ON FINITE GROUPS ADMITTING A FIXED POINT FREE ABELIAN OPERATOR GROUP WHOSE ORDER IS A PRODUCT OF THREE PRIMES

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ABSTRACT

ON FINITE GROUPS ADMITTING A FIXED POINT FREE ABELIAN OPERATOR GROUP WHOSE ORDER IS A PRODUCT OF THREE PRIMES

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A long-standing conjecture states that if A is a finite group acting fixed point freely on a finite solvable group G of order coprime to $|A|$, then the Fitting length of G is bounded by the length of the longest chain of subgroups of A. If A is nilpotent, it is expected that the conjecture is true without the coprimeness condition. We prove that the conjecture without the coprimeness condition is true when A is a cyclic group whose order is a product of three primes which are coprime to 6 and the Sylow 2-subgroups of G are abelian. We also prove that the conjecture without the coprimeness condition is true when A is an abelian group whose order is a product of three primes which are coprime to 6 and $|G|$ is odd.

Keywords: finite groups, fixed point free automorphisms, Fitting length.

ÖZ

MERTEBESİ ÜÇ ASAL SAYININ ÇARPIMI OLAN ABEL GRUPLARI SABİT NOKTASIZ OPERATOR GRUBU KABUL EDEN SONLU GRUPLAR

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Bir sonlu A grubu, $|G|$ ve $|A|$ aralarında asal olan sonlu ve çözülebilir bir G grubu üzerinde sabit noktasız etki ediyorsa, G grubunun Fitting uzunluğunun A grubunun altgrup dizileri içinde en uzun olanının uzunluğu ile sınırlanabilirliği uzun zamandır varolan açık bir sorudur. A grubunun nilpotent olması durumunda bu sorunun aralarında asallık şartı olmaksızın da doğru olması beklenmektedir. Biz mertebesi 6 ile aralarında asal üç asal sayının çarpımı olan döngüsel bir A grubunun Sylow 2-altgrupları abel olan bir sonlu G grubu üzerinde sabit noktasız etki etmesi durumunda bu varsayımın $|G|$ ve $|A|$ aralarında asal olmaksızın da doğru olduğunu kanıtladık. Bu sonucun bir çıkarımı olarak da, mertebesi 6 ile aralarında asal üç asal sayının çarpımı olan abel bir A grubunun, mertebesi tek sayı olan sonlu G grubu üzerinde sabit noktasız etki etmesi durumunda bu varsayımın $|G|$ ve $|A|$ aralarında asal olmaksızın da doğru olduğunu gösterdik.

Anahtar Kelimeler: sonlu gruplar, sabit noktası olmayan otomorfizmalar, Fitting uzunluğu

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To my lovely parents

TABLE OF CONTENTS

LIST OF SYMBOLS

$F(G)$	The Fitting subgroup of G
$\Phi(G)$	The Frattini subgroup of G
$[A, B]$	The commutator of A and B
$C_G(A)$	The centralizer of A in G
$N_G(A)$	The normalizer of A in G
$A \otimes_F B$	The tensor product of A and B over F
$f(G)$	The Fitting length of a group G
special p-group	An elementary abelian p-group, or a nonabelian p-group whose center, commutator subgroup and Frattini subgroup coincide and are elementary abelian
extraspecial p-group	A nonabelian special p-group whose center is of order p
$Stab_G(a)$	The Stabilizer of a in G

CHAPTER 1

INTRODUCTION

Let A be a subgroup of the automorphism group of a finite group G. The centralizer $C_G(A)$ of A in G is defined as $C_G(A) = \{g \in G \mid g^a = g \text{ for all } a \in A\ \}$. We say that A acts fixed point freely on G if the centralizer $C_G(A)$ is trivial. Assume further that $|A|$ and $|G|$ are relatively prime. Then by the classification of finite simple groups [11, Theorem 1.48], G is solvable. For a solvable group G, the Fitting subgroup $F(G)$ is defined as the subgroup generated by all normal nilpotent subgroups of G. When G is a finite solvable group, $F(G)$ is the largest normal nilpotent subgroup of G and it is nontrivial. Then the Fitting series of G is defined by

$$F_0(G) = 1 \text{ and } F_i(G)/F_{i-1}(G) = F(G/F_{i-1}(G))$$

for $i \geq 1$ and the Fitting length $f = f(G)$ is the least integer such that $F_f(G) = G$.

In 1900, Frobenius conjectured that G has to be nilpotent whenever A is a fixed point free automorphism of prime order. Fiftynine years later, Thompson settled this conjecture without assuming the solvability of G [18]. This theorem is the starting point for further investigation of the structure of finite groups G with a subgroup A of Aut$G$ such that $C_G(A) = 1$. Then Frobenious' conjecture has extended to the following form:

Conjecture A. *Let A be a finite group acting fixed point freely on a finite solvable group G of order coprime to $|A|$. Then the Fitting length $f(G)$ is bounded by the length of the longest chain of subgroups of A, denoted by $l(A)$.*

If A is solvable, then $l(A)$ is exactly the number of primes (counted with multiplicities) dividing $|A|$, because of the fact that an abelian series, a series whose factors are abelian for a finite group, can be refined to a composition series whose factors are of prime order. Thus, the length of the longest chain of subgroups of A is exactly the number of primes (not neccesarily distinct) dividing $|A|$.

Berger made a great progress towards the answer of this problem. He settled Conjecture A if A is a nilpotent group which has no section isomorphic to the wreath product of two groups of order p for any prime p, [2]. More generally, for a solvable group G, Thompson proved that $f(G) \leq 5^{l(A)} f(C_G(A))$ where A is a solvable subgroup of Aut$G$ with $(|G|, |A|) = 1$ (see [19]). This result was improved by Kurzweil as $f(G) \leq f(C_G(A)) + 4l(A)$ [21].

In 1984 Turull [24, Theorem 3.3] showed that given A up to isomorphism, we can find infinitely many G's such that $f(G)$ is exactly the number of primes dividing the order of A counted with multiplicities. In other words, if the inequality $f(G) \leq l(A)$ is true, then it is the best possible bound. We shall explain this result in Chapter 2. In the same year, Turull also found a bound including the case where the group of automorphism does have some fixed points. More precisely, he proved that if G is a finite solvable group and A is a solvable group of automorphisms of G with $(|G|, |A|) = 1$, then $f(G) \leq 2l(A) + f(C_G(A))$ as an improvement of Thompson's work (see [25]). Of course as a corollary of this result we have $f(G) \leq 2l(A)$ if A acts fixed point freely on G . In 1986, Turull handled the problem under the assumption that A acts with regular orbits and proved the following: if a finite group A acts fixed point freely and coprimely on a finite (solvable) group G, and also if A acts with regular orbits, then $f(G) \leq l(A)$ [26]. In other words Conjecture A is true when A acts with regular orbits. A finite group A acts with regular orbits if for every proper subgroup B of A and every elementary abelian B-invariant section S of G, B has a regular orbit on S, i.e. there is $v \in S$ such that $C_B(v) = C_B(S)$. For example, an abelian group A acts with regular orbits on G if $(|A|, |G|) = 1$. To see this, it suffices to show that for each $B \leq A$ and for each irreducible elementary abelian B-invariant section S of G, there exists $v \in S$ such that $C_B(v) = C_B(S)$. Let $0 \neq v \in S$. If $C_B(v) \gneqq C_B(S)$, then there exists $b \in C_B(v) - C_B(S)$. Now $C_S(b) = 1$ as $C_S(b)$ is a B-invariant subgroup of S, a contradiction.

Later on Turull obtained a stronger result in 1990, namely he showed in [27] that if A is a finite group acting on a finite solvable group G such that $(|G|, |A|) = 1$ and if A acts with regular orbits, then $f(G) \leq l(A) + l(C_G(A))$.

However without the coprimeness condition, there is still not much known. We should immediately note that some coprimeness condition is necessary in general, since it was shown by Bell and Hartley in [1] that any finite non-nilpotent group can act fixed point freely on solvable groups with arbitrarily large Fitting length with $(|G|, |A|) \neq 1$. Hence we expect that the conjecture is true when the coprimeness condition is replaced by the assumption that A is nilpotent. This question is still unsettled and can be stated as the following modified version of Conjecture A.

Conjecture B. *Let A be a finite nilpotent group acting fixed point freely on a finite* *solvable group G. Then $f(G) \leq l(A)$.*

In [20], Kei-Nah showed that Conjecture B is true in the case where A is a cyclic group whose order is a product of two distinct primes. After almost two decades, Ercan and Güloğlu obtained a result that takes Kei-Nah's work one step further by setting the Conjecture B when A is cyclic of order pqr for pairwise distinct primes p, q and r. Namely they showed that the Fitting length of G is at most 3 in this case [4].

Apart from these, fixed point free automorphisms are closely related to Carter subgroups of solvable groups. A Carter subgroup is any nilpotent self-normalizing subgroup of a finite solvable group. In a rather lengthy and well-known paper, Dade proved that there is an exponential function e such that if G is a finite solvable group and C is its Carter subgroup, then $f(G) \leq e(l(C))$ [3]. Dade conjectured in the same paper that perhaps one could show that there is a linear bound. Even though Dade made this conjecture in 1969, it is still not known whether Dade's conjecture is true or false. A special case of Dade's conjecture is that there is a linear function h such that $f(G) \leq h(l(C))$ whenever C is a nilpotent finite group which acts fixed point freely on the solvable finite group G. Indeed, under these hypothesis, C is a self-normalizing nilpotent subgroup of GC, that is, a Carter subgroup of GC: For if $N = N_G(C)$, then $[N, C] \leq N \cap C = 1$ and so C is self-normalizing in GC. Hence any answer to Dade's conjecture is also an upper bound for $f(G)$ under the hypothesis of Conjecture B. To prove the conjecture of Dade, some additional conditions are imposed on the groups G and C. For example in 1995, Turull proved that if C is a finite abelian group with squarefree odd exponent acting fixed point freely on the finite solvable group G, then $f(G) \leq 5l(C)$ [28]. In 2008, Ercan and Güloğlu improved this result by showing that the truth of Conjecture B under some additional hypothesis. More precisely they obtained the following two results [5]:

- Let A be a finite abelian group acting fixed point freely on a finite group G of odd order. If A has squarefree exponent coprime to 6, then $f(G) \leq l(A)$.
- Let G be a finite (solvable) group of order coprime to 6. If C is a Carter subgroup of G, then $f(G) \leq 2(2^{l(C)} - 1)$.

Note that these two results are established without the coprimeness condition. The first one improves Turull's bound given in Theorem 3.4 of [28] and the second one improves the bound given in Theorem 8.5 of Dade's paper [3].

In this thesis, we studied a minimal configuration in which A has nonsquarefree exponent and prove the following result:

Theorem. *Let G be a finite group admitting a fixed point free automorphism $\langle \alpha \rangle$ whose order is a product of three primes which are coprime to 6. If the Sylow 2-subgroups of G are abelian, then G has Fitting length at most 3.*

As we pointed out the importance of this result is the possibility of allowing nonsquarefree exponents. To be precise, it settles Conjecture B when A is cyclic whose order is a product of three primes under an additional hypothesis on G. However if A is a p-group for some prime p, then because of its fixed point free action on G, $|A|$ and $|G|$ must be coprime (Proposition 2.1.8). Since A acts with regular orbits by a previous remark, a result of Turull shows that $f(G) \leq l(A)$ [26]. Moreover, as we mentioned before, by a result due to Ercan and Güloğlu, if the order of A is a product of three distinct primes, then the Fitting length of G is at most 3 even without assuming that $|A|$ is coprime to 6 and the Sylow 2-subgroups of G are abelian [4]. Also the last result mentioned above due to Ercan and Güloğlu (2008), establishes that $f(G) \leq 3$ when A is abelian of square free exponent coprime to 6 and G is of odd order. Thus, the problem is reduced to the case where A is cyclic of order p^2q where p and q are distinct primes. In our main theorem we study the case under a weaker condition on G by assuming that Sylow 2-subgroups of G are abelian. As an immediate consequence of our main result we state the following Corollary.

Corollary. *Let A be a finite abelian group whose order is a product of three primes coprime to 6. Assume that A acts fixed point freely on a finite group G of odd order. Then $f(G) \leq 3$.*

When $|GA|$ is divisible by the primes 2 and 3, it is a well-known fact that the study of such problems needs much more effort. Therefore our theorem requires $|A|$ to be coprime to 6 and G to have abelian Sylow 2-subgroups.

The outline of the thesis is as follows:

Section 1 and 2 of Chapter 2 give the necessary preparation from group theory and their representations. Most of them are well known results which will be referred throughout the thesis. Section 3 of Chapter 2 contains a collection of three important results due to Shult, Gagola and Gross which will be referred throughout the thesis. Section 4 of Chapter 2 contains an example due to Turull [24], which guarantees that the bound given in our main result is the best possible bound.

Chapter 3 includes our technical results pertaining the proof of the main result.

Finally in Chapter 4, we state and prove the main result. A contradiction will be deduced over a series of steps, from the assumption of the existence of a counterexample. This chapter also contains a corollary.

The main result of this thesis is from the article [6].

CHAPTER 2

PREPARATORY CHAPTER

2.1 Group Theoretical Part

In this section we shall give necessary preparation from group theory. Most of them are well known results which will be referred throughout the thesis.

Proposition 2.1.1. *(a) (Frattini Argument) Let H be a normal subgroup of G and P be a Sylow p-subgroup of H. Then $G = HN_G(P)$.*

(b) (The Three Subgroup Lemma) Let A, B and C be three subgroups of G such that $[A, B, C] = 1$, $[B, C, A] = 1$. Then $[C, A, B] = 1$.

Proof. [10] 1.3.7 and 2.2.3.

Proposition 2.1.2. *For a solvable group G we have $C_G(F(G)) \leq F(G)$ and equality holds if $F(G)$ is an abelian group.*

Proof. [10] 6.1.3.

Proposition 2.1.3. *A minimal normal subgroup of a solvable group G is an elementary abelian p-subgroup for some prime p.*

Proof. [10] 2.4.1 (v)

Proposition 2.1.4. *Let G be a finite solvable group. Then $\Phi(F(G)) \leq \Phi(G)$. Also $F(G/\Phi(G)) = F(G)/\Phi(G)$.*

Proof. [10] 6.1.6 (ii).

Proposition 2.1.5. *The Frattini factor group $P/\Phi(P)$ of a p-group P is elementary abelian. Furthermore, $\Phi(P) = 1$ if and only if P is elementary abelian.*

Proof. [10] 5.1.3.

