# EINSTEIN AETHER GRAVITY

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# ABSTRACT

EINSTEIN AETHER GRAVITY

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In this thesis, we review some basic properties of the Einstein-aether gravity. We derive the field equations from an action and study a subclass of this theory corresponding to the Einstein-Maxwell like theory. We also show that the Gödel type metrics are also exact solutions of this theory. Furthermore, we determine the observational constraints on the dimensionless preferred parameters of this theory using the parametrized post-Newtonian formalism. We stress that none of calculations and discussions are original in this thesis.

Keywords: Einstein-aether theory, Gödel type metrics, Newtonian limit, parametrized post Newtonian formalism.

# ÖZ

EİNSTEİN ETER GRAVİTASYON KURAMI

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Bu tezde, Einstein-eter gravitasyon kuramının bazı temel özellikleri tekrarlanmıştır. Aksiyon prensibinden alan denklemleri bulunmustur ve bu teorinin Einstein-Maxwell benzeri kurama tekabül eden bir alt grubu çalışılmıştır. Ayrıca, Gödel tipi metriklerinin bu teorinin tam çözümü olduğu gösterilmiştir. Parametreli Newton-sonrası formalizmi kullanarak bu kuramın boyutsuz parametrelerin deneysel kısıtlamaları belirlenmiştir. Bu tezdeki tartışmalar ve hesaplar orijinal değildir.

Anahtar Kelimeler: Einstein-aether kuramı, Gödel uzay-zamanı, Newton limiti, parametreli Newton-sonrası formalizmi.

**To my lovely mother and father**

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# CHAPTER 1

# INTRODUCTION

It is well known that the formulation of general relativity is based on the spacetime metric. This theory is Lorentz covariant and locally Lorentz invariant. Moreover, it does not possess preferred coordinate systems. On the other hand, one of the Lorentz-invariance violating theory is the Einstein-aether theory, which represents a gravitational theory with a preferred frame. This frame is introduced by a dynamical unit time-like vector field. This theory contains four arbitrary dimensionless parameters which preserve the Lorentz covariance of general relativity. The Einstein-aether theory provides a simple model for us to see the gravitational and cosmological effects of a preferred frame. The Einstein-aether theory was first studied in [1] and [2] and later revived in [3]. A subclass of Einstein-aether theory, corresponding to the Einstein-Maxwell theory with dust distribution was studied in [3,4]. The structure of this theory, status of observational constraints, and some recent developments have been given in [5].

Recently, a class of metrics, called as Gödel-type metrics, was defined and used for generating new solutions in various dimensions. It is shown that [6,7] in all dimensions Einstein field equations for this class of metrics reduce to the Euclidean source-free Maxwell equations. Most recently, it is shown [8] that Gödel-type metrics are also exact solutions of the Einstein-aether theory and the only field equations, like in general relativity, are the three dimensional Euclidean Maxwell equations with a constraint on two of the preferred frame parameters.

The Newtonian limit of the Einstein-aether theory was examined in [9]. In that work, both fields and sources were taken to be static and weak. Furthermore, the aether vector field was chosen to be a timelike Killing vector. The observational constraints on the parameters of the Einstein-aether theory are determined by parametrized post-Newtonian(PPN) formalism [1, 10, 11, 15]. This formalism is an approximation to general relativity and to all other metric theories. PPN approximation assumes that, as in the Solar System, the sources of the field move slowly and gravitate weakly everywhere. Furthermore, PPN formalism is characterized by ten real parameters. Five of them vanish identically for any theory which is derivable from an action principle. The others measure the nonlinearity, spatial curvature, the preferred frame effects and preffered location effects. To determine these parameters, one solves the approximate field equations with the fluid source in a standard coordinate gauge.

In this thesis, we study some basic properties of the Einstein-aether theory, and show that this and Einstein theories are equivalent when the metric is a subclass of Gödel type metric. We also obtain the Newtonian and PPN expansion of this theory.

In chapter 2, we review the Einstein-aether action principle and derive the field equations.

In chapter 3, we review the Gödel type metrics in general relativity and show that this type of metrics are also exact solutions of the Einstein-aether theory.

In chapter 4, we study the Newtonian limit of the Einstein-aether theory.

In chapter 5, we determine the observational constraints on the parameters of the Einstein-aether theory by using the parametrized post-Newtonian formalism.

# CHAPTER 2

# EINSTEIN-AETHER GRAVITY

## 2.1 Action and Field Equations

In general relativity, the spacetime structure is characterized by the metric tensor $g_{ab}$, and the theory is both diffeomorphism invariant and locally Lorentz invariant. Furthermore in general relativity there are no preferred frames. Einstein-aether theory is a simple extension of general relativity containing a dynamical unit timelike vector field $u^a$ that breaks the local Lorentz symmetry. This dynamical vector field is called the aether and specifies a preferred rest frame at each point of spacetime. Einstein-aether theory contains four free parameters which preserve the lorentz covariance of general relativity.

In this chapter, we review the Einstein-aether action principle and derive the field equations.

The conventional Einstein-aether action is defined as [3]

$$I = \frac{1}{16\pi G_*} \int \sqrt{-g} \left( R - K^{ab}{}_{mn} \nabla_a u^m \nabla_b u^n + \lambda (g_{ab} u^a u^b + 1) \right) \mathtt{d}^4 x, \tag{2.1}$$

where

$$K^{ab}{}_{mn} = c_1 g^{ab} g_{mn} + c_2 \delta^a_m \delta^b_n + c_3 \delta^a_n \delta^b_m - c_4 u^a u^b g_{mn}. \tag{2.2}$$

Here $u^a$ is a time-like unit vector, $R$ is the scalar curvature and $\lambda$ is the Lagrange multiplier field and $c_1, c_2, c_3$ and $c_4$ are dimensionless constants. The metric signature is chosen as $(- + ++)$ and the speed of light defined by the metric $g^{ab}$ is unity. The constant $G_*$ is related to the Newton's gravitational constant $G$. Throughout this work we take Latin indices $a, b, c, ...$ to run from 0 to 4 and Latin indices from the middle of the alphabet $i, j, k, ...$ to run from 1 to 3. We use the notation $c_{14} = c_1 + c_4$,

$c_{123} = c_1 + c_2 + c_3$, etc.

The field equations are obtained by making variation of Eq.(2.1) with respect to the metric $g^{ab}$,the vector field $u^a$ and the Lagrange multiplier field $\lambda$.

For this purposes we may split the action as,

$$I \equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6,$$

where

$$I_1 = \frac{1}{16\pi G_*}\int \sqrt{-g}R\mathtt{d}^4x. \tag{2.3}$$

$$I_2 = -\frac{c_1}{16\pi G_*}\int \sqrt{-g}g^{ab}g_{mn}(\nabla_a u^m)(\nabla_b u^n)\mathtt{d}^4x. \tag{2.4}$$

$$I_3 = -\frac{c_2}{16\pi G_*}\int \sqrt{-g}\delta^a_m\delta^b_n(\nabla_a u^m)(\nabla_b u^n)\mathtt{d}^4x. \tag{2.5}$$

$$I_4 = -\frac{c_3}{16\pi G_*}\int \sqrt{-g}\delta^a_n\delta^b_m(\nabla_a u^m)(\nabla_b u^n)\mathtt{d}^4x. \tag{2.6}$$

$$I_5 = \frac{c_4}{16\pi G_*}\int \sqrt{-g}u^a u^b g_{mn}(\nabla_a u^m)(\nabla_b u^n)\mathtt{d}^4x. \tag{2.7}$$

$$I_6 = \frac{1}{16\pi G_*}\int \sqrt{-g}\lambda(g_{ab}u^a u^b + 1)\mathtt{d}^4x. \tag{2.8}$$

*Variation with respect to* $\lambda$:

It is clear that,

$$\delta I_\alpha \equiv 0, \qquad \alpha = 1\cdots 5$$

Variation of Eq.(2.8)

$$\begin{aligned} \delta I_6 &= \frac{1}{16\pi G_*}\int \sqrt{-g}(g_{ab}u^a u^b + 1)\delta\lambda\mathtt{d}^4x, \\ &\equiv 0, \end{aligned}$$

with respect to $\lambda$ yields constraint equation

$$g_{ab}u^a u^b \equiv -1. \tag{2.9}$$

The Lagrange multiplier field $\lambda$ constrains the vector field $u^a$ to have a lenght -1.

*Variation with respect to $u^a$:*

The variation of Eq.(2.3) is,

$$\delta I_1 = 0,$$

with respect to the field $u^a$.

