RECURSIVE PASSIVE LOCALIZATION METHODS USING TIME DIFFERENCE OF ARRIVAL

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

SEDAT ÇAMLICA

IN PARTIAL FULLFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
ELECTRICAL AND ELECTRONICS ENGINEERING

SEPTEMBER 2009

Approval of the thesis:

RECURSIVE PASSIVE LOCALIZATION USING TIME DIFFERENCE OF ARRIVAL

submitted by **SEDAT ÇAMLICA** in partial fulfillment of the requirements for the degree of **Master of Science in Electrical and Electronics Engineering Department**, **Middle East Technical University** by,

Prof. Dr. Canan Özgen
Dean, Graduate School of **Natural and Applied Sciences**

Prof. Dr. İsmet Erkmen
Head of Department, **Electrical and Electronics Engineering**

Prof. Dr. Yalçın Tanık
Supervisor, **Electrical and Electronics Engineering Dept., METU**

Examining Committee Members:

Prof. Dr. Mete Severcan
Electrical and Electronics Engineering Dept., METU

Prof. Dr. Yalçın Tanık
Electrical and Electronics Engineering Dept., METU

Prof. Dr. Mübeccel Demirekler
Electrical and Electronics Engineering Dept., METU

Assist. Prof. Dr. Çağatay Candan
Electrical and Electronics Engineering Dept., METU

Dr. Toygar Birinci
ASELSAN Inc.

Date: **17.09.2009**

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last name : Sedat Çamlıca

Signature :

ABSTRACT

RECURSIVE PASSIVE LOCALIZATION METHODS USING TIME DIFFERENCE OF ARRIVAL

Çamlıca, Sedat

M.Sc., Department of Electrical and Electronics Engineering
Supervisor : Prof. Dr. Yalçın Tanık

September 2009, 114 pages

In this thesis, the passive localization problem is studied. Robust and recursive solutions are presented by the use of Time Difference of Arrival (TDOA). The TDOA measurements are assumed to be gathered by moving sensors which makes the number of the sensors increase synthetically.

First of all, a location estimator should be capable of processing the new measurements without omitting the past data. This task can be accomplished by updating the estimate recursively whenever new measurements are available. Convenient forms of the recursive filters, such as the Kalman filter, the Extended Kalman filter etc., can be applied. Recursive filter can be divided to two major groups: (a) The first type of recursive estimators process the TDOA measurements directly, and (b) the second type of the recursive estimators is the post processing estimators which process the TDOA indirectly, instead they fuse or smooth available location estimates. In this sense, recursive passive localization methods are presented for both types.

In practice, issues like being spatially distant from each other and/or a radar with a rotating narrow beam may prevent the sensors to receive the same pulse. In such a case, the sensors can not construct common TDOA measurements which means that they can not accomplish the location estimation procedure. Additionally, there may be more than one sensor group making TDOA measurements. An estimator should be capable of fusing the measurements from different sensor groups. A sensor group consists of sensors which are able to receive the same pulse. In this work, solutions of these tasks are also given.

Performances of the presented methods are compared by simulation studies. The method having the best performance, which is based on the Kalman Filter, is also capable of estimating the track of a moving emitter by directly processing the TDOA measurements.

Keywords : Time Difference of Arrival, Kalman Filter, emitter localization, recursive estimators.

ÖZ

VARIŞ ZAMANI FARKI KULLANARAK ÖZYİNELEMELİ PASİF KONUM BELİRLEME

Çamlıca, Sedat

Yüksek Lisans, Elektrik Elektronik Mühendisliği Bölümü
Tez Yöneticisi : Prof. Dr. Yalçın Tanık

Eylül 2009, 114 sayfa

Bu tezde, pasif konum belirme problemi üzerine çalışılmıştır. Bu probleme Varış Zamanları Farkları (VZF) kullanılarak gürbüz ve özyinelemeli çözümler getirilmiştir. VZF ölçümleri hareketli sensorlar tarafından yapılmaktadır. Sensorların hareketli olması nedeniyle sensor sayısı yapay olarak artmaktadır.

Konum kestirimi yapan bir yöntem eski ölçüm bilgisini ihmal etmeden, yeni ölçümleri kullanabilme yeteneğine sahip olmalıdır. Bu gerek, yeni ölçümler alındıkça konum kestirimini özyinelemeli bir şekilde güncelleyerek sağlanabilir. Kalman Süzgeci ve türevleri gibi uygun bir özyinelemli süzgeç yardımıyla bu işlem gerçekleştirilebilir. Özyinelemeli yöntemler iki ana grup altında toplanabilir: (a) Birinci gruptaki yöntemler VZF ölçümlerini doğrudan işleyebilme yeteneğine sahiptirler. (b) İkinci gruptaki yöntemler ise VZF ölçümlerini doğrudan işlemek yerine, önceden yapılmış konum kestirimlerini düzeltirler. Bu bağlamda, her iki grup için de yöntemler verilmiştir.

Pratikte, birbirinden konumsal olarak ayrı olmak ya da dar bir dönen huzmeye sahip bir radar sensorların aynı darbeyi almalarını engelleyebilir. Böyle bir durumda, sensorlar ortak VZF ölçümü oluşturamayıp, konum kestirimini gerçekleştiremezler. Ek olarak, ortamda VZF ölçümü yapan birden fazla sensor grubu olabilir. Bir konum kestirim yöntemi farklı sensor gruplarından alınan ölçümleri aynı anda işleyebilmelidir. Burada, aynı darbeyi alabilen sensorlar, bir sensor grubunu oluşturmaktadırlar. Bu çalışmada, yukarıda anlatılan gereklere de çözümler getirilmiştir.

Verilen yöntemlerin performansları benzetim çalışmaları ile karşılaştırılmıştır. En iyi performansa sahip yöntem Kalman Süzgeci tabanlıdır ve VZF ölçümleri doğrudan işleyerek hareketli bir vericinin izini kestirebilmektedir.

Anahtar Kelimeler : Varış Zamanları Farkları, Kalman Süzgeci, konum belirleme, özyinelemeli kestirim.

Aileme..

ACKNOWLEDGEMENTS

I would like to express my sincere thanks and gratitude to my supervisor Prof. Dr. Yaçın Tanık for his belief, encouragements, complete guidance, advice and criticism throughout this study.

I would like to thank Aselsan Inc. for facilities provided for the completion of this thesis.

I would like to express my thanks to my friends for their support and fellowship.

I would also like to thank TÜBİTAK for its support on scientific and technological researches.

I would like to express my special appreciation to my family for their continuous support and encouragements.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

TDOA	: Time Difference of Arrival
FDOA	: Frequency Difference of Arrival
AOA	: Angle of Arrival
UAV	: Unmanned Aerial Vehicle
GPS	: Global Positioning system
SNR	: Signal to Noise Ratio
ML	: Maximum Likelihood
NLOS	: Non Line of Sight
CRLB	: Cramer Rao Lower Bound
UKS	: Unscented Kalman Filter
EKF	: Extended Kalman Filter
KF	: Kalman Filter
i.i.d.	: Independent Identically Distributed
RDOA	: Range Difference of Arrival
WLS	: Weighted Least Squares
RLS	: Recursive Least Squares

IMM : Interacting Multiple Models

MHT : Multiple Hypothesis Tracking

VZF : Varış Zamanları Farkları

RMS : Root Mean Square

MUSIC : Multiple Signal Classification

ESPRIT : Estimation of Signal Parameters via Rotational Invariance Techniques

CHAPTER 1

INTRODUCTION

In Electronic Warfare (EW), determining Electronic Order of Battle (EOB) and staying undetected are some of the most critical issues. Since; EOB presents intelligence about environment and provides situational awareness. One of the most important requirements of constructing EOB is locating emitters of interest. In addition, staying undetected is vital to avoid engagement. In this context, monitoring environment via passive sensors becomes inevitable. It is a complicated task to locate an emitter which does not cooperate with the locator system, which is the usual case in EW. Location of an emitter can be estimated either by using Time Difference of Arrival (TDOA), Angle of Arrival (AOA) and/or Frequency Difference of Arrival measurements, or any other localization technique. For example, high resolution direction finding methods such as amplitude comparison, phase comparison or subspace techniques like Multiple Signal Classification (MUSIC) or Estimation of Signal Parameters via Rotational Invariance Techniques (ESPRIT) can be applied. On the other hand, these methods require proper antenna calibration which is not necessary with TDOA. This is one of the most important properties of TDOA. In addition, TDOA measurements are constructed by using arrival times of signals received by multiple passive sensors. So that, no knowledge about the waveform of the received signal is required. Because of the stated facts, emitter localization by TDOA is considered as one of the elegant localization methods and it is advantageous among other localization techniques.

In the open literature, there is plenty of work on estimating location of an emitter using TDOA measurements. Studies can be categorized into two major groups: (1) Memoriless and (2) Recursive estimators (with memory). (1) The most studied topic is memoriless TDOA location estimators. For example, the closed form estimators are in the type of the memoriless estimators [2, 5]. Most of the these estimators use only one set of TDOA measurements constructed from a single pulse. Data or measurement accumulation is not performed. Thus, the estimators are memoriless and this phenomenon is the most considerable drawback of this type. Moreover, several memoriless estimators which use Taylor series expansion have also been developed. They are capable of accumulating limited amount of data. However, convergence is not guaranteed [1, 26]. (2) The second type of the estimators is the recursive estimators. This type of estimators uses practically all of the past data. The estimate is updated when the new measurement is available without any information loss about past measurements. Thus, the recursive estimators have memory. The recursive estimators can also be divided into two major groups (a) The first type of recursive estimators process the TDOA measurements directly, and (b) the second type of the recursive estimators is the post processing estimators which process the TDOA indirectly. (a) There are examples of the first type [24, 25]. On the other hand, despite these recursive estimators accumulate TDOA measurements; they may suffer from divergence problems [25]. There are also some examples of post processing estimators [14, 28]. A post processing estimator does not handle TDOA measurements directly, instead it smoothes location estimates. Thus, information loss about measurements is probable.

In EW, emitter location estimation studies are mostly about radar localization applications. In the case of locating a pulsed radar, there is considerable amount of available data that is constructed by using consecutive radar pulses. In spite of good performance of memoriless estimators, ignoring past measurement data is not acceptable in the case of locating pulsed radar, since a lot of useful data is omitted. So, a recursion procedure is needed to process past data. In addition, a recursive estimator should not suffer from divergence problems. From this point of view, there is a need for a robust and recursive algorithm, which uses all of the past data and updates the location estimate when new measurement arrives. The goal of this thesis is to develop robust and recursive estimators which process all of the available data to estimate location of the emitter and update the location estimate when new data is measured.

In this thesis, recursive emitter localization based on the TDOA is studied. In this context, two types of recursive estimators are presented: A Post-Processing estimator and a recursive estimator. Because of the importance of the context for our work, detailed derivation of a post processing estimator is given. Secondly, a recursive estimator is developed which updates the location estimate, using directly TDOA measurements. In addition, modifications are also proposed to improve the performance of a memoriless solution found in the literature. In all estimators, estimates are found under the assumption of fixed emitter location and measurements are taken from moving sensors which are on different moving platforms.

1.1 Outline of the Thesis

The following chapter presents the problem statement. Error sources of TDOA are mentioned. Effects of SNR and sensor location error on TDOA localization are also discussed.

The derivation of a post-processing location estimator is given in Chapter 3. Inputs of the estimator are location estimates from other location estimators and error covariance matrices of these estimates. Then, Kalman Filter equivalent of the post-processing algorithm is derived, expressing post-processor algorithm in recursive form.

Chapter 4 starts with the derivation of a recursive location estimator. The recursive localization algorithm is first derived in Recursive Least Squares sense. Then, it is extended to Kalman Filter form for only one reference sensor. Lastly, recursive localization algorithm is expressed in Kalman Filter form for general case.

The location estimation in the maximum likelihood sense is given in Chapter 5. The estimator uses Taylor Series expansion. Some improvements are made such as pulse accumulation, and fusion of data from different groups of sensors to improve the accuracy of the estimate.

Simulation scenarios are given in Chapter 6 and performance comparison of the methods are presented under these scenarios.

Finally in Chapter 7, the conclusions are drawn and some future work is proposed.

CHAPTER 2

PROBLEM STATEMENT

The problem of estimating location of an emitter can be solved by processing relative arrival time measurements at three or more sensors [1]. Pulse arrival times are measured by the sensors. One of the sensors is chosen as the reference sensor and time of arrival of the pulses are expressed relative to the reference sensor. Thus, these measurements are named as Time Difference of Arrival (TDOA). Location estimators in this manner are regarded as TDOA location systems. In the absence of TDOA errors, the measurements limit possible locations of emitter to a hyperboloid. It can be considered that the emitter location is estimated using intersections of these hyperboloids. Therefore, TDOA localization is also called hyperbolic positioning. Note that, TDOA measurements form hyperboloids in three dimensions, and hyperbolas in two dimensions.

TDOA localization techniques are advantageous because of two important properties of TDOA measurements. Location estimation can be realized effectively without any knowledge about received waveform of the signal. Since, all information about the location of the emitter is contained in relative arrival time measurements. Furthermore, no antenna calibration is needed. On the other hand, as a disadvantage, TDOA location estimation methods require accurate time synchronization between receiver stations.

In most scenarios, there may be more than one radar in the environment. So, radar pulses will be received in a time interleaved fashion. These pulses have to be deinterleaved before processing in the location estimator. If transmitters operate in different frequencies, deinterleaving is a trivial task. On the other hand, if operating frequency bands overlap, Pulse Repetition Interval and Pulse Width parameters of transmitters have to be considered along with frequency. Deinterleaving methods can be found in the literature [32] and the deinterleaving process is beyond the scope of this thesis. In this work, pulses are assumed to be deinterleaved.

2.1 Formulation of TDOA Emitter Localization Problem

Let the column vector $\mathbf{p}$ $(N \times 1)$ represent location of the emitter, and column vector $\mathbf{s}_i$ $(N \times 1)$ represent the location of the i^{th} sensor where N is the dimension of location parameters for $i = 1, 2, \ldots, L$, where L denotes number of sensors. Assume that, a pulse is transmitted at time instance t_0 from the emitter whose location is desired to be estimated.

Distance from i^{th} sensor to emitter (d_i) is found to be:

$$d_i = \sqrt{(\mathbf{p} - \mathbf{s}_i)^T (\mathbf{p} - \mathbf{s}_i)}\,. \tag{2.1}$$

Thus, the transmitted pulse is received by the i^{th} sensor at time instance t_i which is expressed by:

$$t_i = t_0 + \frac{d_i}{c}\,. \tag{2.2}$$

where c is speed of light.

To avoid estimating parameter t_0, which is impossible, relative arrival times ($\tau_{i,j}$) are calculated:

$$
\begin{aligned}
\tau_{i,j} &= t_i - t_j = \frac{d_i - d_j}{c}, \\
&= \frac{\left(\sqrt{(\mathbf{p}-\mathbf{s}_i)^T(\mathbf{p}-\mathbf{s}_i)} - \sqrt{(\mathbf{p}-\mathbf{s}_j)^T(\mathbf{p}-\mathbf{s}_j)}\right)}{c}.
\end{aligned} \tag{2.3}
$$

where i and j represents i^{th} sensor and j^{th} sensor respectively. Relative arrival time parameter $\tau_{i,j}$ stands for TDOA measurement.

The problem of emitter localization using TDOA measurements can be stated as solving desired parameter $\mathbf{p}$ using non-linear quadratic equations which is not a trivial task.

To illustrate the problem of emitter location estimation using TDOA better, consider scenario with four sensors as shown in Figure 2.1. The emitter is located at $[12 \quad 25]^T$ km., the sensors are located at $[0 \quad 0]^T$, $[5 \quad -1]^T$, $[10 \quad 1]^T$ and $[15 \quad 0.5]^T$ km. respectively. The TDOA error free hyperbolas are plotted in the figure.

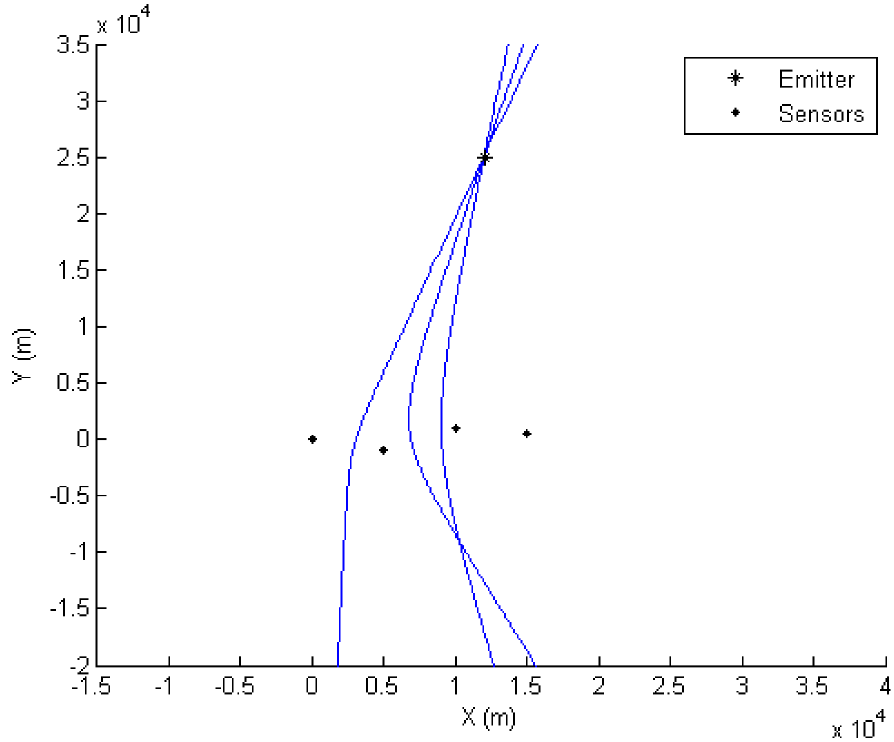

Figure 2.1: TDOA Scenario

The reference sensor is the one located at the origin. Only emitter location related isochrones of hyperbolas are plotted for simplicity. As shown in Figure 2.1, there may be two intersection points with two hyperbolas. So, there must be at least three hyperbolas for unambiguous estimate in two dimensions, which means there must be at least four sensors for unambiguous location estimation in two dimensions [2]. In general case, an unambiguous estimate requires at least $(N+2)$ sensors for N dimensional location estimate.

2.2 Bibliographical Notes

In the open literature, there are numerous work on emitter location finding algorithms using TDOA. Closed-form (memoriless) estimators can be seen to be as the most studied topic [2, 7-9]. In [2], an approximate realization of Maximum Likelihood (ML) estimator is derived and it is shown that the variance approaches CRLB for small TDOA errors. In [7], an algebraic solution to location estimation problem using TDOA is presented. The solution suffers from the requirement for the symmetry in sensor locations.

Frequency Difference of Arrival (FDOA) measurements are also used together with TDOA to estimate location when there is relative motion between source and receivers [12, 13, 15].

TDOA measurements are also combined with Angel of Arrival (AOA) measurements in location finding problems [17-19]. There are also location finding examples which use combination of TDOA, FDOA and AOA measurements all together [20].

TDOA emitter localization problem is also studied under Non-Line-of-Sight (NLOS) environments such as cellular mobile networks [18, 23]. In [23], combination of AOA and TDOA is used to estimate the location. In addition, signal strength parameter is used with AOA and TDOA to solve localization problem in [18].

There are works focusing on sensor location uncertainty as well. From this point of view, [8], [15] and [36] are some examples.

As an example for TDOA localization applications, TDOA positioning systems are also used in search and rescue operations. In [16], TDOA measurements are used to locate 911 callers.

Optimality issues in the geometry must be studied, since the geometry is one of the major factors in TDOA geolocation accuracy. In [21] and [22] optimization of geometry of the sensors is investigated. In [21], relative geometries between the emitter and the sensors are examined to increase localization performance. In [22], Unmanned Aerial Vehicles (UAVs) are used as sensor platforms and automatic formation of UAVs is studied to optimize TDOA localization accuracy.

Closed-form (memoriless) estimators process only one set of TDOA measurement. This phenomenon results in omitting excessive amount of data in the case of pulsed radar. On the other hand, there are closed-form estimators which use more than one set of TDOA measurement [10, 11]. These closed-form estimators are lack of recursion and process all of the data at a single step which may result in unacceptable increase in the process time.

In [28] a post-processing method is proposed which smoothes location estimates of a moving source in Recursive Least Squares Filter sense. The consecutive location estimates are assumed to be pre-calculated by external estimators. Similarly, Kalman Filter is used as a post-processing smoother in [14].

Another topic studied in the literature is recursive estimators. A recursive estimator uses all of the past measurement data and updates location estimate when new measurement arrives. Unscented Kalman Filter (UKF) is studied in [24]. An UKF based emitter localization filter which TDOA measurements is developed. Furthermore, Extended Kalman Filtering (EKF) and (UKF) are also studied in [25]. Due to the fact that EKF and UKF are recursive filters; they suffer from divergence problems [25].

Kalman Filter based estimators are also developed for cellular networks. For example, Kalman Filter based localization is studied for UMTS using TDOA in [35]. A similar method is investigated in [27] for non-line of sight situations.

In addition to recursive filters, some memoriless methods which use Taylor series expansion have also been developed. This type of estimators updates location estimate after each measurement. Using the past estimation as the starting point, they are capable of accumulating limited amount of data. Apart from this, past measurements are ignored. In [26], a memoriless localization algorithm is developed using Taylor-series approximation. This algorithm is capable of using TDOA, FDOA or AOA measurements at the same time. A similar algorithm was developed in [1]. It also uses Taylor-series approximation, but the estimator is derived in ML sense. Requirement of an initial point is a major disadvantage of these algorithms [26]. Another disadvantage is convergence is not guaranteed [26].

2.3 TDOA Error Sources

Some of the most important error sources of TDOA emitter localization systems are listed as follows:

1. The geometry formed by the sensors and the emitter location is one of the main characteristics that determine the performance [3, 5]. A bad geometry of sensors can result in ambiguous or even a false estimate.

2. The sensor location uncertainties affect the accuracy of location estimate [3]. Therefore, the sensor position errors directly result in emitter position estimation errors. The sensor positions can be determined using a GPS system. There are two kinds of sensor location errors due to GPS; a) All of the sensors may have the same position error mostly due to atmospheric effects, in such a case location estimate will be shifted relative to the amount of the error. b) The second type is the independent position errors on sensor locations which is called relative positioning errors. These types of errors are mostly caused by receiver characteristics, SNR, multipath effects etc. In this manner, relative positioning becomes important in the case of TDOA localization. Differential GPS or relative positioning methods can be used between sensor platforms [4]. In addition, maximum relative positioning

error is found to be approximately 20 cm. for a 10 km. distance between platforms in [4].

3. Time synchronization between sensors is one of the most important error sources. It directly affects the accuracy of the estimate. In order to build a successful TDOA localization system, the first and the most important requirement is to be able to calculate TDOA values very accurately. For example, TDOA measurement accuracy has to be better than several nanoseconds for a good location estimate.

4. Accuracy of location estimate is also dependent on other factors such as receiver noise [3], frequency of the signal, received SNR and integration time which is used to estimate TDOA itself [6]. Increase in SNR, frequency of the signal and/or integration time result in better location estimates [6].

A simple and effective way to prevent noise effects on TDOA is averaging consecutive TDOA measurements. Note that; time interval, over which TDOA measurements are taken, must be short enough to avoid bias because sensors move together with their moving platforms.

Another issue that has to be considered is the maximum distance between the sensors. The sensors have to receive same signal in order to construct TDOA measurements. For example, in order to locate a radar, the sensors have to stay in the beam of the radar at the same time to be able to receive same pulse.

2.4 Effect of SNR, Signal Frequency and Integration Time on TDOA

In [6], for a single pulse the Cramer Rao Lower Bound (CRLB) of estimation of TDOA measurements is evaluated under the assumption that both noise and signal have constant power spectral density over the bandwidth specified by f_1 and f_2

($f_2 > f_1$). Using this assumption, the Signal to Noise Ratio (SNR) is expressed as [6]:

$$SNR = \frac{S_0 (f_2 - f_1)}{N_0 (f_2 - f_1)} = \frac{S_0}{N_0}, \tag{2.4}$$

where S_0 and N_0 stand for signal and noise power spectral densities respectively. Furthermore, the noise power spectral densities are assumed to be identical for all of the sensors used to construct TDOA, and the same assumption also applies to the signal power spectral densities.

For low SNR ($SNR << 1$), CRLB of TDOA estimation is found by [6]:

$$\sigma_{TDOA} \geq \sqrt{\frac{3}{8\pi^2 T}} \frac{1}{SNR} \frac{1}{\sqrt{f_2^3 - f_1^3}}, \tag{2.5}$$

where T is the integration time which is used in TDOA estimation.

For high SNR ($SNR >> 1$), the CRLB is found by [6]:

$$\sigma_{TDOA} \geq \sqrt{\frac{3}{8\pi^2 T}} \frac{1}{\sqrt{SNR}} \frac{1}{\sqrt{f_2^3 - f_1^3}}. \tag{2.6}$$

In Figure 2.2 and 2.3, effect of SNR, integration time and frequency of the signal is plotted for bandwidth specified by $f_2 = 9010\, MHz$ and $f_1 = 8090\, MHz$. Maximum limit of the integration time T is $T_{\max} = \frac{1}{f_2 - f_1} = 50\, n\sec$.

Results are plotted in Figure 2.2 for low SNR case:

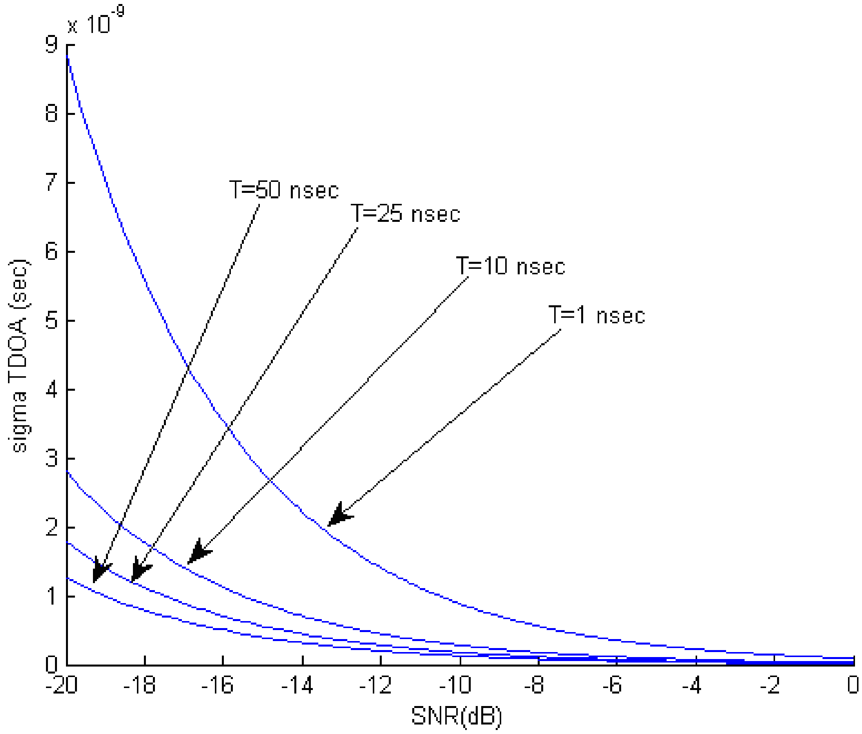

Figure 2.2: Effect of SNR, integration time and frequency on TDOA estimation for low SNR case.

Results are plotted in Figure 2.3 for high SNR case:

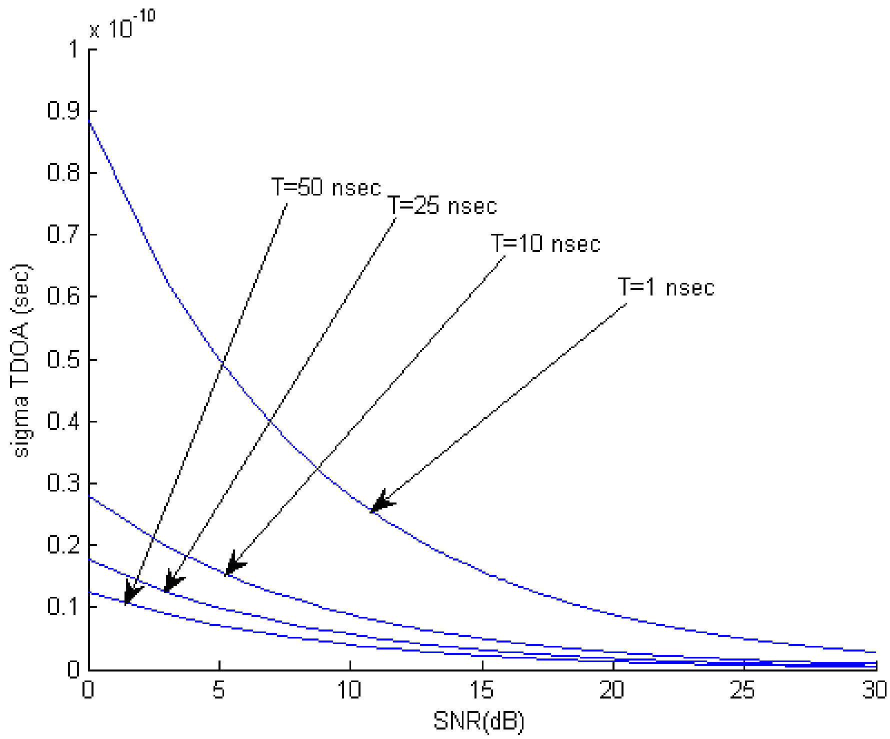

Figure 2.3: Effect of SNR, integration time and frequency on TDOA estimation for high SNR case.

Because of the assumption of constant signal and noise power spectral density, evaluated CRLB expressions are only approximations. Indeed, they are useful tools to analyze effects of SNR, signal frequency and integration time on TDOA.