Proposition 2.1.6. *Let G be a finite solvable group, $K \lhd G$ and $K \leq O_r(G)$. Then $O_r(G)/K = O_r(G/K)$.*

Proof. Let $X/K = O_r(G/K)$. As $K \leq O_r(G)$ we get that X is an r-group. Since $X/K \lhd G/K$ we have $X \lhd G$ and hence $X \leq O_r(G)$. Thus $X/K \leq O_r(G)/K \leq X/K$. Hence $X/K = O_r(G)/K$. $\square$

Proposition 2.1.7. *Let H be a solvable group of Fitting length n. Then $f(H/F_i(H)) = n-i$ and $F_j(H/F_i(H)) = F_{j+i}(H)/F_i(H)$ for all $1 \leq j \leq n-i$, $1 \leq i < n$.*

Proof. Consider the Fitting series of H, which is

$$1 \lhd F(H) \lhd F_2(H) \lhd \ldots \lhd F_i(H) \lhd F_{i+1}(H) \lhd \ldots \lhd F_n(H) = H$$

and let $\overline{H} = H/F_i(H)$. Then we have

$$1 = \overline{F_i(H)} \lhd \overline{F_{i+1}(H)} \lhd \ldots \lhd \overline{F_n(H)} = \overline{H}$$

Now $F(\overline{H}) = F(H/F_i(H)) = F_{i+1}(H)/F_i(H) = \overline{F_{i+1}(H)}$.

Now suppose that $F_{j-1}(\overline{H}) = \overline{F_{j-1+i}(H)}$. Then

$$\begin{aligned}
F_j(\overline{H})/F_{j-1}(\overline{H}) = F(\overline{H}/F_{j-1}(\overline{H})) &= F(\overline{H}/\overline{F_{j-1+i}(H)}) \\
&= F(H/F_i(H)/F_{j-1+i}(H)/F_i(H)) \\
\cong F(H/F_{j-1+i}(H)) &= F_{j+i}(H)/F_{j-1+i}(H) \\
&\cong F_{j+i}(H)/F_i(H)/F_{j-1+i}(H)/F_i(H) \\
&= \overline{F_{j+i}(H)}/\overline{F_{j-1+i}(H)}
\end{aligned}$$

and hence $F_j(\overline{H}) = \overline{F_{j+i}(H)}$. Thus by induction we get that $F_j(\overline{H}) = \overline{F_{j+i}(H)}$, for all $1 \leq j \leq n-i$. Hence $F_{n-i}(\overline{H}) = \overline{F_{n-i+i}(H)} = \overline{F_n(H)} = \overline{H}$ and $f(\overline{H}) = n-i$. This proof also shows that $F_j(H/F_i(H)) = F_{j+i}(H)/F_i(H)$. $\square$

Proposition 2.1.8. *Let A be a p-group of automorphisms of a group G with the property that $C_G(A) = 1$. Then G is a p'-group.*

Proof. [10] 6.2.3.

Proposition 2.1.9. *Let A be a p'-group of automorphisms of the p-group P.*

(a) $[P, A, A] = [P, A]$. *In particular, if* $[P, A, A] = 1$, *then* $A = 1$.

(b) $P = [P, A].C_P(A)$. *In particular if* $[P, A] \leq \Phi(P)$, *then* $A = 1$.

(c) If P is abelian, then $P = [P, A] \oplus C_P(A)$.

Proof. [10] 5.3.6, 5.3.5 and 5.2.3.

Proposition 2.1.10. *Let ϕ be a fixed point free automorphism of a group G. Then every element of G can be expressed in the form $x^{-1}(x\phi)$ and $(x\phi)x^{-1}$ for suitable x in G.*

Proof. [10] 10.1.1.(i).

Proposition 2.1.11. *If ϕ is a fixed point free automorphism of G, then ϕ leaves invariant a unique Sylow p-subgroup P of G for each prime p in $\pi(G)$. Furthermore, P contains every ϕ-invariant p-subgroup of G.*

Proof. [10] 10.1.2.

Proposition 2.1.12. *Let ϕ be a fixed point free automorphism of a group G and let H be a ϕ-invariant normal subgroup of G. Then ϕ induces a fixed point free automorphism of G/H.*

Proof. [10] 10.1.3.

Proposition 2.1.13. *(Thompson) If a group G admits a fixed point free automorphism of prime order then G is a nilpotent group.*

Proof. [10] 10.2.1.

Definition 2.1.1. *(Definition 1.1, [25]) We say that a sequence of B-invariant subgroups of G $(\hat{P}_i)$, $i = 1, \ldots, h$, is a B-tower of G if the following are satisfied:*

(1) $\pi(\hat{P}_i) = \{p_i\}$ consists of a single prime for $i = 1, \ldots, h$;

(2) $\hat{P}_i$ normalizes $\hat{P}_j$, for $i < j$;

(3) We set $P_h = \hat{P}_h$ and $P_i = \hat{P}_i / C_{\hat{P}_i}(P_{i+1})$, $i = 1, \ldots, h-1$, and P_i is not trivial for $i = 1, \ldots, h$;

(4) $p_i \neq p_{i+1}$, $i = 1, \ldots, h-1$.

h is called the height of the tower.

Proposition 2.1.14. *(Theorem 3.1, [25]) Let A be a group of prime order acting on a group G with $(|A|, |G|) = 1$. Let $(\hat{P}_i)$, $i = 1, \ldots, h$, be a A-tower and assume that A centralizes $\hat{P}_k$ (possibly with $k = 0$ and $\hat{P}_k = 1$). Then there exists a $j \geq k$ such that $(C_{\hat{P}_i}(A))$, $i = 1, \ldots, j-1, j+1, \ldots, h$ satisfies conditions (1), (2), (3) of the definition of A-tower. If $2 \nmid |\hat{P}_k$ we may take $j > k$.*

Proposition 2.1.15. *(Thompson $A \times B$ Lemma) Let H be a finite group that acts as an automorphisms on a finite p-group G, and suppose that $H = A \times B$ is an internal direct* *product of a p-group A and a p'-group B. Suppose that B fixes every element of G that A fixes. Then B acts trivially.*

Proof. [10] Theorem 5.3.4.

2.2 Representation Theoretical Part

In this section we shall give necessary preparation from representation theory. Most of them are well known results which will be referred throughout the thesis.

Proposition 2.2.1. *Every irreducible representation of a p-group on a vector space over a field of characteristic p is trivial. Equivalently a nontrivial p-group does not possess a faithful irreducible representation on a vector space over a field of characteristic p.*

Proof. [10] 3.1.2.

Proposition 2.2.2. *If G possesses a faithful irreducible representation on a vector space over a field of characteristic p, then G has no nontrivial normal p-subgroups.*

Proof. [10] 3.1.3.

Proposition 2.2.3. *If D is an irreducible representation of an abelian group G with kernel K, then G/K is a cyclic group. In particular a noncyclic abelian group does not possess a faithful irreducible representation.*

Proof. [10] 3.2.3.

Proposition 2.2.4. *If a group G possesses a faithful irreducible representation, then it has a cyclic center.*

Proof. [10] 3.2.2.

Proposition 2.2.5. *Let ϕ be a representation of G on a vector space V over a field F. Assume that either F is of characteristic 0 or of characteristic prime to $|G|$. Suppose*

$$V = V_1 \supset V_2 \supset \ldots \supset V_{n+1} = 0$$

is sequence of $\phi(G)$-invariant subspaces such that $\phi(G)$ acts trivially on each V_i/V_{i+1}. Then ϕ is the trivial representation on V.

Proof. [10] Theorem 3.3.4.

Proposition 2.2.6. *(Maschke's theorem) Let V be a kG-module and assume that either k is of characteristic 0 or relatively prime to $|G|$. Then V is completely reducible.*

Proof. [10] 3.3.1

Proposition 2.2.7. *(Clifford's theorem) Let V be an irreducible FG-module and let H be a normal sugbroup of G. Then V is the direct sum of H-invariant subspaces V_i, $1 \leq i \leq r$, which satisfy the following conditions:*

(i) $V_i = X_{i1} \oplus X_{i2} \oplus \ldots \oplus X_{it}$, where each X_{ij} is an irreducible H-submodule, $1 \leq i \leq r$, t is independent of i, and X_{ij}, $X_{i'j'}$ are isomorphic H-submodules if and only if $i = i'$.

(ii) For any H-submodule U of V, we have $U = U_1 \oplus U_2 \oplus \ldots \oplus U_r$, where $U_i = U \cap V_i$, $1 \leq i \leq r$. In particular, any irreducible H-submodule of V lies in one of the V_i.

(iii) For x in G, the mapping $\pi(x) : V_i \to V_i x$, $1 \leq i \leq r$, is a permutation of the set $S = \{V_1, V_2, \ldots, V_r\}$ and π induces a transitive permutation representation of G on S. Furthermore, $HC_G(H)$ is contained in the kernel π.

The subspaces V_i, $1 \leq i \leq r$, are often referred to as the Wedderburn components of V with respect to H.

Proof. [10] 3.4.1.

Definition 2.2.1. *Let R be the ring of algebraic integers in $\mathbb{C}$. Fix a prime p and choose a maximal ideal M of R containing pR. Let $F = R/M$ be a field of characteristic p, and let*

$$^* : R \to F$$

be the natural ring homomorphism.

Let us set $U = \{\epsilon \in \mathbb{C} | \, \epsilon^m = 1 \text{ for some integer } m \text{ not divisible by } p\} \subseteq R$, the multiplicative group of p'-roots of unity.

Let G^0 be the set of p-regular elements of a finite group G (that is, the set of elements $g \in G$ such that the order of g is not divisible by p). Suppose that $\chi : G \to GL(n, F)$ is a representation of G. If $g \in G^0$, then since the restriction of * to U is an isomorphism $U \to F^\times$ of multiplicative groups (see [17, Lemma 2.1]), all the eigenvalues of $\chi(g)$ (which lie in $F^\times$ since F is algebraically closed (see [17, Lemma 2.1]) and $\chi(g)$ is an invertible matrix) are of the form $\epsilon_1^*, \ldots, \epsilon_n^*$ for uniquely determined $\epsilon_1, \ldots, \epsilon_n \in U$. We say that $\varphi : G \to \mathbb{C}$ defined for $g \in G^0$ by

$$\varphi(g) = \epsilon_1 + \ldots + \epsilon_n$$

is the Brauer character of G afforded by the representation χ. Notice that φ is uniquely determined (once M has been chosen) by the equivalence class of the representation χ. We say that φ is irreducible if χ is irreducible.

Proposition 2.2.8. *(Fong-Swan Theorem) Let G be a finite p-solvable group and φ be an irreducible Brauer character of G, then there must exist an ordinary irreducible character of*

G such that $\chi^o = \varphi$ where ^{o} denotes restriction to the set of p-regular elements of G. Such character is called a lift of φ.

Proof. [17] Theorem 10.1 (Fong-Swan).

Proposition 2.2.9. *Let P be a p-subgroup of a group G and $Q \lhd A \leq N_G(P)$. Suppose that p does not divide the order of Q and $Q \nleq C_G(P)$. Let P_0 be a minimal element of the set $\{V | V \leq P, A \leq N_G(V), Q \nleq C_G(V)\}$. Then*

(i) $P_0/\Phi(P_0)$ is an irreducible A-module with $[Q, P_0] = P_0$,

(ii) $[\Phi(P_0), Q] = 1$,

(iii) P_0 is a special group.

Proof. For any A-invariant proper subgroup W of P_0 we get $[W, Q] = 1$. In particular $[Q, \Phi(P_0)] = 1$ and (ii) follows. Let $\overline{P} = P_0/\Phi(P_0)$. Then $\overline{P} = [\overline{P}, Q] \oplus C_{\overline{P}}(Q)$ by Proposition 2.1.9 (c). If $[\overline{P}, Q] < \overline{P}$, then by Proposition 2.1.9 (a), $[\overline{P}, Q] = [\overline{P}, Q, Q] < [\overline{P}, Q]$ and so $[\overline{P}, Q] = 1$. It follows that $[P_0, Q] = 1$ by Proposition 2.1.9 (d) which is not possible. Thus $[\overline{P}, Q] = \overline{P}$ and $C_{\overline{P}}(Q) = 1$. Now let $1 \neq \overline{W} = W/\Phi(P_0)$ be an A-invariant submodule of $\overline{P}$. If $\overline{W} < \overline{P}$, then $W < P_0$ and we have $[W, Q] = 1$. This gives that $[\overline{W}, Q] = 1$ and hence $\overline{W} \leq C_{\overline{P}}(Q) = 1$ which is not possible. This shows that $\overline{P}$ is an irreducible A-module. Now $[\overline{P}, Q] = \overline{P}$, so we have $[P_0, Q]\Phi(P_0) = P_0$ and hence $[P_0, Q] = P_0$.

Now suppose that P_0 is not elementary abelian group i.e. $\Phi(P_0) \neq 1$. Since $[Q, \Phi(P_0)] = 1$ we get that $[Q, \Phi(P_0), P_0] = 1$. Also since $[\Phi(P_0), P_0, Q] = 1$ by the three subgroup lemma 2.1.1 (b) we have $[P_0, Q, \Phi(P_0)] = [P_0, \Phi(P_0)] = 1$. Thus $\Phi(P_0) \leq Z(P_0)$. Now $Z(P_0)/\Phi(P_0)$ is an A-invariant submodule of the irreducible A-module $\overline{P} = P_0/\Phi(P_0)$. As P_0 is nonabelian group, we must have $Z(P_0)/\Phi(P_0) = 1$ and hence $Z(P_0) \leq \Phi(P_0)$. Thus $Z(P_0) = \Phi(P_0)$. Since $P_0/\Phi(P_0) = \overline{P}$ is an abelian group, then $P_0' \leq \Phi(P_0)$ As $[P_0, Q] = P_0$ we get $[P_0/P_0', Q] = P_0/P_0'$. But P_0/P_0' is an abelian group, so $P_0/P_0' = [P_0/P_0', Q] \oplus C_{P_0/P_0'}(Q)$ by Proposition 2.1.9 (c) and hence $C_{P_0/P_0'}(Q) = 1$ or $C_{P_0}(Q) \leq P_0'$. Thus $\Phi(P_0) = C_{P_0}(Q) \leq P_0'$ and hence $P_0' = \Phi(P_0)$. Thus $P_0' = \Phi(P_0) = Z(P_0)$.

