Figure 3.24). The monodromy restricted to each annulus is

  $\left(\phi|_{U_1^2}\right)^2 = t_{\delta_1}$, $\left(\phi|_{U_1^{2,3}}\right)^2 = t_{\alpha_1}$, $\left(\phi|_{U_1^{3,4}}\right)^2 = t_{\alpha_1}$, $\left(\phi|_{U_1^{4,5}}\right)^2 = t_{\alpha_1}$, $\left(\phi|_{U_2^{2,3}}\right)^2 = t_{\alpha_2}$, $\left(\phi|_{U_2^{3,4}}\right)^2 = t_{\alpha_2}$, $\left(\phi|_{U_2^{4,5}}\right)^2 = t_{\alpha_2}$, $\phi|_{U_1^6} = t_{\delta_2}$,

$\left(\phi|_{U_1^{5,6}}\right)^2 = t_{\delta_2}$,
Hence, the monodromy ϕ of the open book is

$$\phi = (t_{\delta_2} t_{\alpha_1} t_{\alpha_2})\,(t_{\delta_1} t_{\delta_2} t_{\alpha_1} t_{\alpha_2})^{1/2}$$

which can be written by using Relation (A.8) as

$$\phi = (t_{\delta_2} t_{\alpha_1} t_{\alpha_2})\,(t_{\alpha_1} t_{\alpha_2} t_\beta t_{\alpha_1} t_{\alpha_2} t_\beta t_{\alpha_2})\,,$$

gives the monodromy of the open book by the Theorem 3.2.

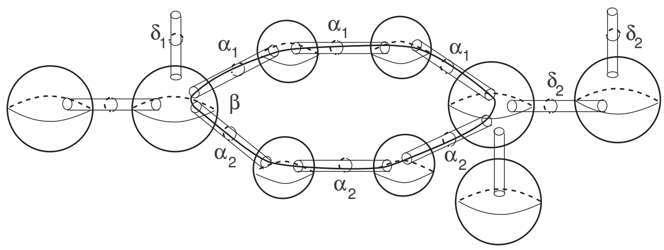


Figure 3.24: Page Σ for Icosahedral Singularities of type (c) Case 1.

- Case 2: if $b > 2$

  The fundamental cycle is $\underline{m} = (1, 1, 1, 1, 1, 1, 1)$.

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -b & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ b-3 \\ 2 \\ 1 \end{bmatrix}$$

As it can be seen in Figure 3.25, page Σ of the open book associated to $\underline{m}$ is a sphere with $b+1$ boundary components, built up as union of the central sphere with b boundary components, and six other spheres corresponding to the other vertices, and annuli. The monodromy restricted to each annulus is

$\phi|_{U_1^1} = t_{\delta_1^1}$, $\phi|_{U_1^{1,2}} = t_{\delta_1^1}$, $\phi|_{U_1^{2,3}} = t_{\delta_1^1}$, $\phi|_{U_1^{3,4}} = t_{\delta_1^1}$, $\phi|_{U_1^{4,5}} = t_{\delta_1^1}$,

$\phi|_{U_i^5} = t_{\delta_i^5}$ for $i = 1, \ldots, b-3$, $\phi|_{U_1^{5,6}} = t_{c_1}$, $\phi|_{U_1^6} = t_{\delta_1^6}$, $\phi|_{U_2^6} = t_{\delta_2^6}$, $\phi|_{U_1^{5,7}} = t_{\delta_1^7}$, $\phi|_{U_1^7} = t_{\delta_1^7}$. Hence the monodromy of the open book is

$$\phi = \left(t_{\delta_1^1}\right)^5 \left(t_{\delta_1^5} \ldots t_{\delta_{b-3}^5}\right) t_{\delta_1^6} t_{\delta_2^6} \left(t_{\delta_1^7}\right)^2 t_{c_1}.$$

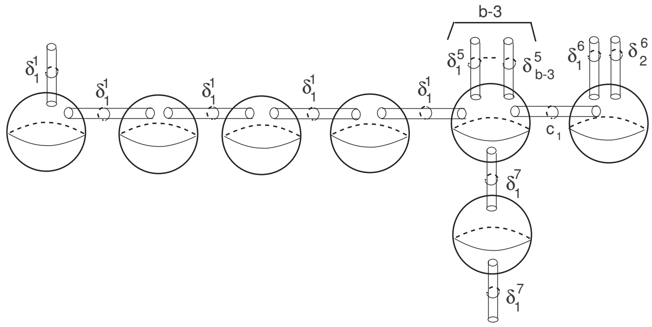


Figure 3.25: Page Σ for Icosahedral Singularities of type (c) Case 2.

(d) Intersection matrix $I(\Gamma)$ for Icosahedral singularity of type (d) is

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix}$$

- Case 1: if $b = 2$

  Consider the 6-tuple of integers $\underline{m} = (1, 1, 2, 2, 1, 1)$. Then we have $\underline{n} = (1, 0, 0, 1, 0, 0)$ from Equation (2.1).

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

  Page Σ of the open book associated to $\underline{m}$ is a torus with two boundary components, built up as union of six spheres and eight annuli. As it is clear from Figure 3.26 after gluing these we end up with Σ being a torus with two boundary components. The monodromy restricted to each annulus is

$\phi|_{U_1^1} = t_{\delta_1}$, $\phi|_{U_1^{1,2}} = t_{\delta_1}$, $\left(\phi|_{U_1^{2,3}}\right)^2 = t_{\delta_1}$, $\left(\phi|_{U_1^{3,4}}\right)^2 = t_{\alpha_1}$, $\left(\phi|_{U_2^{3,4}}\right)^2 = t_{\alpha_2}$, $\left(\phi|_{U_1^4}\right)^2 = t_{\delta_2}$.