Variation of Eq.(2.4) is,

$$\begin{aligned}
\delta I_2 &= -\frac{c_1}{16\pi G_*}\int \sqrt{-g}g^{ab}g_{mn}[\delta(\nabla_a u^m)(\nabla_b u^n) + (\nabla_a u^m)\delta(\nabla_b u^n)\mathtt{d}^4x, \\
&= -\frac{c_1}{8\pi G_*}\int \sqrt{-g}g^{ab}g_{mn}(\nabla_b u^n)\delta(\nabla_a u^m)\mathtt{d}^4x, \\
&= -\frac{c_1}{8\pi G_*}\int \sqrt{-g}g^{ab}g_{mn}(\nabla_b u^n)\nabla_a(\delta u^m)\mathtt{d}^4x, \\
&= -\frac{c_1}{8\pi G_*}\int \sqrt{-g}\nabla_a(g^{ab}g_{mn}(\nabla_b u^n)\delta u^m)\mathtt{d}^4x \\
&\quad +\frac{c_1}{8\pi G_*}\int \sqrt{-g}\delta u^m\nabla_a(g^{ab}g_{mn}(\nabla_b u^n))\mathtt{d}^4x, \\
&= -\frac{c_1}{8\pi G_*}\int \sqrt{-g}\nabla_a[(\nabla^a u_m)\delta u^m]\mathtt{d}^4x \\
&\quad +\frac{c_1}{8\pi G_*}\int \sqrt{-g}\nabla_a(\nabla^a u_m)\delta u^m\mathtt{d}^4x. \qquad (2.10)
\end{aligned}$$

The first term of Eq.(2.10) vanishes due to the Gauss' Theorem,

$$\delta I_2 = \frac{c_1}{8\pi G_*}\int \sqrt{-g}\nabla_a(\nabla^a u_m)\delta u^m\mathtt{d}^4x.$$

Similarly, variations of Eq.(2.5),(2.6) are,

$$\delta I_3 = \frac{c_2}{8\pi G_*}\int \sqrt{-g}\nabla_m(\nabla_n u^n)\delta u^m\mathtt{d}^4x.$$

$$\delta I_4 = \frac{c_3}{8\pi G_*}\int \sqrt{-g}\nabla_n(\nabla_m u^n)\delta u^m\mathtt{d}^4x.$$

Variation of Eq.(2.7) is,

$$\begin{aligned}
\delta I_5 &= \frac{c_4}{16\pi G_*}\int \sqrt{-g}u^b g_{mn}(\nabla_a u^m)(\nabla_b u^n)\delta u^a\mathtt{d}^4x \\
&+ \frac{c_4}{16\pi G_*}\int \sqrt{-g}u^a g_{mn}(\nabla_a u^m)(\nabla_b u^n)\delta u^b\mathtt{d}^4x \\
&+ \frac{c_4}{16\pi G_*}\int \sqrt{-g}u^a u^b g_{mn}(\nabla_b u^n)\nabla_a(\delta u^m)\mathtt{d}^4x \\
&+ \frac{c_4}{16\pi G_*}\int \sqrt{-g}u^a u^b g_{mn}(\nabla_a u^m)\nabla_b(\delta u^n)\mathtt{d}^4x.
\end{aligned}$$

$$\delta I_5 = \frac{c_4}{8\pi G_*}\int \sqrt{-g}[u^b g_{an}(\nabla_m u^a)(\nabla_b u^n) - \nabla_b(u^a u^b g_{mn}\nabla_a u^n)]\delta u^m \mathtt{d}^4x.$$

Variation of Eq.(2.8) is,

$$\delta I_6 = \frac{1}{16\pi G_*}\left(\int \sqrt{-g}\lambda g_{ab}u^b \delta u^a \mathtt{d}^4x \ + \int \sqrt{-g}\lambda g_{ab}u^a \delta u^b \mathtt{d}^4x \right).$$

$$\delta I_6 = \frac{1}{8\pi G_*}\int \sqrt{-g}\lambda g_{mn}u^n \delta u^m \mathtt{d}^4x.$$

Finally, we sum up the results,

$$\delta I \equiv 0.$$

$$\delta I_1 + \delta I_2 + \delta I_3 + \delta I_4 + \delta I_5 + \delta I_6 \equiv 0$$

and obtain the aether field equation,

$$c_4 u^m \nabla_m u^a \nabla_b u_a + \nabla_a J^a{}_b + \lambda u_b = 0 \tag{2.11}$$

where

$$J^a{}_m = K^{ab}{}_{mn}\nabla_b u^n. \tag{2.12}$$

*Variation with respect to* $g^{ab}$:

We note that

$$\delta(\nabla_a u^m) = (\delta\Gamma^m{}_{ad})u^d.$$

The variation of the Christoffel symbol

$$\Gamma^m{}_{ad} = \frac{1}{2}g^{mn}(\partial_a g_{dn} + \partial_d g_{an} - \partial_n g_{ad}), \tag{2.13}$$

yields

$$\begin{aligned}
\delta\Gamma^m{}_{ad} &= \frac{1}{2}\delta g^{mn}(\partial_a g_{dn} + \partial_d g_{an} - \partial_n g_{ad}) + \frac{1}{2}g^{mn}(\delta\partial_a g_{dn} + \delta\partial_d g_{an} - \delta\partial_n g_{ad}), \\
&= \frac{1}{2}\delta g^{mn}\delta^c_n(\partial_a g_{dc} + \partial_d g_{ac} - \partial_c g_{ad}) \\
&+ \frac{1}{2}g^{mn}(\delta\partial_a g_{dn} + \delta\partial_d g_{an} - \delta\partial_n g_{ad}).
\end{aligned}$$

$$\begin{aligned}
\delta\Gamma^m{}_{ad} = &- \frac{1}{2}g^{mn}\delta g_{ne}g^{ce}(\partial_a g_{dc} + \partial_d g_{ac} - \partial_c g_{ad}) \\
&+ \frac{1}{2}g^{mn}(\partial_a g_{dn} + \partial_d g_{an} - \partial_n g_{ad}).
\end{aligned}$$

Using $\delta^c_n = g_{ne}g^{ce}$ and

$$\delta g_{dn} = -g_{cd}g_{en}\delta g^{ce}, \tag{2.14}$$

we obtain,

$$\delta\Gamma^m{}_{ad} = -g^{mn}\delta g_{ne}\Gamma^e{}_{ad} + \frac{1}{2}g^{mn}(\partial_a g_{dn} + \partial_d g_{an} - \partial_n g_{ad}).$$

It can also be written as,

$$\delta\Gamma^m{}_{ad} = \frac{1}{2}g^{mn}(-\delta g_{ne}\Gamma^e{}_{ad} - \delta g_{ne}\Gamma^e{}_{ad} + \partial_a g_{dn} + \partial_d g_{an} - \partial_n g_{ad}). \tag{2.15}$$

This can be further simplified, by adding and subtracting the term $(\delta g_{ed}\Gamma^e{}_{an} + \delta g_{ae}\Gamma^e{}_{nd})$ to the Eq.(2.15), then in covariant form we obtain

$$\delta\Gamma^m{}_{ad} = \frac{1}{2}g^{mn}(\nabla_a\delta g_{dn} + \nabla_d\delta g_{an} - \nabla_n\delta g_{ad}),$$

and

$$\begin{aligned}\delta(\nabla_a u^m) &= \frac{1}{2}u^d g^{mn}(\nabla_a\delta g_{dn} + \nabla_d\delta g_{an} - \nabla_n\delta g_{ad}),\\ &= -\frac{1}{2}u^c(g_{cd}\nabla_a\delta g^{md} + g_{ad}\nabla_c\delta g^{md} + g^{md}\nabla_d\delta g_{ac}),\end{aligned}$$

where we have used Eq.(2.14).

Furthermore,

$$\begin{aligned}K^{ab}{}_{mn}\nabla_a u^m\delta(\nabla_b u^n) = &- \frac{1}{2}K^{ab}{}_{mn}\nabla_a u^m u^c(g_{cd}\nabla_b\delta g^{nd}\\ &+ g_{bd}\nabla_c\delta g^{nd} + g^{nd}\nabla_d\delta g_{bc}).\end{aligned} \tag{2.16}$$

Changing $n \longleftrightarrow a$ , $d \longleftrightarrow b$ for the first and second terms of Eq. (2.16), we have,

$$\begin{aligned}K^{ab}{}_{mn}\nabla_a u^m\delta(\nabla_b u^n) = &- \frac{1}{2}K^{nd}{}_{ma}\nabla_n u^m u^c g_{cb}\nabla_d\delta g^{ab}\\ &- \frac{1}{2}K^{nd}{}_{ma}\nabla_n u^m u^c g_{bd}\nabla_c\delta g^{ab}\\ &+ \frac{1}{2}K^{rs}{}_{mn}\nabla_r u^m u^c g^{nd}g_{as}g_{bc}\nabla_d\delta g^{ab},\end{aligned}$$

where we have used equation(2.14).