2.5 Effect of Sensor Location Uncertainty in TDOA Emitter Localization

In practice, there will be some uncertainty in sensor locations due to noise and other effects. That will directly affect emitter location estimation's accuracy. Therefore, effects of sensor location uncertainty on emitter localization must be analyzed before developing an estimator.

Taking the measurement noise into account (2.2) becomes:

$$t_i = t_0 + \frac{d_i}{c} + n \,, \tag{2.7}$$

where n is the noise.

Letting $t_0 = 0$ for simplicity and expressing (2.7) in 2-dimensional Cartesian space yields:

$$\begin{aligned} f(\mathbf{p},\mathbf{s}_i) = t_i &= \frac{d_i}{c} + n, \\ &= \frac{1}{c}\sqrt{(\mathbf{p}-\mathbf{s}_i)^T(\mathbf{p}-\mathbf{s}_i)} + n, \\ &= \frac{1}{c}\sqrt{(x_e - x_i)^2 + (y_e - y_i)^2} + n, \end{aligned} \tag{2.8}$$

where $\mathbf{p} = [x_e \quad y_e]^T$, and $\mathbf{s}_i = [x_i \quad y_i]^T$.

Sensor locations are also corrupted by noise, hence:

$$\begin{aligned} \tilde{\mathbf{s}}_i &= \mathbf{s}_i + \mathbf{n}_s, \\ &= \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \begin{bmatrix} n_{xi} \\ n_{yi} \end{bmatrix}. \end{aligned} \tag{2.9}$$

where, $[n_{xi} \quad n_{yi}]^T$ expresses sensor location error.

Then, (2.8) can be expressed as:

$$f(\mathbf{p},\mathbf{s}_i) = \frac{1}{c}\sqrt{(x_e - (x_i + n_{xi}))^2 + (y_e - (y_i + n_{yi}))^2} + n, \tag{2.10}$$

Letting $x = (x_e - x_i)$, and $y = y_e - y_i$, (2. 10) becomes:

$$f(\mathbf{p},\mathbf{s}_i) = \frac{1}{c}\sqrt{(x-n_{xi})^2 + (y-n_{yi})^2} + n, \\ = \frac{1}{c}\sqrt{x^2 + y^2 - 2xn_{xi} - 2yn_{yi} + n_{xi}^2 + n_{yi}^2} + n. \tag{2.11}$$

When range-to-baseline ratio is large, if the bias in sensor locations is assumed to be small, the term $n_{xi}^2 + n_{yi}^2$ in (2.11) can be omitted:

$$n_{xi}^2 + n_{yi}^2 \approx 0. \tag{2.12}$$

Using (2.12) in (2.11) yields:

$$f(\mathbf{p},\mathbf{s}_i) = \frac{1}{c}\sqrt{x^2 + y^2 - 2xn_{xi} - 2yn_{yi}} + n, \\ = \frac{1}{c}\sqrt{x^2 + y^2}\sqrt{1 - \frac{2xn_{xi}}{x^2 + y^2} - \frac{2yn_{yi}}{x^2 + y^2}} + n, \tag{2.13}$$

Using the fact that:

$$\sqrt{1+d} \cong 1 + \frac{d}{2}, \text{ when } |d| << 1, \tag{2.14}$$

with (2.14), $f(p, s_i)$ can be written in the following form:

$$f(\mathbf{p},\mathbf{s}_i) = \frac{1}{c}\sqrt{x^2 + y^2}\left(1 - \left(\frac{xn_{xi}}{x^2 + y^2} + \frac{yn_{yi}}{x^2 + y^2}\right)\right) + n, \tag{2.15}$$

For convenience, let

$$A = \frac{x}{x^2 + y^2}, \text{ and } B = \frac{y}{x^2 + y^2}. \tag{2.16}$$

Rewriting (2.15) using (2.16) yields:

$$f(\mathbf{p},\mathbf{s}_i) = \frac{1}{c}\sqrt{x^2 + y^2}\left(1 - \left(An_{xi} + Bn_{yi}\right)\right) + n. \tag{2.17}$$

Assume that n_{xi}, n_{yi} and n are independent. Moreover, n_{xi} and n_{yi} are biased which are defined as $n_{xi} = e_{xi} + \mu_{xi}$, $n_{yi} = e_{yi} + \mu_{yi}$ with means μ_{xi}, μ_{yi} and variances σ_{yi}^2, σ_{yi}^2 respectively where e_{xi} and e_{yi} denote the independent noise components.

Then, finding the expected value of (2.17) results in:

$$
\begin{aligned}
E[f(\mathbf{p},\mathbf{s}_i)] &= E\left[\frac{1}{c}\sqrt{x^2+y^2}\left(1-\left(An_{xi}+Bn_{yi}\right)\right)+n\right], \\
&= \frac{1}{c}\sqrt{x^2+y^2}\left(1-\left(A\mu_{xi}+B\mu_{yi}\right)\right) = \frac{d_i}{c}\left(1-\left(A\mu_{xi}+B\mu_{yi}\right)\right).
\end{aligned}
\tag{2.18}
$$

Notice that, the term $\left(A\mu_{xi}+B\mu_{yi}\right)$ in (2.18) results in bias, and it is assumed to be small in the derivation.

Using (2.18), the variance of $f(\mathbf{p},\mathbf{s}_i)$ is found to be:

$$
\begin{aligned}
\operatorname{var}(f(\mathbf{p},\mathbf{s}_i)) &= E\left[\left(\frac{d_i}{c}\left(1-\left(An_{xi}+Bn_{yi}\right)\right)+n-\frac{d_i}{c}\left(1-\left(A\mu_{xi}+B\mu_{yi}\right)\right)\right)^2\right], \\
&= E\left[\left(n-\frac{d_i}{c}\left(Ae_{xi}+Be_{yi}\right)\right)^2\right],
\end{aligned}
\tag{2.19}
$$

$$
= E\left[n^2+\left(\frac{d_iAe_{xi}}{c}\right)^2+\left(\frac{d_iBe_{yi}}{c}\right)^2-2\frac{d_iA}{c}n\cdot e_{xi}-2\frac{d_iB}{c}n\cdot e_{yi}+2\frac{d_i^2AB}{c}e_{xi}e_{yi}\right].
$$

Since, n_{xi}, n_{yi} and n are independent, expected values of the cross terms are zero. Thus,

$$\begin{aligned}\mathrm{var}(f(\mathbf{p},\mathbf{s}_i)) &= E\left[n^2+\left(\frac{d_iAe_{xi}}{c}\right)^2+\left(\frac{d_iBe_{yi}}{c}\right)^2\right],\\ &= E[n^2]+E\left[\left(\frac{d_iAe_{xi}}{c}\right)^2\right]+E\left[\left(\frac{d_iBe_{yi}}{c}\right)^2\right],\\ &= \sigma^2+\left(\frac{d_iA}{c}\right)^2\sigma_{xi}^2+\left(\frac{d_iB}{c}\right)^2\sigma_{yi}^2.\end{aligned} \tag{2.20}$$

Using (2.16) and rearranging the terms yields:

$$\begin{aligned}\mathrm{var}(f(\mathbf{p},\mathbf{s}_i)) &= \sigma^2+\left(\frac{d_i}{c}\frac{x}{x^2+y^2}\right)^2\sigma_{xi}^2+\left(\frac{d_i}{c}\frac{y}{x^2+y^2}\right)^2\sigma_{yi}^2,\\ &= \sigma^2+\left(\frac{d_i}{c}\frac{x}{d_i^2}\right)^2\sigma_{xi}^2+\left(\frac{d_i}{c}\frac{y}{d_i^2}\right)^2\sigma_{yi}^2,\\ &= \sigma^2+\left(\frac{x_e-x_i}{c\cdot d_i}\right)^2\sigma_{xi}^2+\left(\frac{y_e-y_i}{c\cdot d_i}\right)^2\sigma_{yi}^2.\end{aligned} \tag{2.21}$$

In (2.18) and (2.21), the terms $\left(\left(\frac{x_e-x_i}{c\cdot d_i}\right)^2\sigma_{xi}^2+\left(\frac{y_e-y_i}{c\cdot d_i}\right)^2\sigma_{yi}^2\right)$ and $(A\mu_{xi}+B\mu_{yi})$ show the effect of sensor location uncertainty on TDOA emitter localization. Note that, (2.18) and (2.21) can easily be expended to three dimensional case.

For unbiased sensor locations, (2.18) is found to be:

$$\begin{aligned}E[f(\mathbf{p},\mathbf{s}_i)] &= E\left[\frac{1}{c}\sqrt{x^2+y^2}\,(1-(An_{xi}+Bn_{yi}))+n\right],\\ &= \frac{1}{c}\sqrt{x^2+y^2}=\frac{d_i}{c},\end{aligned} \tag{2.22}$$

which is actually the value of the TOA itself at i^{th} sensor.

It can be seen that, because of parameters x, y and d_i in (2.18) and (2.21), the effect of sensor location uncertainty is dependent on geometry of sensors relative to the emitter. It is an important fact that, bias in sensor locations is not a dominant error source in TDOA emitter localization performance. For low bias levels, the sensor location bias effects can be neglected which is shown in Chapter 6 by simulation studies. On the other hand, it is also shown that, independent noise in sensor locations is one of the major and dominant error sources.

Equation (2.21) only states variance of TOA measurements. However, this result can be used to derive expected value and variance of the TDOA measurements. Assume that there are N sensors and first sensor is the reference sensor. Then, if relative arrival times are determined by subtracting the measured arrival times, then TDOA values are calculated as [1]:

$$\begin{aligned}
&\tau_{2,1} = t_2 - t_1, \\
&\tau_{3,1} = t_3 - t_1, \\
&\vdots \\
&\tau_{L,1} = t_L - t_1,
\end{aligned} \tag{2.23}$$

In matrix form, (2.23) is found to be [1]:

$$\boldsymbol{\tau} = \mathbf{H} \cdot \mathbf{t}, \tag{2.24}$$

where $\boldsymbol{\tau}$ is $(L-1)\times 1$ dimensional TDOA measurement vector, $\mathbf{H}$ is an $(L-1)\times L$ dimensional matrix which implements subtractions and $\mathbf{t}$ is an $L\times 1$ dimensional TOA measurement vector.

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{2,1} \\ \tau_{3,1} \\ \vdots \\ \tau_{L,1} \end{bmatrix}_{(L-1)\times 1}, \quad \mathbf{H} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ \vdots & \cdots & \ddots & \vdots \\ -1 & 0 & 0 & 1 \end{bmatrix}_{(L-1)\times L} \quad \text{and } \mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_L \end{bmatrix}_{L\times 1}. \tag{2.25}$$

Let $\tilde{\mathbf{t}}$ express noise corrupted TOA values. Then, the expected value and variance of TDOA measurements are calculated as follows:

$$\begin{aligned} E[\boldsymbol{\tau}] &= E\left[\mathbf{H}\cdot\tilde{\mathbf{t}}\right], \\ &= \mathbf{H}\cdot E\left[\tilde{\mathbf{t}}\right] = \mathbf{H}\cdot\mathbf{t}. \end{aligned} \tag{2.26}$$

Using (2.25) and (2.26) the variance of TDOA values is found as:

$$\begin{aligned} \operatorname{cov}(\boldsymbol{\tau}) &= E\left[\left(\mathbf{H}\cdot\tilde{\mathbf{t}} - \mathbf{H}\cdot\mathbf{t}\right)\left(\mathbf{H}\cdot\tilde{\mathbf{t}} - \mathbf{H}\cdot\mathbf{t}\right)^T\right], \\ &= \mathbf{H}\cdot E\left[\left(\tilde{\mathbf{t}} - \mathbf{t}\right)\left(\tilde{\mathbf{t}} - \mathbf{t}\right)^T\right]\cdot\mathbf{H}^T, \end{aligned} \tag{2.27}$$

Define σ^2_{TOA-i} as (see (2.21)):

$$\sigma^2_{TOA-i} = \sigma^2 + \left(\frac{x_e - x_i}{c\cdot d_i}\right)^2 \sigma_x^2 + \left(\frac{y_e - y_i}{c\cdot d_i}\right)^2 \sigma_y^2. \tag{2.28}$$

Since, n_{xi}, n_{yi} and n are assumed to be independent:

$$E\left[\left(\tilde{\mathbf{t}} - \mathbf{t}\right)\left(\tilde{\mathbf{t}} - \mathbf{t}\right)^T\right] = \begin{bmatrix} \sigma^2_{TOA-1} & 0 & \cdots & 0 \\ 0 & \sigma^2_{TOA-2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sigma^2_{TOA-L} \end{bmatrix}. \tag{2.29}$$

Then, (2.27) becomes the following $(L-1)\times(L-1)$ dimensional TDOA covariance matrix [1]:

$$\begin{aligned} \operatorname{cov}(\boldsymbol{\tau}) &= \mathbf{H}\cdot E\left[\left(\tilde{\mathbf{t}} - \mathbf{t}\right)\left(\tilde{\mathbf{t}} - \mathbf{t}\right)^T\right]\cdot\mathbf{H}^T \\ &= \begin{bmatrix} \sigma^2_{TOA-1} + \sigma^2_{TOA-2} & \sigma^2_{TOA-1} & \cdots & \sigma^2_{TOA-1} \\ \sigma^2_{TOA-1} & \sigma^2_{TOA-1} + \sigma^2_{TOA-3} & \cdots & \vdots \\ \vdots & & \ddots & \sigma^2_{TOA-1} \\ \sigma^2_{TOA-1} & \cdots & \sigma^2_{TOA-1} & \sigma^2_{TOA-1} + \sigma^2_{TOA-L} \end{bmatrix}. \end{aligned} \tag{2.30}$$

where off diagonal elements are equal to σ^2_{TOA-1} (which is the variance of the TOA estimate of reference sensor), and i^{th} diagonal element in (2.30) is equal to $\sigma^2_{TOA-1} + \sigma^2_{TOA-(i+1)}$.

In summary, the covariance matrix and expected values of the TDOA measurements are given as (2.26) and (2.30) respectively. (2.30) is valid for TDOA values which are calculated by subtracting TOA values.

To demonstrate the effect of sensor location uncertainty, the following scenario was proposed: It is assumed that σ_x and σ_y are equal to σ_L in (2.28) and they are the same for all sensors. Furthermore, the emitter and locations of first and second sensors from Figure 2.1 are used. The results are plotted in Figure 2.4.

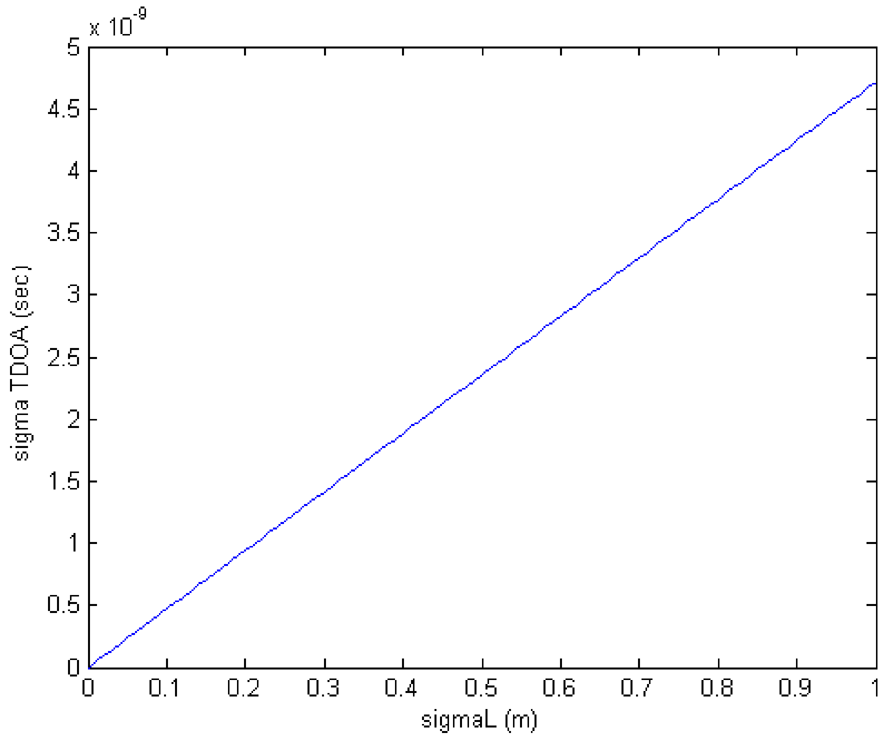

Figure 2.4: Effect of sensor location uncertainty on TDOA

Examining Figure 2.4, it can be stated that, the sensor location uncertainty directly affects the accuracy of the TDOA localization.

CHAPTER 3

EMITTER LOCATION ESTIMATION BY POST PROCESSING METHODS

Derivation of a location estimator which uses post processing methods is given in this chapter. First, closed form of the post processing estimator is expressed. Then, recursive implementation of the method is given.

Suppose that independent location estimates of an emitter are available. Fusion of these independent location estimates will probably result in a better estimate in return. Fusion will probably reduce the noise effects. This type of location estimators which fuse or smooth available estimates are called post-processing location estimators. The concept of post-processing localization is shown in Figure 3.1, where $\hat{p}_i$ indicates successful estimate which is output of the i^{th} estimator. TDOA measurements are used by individual estimators to produce location estimates. Then, these independent estimates are fused by the post-processor to get a better estimate. Thus, a post-processing estimator does not use TDOA measurements directly.

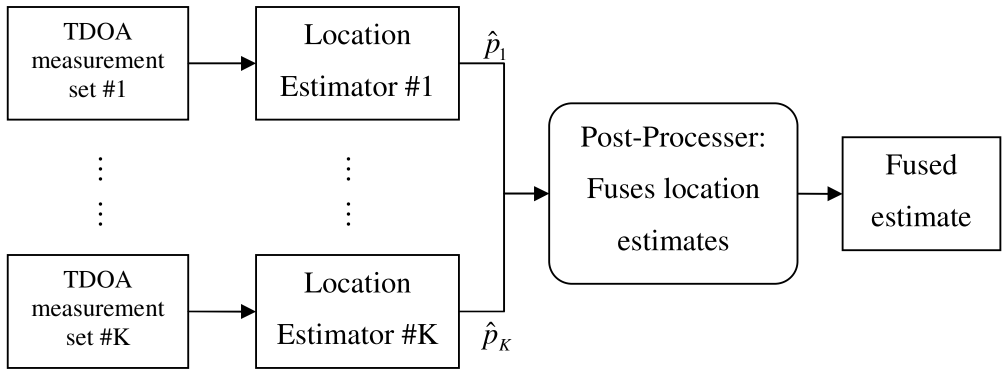

Figure 3.1: Concept of post-processing method

Some examples of post-processing emitter localization methods are given in [14] and [28]. In [28], consecutive location estimates are smoothed in recursive least squares sense to determine the track of a moving source. Kalman filter is used as the post processor in [14]. However, the derivation is presented with no detail. Since the concept of the post-processing estimator is essential in localization problem, a detailed study is needed for further progress. Thus, we provide a detailed derivation of the work in [14] in the next section under the assumption of the fixed emitter location. First, a closed form estimator is derived in ML sense. Then, a recursive implementation of the solution is obtained.

3.1 Derivation of Post-Processing Location Estimator

Assume that location estimates $\hat{\mathbf{p}}_i$ are available for $i = 1, 2, \ldots, K$ with covariance matrices $\mathbf{Q}_i$. $\hat{\mathbf{p}}_i$ are assumed to be corrupted by independent Gaussian noise $\mathbf{n}_i$ where $\mathbf{n}_i \sim N(\mathbf{0}, \mathbf{Q}_i)$ for $i = 1, 2, \ldots, K$. Hence:

$$\hat{\mathbf{p}}_i = \mathbf{p} + \mathbf{n}_i,\ i = 1, 2, \ldots, K. \tag{3.1}$$

As a result, $\hat{\mathbf{p}}_i \sim N(\mathbf{p}, \mathbf{Q}_i)$ and are independent for $i = 1, 2, \ldots, K$. $\mathbf{p}$ and $\hat{\mathbf{p}}_i$ are $D \times 1$ dimensional column vectors, and $\mathbf{Q}_i$ are $D \times D$ dimensional covariance matrices for $i = 1, 2, \ldots, K$. Under the assumption of stationary emitter location, the estimates $\hat{\mathbf{p}}_i$ can be fused as follows.

The joint conditional probability density function of $\hat{\mathbf{p}}_i$ given $\mathbf{p}$ for $i = 1, 2, \ldots, K$ is found as:

$$f(\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2, \ldots, \hat{\mathbf{p}}_K \mid \mathbf{p}) = f(\hat{\mathbf{p}}_1 \mid \mathbf{p}) f(\hat{\mathbf{p}}_2 \mid \mathbf{p}) \cdots f(\hat{\mathbf{p}}_K \mid \mathbf{p}). \tag{3.2}$$

where,

$$f(\hat{\mathbf{p}}_i \mid \mathbf{p}) = \frac{1}{\sqrt{2\pi \mid \mathbf{Q}_i \mid}} \exp\left\{ -0.5(\hat{\mathbf{p}}_i - \mathbf{p})^T \mathbf{Q}_i^{-1} (\hat{\mathbf{p}}_i - \mathbf{p}) \right\}. \tag{3.3}$$

Rewriting (3.2) using (3.3) yields:

$$f(\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2, \ldots, \hat{\mathbf{p}}_K \mid \mathbf{p}) = \prod_{i=1}^{K} \frac{1}{\sqrt{2\pi \mid \mathbf{Q}_i \mid}} \exp\left\{ -0.5(\hat{\mathbf{p}}_i - \mathbf{p})^T \mathbf{Q}_i^{-1} (\hat{\mathbf{p}}_i - \mathbf{p}) \right\}. \tag{3.4}$$

Taking natural logarithm of both sides:

$$\begin{aligned} &\ln\left(f(\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2, \ldots, \hat{\mathbf{p}}_K \mid \mathbf{p}) \right) = \\ &\qquad \sum_{i=1}^{K} \ln\left(\frac{1}{\sqrt{2\pi \mid \mathbf{Q}_i \mid}} \right) - \sum_{i=1}^{K} 0.5(\hat{\mathbf{p}}_i - \mathbf{p})^T \mathbf{Q}_i^{-1} (\hat{\mathbf{p}}_i - \mathbf{p}). \end{aligned} \tag{3.5}$$

Maximum Likelihood estimator of $\mathbf{p}$ is defined as [29]:

$$\mathbf{p}_{ML} = \arg\max_{\mathbf{p}} \left(f(\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2, \ldots, \hat{\mathbf{p}}_K \mid \mathbf{p}) \right). \tag{3.6}$$

Omitting terms unrelated of $\mathbf{p}$ in (3.5), using (3.6) and rearranging terms results:

$$\mathbf{p}_{ML} = \arg\min_{\mathbf{p}} \sum_{i=1}^{K} \left(\mathbf{p}^T \mathbf{Q}_i^{-1} \mathbf{p} - 2\mathbf{p}^T \mathbf{Q}_i^{-1} \hat{\mathbf{p}}_i \right). \tag{3.7}$$

Since $\mathbf{p}$ is independent of sum operator index i, it can be taken outside of the sum. Hence:

$$\mathbf{p}_{ML} = \arg\min_{\mathbf{p}} \left(\mathbf{p}^T \sum_{i=1}^{K} \left(\mathbf{Q}_i^{-1} \right) \mathbf{p} \ - 2\mathbf{p}^T \sum_{i=1}^{K} \left(\mathbf{Q}_i^{-1} \hat{\mathbf{p}}_i \right) \right). \tag{3.8}$$

Define $\mathbf{A}$ and $\mathbf{B}$ as:

$$\mathbf{A} = \sum_{i=1}^{K} \mathbf{Q}_i^{-1}, \quad \mathbf{B} = \sum_{i=1}^{K} \mathbf{Q}_i^{-1} \hat{\mathbf{p}}_i \,. \tag{3.9}$$

To find the argument which minimizes (3.7), gradient of the expression in brackets is computed with respect to $\mathbf{p}$:

$$\nabla_{\mathbf{p}} \left(\mathbf{p}^T \mathbf{A} \mathbf{p} - 2\mathbf{p}^T \mathbf{B} \right) = 2\mathbf{A}\mathbf{p} - 2\mathbf{B} \,. \tag{3.10}$$

Then, equation (3.10) is evaluated where it is equal to zero vector:

$$2\mathbf{A}\mathbf{p} - 2\mathbf{B} = \mathbf{0} \Rightarrow \mathbf{p} = \mathbf{A}^{-1}\mathbf{B} \,. \tag{3.11}$$

Finally, using (3.11) Maximum Likelihood estimate of $\mathbf{p}$ is found as:

$$\mathbf{p}_{ML} = \left(\sum_{i=1}^{K} \mathbf{Q}_i^{-1} \right)^{-1} \left(\sum_{i=1}^{K} \mathbf{Q}_i^{-1} \hat{\mathbf{p}}_i \right). \tag{3.12}$$

(3.12) can be used to fuse location estimates of unrelated localization systems under the assumption of fixed emitter location. Note that, the inputs of the post processing estimator are the location estimates $\hat{\mathbf{p}}_i$ and the covariance matrices ($\mathbf{Q}_i$) of the estimates. In the simplest sense, (3.12) can be regarded as a weighted average of the location estimates. Notice that the weighting coefficients are the covariance matrices. In practice, calculating the covariance matrices requires the knowledge of the actual location of the emitter which is indeed desired to be estimated. So, there is a contradiction which makes (3.12) impossible to realize. On the other hand, a similar problem is faced in [2], and the estimated location of the emitter is used instead of the actual one. The same procedure can be used in (3.12) and the estimated locations $\hat{\mathbf{p}}_i$ can be used to calculate $\mathbf{Q}_i$ for $i = 1, 2, \ldots, K$. Thus, $\mathbf{Q}_i$ becomes an estimate itself resulting a suboptimal solution to (3.12) in return.

3.2 Variance of the Post-Processing Estimator

In order to conclude the maximum likelihood estimator's derivation, the covariance of the estimate must be evaluated, which is found as:

$$\mathrm{var}(\mathbf{p}_{ML} \mid \mathbf{p}) = \left(\sum_{i=1}^{K} \mathbf{Q}_i^{-1} \right)^{-1}. \tag{3.13}$$

The derivation of (3.13) is given in Appendix B.

3.3 Recursive Implementation of Post-Processing Estimator

(3.12) is a powerful tool to fuse location estimates. Indeed, it is a closed-form estimator, but it suffers from processing all of the data at once. It may be inefficient when the amount of the data keeps growing which results in processing redundancy. In short, a recursive procedure is required to compute (3.12) efficiently. In this section, a recursive implementation of (3.12) in Kalman Filter context will be derived. The only assumption in the derivation is the stationary emitter location.

Let us define the estimate error covariance matrix as:

$$\mathbf{P}_K = \left(\sum_{i=1}^{K} \mathbf{Q}_i^{-1} \right)^{-1}. \tag{3.14}$$

For $(K+1)^{th}$ step, (3.14) becomes:

$$\mathbf{P}_{K+1} = \left(\mathbf{P}_K^{-1} + \mathbf{Q}_{K+1}^{-1}\right)^{-1}. \tag{3.15}$$

Let the estimate at K^{th} step $\mathbf{x}^K = \mathbf{p}_{ML}$, then rewriting (3.12) using (3.15) yields:

$$\begin{aligned}
\mathbf{x}^{K+1} &= \mathbf{P}_{K+1}\left(\sum_{i=1}^{K}\mathbf{Q}_i^{-1}\hat{\mathbf{p}}_i + \mathbf{Q}_{K+1}^{-1}\hat{\mathbf{p}}_{K+1}\right) \\
&= \mathbf{x}^K + \mathbf{P}_{K+1}\left((\mathbf{P}_K^{-1} - \mathbf{P}_{K+1}^{-1})\mathbf{x}^K + \mathbf{Q}_{K+1}^{-1}\hat{\mathbf{p}}_{K+1}\right), \\
&= \mathbf{x}^K + \mathbf{P}_{K+1}\mathbf{Q}_{K+1}^{-1}\left(\hat{\mathbf{p}}_{K+1} - \mathbf{x}^K\right).
\end{aligned} \tag{3.16}$$

Using the following matrix identity [30]:

$$\mathbf{A}\mathbf{B}^T(\mathbf{C} + \mathbf{B}\mathbf{A}\mathbf{B}^T)^{-1} = (\mathbf{A}^{-1} + \mathbf{B}^T\mathbf{C}^{-1}\mathbf{B})^{-1}\mathbf{B}^T\mathbf{C}^{-1}, \tag{3.17}$$

where $\mathbf{B} = \mathbf{I}$, $\mathbf{C} = \mathbf{Q}_{K+1}$ and $\mathbf{A} = \mathbf{P}_K$, then (3.16) can be written as:

$$\mathbf{x}^{K+1} = \mathbf{x}^K + \mathbf{P}_K(\mathbf{P}_K + \mathbf{Q}_{K+1})^{-1}\left(\hat{\mathbf{p}}_{K+1} - \mathbf{x}^K\right). \tag{3.18}$$

Furthermore, using Matrix Inversion Lemma (3.15) can be written as [30]:

$$\mathbf{P}_{K+1} = (\mathbf{I} - \mathbf{P}_K(\mathbf{P}_K + \mathbf{Q}_{K+1})^{-1})\mathbf{P}_K. \tag{3.19}$$

Finally, recursive form of the (3.12) is expressed using (3.18) and (3.19) which are in fact Kalman Filter equations for stationary emitter location [30, 31, 37]. Note that, evaluating recursion process, the only assumption is made about location of the emitter and it is assumed to be fixed; there is no restriction about sensor geometry or movements etc.

3.4 Application of Post Processing Algorithm: Tracking Moving Targets

Kalman Filter equations (given in (3.18) and (3.19)) use fixed emitter location system model. These equations can be easily extended to fuse or smooth location estimates of moving targets using different system models. Detailed work about this topic can be found in the target tracking literature e.g. [30, 31, 37] which is beyond the scope of this thesis.