The monodromy of the open book is

$$\phi = (t_{\delta_1})^2 \left(t_{\delta_1} t_{\delta_2} t_{\alpha_1} t_{\alpha_2}\right)^{1/2}.$$

By using Relation (A.8) and Theorem 3.2

$$\phi = (t_{\delta_1})^2 \left(t_{\alpha_1} t_{\alpha_2} t_\beta t_{\alpha_1} t_{\alpha_2} t_\beta t_{\alpha_2}\right).$$

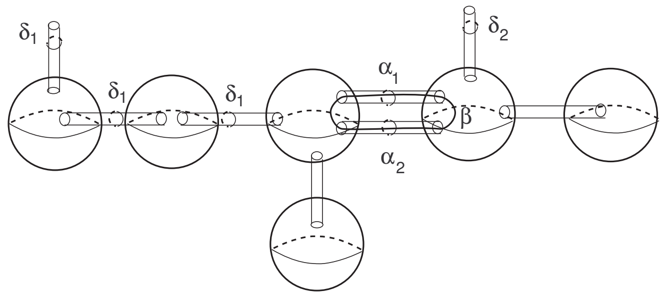


Figure 3.26: Page Σ for Icosahedral Singularities of type (d) Case 1.

- Case 2: if $b > 2$

  Similarly we write page Σ of the open book associated to the fundamental cycle $\underline{m} = (1,1,1,1,1,1)$.

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \\ b-3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

  As it can be seen from Figure 3.27, page Σ of the open book associated to $\underline{m}$ is a sphere with $b+1$ boundary components, built up as union of the central sphere with b boundary components, and five other spheres corresponding to the other vertices, and annuli. The monodromy restricted to each annulus is

$\phi|_{U_1^1} = t_{\delta_1^1}$, $\phi|_{U_1^{1,2}} = t_{\delta_1^1}$, $\phi|_{U_1^2} = t_{\delta_1^2}$, $\phi|_{U_1^{2,3}} = t_{c_1}$, $\phi|_{U_i^3} = t_{\delta_i^3}$ for $i = 1, \ldots, b-3$, $\phi|_{U_1^5} = t_{\delta_1^5}$, $\phi|_{U_1^{3,4}} = t_{\delta_1^5}$, $\phi|_{U_1^{4,5}} = t_{\delta_1^5}$, $\phi|_{U_1^6} = t_{\delta_1^6}$, $\phi|_{U_1^{3,6}} = t_{\delta_1^6}$.

Hence the monodromy of the open book is

$$\phi = \left(t_{\delta_1^1}\right)^2 t_{\delta_1^2} \left(t_{\delta_1^3} \ldots t_{\delta_{b-3}^3}\right) \left(t_{\delta_1^5}\right)^3 \left(t_{\delta_1^6}\right)^2 t_{c_1}.$$

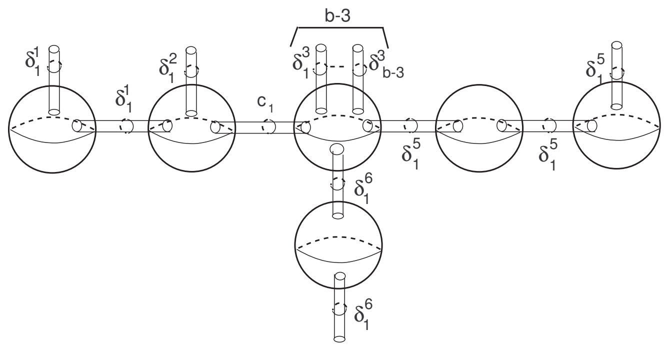


Figure 3.27: Page Σ for Icosahedral Singularities of type (d) Case 2.

(e) Intersection matrix $I(\Gamma)$ for Icosahedral singularity of type (e) is

$$\begin{bmatrix} -3 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -b & 1 & 1 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 & -2 \end{bmatrix}$$

- Case 1: if $b = 2$

$$\begin{bmatrix} -3 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Page Σ of the open book associated to the fundamental cycle $\underline{m} = (1,2,2,1,1)$ is a torus with four boundary components, built up as union of five spheres and nine annuli. As it is clear from Figure 3.28 after gluing these we end up with Σ being a torus with four boundary components. The monodromy restricted to each annulus is
$\phi|_{U_1^1} = t_{\delta_1}$, $\left(\phi|_{U_1^{1,2}}\right)^2 = t_{\delta_1}$, $\left(\phi|_{U_1^3}\right)^2 = t_{\delta_2}$, $\left(\phi|_{U_1^{2,3}}\right)^2 = t_{\alpha_2}$,
$\left(\phi|_{U_2^{2,3}}\right)^2 = t_{\alpha_3}$, $\phi|_{U_1^4} = t_{\delta_3}$, $\left(\phi|_{U_1^{3,4}}\right)^2 = t_{\delta_3}$.
The monodromy of the open book is

$$\phi = t_{\delta_1} t_{\delta_3} \left(t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\alpha_2} t_{\alpha_3}\right)^{1/2}$$

We can use Relation (A.4) for the torus with boundary components $\delta_1, \delta_2, \delta_3$. Hence ϕ is

$$\phi = t_{\delta_1} t_{\delta_3} \left(t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_\beta t_{\alpha_2} t_{\alpha_3} t_\beta\right)$$

by using Theorem 3.2.