Using Gauss' Theorem and changing indices, $n \longleftrightarrow m$ , $a \longleftrightarrow b$ , we obtain

$$\begin{aligned}K^{ab}{}_{mn}\nabla_b u^n\delta(\nabla_a u^m) &= [\frac{1}{2}\nabla_d(K^{md}{}_{nb}\nabla_m u^n u_a)\\ &+ \frac{1}{2}\nabla_c(K^{md}{}_{nb}\nabla_m u^n u^c g_{ad})\\ &- \frac{1}{2}\nabla_d(K^{rs}{}_{mn}\nabla_r u^n u^c g^{md}g_{sb}g_{ca})]\delta g^{ab}.\end{aligned}$$

Then,

$$
\begin{aligned}
K^{ab}{}_{mn}\nabla_b u^n \delta(\nabla_a u^m) &= [\frac{1}{2}\nabla_m(J^m{}_b u_a) \\
&+ \frac{1}{2}\nabla_m(J^d{}_b u^m g_{ad}) \\
&- \frac{1}{2}\nabla_m((J^s{}_d u^c g^{md} g_{sb} g_{ca})]\delta g^{ab}.
\end{aligned}
$$

$$
\begin{aligned}
K^{ab}{}_{mn}\nabla_b u^n \delta(\nabla_a u^m) &= [\frac{1}{2}\nabla_m(J^m{}_b u_a) + \frac{1}{2}\nabla_m(J_{ab} u^m) \\
&- \frac{1}{2}\nabla_m((J^m_b u_a)]\delta g^{ab}.
\end{aligned}
$$

Similarly, we can calculate the following expressions

$$K^{ab}{}_{mn}\nabla_a u^m \delta(\nabla_b u^n) = [\frac{1}{2}\nabla_m(J^m{}_a u_b + J_{ba} u^m - J^m_a u_b)]\delta g^{ab}.$$

$$K^{ab}{}_{mn}\nabla_b u^n \delta(\nabla_a u^m) + K^{ab}{}_{mn}\nabla_a u^m \delta(\nabla_b u^n) = [\nabla_m(J^m{}_{(b} u_{a)} + J_{(ab)} u^m - J_{(b}{}^m u_{a)}]\delta g^{ab}.$$

$$\delta K^{ab}{}_{mn} = c_1(g_{mn}\delta g^{ab} + g^{ab}\delta g_{mn}) - c_4(u^a u^b \delta g_{mn}).$$

$$
\begin{aligned}
\delta(K^{ab}{}_{mn})\nabla_a u^m \nabla_b u^n &= \nabla_a u^m \nabla_b u^n (c_1 g_{mn}\delta g^{ab} + c_1 g^{ab}\delta g_{mn} - c_4 u^a u^b \delta g_{mn}) \\
&= [c_1 \nabla_a u^m \nabla_b u^n g_{mn} - c_1 \nabla_r u^m \nabla_s u^n g^{rs} g_{ma} g_{nb} \\
&+ c_4 \nabla_r u^m \nabla_s u^n u^r u^s g_{ma} g_{nb}]\delta g^{ab},
\end{aligned}
$$

where we have used equation(2.14).

Variation of $(\sqrt{-g}R)$ is,

$$\delta(\sqrt{-g}R) = \sqrt{-g}(R_{ab} - \frac{1}{2}g_{ab}R)\delta g^{ab} = \sqrt{-g}G_{ab}\delta g^{ab}.$$

Substituting all these results to the variation, the field equations can be written in the form $G_{ab} = S_{ab}$ where $G_{ab}$ is the Einstein tensor and $S_{ab}$ is the aether stress tensor:

$$
\begin{aligned}
G_{ab} &= \nabla_m(J^m{}_{(b} u_{a)} - J_{(b}{}^m u_{a)} + J_{(ab)} u^m) - \frac{1}{2}g_{ab} K^{cd}{}_{mn}\nabla_c u^m \nabla_d u^n \\
&+ c_1(\nabla_a u_n \nabla_b u^n - \nabla_r u_a \nabla^r u_b) + \lambda u_a u_b + c_4 \dot{u}_a \dot{u}_b. \qquad (2.17)
\end{aligned}
$$

where $\dot{u}_a = u^b \nabla_b u_a$ and brackets around the indices denote the symmetrization.

Finally, the variational principle(2.1) implies the following field equations

$$g_{ab} u^a u^b \equiv -1.$$

$$c_4 u^m \nabla_m u^a \nabla_b u_a + \nabla_a J^a{}_b + \lambda u_b = 0.$$

$$\begin{aligned} G_{ab} &= \nabla_m (J^m{}_{(b} u_{a)} - J_{(b}{}^m u_{a)} + J_{(ab)} u^m) - \frac{1}{2} g_{ab} K^{cd}{}_{mn} \nabla_c u^m \nabla_d u^n \\ &+ c_1 (\nabla_a u_n \nabla_b u^n - \nabla_r u_a \nabla^r u_b) + \lambda u_a u_b + c_4 \dot{u}_a \dot{u}_b. \end{aligned}$$

## 2.2 Special case: Einstein-Maxwell Theory

In this section we study a special class of Einstein-aether theory which corresponds to the Einstein-Maxwell theory with a dust distribution (without pressure) [3, 8]. Now, let $c_2 \equiv c_4 \equiv 0$ and $c_3 \equiv -c_1$. Then the action contains only the antisymmetrized derivative of the dynamical unit timelike vector field.

The tensor given in Eq.(2.2) can now be calculated as

$$K^{ab}{}_{mn} = c_1 g^{ab} g_{mn} - c_1 \delta^a_n \delta^b_m,$$

and hence Eq.(2.12) becomes,

$$\begin{aligned} J^a{}_m &= c_1 (g^{ab} g_{mn} - \delta^a_n \delta^b_m) \nabla_b u^n, \\ &= c_1 (\nabla^a u_m - \nabla_m u^a). \end{aligned}$$

If we define

$$F_{ab} = \nabla_a u_b - \nabla_b u_a,$$

which has the same form of the electromagnetic field tensor, then the action (2.1) becomes,

$$S = \frac{1}{16\pi G_*} \int \sqrt{-g} (R - F^{ab} F_{ab} + \lambda (g_{ab} u^a u^b + 1)) \mathrm{d}^4 x. \tag{2.18}$$

The field equations are obtained by making variation of Eq.(2.18) with respect to $g_{ab}$, $u^a$ and $\lambda$.

Variation of Eq.(2.18) with respect to $g_{ab}$ is,

$$\begin{aligned} \delta S &= \frac{1}{16\pi G_*} \int \delta(\sqrt{-g}) (R - c_1 F^{ab} F_{ab} + \lambda (g_{ab} u^a u^b + 1)) \mathrm{d}^4 x \\ &+ \frac{1}{16\pi G_*} \int \sqrt{-g} (\delta R - c_1 \delta (F^{ab} F_{ab}) + \lambda (\delta g_{ab} u^a u^b + 1)) \mathrm{d}^4 x. \end{aligned} \tag{2.19}$$

Using Eq.(2.14), the identity

$$\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{ab} \delta g^{ab},$$

and

$$\int \sqrt{-g} g^{ab} \delta R_{ab} \mathtt{d}^4 x = 0,$$

we obtain,

$$\begin{aligned}
\delta S &= -\frac{1}{32\pi G_*}\int \sqrt{-g} g_{ab}\delta g^{ab}(R - c_1 F^{ab}F_{ab} + \lambda(g_{ab}u^a u^b + 1))\mathtt{d}^4 x \\
&+ \frac{1}{16\pi G_*}\int \sqrt{-g} R_{ab}\delta g^{ab}\mathtt{d}^4 x - \frac{c_1}{8\pi G}\int \sqrt{-g} F_{db}F^{d}{}_{a}\delta g^{ab}\mathtt{d}^4 x \\
&- \frac{1}{16\pi G_*}\lambda \int \sqrt{-g} u_a u_b \delta g^{ab}\mathtt{d}^4 x \equiv 0.
\end{aligned}$$

Then, we have

$$-\frac{1}{2} g_{ab} R - \frac{c_1}{2} F^2 g_{ab} - R_{ab} + 2c_1 F_{db}F^{d}{}_{a} + \lambda u_a u_b = 0,$$

where, $F^2 = F^{ab}F_{ab}$. The Einstein tensor $G_{ab}$ is simply given by

$$G_{ab} = \frac{-c_1}{2} g_{ab} F^2 + 2c_1 F_{db}F^{d}{}_{a} + \lambda u_a u_b. \tag{2.20}$$

In terms of the energy momentum tensor of $F_{ab}$ is [8]

$$G_{ab} = 2c_1 \mathcal{T}_{ab} + \lambda u_a u_b,$$

where

$$\mathcal{T}_{ab} = -\frac{1}{4} g_{ab} F^2 + F_{db}F^{d}{}_{a}.$$

Variation with respect to $u^a$ is,

$$\delta S = \int \sqrt{-g}(-c_1 \delta F^2 + \lambda g_{ab}\delta u^a u^b + \lambda g_{ab} u^a \delta u^b)\mathtt{d}^4 x.$$

$$\delta S = -2\int \sqrt{-g} c_1 F^{ab}(\nabla_a \delta u_b - \nabla_b \delta u_a)\mathtt{d}^4 x + 2\int \sqrt{-g}\lambda g_{ab} u^b \delta u^a \mathtt{d}^4 x.$$