Now let $x, y \in P_0$ and $z = [x, y]$, then $z \in P_0' = Z(P_0)$ and hence $[z, \langle x, y \rangle] = 1$. Then $z^p = [x, y^p]$. But $y^p \in Z(P_0)$ as $P_o/Z(P_0)$ is an elementary abelian group. Thus $z^p = [x, y]^p = [x, y^p] = 1$. As P_0' is an abelian group which is generated by elements of order p, then $P_0' = \Phi(P_0) = Z(P_0)$ is an elementary abelian group. Therefore P_0 is special. □

Proposition 2.2.10. *Let $N \lhd G$ with G/N cyclic and let ϑ be an irreducible character of N which is invariant in G. Then ϑ is extendible to G.*

Proof. [16] 11.22.

Proposition 2.2.11. *Let V be a simple kG-module, where k is a field.*

(a) If $N \leq Z(G)$, then $V|_N$ is homogeneous.

(b) If V is faithful for $Z(G)$, then $Z(G)$ is cyclic.

Proof. [12] Corollary B.9.4.

Proposition 2.2.12. *Let $G = HA$ be a finite group where $H \lhd G$ and $H \cap A = 1$, and let V be a finite dimensional, irreducible KG-module for some field K. Let W be a Wedderburn component of V with respect to H. Then the stability group of W in G is HB, where B is a subgroup of A, and W is an irreducible KHB-module. Furthermore, if A acts fixed point freely on V, then B acts fixed point freely on W*

Proof. The first statement follows directly from Theorem 2.2.7. Next assume that $C_W(B) \neq 0$. Let $t = |HA : HB|$ be the number of Wedderburn components of V with respect to H. Since $|HA : HB| = |A : B|$, we can write $V|_H = W^{a_1} \oplus W^{a_2} \oplus \ldots \oplus W^{a_t}$ where $A = \cup_{i=1}^t Ba_i$. Now there is a nontrivial element $w \in C_W(B)$ such that $\sum_{i=1}^t w^{a_i} \neq 0$. Since $(w^{a_1} + w^{a_2} + \ldots + w^{a_t})^a = w^{a_1 a} + w^{a_2 a} + \ldots + w^{a_t a}$ and $A = \cup_{i=1}^t Ba_i$, we can write $w^{a_1 a} + w^{a_2 a} + \ldots + w^{a_t a} = w^{b_1 a_{s_1}} + w^{b_2 a_{s_2}} + \ldots + w^{b_t a_{s_t}} = w^{a_1} + w^{a_2} + \ldots + w^{a_t}$ as $w \in C_W(B)$ and $b_i \in B$ for $i = 1, 2, \ldots, t$. Thus $\sum_{i=1}^t w^{a_i} \in C_V(A)$, which is impossible. □

Proposition 2.2.13. *([15, Theorem]) Let A act on a solvable group H and suppose that A has a normal p-complement for every prime $p \mid |H|$. Then the following are equivalent:*

(1) A fixes an irreducible character $\neq 1$ of H,

(2) A fixes an element $\neq 1$ of H,

(3) A fixes a class $\neq 1$ of H.

Proposition 2.2.14. *([25, Theorem 2.1]) Let $B \neq 1$ be a cyclic r-group for some prime r. Let $A = \Omega_1(B)$. Suppose $G \lhd GB$, $r \nmid |G|$ and $P \subseteq G$, $P \lhd GB$ is a p-group (p a prime) such that*

(1) $\Phi(\Phi(P)) = 1$;

(2) $\Phi(P) \subseteq Z(P)$;

(3) If $p \neq 2$, $exp(P)$=p.

Let k be a field such that $char(k) \nmid rp$. Let M be a kGB-module. Set $P_0 = \cap ker\hat{M}$, where $\hat{M}$ ranges through the irreducible P-submodules $\hat{M}$ of $M|_P$ such that $[A, P]$ is not trivial on

$\hat{M}$. *Assume* $P \neq P_0$. *Let* Ω *be a* GB*-stable subset of* M^* *(the dual of* M*) which linearly spans* M^*. *We set*

$$M_0 = \{v \in M : \text{ for every } f \in \Omega - C_\Omega(A)\ , f(v) = 0\}.$$

Let $T \subseteq C_G(B) \cap C_G(\Phi(P))$ *be a* t*-subgroup (*t *a prime). Assume that* $T \subseteq P$ *and if* $\Phi(P) \neq 1$ *there is* H *an* h*-subgroup (*h *a prime) such that* $H \subseteq C_G(\Phi(P))$*,* B *normalizes* H*,* $H/C_H(P)$ *is elementary abelian and* $[H, P] = P$. *We set* $T_0 = P_0 \cap T$.

Then $C_M \nsubseteq M_0$ *and* $T_0 \supseteq C_T(C_M(B)/C_{M_0}(B))$.

Proposition 2.2.15. *([12, Theorem 9.18]) Let* E *be an extraspecial* p*-group of order* p^{2t+1}*,* H *be a cyclic* p'*-subgroup of* $AutG$*, and* G *denote the semi direct product of* EH. *Assume that* H *acts regularly on* $E/Z(E)$ *and trivially on* $Z(E)$. *Let* K *be a field containing a primitive pth root of unity whose characteristic does not divide* $|G|$*, and let* V *be a* KG*-module such that* V_E *is irreducible and faithful for* E. *Then there exists* $\delta = \pm 1$ *such that* $|H|$ *divides* $p^t - \delta$ *and a* 1*-dimensional* KH*-module* U *such that*

(i) if $\delta = 1$*, then* $V_H \cong m(KH) \oplus U$*, and*

(ii) if $\delta = -1$*, then* $V_H \oplus U \cong m(KH)$

where $m = (p^t - \delta)/|H|$ *and* $m(KH)$ *denotes the direct sum of* m *copies of the regular module* KH.

Definition 2.2.2. *Let* G *be a solvable group and* A *act on* G. *A subgroup* P *is called a generating* A*-support subgroup of* G *if:*

1) $P \lhd GA$*,* $P \subseteq G$ *and* P *is a* p*-group for some prime* p.

2) There are GA*-invariant subgroups* P_1 *and* H *such that*

A) $P_1 \subseteq Z(P)$*,* P/P_1 *is elementary abelian and* GA*-completely reducible,*

B) $H \subseteq C_G(P_1)$*,*

C) $H/H \cap C_G(P/P_1)$ *is elementary abelian for some prime* r*,*

D) H *acts nontrivially on each* H*-chief factor of* P/P_1.

We call the A*-support of* G *(denoted* $supp_A(G)$*) the subgroup generated by all subgroups* $S \subseteq G$ *such that* $S \lhd GA$ *and either* S *is abelian or a generating* A*-support subgroup of* G.

Definition 2.2.3. *Let* A *be a finite group and* π *a set of primes. We say that* A *is* π*-regular if:*

1) $\pi(A) \cap \pi = \emptyset$*;*

2) For any $p \in \pi$ *and any elementary abelian* p*-group* H *on which* A *acts and any section* S *of* HA*, if all abelian normal subgroups of* S *are cyclic,* S *has a self-centralizing cyclic* *normal subgroups;*

3) For any section S of A and any chief factor X of S, $S/C_S(X)$ has a regular orbit on X;

4) If $\{3,5\} \subseteq \pi$, any chief 2-factor of A is cyclic and if further $8||A|$, either A is supersolvable or it has a normal Sylow 2-subgroups;

5) No section of A is isomorphic to $\mathbb{Z}_r \int \mathbb{Z}_s$ (any $r, s > 1$) or to $GN(\epsilon, p^n, q)$ where $p \in \pi$, $n \geqq 1$ is an integer, $q \mid |Gal(\epsilon, p^n)|$ is a prime and if $\epsilon \neq 1$ $\pi(F_\epsilon(p^n)) \cap \pi \neq \emptyset$.

Proposition 2.2.16. *[22, Proposition 4.5] Let GA be a finite group where $G \triangleleft GA$ is solvable and V a kGA-module. Assume the following:*

1) k is a splitting field for all subgroups of GA;

2) $V|_G$ is homogeneous and faithful;

3) $C_V(A) = 0$;

4) A is $char(k) \cup \pi(G)$-regular.

Then

$$C_V(C_A(supp_A(G))) = 0.$$

Proposition 2.2.17. *Let G be a group, k be a field and V be a faithful and irreducible kG-module. Let $\overline{k}$ be an extension of k and $\overline{V} = V \otimes_k \overline{k}$. Let N be an irreducible $\overline{k}G$-submodule of $\overline{V}$. Then*

1) Let $H \leq G$. Then $C_{\overline{V}}(H) = C_V(H) \otimes_k \overline{k}$. In particular, $C_{\overline{V}}(x) = C_V(x) \otimes_k \overline{k}$ for any $x \in G$.

2) N is faithful as $\overline{k}G$-module.

Proof. 1) $\overline{V} = V \otimes_k K$ is a left $\overline{k}G$-module, where $(ag)(v \otimes x) = gv \otimes ax$, for any $a, x \in \overline{k}$, $g \in G$, $v \in V$. Let $\{v_1, \dots v_n\}$ be a basis for V over k and let B be a basis for $\overline{k}$ over k. Then we have

$$\overline{V} = V \otimes_k (\sum_{b \in B} bk) = \sum_{b \in B} (V \otimes_k bk) = \sum_{b \in B} (V \otimes_k b)$$

Let now $\overline{v} \in C_{\overline{V}}(H)$. Then there exist pairwise different elements $b_1, \dots, b_s$ in B such that $\overline{v} = \sum_{i=1}^s x_i \otimes b_i$ with suitable $x_i \in V$, $i = 1, \dots, s$ and we have $h\overline{v} = \sum_{i=1}^s hx_i \otimes b_i = \sum_{i=1}^s x_i \otimes b_i$ for any $h \in H$.

Because of the direct sum decomposition $\overline{V} = \sum_{b \in B} (V \otimes_k b)$, the above equation yields that $hx_i \otimes b_i = x_i \otimes b_i$. Since we are considering the tensor product of two vector spaces, the equation $hx_i \otimes b_i - x_i \otimes b_i = (hx_i - x_i) \otimes b_i = 0$ gives that $hx_i - x_i = 0$ for any $i = 1, \dots, s$;

i.e. $x_i \in C_V(h)$ for any $h \in H$. Thus we get $\overline{v} \in C_V(H) \otimes_k \overline{k}$. Obviously we also have $C_V(H) \otimes_k \overline{k} \leq C_{\overline{V}}(H)$. Hence $C_{\overline{V}}(H) = C_V(H) \otimes_k \overline{k}$.

2) Let K be the kernel of the representation of G afforded by N. For $x \in K$, $N \leq C_{\overline{V}}(x) = C_V(x) \otimes_k \overline{k}$ by part 1). Hence $C_V(K) \neq 1$. As $C_V(K)$ is normalized by G, we have $C_V(K) = V$, because of the irreducibility of V. As V is a faithful kG-module, $K = 1$, i.e. N is a faithful $\overline{k}G$-module. □

Definition 2.2.4. *Let X, Y be subgroups of a group G. For an element $g \in G$ a set of the form*

$$XgY = \{xgy : x \in X, y \in Y\}$$

is called an (X, Y)-double coset of G.

Proposition 2.2.18. *(Mackey's theorem) Let X and Y be subgroups of a group G, and let $\{g_1, \ldots, g_m\}$ be a full set of (X, Y)-coset representatives of G. If V is a KX-module, then*

$$(V^G)_Y \cong \bigoplus_{i=1}^{m} ((V \otimes g_i)_{X^{g_i} \cap Y})^Y.$$

Here $V \otimes g_i$ is viewed as a $K(X^{g_i} \cap Y)$-module via the action

$$(v \otimes g_i)x^{g_i} = vx \otimes g_i$$

for all $x^{g_i} \in X^{g_i} \cap Y$.

Proof. [12] B.6.20.

2.3 Theorems which will be used througout the thesis

A theorem due to Shult

Lemma 2.3.1. *Let N be a normal subgroup of a group G and $(|G/N|, |N|) = 1$. Then there exists a subgroup B of G such that $G = NB$ and $N \cap B = 1$. Furthermore if both G and N act on a set Ω transitively, then $B \leq Stab_G(\omega)$ for some ω.*

Proof. The first statement is a consequence of Schur-Zassenhaus theorem (see [10, Theorem 6.2.1]). Assume that both G and N act on a set Ω transitively. Since $|\Omega| = |G : \mathrm{Stab}_G(\omega)| = |N : \mathrm{Stab}_N(\omega)|$, $\mathrm{Stab}_N(\omega) \lhd \mathrm{Stab}_G(\omega)$ and $(|\mathrm{Stab}_G(\omega)/\mathrm{Stab}_N(\omega)|, |\mathrm{Stab}_N(\omega)|) = 1$, we have by Schur-Zassenhaus theorem (see [10, Theorem 6.2.1]), $\mathrm{Stab}_G(\omega) = \mathrm{Stab}_N(\omega)K$ for some $K \leq \mathrm{Stab}_G(\omega)$ and $|K| = |B|$. Then K is a complement of N in G and so there is $g \in G$ with $K^g = B$. Thus $B \leq \mathrm{Stab}_G(\omega^g)$.

Lemma 2.3.2. *Suppose that V is a kGA-module where k is a field, $G \lhd GA$, $(|G|, |A|) = 1$ and A acts faithfully on G. Then if $V|_G$ is faithful, V is a faithful kGA-module.*

Proof. Let $K = \mathrm{Ker}(GA \text{ on } V)$. Since $(|G|, |A|) = 1$, $K = (G \cap K)(A \cap K)$. By the hypothesis $K = A \cap K$, that is $K \leq A$. On the other hand, $[K, G] \leq K \cap G$, as $K \lhd GA$ and $G \lhd GA$. It follows that $K \cap G = 1$ and so $[K, G] = 1$. This forces that $K = 1$ as A acts faithfully on G. ☐

Theorem 2.3.1. *(Shult's Theorem) (Theorem 4.1, [30]) . Let A be an abelian group of operators acting on a solvable group G of order prime to $|A|$, and suppose that $|G|$ is not divisible by any prime p such that $p^f + 1$ is a divisor of the exponent of A for some positive integer f. Form the semidirect product $H = GA$ and let V be a faithful KH-module, where K is a splitting field for all subgroups of H and which has characteristic not dividing $|A|$. Suppose further that*

(i) V is a sum of equivalent indecomposable KH-modules,

(ii) A acts in fixed point free manner on the elements of V in this representation,

(iii) G has no normal p-groups, where $p = charK$. (If $charK=0$, this requirement can be ignored.)