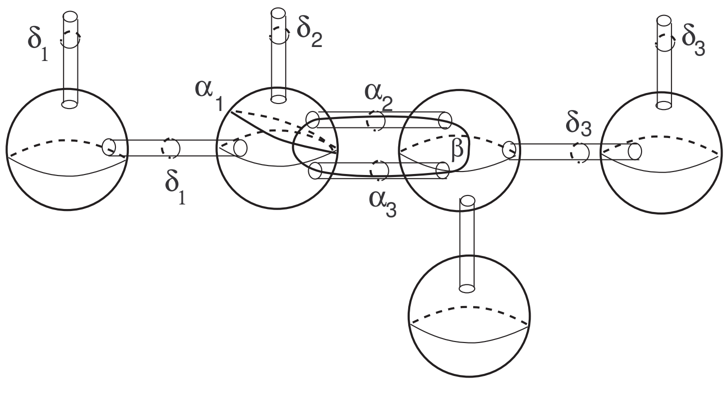


Figure 3.28: Page Σ for Icosahedral Singularities of type (e) Case 1.

- Case 2: if $b > 2$

$$\begin{bmatrix} -3 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -b & 1 & 1 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 2 \\ 0 \\ b-3 \\ 2 \\ 1 \end{bmatrix}$$

As it can be seen from Figure 3.29, page Σ of the open book associated to the fundamental cycle $\underline{m} = (1,1,1,1,1)$ is a sphere with $b+2$ boundary components, built up as union of the central sphere with b boundary components, and four other spheres corresponding to the other vertices, and annuli. We find the monodromy restricted to each annulus.
$\phi|_{U_1^1} = t_{\delta_1^1}$, $\phi|_{U_2^1} = t_{\delta_2^1}$, $\phi|_{U_1^{1,2}} = t_{c_1}$, $\phi|_{U_1^{2,3}} = t_{c_1}$, $\phi|_{U_i^3} = t_{\delta_i^3}$ for $i = 1, \ldots, b-3$, $\phi|_{U_1^{3,4}} = t_{c_2}$, $\phi|_{U_1^4} = t_{\delta_1^4}$, $\phi|_{U_2^4} = t_{\delta_2^4}$, $\phi|_{U_1^5} = t_{\delta_1^5}$, $\phi|_{U_1^{3,5}} = t_{\delta_1^5}$.
Hence the monodromy of the open book is

$$\phi = t_{\delta_1^1} t_{\delta_2^1} \left(t_{\delta_1^3} \ldots t_{\delta_{b-3}^3} \right) t_{\delta_1^4} t_{\delta_2^4} \left(t_{\delta_1^5} \right)^2 t_{c_1}{}^2 t_{c_2}.$$

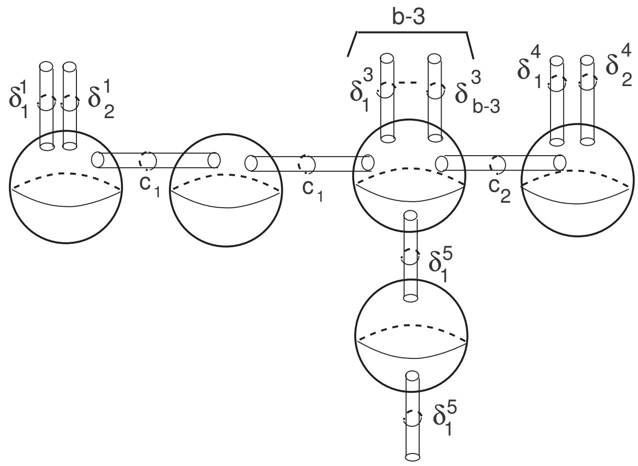


Figure 3.29: Page Σ for Icosahedral Singularities of type (e) Case 2.

(f) Intersection matrix $I(\Gamma)$ for Icosahedral singularity of type (f) is

$$\begin{bmatrix} -5 & 1 & 0 & 0 & 0 \\ 1 & -b & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & -2 \end{bmatrix}$$

- Case 1: if $b = 2$

$$\begin{bmatrix} -5 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Page Σ of the open book associated to the fundamental cycle $\underline{m} = (1,2,2,1,1)$ is a torus with four boundary components, built up as union of five spheres and nine annuli. As it is clear from Figure 3.30 after gluing these we end up with Σ being a torus with three boundary components. The monodromy restricted to each annulus is
$\phi|_{U_1^1} = t_{\delta_1}$, $\phi|_{U_2^1} = t_{\delta_2}$, $\phi|_{U_3^1} = t_{\delta_3}$, $\left(\phi|_{U_1^{1,2}}\right)^2 = t_{c_1}$, $\left(\phi|_{U_1^{2,3}}\right)^2 = t_{\alpha_1}$, $\left(\phi|_{U_2^{2,3}}\right)^2 = t_{\alpha_2}$, $\left(\phi|_{U_1^3}\right)^2 = t_{\delta_4}$.
The monodromy of the open book is

$$\phi = t_{\delta_1} t_{\delta_2} \left(t_{c_1} t_{\delta_4} t_{\alpha_1} t_{\alpha_2}\right)^{1/2}$$