Using Gauss' theorem we have,

$$\delta S = 4c_1 \int \sqrt{-g}(\nabla_b F^{b}{}_{a})\delta u^a + 2\int \sqrt{-g}\lambda u_a \delta u^a \mathtt{d}^4 x = 0,$$

then we obtain

$$4c_1\nabla_b F^b{}_a + 2\lambda u_a = 0,$$
$$\nabla_a F^{ab} = -\frac{\lambda}{2c_1}u^b, \tag{2.21}$$

and variation with respect to $\lambda$ gives the normalization condition

$$g_{ab}u^a u^b = -1. \tag{2.22}$$

Now if we identify the rest energy density of dust $\rho$ and vector potential $A_n$ as [3]

$$\rho \leftrightarrow \lambda$$

$$A_n \leftrightarrow \sqrt{2c_1}u_n,$$

Eq.(2.21) can be interpreted as the Maxwell equation with 4-velocity $u_b$ and charge density $-\lambda/\sqrt{2c_1}$ of a charged dust fluid. We can easily find that charge to mass ratio is $-1/\sqrt{2c_1}$. Furthermore, in terms of the vector potential the constraint equation (2.22) turns out to be

$$A_n A^n = -2c_1.$$

It can be easily seen that $c_1$ and $\lambda$ are positive.

In order to obtain an explicit equation of motion for the dust, we take the gradient of the constraint equation(2.22),

$$\begin{aligned} \nabla_a(u^b u_b) &= 0, \\ &= 2u^b\nabla_a u_b, \\ &= 2(u^b\nabla_b u_a + u^b F_{ab}). \end{aligned} \tag{2.23}$$

Defining $\tilde{F_{ab}} = \sqrt{2c_1}F_{ab}$, Eq.(2.23) becomes,

$$u^b\nabla_b u_a = -\frac{1}{\sqrt{2c_1}}\tilde{F_{ab}}u^b. \tag{2.24}$$

This is the equation of motion for a particle in the electromagnetic field $\tilde{F_{ab}}$ with charge to mass ratio $-\frac{1}{\sqrt{2c_1}}$. On the other hand, this Maxwell-like special case of Einstein-aether theory is different from the usual Einstein-Maxwell theory due to the constraint equation (2.22) which breaks the gauge invariance of the theory.

# CHAPTER 3

# GÖDEL-TYPE METRICS IN EINSTEIN-AETHER THEORY

## 3.1 Gödel-Type Metrics in General Relativity

In this chapter we briefly review the Gödel-type metrics in general relativity and show that these type of metrics are also exact solutions of the Einstein-aether theory.

The Gödel metric

$$ds^2 = -(dx^0)^2 + (dx^1)^2 - \frac{1}{2}e^{2x^1}(dx^2)^2 + (dx^3)^2 - 2e^{x^1}dx^0dx^2, \tag{3.1}$$

which describes a pressure-free perfect fluid solution in general relativity with a negative cosmological constant, was introduced by Kurt Gödel in 1949 [12]. It possesses closed timelike and null curves. This metric can be put into the form [6, 13]

$$g_{ab} = h_{ab} - u_a u_b,$$

in two different ways. First, if we consider the background metric $h_{ab}$ to be a 3-dimensional non-flat metric then Eq.(3.1) takes the form

$$ds^2 = (dx^1)^2 + \frac{1}{2}e^{2x^1}(dx^2)^2 + (dx^3)^2 - (dx^0 + e^{x^1}dx^2)^2,$$

with a timelike unit vector

$$u_a = \delta_a^0 + e^{x^1}\delta_a^2.$$

Second, if we consider the background metric $h_{ab}$ to be the 3-dimensional flat metric then Eq.(3.1) takes the form

$$ds^2 = (dx^0)^2 + (dx^1)^2 + (dx^3)^2 - (\sqrt{2}dx^0 + \frac{1}{\sqrt{2}}e^{x^1}dx^2)^2,$$

with a unit timelike vector

$$u_a = \sqrt{2}\delta_a^0 + (1/\sqrt{2})e^{x^1}\delta_a^2.$$

Motivated by the Gödel metric, a class of metrics of the form

$$g_{ab} = h_{ab} - u_a u_b, \tag{3.2}$$

are called Gödel-type metrics if the following conditions are satisfied.

1. $h_{ab}$ is a degenerate $(D \times D)$ matrix with rank equal to $D - 1$,
2. $h_{0a} = 0$, $x^0$ is a fixed coordinate,
3. $\partial_0 h_{ab} = 0$,
4. $h_{ij}$ is the metric of $(D - 1)$ dimensional Euclidian space,
5. $u^a$ is a timelike unit vector, i.e. $u^a u_a = -1$,
6. $\partial_0 u_a = 0$.

In [6, 7] it was shown that Gödel-type metrics can be used in constructing solutions in various dimensions.

Now, we show that in four dimensions Gödel-type metrics form an exact solution of the Einstein equations with charged dust source provided that a simple 3-dimensional Euclidian source-free Maxwell's equation is satisfied.
Let

$$u^a = -\frac{1}{u_0}\delta_0^a$$

be a timelike vector with $u_0 = 1$ and $u_k$=constant. Then, defining an antisymmetric tensor $f_{ab}$,

$$f_{ab} = u_{b,a} - u_{a,b} = 2\nabla_{[a}u_{b]},$$

which is closely analogous with the electromagnetic field tensor, the Christoffel symbols can be found as

$$\Gamma^m{}_{ab} = \frac{1}{2}(u_a f^m{}_b + u_b f^m{}_a) - \frac{1}{2}(u_{a,b} + u_{b,a})u^m.$$

The following identities are useful in the derivation of Einstein tensor:

$$u^a \partial_a u_b = 0,$$

$$u^a f_{ab} = 0,$$

$$\dot{u}^a = 0.$$

The Ricci tensor is given by

$$R_{bd} = \partial_c \Gamma^c_{bd} - \partial_d \Gamma^c_{bc} + \Gamma^e_{bd}\Gamma^c_{ec} - \Gamma^e_{bc}\Gamma^c_{ed}$$

and the corresponding Christoffel symbols can be calculated as

$$\Gamma^c{}_{bc} = \frac{1}{2}(u_b f^c{}_b + u_c f^c{}_b) - \frac{1}{2}(u_{b,c} + u_{c,b})u^c.$$

On the other hand, we calculate the following expressions

$$\begin{aligned}
\partial_c \Gamma^c_{bd} &= \frac{1}{2}\partial_c(u_b f^c{}_d + u_d f^c{}_b) - \frac{1}{2}\partial_c(u^c(u_{b,d} + u_{d,b})), \\
&= \frac{1}{2}(u_{b,c} f^c{}_d - u_b \partial_c f_d{}^c + u_{d,c} f^c{}_b - u_d \partial_c f_b{}^c), \\
&= \frac{1}{2}(u_{b,c} f^c{}_d - u_b j_d + u_{d,c} f^c{}_b - u_d j_b),
\end{aligned}$$

where $j_b = \partial_a f_b{}^a$.

$$\begin{aligned}
\Gamma^e_{bc}\Gamma^c_{ed} &= \frac{1}{4}(u_b f^e{}_c + u_c f^e{}_b - u^e u_{b,c} - u^e u_{c,b})(u_e f^c{}_d + u_d f^c{}_e - u^c u_{e,d} - u^c u_{d,e}), \\
&= \frac{1}{4}\left(-u_b u_d f^2 + f^e{}_b(u_{e,d} + u_{d,e}) + f^c{}_d(u_{b,c} + u_{c,b})\right). \qquad (3.3)
\end{aligned}$$

Then we get the Ricci tensor,

$$\begin{aligned}
R_{bd} &= \frac{1}{2}(u_{b,c} f^c{}_d - u_b j_d + u_{d,c} f^c{}_b - u_d j_b) \\
&- \frac{1}{4}\left(-u_b u_d f^2 + f^e{}_b(u_{e,d} + u_{d,e}) + f^c{}_d(u_{b,c} + u_{c,b})\right) \\
&= \frac{1}{2}(f_{de} f_b{}^e - u_b j_d - u_d j_b) + \frac{1}{4}u_b u_d f^2.
\end{aligned}$$

The Ricci scalar is obtained as,

$$\begin{aligned}
R &= g^{bd} R_{bd} \\
&= \frac{1}{2}(g^{bd} f_{de} f_b{}^e - g^{bd} u_b j_d - g^{bd} u_d j_b) + \frac{1}{4} g^{bd} u_b u_d f^2 \\
&= \frac{1}{4} f^2 - u^d j_d.
\end{aligned}$$

Finally the Einstein tensor, with $j_d = 0$, is simply given by,

$$\begin{aligned} G_{bd} &= R_{bd} - \frac{1}{2}g_{bd}R \\ &= \frac{1}{2}f_{de}f_b{}^{\ e} + \frac{1}{4}u_bu_df^2 - \frac{1}{8}g_{bd}f^2 \\ &= \frac{1}{2}T^f_{bd} + \frac{1}{4}f^2u_bu_d, \end{aligned} \tag{3.4}$$

where the Maxwell energy momentum tensor $T^f_{bd}$ is

$$T^f_{bd} = f_{de}f_b{}^{\ e} - \frac{1}{4}g_{bd}f^2.$$

Eq.(3.4) implies that the Gödel-type metric(3.2) is a solution of the charged dust field equations in 4-dimensions. The energy density of the dust fluid is $\frac{1}{4}f^2$. Furthermore, we have

$$j_i = \partial_k f_i{}^{\ k} = 0, \qquad i = 1, 2, 3$$

since $j_0 = 0$.