3.5 Discussion

(3.18) is a powerful tool to fuse location estimates. On the other hand, optimality issues must be considered due to assumptions. First of all, consecutive location estimates are assumed to be corrupted by independent. Most probably, in practice, location estimate noise will be correlated, because of using similar equipment or operating in the nearby area etc. Secondly, $\mathbf{Q}_i$ are assumed to be perfectly known for $i = 0, 1, 2, \ldots, K$. In most of the cases, exact location of the emitter and exact knowledge about estimation error is required to calculate $\mathbf{Q}_i$ which is impossible when estimating the unknown location. Instead, estimated values of these factors are used in calculating $\mathbf{Q}_i$, resulting that, $\mathbf{Q}_i$ itself becomes an estimate. Because of the stated facts, post-processing estimator (3.18) is suboptimal. Despite suboptimality issues, a powerful aspect of (3.18) is that it can take input from any type of location estimator as long as estimates and their covariances are provided.

CHAPTER 4

A RECURSIVE ESTIMATOR

In the case of locating a pulsed radar, there is plenty of available measurements from the abundant number of pulses. A simple and possible way to fuse these measurements is to use a post-processing estimator as described in the previous chapter. On the other hand, post processing results in suboptimal estimates. From this point of view, directly processing the measurements is a more effective and elegant way. An estimator is desired to update the estimate when the new measurements become available without any loss of the knowledge about the past data. A recursive estimator (with memory) can accomplish the task.

In the open literature, there are various studies about recursive estimators. Some examples are given in the EKF and the UKF sense using TDOA in [24, 25]. On the other hand, these estimators suffer from divergence problems [24]. This issue makes the estimators unusable. Furthermore, a Kalman filter based solution is also given for cellular network applications in [35].

The aim of this chapter is to derive a recursive and robust location estimator which is capable of processing the TDOA measurements directly. The measurements are assumed to be collected by moving sensors (Thus, the number sensors increases artificially). The location estimate is updated using the new measurements without any loss of information about past data. For this reason, firstly, a simple closed-form location estimator is derived using Law of Cosines which uses only one set of

TDOA measurement. After that, it is extended into a multi-pulse closed-form solution which is capable of processing multiple-set of TDOA measurements without recursion. Then, the multi-pulse location estimator is represented in recursive sense by the Kalman filter equations. Finally, a solution to the passively tracking of a moving emitter is presented.

4.1 A Basic Closed-Form Solution to TDOA Localization Problem

In [8], a simple closed-form solution to TDOA emitter localization problem is presented. Derivation of the solution is constructed using Law of Cosines. In this section, we review the derivation because it constitutes one of the basic concepts in the TDOA localization problem, and it is the origin of the recursive estimator.

Consider the scenario shown in Figure 4.1 [8];

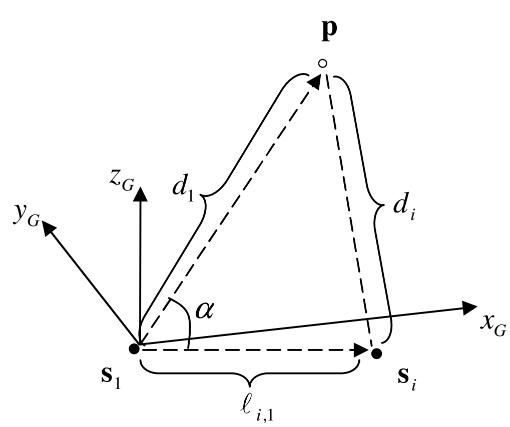

Figure 4.1: Example scenario

where (x_G, y_G, z_G) denotes global coordinates, and $\ell_{i,1}$ is the distance between $\mathbf{s}_1$ and $\mathbf{s}_i$ for $i = 2, 3, \ldots, L$. $\mathbf{s}_1$ is the reference sensor and it is assumed to be located at origin for simplicity. Then, the position vector of the emitter is denoted by $\mathbf{p} = [x_e \quad y_e \quad z_e]^T$. Similarly, the position vector of i^{th} sensor is denoted by $\mathbf{s}_i = [x_i \quad y_i \quad z_i]^T$ for $i = 1, 2, 3, \ldots, L$. d_i is the distance between the emitter and the i^{th} sensor.

d_i can be written in terms of d_1:

$$d_i = d_1 + r_{i,1}, \tag{4.1}$$

where $r_{i,1}$ accounts for Range Difference of Arrival (RDOA) measurement between the first and the i^{th} sensors. RDOA values ($r_{i,j}$) can trivially expressed using TDOA measurements ($\tau_{i,j}$):

$$r_{i,j} = c\tau_{i,j}, \tag{4.2}$$

where c is speed of light.

Applying law of cosines to triangle constructed by $\mathbf{s}_1$, $\mathbf{s}_i$ and $\mathbf{p}$ yields:

$$(d_1 + r_{i,1})^2 = d_1^2 + \ell_{i,1}^2 - 2|d_1||l_{i,1}|\cos\alpha. \tag{4.3}$$

It is a fact that [33]:

$$\mathbf{a}^T\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\alpha. \tag{4.4}$$

Using (4.4), equation (4.3) becomes:

$$(d_1 + r_{i,1})^2 = d_1^2 + \ell_{i,1}^2 - 2\mathbf{s}_i^T\mathbf{p}. \tag{4.5}$$

Rearranging terms in (4.5) results in [8]:

$$\ell_{i,1}^2 - 2d_1 r_{i,1} - r_{i,1}^2 - 2\mathbf{s}_i^T\mathbf{p} = 0. \tag{4.6}$$

Equation (4.6) is the key to the closed form estimator. On the other hand, sensor and emitter locations are expressed relative to the reference sensor location. This dependency must be avoided before proceeding further. Now, consider that reference sensor is not located at the origin, (4.6) simply becomes [10]:

$$\begin{aligned} 0 &= \ell_{i,1}^2 - 2d_1 r_{i,1} - r_{i,1}^2 - 2(\mathbf{s}_i - \mathbf{s}_1)^T(\mathbf{p} - \mathbf{s}_1), \\ &= \ell_{i,1}^2 - 2d_1 r_{i,1} - r_{i,1}^2 - 2(\mathbf{s}_i - \mathbf{s}_1)^T\mathbf{p} + 2(\mathbf{s}_i - \mathbf{s}_1)^T\mathbf{s}_1. \end{aligned} \tag{4.7}$$

In matrix form, for $i = 1, 2, 3, \ldots, L$ (4.7) becomes [10]:

$$\mathbf{A}\mathbf{x} = \mathbf{b}\,, \tag{4.8}$$

where;

$$\mathbf{A} = \begin{bmatrix} (\mathbf{s}_2 - \mathbf{s}_1)^T & r_{2,1} \\ (\mathbf{s}_3 - \mathbf{s}_1)^T & r_{3,1} \\ \vdots & \vdots \\ (\mathbf{s}_L - \mathbf{s}_1)^T & r_{L,1} \end{bmatrix},\ \mathbf{x} = \begin{bmatrix} \mathbf{p} \\ d_1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 0.5(\ell_{2,1}^2 - r_{2,1}^2) + (\mathbf{s}_2 - \mathbf{s}_1)^T\mathbf{s}_1 \\ 0.5(\ell_{3,1}^2 - r_{3,1}^2) + (\mathbf{s}_3 - \mathbf{s}_1)^T\mathbf{s}_1 \\ \vdots \\ 0.5(\ell_{L,1}^2 - r_{L,1}^2) + (\mathbf{s}_L - \mathbf{s}_1)^T\mathbf{s}_1 \end{bmatrix}. \tag{4.9}$$

Since $\mathbf{A}$ is not invertible, its pseudo inverse is used in order to find the Least Squares (LS) solution to (4.8), which is:

$$\hat{\mathbf{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}. \tag{4.10}$$

(4.10) can be applied to a TDOA localization problem where TDOA measurements have identical noise variance. On the other hand, if TDOA measurements have different variances, a weighting procedure must be taken into account in (4.8). The Weighted Least Squares (WLS) solution to (4.10) is given as [10]:

$$\hat{\mathbf{x}} = (\mathbf{A}^T\mathbf{N}^{-1}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{N}^{-1}\mathbf{b}, \tag{4.11}$$

where $\mathbf{N}$ is the to RDOA noise covariance matrix for large range-to-baseline situations [10] which is the case for the problem investigated in this work.

The equation given in (4.11) is a closed-form solution to the TDOA emitter localization problem. It uses only one set of TDOA measurements. On the other hand, there are plenty of available TDOA measurement sets in the case of locating a pulsed radar. Hence, there is a need for some methodology to process all of the available data. The following section describes a solution to this problem.

4.2 WLS Localization Using Multiple TDOA Sets [10]

In this section, the WLS solution [10] of the localization problem using multiple TDOA sets is derived. It is one of the major parts of the recursive estimator.

Suppose that there are L moving sensors which take TDOA measurements from a fixed location pulsed radar. When the k^{th} pulse is received (4.8) becomes:

$$\begin{bmatrix} \mathbf{s}_{2,1}^T(k) & r_{2,1}(k) \\ \mathbf{s}_{3,1}^T(k) & r_{3,1}(k) \\ \vdots & \vdots \\ \mathbf{s}_{L,1}^T(k) & r_{L,1}(k) \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ d_1(k) \end{bmatrix} = \begin{bmatrix} 0.5(\ell_{2,1}^2(k) - r_{2,1}^2(k)) + \mathbf{s}_{2,1}^T(k)\mathbf{s}_1(k) \\ 0.5(\ell_{3,1}^2(k) - r_{3,1}^2(k)) + \mathbf{s}_{3,1}^T(k)\mathbf{s}_1(k) \\ \vdots \\ 0.5(\ell_{L,1}^2(k) - r_{L,1}^2(k)) + \mathbf{s}_{L,1}^T(k)\mathbf{s}_1(k) \end{bmatrix}. \tag{4.12}$$

where k is the time index of the k^{th} pulse. $\mathbf{s}_{i,1}^T(k)$ is defined as $\mathbf{s}_{i,1}^T(k) = (\mathbf{s}_i(k) - \mathbf{s}_1(k))^T$ for $i = 2, 3, \ldots, L$. Thus, $\mathbf{A}$, $\mathbf{x}$ and $\mathbf{b}$ in (4.8) can be expressed as:

$$\begin{gathered} \mathbf{A}(k) = \begin{bmatrix} \mathbf{s}_{2,1}^T(k) & r_{2,1}(k) \\ \mathbf{s}_{3,1}^T(k) & r_{3,1}(k) \\ \vdots & \vdots \\ \mathbf{s}_{L,1}^T(k) & r_{L,1}(k) \end{bmatrix}, \quad \mathbf{x}(k) = \begin{bmatrix} \mathbf{p} \\ d_1(k) \end{bmatrix}, \\ \mathbf{b}(k) = \begin{bmatrix} 0.5(\ell_{2,1}^2(k) - r_{2,1}^2(k)) + \mathbf{s}_{2,1}^T(k)\mathbf{s}_1(k) \\ 0.5(\ell_{3,1}^2(k) - r_{3,1}^2(k)) + \mathbf{s}_{3,1}^T(k)\mathbf{s}_1(k) \\ \vdots \\ 0.5(\ell_{L,1}^2(k) - r_{L,1}^2(k)) + \mathbf{s}_{L,1}^T(k)\mathbf{s}_1(k) \end{bmatrix}. \end{gathered} \tag{4.13}$$

For convenience, we define the followings:

$$\mathbf{S}(k) = \begin{bmatrix} \mathbf{s}_{2,1}^T(k) \\ \mathbf{s}_{3,1}^T(k) \\ \vdots \\ \mathbf{s}_{L,1}^T(k) \end{bmatrix} \text{ and } \mathbf{r}(k) = \begin{bmatrix} r_{2,1}(k) \\ r_{3,1}(k) \\ \vdots \\ r_{L,1}(k) \end{bmatrix}. \tag{4.14}$$

Then (4.12) becomes:

$$[\mathbf{S}(k) \quad \mathbf{r}(k)] \begin{bmatrix} \mathbf{p} \\ d_1(k) \end{bmatrix} = \mathbf{b}(k). \tag{4.15}$$

Since we are trying to accumulate TDOA measurements to get a better location estimate, suppose that consecutive measurements are stacked in equation (4.15), resulting in:

$$\underbrace{\begin{bmatrix} \mathbf{S}(1) & \mathbf{r}(1) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{S}(2) & \mathbf{0} & \mathbf{r}(2) & & \vdots \\ \vdots & & & \ddots & \mathbf{0} \\ \mathbf{S}(k) & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{r}(k) \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} \mathbf{p} \\ d_1(1) \\ d_1(2) \\ \vdots \\ d_1(k) \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \mathbf{b}(1) \\ \mathbf{b}(2) \\ \mathbf{b}(3) \\ \vdots \\ \mathbf{b}(k) \end{bmatrix}}_{\mathbf{d}}, \tag{4.16}$$

where $\mathbf{C}$ is of size $k \cdot (L-1) \times (L+2)$ for 2 dimensional emitter localization.

The WLS solution to (4.16) is given by:

$$\hat{\mathbf{y}} = (\mathbf{C}^T \mathbf{W}^{-1} \mathbf{C})^{-1} \mathbf{C}^T \mathbf{W}^{-1} \mathbf{d}, \tag{4.17}$$

where $\mathbf{W}$ is a block diagonal matrix consisting of RDOA covariance matrices under the assumption of long range-to-baseline ratio:

$$\mathbf{W} = \begin{bmatrix} \mathbf{N}(1) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{N}(2) & & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{N}(k) \end{bmatrix}. \tag{4.18}$$

It is assumed that noise is independent at different time instances.

Examining (4.16), it can be stated that because of time varying parameter $d_1(k)$, (4.17) suffers from dimension growth in $\mathbf{y}$ when a new measurement arrives. As a result, computational complexity increases.

A solution to this problem is to model the range variation with a constant velocity model which is given as:

$$d_1(k) = d_1(0) + v_1(0)t(k), \tag{4.19}$$

where $d_1(0)$ and $v_1(0)$ are initial range and rate of change in range respectively. In other words, $v_1(0)$ is the velocity of the initial range, and $t(k)$ is the arrival time (TOA) of the k^{th} pulse. Note that even if the sensors move with constant velocity, $d_1(0)$ changes nonlinearly with $t(k)$. Thus, (4.19) is an approximation.

Using (4.19), (4.16) becomes:

$$\begin{bmatrix} \mathbf{S}(1) & \mathbf{r}(1) & t(1)\mathbf{r}(1) \\ \mathbf{S}(2) & \mathbf{r}(2) & t(2)\mathbf{r}(2) \\ \vdots & \vdots & \vdots \\ \mathbf{S}(k) & \mathbf{r}(k) & t(k)\mathbf{r}(k) \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{p} \\ d_1(0) \\ v_1(0) \end{bmatrix}}_{\mathbf{z}} = \begin{bmatrix} \mathbf{b}(1) \\ \mathbf{b}(2) \\ \mathbf{b}(3) \\ \vdots \\ \mathbf{b}(k) \end{bmatrix}. \tag{4.20}$$

Then the WLS solution to the emitter localization using multiple TDOA sets is given by:

$$\hat{\mathbf{z}}_k = (\mathbf{F}_k^T \mathbf{W}_k^{-1} \mathbf{F}_k)^{-1} \mathbf{F}_k^T \mathbf{W}_k^{-1} \mathbf{d}_k. \tag{4.21}$$

where $\mathbf{F}_k$, $\mathbf{d}_k$ and $\mathbf{W}_k$ are defined as:

$$\mathbf{F}_k = \begin{bmatrix} \mathbf{S}(1) & \mathbf{r}(1) & t(1)\mathbf{r}(1) \\ \mathbf{S}(2) & \mathbf{r}(2) & t(2)\mathbf{r}(2) \\ \vdots & \vdots & \vdots \\ \mathbf{S}(k) & \mathbf{r}(k) & t(k)\mathbf{r}(k) \end{bmatrix}, \mathbf{d}_k = \begin{bmatrix} \mathbf{b}(1) \\ \mathbf{b}(2) \\ \mathbf{b}(3) \\ \vdots \\ \mathbf{b}(k) \end{bmatrix}, \mathbf{W}_k = \begin{bmatrix} \mathbf{N}(1) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{N}(2) & & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{N}(k) \end{bmatrix}, \quad (4.22)$$

and the subscript k indicates being constructed when k^{th} pulse is received.

Finally, the WLS emitter localization using multiple TDOA sets is realized by (4.21).

4.3 Recursive Estimator with Multiple TDOA Sets

(4.21) is a useful estimator, since it is capable of processing multiple TDOA sets. On the hand, it is in the form of a closed-form estimator, which means it has drawbacks in processing data: All of the measurements have to be processed again when new measurement arrives. This phenomenon leads to an undesired increment in computational cost. A recursion process is needed to overcome this problem.

Simply, a Recursive Least Squares (RLS) filter can be used to solve (4.21) iteratively [34]. (4.21) gives the WLS solution when k^{th} pulse is received. On the other hand, when $(k+1)^{th}$ pulse arrives (4.21) becomes:

$$\begin{aligned} &\mathbf{F}_{k+1} = \begin{bmatrix} \mathbf{F}_k \\ \mathbf{f}_{k+1} \end{bmatrix}, \mathbf{d}_{k+1} = \begin{bmatrix} \mathbf{d}_k \\ \mathbf{b}(k+1) \end{bmatrix}, \mathbf{W}_{k+1} = \begin{bmatrix} \mathbf{W}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{N}(k+1) \end{bmatrix}, \\ &\hat{\mathbf{z}}_{k+1} = (\mathbf{F}_{k+1}^T \mathbf{W}_{k+1}^{-1} \mathbf{F}_{k+1})^{-1} \mathbf{F}_{k+1}^T \mathbf{W}_{k+1}^{-1} \mathbf{d}_{k+1}, \end{aligned} \quad (4.23)$$

where $\mathbf{f}_{k+1}$ is defined as:

$$\mathbf{f}_{k+1} = [\mathbf{S}(k+1) \quad \mathbf{r}(k+1) \quad t(k+1)\mathbf{r}(k+1)]. \quad (4.24)$$

In order to realize a recursion process, let us try to express $\hat{\mathbf{z}}_{k+1}$ in terms of $\hat{\mathbf{z}}_k$ [30]:

$$
\begin{aligned}
\hat{\mathbf{z}}_{k+1} &= (\mathbf{F}_{k+1}^T \mathbf{W}_{k+1}^{-1} \mathbf{F}_{k+1})^{-1} \mathbf{F}_{k+1}^T \mathbf{W}_{k+1}^{-1} \mathbf{d}_{k+1}, \\
&= \left(\begin{bmatrix} \mathbf{F}_k^T & \mathbf{f}_{k+1}^T \end{bmatrix} \begin{bmatrix} \mathbf{W}_k^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{N}(k+1))^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{F}_k \\ \mathbf{f}_{k+1} \end{bmatrix} \right)^{-1} \cdot \\
&\qquad \begin{bmatrix} \mathbf{F}_k^T & \mathbf{f}_{k+1}^T \end{bmatrix} \begin{bmatrix} \mathbf{W}_k^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{N}(k+1))^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{d}_k \\ \mathbf{b}(k+1) \end{bmatrix}, \\
&= \left(\mathbf{F}_k^T \mathbf{W}_k^{-1} \mathbf{F}_k + \mathbf{f}_{k+1}^T (\mathbf{N}(k+1))^{-1} \mathbf{f}_{k+1} \right)^{-1} \left(\mathbf{F}_k^T \mathbf{W}_k^{-1} \mathbf{d}_k + \mathbf{f}_{k+1}^T (\mathbf{N}(k+1))^{-1} \mathbf{b}(k+1) \right).
\end{aligned}
\tag{4.25}
$$

In (4.25), it is assumed that noise is independent at distinct time instances.

The estimate covariances have been found as [30, 31]:

$$
\begin{aligned}
\mathbf{P}_k &= \left(\mathbf{F}_k^T \mathbf{W}_k^{-1} \mathbf{F}_k \right)^{-1}, \\
\mathbf{P}_{k+1} &= \left(\mathbf{F}_{k+1}^T \mathbf{W}_{k+1}^{-1} \mathbf{F}_{k+1} \right)^{-1}, \\
&= \left(\mathbf{F}_k^T \mathbf{W}_k^{-1} \mathbf{F}_k + \mathbf{f}_{k+1}^T (\mathbf{N}(k+1))^{-1} \mathbf{f}_{k+1} \right)^{-1}, \\
&= \left(\mathbf{P}_k^{-1} + \mathbf{f}_{k+1}^T (\mathbf{N}(k+1))^{-1} \mathbf{f}_{k+1} \right)^{-1}.
\end{aligned}
\tag{4.26}
$$

Using (4.21), one obtains:

$$
\mathbf{P}_k^{-1} \hat{\mathbf{z}}_k = \mathbf{F}_k^T \mathbf{W}_k^{-1} \mathbf{d}_k. \tag{4.27}
$$

Then, using (4.25), (4.26) and (4.27) $\hat{\mathbf{z}}_{k+1}$ can be written as:

$$
\hat{\mathbf{z}}_{k+1} = \mathbf{P}_{k+1} \left(\mathbf{P}_k^{-1} \hat{\mathbf{z}}_k + \mathbf{f}_{k+1}^T (\mathbf{N}(k+1))^{-1} \mathbf{b}(k+1) \right). \tag{4.28}
$$

(4.26) and (4.28) yields a method of combining previous estimates and the new measurement to construct new updated estimate.

Now, let us write equation (4.28) in the following form [30]:

$$
\hat{\mathbf{z}}_{k+1} = \hat{\mathbf{z}}_k + \mathbf{P}_{k+1} \left((\mathbf{P}_k^{-1} - \mathbf{P}_{k+1}^{-1}) \hat{\mathbf{z}}_k + \mathbf{f}_{k+1}^T (\mathbf{N}(k+1))^{-1} \mathbf{b}(k+1) \right). \tag{4.29}
$$

Inserting definition of $\mathbf{P}_{k+1}$ (4.26) into (4.29) yields:

$$\begin{aligned}
\hat{\mathbf{z}}_{k+1} &= \hat{\mathbf{z}}_k + \mathbf{P}_{k+1}\left((\mathbf{P}_k^{-1} - \mathbf{P}_{k+1}^{-1})\hat{\mathbf{z}}_k + \mathbf{f}_{k+1}^T(\mathbf{N}(k+1))^{-1}\mathbf{b}(k+1)\right), \\
\hat{\mathbf{z}}_{k+1} &= \hat{\mathbf{z}}_k + \mathbf{P}_{k+1}\left((\mathbf{P}_k^{-1} - \mathbf{P}_k^{-1} - \mathbf{f}_{k+1}^T(\mathbf{N}(k+1))^{-1}\mathbf{f}_{k+1})\hat{\mathbf{z}}_k + \mathbf{f}_{k+1}^T(\mathbf{N}(k+1))^{-1}\mathbf{b}(k+1)\right), \\
\hat{\mathbf{z}}_{k+1} &= \hat{\mathbf{z}}_k + \mathbf{P}_{k+1}\left(\mathbf{f}_{k+1}^T(\mathbf{N}(k+1))^{-1}\mathbf{b}(k+1) - \mathbf{f}_{k+1}^T(\mathbf{N}(k+1))^{-1}\mathbf{f}_{k+1}\hat{\mathbf{z}}_k\right).
\end{aligned} \tag{4.30}$$

Now, rearranging terms in (4.30) results in:

$$\hat{\mathbf{z}}_{k+1} = \hat{\mathbf{z}}_k + \mathbf{P}_{k+1}\mathbf{f}_{k+1}^T(\mathbf{N}(k+1))^{-1}\left(\mathbf{b}(k+1) - \mathbf{f}_{k+1}\hat{\mathbf{z}}_k\right). \tag{4.31}$$

Equation (4.31) and $\mathbf{P}_{k+1}$ (given in (4.26)) constitutes a recursive update process.

In addition, using the Matrix Inversion Lemma, $\mathbf{P}_{k+1}$ can be written in the following form [30] to increase computational efficiency:

$$\begin{aligned}
\mathbf{P}_{k+1} &= \left(\mathbf{P}_k^{-1} + \mathbf{f}_{k+1}^T(\mathbf{N}(k+1))^{-1}\mathbf{f}_{k+1}\right)^{-1}, \\
&= \mathbf{P}_k - \mathbf{P}_k\mathbf{f}_{k+1}^T\left(\mathbf{f}_{k+1}\mathbf{P}_k\mathbf{f}_{k+1}^T + \mathbf{N}(k+1)\right)^{-1}\mathbf{f}_{k+1}\mathbf{P}_k\ .
\end{aligned} \tag{4.32}$$

In summary, recursive estimation of the emitter localization using multiple TDOA measurement sets is realized in the RLS sense by equations (4.31) and (4.32) under the assumption of fixed emitter location [34]. Derivation continues with expressing (4.31) and (4.32) in the Kalman Filter form to make the estimator more adaptable to the scenarios other than the stationary emitter location..

Define the Kalman Gain $\mathbf{K}_{k+1}$ as [30]:

$$\begin{aligned}
\mathbf{K}_{k+1} &= \left(\mathbf{P}_{k+1}\mathbf{f}_{k+1}^T(\mathbf{N}(k+1))^{-1}\right)\mathbf{f}_{k+1}^T(\mathbf{N}(k+1))^{-1}, \\
&= \left(\mathbf{P}_k^{-1} + \mathbf{f}_{k+1}^T(\mathbf{N}(k+1))^{-1}\mathbf{f}_{k+1}\right)^{-1}\mathbf{f}_{k+1}^T(\mathbf{N}(k+1))^{-1}.
\end{aligned} \tag{4.33}$$

Using the following matrix identity:

$$\mathbf{A}\mathbf{B}^T(\mathbf{C} + \mathbf{B}\mathbf{A}\mathbf{B}^T)^{-1} = (\mathbf{A}^{-1} + \mathbf{B}^T\mathbf{C}^{-1}\mathbf{B})^{-1}\mathbf{B}^T\mathbf{C}^{-1}, \tag{4.34}$$

The gain can be written as [30]:

$$\mathbf{K}_{k+1} = \mathbf{P}_k \mathbf{f}_{k+1}^T \left(\mathbf{N}(k+1) + \mathbf{f}_{k+1} \mathbf{P}_k \mathbf{f}_{k+1}^T\right)^{-1}. \tag{4.35}$$

Using (4.35), (4.32) turns into:

$$\mathbf{P}_{k+1} = \mathbf{P}_k - \mathbf{K}_{k+1} \mathbf{f}_{k+1} \mathbf{P}_k \ . \tag{4.36}$$

Similar to (4.36), equation (4.31) is expressed using (4.35) as:

$$\hat{\mathbf{z}}_{k+1} = \hat{\mathbf{z}}_k + \mathbf{K}_{k+1} \left(\mathbf{b}(k+1) - \mathbf{f}_{k+1} \hat{\mathbf{z}}_k\right). \tag{4.37}$$

At last, the expressions (4.35), (4.36) and (4.37) form the Kalman Filter equations of the estimator given in (4.31) [30, 34], where $\hat{\mathbf{z}}_k$ is defined as the state, $\mathbf{P}_k$ is the state error covariance matrix and $\mathbf{K}_k$ is the Kalman Gain at time step k [31].

4.3.1 Expressing Recursive Estimator in Kalman Filter Equations for General Case

The recursive estimator (4.31) is advantageous since it can accumulate the data, and in addition, because of the RLS structure, obviously it can update the previous estimate when the new measurement arrives. On the other hand, it has some drawbacks due to the assumption of the constant range velocity model given in (4.19). First of all, only the reference sensor's range variation is modeled. Under this assumption, the reference sensor can not be changed through the operation, the same sensor has to remain as the reference, which means if the reference sensor can not receive a pulse, then no measurement update can be accomplished. Furthermore, suppose that there are several groups of receivers which can not always measure the same pulse. In the case of using constant range-velocity model (4.19), these groups of receivers have to estimate the emitter location separately which results in wasting appropriate chance of fusing data to get a better estimate. Secondly, constant range velocity model restricts the sensors to perform maneuvers. This type of operation can not be accepted in practice. Because of the stated problems, some modifications are necessary to (4.31).

As the first step, the range-velocity model given in (4.19) will be modified. First of all and most importantly, initial range dependency must be avoided. For this purpose, let us write $d_1(k)$ and $d_1(k-1)$ using (4.19):

$$\begin{aligned} d_1(k-1) &= d_1(0) + v_1(0)t(k-1), \\ d_1(k) &= d_1(0) + v_1(0)t(k). \end{aligned} \tag{4.38}$$

If we subtract $d_1(k-1)$ from $d_1(k)$ we get:

$$d_1(k) - d_1(k-1) = v_1(0)(t(k) - t(k-1)). \tag{4.39}$$

Rearranging terms in (4.39) yields:

$$d_1(k) = d_1(k-1) + v_1(0)(t(k) - t(k-1)). \tag{4.40}$$

Deriving (4.40), initial range dependency is removed. Now, for generality, we substitute initial range velocity ($v_1(0)$) with the current one ($v_1(k)$). Then (4.40) becomes:

$$d_1(k) = d_1(k-1) + v_1(k)(t(k) - t(k-1)). \tag{4.41}$$

Equation (4.41) is the modified version of range-velocity model given in (4.19).

After derivation of (4.41), recursive estimator expressions must be prepared in order to use proposed range-velocity model.