We can use Relation (A.8) and Theorem 3.2 for the torus with boundary components c_1, δ_4. Hence ϕ is

$$\phi = t_{\delta_1} t_{\delta_2} \left(t_{\alpha_1} t_{\alpha_2} t_\beta t_{\alpha_1} t_{\alpha_2} t_\beta t_{\alpha_2}\right).$$

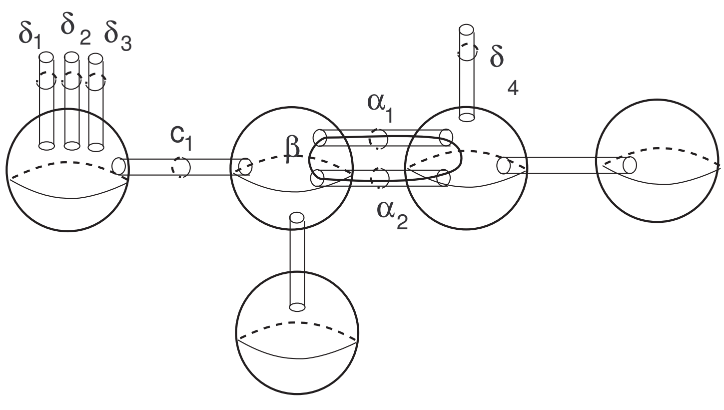


Figure 3.30: Page Σ for Icosahedral Singularities of type (f) Case 1.

- Case 2: if $b > 2$

$$\begin{bmatrix} -5 & 1 & 0 & 0 & 0 \\ 1 & -b & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 4 \\ b-3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Page Σ of the open book associated to the fundamental cycle $\underline{m} = (1,1,1,1,1)$ is a sphere with $b+3$ boundary components, built up as union of the central sphere with b boundary components, and four other spheres corresponding to the other vertices, and annuli. As it can be seen from Figure 3.31 after gluing these we end up with Σ being a sphere with $b+3$ boundary components. The monodromy restricted to each annulus is

$\phi|_{U_1^1} = t_{\delta_1^1}$, $\phi|_{U_2^1} = t_{\delta_2^1}$, $\phi|_{U_3^1} = t_{\delta_3^1}$, $\phi|_{U_4^1} = t_{\delta_4^1}$, $\phi|_{U_i^2} = t_{\delta_i^2}$ for $i = 1, \ldots, b-3$, $\phi|_{U_1^4} = t_{\delta_1^4}$, $\phi|_{U_1^5} = t_{\delta_1^5}$, $\phi|_{U_1^{1,2}} = t_{c_1}$, $\phi|_{U_1^{2,3}} = t_{\delta_1^4}$, $\phi|_{U_1^{3,4}} = t_{\delta_1^4}$, $\phi|_{U_1^{2,5}} = t_{\delta_1^5}$.

Hence the monodromy ϕ of the open book is

$$\phi = t_{\delta_1^1} t_{\delta_2^1} t_{\delta_3^1} t_{\delta_4^1} \left(t_{\delta_1^2} \ldots t_{\delta_{b-3}^2} \right) \left(t_{\delta_1^4} \right)^3 \left(t_{\delta_1^5} \right)^2 t_{c_1}.$$

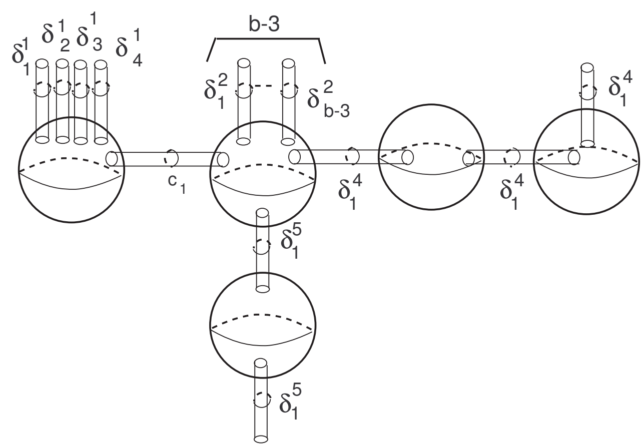


Figure 3.31: Page Σ for Icosahedral Singularities of type (f) Case 2.

(g) Intersection matrix $I(\Gamma)$ for Icosahedral singularity of type (g) is

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -b & 1 & 1 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 & -2 \end{bmatrix}$$

- Case 1: if $b = 2$

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Page Σ of the open book associated to the fundamental cycle $\underline{m} = (1, 1, 2, 1, 1)$ is built up as union of the central torus with four boundary components, and four other spheres corresponding to the other vertices, and annuli. As it can be seen from Figure 3.32 after gluing these we end up with Σ being a torus with three boundary components. The monodromy restricted to each annulus is

$\phi|_{U_1^1} = t_{\delta_1}$, $\phi|_{U_1^3}{}^2 = t_{\delta_2}$, $\phi|_{U_1^4} = t_{\delta_3}$, $\phi|_{U_1^{1,2}} = t_{\delta_1}$, $\phi|_{U_1^{2,3}}{}^2 = t_{\delta_1}$, $\phi|_{U_1^{3,4}}{}^2 = t_{\delta_3}$.