In covariant form the above equation can also be written as

$$\nabla_a f^{ab} = \frac{1}{2}f^2u^b. \tag{3.5}$$

## 3.2 Gödel-Type Metrics in Einstein-Aether Theory

In this section we show that Gödel-type of metrics of general relativity are also exact solutions of the Einstein-aether theory. We use the Gödel-type metric and its time-like vector $u^a$ in Einstein-aether theory [8].
Using

$$f_{ab} = u_{b,a} - u_{a,b} = 2\nabla_{[a}u_{b]}$$

and $u^a\nabla_a u^b = 0(\dot{u}^b = 0)$, we find

$$J^a{}_m = \frac{1}{2}(c_1 - c_3)f^a{}_m. \tag{3.6}$$

To calculate $\lambda$,

$$\lambda = c_4\dot{u}^a\dot{u}_a + u^a\nabla_b J^b{}_a,$$

we specifically calculate,

$$\begin{aligned} \nabla_b J^b{}_a &= \frac{1}{2}(c_1 - c_3)(\partial_b f^b{}_a - \Gamma^e_{ab}f^b{}_e) \\ &= \frac{1}{2}(c_1 - c_3)(\partial_b f^b{}_a + \frac{1}{2}f^2u_a). \end{aligned}$$

$$\begin{aligned} u^a\nabla_b J^b{}_a &= \frac{1}{2}(c_1-c_3)(u^a\partial_b f^b{}_a - \frac{1}{2}f^2) \\ &= -\frac{1}{4}(c_1-c_3)f^2. \end{aligned}$$

Then we obtain,

$$\lambda = -\frac{1}{4}(c_1-c_3)f^2.$$

In order to calculate the field equations the following derivations are useful:

$$\begin{aligned} L &= K^{ab}_{mn}(\nabla_a u^m)(\nabla_b u^n) \\ &= \frac{1}{2}(c_1 g^{ab}g_{mn} + c_2\delta^a_m\delta^b_n + c_3\delta^a_n\delta^b_m + c_4 u^a u^b g_{mn})(-f^m{}_a)(\nabla_b u^n) \\ &= \frac{1}{4}(c_1-c_3)f^2. \end{aligned} \tag{3.7}$$

The Maxwell equation becomes,

$$\nabla_e f^e{}_a = \frac{1}{2}f^2 u_a. \tag{3.8}$$

To obtain the field equations we calculate

$$\begin{aligned} \nabla_e(J^e{}_{(a}u_{b)} - J_{(a}{}^e u_{b)} + J_{(ab)}u^e) &= \nabla_e(J^e{}_a u_b + J^e{}_b u_a) \\ &= (\nabla_e J^e{}_a)u_b + J^e{}_a\nabla_e u_b + (\nabla_e J^e{}_b)u_a + J^e{}_b\nabla_e u_a \\ &= \frac{1}{2}(c_1-c_3)(f^2 u_a u_b + f^e{}_a\nabla_e u_b + f^e{}_b\nabla_e u_a) \end{aligned}$$

and

$$\begin{aligned} c_1(\nabla_a u_e\nabla_b u^e - \nabla_e u_a\nabla^e u_b) &= c_1(f_{ae} + \nabla_e u_a)\nabla_b u^e - c_1\nabla_e u_a\nabla^e u_b \\ &= c_1 f_{ae}\nabla_b u^e + c_1\nabla_e u_a(f_b{}^e), \end{aligned}$$

where we have used Eq.(3.6) and (3.8). Therefore, the Einstein field equations comes out to be

$$G_{ab} = (c_1-c_3)(\frac{1}{2}T^f_{ab} + \frac{1}{4}f^2 u_a u_b). \tag{3.9}$$

Now, a comparison of Eq.(3.8) and (3.9) with Eq.(3.4) and (3.5) implies that $c_1 - c_3 = 1$. Hence the only field equations remaining for the Einstein-aether theory are given in Eq.(3.8).

# CHAPTER 4

# NEWTONIAN LIMIT OF THE EINSTEIN-AETHER THEORY

In this chapter, we will study the Newtonian limit of the Einstein-aether theory [9] which will enable us to see the observable effects of the aether field(for example, Newtonian theory works well in the Solar System experiments).We assume that the metric field is so weak that we can consider it as nearly flat. And therefore, we can split the metric into two:

$$g_{ab} = \eta_{ab} + h_{ab},$$

where $\eta_{ab}$ is the Minkowski metric and $h_{ab}$ is the small metric perturbation such that $\mid h_{ab} \mid \ll 1$.

To linearize the field equations we keep only the first order terms of $h_{ab}$. Then, the linearized Christoffel symbols can be written as,

$$\Gamma^m_{\ ab} = \frac{1}{2}(\partial_a h_b{}^m + \partial_b h_a^{\ m} - \partial^m h_{ab})$$

and the Ricci tensor is,

$$\begin{aligned} R_{ab} &= \partial_c \Gamma^c_{ab} - \partial_b \Gamma^c_{ac} \\ &= \frac{1}{2}(\partial^c \partial_a h_{bc} - \Box h_{ab} - \partial_b \partial_a h + \partial_b \partial^c h_{ac}) \end{aligned} \tag{4.1}$$

where, $\Box \equiv \partial_a \partial^a$, the d'Alembertian operator and $h \equiv h^a_{\ a} = \eta^{ab} h_{ab}$, trace of the $h_{ab}$.

The Ricci scalar is,

$$R = \partial_a \partial_b h^{ab} - \Box h.$$

Finally, we obtain the linearized Einstein tensor,

$$G_{ab} = \frac{1}{2}(\partial^c \partial_a h_{bc} - \Box h_{ab} - \partial_b \partial_a h + \partial_b \partial^c h_{ac} - \eta_{ab} \partial_c \partial_d h^{cd} + \eta_{ab} \Box h). \tag{4.2}$$

For simplicity, we define a new quantity [14],

$$\gamma_{ab} \equiv h_{ab} - \frac{1}{2}\eta_{ab}h,$$

in terms of which $g_{ab}$ is given by,

$$g_{ab} = \eta_{ab} + \gamma_{ab} - \frac{1}{2}\eta_{ab}\gamma, \tag{4.3}$$

where $\gamma$ is the trace of the $\gamma_{ab}$.

Then, linearized field equations in terms of the defined quantity are,

$$G_{ab} = \frac{1}{2}(\partial^c\partial_a\gamma_{bc} + \partial^c\partial_b\gamma_{ac} - \eta_{ab}\partial_c\partial_d\gamma^{cd} - \Box\gamma_{ab}). \tag{4.4}$$

Using the Hilbert gauge [14],

$$\partial_b\gamma^{ab} = 0,$$

our linearized field equations reduce to,

$$G_{ab} = -\frac{1}{2}\Box\gamma_{ab}. \tag{4.5}$$

Eq.(4.5) can also be written as,

$$\Box\gamma_{ab} = -16\pi G\, T_{ab}. \tag{4.6}$$

We consider the retarded solution of Eq.(4.6) [14],

$$\gamma_{ab}(x) = 4G_N \int d^3x' \frac{T_{ab}(x^0 - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}. \tag{4.7}$$

In the Newtonian limit we consider small velocities, which means $T_{00} \gg |T_{0i}|$ and $T_{00} \gg |T_{ij}|$. Then Eq.(4.7) becomes,

$$\gamma_{00} = -4\phi,$$

$$\gamma_{0i} = \gamma_{ij} = 0,$$

where $\phi$ is the Newtonian potential,

$$\phi(\mathbf{x}) = -G_N \int d^3x' \frac{T_{00}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}.$$

Using Eq.(4.3) we can write the Newtonian limit metric,

$$ds^2 = -(1 + 2\phi)dt^2 + (1 - 2\phi)(dx^2 + dy^2 + dz^2). \tag{4.8}$$

We can choose, without loss of generality, the dynamical vector $u^a$ as,

$$u^a = (u^0, 0, 0, 0) = u^0 \delta^a_0, \tag{4.9}$$

with the condition,

$$u^a u_a = -1.$$

The aether field in Newtonian limit becomes,

$$u^0 g_{00} u^0 = -1.$$

$$u^0 = (1 + 2\phi)^{-\frac{1}{2}} \cong 1 - \phi.$$

Then we can write,

$$u^a = (1 - \phi)\delta^a_0$$

and also we have

$$u_a = g_{ab} u^b = g_{a0} u^0 = g_{00} u^0 \delta^0_a = (-1 - 2\phi)(1 - \phi)\delta^0_a \cong -(1 + \phi)\delta^0_a. \tag{4.10}$$

Using Eq.(2.11), we obtain the value of Lagrange multiplier field $\lambda$ in Newtonian limit,

$$\lambda = c_3 \nabla^2 \phi,$$

where $\nabla^2$ is the three-dimensional Laplace operator.