Then there exists a non-trivial subgroup $B \lhd A$ such that B fixes G elementwise.

Proof. We use induction on $|GA| + dim_K V$.

(1) V is an indecomposable KGA-module:

Let $V = V_1 \oplus \ldots \oplus V_t$ where V_i are equivalent indecomposable KGA-modules for $i = 1, \ldots, t$. As Ker(GA on V_i)=1 for $i = 1, \ldots, t$, we can apply induction to V_iGA and we get a contradiction. Thus V is an indecomposable KGA-module.

(2) V is an irreducible GA-module:

Assume not. Let W be a maximal KGA-submodule of V and let K=Ker(G on V/W). Now $K \lhd G$, and so $O_p(K) \leq O_p(G) = 1$, i.e., K has no normal p-subgroup. Let $K_0 = O_{p'}(K)$. Since $O_{p'}(K)\mathrm{char}K \lhd GA$, K_0 is GA-invariant. Now K_0 can be regarded as a group of operators of order prime to p, acting on an elementary abelian group V. Now $C_{V/W}(K_0) = V/W$ as $K_0 \subseteq$Ker(G on V/W). Then $C_V(K_0)W/W = V/W$ since $(|K_0|, |V|) = 1$. By the same reason $V = [V, K_0] \oplus C_V(K_0)$. Since $K_0 \lhd GA$, each component is GA-invariant. Then (1) implies that $C_V(K_0) = 1$ or V. If it is not 1, then $V = W$, impossible. Also $C_V(K_0) \neq V$ because V is faithful as a GA-module and $K_0 \neq 1$. This contradicts the indecomposibility of V. Thus V is irreducible as a GA-module.

(3) $V|_G$ is homogeneous:

By Clifford's theorem 2.2.7 we may write as $V|_G = W_1 \oplus \ldots \oplus W_s$ where W_i are homogeneous KG-modules for $i = 1, \ldots, s$. Now A permutes them transitively. Let $S =\mathrm{Stab}_A(W_1)$. Then $|A : S| = s$. If $S = 1$, then $|A| = s$ and $V = \oplus_{a\in A} W_1^a$. It follows that $C_V(A) \neq 0$ because for any $0 \neq \omega \in W_1$, $0 \neq \sum_{a\in A} \omega^a \in C_V(A)$, a contradiction. Thus $S \neq 1$. If $S < A$, then $C_{W_1}(S) = 0$, and so $C_{W_i}(S) = 0$ for each $i = 1, \ldots, s$ by Proposition 2.2.12. Let $\overline{G} = G/$Ker(G on W_i). We will see that $O_p(\overline{G}) = 1$: Let $T_i = O_p(\overline{G}) \neq 1$. Now $T_i\mathrm{char}\overline{G} \lhd \overline{G}A$. Since W_i is an irreducible KGS-module and $T_i \lhd \overline{G}S$, by Clifford's theorem 2.2.7, W_i is a sum of conjugate irreducible T_i-modules. Since T_i is a p-group and char$K$=p, we see that any irreducible T_i-module in W_i is trivial by Proposition 2.2.1. Thus $T_i \leq$Ker($\overline{G}$ on W_i)=1. Now we are ready to apply induction to the action of $\overline{G}S$ on W_i. Then there is $1 \neq a \in A$ such that $[\overline{G}, a] = 1$. Note that if $A = \sum_{i=1}^{s} Sa_i$, then $V = W_1^{a_1} \oplus \ldots \oplus W_1^{a_s}$. Hence Ker($G$ on W_i)=Ker(G on $W_1^{a_i}$)=Ker(G on W_1)^{a_i}. Hence $[G, a] = [G, a]^{a_i^{-1}} \leq$Ker($G$ on W_1) as A is abelian, and so $[G, a] \subseteq$Ker(G on W_j) for $j = 1, \ldots, s$. Thus $[G, a] = 1$, a contradiction. This shows that $S = A$ and so $V|_G$ is homogeneous.

(4) Let M be a maximal normal A-invariant subgroup of G, containing G'. Then $V|_M$ is homogeneous.

Since G is solvable, there exists a maximal normal A-invariant subgroup M, necessarily containing G'. We show that G/M is elementary abelian: There is a prime q such that $O_q(G/M) \neq 1$. Let $X/M = O_q(G/M)$. Now X is a normal A-invariant subgroup of G. It follows that $X = G$ by the maximality of M. So G/M is a q-group. Similarly if

$\Phi(G/M) = Y/M \neq 1$, then Y is a normal A-invariant subgroup of G and so $Y/M = 1$. Thus G/M is elementary abelian and it may be regarded as an A-module over the field of q elements. By the maximality of M, it is an irreducible A-module. Since A is abelian, A/B is cyclic where B =Ker(A on G/M). Since V is irreducible and $M \lhd GA$, $V|_M = U_1 \oplus \ldots \oplus U_k$ where $U_1, \ldots, U_k$ are homogeneous modules by Clifford's theorem 2.2.7. Now GA/M permutes $U_1, \ldots, U_k$ transitively. As $V|_G \cong X \oplus \ldots \oplus X$ where X is an irreducible KG-module, $V|_M \cong X|_M \oplus \ldots \oplus X|_M \cong (T_1 \oplus \ldots \oplus T_f) \oplus \ldots \oplus (T_1 \oplus \ldots \oplus T_f)$ where $T_1, \ldots, T_f$ are homogeneous M-components of $X|_M$. Hence $f = k$ and $V|_M = U_1 \oplus \ldots \oplus U_k \cong (T_1 \oplus \ldots \oplus T_1) \oplus \ldots \oplus (T_k \oplus \ldots \oplus T_k)$ and $U_1, \ldots, U_k$ are permuted transitively by G/M because $V|_G$ is homogeneous. Let $\overline{x} \in G/M$ with $\overline{x} \in \mathrm{Stab}_{G/M}(U_1)$, and let $a \in A$. Now $U_1^{\overline{x}} = U_1$ implies that $U_i^{\overline{x}} = U_1^{\overline{y}\ \overline{x}} = U_1^{\overline{x}\ \overline{y}} = U_1^{\overline{y}} = U_i$ for each i and for some $\overline{y} \in G/M$, which is abelian. Then for any $a \in A$, $U_1^{a^{-1}\overline{x}a} = U_j^{\overline{x}a} = U_j^a = U_1$ for some j. Then X/M=$\mathrm{Stab}_{G/M}(U_1)$ is A-invariant. Since G/M is abelian, $[X/M, G/M] = 1$ in other words, $[X, G] \leq M \leq X$. Thus $X \lhd G$ and also X is A-invariant, giving $X = M$ or $X = G$ by the maximality of M. Hence either $k = |G/M :\mathrm{Stab}_{G/M}(U_1)|$=$|G : M|$ or $k = 1$. First assume that $k = |G : M|$. Write $G = \cup_{i=1}^{k} Mt_i$, $t_1 = 1$ and $U_i = U_1^{t_i}$.

Since both GA/M and G/M acts transitively on $\{U_1, \ldots, U_k\}$ and $(|G|, |A|) = 1$, by Lemma 2.3.1, A leaves invariant some component, say U_1, of $V|_M$. Now B =Ker(A on G/M) leaves each U_i fixed: for all $b \in B$, $U_i^b = U_1^{t_i b} = U_1^{bb^{-1}t_i b} = U_1^{t_i} = U_i$. Moreover for any $a \in A$, $U_i^a = U_1^{t_i a} = U_1^{aa^{-1}t_i a} = U_1^{t_i^a}$. That is, the elements of A permute $\{U_1, \ldots, U_k\}$ in exactly in the same manner in which the elements of A permute the elements, t_i of G/M. Now G/M represents A/B irreducibly and faithfully where A/B is cyclic of order n.

Any A-orbit of nonidentity element of G/M with respect to this action has length $|A/B : \mathrm{Stab}_{A/B}(\overline{x})|$ for $\overline{x}$ lying in this orbit. If $1 \neq a \in \mathrm{Stab}_A(\overline{x})$, then $1 \neq \overline{x} \in C_{G/M}(a)$ is A/B-invariant and so $[G/M, a] = 1$ giving that $a \in B$. Hence $\mathrm{Stab}_{A/B}(\overline{x}) = 1$ and so the length of any such orbit is n. Since $U_i^a = U_1^{a^{-1}t_i a}$ for any $a \in A$ we see that all but one of the U_i's are permuted in cycles of length n (Recall that U_1 is A-invariant). If $n = 1$, then $A = B$ and so each U_i is A-invariant. If $n > 1$, $B \neq 1$ since otherwise for any $0 \neq \sum_{a \in A} u^a \in C_V(A)$ for an $u \in U_i$, a contradiction. Thus if $n > 1$, B =$\mathrm{Stab}_A(U_i)$, because we know that $B \leq \mathrm{Stab}_A(U_i)$ and also that U_i's are permuted in cycles of length n, that is $|A : \mathrm{Stab}_A(U_i)| = n$ giving that B =$\mathrm{Stab}_A(U_i)$. In both cases when $n = 1$ and $n > 1$, B is fixed point free on each U_i. Because otherwise $u \in C_{U_i}(B)$ we have $0 \neq \sum_{j=1}^{s} u^{a_j} \in C_V(A)$ where $A = \cup_{j=1}^{s} Ba_j$, a contradiction. We have $N_{GA}(U_i) = MB$ for all i. Then by Clifford's theorem 2.2.7 U_i is an irreducible MB-module such that $U_i|_M$ is homogeneous.

Now consider $M_i = O_p(M/\mathrm{Ker}(M \text{ on } U_i))$. Since M_i is a p-group and char$K = p$, we have $C_X(M_i) \neq 1$ where $U_i \cong X \oplus \ldots \oplus X$, a sum of irreducible M-modules. Thus $[X, M_i] = 1$ and so $[U_i, M_i] = 1$ which implies that $M_i = 1$ as U_i is faithful for $M/\mathrm{Ker}(M$ on $U_i)$.

Since $k = [G : M] > 1$ we can apply induction to the action of $(M/\mathrm{Ker}(M \text{ on } U_i))B$ on U_i and get a nontrivial subgroup B_0 of B such that $[M/\mathrm{Ker}(M \text{ on } U_i), B_0] = 1$. Since $[G/M, B] = 1$ and $(|G|, |B|) = 1$, $G/M = C_{G/M}(B) = C_G(B)M/M$. Hence we see that U_j $(j = 1, \ldots, k)$ are conjugate by the elements of $C_G(B)$. Now $[C_G(B), B_0] = 1$. Say $U_j = U_i^x$ for some $x \in C_G(B)$. As $M \lhd G$, $M^x = M$. As $[M, B_0] \subseteq \mathrm{Ker}(M \text{ on } U_i)$, we have $[M, B_0]^x = [M^x, B_0^x] = [M, B_0] \subseteq \mathrm{Ker}(M^x \text{ on } U_i^x) = \mathrm{Ker}(M \text{ on } U_j)$. Then $[M, B_0] \subseteq \cap_{j=1}^{k}$ $\mathrm{Ker}(M \text{ on } U_j) = 1$. Since $[G, B_0] = [G, B_0, B_0] \leq [M, B_0] = 1$, we get a contradiction. Thus $k = 1$ and so $V|_M$ is homogeneous.

(5) B =Ker $(A$ on $G/M) = 1$, $G = Q$ is an extraspecial q-group with $[Z(Q), A] = 1$ and $V|_Q$ is homogeneous.

$O_p(G) = 1$ implies that $O_p(M) = 1$ and $\mathrm{Ker}(M \text{ on } V)$=1, that is, V is a faithful KM-module on which A acts fixed point freely. By hypothesis $|M|$ is not divisible by a prime p such that $p^f + 1$ is a divisor of $|A|$ for some positive integer f because $|G|$ is not. Since $|MA|+\dim_K V < |GA|+\dim_K V$, we may apply induction to the action of MA on V, and get $[M, a] = 1$ for some $1 \neq a \in A$. If $[G/M, a] = 1$ then we are done. So we need to show that the following claim is true.

Claim. $[G/M, a] = 1$

Assume that $[G/M, a] \neq 1$. Then $C_{G/M}(a) \neq G/M$ and so $C_{G/M}(a) = 1$ as G/M is an irreducible A-module and A is abelian. Then $C_G(a) \geq M \geq C_G(a)$ that is $M = C_G(a)$. Since $M \lhd G$, $C_G(M) \lhd G$ and so $MC_G(M)$ is a normal A-invariant subgroup. By the maximality of M we get $C_G(M) \leq M$ or $MC_G(M) = G$. The former is not true because otherwise $[M, a] = 1$ implies by the three subgroup lemma 2.1.1 that $[G, a] \leq C_G(M) \leq M$. Thus $MC_G(M) = G$. In fact, if $(C_G(M))_q$ is an A-invariant Sylow q-subgroup of $C_G(M)$, then $M(C_G(M))_q = G$ because $G/M = M(C_G(M))_q/M$ is a q-group. Let Q be a minimal with respect to being an A-invariant subgroup of $(C_G(M))_q$ such that $MQ = G$. Then since $[M, Q] = 1$ we get $Q \lhd G$. Also, then $[Z(Q), MQ] = 1$ and so $1 \neq Z(Q) \leq Z(G)$. Since V is a homogeneous KG-module, where K is a splitting field for every subgroup of GA, $Z(G)$ acts by scalars on V. Thus $[Z(Q), A] = 1$ and $Z(Q) \leq C_G(A) \leq C_G(a) = M$. It is evident that Q is nonabelian.

We shall observe that Q is an extraspecial q-group. Let D be any proper A-invariant subgroup of Q. By the minimality of Q, and the maximality of M, $MD = M$ and so $D \subseteq M$. Since $[M, Q] = 1$, $D \subseteq Z(Q)$. It follows that $Z(Q)$ is the unique maximal A-invariant subgroup of Q and so $Z(Q)$ is the intersection of all maximal A-invariant subgroups of Q. Now $\Phi(Q)\mathrm{char}Q \lhd GA$ and so $\Phi(Q)$ is a proper A-invariant subgroup of Q. Thus $\Phi(Q) \leq Z(Q)$. Since $Q/\Phi(Q)$ is an irreducible A-module we have $\Phi(Q) = Z(Q)$. Consider $V|_{Z(Q)}$. We know that V is an irreducible GA-module and $Z(Q) \subseteq Z(GA)$. Thus $V|_{Z(Q)}$ is homogeneous by Clifford's theorem 2.2.7, that is, $V_{Z(Q)} \cong X \oplus \ldots \oplus X$ where X is an irreducible $Z(Q)$-module. Now 1=Ker($Z(Q)$ on V)=Ker($Z(Q)$ on X), and so V has a faithful and irreducible $Z(Q)$-module. Hence $Z(Q)$ is cyclic.