Hence the monodromy ϕ of the open book is

$$\phi = (t_{\delta_1})^2 \, t_{\delta_3} \, (t_{\delta_1} t_{\delta_2} t_{\delta_3})^{1/2}$$

and using Relation (A.2) and Theorem 3.2 for the torus with boundary components δ_1, δ_2 and δ_3,

$$\phi = (t_{\delta_1})^2 \, t_{\delta_3} \, (t_{\alpha_1} t_{\alpha_3} t_{\beta} t_{\alpha_2} t_{\alpha_3} t_{\beta}) \, .$$

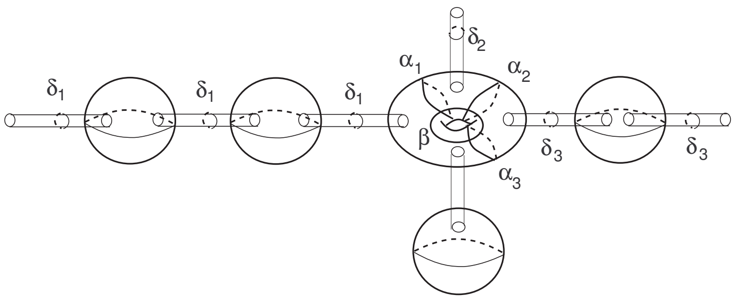


Figure 3.32: Page Σ for Icosahedral Singularities of type (g) Case 1.

- Case 2: if $b > 2$

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -b & 1 & 1 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \\ b-3 \\ 2 \\ 1 \end{bmatrix}$$

As it can be seen in Figure 3.33, page Σ of the open book associated to the fundamental cycle $\underline{m} = (1, 1, 1, 1, 1)$ is a sphere with $b+2$ boundary components, built up as union of the central sphere with b boundary components, and four other spheres corresponding to the other vertices, and annuli. We find the monodromy restricted to each annulus.

$\phi|_{U_1^1} = t_{\delta_1^1}$, $\phi|_{U_1^{1,2}} = t_{\delta_1^1}$, $\phi|_{U_1^2} = t_{\delta_1^2}$, $\phi|_{U_1^{2,3}} = t_{c_1}$, $\phi|_{U_i^3} = t_{\delta_i^3}$ for $i = 1, \ldots, b-3$, $\phi|_{U_1^{3,4}} = t_{c_2}$, $\phi|_{U_1^4} = t_{\delta_1^4}$, $\phi|_{U_2^4} = t_{\delta_2^4}$, $\phi|_{U_1^5} = t_{\delta_1^5}$, $\phi|_{U_1^{3,5}} = t_{\delta_1^5}$.

Hence the monodromy of the open book is

$$\phi = \left(t_{\delta_1^1}\right)^2 t_{\delta_1^2} \left(t_{\delta_1^3} \ldots t_{\delta_{b-3}^3}\right) t_{\delta_1^4} t_{\delta_2^4} \left(t_{\delta_1^5}\right)^2 t_{c_1} t_{c_2}.$$

(h) Intersection matrix $I(\Gamma)$ for Icosahedral singularity of type (h) is

$$\begin{bmatrix} -5 & 1 & 0 & 0 \\ 1 & -b & 1 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix}$$

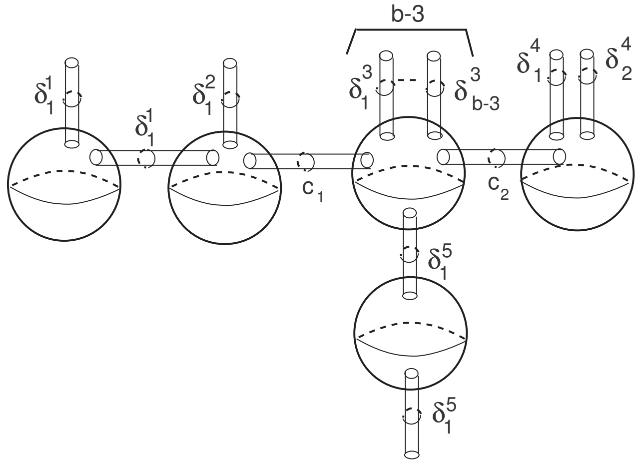


Figure 3.33: Page Σ for Icosahedral Singularities of type (g) Case 2.

- Case 1: if $b = 2$

$$\begin{bmatrix} -5 & 1 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Page Σ of the open book associated to $\underline{m}$ is built up as union of the central torus with four boundary components, and three other spheres corresponding to the other vertices, and annuli. As it can be seen from Figure 3.34 after gluing these we end up with Σ being a torus with five boundary components. The monodromy restricted to each annulus is $\phi|_{U_1^1} = t_{\delta_1}$, $\phi|_{U_2^1} = t_{\delta_2}$, $\phi|_{U_3^1} = t_{\delta_3}$, $\left(\phi|_{U_1^2}\right)^2 = t_{\delta_4}$, $\phi|_{U_1^3} = t_{\delta_5}$, $\left(\phi|_{U_1^{1,2}}\right)^2 = t_{c_1}$, $\left(\phi|_{U_1^{2,3}}\right)^2 = t_{\delta_5}$.
Hence, monodromy ϕ of the open book is

$$\phi = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_5} \left(t_{c_1} t_{\delta_4} t_{\delta_5}\right)^{1/2},$$

and we can use 3-holed torus Relation (A.2) and Theorem 3.2 for the torus with boundary components c_1, δ_4 and δ_5 then ϕ is

$$\phi = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_5} \left(t_{\alpha_1} t_{\alpha_3} t_{\beta} t_{\alpha_2} t_{\alpha_3} t_{\beta}\right).$$

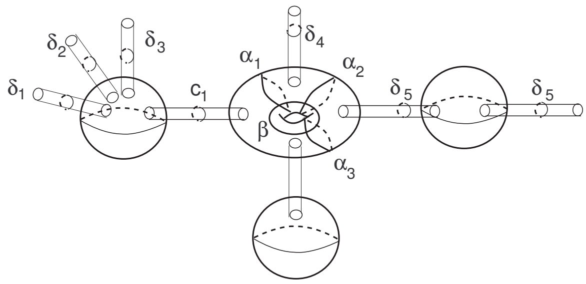


Figure 3.34: Page Σ for Icosahedral Singularities of type (h) Case 1.