The energy-momentum tensor for the vector field at linear order is,

$$S_{00} = c_{14} \nabla^2 \phi,$$

and other components are zero.

The linearized field equations are,

$$G_{ab} = R_{ab} - \frac{1}{2}\eta_{ab} R.$$

Then we have,

$$G_{ab} = \frac{1}{2}(\partial_c \partial_b h^c{}_a + \partial_c \partial_a h^c_b - \partial_a \partial_b h - \partial_d \partial^d h_{ab} - \eta_{ab} \partial_c \partial_d h^{cd} + \eta_{ab} \partial_d \partial^d h). \tag{4.11}$$

The Einstein field equations, with matter field, can be written as,

$$G_{ab} = S_{ab} + 8\pi G_* T_{ab}. \tag{4.12}$$

The time-time component of Eq.(4.12) becomes

$$S_{00} + 8\pi G_* T_{00} = 2\nabla^2 \phi, \tag{4.13}$$

where $T_{00}$ is the matter energy-momentum tensor for a dust distribution.

We assume $T_{00} = \rho_m(\mathbf{x})$ where $\rho_m(\mathbf{x})$ is the matter density.

Rewriting Eq.(4.13) as a Poisson's equation we have,

$$\nabla^2 \phi = 4\pi G_N \rho_m,$$

where $G_N$ is the Newton's constant,

$$G_N = \left(1 - \frac{c_{14}}{2}\right)^{-1} G_* \tag{4.14}$$

which is seen to be rescaled.

# CHAPTER 5

# POST-NEWTONIAN PARAMETERS AND CONSTRAINTS IN EINSTEIN-AETHER THEORY

There are alternative theories of gravity to explain the geometry of the universe. We can test a candidate theory by comparing it with the Solar System observations. In the Solar System the gravity is weak and stress, internal energy and velocity of the matter are low, which enable us to simplify our theory without losing any accuracy. It can be considered as a correction to the Newtonian theory. This correction is called as 'Parametrized Post Newtonian(PPN) analysis'. It has 10 parameters, $\gamma$(related to spatial curvature), $\beta$(related to nonlinearity), $\xi$(related to preferred location effects), $\zeta_1, \zeta_2, \zeta_3, \zeta_4$(related to total momentum conservation), $\alpha_1, \alpha_2$(related to preferred frame effects), $\alpha_3$(related to total momentum conservation and preferred frame effects), named as 'PPN parameters', to be specified [1, 10, 11].

In this chapter, we examine PPN parameters in Einstein-aether theory [15]. Before calculation of the PPN parameters of the Einstein-aether theory, we will give necessary introductive calculations and definitions.

It is convenient to write the Einstein's field equations

$$T^G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R, \tag{5.1}$$

in terms of the total energy momentum tensor as

$$T^G_{ab} = R_{ab} + \frac{1}{2} g_{ab} T^G,$$

where $T^G_{ab} = S_{ab} + T_{ab}$ and $T^G = T^{G\,a}_{\ \ a}$.

We rewrite the Einstein's equations in a nonstandard form as

$$R_{ab} = (S_{cd} + 8\pi G_* T_{cd})(\delta^c_a \delta^d_b - \frac{1}{2} g_{ab} g^{cd}), \tag{5.2}$$

where

$$T^{ab} = (\rho + \rho\Pi + p)\vartheta^a\vartheta^b + pg^{ab}. \tag{5.3}$$

$T^{ab}$ is the stress tensor for the perfect fluid, $\rho$ is the rest mass energy density, $\Pi$ is the internal energy density, $p$ is the isotropic pressure and $\vartheta$ is the four-velocity of the fluid.

In the PPN analysis, order is a crucial point. We take $\rho \sim \Pi \sim p/\rho \sim (\vartheta^i)^2 \sim O(1)$. If a quantity's order is $X$, after taking its time derivative its order becomes $X + 1/2$. For the metric perturbations $h_{ab}$ we assume, $h_{00} \sim O(1) + O(2)$, $h_{ij} \sim O(1)$, $h_{0i} \sim O(1, 5)$.

The following relations are useful for later use

$$\begin{aligned} \nabla^2\Phi_1 &= -4\pi\rho G_N\vartheta^2, \\ \nabla^2\Phi_2 &= -4\pi\rho G_N\phi, \\ \nabla^2\Phi_3 &= -4\pi\rho G_N\Pi, \\ \nabla^2\Phi_4 &= -4\pi p G_N. \end{aligned} \tag{5.4}$$

The explicit definitions of functions $\Phi_1, \Phi_2, \Phi_3$ and $\Phi_4$, are given in ( [1],section 4.1). We define the 'Superpotential' $\chi(\mathbf{x}, t)$ as

$$\chi(\mathbf{x}, t) = -G_N \int \mathrm{d}^3x' \rho(\mathbf{x}', t)|\mathbf{x} - \mathbf{x}'|, \tag{5.5}$$

which satisfies

$$\chi_{,ii} = -2\phi. \tag{5.6}$$

We also use the relation,

$$\chi_{,0i} = V_i - W_i, \tag{5.7}$$

where

$$V_i = \int \mathrm{d}^3x' \frac{\rho(\mathbf{x}', t)\vartheta'_i}{|\mathbf{x} - \mathbf{x}'|},$$

$$W_i = \int \mathrm{d}^3x' \frac{\rho(\mathbf{x}', t)\mathbf{v}' \cdot (\mathbf{x} - \mathbf{x}')(x - x')_i}{|\mathbf{x} - \mathbf{x}'|^3}.$$

These potentials are the results of the work for a general, reasonable and simple post-Newtonian metric [15],

$$
\begin{aligned}
g_{00} = & - 1 - 2\phi + 2\beta\phi^2 + 2\xi\Phi_W - (2\gamma + 2 + \alpha_3 + \zeta_1 - 2\xi)\Phi_1 \\
& - 2(3\gamma - 2\beta + 1 + \zeta_2 + \xi)\Phi_2 - 2(1 + \zeta_3)\Phi_3 \\
& - 2(3\gamma + 3\zeta_4 - 2\xi)\Phi_4 + (\zeta_1 - 2\xi)A,
\end{aligned}
$$

$$
\begin{aligned}
g_{0i} &= \frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1 - 2\xi)V_i + \frac{1}{2}(1 + \alpha_2 + \zeta_1 + 2\xi)W_i, \\
g_{ij} &= (1 - 2\gamma\phi)\delta_{ij}.
\end{aligned} \tag{5.8}
$$

The solving procedure involves 6 steps:

**1. $\mathbf{u^0}$ to $O(\mathbf{1})$**

We solve the constraint equation(2.9) for $u^0$ to $O(1)$,

$$u^0 = 1 + \frac{1}{2}h_{00}$$

and also we have

$$u_0 = g_{00}u^0 = -1 + \frac{1}{2}h_{00},$$

$$u_i = u^a g_{ai} = -u^i + h_{0i} = n_i + h_{0i},$$

where

$$n_i = u_i - h_{0i}. \tag{5.9}$$

From the constraint equation(2.9),

$$g_{00}u^0 u^0 = -1,$$

we obtain the covariant derivative of $u_0$ as

$$\nabla_a u_0 = 0, \tag{5.10}$$

to $O(2)$.