The Kalman Filter equations (4.35), (4.36) and (4.37) have been derived under the assumptions of the fixed emitter location and constant range and range velocity. In other words, the state transition matrix ($\mathbf{A}_k$) is the identity matrix where the state is $\hat{\mathbf{z}}_k$ itself. So, it is not specified particularly in the filter equations. From this point of view, if the filter equations are rewritten especially indicating state transition matrix, (4.35), (4.36) and (4.37) becomes [30, 31]:

$$
\begin{aligned}
\mathbf{K}_{k+1} &= \mathbf{A}_k \mathbf{P}_k \mathbf{A}_k^T \mathbf{f}_{k+1}^T \left(\mathbf{N}(k+1) + \mathbf{f}_{k+1} \mathbf{A}_k \mathbf{P}_k \mathbf{A}_k^T \mathbf{f}_{k+1}^T \right)^{-1}, \\
\mathbf{P}_{k+1} &= (\mathbf{I} - \mathbf{K}_{k+1} \mathbf{f}_{k+1}) \mathbf{A}_k \mathbf{P}_k \mathbf{A}_k^T, \\
\hat{\mathbf{z}}_{k+1} &= \mathbf{A}_k \hat{\mathbf{z}}_k + \mathbf{K}_{k+1} \left(\mathbf{b}(k+1) - \mathbf{f}_{k+1} \mathbf{A}_k \hat{\mathbf{z}}_k \right),
\end{aligned}
\tag{4.42}
$$

where $\mathbf{f}_{k+1} = [\mathbf{S}(k+1) \quad \mathbf{r}(k+1) \quad \mathbf{0}]$, the state vector $\hat{\mathbf{z}}_k$ and the state transition matrix ($\mathbf{A}_k$) are defined as:

$$
\hat{\mathbf{z}}_k = \begin{bmatrix} \mathbf{p} \\ d_1(0) \\ v_1(0) \end{bmatrix} \text{ and } \mathbf{A}_k = \mathbf{I}. \tag{4.43}
$$

Using the range and range velocity model given in (4.41), under the assumption of two dimensional fixed emitter location where $\mathbf{p} = [x_e \quad y_e]^T$, (4.43) becomes:

$$
\hat{\mathbf{z}}_k = \begin{bmatrix} x_e \\ y_e \\ d_1(k) \\ v_1(k) \end{bmatrix} \text{ and } \mathbf{A}_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T_k \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.44}
$$

where $T_k = (t(k) - t(k-1))$.

Using (4.44) along with (4.42) is the solution for the restriction of the single reference. Furthermore, (4.42) and (4.44) are also a solution for the sensor maneuvers.

Similarly, for three dimensional localization $\hat{\mathbf{z}}_k$ and $\mathbf{A}_k$ are given as:

$$\hat{\mathbf{z}}_k = \begin{bmatrix} x_e \\ y_e \\ z_e \\ d_1(k) \\ v_1(k) \end{bmatrix} \text{ and } \mathbf{A}_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & T_k \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.45}$$

Modeling the state variable ($\hat{\mathbf{z}}_k$) with (4.45) leads to uncertainties in $d_1(k)$ and $v_1(k)$ which arises from maneuvers of the sensors. At this point, this uncertainty has to be considered. Furthermore, in tracking literature, this condition is handled by taking into account the process noise with covariance matrix ($\mathbf{Q}_k$) which is indeed used to encompass uncertainties in the state model. Using $\mathbf{Q}_k$, (4.42) is rewritten as [30, 31]:

$$\begin{aligned}
\mathbf{P}'_{k+1} &= (\mathbf{A}_k \mathbf{P}_k \mathbf{A}_k^T + \mathbf{Q}_k), \\
\mathbf{K}_{k+1} &= \mathbf{P}'_{k+1}\mathbf{f}_{k+1}^T \left(\mathbf{N}(k+1) + \mathbf{f}_{k+1}\mathbf{P}'_{k+1}\mathbf{f}_{k+1}^T\right)^{-1}, \\
\mathbf{P}_{k+1} &= (\mathbf{I} - \mathbf{K}_{k+1}\mathbf{f}_{k+1})\mathbf{P}'_{k+1}, \\
\hat{\mathbf{z}}_{k+1} &= \mathbf{A}_k \hat{\mathbf{z}}_k + \mathbf{K}_{k+1}\left(\mathbf{b}(k+1) - \mathbf{f}_{k+1}\mathbf{A}_k \hat{\mathbf{z}}_k\right).
\end{aligned} \tag{4.46}$$

where $\mathbf{b}(k+1)$ is defined in (4.13). Note that the output equation of the system is obtained from (4.7) by considering ($0.5(\ell_{i,1}^2 - r_{i,1}^2) + (\mathbf{s}_i - \mathbf{s}_1)^T \mathbf{s}_1$ where $i = 2, 3, \ldots, L$) as the measurement. Expressions in (4.46) are the Kalman Filter form of the recursive estimator.

For the stationary emitter case, $\mathbf{Q}_k$ is found to be:

$$\mathbf{Q}_k = \sigma_V^2 \mathbf{G} \cdot \mathbf{G}^T, \tag{4.47}$$

where σ_V^2 is process noise variance and defined as filter parameter. For two and three dimensional state variable $\hat{\mathbf{z}}_k$ in (4.44) $\mathbf{G}$ is respectively defined as:

$$\mathbf{G} = \begin{bmatrix} 0 \\ 0 \\ 0.5T_k^2 \\ T_k \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5T_k^2 \\ T_k \end{bmatrix}. \tag{4.48}$$

Note that $\mathbf{Q}_k$ is used to model uncertainties in $d_1(k)$ and $v_1(k)$ as noise. $\mathbf{G}$ is a simple but efficient model. Besides, more sophisticated models can be used as well. On the other hand, the target tracking concepts are out of the scope of this thesis and detailed information about the tracking methods can be found in the target tracking literature [30, 31, 37].

To initialize the recursive estimator given in (4.46), first (4.21) is used to calculate initial values of the state vector and the state error covariance matrix which are $\hat{\mathbf{z}}_0$ and $\mathbf{P}_0$ respectively. Then, updates are performed when new measurement arrives.

4.3.2 Location Estimation of a Moving Emitter Using TDOA

In passive localization systems, there are generally two methods to estimate location of moving emitters.

1. This method uses location estimates as the inputs of a tracking filter as discussed in Chapter 3. These localization techniques are in the class of post processing methods. Several examples can be found in the literature [14, 28]. Post-processing methods are suboptimal because they process information that was processed nonlinearly which suffers from the threshold effect.

2. In concept of location estimation of moving emitters by passive sensors, this type uses the measurements directly as the inputs of the estimator. This class has the advantage of prossessing all the information involved in the measurements which post-processing methods can not attain.

(4.46) can be easily adopted to estimate the location of moving emitters. In order to achieve this goal, only slight modifications are required for the filter variables. Filter equations are simply extended to the moving target case. Information about the target motion models can be found the target tracking literature [30, 31, 37].

First and most importantly, $\mathbf{z}_k$ must include the information about the velocity of the emitter:

$$\hat{\mathbf{z}}_k = \begin{bmatrix} x_e \\ y_e \\ d_1(k) \\ v_1(k) \\ v_x(k) \\ v_y(k) \end{bmatrix}, \tag{4.49}$$

where $v_x(k)$ and $v_y(k)$ are the velocity components of the emitter in x and y directions respectively.

The initial value of $\hat{\mathbf{z}}_k$ is:

$$\hat{\mathbf{z}}_0 = \begin{bmatrix} \hat{\mathbf{z}}_0^{'} \\ 0 \\ 0 \end{bmatrix}, \tag{4.50}$$

where $\hat{\mathbf{z}}_0^{'}$ is calculated from (4.21). When only TDOA is used, there is no available priori information about the velocity of the emitter. So, the initial values of the velocity components of the emitter are set to zero.

Then, the state transition matrix ($\mathbf{A}_k$) becomes:

$$
\mathbf{A}_k = \begin{bmatrix} 1 & 0 & 0 & 0 & T_k & 0 \\ 0 & 1 & 0 & 0 & 0 & T_k \\ 0 & 0 & 1 & T_k & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.51}
$$

Of course, $\mathbf{f}_k$ is also modified. Since, measurements do not include any information about the emitter's velocity, we simply add zeros to relevant parts of $\mathbf{f}_k$:

$$
\mathbf{f}_{k+1} = [\mathbf{S}(k+1) \quad \mathbf{r}(k+1) \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0}]. \tag{4.52}
$$

Process noise covariance matrix takes the following form:

$$
\mathbf{Q}_k = \mathbf{G} \cdot \mathbf{Q}^{'} \cdot \mathbf{G}^T, \tag{4.53}
$$

where $\mathbf{G}$ and $\mathbf{Q}^{'}$ are defined as:

$$
\mathbf{G} = \begin{bmatrix} 0.5T_k^2 & 0 & 0 \\ 0 & 0.5T_k^2 & 0 \\ 0 & 0 & 0.5T_k^2 \\ 0 & 0 & T_k \\ T_k & 0 & 0 \\ 0 & T_k & 0 \end{bmatrix}, \quad \mathbf{Q}^{'} = \begin{bmatrix} \sigma_{v1}^2 & 0 & 0 \\ 0 & \sigma_{v2}^2 & 0 \\ 0 & 0 & \sigma_{v3}^2 \end{bmatrix}. \tag{4.54}
$$

Similar to (4.47), $\mathbf{Q}^{'}$ is filter parameter.

Notice that the state error covariance matrix $\mathbf{P}_k$ also has to be modified. Since, there is no prior knowledge about the target velocity; some artificial extension to initial value of $\mathbf{P}_k$ is required. Simply, zeros or some convenient noise variance can be added to relevant parts of $\mathbf{P}_0$.

For example, following can be used as the initial value of the state error covariance matrix $\mathbf{P}_k$:

$$\mathbf{P}_0 = \begin{bmatrix} \mathbf{P}_0^{'} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_0^{''} \end{bmatrix}. \tag{4.55}$$

Note that, $\mathbf{P}_0^{''}$ represents uncertainties in the velocity of the emitter. $\mathbf{P}_0^{''}$ must be chosen large enough to encompass those uncertainties, and it can be determined according to some priori statistics or knowledge about the emitter or simulation studies.

Using (4.49), (4.51), (4.53) and (4.55) along with (4.46), moving emitter location estimation is accomplished. Consequently, recursive filter directly uses TDOA measurements as input.

Notice that, emitter's motion is modeled by a simple constant-velocity motion model. Other motion models can be used or $\mathbf{Q}_k$ can be altered as well. For example, Interacting Multiple Models (IMM) can be used with more sophisticated motion models to improve accuracy of the recursive filter. Multi Hypothesis Tracking (MHT) can also be used to get better location estimates. Detailed information about these methods can be found in the target tracking literature [30, 31, 37]. Expressed tracking technique can be easily extended in to those more sophisticated ones. On the other hand, in order to not to exceed the scope of this thesis, possible extensions are not given here.

4.3.3 Reference Sensor Shifting by Heuristic Methods

In practice, all of the sensors may not always be capable of receiving the same pulse. For example, being spatially distant from each other and/or a radar with a rotating narrow beam, which occurs as a common case in radar location finding systems, may prevent the sensors to receive the same pulse. In such a case, the sensors can not construct a common TDOA measurement which means that they can not accomplish the location estimation procedure. Using the unmodified version of the recursive estimator (4.31), each group of sensors have to perform the localization separately, unable to share data which can be very useful in increasing the estimator's performance. Notice that; we refer to a sensor group that consists of sensors which are able to receive the same pulse. Besides, using (4.31), TDOA measurements have to be constructed according to the same reference sensor throughout all the operation. On the other hand, the reference sensor simply may not always receive the pulse. In such a case, the localization procedure can not be realized under the restriction of the same reference.

In this section, a reference sensor shifting technique is proposed. Using this method spatially distant sensor groups are able to share and fuse location estimate knowledge resulting increase in estimator's accuracy. The same procedure can be used by a group of sensors to alter the reference of the group in the case of a failure.

Consider the scenario shown in Figure 4.2 where there are two groups of sensors somewhat spatially distant enough so as not to receive the same pulse.

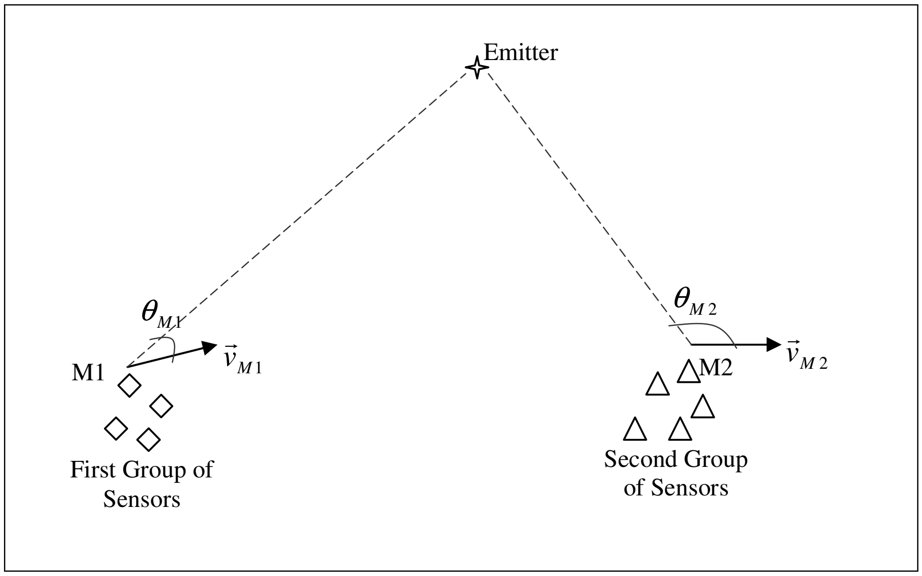

Figure 4.2: Spatially distant sensor groups

In Figure 4.2, M1 and M2 are the reference sensors of the first and the second groups, respectively. $\vec{v}_{Mi}$ is the velocity of the i^{th} reference sensor for $i=1,2$. θ_{Mi} is the angle between the emitter position and $\vec{v}_{Mi}$ for $i=1,2$.

Each sensor group has its own individual reference sensor which means range and range velocity components of state variable ($\hat{\mathbf{z}}_k$) is different. Let $\hat{\mathbf{z}}_{k,1}$ and $\hat{\mathbf{z}}_{k,2}$ denote state variables of the first and the second sensor groups, respectively. For moving emitter case, the state variables are given as:

$$\hat{\mathbf{z}}_{k,1} = \begin{bmatrix} x_e \\ y_e \\ d_{1,1}(k) \\ v_{1,1}(k) \\ v_x(k) \\ v_y(k) \end{bmatrix} \text{ and } \hat{\mathbf{z}}_{k,2} = \begin{bmatrix} x_e \\ y_e \\ d_{1,2}(k) \\ v_{1,2}(k) \\ v_x(k) \\ v_y(k) \end{bmatrix}. \tag{4.56}$$

In fact, only $d_{1,i}(k)$ and $v_{1,i}(k)$ must be processed to perform data fusion between the sensor groups or to alter the reference sensor in the same group.

Suppose that the i^{th} group performed localization ($\hat{\mathbf{z}}_{k,i}$ is available), and data fusion is desired to be performed with j^{th} group. Then, using estimated emitter location by the i^{th} sensor group, $d_{1,j}(k)$ and $v_{1,j}(k)$ are computed in the following way:

$$\begin{aligned} d_{1,j}(k) &= \sqrt{\left(x_e - x_{M,j}\right)^2 + \left(y_e - y_{M,j}\right)^2}, \\ v_{1,j}(k) &= \cos(\theta_{M\,j})\vec{v}_{M\,j}, \end{aligned} \tag{4.57}$$

where $(x_{M,j}, y_{M,j})$ is the coordinate of the j^{th} group's reference sensor in Cartesian coordinate system. Note that $v_{1,j}(k)$ is the radial component of the j^{th} reference sensor velocity ($\vec{v}_{M\,j}$), thus, it is independent from $\theta_{M\,i}$ where $i \neq j$.

Using (4.57) $\hat{\mathbf{z}}_{k,j}$ is found as:

$$\hat{\mathbf{z}}_{k,j} = \begin{bmatrix} x_e \\ y_e \\ \sqrt{\left(x_e - x_{M,j}\right)^2 + \left(y_e - y_{M,j}\right)^2} \\ \cos(\theta_{M,j})\vec{v}_{M,j} \\ v_x(k) \\ v_y(k) \end{bmatrix}. \tag{4.58}$$

In summary, (4.58) implements of the reference sensor shifting mechanism. (4.58) defines the state variable $\hat{\mathbf{z}}_{k,j}$ which can be used by the j^{th} group in the localization. Remaining filter variable, the state error covariance matrix, $\mathbf{P}_k$ is used without any modification.

Spatially disjoint sensor groups can make data fusion using (4.58) resulting in better accuracy in the estimation. Furthermore, (4.58) can be used by the sensors which are in the same group in order to change the reference sensor.

Having developed a heuristic reference sensor shifting procedure, a simple and effective solution is proposed to a subtle but relevant practical restriction.

4.4 Cramer Rao Lower Bound

In [2] CRLB for location estimation for single TDOA measurement set has been obtained for fixed emitter location. Here, an extension of the bound to multiple TDOA sets is derived. The derivation follows almost the same lines as that of [2].

For a single TDOA measurement set $\boldsymbol{\tau}$, the conditional probability density function of $\boldsymbol{\tau}$ given $\mathbf{p}$ is

$$f(\boldsymbol{\tau} \mid \mathbf{p}) = \frac{1}{(2\pi)^{0.5(L-1)} \left|\mathbf{Q}\right|^{0.5}} \exp\left\{-0.5\left(\boldsymbol{\tau} - \frac{\mathbf{r}^0}{c}\right)^T \mathbf{Q}^{-1}\left(\boldsymbol{\tau} - \frac{\mathbf{r}^0}{c}\right)\right\}, \tag{4.59}$$

where $\mathbf{r}^0$ is the noise free RDOA vector and is a function of $\mathbf{p}$; $\mathbf{Q}$ is the TDOA noise covariance matrix.

Now assume that the TDOA measurements ($\boldsymbol{\tau}_i$, $i = 1, 2, ..., k$) are Gaussian and independent for different time instances i, $i = 1, 2, ..., k$. Then, the joint conditional probability density function of $\boldsymbol{\tau}_i$, $i = 1, 2, ..., k$ given $\mathbf{p}$ is expressed by

$$\begin{aligned} f(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, ..., \boldsymbol{\tau}_k \mid \mathbf{z}) &= \prod_{i=1}^{k} f(\boldsymbol{\tau}_i \mid \mathbf{p}), \\ &= \prod_{i=1}^{k} \frac{1}{(2\pi)^{0.5(L-1)} \left|\mathbf{Q}_i\right|^{0.5}} \exp\left\{-0.5\left(\boldsymbol{\tau}_i - \frac{\mathbf{r}_i^0}{c}\right)^T \mathbf{Q}_i^{-1}\left(\boldsymbol{\tau}_i - \frac{\mathbf{r}_i^0}{c}\right)\right\}. \end{aligned} \tag{4.60}$$

If the errors in the TDOA measurements are small enough so that the square of the bias is negligible as compared to the variance, the CRLB of the estimation of $\mathbf{p}$ is given by;

$$CRLB = \left[E\left\{ \left(\nabla_{\mathbf{p}} \left(\ln f(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, ..., \boldsymbol{\tau}_k \mid \mathbf{p}) \right) \right) \left(\nabla_{\mathbf{p}} \left(\ln f(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, ..., \boldsymbol{\tau}_k \mid \mathbf{p}) \right) \right)^T \right\}_{\mathbf{p} = \mathbf{p}^0} \right]^{-1}, \qquad (4.61)$$

where $\mathbf{p}^0$ is the true location of the emitter. The gradient of $\ln f((\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, ..., \boldsymbol{\tau}_k \mid \mathbf{p}))$ with respect to $\mathbf{p}$ is;

$$\nabla_{\mathbf{p}} \left(\ln f(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, ..., \boldsymbol{\tau}_K \mid \mathbf{p}) \right) = \frac{1}{c} \sum_{i=1}^{k} \left(\left(\nabla_{\mathbf{p}} \left(\mathbf{r}_i^0 \right) \right) \mathbf{Q}_i^{-1} \left(\mathbf{d}_i - \frac{\mathbf{r}_i^0}{c} \right) \right). \qquad (4.62)$$

Hence, the CRLB is given as

$$CRLB = c^2 \left(\sum_{i=1}^{k} \mathbf{G}_i \mathbf{Q}_i^{-1} \mathbf{G}_i^T \right)^{-1} \Bigg|_{\mathbf{p} = \mathbf{p}^0}, \qquad (4.63)$$

where $\mathbf{G}_i = \nabla_{\mathbf{p}} \left(\mathbf{r}_i^0 \right)$ is found by using the definition of $\mathbf{p}$, which is given in [2] as

$$\mathbf{G}_i = \left[\frac{(\mathbf{p} - \mathbf{s}_1(i))}{\|\mathbf{p} - \mathbf{s}_1(i)\|} - \frac{(\mathbf{p} - \mathbf{s}_2(i))}{\|\mathbf{p} - \mathbf{s}_2(i)\|} \quad \cdots \quad \frac{(\mathbf{p} - \mathbf{s}_1(i))}{\|\mathbf{p} - \mathbf{s}_1(i)\|} - \frac{(\mathbf{p} - \mathbf{s}_L(i))}{\|\mathbf{p} - \mathbf{s}_L(i)\|} \right]. \qquad (4.64)$$

Finally, using (4.64) in (4.63) gives the Cramer Rao Lower Bound for the localization using multiple TDOA sets.

4.5 CRLB with Erroneous Sensor Positions

The CRLB expression (4.63) was obtained under the assumption of no sensor location uncertainty. On the other hand, erroneous sensor positions obviously affect the accuracy of the estimate. So, sensor location uncertainties have to be included in CRLB. In this sense, a CRLB derivation is given in [15] and [36] for a single measurement set. In this section, the extension for multiple measurement sets is considered.

To evaluate the CRLB, suppose that both the position of the emitter ($\mathbf{p}$) and the positions of the sensors are desired to be estimated. Let $\mathbf{K} = \begin{bmatrix} \mathbf{s}_1^T & \mathbf{s}_2^T & \cdots & \mathbf{s}_L^T \end{bmatrix}^T$ be the vector that contains the positions of the sensors and define $\boldsymbol{\theta}$ as:

$$\boldsymbol{\theta} = \begin{bmatrix} \mathbf{p}^T & \mathbf{K}^T \end{bmatrix}^T, \tag{4.65}$$

which is desired to be estimated.

Let $\mathbf{v}_i = \begin{bmatrix} \mathbf{r}_i & \mathbf{K}_i \end{bmatrix}^T$ be the measurement vector which contains both the RDOA values and the erroneous sensor locations $\mathbf{K}_i = \begin{bmatrix} \mathbf{s}_1(i)^T & \mathbf{s}_2(i)^T & \cdots & \mathbf{s}_L(i)^T \end{bmatrix}^T$. Assume that, both the RDOA measurements and the sensor locations are Gaussian distributed and independent from each other and also independent at different time instances i for $i = 1, 2, ..., k$. Then, similar to single measurement case [15], the joint conditional probability density function of $\mathbf{v}_i$, $i = 1, 2, ..., k$ given $\boldsymbol{\theta}$ is expressed by

$$f(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k \mid \boldsymbol{\theta}) = \prod_{i=1}^{k} f(\mathbf{r}_i \mid \boldsymbol{\theta}) f(\mathbf{K}_i \mid \boldsymbol{\theta}), \tag{4.66}$$

Where the subscript *i* denotes being constructed from the i^{th} pulse.

Using (4.66), $\ln\left(f(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k \mid \boldsymbol{\theta})\right)$ is found as:

$$
\begin{aligned}
& f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \mid \boldsymbol{\theta}) = \\
& \quad \prod_{i=1}^{k} \left(\frac{1}{(2\pi)^{0.5(L-1)} \left(\mid \mathbf{Q}_{\mathbf{r}}^{i} \mid\right)^{0.5}} \exp\left\{ -0.5(\mathbf{r}_i - \mathbf{r}_i^0)^T \left(\mathbf{Q}_{\mathbf{r}}^{i}\right)^{-1} (\mathbf{r}_i - \mathbf{r}_i^0) \right\} \cdot \right. \\
& \quad \left. \frac{1}{(2\pi)^{DL} \left(\mid \mathbf{Q}_{\mathbf{K}}^{i} \mid\right)^{0.5}} \exp\left\{ -0.5(\mathbf{K}_i - \mathbf{K}_i^0)^T \left(\mathbf{Q}_{\mathbf{K}}^{i}\right)^{-1} (\mathbf{K}_i - \mathbf{K}_i^0) \right\} \right), \\
& \ln\left(f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \mid \boldsymbol{\theta})\right) = \sum_{i=1}^{k} -0.5\left(\ln\left(2\pi \mid \mathbf{Q}_{\mathbf{r}}^{i} \mid\right) + \ln\left(2\pi \mid \mathbf{Q}_{\mathbf{K}}^{i} \mid\right)\right) + \\
& \quad -0.5 \sum_{i=1}^{k} \left((\mathbf{r}_i - \mathbf{r}_i^0)^T \left(\mathbf{Q}_{\mathbf{r}}^{i}\right)^{-1} (\mathbf{r}_i - \mathbf{r}_i^0) \right) + \\
& \quad -0.5 \sum_{i=1}^{k} \left((\mathbf{K}_i - \mathbf{K}_i^0)^T \left(\mathbf{Q}_{\mathbf{K}}^{i}\right)^{-1} (\mathbf{K}_i - \mathbf{K}_i^0) \right).
\end{aligned} \tag{4.67}
$$

where ‘ ^{0} ’ denotes true value of a vector, D is dimension of the sensor locations, and $\mathbf{Q}_{\mathbf{r}}^{i}$ is the RDOA noise covariance matrix

$$
\mathbf{Q}_{\mathbf{r}}^{i} = c^2 \mathbf{Q}_i , \tag{4.68}
$$

c is the speed of light and $\mathbf{Q}_i$ is the TDOA covariance matrix for the time instance i.

Then, similar to the single measurement case [15], for multiple measurements the CRLB can be found by:

$$
CRLB(\boldsymbol{\theta}) = \left[-E\left\{ \nabla_{\boldsymbol{\theta}}^{2} \left(\ln f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \mid \boldsymbol{\theta}) \right) \right\} \Big|_{\boldsymbol{\theta} = \boldsymbol{\theta}^0} \right]^{-1}, \tag{4.69}
$$

where $\boldsymbol{\theta}^0$ denotes the noise free value of the vector $\boldsymbol{\theta}$.

Then the $CRLB(\boldsymbol{\theta})$ is given by:

$$
CRLB(\boldsymbol{\theta}) = \begin{bmatrix} \mathbf{X} & \mathbf{Y}^T \\ \mathbf{Y} & \mathbf{Z} \end{bmatrix}^{-1}, \tag{4.70}
$$

where

$$
\begin{aligned}
\mathbf{X}_{ij} &= E\left\{\frac{\partial \ln f(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k \mid \boldsymbol{\theta})}{\partial \mathbf{p}_i}\frac{\partial \ln f(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k \mid \boldsymbol{\theta})}{\partial \mathbf{p}_j}\right\},\\
\mathbf{Y}_{ij} &= E\left\{\frac{\partial \ln f(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k \mid \boldsymbol{\theta})}{\partial \mathbf{K}_i}\frac{\partial \ln f(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k \mid \boldsymbol{\theta})}{\partial \mathbf{p}_j}\right\},\\
\mathbf{Z}_{ij} &= E\left\{\frac{\partial \ln f(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k \mid \boldsymbol{\theta})}{\partial \mathbf{K}_i}\frac{\partial \ln f(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k \mid \boldsymbol{\theta})}{\partial \mathbf{K}_j}\right\}.
\end{aligned}
\tag{4.71}
$$

Then using (4.67), $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ are calculated as [15, 36]:

$$
\begin{aligned}
\mathbf{X} &= E\left\{\nabla_{\mathbf{p}}^2\left(\ln f(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k \mid \boldsymbol{\theta})\right)\right\} = -\sum_{i=1}^{k}\left(\mathbf{G}_i^{\mathbf{r},\mathbf{p}}\right)\left(\mathbf{Q}_{\mathbf{r}}^i\right)^{-1}\left(\mathbf{G}_i^{\mathbf{r},\mathbf{p}}\right)^T,\\
\mathbf{Y} &= E\left\{\nabla_{\mathbf{K}}\nabla_{\mathbf{p}} \ln f(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k \mid \boldsymbol{\theta})\right\} = -\sum_{i=1}^{k}\left(\mathbf{G}_i^{\mathbf{r},\mathbf{K}}\right)\left(\mathbf{Q}_{\mathbf{r}}^i\right)^{-1}\left(\mathbf{G}_i^{\mathbf{r},\mathbf{p}}\right)^T,\\
\mathbf{Z} &= E\left\{\nabla_{\mathbf{K}}^2 \ln f(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k \mid \boldsymbol{\theta})\right\} = -\sum_{i=1}^{k}\left(\mathbf{G}_i^{\mathbf{r},\mathbf{K}}\right)\left(\mathbf{Q}_{\mathbf{r}}^i\right)^{-1}\left(\mathbf{G}_i^{\mathbf{r},\mathbf{K}}\right)^T\\
&\qquad -\sum_{i=1}^{k}\left(\mathbf{G}_i^{\mathbf{K},\mathbf{K}}\right)\left(\mathbf{Q}_{\mathbf{K}}^i\right)^{-1}\left(\mathbf{G}_i^{\mathbf{K},\mathbf{K}}\right)^T.
\end{aligned}
\tag{4.72}
$$

$\mathbf{G}_i^{\mathbf{r},\mathbf{p}} = \nabla_{\mathbf{p}}(\mathbf{r}_i^0)$, $\mathbf{G}_i^{\mathbf{r},\mathbf{K}} = \nabla_{\mathbf{K}}(\mathbf{r}_i^0)$ and $\mathbf{G}_i^{\mathbf{K},\mathbf{K}} = \nabla_{\mathbf{K}}(\mathbf{K}_i^0)$ are given as [36]:

$$
\begin{aligned}
\nabla_{\mathbf{p}}(\mathbf{r}_i^0) &= \begin{bmatrix} \dfrac{(\mathbf{p}-\mathbf{s}_1(i))}{d_1^0(i)} - \dfrac{(\mathbf{p}-\mathbf{s}_2(i))}{d_2^0(i)} & \cdots & \dfrac{(\mathbf{p}-\mathbf{s}_1(i))}{d_1^0(i)} - \dfrac{(\mathbf{p}-\mathbf{s}_L(i))}{d_L^0(i)} \end{bmatrix},\\
\nabla_{\mathbf{K}}(\mathbf{r}_i^0) &= \begin{bmatrix} -\dfrac{(\mathbf{p}-\mathbf{s}_1(i))}{d_1^0(i)} & \cdots & -\dfrac{(\mathbf{p}-\mathbf{s}_1(i))}{d_1^0(i)} \\ \dfrac{(\mathbf{p}-\mathbf{s}_2(i))}{d_2^0(i)} & \cdots & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \dfrac{(\mathbf{p}-\mathbf{s}_L(i))}{d_L^0(i)} \end{bmatrix},\\
\nabla_{\mathbf{K}}(\mathbf{K}_i^0) &= \mathbf{I}.
\end{aligned}
\tag{4.73}
$$

Using (4.72) and (4.73), (4.70) gives the CRLB for the emitter localization with the erroneous sensor locations. Notice that, the upper left $D \times D$ dimensional submatrix of (4.70) gives the CRLB for the estimation $\mathbf{p}$.