- Case 2: if $b > 2$

$$\begin{bmatrix} -5 & 1 & 0 & 0 \\ 1 & -b & 1 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 4 \\ b-3 \\ 2 \\ 1 \end{bmatrix}$$

Page Σ of the open book associated to the fundamental cycle $\underline{m} = (1,1,1,1)$ is a sphere with $b+4$ boundary components, built up as union of the central sphere with b boundary components, and three other spheres corresponding to the other vertices, and annuli. As it can be seen from Figure 3.35 after gluing these we end up with Σ being a sphere with $b+4$ boundary components. We find the monodromy restricted to each annulus.

$\phi|_{U_1^1} = t_{\delta_1^1}$, $\phi|_{U_2^1} = t_{\delta_2^1}$, $\phi|_{U_3^1} = t_{\delta_3^1}$, $\phi|_{U_4^1} = t_{\delta_4^1}$, $\phi|_{U_i^2} = t_{\delta_i^2}$ for $i = 1, \ldots, b-3$, $\phi|_{U_1^3} = t_{\delta_1^3}$, $\phi|_{U_2^3} = t_{\delta_2^3}$, $\phi|_{U_1^4} = t_{\delta_1^4}$, $\phi|_{U_1^{1,2}} = t_{c_1}$, $\phi|_{U_1^{2,3}} = t_{c_2}$, $\phi|_{U_1^{2,4}} = t_{\delta_1^4}$.

Hence the monodromy ϕ of the open book is

$$\phi = t_{\delta_1^1} t_{\delta_2^1} t_{\delta_3^1} t_{\delta_4^1} \left(t_{\delta_1^2} \ldots t_{\delta_{b-3}^2} \right) t_{\delta_1^3} t_{\delta_2^3} \left(t_{\delta_1^4} \right)^2 t_{c_1} t_{c_2}.$$

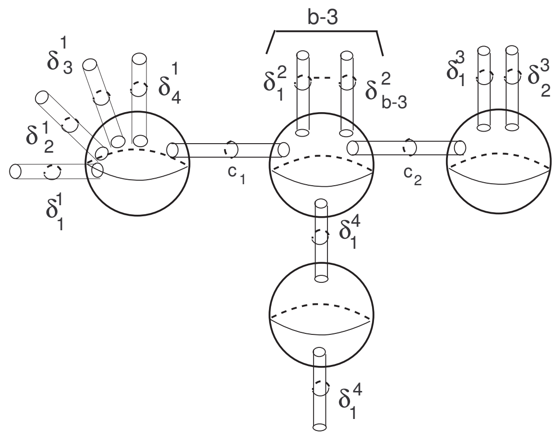


Figure 3.35: Page Σ for Icosahedral Singularities of type (h) Case 2.

3.3.2 Support Genus Problem

The unique Milnor fillable contact structures on the links of quotient surface singularities of Cyclic, Dihedral Case 1, Tetrahedral of types (a), (b) and (c) for Case 2, Octahedral of types (a), (b) and (c) for Case 2, Icosahedral of types (a), (b) and (c) for Case 2 are all supported by planar open books. Hence the minimum page-genus for Milnor open books gives the support genus for these types.

In this section, we show that the unique Milnor fillable contact structures on the links of quotient surface singularities of Tetrahedral of type (a) Case 1, Octahedral of type (a) Case 1, Icosahedral of type (a) Case 1 and 2; cannot be supported by planar open book decompositions. (For Icosahedral singularity of of type (a) Case 1, it is also shown in [9]). These have Milnor genus-1 open book decompositions. Therefore we will be able to say that for these types Milnor genus gives the support genus also.

If X is a symplectic filling of a contact 3-manifold (M, ξ) and ξ is supported by a planar open book then X can be embedded in $\#_n\overline{CP}^2$, connected sum of n copies of $\overline{CP}^2$ by the (proof of) Theorem 1.2 of [9]. Hence, in order to show that the natural contact structure on the links stated above have support genus one, we show that their symplectic fillings cannot be embedded in $\#_n\overline{CP}^2$.

Let v_1, v_2, v_3, v_4 be the standard generators of the intersection lattice $(\mathbb{Z}^4, D_4)$ having self-intersection -2, and $e_1, \ldots, e_n$ be the standard generators of $(\mathbb{Z}^n, \mathbb{D}_n = \oplus_n \langle -1 \rangle)$

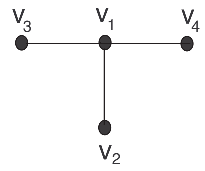


Figure 3.36: Intersection lattice $(\mathbb{Z}^4, D_4)$

diagonal intersection lattice with self-intersection -1. By Lemma 3.1 of [14] (see also Proof of Theorem 4.2 of [15]), there exists only one, up to composing with an automorphism of $(\mathbb{Z}^n, \mathbb{D}_n)$), isometric embedding from $(\mathbb{Z}^4, D_4)$ to $(\mathbb{Z}^n, \mathbb{D}_n)$ which sends v_1 to $e_1 + e_2$, v_2 to $-e_2 + e_3$, v_3 to $-e_1 + e_3$ and v_4 to $-e_2 - e_3$. The proof follows from the fact that, each v_i has self-intersection -2, so their image must be of the form $e_i + e_j$ and from the intersection form D_4, one can only get the embedding above (up to sign changes and permutations of generators of $(\mathbb{Z}^n, \mathbb{D}_n)$).