Also to $O(2)$ we calculate

$$
\begin{aligned}
\nabla_0 u_i &= \partial_0 u_i - \frac{1}{2}g^{ab}(\partial_i g_{b0} + \partial_0 g_{bi} - \partial_b g_{i0})u_a \\
&= \partial_0 u_i - \frac{1}{2}(\partial_i h_{b0} + \partial_0 h_{bi} - \partial_b h_{i0})u^b \\
&= u_{i,0} - \frac{1}{2}(\partial_i h_{00} + \partial_0 h_{0i} - \partial_0 h_{i0})u^0,
\end{aligned} \tag{5.11}
$$

where $u_{i,0} = \partial_0 u_i$. It simplifies to

$$\begin{aligned} \nabla_0 u_i &= \partial_0 u_i - \frac{1}{2}h_{00,i}(1 + \frac{1}{2}h_{00}) \\ &= n_{i,0} + h_{0i,0} - \frac{1}{2}h_{00,i}(1 + \frac{1}{2}h_{00}), \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} \dot{u}_i = u^a \nabla_a u_i = u^0 \nabla_0 u_i &= (1 + \frac{1}{2}h_{00})[n_{i,0} + h_{0i,0} - \frac{1}{2}h_{00,i}(1 + \frac{1}{2}h_{00})] \\ &= n_{i,0} + h_{0i,0} - \frac{1}{2}h_{00,i}(1 + \frac{1}{2}h_{00})^2 \\ &= n_{i,0} + h_{0i,0} - \frac{1}{2}h_{00,i}(1 + h_{00}). \end{aligned} \tag{5.13}$$

To $O(1.5)$,we calculate

$$\begin{aligned} \nabla_j u_i &= u_{i,j} - \Gamma^k_{ij} u_k \\ &= u_{i,j} - \frac{1}{2}(h_{0j,i} + h_{0i,j} - h_{ij,0}) \\ &= n_{i,j} + \frac{1}{2}h_{ij,0} + h_{0[i,j]}. \end{aligned} \tag{5.14}$$

**2. $g_{00}$ to $O(1)$**

Time-time component of Ricci tensor at $O(2)$ is [1]

$$R_{00} = \frac{1}{2}(-h_{00,ii} - (h_{ii,00} - 2h_{i0,i0}) + h_{00,j}(h_{ji,i} - \frac{1}{2}h_{ii,j}) - \frac{1}{2}h_{00,i}h_{00,i} + h_{ij}h_{00,ij}). \tag{5.15}$$

At $O(1)$ it reduces to

$$R_{00} = -\frac{1}{2}h_{00,ii}.$$

Also, to $O(1)$, we evaluate

$$T_{00} = \rho\vartheta_0^2 = \rho(1 - 2\phi + \vartheta^2) = \rho,$$

$$T_{ij} = 0.$$

To compute the $S_{00}$ term we need to evaluate

$$\begin{aligned} \nabla_m({J_0}^m u_0) &= u_0(\partial_i {J_0}^i + \Gamma^i_{ai}{J_0}^a - \Gamma^a_{0i}{J_a}^i \\ &= u_0 \partial_i {J_0}^i \\ &= \frac{c_{14}}{2}h_{00,ii}, \end{aligned} \tag{5.16}$$

and then to $O(1)$ ,

$$S_{00} = -\nabla_m({J_0}^m u_0) = -\frac{c_{14}}{2}h_{00,ii},$$

and simply

$$S_{ij} = 0.$$

Then Eq.(5.2) becomes,

$$\begin{aligned} 2R_{00} - S_{00} &= 8\pi G_* \rho, \\ \left(-1 + \frac{c_{14}}{2}\right) h_{00,ii} &= 8\pi G_* \rho, \\ \nabla^2 \phi &= 4\pi G_* \rho \left(1 - \frac{c_{14}}{2}\right)^{-1}, \end{aligned} \tag{5.17}$$

giving the Newton's constant as

$$G_N = \left(1 - \frac{c_{14}}{2}\right)^{-1} G_*, \tag{5.18}$$

which is in agreement with Eq.(4.14).

### 3. $\mathbf{g_{ij}}$ to $O(\mathbf{1})$

Space-space component of Ricci tensor at $O(1)$ is [1]

$$R_{ij} = \frac{1}{2}(-h_{ij,kk} + h_{00,ij} - h_{kk,ij} + h_{ki,kj} + h_{kj,ki}).$$

Imposing the gauge condition

$$h_{ij,j} = -\frac{1}{2}(h_{00,i} - h_{jj,i}), \tag{5.19}$$

we have

$$R_{ij} = -\frac{1}{2} h_{ij,kk}.$$

Also from Eq.(5.2) we can write

$$R_{ij} = -4\pi G_N \rho \delta_{ij}, \tag{5.20}$$

where we have used Eq.(5.17) and (5.18).

Then Eq.(5.20) becomes

$$h_{ij,kk} = 8\pi G_N \rho \delta_{ij}.$$

### 4. $\mathbf{u^i}$ to $O(\mathbf{1,5})$

We solve the space components of the aether field Eq.(2.11) for $u^i$ to $O$(1,5).

$$\nabla_a J^a{}_i = \partial_a J^a{}_i = 0.$$

$$\partial_0 J_{0i} - \partial_j J_{ji} = 0. \tag{5.21}$$

To $O(1.5)$ we have

$$\partial_0 J_{0i} = c_{14}\partial_0(\nabla_0 u_i) = -\frac{c_{14}}{2}h_{00,0i} = -\frac{c_{14}}{2}\chi_{,0ijj}, \tag{5.22}$$

and

$$\partial_j J_{ji} = c_1\partial_j(\nabla_j u_i) + c_2\partial_i(\nabla_k u_k) + c_3\partial_j(\nabla_i u_j).$$

Making use of Eq.(5.14), gauge condition(5.19), we have

$$\partial_j J_{ji} = c_1 n_{i,jj} + c_{23}n_{j,ji} + (c_1 - c_3)h_{0[i,j]j} + \left(\frac{c_{13}}{2} + \frac{3c_2}{2}\right)\chi_{,0ikk}.$$

The aether field equation(5.21) then becomes

$$\left(c_1 n_i + \frac{c_-}{2}h_{0i} + \frac{1}{2}(2c_1 + 3c_2 + c_3 + c_4)\chi_{,i0}\right)_{,jj} - \left(\frac{c_-}{2}h_{0j,j} - c_{23}n_{j,j}\right)_{,i} = 0. \tag{5.23}$$

where $c_- = c_1 - c_3$.

Taking the spatial divergence of (5.23), we obtain

$$c_1 n_{i,ijj} + \frac{1}{2}(2c_1 + 3c_2 + c_{34})\chi_{,0iijj} + c_{23}n_{j,jii} = 0.$$

This can be further written as

$$n_{i,ijj} = C\chi_{,0iijj}, \tag{5.24}$$

here

$$C = -\frac{2c_1 + 3c_2 + c_{34}}{2c_{123}}.$$

Using Eq.(5.24) and gauge condition

$$h_{oi,i} = -3U_{,0} + \theta n_{i,i} \tag{5.25}$$

Eq.(5.23) can be written as

$$n_i = -\frac{1}{2c_1}\left(c_- h_{0i} - (2c_1 C + c_-(3/2 + C\theta))\chi_{,0i}\right) \tag{5.26}$$

where $\theta$ is an arbitrary parameter.

### 5. $\mathbf{g_{0i}}$ to $O(1,5)$

We solve the time-space components of Eq.(5.2) for $g_{0i}$ to $O(1,5)$.

We have [1]

$$R_{0i} = -\frac{1}{2}(h_{0i,kk} - h_{k0,ik} + h_{kk,0i} - h_{ki,0k}), \tag{5.27}$$

to $O(1,5)$.

Using Eq.(5.19), (5.24) and, (5.25) we obtain

$$R_{0i} = -\frac{1}{2}\left(h_{0i} + \frac{1}{2}(1-2\theta C)\chi_{,0i}\right)_{,kk} \tag{5.28}$$

and

$$T_{0i} = -\rho\vartheta_i,$$

$$S_{0i} = J_{0i,0} - \frac{1}{2}(-J_{ij,j} + J_{ji,j}).$$

Using Eq.(5.21), we have

$$S_{0i} = \frac{1}{2}(J_{0i,0} + J_{ij,j}).$$

$$\begin{aligned} J_{ij,j} &= (c_1\nabla_i u_j + c_2\delta_{ij}\nabla_k u_k + c_3\nabla_j u_i)_{,j} \\ &= c_{12}n_{j,ji} + c_3 n_{i,jj} + \frac{1}{2}(c_{13}h_{ij,0j} + c_2 h_{jj,0i} + 2c_- h_{0[j,i]j}). \end{aligned} \tag{5.29}$$

Using Eq.(5.26) we have

$$\begin{aligned} J_{ij,j} = &- \frac{c_-c_+}{2c_1}h_{0i,jj} + \left(\frac{-c_{12}c_-}{2c_1} + \frac{c_-}{2}\right)h_{0j,ij} \\ &+ \left(C + \frac{c_-}{2c_1}(\frac{3}{2} + C\theta)\right)(c_{12}+c_3)\chi_{,0ijj} \\ &- (c_+ + 3c_2)\phi_{,0i}, \end{aligned}$$

where $c_+ = c_{13}$.