4.6 Discussion

In this chapter, a Kalman Filter based recursive location estimator was developed. TDOA measurements are gathered by moving sensors and the method is capable of tracking a moving emitter. The estimator directly processes the TDOA measurements. Finally, a reference sensor shifting method was described. One of the most powerful aspects of the method is that filter structure does not lead to divergence which is the case in the Extended Kalman Filter and the Unscented Kalman Filter [25]. Additionally, no external initialization is required.

Despite all of the capabilities of the estimator, it suffers from bias problems [10]. The square operation on the TDOA measurements in (4.7) results in a bias, and its effects become more severe with high noise levels. A bias compensation method is proposed in [11]. On the other hand, the bias compensation method requires the post processing of all of the measurements which contradicts with recursion. Consequently, the bias compensation is not handled in this thesis and left as a future work. Fortunately, the bias becomes negligible after the processing of sufficient number of measurements.

As stated in Section 4.3.2, in the case of the tracking a moving emitter, the velocity of the emitter is unobservable using only TDOA measurements. FDOA measurements can be used along with the TDOA to overcome this situation.

CHAPTER 5

LOCATION ESTIMATION IN ML SENSE

In this chapter, a location estimator is derived in Maximum Likelihood (ML) estimation sense using TDOA measurements. Examining the ML solution is important and beneficial to fully cover the concept of the passive localization. In this context, after expressing the derivation given in [1], some modifications are given to improve the performance of the estimator. Although the same estimator is derived in a different manner in [26], this chapter follows [1] closely. The derivation uses Taylor Series expansion which requires an initial point. Thus, the ML solution is not in a closed form, it is iterative. The method updates the previous location estimate when a new measurement arrives. However, the measurement data are not stored. Hence, the solution is memoriless.

Note that, the estimator given in [1] is regarded as one of the best passive localization algorithms [2, 3, 10].

5.1 Location Estimator in ML Sense

Suppose that a set of L measurements q_i, for $i = 1, 2, ..., L$ are gathered at several locations where L is the number sensors. When there is no measurement error, it is known that q_i equals to a known function $f_i(\mathbf{p})$ where $\mathbf{p}$ is a $D \times 1$ vector denoting the emitter's location which is desired to be estimated. When q_i is subject to additive noise, it is written as [1]:

$$q_i = f_i(\mathbf{p}) + n_i \text{, for } i = 1, 2, ..., L. \tag{5.1}$$

If we write L equations as column vectors:

$$\mathbf{q} = \mathbf{f}(\mathbf{p}) + \mathbf{n}. \tag{5.2}$$

The measurement error $\mathbf{n}$ is assumed to have a Gaussian distribution with zero mean and $L \times L$ dimensional covariance matrix $\mathbf{N}$ [1].

If $\mathbf{p}$ is regarded as an unknown but nonrandom vector, then the conditional probability density function of $\mathbf{q}$ given $\mathbf{p}$ is expressed as:

$$p(\mathbf{q} \mid \mathbf{p}) = \frac{1}{(2\pi)^{k/2}\sqrt{|\mathbf{N}|}} \exp\left\{-0.5[\mathbf{q} - \mathbf{f}(\mathbf{p})]^T \mathbf{N}^{-1}[\mathbf{q} - \mathbf{f}(\mathbf{p})]\right\}, \tag{5.3}$$

where $|\mathbf{N}|$ is the determinant of the covariance matrix $\mathbf{N}$ [1].

Taking natural logarithm of both sides in (5.3) yields:

$$\ln\left(p(\mathbf{q} \mid \mathbf{p})\right) = \ln\left(\frac{1}{(2\pi)^{k/2}\sqrt{|\mathbf{N}|}}\right) - 0.5[\mathbf{q} - \mathbf{f}(\mathbf{p})]^T \mathbf{N}^{-1}[\mathbf{q} - \mathbf{f}(\mathbf{p})]. \tag{5.4}$$

The maximum likelihood estimator of $\mathbf{p}$ is the value that maximizes (5.4). Equivalently, the maximum likelihood estimator minimizes the following quadratic equation [1]:

$$Q(\mathbf{p}) = [\mathbf{q} - \mathbf{f}(\mathbf{p})]^T \mathbf{N}^{-1}[\mathbf{q} - \mathbf{f}(\mathbf{p})]. \tag{5.5}$$

In case of emitter localization using TDOA, it is clear that $\mathbf{f}(\mathbf{p})$ is a nonlinear function. However, derivation of an estimator can be accomplished by linearizing $\mathbf{f}(\mathbf{p})$ using Taylor series expansion about a reference point $\mathbf{p}_0$. If only the first two terms of Taylor series are used, then $\mathbf{f}(\mathbf{p})$ can be expressed as:

$$\mathbf{f}(\mathbf{p}) \cong \mathbf{f}(\mathbf{p}_0) + \mathbf{G}(\mathbf{p} - \mathbf{p}_0)\,, \tag{5.6}$$

where $\mathbf{G}$ is the $L \times D$ sized matrix which contains derivatives evaluated at $\mathbf{p}_0$ [1]:

$$\mathbf{G} = \begin{bmatrix} \left.\dfrac{\partial f_1}{\partial p_1}\right|_{\mathbf{p}=\mathbf{p}_0} & \cdots & \left.\dfrac{\partial f_1}{\partial p_N}\right|_{\mathbf{p}=\mathbf{p}_0} \\ \vdots & \ddots & \vdots \\ \left.\dfrac{\partial f_L}{\partial p_1}\right|_{\mathbf{p}=\mathbf{p}_0} & \cdots & \left.\dfrac{\partial f_1}{\partial p_N}\right|_{\mathbf{p}=\mathbf{p}_0} \end{bmatrix}. \tag{5.7}$$

The initial point $\mathbf{p}_0$ can be estimated using a closed form estimator or it can be an estimate of $\mathbf{p}$ from a previous iteration. It is assumed that $\mathbf{p}_0$ is sufficiently close to $\mathbf{p}$, so that (5.6) is an adequate approximation.

For convenience, let $\mathbf{q}_1$ be defined as:

$$\mathbf{q}_1 = \mathbf{q} - \mathbf{f}(\mathbf{p}_0) + \mathbf{G}\mathbf{p}_0, \tag{5.8}$$

and using (5.8), let us combine (5.5) and (5.6):

$$Q(\mathbf{p}) = [\mathbf{q}_1 - \mathbf{G}\mathbf{p}]^T \mathbf{N}^{-1} [\mathbf{q}_1 - \mathbf{G}\mathbf{p}]. \tag{5.9}$$

To find the estimate ($\hat{\mathbf{p}}$) which minimizes $Q(\mathbf{p})$, first gradient of $Q(\mathbf{p})$ is calculated, then it is solved for $\nabla_{\mathbf{p}} Q(\mathbf{p}) = 0$ where the gradient of $Q(\mathbf{p})$ is expressed as:

$$\nabla_{\mathbf{p}} Q(\mathbf{p}) = \left[\frac{\partial Q}{\partial p_1} \frac{\partial Q}{\partial p_2} \cdots \frac{\partial Q}{\partial p_N} \right]^T. \tag{5.10}$$

To evaluate its gradient, let us expand (5.9):

$$Q(\mathbf{p}) = \mathbf{q}_1^T \mathbf{N}^{-1} \mathbf{q}_1 - \mathbf{p}^T \mathbf{G}^T \mathbf{N}^{-1} \mathbf{q}_1 - \mathbf{q}_1^T \mathbf{N}^{-1} \mathbf{G}\mathbf{p} + \mathbf{p}^T \mathbf{G}^T \mathbf{N}^{-1} \mathbf{G}\mathbf{p}. \tag{5.11}$$

The gradient of $Q(\mathbf{p})$ at $\mathbf{p}=\hat{\mathbf{p}}$ is

$$\nabla_{\mathbf{p}} Q(\mathbf{p})\Big|_{\mathbf{p}=\hat{\mathbf{p}}} = 2\mathbf{G}^T\mathbf{N}^{-1}\mathbf{G}\hat{\mathbf{p}} - 2\mathbf{G}^T\mathbf{N}^{-1}\mathbf{q}_1. \tag{5.12}$$

Now, if we solve (5.12) for $\nabla_{\mathbf{p}} Q(\mathbf{p})\Big|_{\mathbf{p}=\hat{\mathbf{p}}} = \mathbf{0}$, we get:

$$\begin{aligned}
&\nabla_{\mathbf{p}} Q(\mathbf{p})\Big|_{\mathbf{p}=\hat{\mathbf{p}}} = 2\mathbf{G}^T\mathbf{N}^{-1}\mathbf{G}\hat{\mathbf{p}} - 2\mathbf{G}^T\mathbf{N}^{-1}\mathbf{q}_1 = \mathbf{0}, \\
&\Rightarrow \hat{\mathbf{p}} = (\mathbf{G}^T\mathbf{N}^{-1}\mathbf{G})^{-1}\mathbf{G}^T\mathbf{N}^{-1}\mathbf{q}_1.
\end{aligned} \tag{5.13}$$

Inserting the definition of $\mathbf{q}_1$ (5.8) into (5.13) and rearranging terms yields:

$$\begin{aligned}
\hat{\mathbf{p}} &= (\mathbf{G}^T\mathbf{N}^{-1}\mathbf{G})^{-1}\mathbf{G}^T\mathbf{N}^{-1}(\mathbf{q}-\mathbf{f}(\mathbf{p}_0)+\mathbf{G}\mathbf{p}_0), \\
&= (\mathbf{G}^T\mathbf{N}^{-1}\mathbf{G})^{-1}\mathbf{G}^T\mathbf{N}^{-1}\mathbf{G}\mathbf{p}_0 + (\mathbf{G}^T\mathbf{N}^{-1}\mathbf{G})^{-1}\mathbf{G}^T\mathbf{N}^{-1}(\mathbf{q}-\mathbf{f}(\mathbf{p}_0)), \\
&= \mathbf{p}_0 + (\mathbf{G}^T\mathbf{N}^{-1}\mathbf{G})^{-1}\mathbf{G}^T\mathbf{N}^{-1}(\mathbf{q}-\mathbf{f}(\mathbf{p}_0)).
\end{aligned} \tag{5.14}$$

where $\hat{\mathbf{p}}$ is the estimate which minimizes quadratic equation (5.5) [1].

Using (5.2), (5.14) can be written as:

$$\hat{\mathbf{p}} = \mathbf{p}_0 + (\mathbf{G}^T\mathbf{N}^{-1}\mathbf{G})^{-1}\mathbf{G}^T\mathbf{N}^{-1}(\mathbf{f}(\mathbf{p})-\mathbf{f}(\mathbf{p}_0)+\mathbf{n}). \tag{5.15}$$

Now, since $\mathbf{n}$ is assumed to have zero mean, expected value of the estimate is given by:

$$E\{\hat{\mathbf{p}}\} = \mathbf{p}_0 + (\mathbf{G}^T\mathbf{N}^{-1}\mathbf{G})^{-1}\mathbf{G}^T\mathbf{N}^{-1}(\mathbf{f}(\mathbf{p})-\mathbf{f}(\mathbf{p}_0)). \tag{5.16}$$

Using (5.15) and (5.16) the covariance of the estimate is found as [1]:

$$\begin{aligned}
E\left\{(\hat{\mathbf{p}}-E\{\hat{\mathbf{p}}\})(\hat{\mathbf{p}}-E\{\hat{\mathbf{p}}\})^T\right\} &= E\left\{(\mathbf{G}^T\mathbf{N}^{-1}\mathbf{G})^{-1}\mathbf{G}^T\mathbf{N}^{-1}\mathbf{n}\mathbf{n}^T\mathbf{N}^{-1}\mathbf{G}(\mathbf{G}^T\mathbf{N}^{-1}\mathbf{G})^{-1}\right\}, \\
&= (\mathbf{G}^T\mathbf{N}^{-1}\mathbf{G})^{-1}\mathbf{G}^T\mathbf{N}^{-1}E\left\{\mathbf{n}\mathbf{n}^T\right\}\mathbf{N}^{-1}\mathbf{G}(\mathbf{G}^T\mathbf{N}^{-1}\mathbf{G})^{-1}, \\
&= (\mathbf{G}^T\mathbf{N}^{-1}\mathbf{G})^{-1}\mathbf{G}^T\mathbf{N}^{-1}\mathbf{N}\mathbf{N}^{-1}\mathbf{G}(\mathbf{G}^T\mathbf{N}^{-1}\mathbf{G})^{-1}, \\
&= (\mathbf{G}^T\mathbf{N}^{-1}\mathbf{G})^{-1}.
\end{aligned} \tag{5.17}$$

Note that, the expression of the covariance of the estimator is included as a term in estimator itself. Hence, no extra effort is necessary to calculate the covariance.

The estimator in (5.14) is capable of taking AOA, TDOA and/or FDOA measurements as input. Of course, individual matrix expressions differ for different measurement types. In the next section, estimator expressions for TDOA location estimation are derived.

5.2 Estimator Expressions for TDOA Measurements

Suppose that there are L sensors which collect TDOA measurements, and recall from Chapter 2 that, the time of arrival of the signal at the i^{th} sensor is given as:

$$t_i = t_0 + \frac{d_i}{c} + n_i \text{ for } i = 1, 2, ..., L, \tag{5.18}$$

and d_i is as the distance between the i^{th} sensor and the emitter:

$$d_i = \sqrt{(\mathbf{p} - \mathbf{s}_i)^T (\mathbf{p} - \mathbf{s}_i)} \text{ for } i = 1, 2, ..., L, \tag{5.19}$$

where $\mathbf{p}$ and $\mathbf{s}_i$ are size of ($D \times 1$) and denote location of the emitter and the i^{th} sensor respectively, and c denotes the speed of light. If we write (5.18) in matrix form, we get:

$$\mathbf{t} = \mathbf{t}_0 + \frac{\mathbf{d}}{c} + \mathbf{n}, \tag{5.20}$$

where $\mathbf{t}$, $\mathbf{d}$ and $\mathbf{n}$ are ($L \times 1$) sized column vectors with components t_i, d_i and n_i for $i = 1, 2, ..., L$ respectively. $\mathbf{t}_0$ is also ($L \times 1$) sized column vector whose all components are equal to t_0.

Now suppose that both t_0 and $\mathbf{p}$ are desired to be estimated. If we define $\mathbf{f}(t_0,\mathbf{p})$ as:

$$\begin{aligned}\mathbf{f}(t_0,\mathbf{p}) &= \mathbf{t}, \\ &= \mathbf{t}_0 + \frac{\mathbf{d}}{c} + \mathbf{n},\end{aligned} \tag{5.21}$$

then the gradient matrix $\mathbf{G}'$ is found as:

$$\mathbf{G}' = \left[\left.\frac{\partial \mathbf{f}}{\partial t_0}\right|_{t_0=t_0^{'}} \quad \nabla_{\mathbf{p}}(\mathbf{f})\big|_{\mathbf{p}=\mathbf{p}_0} \right], \tag{5.22}$$

Clearly,

$$\left.\frac{\partial \mathbf{f}}{\partial t_0}\right|_{t_0=t_0^{'}} = \mathbf{1}, \tag{5.23}$$

where $\mathbf{1}$ is a column vector of ones.

Moreover, define $f_i^{'}(\mathbf{p})$ as:

$$\begin{aligned}f_i^{'}(\mathbf{p}) &= \frac{d_i}{c}, \\ &= \frac{\sqrt{(\mathbf{p}-\mathbf{s}_i)^T(\mathbf{p}-\mathbf{s}_i)}}{c}.\end{aligned} \tag{5.24}$$

Now, taking the gradient of $f_i^{'}(\mathbf{p})$ where $\mathbf{p}=\mathbf{p}_0$ yields:

$$\nabla_{\mathbf{p}}\left(f_i^{'}(\mathbf{p})\right)\Big|_{\mathbf{p}=\mathbf{p}_0} = \frac{(\mathbf{p}_0-\mathbf{s}_i)^T}{c}\left(\sqrt{(\mathbf{p}_0-\mathbf{s}_i)^T(\mathbf{p}_0-\mathbf{s}_i)}\right)^{-1}. \tag{5.25}$$

where $\mathbf{p}_0$ is the reference point which the Taylor Series expansion is applied.

Using (5.19), (5.25) can be expressed as:

$$\nabla_{\mathbf{p}}\left(f_i^{'}(\mathbf{p})\right)\Big|_{\mathbf{p}=\mathbf{p}_0} = \frac{(\mathbf{p}_0 - \mathbf{s}_i)^T}{d_i^0}, \tag{5.26}$$

and d_i^0 is the distance between the i^{th} sensor and the reference point $\mathbf{p}_0$.

Finally, using (5.23) and (5.26) the gradient matrix $\mathbf{G}'$ in (5.22) is found to be [1]:

$$\mathbf{G}' = \begin{bmatrix} \mathbf{1} & \frac{\mathbf{F}}{c} \end{bmatrix}, \tag{5.27}$$

where $\mathbf{F}$ is a ($L \times D$) matrix with components which are defined in (5.26) [1]:

$$\mathbf{F} = \begin{bmatrix} \frac{(\mathbf{p}_0 - \mathbf{s}_1)^T}{d_1^0} \\ \frac{(\mathbf{p}_0 - \mathbf{s}_2)^T}{d_2^0} \\ \vdots \\ \frac{(\mathbf{p}_0 - \mathbf{s}_L)^T}{d_L^0} \end{bmatrix}. \tag{5.28}$$

In TDOA localization systems, it is practically impossible to estimate t_0 and obviously TDOA measurements are used instead. Hence, t_0 is eliminated:

$$t_i - t_{i+1} = \frac{d_i - d_{i+1}}{c} + e_i, \text{ for } i = 1, 2, \ldots, L-1, \tag{5.29}$$

where e_i is the TDOA measurement error. In matrix form (5.29) becomes:

$$\mathbf{Ht} = \frac{1}{c}\mathbf{Hd} + \mathbf{e}, \tag{5.30}$$

where $\mathbf{e}$ is the vector containing the noise components and the $(L-1) \times L$ matrix $\mathbf{H}$ is defined as [1]:

$$\mathbf{H} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}. \qquad (5.31)$$

Since, it is our goal to estimate emitter location $\mathbf{p}$, (5.30) is written in the form of (5.2):

$$\mathbf{q} = \mathbf{Ht}, \text{ and } \mathbf{f}(\mathbf{p}) = \frac{1}{c}\mathbf{Hd} + \mathbf{e}. \qquad (5.32)$$

Then, using (5.32) the gradient matrix $\mathbf{G}$ of $\mathbf{f}(\mathbf{p})$ is expressed as [1]:

$$\begin{aligned} \mathbf{G} &= \nabla_{\mathbf{p}}\left(\mathbf{f}(\mathbf{p})\right)\Big|_{\mathbf{p}=\mathbf{p}_0}, \\ &= \frac{1}{c}\mathbf{HF}. \end{aligned} \qquad (5.33)$$

Using (5.14) and (5.33), the estimate $\hat{\mathbf{p}}$ is found as:

$$\hat{\mathbf{p}} = \mathbf{p}_0 + c(\mathbf{F}^T\mathbf{H}^T\mathbf{N}_e^{-1}\mathbf{HF})^{-1}\mathbf{F}^T\mathbf{H}^T\mathbf{N}_e^{-1}(\mathbf{Ht} - \frac{\mathbf{Hd}_0}{c}), \qquad (5.34)$$

where $\mathbf{d}_0$ is the vector $\mathbf{d}$ in (5.20) which is constructed using the reference point $\mathbf{p}_0$.

In (5.34), $\mathbf{N}_e$ is the covariance matrix of the TDOA measurement errors. If the TDOA values are calculated by subtracting the TOA values using (5.29), then e_i is found as:

$$e_i = n_i - n_{i+1}, \text{ for } i = 1, 2, ..., L-1. \qquad (5.35)$$

Moreover, expressing (5.35) in matrix form yields:

$$\mathbf{e} = \mathbf{Hn}, \qquad (5.36)$$

Hence, $\mathbf{N}_e$ is found as [1]:

$$\mathbf{N}_e = \mathbf{H}\mathbf{N}\mathbf{H}^T . \tag{5.37}$$

Finally, if (5.37) is taken into account, then covariance of $\hat{\mathbf{p}}$ in the case TDOA measurements is given as:

$$\mathrm{var}(\hat{\mathbf{p}}) = c^2(\mathbf{F}^T\mathbf{H}^T\mathbf{N}_e^{-1}\mathbf{H}\mathbf{F})^{-1}. \tag{5.38}$$

The expressions (5.34) and (5.38) are the location estimator and its covariance respectively. In fact, (5.34) is TDOA measurement form of (5.15). However, equation (5.34) covers only one set of TDOA. It can be improved to process more than one set of TDOA measurements. This operation will increase the immunity of the filter against noise.

5.3 Fusion of TDOA Measurements

Suppose that, there are k set of TDOA measurements. Let us define the following matrices:

$$\mathbf{H}_i,\ \mathbf{F}_i,\ \mathbf{d}_0^i,\ \mathbf{t}_i \text{ and } \mathbf{N}_i \text{ for } i = 1, 2, \ldots, k . \tag{5.39}$$

The matrices in (5.39) are in the form of those defined in Section 5.2 (previous section), for the i^{th} TDOA measurement.

Using definitions in (5.39), we define the global matrix expressions as follows:

$$\mathbf{H}_k^G = \begin{bmatrix} \mathbf{H}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_2 & \dots & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{H}_k \end{bmatrix}, \mathbf{F}_k^G = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \vdots \\ \mathbf{F}_k \end{bmatrix}, \mathbf{t}_k^G = \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \vdots \\ \mathbf{t}_k \end{bmatrix},$$

$$\mathbf{N}_k^{e,G} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_2 & & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{N}_k \end{bmatrix}, \mathbf{d}_0^{k,G} = \begin{bmatrix} \mathbf{d}_0^1 \\ \mathbf{d}_0^2 \\ \vdots \\ \mathbf{d}_0^k \end{bmatrix}, \tag{5.40}$$

where superscript G stands for global. In fact, fusion of different set of TDOA measurements is completed with the definition of (5.40). Remaining matter is only to draw the usage of (5.40), which is indeed defined in (5.34).

Modifying (5.34), the fusion of different TDOA sets can be accomplished by:

$$\hat{\mathbf{p}} = \mathbf{p}_0 + c((\mathbf{H}_k^G \mathbf{F}_k^G)^T (\mathbf{N}_k^{e,G})^{-1} \mathbf{H}_k^G \mathbf{F}_k^G)^{-1} (\mathbf{H}_k^G \mathbf{F}_k^G)^T (\mathbf{N}_k^{e,G})^{-1} (\mathbf{H}_k^G \mathbf{t}_k^G - \frac{\mathbf{H}_k^G \mathbf{d}_k^{k,G}}{c}). \tag{5.41}$$

Similar to (5.38), the covariance of (5.41) is found as:

$$\mathrm{var}(\hat{\mathbf{p}}) = c^2 ((\mathbf{H}_k^G \mathbf{F}_k^G)^T (\mathbf{N}_k^{e,G})^{-1} \mathbf{H}_k^G \mathbf{F}_k^G)^{-1}. \tag{5.42}$$

Notice that, dimension of matrices grows with increasing k. This issue has the potential of making implementation of the estimator very difficult. Fortunately, recursive least squares algorithm can be used to overcome this problem. A similar derivation was also given in Chapter 4. In this case, only the second term on the right hand side of the equation (5.41) will be expressed using RLS.

Now, express $\hat{\mathbf{p}}$ in the following way:

$$\hat{\mathbf{p}} = \mathbf{p}_0 + \mathbf{z}_k, \tag{5.43}$$

where $\mathbf{z}_k$ is defined by:

$$\mathbf{z}_k = c((\mathbf{H}_k^G\mathbf{F}_k^G)^T(\mathbf{N}_k^{e,G})^{-1}\mathbf{H}_k^G\mathbf{F}_k^G)^{-1}(\mathbf{H}_k^G\mathbf{F}_k^G)^T(\mathbf{N}_k^{e,G})^{-1}(\mathbf{H}_k^G\mathbf{t}_k^G - \frac{\mathbf{H}_k^G\mathbf{d}_0^{k,G}}{c}). \quad (5.44)$$

We try to express $\mathbf{z}_{k+1}$ in terms of $\mathbf{z}_k$ recursively in order to avoid dimension growth. Let us define following matrices:

$$\begin{aligned}\mathbf{P}_k &= \left((\mathbf{H}_k^G\mathbf{F}_k^G)^T(\mathbf{N}_k^{e,G})^{-1}\mathbf{H}_k^G\mathbf{F}_k^G\right)^{\mathbf{-1}},\\ \mathbf{P}_{k+1} &= \left((\mathbf{H}_{k+1}^G\mathbf{F}_{k+1}^G)^T(\mathbf{N}_{k+1}^{e,G})^{-1}\mathbf{H}_{k+1}^G\mathbf{F}_{k+1}^G\right)^{\mathbf{-1}}.\end{aligned} \quad (5.45)$$

In (5.45), if we let $\mathbf{H}_k^G = \mathbf{H}_0$, then $\mathbf{P}_k$ becomes ($D \times D$) dimensional.

Using (5.45) $\mathbf{z}_{k+1}$ can be written as [30]:

$$\mathbf{z}_{k+1} = \mathbf{z}_k + \mathbf{P}_{k+1}(\mathbf{H}_{k+1}\mathbf{F}_{k+1})^T(\mathbf{N}_{k+1}^e)^{-1}(c\mathbf{H}_{k+1}\mathbf{t}_{k+1} - \mathbf{H}_{k+1}\mathbf{d}_0^k - \mathbf{H}_{k+1}\mathbf{F}_{k+1}\mathbf{z}_k), \quad (5.46)$$

where $\mathbf{P}_{k+1}$ is defined as [30]:

$$\mathbf{P}_{k+1} = \mathbf{P}_k - \mathbf{P}_k(\mathbf{H}_{k+1}\mathbf{F}_{k+1})^T\left(\mathbf{H}_{k+1}\mathbf{F}_{k+1}\mathbf{P}_k(\mathbf{H}_{k+1}\mathbf{F}_{k+1})^T + \mathbf{N}_{k+1}^e\right)^{-1}\mathbf{H}_{k+1}\mathbf{F}_{k+1}\mathbf{P}_k. \quad (5.47)$$

Expressions (5.46) and (5.47) give a solution to the dimension growth problem in (5.41). The initial values $\mathbf{P}_1$ and $\mathbf{z}_1$ are given as:

$$\begin{aligned}\mathbf{z}_1 &= ((\mathbf{H}_1\mathbf{F}_1)^T(\mathbf{N}_1^e)^{-1}\mathbf{H}_1\mathbf{F}_1)^{-1}(\mathbf{H}_1\mathbf{F}_1)^T(\mathbf{N}_1^e)^{-1}(c\mathbf{H}_1\mathbf{t}_1 - \mathbf{H}_1\mathbf{d}_0^1),\\ \mathbf{P}_1 &= \left((\mathbf{H}_1\mathbf{F}_1)^T(\mathbf{N}_1^e)^{-1}\mathbf{H}_1\mathbf{F}_1\right)^{\mathbf{-1}}.\end{aligned} \quad (5.48)$$

Notice that, updating $\mathbf{z}_k$ does not affect $\mathbf{p}_0$ in (5.43). Dependent on the scenario, $\mathbf{p}_0$ can be updated by replacing with the calculated $\hat{\mathbf{p}}$ after gathering sufficient number of measurements. Then, the new $\mathbf{z}_k$ is constructed when the new measurements arrive.

It is an important fact that the measurement fusion is developed for general case. It can be either used for fusing consecutive TDOA sets which are taken by a single sensor group, or the same procedure can be applied to fuse the measurement sets from different group of sensors. There is no need for an additional process. This result is one of the most powerful properties of the estimator (5.34).

5.4 Discussion

As mentioned before, the estimator (5.14) is regarded as one of the best location estimators [2, 3, 10]. It has the basic advantages [1, 26]:

1. Multiple types of measurements can be used at the same time; such as AOA, TDOA and/or FDOA.
2. Multiple measurement sets can be fused easily without a need for extra operation.
3. Estimation is realized by reasonable computational complexity.

However, several practical issues have to be considered [26]:

1. The method is iterative and requires an initial condition.
2. Its convergence is not guaranteed.

In summary, noting the stated problems, although the estimator in (5.14) is a powerful localization tool, it has to be used with caution.

CHAPTER 6

SIMULATIONS

In this chapter, the simulation studies of the estimators which are described in the previous chapters are presented. First, the recursive estimator in Chapter 4 is examined in detail. The simulation results are given for the cases of maneuvering sensors, shifting the reference sensor and for estimating location of a moving source. Then, the performance of the estimator in Chapter 5 is examined with data fusion. Since it is capable of accumulating limited amount of data, for convenience this estimator will be referred as the pseudo recursive estimator for the remaining part of the text. Finally, performance comparison simulation results for all of the estimators including the post-processing, the recursive and the pseudo recursive types are presented.