Let L be any intersection lattice containing the sublattice with vertices $v_1, \ldots, v_6$ as shown in Figure 3.37 where v_1, v_2, v_3, v_4 have self-intersections -2. We prove for any $n \geq 1$, there exists no isometric embedding from L into $(\mathbb{Z}^n, \mathbb{D}_n)$. Suppose there

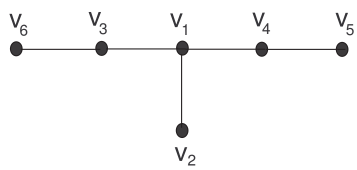


Figure 3.37: The Sublattice

is an isometric embedding φ. From Lemma 3.1 of [14] one knows the images of four generators v_1, v_2, v_3, v_4. That is, as stated above $\varphi(v_1) = e_1 + e_2$, $\varphi(v_2) = -e_2 + e_3$, $\varphi(v_3) = -e_1 + e_3$ and $\varphi(v_4) = -e_2 - e_3$. From the intersection form of L one can see that, v_5 has an intersection with v_4 which is sent to $-e_2 - e_3$. On the otherhand v_5 does not intersect v_2 which has an image $-e_2 + e_3$. Then one can get the equalities below:

$1 = \varphi(1) = \varphi(v_5 \cdot v_4) = \varphi(v_5) \cdot \varphi(v_4) = \varphi(v_5) \cdot (-e_2 - e_3)$ and

$0 = \varphi(0) = \varphi(v_5 \cdot v_2) = \varphi(v_5) \cdot \varphi(v_2) = \varphi(v_5) \cdot (-e_2 + e_3)$. Summing them, one has

$1 = 2 \cdot \varphi(v_5) \cdot -e_2$, which is impossible.

Hence, if one considers the intersection lattice L as stated above and the natural contact structure on the link of that plumbing, then its symplectic filling cannot be embedded in $\#_n \overline{CP}^2$. So that contact structure cannot be supported by a planar open book decomposition. Therefore the unique Milnor fillable contact structures on the links of quotient surface singularities of Tetrahedral of type (a) Case 1, Octahedral of type (a) Case 1, Icosahedral of type (a) Case 1 and 2; cannot be supported by planar open book decompositions. These contact structures have support genus one.

For the remaining cases, we constructed minimal page-genus Milnor open books, and the pages are all genus one surfaces. Hence, we conclude that support genus is at most one for the corresponding contact structures. $\square$

Remark 3.3 *This method we used above, can be used to prove for some other symplectically fillable contact structures on different types of singularities have support genus one.*

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APPENDIX A

Relations

A.1 4-holed torus relation

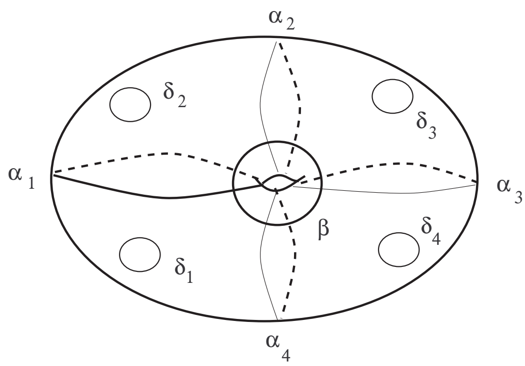

Figure A.1: Four Holed Torus.

$$t_{\delta_1}t_{\delta_2}t_{\delta_3}t_{\delta_4} = (t_{\alpha_1}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_4}t_\beta)^2 \,. \tag{A.1}$$

A.2 3-holed torus relation

We will write the relation from the 4-holed torus relation. If δ_4 bounds a disc, then 4-holed torus relation becomes

$$t_{\delta_1}t_{\delta_2}t_{\delta_3} = (t_{\alpha_1}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta)^2 \,. \tag{A.2}$$

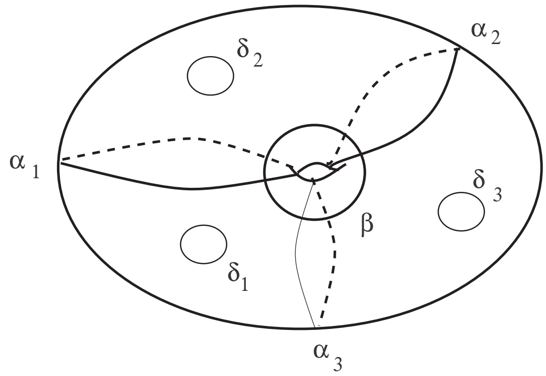

Figure A.2: Three Holed Torus.