We see also that $V|_Q$ is homogeneous: Since $V|_G \cong Y \oplus \ldots \oplus Y$ where Y is an irreducible G-module and $Q \lhd G$, we can write $V|_Q \cong Y|_Q \oplus \ldots \oplus Y|_Q$ with $Y|_Q = W_1 \oplus \ldots \oplus W_s$ where $W_1, \ldots, W_s$ are homogeneous Q-modules. Thus $V|_Q \cong (W_1 \oplus \ldots \oplus W_1) \oplus \ldots \oplus (W_s \oplus \ldots \oplus W_s) = T_1 \oplus T_2 \oplus \ldots \oplus T_s$ where $T_1, T_2, \ldots, T_s$ are homogeneous Q-submodules. Now G/Q acts transitively on $\{T_1, T_2, \ldots, T_s\}$. By Clifford's Theorem 2.2.7, $C_G(Q)$ stabilizes each T_i. Hence $QM = G$ stabilizes T_i. It follows that $V|_Q$ is homogeneous.

Now B fixes G/M elementwise. Since $G/M \cong Q/\Phi(Q)$ as an A-isomorphism, we have $[Q/\Phi(Q), B] = 1$. Then $[Q, B] = [Q, B, B] \leq [\Phi(Q), B] = [Z(Q), B] \leq [Z(Q), A] = 1$. Thus $[Q, B] = 1$. Since $C_G(a) = M$ for some $a \in A$, we have $B \neq A$, because otherwise $[Q, A] = 1$ would imply that $Q \subseteq M$ and so $G = M$, a contradiction. If $C_V(B) = 1$, then $B \neq 1$. Since $V|_{GB}$ is completely reducible as a sum of homogeneous GB-modules conjugate under A, we may apply induction to GB on U, one of these components, and get $a \in A$ such that $[G, a]$ is trivial on U. It follows that $[G, a] = 1$ as A is abelian. Hence we have $C_V(B) \neq 1$. Since $[Q, B] = 1$ and A is abelian, $C_V(B)$ is a KQA-module. Since $V|_Q$ is homogeneous $C_V(B)|_Q$ is also homogeneous and so $C_V(B)$ is faithful KQ-module. Also $C_{C_V(B)}(A/B) = 1$. We may apply induction to the action of $Q(A/B)$ on $C_V(B)$ and get a subgroup $A_1/B \lhd A/B$ such that $[Q, A_1/B] = 1$. Now $[Q, A_1] = 1$ as $[Q, B] = 1$. But then $[G/M, A_1] = 1$ and so $A_1 \leq B =$Ker(A on G/M)=Ker(A on $Q/\Phi(Q)$). So $B = A_1$, a contradiction. Thus $C_V(B) = V$ and $G = Q$. Now B fixes all of G and so $B = 1$.

(6) $V|_Q$ is irreducible.

Since A/B is cyclic and $B = 1$ by (5), A is cyclic. Also since $V|_Q = V|_G$ is homogeneous, we can write $V|_Q \cong U \oplus \ldots \oplus U$ where U is an irreducible KQ-module. Now since $QA/Q \simeq A$ and U is an irreducible KQ-module, by Proposition 2.2.10, U is an irreducible KQA-module. But by (2) we know that V is an irreducible KQA-module, a contradiction. Thus $V|_Q$ is irreducible.

(7) Final contradiction.

Since $Q/\Phi(Q)$ is an irreducible A-module, $|Q/\Phi(Q)| = q^e$ where e is the exponent of q modulo $n(=|A|)$ and e is even. Let $e = 2k$. Then by Proposition 2.2.15, there exists $\delta = \mp 1$ such that $n \mid q^k - \delta$ and a one dimensional KA-module U such that

(a) if $\delta = 1$, then $V_A \cong m(KA) \oplus U$ and

(b) if $\delta = -1$, then $V_A \oplus m(KA) \cong U$

where $m = (q^k - \delta)/|A|$ and $m(KA)$ denotes the direct sum of m copies of the regular module KA. If $\delta = -1$, then $n \mid q^k + 1$, that is, $q^k + 1 \geq n$. If $q^k + 1 = n$, then we get a contradiction. Hence $q^k + 1 > n$. That is $m \geq 2$. Hence if (b) holds, we have $V_A \supseteq KA$. If (a) holds we have already $V_A \supseteq KA$. Both are impossible as $C_V(A) = 0$. □

Corollary 2.3.1. *(A corollary of Shult's Theorem) [8, Lemma 3.3] Let G be a solvable group admitting an abelian group A such that $(|G|, |A|) = 1$. Suppose V is a finite-dimensional KGA-module, where K is a splitting field for all subgroups of GA and char$K \nmid |A|$. Suppose further that*

(1) V_G is a faithful, homogeneous KG-module;

(2) A acts fixed point freely on V;

(3) $|G|$ is not divisible by any prime p such that $p^f + 1$ is a divisor of the exponent of A for some positive integer f.

Then there exists a nontrivial normal subgroup of A centralizing G.

Proof. We proceed by induction on $\dim_K V$. Consider a counterexample to the corollary minimal with respect to this number. Then V is an irreducible KGA-module. For let $W \neq 0$ be an irreducible KGA-submodule of V. By Clifford's theorem 2.2.7, W_G is completely reducible; so W_G is a homogeneous KG-module since V_G is by [10, Corollary 3.4.2], and W_G is a faithful KG-module since V_G is a faithful, homogeneous KG-module. Thus W satisfies hypothesis (1) of the corollary. Clearly hypotheses (2) and (3) are satisfied; so by minimality of the counterexample, $W = V$.

Now since we have a counterexample to the corollary, A acts faithfully on G. Then by Lemma 2.3.2 since V_G is faithful KG-module, V is a faithful KGA-module. Finally, since V is faithful, irreducible KGA-module, if char$K = r \neq 0$, $O_r(GA) = 1$ by Proposition 2.2.2. But $O_r(G)$char$G \lhd GA$; so $O_r(G) \lhd GA$ and $O_r(G) \leq O_r(GA) = 1$. Thus all the hypothesis of Theorem 2.3.1 (Shult) are satisfied, while the conclusion fails to hold. This contradiction establishes Corollary 2.3.1. □

A theorem due to Gagola

Proposition 2.3.1. *(Thompson) Let $G = PA$ be a semidirect product of a p-group $P \neq 1$ with a group A (here $P \lhd G$) with the following properties:*

1) $(|P|, |A|) = 1$,

2) $[P, A] = P$,

3) $[A, K] = 1$ *for any abelian, characteristic subgroup K of P.*

Then P is a special group.

Proposition 2.3.2. *Let P be a p-group and Q be a noncyclic abelian group of automorphisms of P for some prime $q \neq p$. Then $P = \langle\ C_P(a) \mid 1 \neq a \in Q\ \rangle$.*

Proof. [10] 5.3.16.

Theorem 2.3.2. *[9, Lemma 2.2](Gagola's Theorem) Let p, q, r be three distinct primes and $G = QA$ where A is a cyclic group of order p and Q is a nontrivial q-group with $[Q, A] = Q$. Assume further that G acts on a vector space V over $k = GF(r)$ in such a way that $[V, A] = V$. Then*

a) If $[V, Q] \neq 0$, then $q = 2$ and p is a Fermat prime

b)If G is faithful and irreducible on V, then Q is an extraspecial 2-group of order $2(p-1)^2$.

Proof. $[V, Q]$ is a kG-submodule of V, since $Q \lhd G$. Let U be an irreducible kG-submodule of V contained in $[V, Q]$. As $C_{[V,Q]}(Q) = 1$, we have $C_U(Q) = 1$. Then $C_G(U)$ is properly contained in Q. Thus U is a faithful and irreducible $k(G/C_G(U))$-module and $G/C_G(U)$ is a group satisfying the hypothesis of the proposition. So to complete the proof, it suffices to prove b).

Let $H = VG$ be the natural semidirect product of G by V. Then $V = F(H)$ is an irreducible kH-module as V is a faithful and irreducible kG-module.

Let Q_1 be any characteristic abelian subgroup of Q. Then $Q_1 = [Q_1, A] \times C_{Q_1}(A)$ by Proposition 2.1.9 and hence $C_{V[Q_1,A]}(A) = C_V(A)C_{[Q_1,A]}(A) = 1$ since $(|A|, |VQ|) = 1$. So the group $V[Q_1, A]$ is nilpotent by Proposition 2.1.13 and hence $[V, [Q_1, A]] = 1$. But since V is a faithful G-module, we see that $[Q_1, A] = 1$. Then by Proposition 2.3.1, Q is a special q-group with $[Z(Q), A] = 1$. Since $C_V(x)$ is normalized by H for any $x \in Z(Q) \leq Z(G)$ and $V = F(H)$ is an irreducible kG-module we see that $C_V(x) = 1$ for any $1 \neq x \in Z(Q)$. If $Z(Q)$ is noncyclic, then $V = \langle C_V(x) \mid 1 \neq x \in Z(Q)\rangle$ by Proposition 2.3.2 and so $V = 1$, which is impossible. Thus $Z(Q)$ is cyclic and Q is an extraspecial q-group, A acts fixed point freely and irreducibly on $Q/\Phi(Q)$, since $[Q, A] = Q$.

Let $\overline{k}$ be a finite extension of k which is a splitting field for all subgroups of H. Let $\overline{V} = V \otimes_k \overline{k}$. Since $[V, A] = V$, we see that $[\overline{V}, A] = \overline{V}$.

Let M be an irreducible G-submodule of $\overline{V}$. Then by Proposition 2.2.17 M is a faithful G-module and $[M, A] = M$. If χ denotes the character of the representation of G on M, then we have by Proposition 2.2.15, $\dim_{\overline{k}}(M) = \chi(1) = q^m$, if $|Q| = q^{2m+1}$, and $\chi|_A = n\rho + s\mu$. Since $[M, A] = M$, $\chi|_A$ does not contain the trivial character of A as a constituent. Since $s = \pm 1$, this is possible only if, $n = 1$, $s = -1$ and μ is the trivial character of A. In particular, $q^m = \chi(1) = \rho(1) - 1 = p - 1$. Since $q^m \neq 1$, p is odd and hence $q = 2$. Then $p = 2^m + 1$ is a Fermat prime. Furthermore, $|Q| = q^{2m+1} = 2(p-1)^2$. □

A theorem due to Gross

Lemma 2.3.3. *Suppose that $p^a = q^b + 1$, where p, q are primes and a, b are positive integers. Then either*

a) $p = 2$, $b = 1$, a is a prime and $q = 2^a - 1$ is a Mersenne prime, or

b) $q = 2$, $a = 1$, $b = 2^m$ and $p = 2^{2^m} + 1$ is a Fermat prime, or

c) $p^a = 9$, $q^b = 8$.

Proof. [13, IX. 2. 4].

Lemma 2.3.4. *Let V be a vector space of dimension n over a field K of q^f elements. Let A be a cyclic group of K linear transformations of V and V be a faithful module. Then $|A|$ divides $q^{nf} - 1$. If V is irreducible then n is the least such integer.*

Proof. [14] pages 165-166.

Lemma 2.3.5. *[10, Lemma 11.1.3]. The $p^n \times p^n$ permutation matrix*

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

with coefficient in Z_p has minimal polynomial $(X-1)^{p^n}$.

Theorem 2.3.3. *(Gross' Theorem) Let $G = Q\langle y \rangle$ be a group where y is a p-element of order p^n and Q is a nontrivial normal q-subgroup for distinct primes p and q, p odd or $p = 2$, $p^n = 4$. Assume that G acts faithfully and irreducibly on a kG-module V where k is a field* *of characteristic r, r a prime, and that $z = y^{p^{n-1}}$ does not centralize Q. Then $C_Q(y) \neq 1$ provided one of the following conditions hold:*

(i) $r \neq p$, and y acts fixed-point-freely on V,

(ii) $r = p$, and the minimum polynomial of y is not $(X-1)^{p^n}$.

Proof. Consider the case $p \neq 2$. Assume that first condition (i) holds. Since the conditions remain unchanged under an extension of k, we may assume that k is algebraically closed. We suppose that the theorem is false and choose a counterexample for which $\dim_k(V) + |G|$ is minimal.

(1) Q is a special group, $[Q', z] = 1$, $C_{Q/Q'}(z) = 1$ and $\langle y \rangle$ acts irreducibly Q/Q'.

Let Q_1 be the minimal element of $\{H \mid H \leq Q\ ,\ H$ is $\langle y \rangle$-invariant $,\ [H, z] \neq 1\}$. Now Q_1/Q_1' is an irreducible $\langle y \rangle$-module with $[Q_1, z] = Q_1$, $Q_1' = \Phi(Q_1)$, $[\Phi(Q_1), z] = 1$, Q_1 is special, $C_{Q_1/Q_1'}(z) = 1$ by Proposition 2.2.9.

Set $G_1 = Q_1\langle y \rangle$. By Clifford's theorem 2.2.7, $V|_{G_1}$ is completely reducible. Choose an irreducible kG_1-submodule of V. Apply induction to the action of $k\overline{G_1}$ on V where $\overline{G_1} = G_1/\mathrm{Ker}(G_1 \text{ on } W)$. Then $C_{\overline{Q_1}}(y) \neq 1$ which implies that $C_{Q_1}(y) \neq 1$ as $(|y|, |Q|) = 1$, which is impossible. Thus $Q_1 = Q$.

(2)

Let $V|_Q = W_1 \oplus \ldots \oplus W_l$ be the Wedderburn decomposition of V into homogeneous Q-components. Since V is irreducible as kG-module, $\langle y \rangle$ acts transitively on $\Omega = \{W_1, \ldots, W_l\}$. Let $\langle y^{p^m} \rangle = stab_G(W_1) \cap \langle y \rangle$. Put $c = y^{p^m}$. Now $|\langle y \rangle : \langle c \rangle| = p^m = l$ is the number of Wedderburn components of $V|_Q$ and W_i is an irreducible $kQ\langle c \rangle$-module for $i = 1, \ldots, l$.

(3) $\langle c \rangle$ is nontrivial, i.e. z stabilizes each W_i for $i \in \{1, \ldots, l\}$.