Other than the stated cases, by the simulation studies the effects of the following factors on the performance of the estimators are also examined:

1. The sensor and the emitter geometry,

2. The measurement noise level,

3. The sensor location uncertainty.

6.1 Assumptions

First of all, the simulations are performed for two dimensional localization for simplicity. Besides, the emitter is assumed to be a search radar having a rotating antenna with scanning period of 2 sec., 3 dB beam width of 3° and pulse repetition interval (PRI) of 1 msec. Since, the emitter has a rotating beam; the sensors do not receive pulse continuously. Consequently, the number of received pulses in each antenna scanning period is assumed to be 10. Measurements are constructed from these pulses, and then the TDOAs which are taken in the same antenna scanning period are averaged to result in an improved TDOA estimate. Furthermore, as described in equation (2.22), the TDOA measurements are constructed from the TOA measurements which are assumed to be corrupted by additive zero mean Gaussian noise with a standard deviation of σ_{TOA} which is adjusted according to the current scenario. All of the other error sources, such as the multi-path effects, the path loss effects or the hardware effects, are ignored.

The geometry of the simulation scenarios are given in Figure 6.1. The figure is not scaled for ease in drawing.

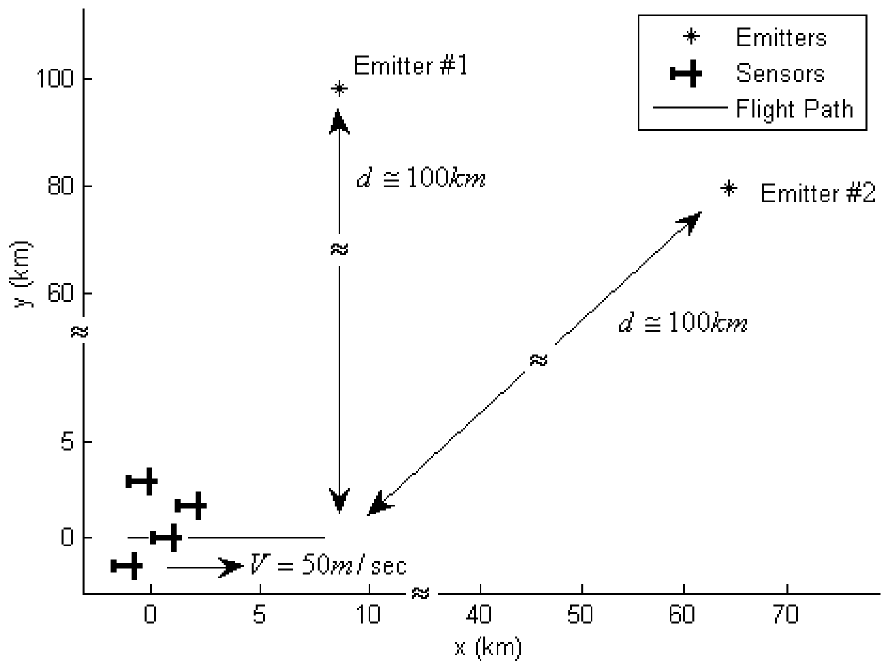

Figure 6.1: The geometry of the simulation scenarios.

In all scenarios, the distance of the emitter is about 100 km. There are 4 sensors which are located at four flying platforms separate from each other at a maximum distance of 3.5km. which permits all of the sensors stay in 3 dB beam width of the radar at the same time. Initial sensor locations are $\mathbf{s}_1 = [1 \quad 0]^T$, $\mathbf{s}_2 = [0 \quad 3]^T$, $\mathbf{s}_3 = [2 \quad 1.5]^T$ and $\mathbf{s}_4 = [-1 \quad -1.5]^T$ km., respectively. The sensors move along the x-axis with a velocity of 50 meters per second for 120 seconds. Unless stated otherwise, sensor locations are assumed to be known perfectly.

There are two emitters which are located at $[9 \quad 100]^T$ (Emitter #1) and $[65 \quad 80]^T$ (Emitter #2) km. respectively. Unless it is stated otherwise, the emitter locations are fixed in all simulations.

All scenarios are simulated for 1000 Monte-Carlo runs, and the distance RMS of the location error is plotted.

For the recursive estimator, the process noise standard deviation (σ_V) is chosen as $\sigma_V = 30cm$ in all scenarios unless it is stated otherwise. Recall from the expression (4.46) that σ_V is used to model the uncertainties only in the range and its rate of change. It must be chosen according to the sensor and/or the emitter dynamics. Some priori knowledge about these is required to determine the appropriate value of σ_V which is out of the scope of this thesis. Here, it is chosen according to the experimental simulation results with sufficiently good performance. Any other value, which would give similar results, can be chosen as well.

6.2 Simulation Results for the Recursive Method

In this section, the behavior of the recursive estimator developed in Chapter 4 is examined through simulations that include the reference sensor shifting and the maneuvering sensors. In all the scenarios, the recursive estimator is initialized using the first 4 TDOA measurement sets. In addition, σ_{TOA} is taken as 1 nano second in all scenarios.

6.2.1

6.2.2 Non-maneuvering Sensor Movement Case

In this scenario, performances of the original version of the recursive estimator in equation (4.31) and the modified version in (4.46) are compared. Remember that, the estimator in (4.46) is immune to sensor maneuvers. The simulation results for Emitter #1 and #2 are plotted in Figure 6.2 and Figure 6.3 respectively;

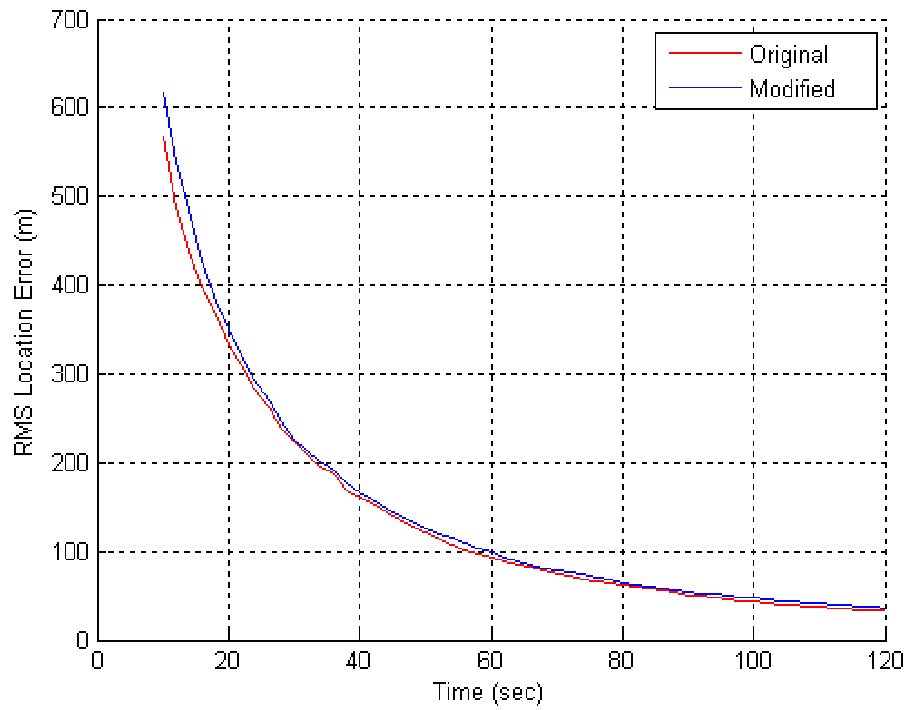

Figure 6.2: The RMS location error for non manoeuvring case for Emitter #1.

In this scenario, the sensors do not have any maneuver. In such a case, there is no significant difference in the estimators' performance. Clearly, examining Figure 6.2 and Figure 6.3, it can be stated that original and modified versions have almost identical performance. Notice that, the average RMS error is larger in Figure 6.3. This increase in error is due to the unfavorable geometry of the sensors relative to the emitter.

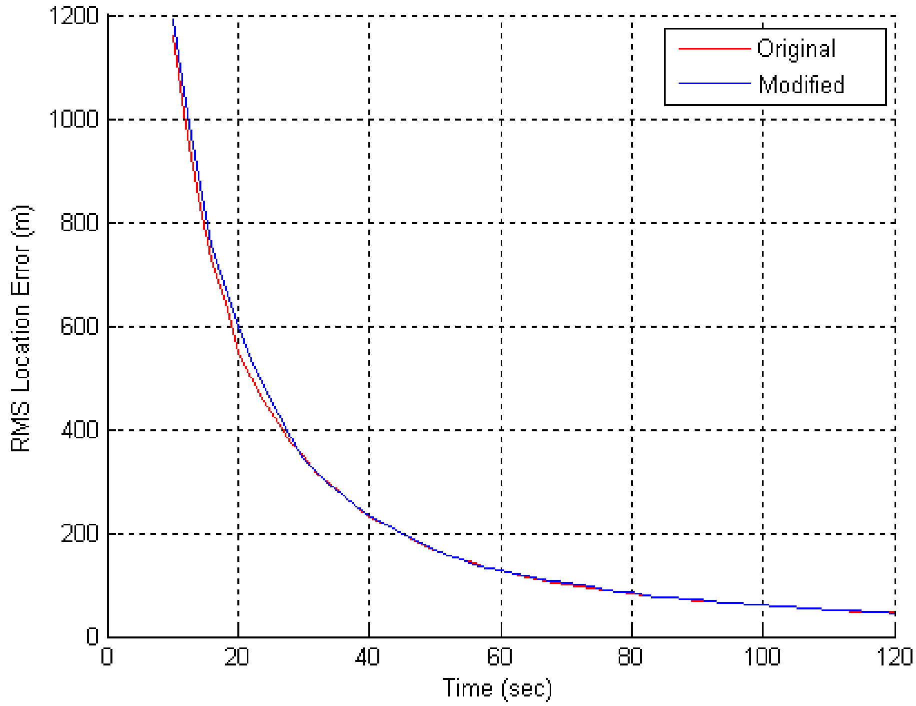

Figure 6.3: The RMS location error for non-manoeuvring case for Emitter #2.

6.2.3 Maneuvering Sensor Case

In this scenario, the sensors move with a velocity of 50 meters per second in x direction for the first 120 seconds, and then they turn backwards and move with the same velocity in negative x direction for another 120 seconds. Under this scenario, RMS location errors of the original and the modified versions of the estimator are plotted in Figure 6.4 and Figure 6.5.

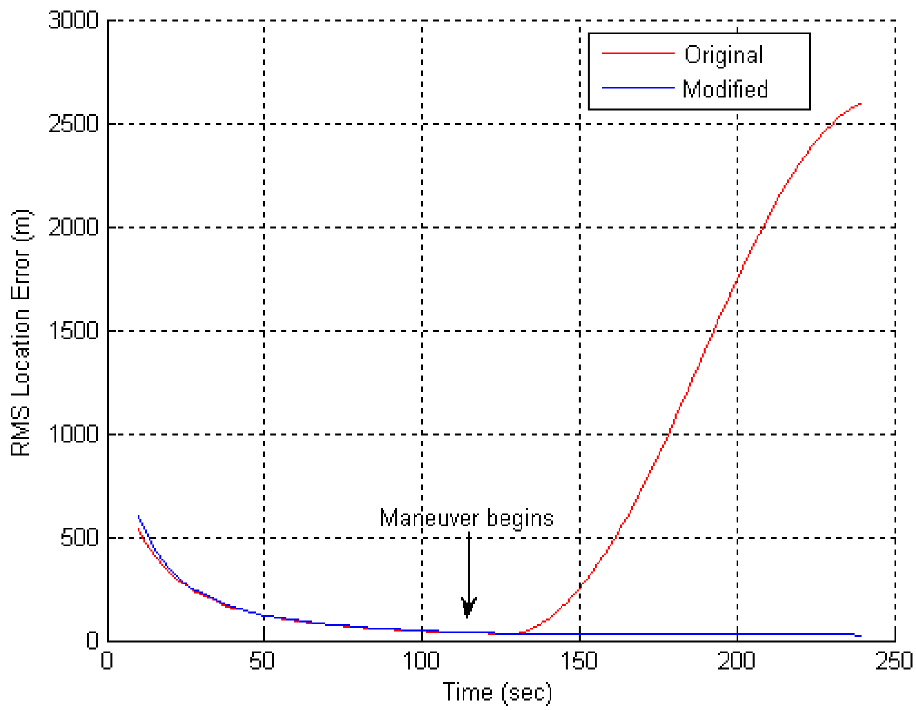

Figure 6.4: The RMS location error for the manoeuvring case for Emitter #1.

For the maneuvering case, the RMS location error performance of the modified estimator is similar to non-maneuvering case. On the other hand, there occurs a dramatic increase in the RMS location error of the original version of the estimator when the sensors are maneuvering. Along with the effect of the sensor-emitter geometry, this performance degradation becomes more severe as shown in Figure 6.5. As a result, it is obvious that, the original version of the recursive estimator can not be used when the sensors are maneuvering which is unacceptable in practice.

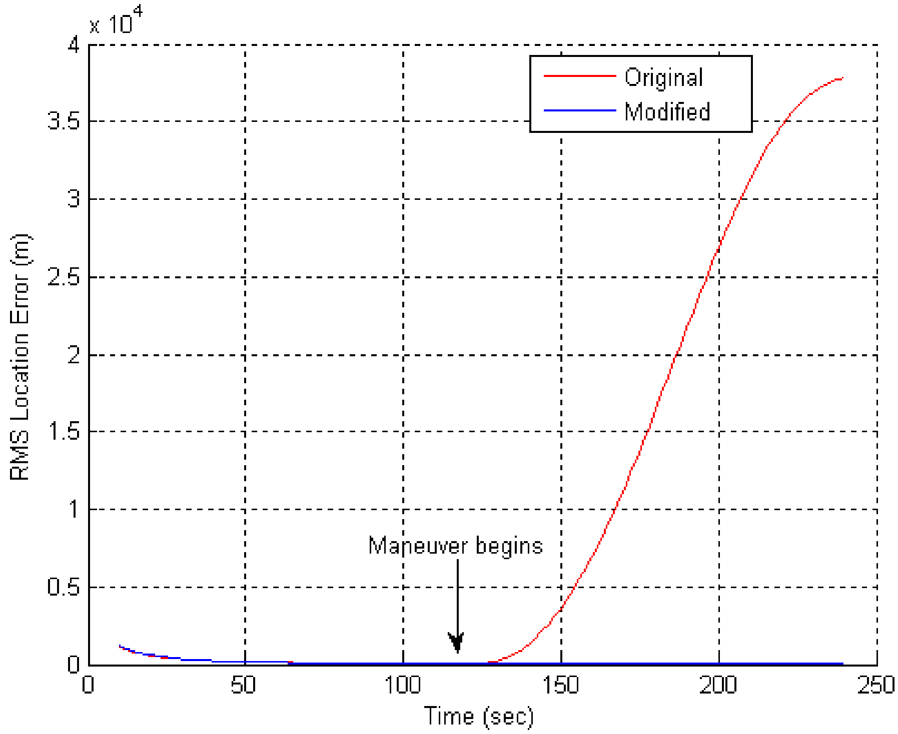

Figure 6.5: The RMS location error for manoeuvring sensor case for Emitter #2.

6.2.4 Reference Sensor Shifting

In this subsection, the effect of the reference sensor shifting is studied on the localization performance. For this purpose, assume that there are two groups of sensors. The first group's initial sensor locations are $\mathbf{s}_1^1 = [1 \quad 0]^T$, $\mathbf{s}_2^1 = [0 \quad 3]^T$, $\mathbf{s}_3^1 = [2 \quad 1.5]^T$ and $\mathbf{s}_4^1 = [-1 \quad -1.5]^T$ km., and the second group's initial locations are $\mathbf{s}_1^2 = [10 \quad 0]^T$, $\mathbf{s}_2^2 = [8 \quad 1]^T$, $\mathbf{s}_3^2 = [9 \quad -0.5]^T$ and $\mathbf{s}_4^2 = [11 \quad -1.5]^T$ km., respectively, where the superscript indicates the sensor group number. The two sensor groups both have a velocity of 50 meters per second in the positive x direction. Suppose that the sensors from different groups can not receive the same pulse at the same time and the reference sensor shifting mechanism, which is described in Section 4.3.3, is used in order to improve the localization performance.

The scenario steps of the reference sensor shifting simulation are described below:

A. Initialization:

1. One of the sensor groups performs the first location estimation. There is no past measurement or priori location estimate at this step. Suppose that first sensor group receives a pulse and estimates the location. Let the sensor group indices be $i = 1$ and $j = 2$.

B. Reference Sensor Shifting: Recall the assumption that the sensors from different groups can not receive the same pulse.

2. At this step, the j^{th} sensor group receives a pulse and is able to construct a TDOA measurement set while the i^{th} sensor group can not receive the pulse.

3. The j^{th} sensor group performs the reference sensor shifting as described in Section 4.3.3 and updates the location estimate using the measured TDOA set.

4. The sensor group indices are swapped at this step. For example, j becomes 1 and i becomes 2 for the values given in the step 1. Then, the scenario continues with the step 2.

In Figure 6.6 and Figure 6.7, the results of this scenario are plotted.

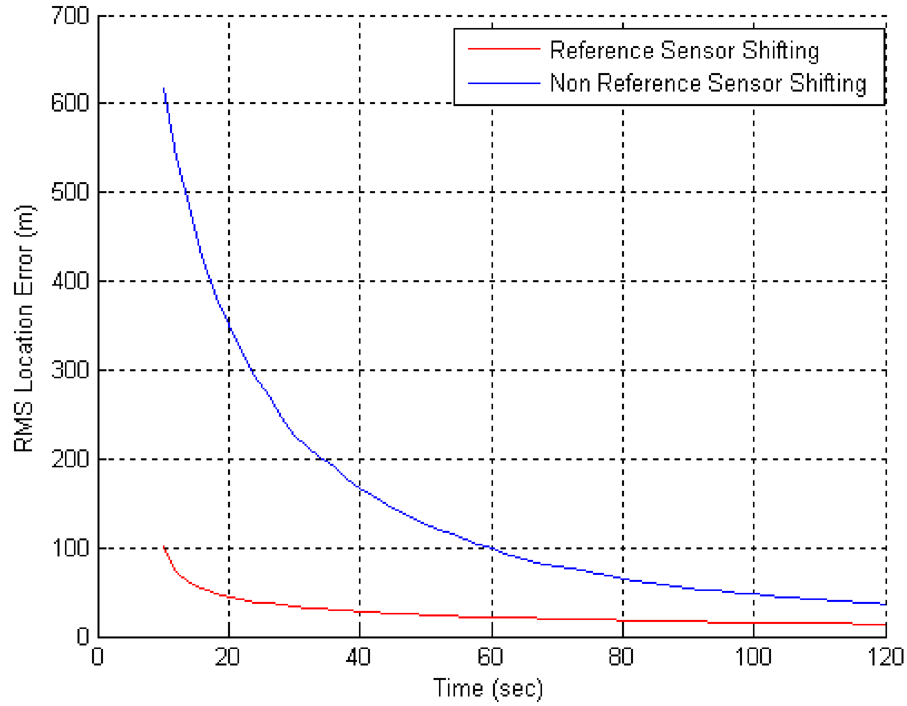

Figure 6.6: The RMS location error with the Reference Sensor Shifting for Emitter #1.

Notice that, in non reference sensor shifting case, only the first sensor group is performing localization. Examining Figures 6.6 and 6.7, it can be figured out that; the reference sensor shifting results in notable performance increase. Of course, this situation is a result of the fusion of the measurements which are taken from more than one group of sensors. The fusion of measurements brings more information about the location of the emitter.

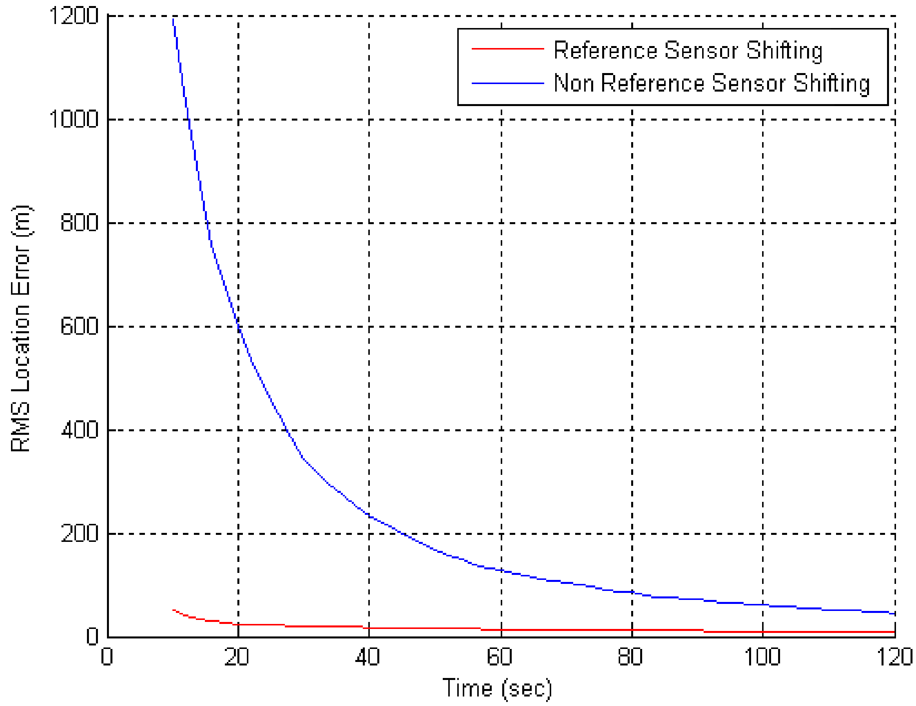

Figure 6.7: The RMS location error with the Reference Sensor Shifting for Emitter #2.

6.2.5 Estimating the Track of a Moving Emitter

In this subsection, the performance of the estimator in Section 4.3.2 is studied. Track estimation of a moving emitter with constant velocity is simulated for different velocity values and the results are plotted. The simulation studies are performed for the emitter velocities of $[-40 \quad 0]$, $[-20 \quad 0]$, $[30 \quad 0]$, $[60 \quad 0]$, $[90 \quad 0]$, $[0 \quad 30]$, $[0 \quad 50]$, $[0 \quad -30]$ m/sec. respectively where $[v_x \quad v_y]$ denote the emitter's velocity vector in x and y directions. In addition the sensor velocities are chosen as $[50 \quad 0]$ m/sec.

Furthermore, filter parameter $\mathbf{Q}^{'}$ (defined in Section 4.3.2) is chosen as:

$$\mathbf{Q}^{'} = 10^{-10}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}. \tag{6.1}$$

Note that, (6.1) is dependent on the dynamics of the sensors and the emitter. Here, it was chosen according to the experimental simulation results.

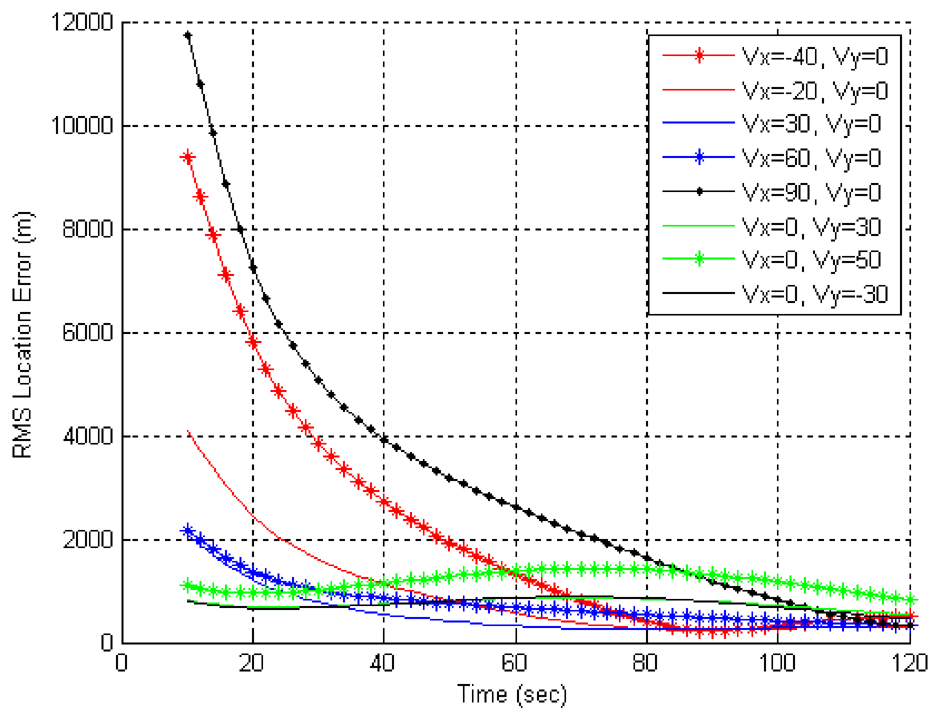

Figure 6.8: The track estimation of a moving source for Emitter #1.

Examining Figures 6.8 and 6.9, it can be stated that the estimator is in general capable of estimating the track of a moving emitter. On the other hand, the error performance is low compared to the estimating fixed location. Moreover, the localization error grows when the relative velocity between the emitter and the sensors is high. Including the effects of the relative sensor geometry, the location error is considerable high in the beginning of the scenario which is shown in Figure 6.9. Fortunately, the location error decreases and reaches reasonable levels after sufficiently many measurements are processed. Remember that the method is only a basic solution of the estimating the track of a moving source. For example, FDOA measurements can be used to achieve better results, since they carry information about the velocity of the emitter which is unobservable by TDOA measurements. Furthermore, the performance of the method can probably be improved by using different tracking filters which can be found in the tracking literature [30, 31, 37].

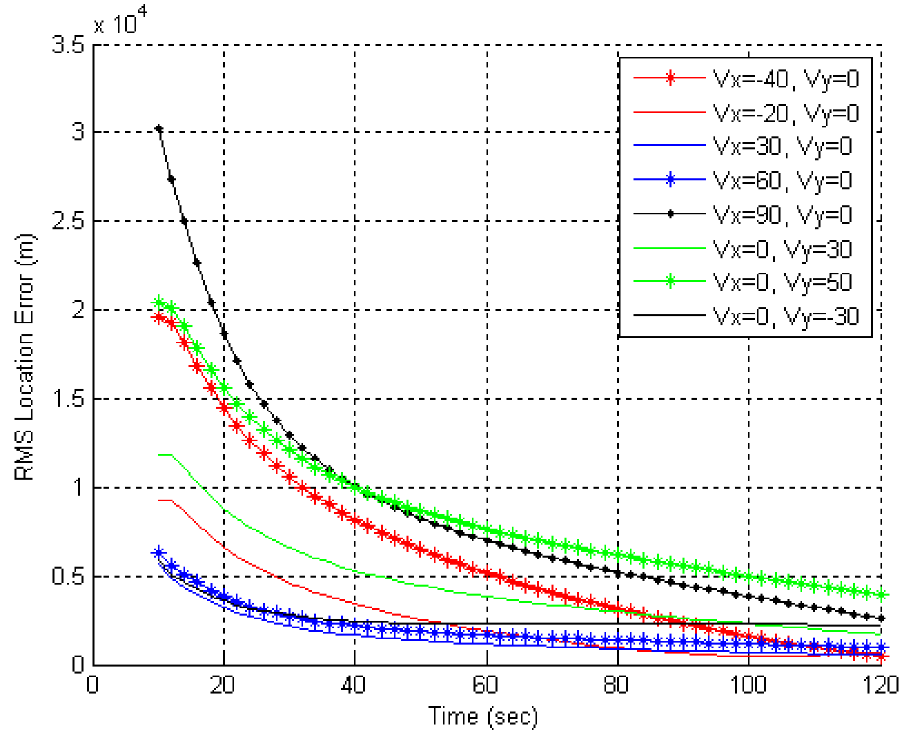

Figure 6.9: The track estimation of a moving source for Emitter #2.

6.3 Simulation Results for the Pseudo Recursive Estimator

In this section, the simulation results of the pseudo recursive method in Chapter 5 are presented. Performances of the original method and the measurement fusion integrated version are compared. In other words, the effect of data accumulation is studied. As in Section 6.2, σ_{TOA} is taken as 1 nano second.

In practice, the initial point of the iteration can be found by using a closed form estimator which will probably result in many different initial points. This will mislead the results. On the other hand, in order to realize a controlled experiment, the initial point is given a constant value of $[x_e + 4000 \quad y_e + 4000]^T$ m., where $[x_e \quad y_e]^T$ is the true coordinates of the emitter.

The simulations are performed for no fusion, fusion of 5, 10 and 20 measurements cases, respectively.

The results are plotted in Figures 6.10 and 6.11;

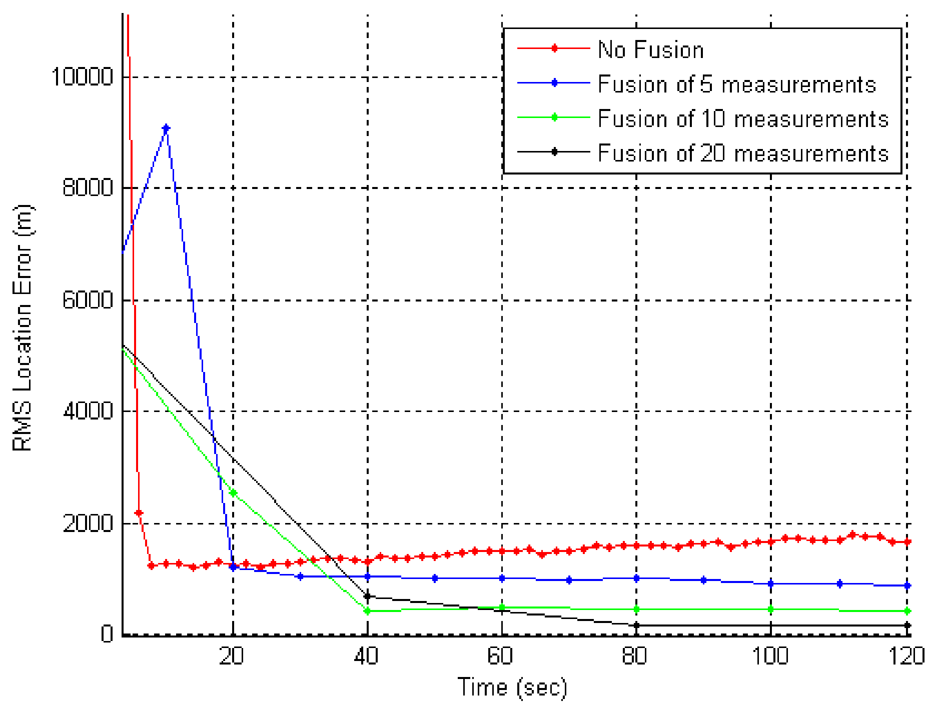

Figure 6.10: The simulation results for Emitter #1.