Equivalently, we can write another 3-holed torus relation, from (A.2).

$$
\begin{aligned}
(t_{\delta_1}t_{\delta_2}t_{\delta_3}) &= (t_{\alpha_1}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta)^2\\
&= t_{\alpha_1}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_1}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta\\
&= t_{\alpha_1}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_3}t_{\alpha_1}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta\\
&= t_{\alpha_1}t_{\alpha_3}t_\beta t_{\alpha_2}t_\beta t_{\alpha_3}t_\beta t_{\alpha_1}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta\\
&= t_{\alpha_1}t_{\alpha_3}t_{\alpha_2}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_1}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta\\
&= t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_1}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta\\
&= t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_3}t_{\alpha_1}t_\beta t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta\\
&= t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta\\
&= (t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta)^3\,. \qquad \text{(A.3)}
\end{aligned}
$$

From Relation (A.3) we have

$$\begin{aligned}
t_{\delta_1}t_{\delta_2}t_{\delta_3}t_{\alpha_2}t_{\alpha_3} &= t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_3} \\
&= t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_3}t_{\alpha_1}t_\beta t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_3} \\
&= t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_1}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_3} \\
&= t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_1}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_3}t_{\alpha_2} \\
&= t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_1}t_\beta t_{\alpha_2}t_\beta t_{\alpha_3}t_\beta t_{\alpha_2} \\
&= t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_1}t_{\alpha_2}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_2} \\
&= t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_1}t_{\alpha_2}t_\beta t_{\alpha_3}t_{\alpha_2}t_\beta t_{\alpha_2} \\
&= t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_1}t_{\alpha_2}t_\beta t_{\alpha_3}t_\beta t_{\alpha_2}t_\beta \\
&= t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_3}t_{\alpha_2}t_\beta \\
&= (t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_3}t_\beta)^2 . \qquad \text{(A.4)}
\end{aligned}$$

A.3 2-holed torus relation

We will write the relation from the 3-holed torus relation. If δ_3 bounds a disc, then 3-holed torus relation (A.2) becomes

$$t_{\delta_1}t_{\delta_2} = \left(t_{\alpha_1}t_{\alpha_2}t_\beta(t_{\alpha_2})^2t_\beta\right)^2 . \qquad \text{(A.5)}$$

Similarly we have from 3-holed torus relation (A.3)

$$t_{\delta_1}t_{\delta_2} = \left(t_{\alpha_1}(t_{\alpha_2})^2t_\beta\right)^3 . \qquad \text{(A.6)}$$

Equivalently we have from two holed torus relation (A.6)

$$
\begin{aligned}
t_{\delta_1}t_{\delta_2} &= t_{\alpha_1}t_{\alpha_2}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\alpha_2}t_{\beta} \\
&= t_{\alpha_1}t_{\alpha_2}t_{\alpha_2}t_{\beta}t_{\alpha_2}t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\alpha_2}t_{\beta} \\
&= t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\alpha_2}t_{\beta} \\
&= t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_2}t_{\alpha_1}t_{\alpha_2}t_{\beta} \\
&= t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\beta}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\beta} \\
&= t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_2}t_{\alpha_1}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\beta} \\
&= t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\beta} \\
&= (t_{\alpha_1}t_{\alpha_2}t_{\beta})^4 . \qquad \text{(A.7)}
\end{aligned}
$$

Then from (A.7) we have

$$
\begin{aligned}
t_{\delta_1}t_{\delta_2}t_{\alpha_1}t_{\alpha_2} &= t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2} \\
&= t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_2}t_{\alpha_1}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2} \\
&= t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_2}t_{\alpha_1}t_{\beta}t_{\alpha_2}t_{\alpha_1}t_{\beta}t_{\alpha_1}t_{\alpha_2} \\
&= t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_2}t_{\alpha_1}t_{\beta}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\beta}t_{\alpha_2} \\
&= t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_2}t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_2}t_{\alpha_1}t_{\beta}t_{\alpha_2} \\
&= t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_2}t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_2} \\
&= (t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_1}t_{\alpha_2}t_{\beta}t_{\alpha_2})^2 . \qquad \text{(A.8)}
\end{aligned}
$$

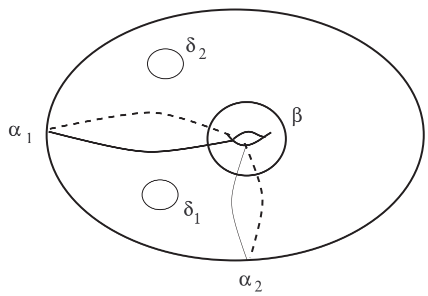

Figure A.3: Two Holed Torus.

A.4 1-holed torus relation

We will write the relation from 2-holed torus relation (A.7). If δ_2 bounds a disc, and then α_1 will be isotopic to α_2. Let us denote the only boundary component δ_2 by δ and α_1 by α.

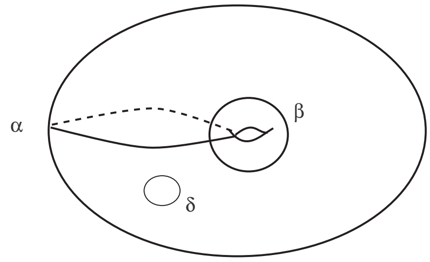

Figure A.4: One Holed Torus.

$$\begin{aligned} t_\delta &= t_\alpha t_\alpha t_\beta t_\alpha t_\alpha t_\beta t_\alpha t_\alpha t_\beta t_\alpha t_\alpha t_\beta \\ &= t_\alpha t_\beta t_\alpha t_\beta t_\alpha t_\beta t_\alpha t_\beta t_\alpha t_\beta t_\alpha t_\beta \\ &= (t_\alpha t_\beta)^6 . \end{aligned} \tag{A.9}$$

VITA

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A note on overtwisted contact structures, to appear in Studia Scientiarum Mathematicarum Hungarica
Right- Veering Diffeomorphisms, Master thesis