Now c acts fixed point freely on each W_i because otherwise for each $0 \neq v \in C_{W_i}(c)$ we have $v + v^y + \ldots + v^{y^{p^m-1}}$ as a nontrivial fixed point of y, a contradiction. Then $m < n$, i.e. $c \neq 1$.

(4) Let K_i=Ker($Q\langle c \rangle$ on W_i). Now $K_i \leq Q$ and Q/K_i is extraspecial and W_i is a faithful and irreducible $k(Q/K_i)$-module for each i.

Note that $\cap_{i=1}^{l} K_i$=Ker($Q\langle c \rangle$ on V)=1. Assume that $K_1 \cap \langle c \rangle \neq 1$. So $z \in K_1$. As $\langle y \rangle$ is transitive on Ω, $z \in K_i$ for each i and so $z = 1$, a contradiction. Thus $K_1 \cap \langle c \rangle = 1$. This means that $K_1 \leq Q$ and so $K_i \leq Q$ for each i. Let $\overline{Q} = Q/K_i$. We know that W_i is an irreducible and faithful $k\overline{Q}\langle c \rangle$-module for each i. By Proposition 2.2.10, we get W_i is an irreducible and faithful $k\overline{Q}$-module. Since k is an algebraically closed field, $Z(\overline{Q})$ acts scalarly on W_i. Thus $[Z(\overline{Q}), c] = 1$ and $[Z(\overline{Q}), z] = 1$. Since $C_{Q/Q'}(z) = 1$, we get $C_{Q/Q'K_i}(z) = 1$ and so $Z(\overline{Q}) \leq \overline{Q'}$. Since Q is special, $\overline{Q'} \leq Z(\overline{Q}) = Z(\overline{Q})$ and $\Phi(\overline{Q}) = \overline{\Phi(Q)} = \overline{Q'} = \overline{Q}'$. Hence $Z(\overline{Q}) = \overline{Q}' = \Phi(\overline{Q})$. Since $\overline{Q} \neq \overline{1}$, we have that $\overline{Q}$ is extraspecial. Also $\overline{Q}/\overline{Q}' \cong Q/Q'K_i$ gives that $C_{\overline{Q}/\overline{Q}'}(z) = 1$.

Let $|\overline{Q}| = q^{2d+1}$.

(5) p is a Fermat prime and $q = 2$. Also $|Q/Q'| = 2^{2dp^m}$ and $\overline{Q}/\overline{Q}'$ is irreducible $\langle c\rangle$-module.

By Proposition 2.2.15, there exists $\delta = \pm 1$ such that $|\langle c\rangle| = p^{n-m}$ divides $q^d - \delta$ and a 1-dimensional $k\langle c\rangle$-module U such that

(i) If $\delta = 1$, then $W_i|_{\langle c\rangle} \cong s(k\langle c\rangle) \oplus U$, and

(ii) If $\delta = -1$, then $W_i|_{\langle c\rangle} \oplus U \cong s(k\langle c\rangle)$

where $s = \frac{q^d-\delta}{p^{n-m}}$ and $s(k\langle c\rangle)$ denotes the direct sum of s copies of $k\langle c\rangle$.

Since $C_{W_i}(c) = 0$ for each i, $i \in \{1, \ldots, l\}$, W_i has no component isomorphic to $k\langle c\rangle$. Thus $\delta = -1$ and so $q^d + 1 = p^{n-m} = |c|$. In particular $q < p^n$. Since p is odd, we get $q = 2$ and p is a Fermat prime by Lemma 2.3.3.

Any irreducible component of $\langle c\rangle$ on $\overline{Q}/\overline{Q}'$ is faithful, because otherwise z fixes a nontrivial element of $\overline{Q}/\overline{Q}'$ which is not the case as $C_{\overline{Q}/\overline{Q}'}(z) = 1$. Let U be any faithful irreducible $\langle c\rangle$-module. By Lemma 2.3.4 above $|c|$ divides $2^{dimU} - 1$ and $\dim U$ is the least such integer. Now $p^{n-m} = 2^d + 1$ gives that $p^{n-m} \mid 2^{2d} - 1 = (2^d + 1)(2^d - 1)$. Since $p^{n-m} \nmid 2^d - 1$, we have the fact that any faithful and irreducible representation of $\langle c\rangle$ over $\mathbb{Z}_2$ has degree $2d$.

Similarly again by Lemma 2.3.4, any faithful and irreducible representation of $\langle y\rangle$ over $\mathbb{Z}_2$ has degree $2dp^m$. It follows that $|Q/Q'| = 2^{2dp^m}$.

(6) $\langle c\rangle < \langle y\rangle$.

If $\langle c\rangle = \langle y\rangle$, then $l = 1$, i.e. $V = W_1$ is irreducible as a $k\overline{Q}$-module by Proposition 2.2.7, and so K_1=Ker($Q\langle y\rangle$ on W_1)=1 giving $K_i = 1$ for each i. Thus $\overline{Q} = Q/K_i = Q$ and by (4), $Z(Q) \leq Z(G)$ as $[Z(\overline{Q}), z] = 1$. This leads to a contradiction because $C_Q(y) \neq 1$.

(7) Let $L_i = \cap_{j\neq i} Q'K_j$ $(i = 1, \ldots, p^m)$. Then $Q' = L_1' \times \ldots \times L_{p^m}'$, $(j = 1, \ldots, l)$.

Observe that $[L_i, Q] \leq [Q'K_j, Q] = [K_j, Q] \leq K_j$ for each $j \neq i$ and the equality holds because $Q' \leq Z(Q)$. Thus $[L_i, Q]$ acts trivially on W_j for each $j \neq i$.

Since $|Q : Q'K_j| = |\overline{Q} : \overline{Q}'| = 2^{2d}$, we have $|Q : L_i| \leq 2^{2d(p^m-1)}$. If $L_i \leq Q'$, then $2^{2dp^m} = |Q : Q'| \leq 2^{2dp^m-2d}$, not the case. Thus $L_i > Q'$ and so $Q \geq L_1 \ldots L_{p^m} > Q'$. But $L_i^y = \cap_{j\neq i} Q'K_j^y = \cap_{j\neq i} Q'\mathrm{Ker}(Q$ on $W_j^y)$. Then $(L_1 \ldots L_{p^m})^y = (L_1 \ldots L_{p^m})$ and so $Q = L_1 \ldots L_{p^m}$ as Q/Q' is an irreducible $\langle y\rangle$-module.

Hence $Q' = L_1' \ldots L_{p^m}'(\prod_{i\neq j}[L_i, L_j])$. Now with $i \neq j$, $[L_i, L_j] \subseteq$Ker(Q on W_k) for all k. So $|L_i, L_j| = 1$ and $Q' = L_1' \ldots L_{p^m}'$. Also $L_i' \cap \prod_{j\neq i} L_j' \subseteq$Ker($Q$ on W_k) for each k

because $L_i' \subseteq$Ker(Q on W_j) for all $j \neq i$ and so $\prod_{j\neq i} L_j' \subseteq$Ker($Q$ on W_i). It follows that $L_i' \cap \prod_{j\neq i} L_j' = 1$, i.e. $Q' = L_1' \times \ldots \times L_{p^m}'$.

(8) Final contradiction.

If $x \in L_i'$, then $x \in Q'$, $\overline{x} \in \overline{Q}'$ and so $\langle \overline{x} \rangle = \overline{Q}'$ as $|\overline{Q}'| = 2$ by (5). It follows that $\overline{x}$ is represented on W_i by ± 1. Since x is represented on W_j by 1 for all $j \neq i$, $x^2 \in \cap_{j=1}^{l} K_j = 1$. Thus $|L_i'| \leq 2$. If $L_i' = 1$ for some i, then $L_i' = 1$ for all i, $i \in \{1, \ldots, p^m\}$, because they are all conjugate and so $Q' = 1$, not the case. Thus $|L_i'| = 2$. Now $|Q'| = 2^{p^m} \equiv 2 \pmod p$.

If the lengths of all the orbits of $\langle y \rangle$ on Q' other than $\{1\}$ are divisible by p, then $|Q'| = sp + 1 \equiv 2 \pmod p$ for some $s \in \mathbb{N}$, not possible. Thus there is an orbit of $\langle y \rangle$ on Q' other than $\{1\}$ whose length is not divisible by p. Now such an orbit has length 1 and so there is $1 \neq x \in C_Q(y)$, a contradiction.

Now we assume condition (ii) holds when p is odd. Again we suppose that the theorem is false and choose a counterexample for which $\dim_k(V) + |G|$ is minimal.

(1) We may assume that k is algebraically closed.

If K is an algebraic closure of k, then G can be regarded as a group of linear transformation of $V_K = V \otimes_k K$. Since a basis of V is also a basis of V_K, y is represented by the same matrix on V_K as it is on V and consequently y has the same minimal polynomial on V_K as it does on V. Hence we may assume that k is algebraically closed.

(2) By Hall-Higman reduction we may assume that $Q = [Q, y^{p^{n-1}}]$, Q/Q' is irreducible under the action of $\langle y \rangle$, $C_{Q/Q'}(z) = 1$, Q is a special group.

Let Q_1 be the minimal element of

$$\{H \mid H \leq Q \text{ , } H \text{ is } \langle y \rangle\text{-invariant , } [H, z] \neq 1\}$$

Now Q_1/Q_1' is an irreducible $\langle y \rangle$-module with $[Q_1, z] = Q_1$, $Q_1' = \Phi(Q_1)$, $[\Phi(Q_1), z] = 1$, $C_{Q_1/Q_1'}(z) = 1$ and Q_1 is special by Proposition 2.2.9. Set $G_1 = Q_1\langle y \rangle$ and assume that $Q \neq Q_1$. By Clifford's theorem 2.2.7 $V|_{G_1}$ is a sum of irreducible kG_1-modules. Let W be an irreducible kG_1-submodule of V. Now apply induction argument to the action of $\overline{G_1}$ on W, where $\overline{G_1} = G_1/$Ker(G_1 on W). Now W is a faithful and irreducible $k\overline{G_1}$-module. Then $C_{\overline{Q_1}}(y) \neq 1$. As $(|Q|, |y|) = 1$, $C_{Q_1}(y) \neq 1$. This leads to a contradiction $C_Q(y) \neq 1$. Thus $Q_1 = Q$.

(3)

Let $V|_Q = W_1 \oplus \ldots \oplus W_l$ be the Wedderburn decomposition of V into homogeneous Q-components by Clifford's theorem 2.2.7. Since V is irreducible as a kG-module, $\langle y \rangle$ acts transitively on $\{W_1, \ldots, W_l\}$. Let $\langle y^{p^m} \rangle = stab_G(W_1) \cap \langle y \rangle$. Put $c = y^{p^m}$. By Clifford's theorem 2.2.7 W_i is an irreducible $kQ\langle y\rangle$-module for $1 \le i \le l$.

(4) $c \neq 1$ and so $z = y^{p^{n-1}}$ stabilizes each W_i, $i \in \{1, \dots, l\}$.

If $c = 1$, then $l = p^n$. This means that if $\{v_i|,\ 1 \le i \le h\}$ is a basis of W_1, then $\mathcal{B} = \{v_i y^j | 1 \le i \le h, 1 \le j \le p^n\}$ is a basis of V. With respect to this basis, the matrix of y is

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_h \end{pmatrix}$$

where A_i, $1 \le i \le h$, is the $p^n \times p^n$ permutation matrix of the Lemma 2.3.5. Hence by that lemma, the minimal polynomial of y on V is $(Y-1)^{p^n}$, which is impossible. Thus $c \neq 1$.

(5) Let K_i=Ker($Q\langle c\rangle$ on W_i). Now $K_i \le Q$ and Q/K_i is extraspecial and W_i is a faithful and irreducible $k(Q/K_i)$-module for each i.

This follows exactly in the same way as in (4) of the case when p is odd and the condition (i) holds.

Let $|\overline{Q}| = q^{2d+1}$.

(6) p is a Fermat prime and $q = 2$.

Let W be a homogeneous Q-component of V. Consider $G_1 = \overline{Q}\langle y\rangle$. Since W is an irreducible $k\overline{Q}\langle y\rangle$-module and faithful for $\overline{Q}$ and since the minimal polynomial of c is not $(Y-1)^{p^{n-m}} = Y^{p^{n-m}} - 1$, by Hall-Higman theorem [13, Theorem IX.2.6], $q^d + 1 = p^{n-m}$. Since p is odd, we get $q = 2$ and p is a Fermat prime by Lemma 2.3.3.

Therefore we show that when p is odd and condition (ii) holds, by Hall-Higman reduction we may assume that $Q = [Q, y^{p^{n-1}}]$, Q/Q' is irreducible under the action of $\langle y\rangle$, Q is a special group, p is Fermat prime and $q = 2$. Now the arguments (5), (6), (7) and (8) in the proof of case (i) when p is odd, follow $C_Q(y) \neq 1$. But this is impossible by the minimality of $|G|+\dim_k(V)$.

Thus let $p^n = 4$. We may assume that k is algebraically closed because taking $p = 2$ does not affect the reduction steps when p is odd. We suppose that the theorem is false and choose a counterexample for which $\dim_k(V) + |G|$ is minimal.

Let $V|_Q = W_1 \oplus \ldots \oplus W_s$ be the Wedderburn decomposition of V into homogeneous Q-components. Let $\langle c\rangle = stab_G(W_1) \cap \langle y\rangle$. Then $\langle c\rangle$ can be either 1, $\langle y^2\rangle$ or $\langle y\rangle$.

Assume that V is a homogeneous Q-module. Then after the usual reductions as in the case when p is odd, $Q = [Q, y^2]$ is extraspecial. Since k is algebraically closed, Proposition 2.2.10 implies that V is an irreducible and faithful kQ-module. Then Schur's lemma gives that $Z(Q)$ acts scalarly on V. Hence $[Z(Q), y] = 1$. In particular $Z(Q) \leq Z(G)$. Thus $C_Q(y) \neq 1$, a contradiction.

Assume that V is decomposed into the sum of four Q-homogeneous components permuted by $\langle y \rangle$. Then $\langle y \rangle$ is represented regularly on V, because V is an irreducible kG-module. Then $C_V(y) \neq 0$. This implies that (i) does not holds. Assume that (ii) holds. Then following the arguments of proof when p is odd and (ii) holds we find that p must be a Fermat prime and $q = 2$. But in our case $2 = p \neq q$.