Localization performance increases when more number of measurements are fused, while, convergence is delayed at the same time. The localization performance is better for Emitter #1. Notice that in Figure 6.10, for the no fusion measurement case, which is the most vulnerable one to noise, the estimator tends to diverge from the true location. This is because of the iterative approach of the estimator; convergence is not guaranteed [26]. On the contrary, the measurement fusion makes the estimation process more robust to noise; which can be seen in Figures 6.11 and 6.12.

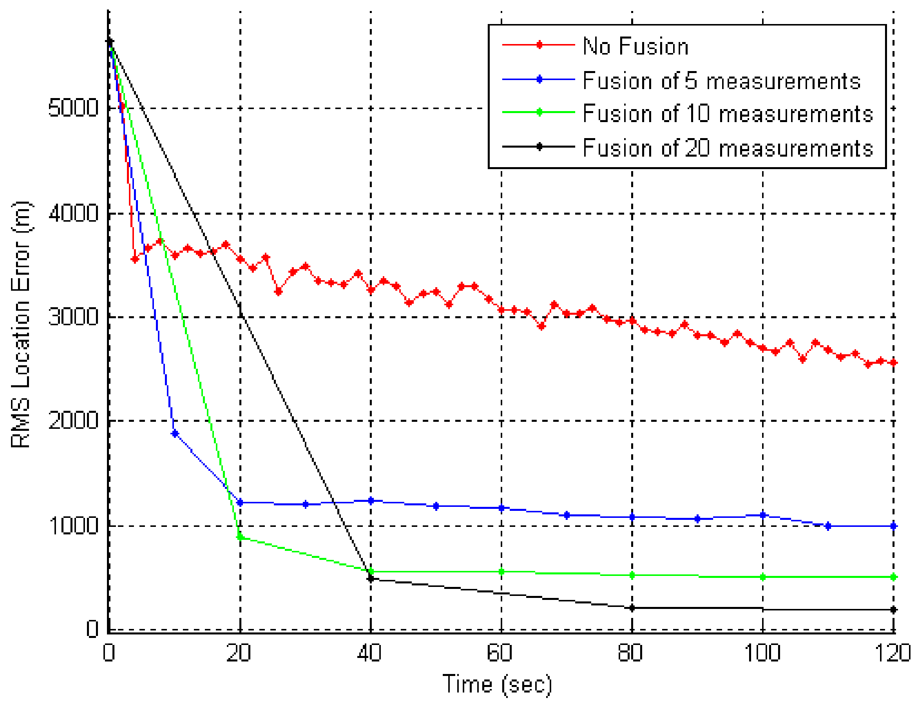

Figure 6.11: The simulation results for Emitter #2.

Although the no measurement fusion case tends to diverge for Emitter #1, it is in convergence behavior for Emitter #2; despite the overall location error is larger. This behavior arises from the different changes of the relative geometry between the sensors and the emitter.

In practice, convergence delay is probably desired to be avoided. For this purpose, one can use a method with no fusion for a short interval, and then switch to the measurement fusion operation. The same scenario is simulated to demonstrate the use of this approach, first 5 measurements are processed without fusion to get a faster convergence, and after that the fusion process is activated. The results are given in Figures 6.12 and 6.13.

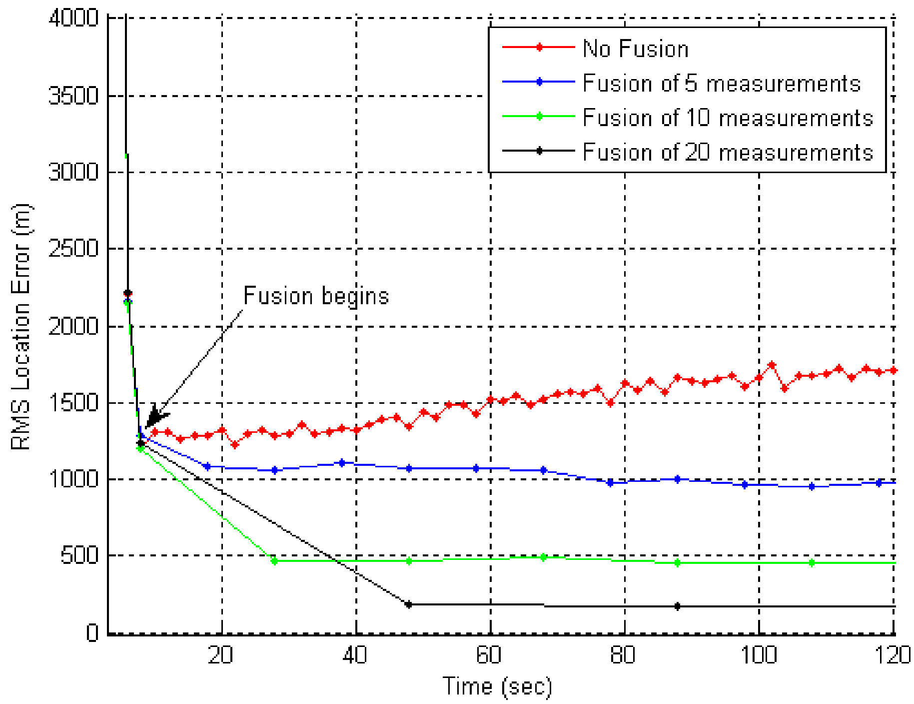

Figure 6.12: Convergence delay avoidance study for Emitter #1.

From Figures 6.13 and 6.14, it can be seen that the proposed procedure brings shorter convergence times.

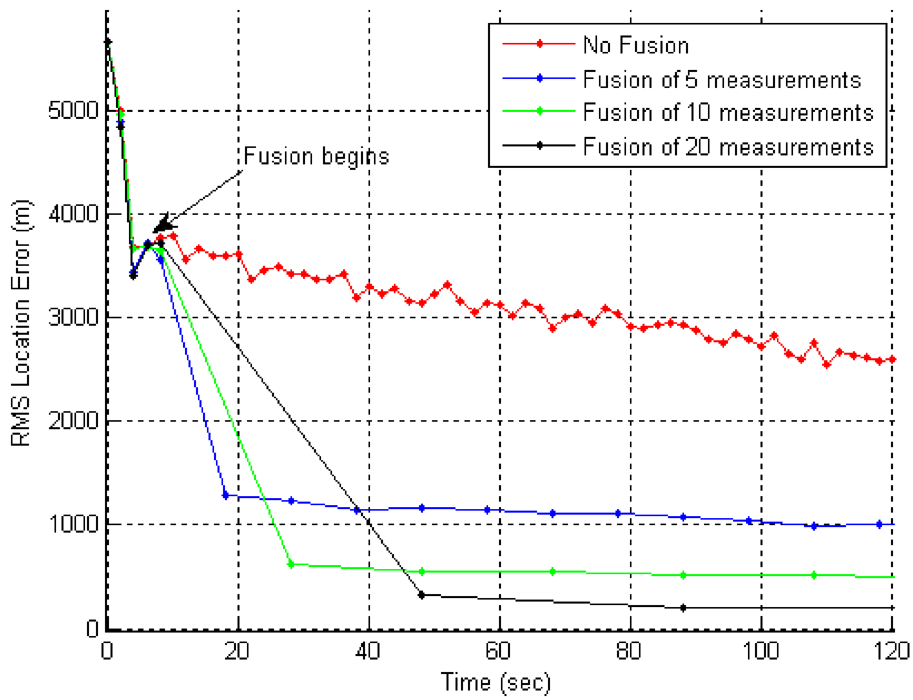

Figure 6.13: Convergence delay avoidance study for Emitter #2.

6.4 Comparative Simulations

In this section, comparative simulation results on the post processing, the recursive and the pseudo recursive estimators are presented for the fixed emitter localization. The simulations are performed under the two cases of (1) the increasing TOA noise with no sensor location error and (2) the increasing sensor location error with constant TOA noise. Remember that, the post processing method requires location estimates to fuse. In this sense, the closed form estimator in [2] is used (The derivation of the used closed form method is given in Appendix A). The appropriate Cramer Rao Lower Bound, mentioned in Section 4.4 and 4.5 is also calculated for each time step. The square root of the sum of the first two diagonal elements of the

CRLB is plotted. Note that, the first two elements are related to the coordinates of the emitter.

The RMS of the location error is plotted for the estimators for each case. In the pseudo recursive estimator, the fusion of 20 measurements is used.

6.4.1 Case 1: Increasing the TOA Noise with No Sensor Location Uncertainty

In this simulation scenario, the TOA measurements are corrupted with independent and identically distributed zero mean Gaussian noise with σ_{TOA} standard deviation. Then, using these TOA measurements, the TDOA sets are constructed. σ_{TOA} is selected as 1, 5 and 10 nano seconds, respectively.

For 1 nano second standard deviation in TOA measurements, results are drawn in Figures 6.14 and 6.15, in order to emphasize RMS location error more clearly.

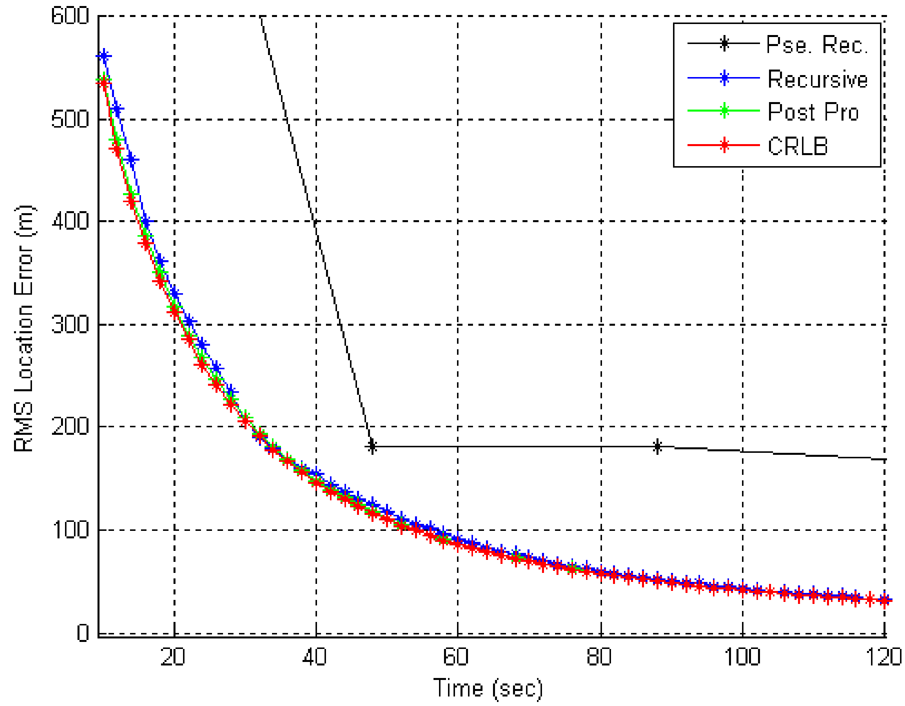

Figure 6.14: Simulation results with 1 nsec standard deviation for Emitter #1.

Notice that, the location RMS of the post processing and recursive methods are very close to the CRLB (The post processing method has greater performance for Emitter #2 in the beginning of the scenario). Clearly, the post processing and the recursive methods have both better performances than the pseudo recursive method.

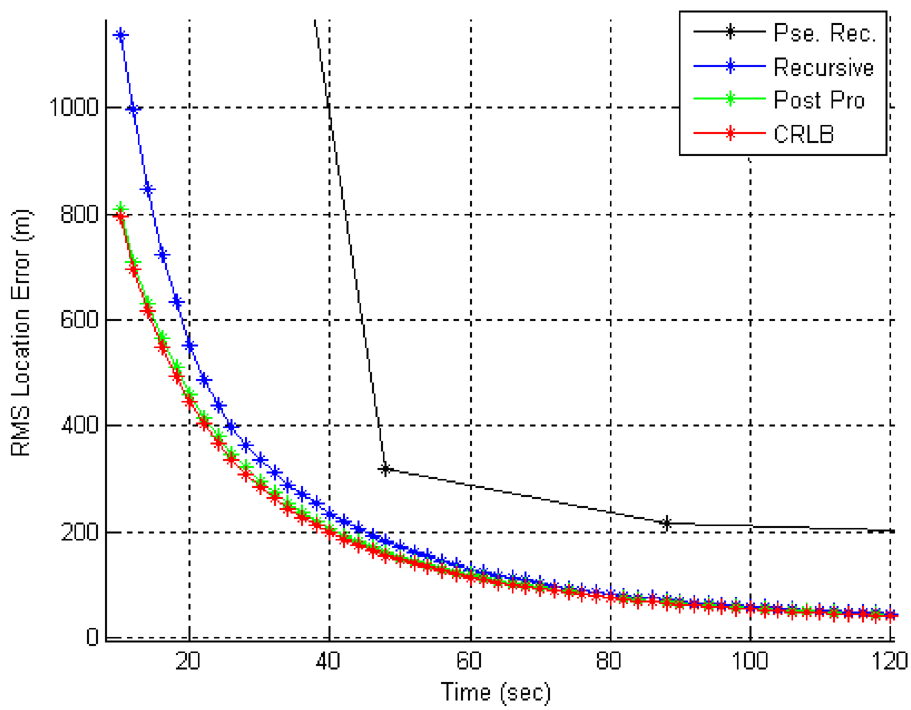

Figure 6.15: Simulation results with 1 nsec standard deviation for Emitter #2.

The results for 5 and 10 nsec TOA noise standard deviations cases are shown in Figures 6.16 and 6.17 for Emitter #1 and #2, respectively.

The pseudo recursive method has diverged for 5 and 10 nsec TOA noise standard deviations for both of the emitters. Moreover, nearly half of the simulation runs have diverged for 10 nsec noise standard deviation.

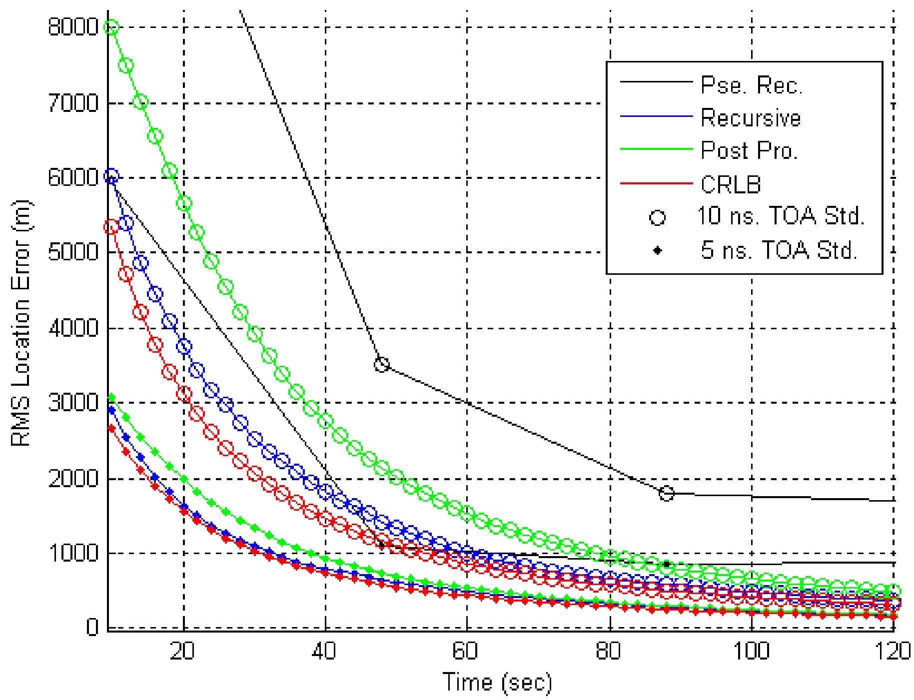

Figure 6.16: Simulation results with 5 and 10 nsec TOA noise standard deviation for Emitter #1.

Examining Figures 6.14, 6.15, 6.16 and 6.17, it can be seen that the difference between the CRLB and the simulated methods becomes more distinguishable when the noise variance increases. Notice that, although the post processing method has better performance for 1 nsec noise standard deviation, the recursive method becomes superior with higher the noise levels. From this point of view, it can be stated that, data or measurement accumulation results in more immunity to noise.

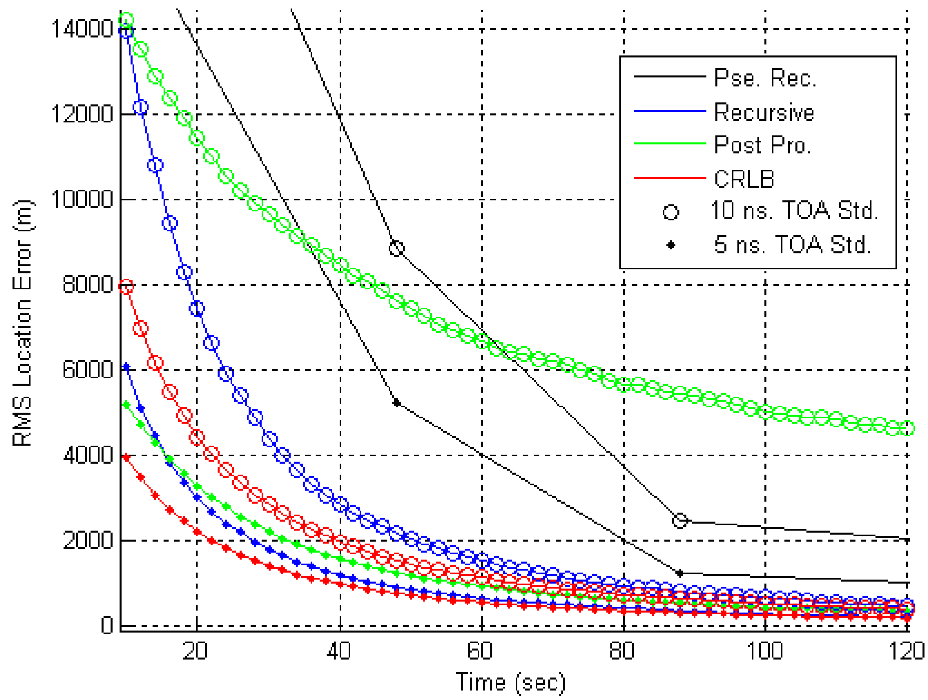

Figure 6.17: Simulation results with 5 and 10 nsec TOA noise standard deviation for Emitter #2.

The results for the last time step of 120 seconds are given in Table 1,

Table 1: Simulation results for the last time step of 120 seconds.

	RMS Location Error (m) for Emitter #1			RMS Location Error (m) for Emitter #2		
TOA noise Std. Dev.	σ =1 ns	σ =5 ns	σ =10 ns	σ =1 ns	σ =5 ns	σ =10 ns
Pseudo Recursive	169	874	1703	202	1023	2046
Recursive	33	176	359	46	229	484
Post Processing	32	185	483	43	365	4654
CRLB	32	160	319	41	206	413

6.4.2 Case 2: Increasing Sensor Location Error with Constant TOA Noise Level

In this subsection, the effects of the sensor location uncertainty are studied. The simulations are performed for the different sensor location noise levels while the TOA noise standard deviation is constant.

x and y components of the sensor locations are assumed to be corrupted by additive independent identically distributed zero mean Gaussian noise. Moreover, the sensor location noise is assumed to be i.i.d. for the different time instances. In the simulations, sensor location noise standard deviations of 0.1, 0.75 and 1.5 m are used, respectively. The TOA noise standard deviation is chosen as 1 nsec. The CRLB is also plotted.

The results for the sensor location noise standard deviation of 0.1 m. are given in Figures 6.18 and 6.19 for emitters #1 and #2, respectively.

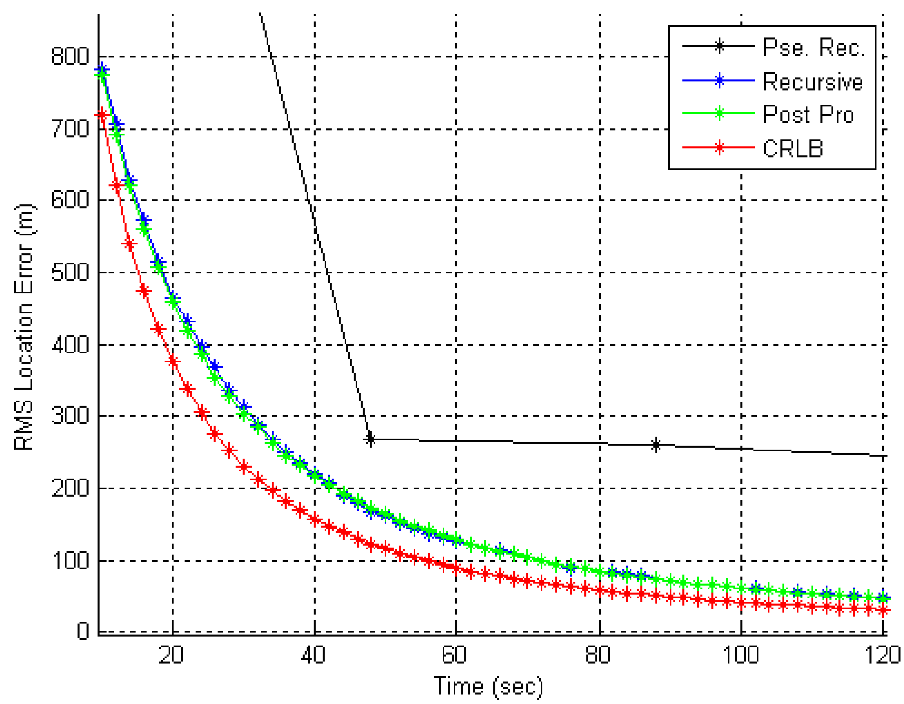

Figure 6.18: RMS Location Error with sensor location uncertainty of standard deviation 0.1 m. for Emitter #1.

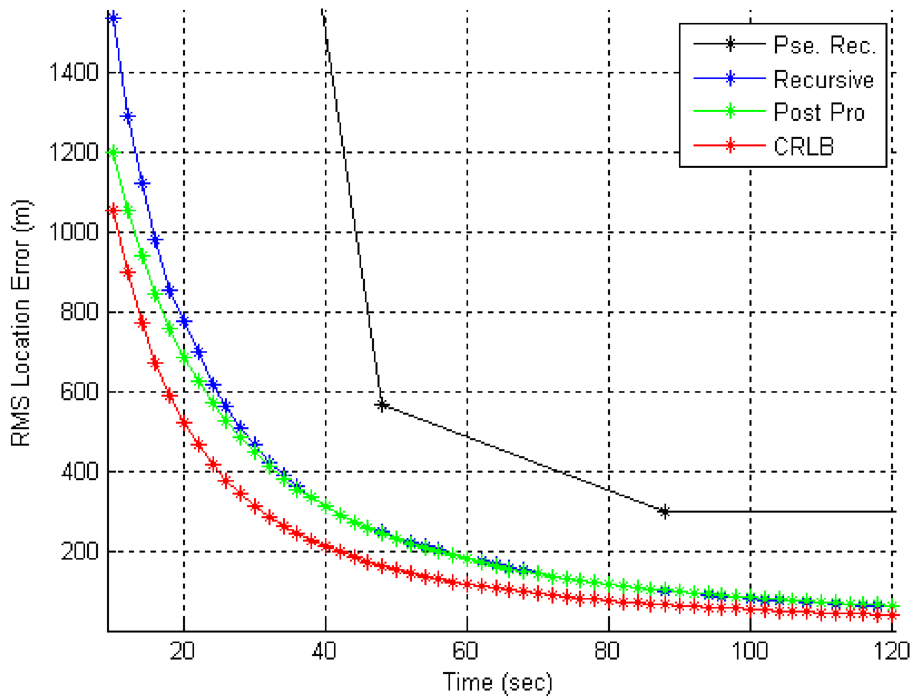

Figure 6.19: RMS Location Error with sensor location uncertainty of standard deviation 0.1 m. for Emitter #2.

Figures 6.20 and 6.21 show the results for the sensor location noise standard deviations of 0.75 and 1.5 m. for emitters #1 and #2, respectively.

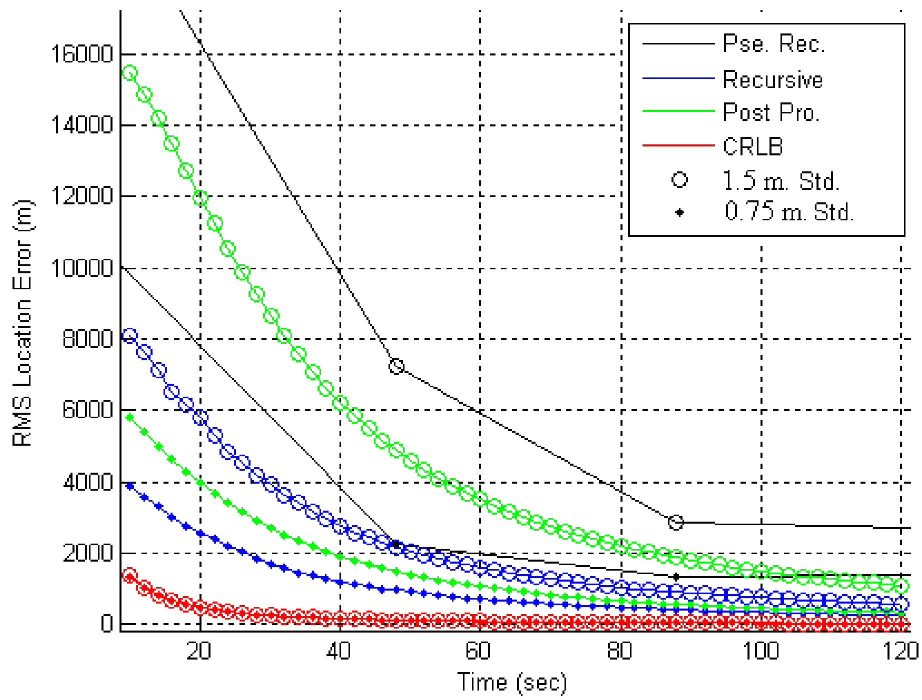

Figure 6.20: RMS Location Error with sensor location uncertainty of standard deviations 0.75 and 1.5 m. for Emitter #1.

From Figures 6.20 and 6.21, it is clear that there is a dramatic performance degradation with increasing sensor location uncertainty. On the other hand, the recursive method again becomes superior with high noise levels. Meanwhile, there is a slight difference, which is not distinguishable in the figures, between the Cramer Rao Lower Bounds for the standard deviations of 0.75 and 1.5 m for both emitters #1 and #2.

Robustness of the recursive method to the sensor location uncertainties comes from data accumulation. Since the data accumulation is not used in the closed form estimator which is used prior to the post processing method, it becomes more sensitive to noise. As a result, degradation in performance occurs for post the processing method.

It must be emphasized that the purpose of the simulations for the post processing method is to show its capability to improve accuracy of the location estimates. Performance of the post processing method is directly related and dependent on the prior location estimator which supplies the location estimates to be fused. Besides, the post processing method is proven to be able to improve accuracy of the location estimate as long as there are successive location estimates to be fused.

Combining the results, it can be stated that similar to measurement noise case, recursion procedure again provides a certain level of immunity to the sensor location noise.

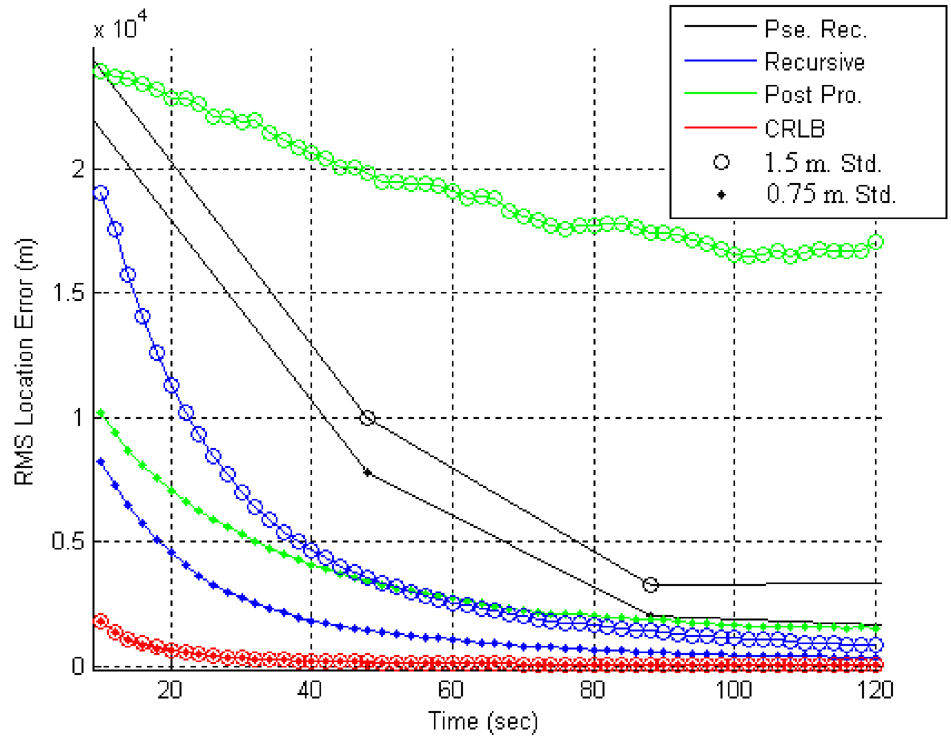

Figure 6.21: RMS Location Error with sensor location uncertainty of standard deviations 0.75 and 1.5 m. for Emitter #2.