Using the gauge condition(5.25) and Eq.(5.24) we obtain

$$J_{ij,j} = \left(-\frac{c_-c_+}{2c_1}h_{0i} + (\frac{c_{14}}{2} - C^*)\chi_{,0i}\right)_{,jj}, \tag{5.30}$$

where

$$C^* = \frac{1}{4c_1}(c_1^2 + 3c_3^2 + 4c_1c_4 - 2c_-c_+C\theta).$$

Combining with Eq.(5.22) we can write

$$S_{0i} = -\left(\frac{c_-c_+}{4c_1}h_{0i} + \frac{C^*}{2}\chi_{,0i}\right)_{,jj}.$$

Then, field equation becomes

$$R_{0i} = S_{0i} + 8\pi G T_{0i},$$

$$\left(h_{0i} + \frac{1}{2}(1-2\theta C)\chi_{,0i}\right)_{,kk} = \left(-\frac{c_-c_+}{2c_1}h_{0i} - C^*\chi_{,0i}\right)_{,kk} + 16\pi G\rho\vartheta_i,$$

$$\left(1-\frac{c_-c_+}{2c_1}\right)h_{0i,kk}=16\pi G\rho\vartheta_i-\left(C^*+\theta C-\frac{1}{2}\right)\chi_{0i,kk}$$

giving

$$h_{0i}=\left(1-\frac{c_-c_+}{2c_1}\right)^{-1}\left((C^*+\theta C-\frac{1}{2})\chi_{,0i}+4(1-\frac{c_{14}}{2})V_i\right),$$

where we have used [15],

$$V_{i,jj}=-4\pi G_N\rho\vartheta_i.$$

**6. $\mathbf{g_{00}}$ to $O(\mathbf{2})$**

We solve the time-time component of Eq.(5.2) for $g_{00}$ to $O(2)$. From Eq.(5.15), we have

$$R_{00}=\frac{1}{2}\left(\tilde{h}_{00}+2\phi-2\phi^2+8\Phi_2+2C\theta\chi_{,00}\right)_{,ii}, \tag{5.31}$$

where we have defined $\tilde{h}_{00}=g_{00}+1+2\phi$ and used Eq.(5.19), (5.24), (5.25) and relation [1]

$$|\nabla\phi|^2=\nabla^2\left(\frac{1}{2}\phi^2-\Phi_2\right).$$

We also evaluate the components of matter energy momentum tensor

$$\begin{aligned}T_{00} &= (\rho+\rho\Pi+p){\vartheta_0}^2+pg_{00}\\ &= (\rho+\rho\Pi+p)(1+\vartheta^2-2\phi)+p(-1-2\phi)\\ &= \rho(1+\Pi+\vartheta^2-2\phi),\end{aligned} \tag{5.32}$$

$$T_{ij}=\rho\vartheta_i\vartheta_j+p\delta_{ij},$$

to $O(2)$ and,

$$\begin{aligned}T_{00}-\frac{1}{2}g_{00}(T_{ab}g^{ab}) &= \frac{1}{2}(T_{00}+T_{ii})\\ &= \frac{1}{2}\rho(1+\Pi+2(\vartheta^2-\phi))+\frac{3}{2}p\\ &= -\frac{(1-c_{14}/2)}{8\pi G}(\phi+2\Phi_1-2\Phi_2+\Phi_3+3\Phi_4)_{,ii},\end{aligned} \tag{5.33}$$

where we have used the equations (5.4).

Before beginning to compute the $S$ tensor we see that

$$\begin{aligned}\nabla_i u_i &= u_{i,i}-\frac{1}{2}(2h_{mi,i}-h_{ii,m})u^m\\ &= u_{i,i}-\frac{1}{2}(2h_{0i,i}-h_{ii,0}),\end{aligned}$$

to $O(2)$. Using Eq.(5.9) we have

$$\nabla_i u_i = \left(\frac{3}{2} + C\right)\chi_{,0ii}, \tag{5.34}$$

and from Eq.(5.12) we calculate

$$\begin{aligned}(\nabla_0 u_i)_{,i} &= -\frac{1}{2}h_{00,ii} - \frac{1}{4}h_{00,ii}h_{00} - \frac{1}{4}h_{00,i}h_{00,i} + h_{0i,0i} + n_{i,0i} \\ &= \phi_{,ii} - \phi\phi_{,ii} - \phi_{,i}\phi_{,i} + \frac{3}{2}\chi_{,00ii} + (\theta+1)n_{i,0i} \\ &= \left(\phi - \frac{1}{2}\phi^2\right)_{,ii} + \left(\frac{3}{2} + (\theta+1)C\right)\chi_{,00ii} \\ &= -\frac{1}{2}\left[-2\phi + \phi^2 + \tilde{h}_{00} - 2\left(\frac{3}{2} + (\theta+1)C\right)\chi_{,00}\right]_{,ii}. \end{aligned} \tag{5.35}$$

Then we obtain

$$S_{00} = \frac{c_{14}}{2}\left(2\phi + \tilde{h}_{00} - \frac{5}{2}\phi^2 + 9\Phi_2\right)_{,ii} + c_{14}\left(\frac{3}{2} + (\theta+1)C\right)\chi_{,00ii}, \tag{5.36}$$

and

$$S_{ii} = \frac{1}{2}c_{14}\left(\frac{1}{2}\phi^2 - \Phi_2\right)_{,ii} + (c_+ + 3c_2)\left(\frac{3}{2} + C\right)\chi_{,00ii}. \tag{5.37}$$

Then we have

$$\begin{aligned} S_{00} - \frac{1}{2}g_{00}S_{ab}g^{ab} &= \frac{c_{14}}{4}(2\phi + \tilde{h}_{00} - 2\phi^2 + 8\Phi_2)_{,ii} \\ &+ \frac{1}{2}\left(\left(\frac{3}{2} + C\right)(2c_1 + 3c_2 + c_3 + c_4) + c_{14}\theta C\right)\chi_{,00ii}. \end{aligned} \tag{5.38}$$

Solving the field equation we have

$$\tilde{h}_{00} = 2\phi^2 - 4\Phi_1 - 4\Phi_2 - 2\Phi_3 - 6\Phi_4 - Q\chi_{,00},$$

where

$$Q = \left(1 - \frac{c_{14}}{2}\right)^{-1}\left((2 - c_{14})\theta + (c_1 + 2c_3 - c_4)\right)C. \tag{5.39}$$

Using the standard gauge Q vanishes and gives $\theta$ as,

$$\theta_0 = -\frac{c_1 + 2c_3 - c_4}{2 - c_{14}}.$$

Now, we write the metric components as

$$g_{00} = -1 - 2\phi + 2\phi^2 - 4\Phi_1 - 4\Phi_2 - 2\Phi_3 - 6\Phi_4,$$

with $\theta = \theta_0$.

$$g_{ij} = (1 - 2\phi)\delta_{ij},$$

$$
\begin{aligned}
g_{i0} &= \left(1 - \frac{c_- c_+}{2c_1}\right)^{-1} \left((C^* + \theta C - \frac{1}{2})\chi_{,0i} + 4(1 - \frac{c_{14}}{2})V_i\right) \\
&= \frac{2c_1}{2c_1 - c_1^2 + c_3^2} \left((C^* + \theta C - \frac{1}{2} + 2(2 - c_{14}))V_i - (C^* + \theta C - \frac{1}{2})W_i\right)
\end{aligned}
$$

where we have used the relation given in Eq.( 5.7).

Comparing with the Post-Newtonian metric (5.8) we can determine the parameters as

$$\gamma \equiv \beta = 1,$$

$$\xi \equiv \zeta_1 \equiv \zeta_2 \equiv \zeta_3 \equiv \zeta_4 \equiv \alpha_3 = 0,$$

$$\alpha_1 = \frac{-8(c_3^2 + c_1 c_4)}{2c_1 - c_1^2 + c_3^2},$$

$$\alpha_2 = \frac{2(-2\theta_0 C c_1 + C\theta_0 c_1^2 - 2c_1 c_4 - 2c_3^2 - C\theta_0 c_3^2)}{2c_1 - c_1^2 + c_3^2}.$$

We notice that all the PPN parameters of Einstein-aether theory agree with those of general relativity except the preferred frame parameters $\alpha_1$ and $\alpha_2$.

# CHAPTER 6

# CONCLUSION

In this thesis, we have derived the field equations of the Einstein-aether theory from the action principle. We have obtained the observational constraints on the parameters of this theory by using the parametrized post-Newtonian approximation. We have also shown that Gödel-type metrics with constant $u_k$(and $u_0 = 1$) are exact solutions of this theory. It would be worth studying to seek whether Gödel-type metrics with non-constant $u_k$(and $u_0 \neq 1$) provide exact solutions to the theory.

The Einstein-aether theory is an extension of the general relativity with a preferred frame. This frame is described by a dynamical unit timelike vector. As previously mentioned in [8], it would be interesting to analyze the theory with a dynamical null vector instead of a timelike vector. In that case, one would try to seek whether Kerr-Schild metrics provide solutions to the theory.

Finally, even though we have not discussed this in the thesis, one might further study the relation of this theory to the scalar-tensor theories. One might speculate that this theory will play a role in the solution of some fundamental problems such as dark energy and quantum gravity [5, 15].

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