Assume $\langle c \rangle = \langle y^2 \rangle$. Then V is a direct sum of two homogeneous Q-components, say W_1 and W_2. Then W_i is an irreducible $kQ\langle c \rangle$-module for $i = 1, 2$ as V is an irreducible kG-module. Let $1 \neq a \in C_{W_1}(y^2)$, then $a + a^y$ is in $C_V(y)$. Thus (i) does not hold. So assume (ii) holds. The same arguments in the case p is odd follow that $2 - 1 = p - 1 = q^d$ where $|Q/Ker(Q\langle c \rangle on W_1)| = q^{2d+1}$. So $Q = 1$, which is impossible. Thus $C_{W_i}(y^2) = 1$ for $i = 1, 2$. Since W_1 is an irreducible $kQ\langle y^2 \rangle$-module and $[W_1, y^2] = W_1$, Theorem 2.3.2 implies that $q = 2$, which is also impossible as $q \neq p$. $\square$

2.4 An example due to Turull

In order to show that the bound we give in our main result is the best possible one, we shall present an example due to Turull ([24]). This shows that for any finite group A, there exists a solvable group G having coprime order to A, such that A acts fixed point freely on G and the Fitting height of G is exactly the number of primes dividing the order of $|A|$.

Definition 2.4.1. *Let A be a finite group and B any subgroup of A. Define $l(A : B)$ to be the largest integer n such that there is a sequence of subgroups $B = C_0 \subset C_1 \subset \dots C_n = A$ each properly contained in the following one.*

It should be noted that if A is solvable, then $l(A) = l(A : 1)$ is the number of primes dividing $|A|$ counted with multiplicities. This is because an abelian series whose factors are abelian can be refined to a composition series whose factors are of prime order.

Theorem 2.4.1. *Let A be any finite group and B a proper subgroup of A. Let $n = l(A : B)$ (Definition 2.4.1) and let $p_1, p_2, \dots, p_n$ be a sequence of primes such that $p_i \neq p_{i+1}$, for $i = 1, \dots, n-1$ and $p_i \nmid |A|$, for $i = 1, \dots, n$.*

Then there exists a solvable group G, and an action of A on G, and subgroups P_i, $i = 1, \dots, n$ of G such that:

1) For each $i = 1, \dots, n$, P_i is an elementary abelian p_i-subgroup of G which is invariant under $P_{i+1} \dots P_n A$ and if $i > 1$, $[P_{i-1}, P_i] = P_{i-1}$;

2) $G = P_1 \dots P_n$;

3) $C_G(A) = 1$;

4) $\cap_{a \in A}[B, G]^a = 1$;

5)$P_1 \neq 1$ and $f(G) = n$.

We shall need the following lemma and theorem in the proof of Theorem 2.4.1.

Lemma 2.4.1. *Let G be a finite solvable group and assume that, for $i = 1, \dots, h$, P_i is a p_i-subgroup of G (p_i a prime) and for $i = 1, \dots, h-1$ we have $p_i \neq p_{i+1}$ and $[P_i, P_{i+1}] = P_i$. Then if $P_1 \neq 1$, we have $f(G) \geq h$.*

Proof. Use induction on $|G|$. If $h = 1$, then the lemma follows. Thus we may take G satisfying the hypothesis of the lemma with $h \geq 2$. Assume that $P_2 \subseteq F(G)$, then $P_1 = [P_2, P_1] \subseteq [F(G), P_1] \subseteq F(G)$. Thus both P_2 and P_1 are in $F(G)$. Since $F(G)$ is nilpotent and $p_1 \neq p_2$, $[P_2, P_1] = 1$, which is impossible. So $P_2 \not\subseteq F(G)$. Now $P_iF(G)/F(G)$ is a sequence of subgroups of $G/F(G)$ for $i = 1, \dots, h$. Since $[P_iF(G)/F(G), P_{i+1}F(G)/F(G)] =$

$[P_i, P_{i+1}]F(G)/F(G) = P_iF(G)/F(G)$ we can apply induction to $G/F(G)$, and get $f(G/F(G)) \geq h-1$. Hence the lemma.

Theorem 2.4.2. *Let A be any finite group, B_1 any subgroup of A and B a subgroup of B_1, $B_1 \neq B$. Suppose G is a finite group and A acts on G, $Q \subseteq G$ is a normal subgroup of GA such that $[B_1, G] \not\supseteq Q$. Then for any prime $p \nmid |QA|$ there exists an $\mathbb{F}_pGA$-module M with the following properties.*

1) $[Q, M] = M$;

2) $C_M(A) = 0$;

3) $[B, GM] \not\supseteq M$.

Proof. Let V be an $\mathbb{F}_p(G/[B_1, G])$-module on which Q acts nontrivially. Consider the $G/[G, B_1]$-composition series for V. If Q acts trivially on each composition factor, then Q acts trivially on V as $p \nmid |Q|$(see Proposition 2.2.5), which is a contradiction. Thus there exists an irreducible $\mathbb{F}_p(G/[B_1, G])$-module N on which Q acts nontrivially. Since $B_1 \neq B$ we may take R a cyclic subgroup of B_1 not contained in B. Let N_0 be a nontrivial irreducible $\mathbb{F}_p(R/R \cap B)$-module. Set $N_1 = N_0^{B_1}$ and take $M_0 = N_1 \otimes N$. Since $[B_1, G]$ acts trivially on N we may consider M_0 as an irreducible $\mathbb{F}_pGB_1$-module such that $[G, B_1] \subseteq \mathrm{Ker} M_0$. It satisfies the following:

a) $[Q, M_0] = M_0$.

Since Q does not lie in $[B_1, G]$ and acts nontrivially on N, $[N, Q] = N$, and so $[Q, M_0] = M_0$.

b) $C_{M_0}(B_1) = 0$.

First, $C_{N_0}(R) = 0$ because otherwise by the irreducibility of N_0 as a $(R/R \cap B)$-module, $C_{N_0}(R) = N_0$, and so $[N_0, R] = 1$ which is impossible. Now by Frobenius reciprocity $C_{N_0}(R) = 0$ implies that $C_{N_1}(B_1) = 0$ as $\mathrm{Hom}_{\mathbb{F}_pB_1}(N_0^{B_1}, Tr) \cong \mathrm{Hom}_{\mathbb{F}_pR}(N_0, Tr|_R)$ where Tr denotes the trivial B_1-module. Since the action of B_1 on M_0 is defined by $(n \otimes m)b_1 = nb_1 \otimes m$ for all $n \in N_1$, $m \in N$ and $b_1 \in B_1$, we have $C_{M_0}(B_1) = 0$, as $C_{N_1}(B_1) = 0$.

c) $C_{N_1}(B) \neq 0$ and so $C_{M_0}(B)$ is a nonzero GB-module.

Let $K = \mathrm{Ker}(M_0)$. $C_{M_0}(B)$ is $G/[G, B]$-invariant because $G/[G, B]$ is centralized by B. By Mackey's theorem 2.2.18, $N_1|_B \cong \sum N_0^x|_{R^x \cap B}|^B$ where the sum is over x, representatives of the double cosets RxB in B_1. In particular $N_1|_B \supseteq N_0|_{R \cap B}|^B$ and hence $C_{N_1}(B) \neq 0$. So $C_{M_0}(B)$ is a nonzero GB-module.

d) The theorem follows.

Define $M = M_0^{GA}$. 1) follows from a), 2) follows from b) as

$$0 = Hom_{\mathbb{F}_pB_1}(M_0, Tr|_{B_1}) \cong Hom_{\mathbb{F}_pA}(M, Tr)$$

where Tr is the trivial A-module. By the definition of M and Mackey's theorem 2.2.18 applied to (GB_1, G) we see that $M|_G$ is completely reducible. More precisely,

$$M|_G = {M_0}^{GA}|_G = \bigoplus_{i=1}^{m}(M_0 \otimes x_i)_{(GB_1)^{x_i}\cap G}|^G = \bigoplus_{i=1}^{m}(M_0 \otimes x_i)_G$$

as x_i, $i = 1, \ldots, m$ form a set of double coset representatives for (GB_1, G). As M_0 is irreducible as GB_1module we have the result. So by Maschke's theorem 2.2.6 $M|_{GB}$ is completely reducible: To see this let W be a GB-submodule of $M|_{GB}$. Then $M|_{GB} = W \oplus U$ for some G-submodule U. By Maschke's theorem 2.2.6 there exists a GB-invariant subspace U_0 such that $M = W \oplus U_0$. Set $C = C_{M_0}(B)$. By c), C is a nonzero GB-submodule of M. We can take a GB-submodule S of M such that $C \oplus S = M$ because of the completely reducibility of M. We have $S[B,G] \lhd SGB$. Since $C \subseteq M_0$ and $[G, B_1] \subseteq \mathrm{Ker} M_0$, we get $S[B,G] \subseteq C_{MG}(C)$ so that $S[B,G] \lhd MGB$. Now $[B,M] \subseteq S$ since $C \subseteq C_M(B)$. Hence $[B, MG] \subseteq S[B,G]$ and $[B, MG] \cap M \subseteq S$. By c) $S \neq M$. This concludes the proof of the theorem. □

Proof of Theorem 2.4.1

Use induction on $l(A : B)$. If $l(A : B) = 1$, take a cyclic subgroup R of A which is not in B, a nontrivial irreducible $\mathbb{F}_{p_1}(R/R \cap B)$-module N and set $N^A = M$. Then by Frobenius reciprocity $\mathrm{Hom}_{\mathbb{F}_{p_1}A}(N^A, Tr) \cong \mathrm{Hom}_{\mathbb{F}_{p_1}R}(N, Tr|_R)$, where Tr is a trivial A-module and so $C_M(A) = 0$ as $C_N(R) = 0$ and $C_M(B) \neq 0$. Take $G = P_1 = M/\cap_{a\in A}[B,M]^a$. Since $C_M(B) \neq 0$ we have $[M,B] \neq M$ and so $P_1 \neq 0$. Thus 1)-5) are satisfied in this case.

Hence we may assume that $n > 1$. Take $B_1 > B$ a subgroup of A such that $l(A : B_1) = l(A : B) - 1$. Then by induction, we may take $G = P_2 \ldots P_n$ satisfying the conclusion of the theorem for A and B_1 and the sequence $p_2, \ldots, p_n$. Since $\cap_{a\in A}[B_1, G]^a = 1$ and $1 \neq P_2 \lhd GA$ we have that $P_2 \nsubseteq [B_1, G]$ because otherwise $P_2^a = P_2 \subseteq [B_1, G]^a$ for all $a \in A$, which is impossible. Set $Q = P_2$ and $p = p_1$. By Theorem 2.4.2, there is an $\mathbb{F}_pP_2 \ldots P_n$-module M with $[M, Q] = [M, P_2] = M$. Now $C_G(A) = 1$ by induction and $C_M(A) = 0$ by Theorem 2.4.2 so that we have $C_{MG}(A) = 1$. Take $H = MG/\cap_{a\in A}[B, MG]^a$. Set P_1 to be the image of M in H. Now $P_1 \neq 1$ by 3) of Theorem 2.4.2. We consider the image of $P_2, \ldots, P_n$ in H and see that H provides a group of the form described in the statement of the theorem except possibly $f(H) = n$. By Lemma 2.4.1 since $P_1 \neq 1$ and so $f(H) \geq n$. Also $f(H) \leq n$ because $f(G) = n - 1$. Thus $f(G) = l(A : B) = n$. □

CHAPTER 3

SOME TECHNICAL LEMMAS PERTAINING THE MAIN RESULT

In this section we shall present some technical results pertaining the proof of our main result. All of them are new except the first one.

Let G be a finite group. If S is a subgroup of G and $a \in G$, then for any positive integer n we denote by $[S,a]^n$ the commutator subgroup $[S,a,\dots,a]$ with a repeated n times.

Lemma 3.0.2. *([4, Lemma 2]) Let $H = ST$, where $S \lhd H$, S is a p-group and T is a t-group for distinct primes p and t, and let α be an automorphism of H of order p^n which leaves T invariant. Assume that $C_{T/T_0}(z) = 1$ where $T_0 = C_T(S)$ and $z = \alpha^{p^{n-1}}$. Let V be a $kH\langle\alpha\rangle$-module on which S acts faithfully, and k a field of characteristic different from p. If $[C_V(z), C_S(z)] = 1$, then $[S,T] = 1$.*

Lemma 3.0.3. *Let $S \lhd SA$ where A is an abelian group and S is an s-group for some prime s which is coprime to $|A|$. Assume that S is abelian when $s = 2$. Let V be an irreducible kSA-module where k is a splitting field for all subgroups of SA and has characteristic not dividing $|SA|$. Suppose that S acts nontrivially and A acts fixed point freely on V. Then there is a nontrivial subgroup D of A such that $[S,D]$ acts trivially on V.*

Proof. Assume that S is abelian. By induction on $|S|$, we may assume that S is faithful on V. Since S is abelian, S acts scalarly on V over k. Hence $[Z(S),D] = [S,D]$ acts trivially on V. So we may assume that S is a nonabelian group of odd order. By induction on $|S|$, we may also assume that S is faithful on V. Since V is a irreducible kSA-module and $S \lhd SA$, by Clifford's theorem 2.2.7, V is a sum of homogenous S-modules. Consider a homogeneous S-submodule W of V. S acts faithfully on W since V is a faithful S-module. Now we can apply Corollary 2.3.1 to the action of SA on W and so there exists a nontrivial normal subgroup of A centralizing S. This completes the proof. $\square$

The following theorem is a sligth modification of Theorem 2 in [5].

Lemma 3.0.4. *Let $S\langle\alpha\rangle$ be a group such that $S \lhd S\langle\alpha\rangle$, S is an s-group, $\langle\alpha\rangle$ is cyclic of order p for distinct primes p and s, $\Phi(\Phi(S)) = 1$, $\Phi(S) \leq Z(S)$. Assume that S is abelian whenever $s = 2$. Let V be an irreducible $kS\langle\alpha\rangle$-module on which $[S, \alpha]$ acts nontrivially where k is a field of characteristic different from s. Then $[V, \alpha]^{p-1} \neq 0$ and Ker($C_S(\alpha)$ on V)= Ker($C_S(\alpha)$ on $[V, \alpha]^{p-1}$).*

Proof. Assume that lemma is false and consider a counterexample with dim$V + |S\langle\alpha\rangle|$ is minimal. Set $C = C_S(\alpha)$.

Claim 1. We may assume that k is a splitting field for all subgroups of $S\langle\alpha\rangle$.