The results for the last time step of 120 seconds are given in Table 2,

Table 2: Simulation results for the last time step of 120 seconds.

	RMS Location Error (m) for Emitter #1			RMS Location Error (m) for Emitter #2		
Sensor location Std. Dev.	σ = 0.1 m	σ = 0.75 m	σ = 1.5 m	σ = 0.1 m	σ = 0.75 m	σ = 1.5 m
Pseudo Recursive	247	1378	2705	300	1681	3316
Recursive	49	261	579	61.2	357	853
Post Processing	47	342	1099	66	1535	17060
CRLB	32.3	33.3	35.2	41.7	42.6	44

6.4.3 Case 3: Biased Sensor Location Error

In this subsection, the effects of the biased sensor location errors on the TDOA localization are studied. The simulations are performed under different sensor location error bias levels with a constant TOA noise standard deviation of 1 nsec. The independent noise components of the sensor location errors are assumed to be non-existent for simplicity.

x and y components of the locations of the sensors are affected by the same amount of bias. The bias levels of the sensor locations are chosen as 5, 20 and 50 m., respectively. The results are plotted in the figures 6.22 and 6.23.

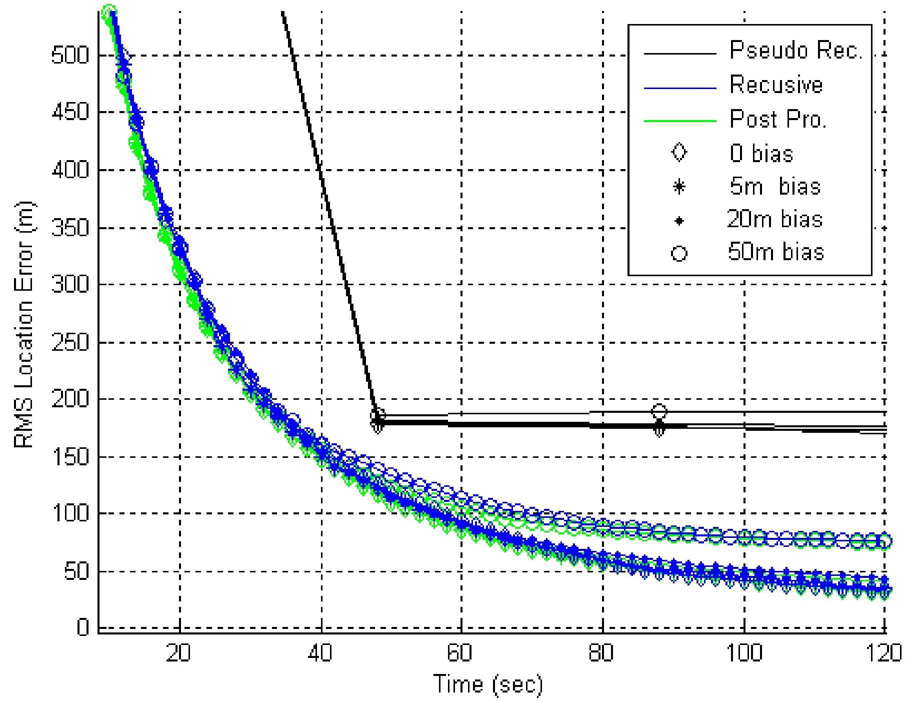

Figure 6.22: The simulation results for biased sensor locations for Emitter #1.

Examining the figures 6.22 and 6.23, the bias levels of 5 and 20 m. result tolerable error levels on the location estimate. On the other hand, the errors caused by the bias level of 50 m. are more distinguishable, yet they can also be regarded as tolerable.

Comparing the results of the simulations of the biased and the independent sensor location errors (see Section 6.4.2), it can be stated that the bias on the sensor locations is a minor error source on the TDOA localization.

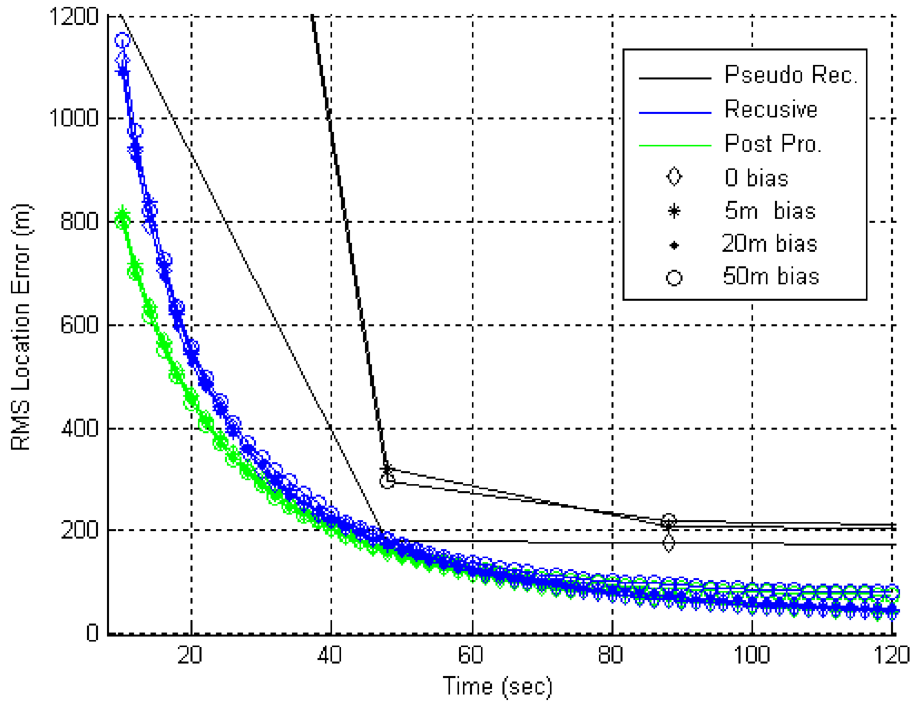

Figure 6.23: The simulation results for biased sensor locations for Emitter #2.

6.5 Discussion

First of all, the recursive method in Chapter 4 was examined. It was seen to provide sufficient emitter localization accuracy when the sensors are maneuvering. Furthermore, the reference sensor shifting mechanism was also seen to improve the accuracy of the estimate considerably. It was observed that, the recursive method approaches the CRLB for low noise levels. The track estimation of a moving source is also simulated. Examining the results, for the moving emitter case, there is some degradation in the localization accuracy compared to the estimating fixed location. There are two main causes of this problem: (1) The data accumulation is not so effective compared to the fixed location case, since there is not much data about an individual state of the emitter's location (because it is moving). (2) The velocity of the emitter is unobservable with using only the TDOA measurements. FDOA measurements can be used along with the TDOA to overcome this issue and yield more accurate location estimates.

Examining the simulation results, the measurement accumulation procedure for the pseudo recursive method is also proved to improve the accuracy of the estimates. However, it must be noted that the pseudo recursive method has divergence problems dependent on the conditions of measurement noise, the sensor location uncertainty and the geometry.

The post processing method is seen to accomplish its mission; improving the accuracy of the estimates. On the other hand, the accuracy of the post processing estimator is highly dependent on the accuracy of the prior location estimator.

The performance comparative simulation studies for all of the three methods have also been presented. From the simulation results, it is seen that the accuracy of the estimation is highly dependent on the measurement noise, the independent sensor location errors and the relative geometry between the sensors and the emitter. The bias on the sensor locations is also an error source, but it is a minor one. Note that, the recursive method generally has better performance than the others. It is also shown that the recursion structure provides a certain level of immunity to noise both in the measurements and the sensor locations.

CHAPTER 7

SUMMARY AND CONCLUSIONS

In this thesis, robust and recursive solutions have been developed for the emitter localization problem by the use of TDOA measurements provided by groups of moving sensors. In this context, a post processing estimator and a recursive estimator were developed. Maximum likelihood approach of the emitter localization has also been explored.

The structure of the recursive estimator is based on

i. A basic localization algorithm to start with,

ii. Developing an algorithm which updates the location estimate as a new set of measurements is made.

We have used the simple closed form solution (equation (4.11)) and the multi-pulse closed form solution (equation (4.21)) as the basic localization algorithms and developed a recursive update method as given in equation (4.46). Note that, the estimator processes directly the TDOA measurements. The simulation results show that the method yields a performance very near to the CRLB. However, as the inaccuracy of the measurements or the sensor locations become bigger, the estimation error becomes quite larger than the CRLB, especially in the initial periods of the recursion. It is thought that the sufficiently large amount of data is not yet accumulated in the initial periods, resulting in considerable estimation errors. It is also observed that the recursive estimator suffers from bias problem, it is thought that the bias is caused by the square operation on the measurements (see equation (4.7)) and using an approximate model in the range and its rate of change (see equation (4.41)). The recursive estimator also has been extended to estimate the track of a moving source. On the other hand, there occurs performance degradation when the emitter is moving. This problem has two main causes:

1. Sufficient amount of data of an individual state of the emitter can not be accumulated, since the emitter is moving,

2. The velocity of the emitter is unobservable using only the TDOA.

A heuristic reference sensor shifting mechanism is also proposed to the recursive estimator, and it is observed to improve the accuracy of the location estimates by simulation results. Finally, by comparative simulation results, the recursive estimator has performed better in general, and the recursion structure provides a certain level of immunity to noise both in the measurements and the sensor locations.

A post-processing estimator has also been presented. The method does not process the TDOA measurements, instead smoothes the location estimates. First, it is given in closed-form, and then implemented recursively. By simulation results, the post processing estimator is seen to improve the accuracy of the location estimate.

Maximum likelihood approach of the emitter localization problem by passive sensors has also been explored. Because of the nonlinear structure of the problem, the maximum likelihood solution can not be expressed in a closed-form. The solution requires Taylor Series expansion about an initial point, and hence becomes iterative. A data accumulation method is proposed, and by simulation results it is shown to perform quite well. On the other hand, because of being a local correction procedure, divergence may occur.

The Cramer Rao Lower Bound of the emitter localization by using multiple TDOA measurement sets is also derived for the measurement noise and the erroneous sensor locations, and it was compared with the performance of the stated algorithms.

As future work,

- The recursive estimator can be extended to use FDOA and/or AOA along with TDOA to improve the accuracy,
- Target tracking algorithms such as IMM or MHT can be used to improve performance of the recursive estimator for estimating the track of a moving source,
- Bias compensation methods can be investigated to have better location estimates by the recursive estimator,
- Sensor placement strategies can be studied to avoid degrading effects of relative geometry between the emitter and the sensors.

REFERENCES

[1] D. J. Torrieri, “Statistical Theory of Passive Location Systems”, IEEE Transactions on Aerospace and Electronic Systems, Vol. AES-20, No. 2, March 1984.

[2] Y. T. Chan and K. C. Ho, “A Simple and Efficient Estimator for Hyperbolic Location”, IEEE Transactions on Signal Processing, Vol. 42, No. 8, August 1994.

[3] Richard A. Poisel, “Electronic Warfare Target Location Methods”, Boston, Artech House, 2005.

[4] Ning Luo and Ge´Rard Lachapelle, “Relative Positioning of Multiple Moving Platforms Using GPS”, IEEE Transactions on Aerospace and Electronic Systems Vol. 39, No. 3 July 2003.

[5] J.D. Bard and F. M. Ham, “Time difference of arrival dilution of precision and applications”, IEEE Transactions on Signal Processing, Volume 47, Issue 2, pp.521 – 523, Feb. 1999.

[6] A.H. Quazi, “An Overview on the Time Delay Estimate in Active and Passive Systems for Target Localization”, IEEE Transactions on Acoustics, Speech, and Signal Processing, Vol. ASSP-29, No. 3, June 1981.

[7] J. D. Bard, F. M. Ham, and W. L. Jones, “An Algebraic Solution to the Time Difference of Arrival Equations”, Proceedings of the IEEE Southeastcon '96. 'Bringing Together Education, Science and Technology', Tampa, FL, USA, April 1996, pp.313-319.

[8] K. E. Lee, D. M. Ahn, Y. Lee, S. Cho and J. Chun, “A Total Least Squares Algorithm For The Source Location Estimation Using Geo Satellites”, MILCOM 2000. 21st Century Military Communications Conference Proceedings, Los Angeles, California, USA, Vol. 1, Oct. 2000, pp.271 - 275.

[9] E. G. Bakhoum, “Closed-Form Solution of Hyperbolic Geolocation Equations”, IEEE Transactions on Aerospace And Electronic Systems Vol. 42, No. 4 October 2006.

[10] K. Doğançay and D. A. Gray, “Closed-Form Estimators For Multi-Pulse TDOA Localization”, Proceedings of the Eighth International Symposium on Signal Processing and Its Applications, Sydney, Australia, August 2005, Vol. 2 pp.543 – 546.

[11] K. Doğançay and D. A. Gray, “Bias Compensation for Least-Squares Multi-Pulse TDOA Localization Algorithms”, Intelligent Sensors, Sensor Networks and Information Processing Conference, Melbourne, Australia, Dec. 2005, pp.51 – 56.

[12] K. C. Ho and W. Xu, “An accurate algebraic solution for moving source location using TDOA and FDOA measurements”, IEEE Transactions on Signal Processing, Vol. 52, no. 9, September 2004.

[13] K. C. Ho and W. Xu, “Localization of A Moving Source Using TDOA and FDOA Measurements”, Proceedings of the 2003 International Symposium on Circuits and Systems, ISCAS '03, Bangkok, Thailand, May 2003, pp.IV-17 - IV-20.

[14] N. J. Thomas, D. G. M. Cruickshank and D. I. Laurenson, “A Robust Location Estimator Architecture with Biased Kalman Filtering of TOA Data for Wireless Systems”, IEEE 6th Int. Sypm. on Spread-Spectrum Tech. & Appli., NJIT. New Jersey, USA, Sept. 6-8, 2000.

[15] K. C. Ho, X. Lu and L. Kovavisaruch, “Source Localization Using TDOA and FDOA Measurements in the Presence of Receiver Location Errors: Analysis and Solution”, IEEE Transactions On Signal Processing, Vol. 55, No. 2, February 2007.

[16] H. Koorapaty, H. Grubeck and M. Cedervall, “Effect of biased measurement errors on accuracy of position location methods”, IEEE Global Telecommunications Conference, Sydney, Australia, Nov. 1998, pp.1497 - 1502 vol.3.

[17] W. Li and P. Liu, “3D AOA-TDOA emitter location by integrated passive radar-GPS-INS systems”, IEEE Int. Workshop VLSI Design & Video Tech. Suzhou, China, May 2005.

[18] Y. Qi, H. Kobayashi and H. Suda, “Analysis of wireless geolocation in a non-line-of-sight environment”, IEEE Transactions On Wireless Communications, Vol. 5, No. 3, March 2006.

[19] G. Gu, “Constrained Kalman Filtering and Its Application to Tracking of Ground Moving Targets”, Wireless Sensing and Processing II, Proc. of SPIE Vol. 6577, 2007.

[20] K. F. McDonald and W. S. Kuklinski, “Track Maintenance and Positional Estimation Via Ground Moving Target Indicator and Geolocation Data Fusion”, IEEE Radar Conference, Atlanta, Georgia, USA, May 2001, pp.239 - 245.

[21] A. N. Bishop, B. Fidan, B. D.O. Anderson, P. N. Pathirana and K. Doğançay, “Optimality Analysis of Sensor-Target Geometries in Passive Localization: Part 2 - Time-of-Arrival Based Localization”, 3rd International Conference on Intelligent Sensors, Sensor Networks and Information, ISSNIP 2007, Melbourne, Australia, Dec. 2007, pp.13 - 18.

[22] D. R. Van Rheeden, B. C. Brown, J. C. Price, B. A. Abbott, G. C. Willden, K. Chhokra, J. Scott and T. Bapty, “Automatic positioning of UAVs to optimize TDOA geolocation performance”, Digital Avionics Systems Conference,Utah, USA, Oct. 2004, pp.2.E.2 - 2.1-9 Vol.2.

[23] L. Cong and W. Zhuang, “Nonline-of-Sight Error Mitigation in Mobile Location”, IEEE Transactions on Wireless Communications, Vol. 4, No. 2, March 2005.

[24] C. O. Savage, R. L. Cramer, H. A. Schmitt, “TDOA Geolocation with the Unscented Kalman Filter”, IEEE International Conference on Networking, Sensing and Control, Florida, USA, 2006, pp.602 – 606.

[25] F. Fletcher, B. Ristic and D. Musicki, “Recursive Estimation of Emitter Location using TDOA Measurements from Two UAVs”, International Conference on Information Fusion, Québec City, Canada, July 2007, pp.1 – 8.

[26] W. H. Foy, “Position-Location Solutions by Taylor-Series Estimation”, IEEE Transactions on Aerospace and Electronic Systems Vol. AES-12, No. 2 March 1976.

[27] J. Li, J. L. H. Xu and J. Sun, “NLOS Error Mitigation and Mobile Tracking”, International Conference on Signal Processing, Beijing, China, 2004, pp.2453 - 2456 vol.3.

[28] A. Hashemi-Sakhtsari and K. Doğançay, “Recursive Least Squares Solution to Source Tracking Using Time Difference of Arrival”, IEEE International Conference on Acoustics, Speech, and Signal Processing, Montreal, Canada, 2004, pp.ii - 385-8 vol.2.

[29] H.L. Van Trees, “Detection, Estimation and Modulation Theory: Part I. Detection, Estimation and Linear Modulation Theory”, New York, John Wiley&Sons, 2001.

[30] R. Johansson, “System Modeling and Identification”, NJ, Prentice Hall, 1993.

[31] Y. Bar-Shalom and X. R. Li, “Multitarget-Multisensor Tracking: Principles and Techniques”, MA, Clearance Center, 1995.

[32] R. G. Wiley, “Electronic Intelligence: The Analysis of Radar Signals”, Second Edition, MA, Artech House, 1993.

[33] M. H. Hayes, “Statistical Digital Signal Processing and Modeling”, NJ, John Wiley&Sons, 1996.

[34] S. Çamlıca and Y. Tanık, “Varış Zamanları Farkları ile Özyinelemeli Konum Belirleme”, IEEE Signal Processing and Communications Applications Conference (SIU 2009), Antalya, Turkey, 2009.

[35] M. Najar and J. Vidal, “Kalman Tracking Based on TDOA for UMTS Mobile Location”, IEEE International Symposium on Personal, Indoor and Mobile Radio Communications, San Diego, USA, 2001, pp.B-45 - B-49 vol.1.

[36] K. C. Ho, L. Kovavisaruch and H. Parikh, “Source Localization Using TDOA with Erroneous Receiver Positions”, Proc. of the 2004 International Symposium on Circuits and Systems, Vancouver, Canada, May 2004, pp.III-453-6.

[37] B. Yıldırım, "A Comparative Evaluation of Conventional and Particle Filter Based Radar Target Tracking", M.Sc. Thesis, Middle East Technical University, November 2007.

APPENDIX A

A CLOSED FORM ESTIMATOR

In [2], a closed form estimator for TDOA localization problem has been proposed. It is stated in [1] that; the proposed solution is an approximate realization of the maximum likelihood estimator. For small TDOA errors, it is shown that the estimator approaches the CRLB [1]. The closed form estimator [1] is summarized here for large range-to-baseline ratios and the two dimensional localization.

The derivation begins with a priori location estimate $\mathbf{z}_a$, which is given by:

$$\mathbf{z}_a \approx (\mathbf{G}_a^T \mathbf{N}^{-1} \mathbf{G}_a)^{-1} \mathbf{G}_a^T \mathbf{N}^{-1} \mathbf{h}\,, \tag{A-1}$$

where $\mathbf{h}$ and $\mathbf{G}_a$ are defined as:

$$\mathbf{h} = 0.5 \begin{bmatrix} r_{2,1}^2 - (\mathbf{s}_2 - \mathbf{s}_1)^T \mathbf{s}_1 \\ r_{3,1}^2 - (\mathbf{s}_3 - \mathbf{s}_1)^T \mathbf{s}_1 \\ \vdots \\ r_{L,1}^2 - (\mathbf{s}_L - \mathbf{s}_1)^T \mathbf{s}_1 \end{bmatrix}, \quad \mathbf{G}_a = -\begin{bmatrix} (\mathbf{s}_2 - \mathbf{s}_1)^T & r_{2,1} \\ (\mathbf{s}_3 - \mathbf{s}_1)^T & r_{3,1} \\ \vdots & \vdots \\ (\mathbf{s}_L - \mathbf{s}_1)^T & r_{L,1} \end{bmatrix}. \tag{A-2}$$

where the position vector of the i^{th} sensor is denoted by $\mathbf{s}_i = [x_i \quad y_i \quad z_i]^T$ for $i = 1, 2, 3, \ldots, L$, $r_{j,i}$ is the RDOA measurement between the i^{th} and the j^{th} sensors respectively. $\mathbf{N}$ is the RDOA measurement covariance matrix.

Using $\mathbf{z}_a$, the posteriori location estimate $\mathbf{z}_a^{'}$ is calculated:

$$\mathbf{z}_a^{'} \approx (\mathbf{G}_a^{'T} \mathbf{B}^{'-1} \mathbf{G}_a \mathbf{N}^{-1} \mathbf{G}_a \mathbf{B}^{'-1} \mathbf{G}_a^{'})^{-1} (\mathbf{G}_a^{'T} \mathbf{B}^{'-1} \mathbf{G}_a \mathbf{N}^{-1} \mathbf{G}_a \mathbf{B}^{'-1}) \mathbf{h}^{'}, \tag{A-3}$$

where the quantities in (A-3) are:

$$\mathbf{G}_a^{'} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \mathbf{h}^{'} = \begin{bmatrix} (\mathbf{z}_{a,1} - x_1)^2 \\ (\mathbf{z}_{a,2} - y_1)^2 \\ \mathbf{z}_{a,3}^2 \end{bmatrix}, \mathbf{B}^{'} = \begin{bmatrix} \mathbf{z}_{a,1} - x_1 & 0 & 0 \\ 0 & \mathbf{z}_{a,2} - y_1 & 0 \\ 0 & 0 & \mathbf{z}_{a,3} \end{bmatrix}, \qquad \text{(A-4)}$$

where $\mathbf{z}_{a,i}$ denotes the i^{th} component.

Then using A-3, the final location estimate $\mathbf{z}_p$ is given by:

$$\mathbf{z}_p = sqrt(\mathbf{z}_a^{'}) + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \text{ or } \mathbf{z}_p = -sqrt(\mathbf{z}_a^{'}) + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}. \qquad \text{(A-5)}$$

where $\begin{bmatrix} x_1 & y_1 \end{bmatrix}^T$ denotes the location of the reference sensor and the sqrt() operator stands for the element wise square root operation. The value of $\mathbf{z}_p$, which lies in the region of interest, is selected as the solution.

In addition, the covariance of $\mathbf{z}_p$ is given as:

$$\mathrm{cov}(\mathbf{z}_p) = c^2 (\mathbf{B}^{''} \mathbf{G}_a^{'T} \mathbf{B}^{'-1} \mathbf{G}_a^{0T} \mathbf{B}^{-1} \mathbf{N}^{-1} \mathbf{B}^{-1} \mathbf{G}_a^{0} \mathbf{B}^{'-1} \mathbf{G}_a^{'} \mathbf{B}^{''})^{-1}, \qquad \text{(A-6)}$$

where $\mathbf{G}_a^0$ is the $\mathbf{G}_a$ matrix which is constructed using the noise free RDOAs. $\mathbf{B}^{''}$ and $\mathbf{B}$ are given as:

$$\mathbf{B}^{''} = \begin{bmatrix} (x^0 - x_1) & 0 \\ 0 & (y^0 - y_1) \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} l_2^0 & 0 & \cdots & 0 \\ 0 & l_3^0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & l_L^0 \end{bmatrix}, \qquad \text{(A-7)}$$

where $\begin{bmatrix} x^0 & y^0 \end{bmatrix}^T$ denotes the true position of the emitter and l_i^0 is the true distance between the i^{th} sensor and the emitter.

Note that, the expressions in the covariance of the estimator require the knowledge of the emitter's true location. In practice, the estimated location of the emitter can be used to calculate the covariance. As a result, the calculated covariance becomes an estimate of the true variance itself.

APPENDIX B

THE COVARIANCE OF THE POST PROCESSING ESTIMATOR

In the following, the expected value and the covariance of (3.12) is evaluated.

Using (3.12), the expected value of the estimate is found to be:

$$
\begin{aligned}
E\{\mathbf{p}_{ML} \mid \mathbf{p}\} &= E\left\{\left(\sum_{i=1}^{K}\mathbf{Q}_i^{-1}\right)^{-1}\left(\sum_{i=1}^{K}\mathbf{Q}_i^{-1}\hat{\mathbf{p}}_i\right)\right\}, \\
&= \left(\sum_{i=1}^{K}\mathbf{Q}_i^{-1}\right)^{-1}\left(\sum_{i=1}^{K}\mathbf{Q}_i^{-1}E\{\hat{\mathbf{p}}_i\}\right), \\
&= \mathbf{p}\cdot\left(\sum_{i=1}^{K}\mathbf{Q}_i^{-1}\right)^{-1}\left(\sum_{i=1}^{K}\mathbf{Q}_i^{-1}\right), \\
&= \mathbf{p}.
\end{aligned}
\tag{B-1}
$$

Since, the expected value of the estimator is equal to $\mathbf{p}$ itself, (3.12) is an unbiased estimator.

The covariance of the estimate in (3.12) can be calculated in the following way. Using (3.1), (3.12) can be expressed as:

$$
\begin{aligned}
\mathbf{p}_{ML} &= \left(\sum_{i=1}^{K}\mathbf{Q}_i^{-1}\right)^{-1}\left(\sum_{i=0}^{K}\mathbf{Q}_i^{-1}(\mathbf{p}+\mathbf{n}_i)\right), \\
&= \left(\sum_{i=1}^{K}\mathbf{Q}_i^{-1}\right)^{-1}\left(\sum_{i=1}^{K}\mathbf{Q}_i^{-1}\mathbf{p}\right)+\left(\sum_{i=1}^{K}\mathbf{Q}_i^{-1}\right)^{-1}\left(\sum_{i=1}^{K}\mathbf{Q}_i^{-1}\mathbf{n}_i\right).
\end{aligned}
\tag{B-2}
$$

Define $\boldsymbol{\eta}$ as:

$$\boldsymbol{\eta} = \left(\sum_{i=1}^{K} \mathbf{Q}_i^{-1} \right)^{-1} \left(\sum_{i=1}^{K} \mathbf{Q}_i^{-1} \mathbf{n}_i \right). \tag{B-3}$$

Since, summation with a constant does not affect statistical characteristics, the first term on the right hand side in (B.2) can be omitted. As a result, covariance of $\mathbf{p}_{ML}$ is equal to covariance of $\boldsymbol{\eta}$. Expected value of $\boldsymbol{\eta}$ is found as:

$$E\{\boldsymbol{\eta}\} = \left(\sum_{i=1}^{K} \mathbf{Q}_i^{-1} \right)^{-1} \left(\sum_{i=1}^{K} \mathbf{Q}_i^{-1} E\{\mathbf{n}_i\} \right) = 0. \tag{B-4}$$

Then, covariance of $\boldsymbol{\eta}$ is equal to:

$$\begin{aligned} E\{\boldsymbol{\eta}\boldsymbol{\eta}^T\} &= E\left\{ \left(\sum_{i=1}^{K} \mathbf{Q}_i^{-1} \right)^{-1} \left(\sum_{i=1}^{K} \mathbf{Q}_i^{-1} \mathbf{n}_i \right) \left(\left(\sum_{j=1}^{K} \mathbf{Q}_j^{-1} \right)^{-1} \left(\sum_{j=1}^{K} \mathbf{Q}_j^{-1} \mathbf{n}_j \right) \right)^T \right\}, \\ &= E\left\{ \left(\sum_{i=1}^{K} \mathbf{Q}_i^{-1} \right)^{-1} \left(\sum_{i=1}^{K} \mathbf{Q}_i^{-1} \mathbf{n}_i \right) \left(\sum_{j=1}^{K} \mathbf{n}_j^T (\mathbf{Q}_j^{-1})^T \right) \left(\left(\sum_{j=1}^{K} \mathbf{Q}_j^{-1} \right)^{-1} \right)^T \right\}. \end{aligned} \tag{B-5}$$

Since $\mathbf{n}_i$ are assumed to be (i.i.d) for $i = 0, 1, 2, \ldots, K$, cross terms are equal to zero when $i \neq j$ in (B-5). Hence:

$$\begin{aligned} E\{\boldsymbol{\eta}\boldsymbol{\eta}^T\} &= E\left\{ \left(\sum_{i=1}^{K} \mathbf{Q}_i^{-1} \right)^{-1} \left(\sum_{i=1}^{K} \mathbf{Q}_i^{-1} \mathbf{n}_i \mathbf{n}_i^T \mathbf{Q}_i^{-1} \right) \left(\sum_{j=1}^{K} \mathbf{Q}_j^{-1} \right)^{-1} \right\}., \\ &= \left(\sum_{i=1}^{K} \mathbf{Q}_i^{-1} \right)^{-1} \left(\sum_{i=1}^{K} \mathbf{Q}_i^{-1} E\{\mathbf{n}_i \mathbf{n}_i^T\} \mathbf{Q}_i^{-1} \right) \left(\sum_{j=1}^{K} \mathbf{Q}_j^{-1} \right)^{-1}, \\ &= \left(\sum_{i=1}^{K} \mathbf{Q}_i^{-1} \right)^{-1}. \end{aligned} \tag{B-6}$$

Using (B-6), covariance of (3.12) is found as:

$$\operatorname{var}(\mathbf{p}_{ML} \mid \mathbf{p}) = \operatorname{var}(\boldsymbol{\eta}) = \left(\sum_{i=1}^{K} \mathbf{Q}_i^{-1} \right)^{-1}. \tag{B-7}$$