Since $S\langle\alpha\rangle$ has only a finite number of subgroups, we may choose a finite algebraic extension field K of k so that K is a splitting field for all subgroups of $S\langle\alpha\rangle$. Let U be an irreducible $KS\langle\alpha\rangle$-submodule of the extension $V \otimes_k K$ of V to a $KS\langle\alpha\rangle$-module. If dim$U$<dim$V \oplus_k K$=dim$V$, then we get that $[U, \alpha]^{p-1}$ is a nonzero $KS\langle\alpha\rangle$-module and Ker($C_S(\alpha)$ on $[U, \alpha]^{p-1}$)=Ker($C_S(\alpha)$ on U) by induction. Since V is irreducible, $U \cong n \times V$ as a $kS\langle\alpha\rangle$-module for some positive integer n. So we have $[U, \alpha]^{p-1} = n \times [V, \alpha]^{p-1}$ and Ker($C_S(\alpha)$ on V)=Ker($C_S(\alpha)$ on U). Hence we have $[V, \alpha]^{p-1} \neq 0$ and Ker($C_S(\alpha)$ on V)= Ker($C_S(\alpha)$ on $[V, \alpha]^{p-1}$), a contradiction. Thus we may now assume that dim$U$=dim$V \oplus_k K$=dim$V$. Now V is an irreducible $KS\langle\alpha\rangle$-module. And all the hypothesis of lemma are satisfied by the action of $KS\langle\alpha\rangle$ on V, an irreducible $KS\langle\alpha\rangle$-module. Thus we may assume that k is a splitting field for all subgroups of $S\langle\alpha\rangle$.

Claim 2. $[Z(S), \alpha] = 1$.

Assume that $[Z(S), \alpha] \neq 1$. Then the stabilizer of any homogeneous component of $V|_{Z(S)}$ is S. By Clifford's theorem 2.2.7, V is induced from an irreducible S-module W. Then by Mackey's theorem 2.2.18 we have that $V|_{C\times\langle\alpha\rangle}$ and $W|_C \otimes k\langle\alpha\rangle$ are $C \times \langle\alpha\rangle$-isomorphic and so $[V, \alpha]^{p-1} \cong W_C \otimes [k\langle\alpha\rangle, \alpha]^{p-1}$. It follows that $[V, \alpha]^{p-1} \neq 0$ and Ker(C on $[V, \alpha]^{p-1}$)=Ker(C on V) because $[k\langle\alpha\rangle, \alpha]^{p-1} \neq 0$ and both $[V, \alpha]^{p-1}$ and V are multiples of the same C-module, namely $W|_C$. This contradiction shows that $[Z(S), \alpha] = 1$.

Let $\overline{S} = S/$Ker(S on V). Now $C \neq 1$ and S is nonabelian with $\Phi(S) \leq Z(S\langle\alpha\rangle)$. Then s is an odd prime because S is nonabelian.

Now we show that $\overline{S}\langle\alpha\rangle$ is a central product of $[\overline{S}, \alpha]\langle\alpha\rangle$ and $\overline{C} = C_{\overline{S}}(\alpha)$. by the coprimeness, $\overline{S} = [\overline{S}, \alpha]C_{\overline{S}}(\alpha)$. Let $[\overline{S}, \alpha] = \langle x^{-1}x^a \rangle$ and take $y \in C_{\overline{S}}(\alpha)$. Then

$$[x^{-1}x^a, y] = [x^{-1}, y]^{x^a}[x^a, y] = [x^{-1}, y][x, y^{a^{-1}}]^a = [x^{-1}, y][x, y] = 1$$

That is, $[C_{\overline{S}}(\alpha), [\overline{S}, \alpha]] = 1$. Let $\tilde{\overline{S}} = \overline{S}/\Phi(\overline{S})$. Also since $\tilde{\overline{S}} = [\tilde{\overline{S}}, \alpha] \oplus C_{\tilde{\overline{S}}}(\alpha)$ and by coprimeness, $[\overline{S}, \alpha] \cap C_{\overline{S}}(\alpha) \subseteq \Phi(\overline{S}) \subseteq Z(S\langle \alpha \rangle)$. Thus $\overline{S}\langle \alpha \rangle$ is a central product of $[\overline{S}, \alpha]\langle \alpha \rangle$ and $\overline{C} = C_{\overline{S}}(\alpha)$.

As V is irreducible as $\overline{S}\langle \alpha \rangle$-module and $C \lhd S\langle \alpha \rangle$, we can write V as a sum of homogeneous $\overline{C}$-modules by Clifford's theorem 2.2.7. Since $\overline{C}$ centralizes $[\overline{S}, \alpha]\langle \alpha \rangle$, $V|_{\overline{C}}$ is homogeneous. This would supply Ker($\overline{C}$ on $[V, \alpha]^{p-1}$)=Ker($\overline{C}$ on V) $= 1$ if $[V, \alpha]^{p-1} \neq 0$ and so Ker(C on $[V, \alpha]^{p-1}$)=Ker(C on V) if $[V, \alpha]^{p-1} \neq 0$. Hence we may assume that $[V, \alpha]^{p-1} = 0$. If char$k \neq p$, then $[V, \alpha] = 0$ and since $[S, \alpha]$ acts nontrivially on V we get $[S, \alpha] = 1$, which is impossible. Thus p is the characteristic of the field k.

Since V is an irreducible $S\langle \alpha \rangle$-module on which $\overline{S}$ acts faithfully, $\Phi(\overline{S})$ is cyclic by Proposition 2.2.4. Since $\Phi(\Phi(S)) = 1$, $\Phi(\overline{S})$ is cyclic of order s.

Set $S_1 = [\overline{S}, \alpha]$ and $Z = [Z(S_1), \alpha]$. We claim that $Z = 1$. Otherwise we consider $V|_Z$. It is completely reducible as $Z \lhd \overline{S}\langle \alpha \rangle$ and α can not stabilize a homogeneous component as $[Z, \alpha] = [Z(S_1), \alpha, \alpha] = Z \neq 1$. Since $[C_{\overline{S}}(\alpha), Z] = 1$ and $[S_1, Z] = 1$ by the three subgroup lemma 2.1.1 and by Proposition 2.1.9 and since $\overline{S} = [\overline{S}, \alpha]C_{\overline{S}}(\alpha)$ by the coprimeness, $\overline{S}$ centralizes Z. Then by Clifford's theorem 2.2.7, V is induced from an irreducible $\overline{S}$-module W. Then by Mackey's theorem 2.2.18 we obtain that $V|_{\overline{C} \times \langle \alpha \rangle} \simeq W|_{\overline{C}} \otimes k\langle \alpha \rangle$ as $\overline{C} \times \langle \alpha \rangle$-modules and so $[V, \alpha]^{p-1} \cong W_{\overline{C}} \otimes [k\langle \alpha \rangle, \alpha]^{p-1}$. Thus $[V, \alpha]^{p-1} \neq 0$ because $[k\langle \alpha \rangle, \alpha]^{p-1} \neq 0$ and both $[V, \alpha]^{p-1}$ and V are multiples of the same $\overline{C}$-module, namely $W|_{\overline{C}}$. Then again Ker($\overline{C}$ on $[V, \alpha]^{p-1}$)=Ker($\overline{C}$ on V) as $V|_{\overline{C}}$ is homogeneous. This supplies a contradiction and hence $Z = 1$.

In particular, S_1 is nonabelian and $Z(S_1) \leq C_{S_1}(\alpha) \leq \Phi(S_1) \leq Z(S_1)$ deducing that $Z(S_1) = \Phi(S_1) = C_{S_1}(\alpha) \geq S_1' \neq 1$. Since $\Phi(S_1) \leq \Phi(\overline{S})$, we also have $S_1' = \Phi(S_1) = Z(S_1)$ is cyclic of order s. That is S_1 is extraspecial. Recall that $\overline{S}\langle \alpha \rangle$ is a central product of $[\overline{S}, \alpha]\langle \alpha \rangle$ and $\overline{C} = C_{\overline{S}}(\alpha)$. Then $V = U_1 \otimes_k U_2$ for an irreducible $S_1\langle \alpha \rangle$-module U_1 and an irreducible $\overline{C}$-module U_2 by [10, Theorem 3.7.1]. Since $[V, \alpha]^{p-1} = 0$, the degree of the minimum polynomial is less than p. And it is also true for U_1. Appealing to Theorem IX.3.2 in [13] for the action of $S_1\langle \alpha \rangle$ on U_1 together with the fact that S_1 is extraspecial with $[Z(S_1), \alpha] = 1$, we get that $s = 2$, a contradiction. □

Lemma 3.0.5. *[5, Lemma 2] Let $S \lhd S\langle \alpha \rangle$ where $\langle \alpha \rangle$ is cyclic of prime order and let V be an irreducible $kS\langle \alpha \rangle$-module. If E is an $\langle \alpha \rangle$-invariant subgroup of $Z(S)$ and U is a nonzero*

$E\langle\alpha\rangle$-submodule of V, then Ker(E on V)=Ker(E on U).

Lemma 3.0.6. *[5, Lemma 3] Let $S\langle\alpha\rangle$ be a group such that $S \lhd S\langle\alpha\rangle$ where $\langle\alpha\rangle$ is of prime order p. Suppose that V is a $kS\langle\alpha\rangle$-module for a field k of characteristic different from p, and Ω is an $S\langle\alpha\rangle$-stable subset of V^*. Set $V_0 = \cap\{Kerf \mid f \in \Omega - C_\Omega(\alpha)\}$. If there exists a nonzero f in Ω and $x \in S$ such that $f(V_0) \neq 0$ and $[x, a, \alpha] \notin C_S(f)$ for each $1 \neq a \in \langle\alpha\rangle$, then $C_V(\alpha) \nsubseteq V_0$.*

Lemma 3.0.7. *Let $A = \langle a\rangle$ be a cyclic group of order p^n for some prime p, and let G be a group acted on by A. Suppose that $S \lhd GA$ is an s-group and T is an A-invariant t-subgroup of G for distinct primes s and t which are both different from p such that $[S, T] \neq 1$ and $[T, z] = T$ where $z = a^{p^{n-1}}$. If $\Phi(T/T_0) \neq 1$ where $T_0 = C_T(S)$, then assume that there is an A-invariant h-subgroup H of G for a prime h different from p such that $H \leq C_G(\Phi(T/T_0))$, $H/C_H(T/T_0)$ is elementary abelian and $[T/T_0, H] = T/T_0$. Let V be a kGA-module on which S acts nontrivially and k is a field of characteristic not dividing $|STHA|$. Then $[C_V(A), C_S(A)] \neq 1$.*

Proof. We use induction on $|SA|$+$\dim_k V$. Set $\overline{S} = S/\text{Ker}(S \text{ on } V)$. Let $\overline{S_1}$ be a minimal $T\langle\alpha\rangle$-invariant subgroup of $\overline{S}$ on which T acts nontrivially. As $s \neq t$, $[\overline{S_1}, T] = \overline{S_1}$, $\overline{S_1}/\Phi(\overline{S_1})$ is an irreducible $T\langle\alpha\rangle$-module, $[\Phi(\overline{S_1}), T] = 1$ and $\overline{S_1}$ is special by Proposition 2.2.9. If $|\overline{S_1}| < |S|$, an induction argument gives that $[C_V(A), C_{\overline{S_1}}(A)] = [C_V(A), \overline{C_{S_1}(A)}] \neq 1$, that is $[C_V(A), C_S(A)] \neq 1$. Thus $S_1 = S$.

We may also assume that $G = STH$. Let $\tilde{S} = S/\Phi(S)$. Notice that $[T, z] = T$ acts nontrivially on each irreducible component of $\tilde{S}|_T$. It is easy to see that $\Omega = \{\ f \in (\tilde{S}) \mid$ there exists an irreducible component N of $V|_S$ such that $\text{Ker}(S \text{ on } N)\Phi(S)/\Phi(S) \subseteq \text{Ker}f\}$ is a GA-invariant subset which linearly spans the dual space $(\tilde{S})^*$. Apply Proposition 2.2.14 to the action of $(T/T_0)A$ on $\tilde{S}$ with Ω we see that $C_{\tilde{S}}(A) \nsubseteq \cap\{\text{Ker}f \mid f \in \Omega - C_\Omega(z)\}$. This gives an $\tilde{x} \in C_{\tilde{S}}(A)$ and an $f \in \Omega - C_\Omega(z)$ such that $f(\tilde{x}) \neq 0$. Now $\tilde{x} \notin$Ker($\tilde{S}$ on N) for some irreducible component N of $V|_S$ by the definition Ω.

On the other hand, one more application of Proposition 2.2.14 to the action of GA on V gives that $C_S(A) = C_{C_S(A)}(C_V(A))$ is contained in the kernel of each irreducible component of $V|_S$ on which $[S, z]$ acts nontrivially. It follows that $[S, z]$ is trivial on N, that is, $[S, z] \subseteq$Ker$f$ and so $f \in C_\Omega(z)$, a contradiction. □

Lemma 3.0.8. *[5, Lemma 1] Let $S\langle\alpha\rangle$ be a group such that $S \lhd S\langle\alpha\rangle$, S is an s-group for some prime s, $\Phi(S) \leq Z(S)$, $\langle\alpha\rangle$ is cyclic of order p for some odd prime p. Suppose that V*

is a $kS\langle\alpha\rangle$-module for a field of characteristic different from s. Then $C_V(\alpha) \neq 0$ if one of the following is satisfied:

(i) $[Z(S), \alpha]$ is nontrivial on V.

(ii) $[S, \alpha]^{p-1}$ is nontrivial on V and $p = s$.

Furthermore, if $S\langle\alpha\rangle$ acts irreducibly on V or the characteristic of k is different from p, then we also have Ker(C on $C_V(\alpha)$)= Ker(C on V) where $C = C_D(\alpha)$ for

$$D = \begin{cases} [S, \alpha]^{p-1} & \text{when (ii) holds,} \\ S & \text{when (i) holds.} \end{cases}$$

Lemma 3.0.9. *Let SA be a group where $S \lhd SA$, S is a q-group for an odd prime q, $\Phi(S) \leq Z(S)$, A is cyclic of order pq for some prime p. Suppose that $[S, A_q]^{q-1} \nleq \Phi(S)$ and $[S, A_p] = S$ where A_p and A_q denote the Sylow p- and q-subgroups of A respectively. Let V be a $\mathbb{C}SA$-module on which $[S, A_q]^{q-1}$ acts nontrivially. Then $C_V(A) \neq 0$.*

Proof. Assume the contrary. Set $\overline{S} = S/\operatorname{Ker}(S \text{ on } V)$. By Lemma 3.0.8 applied to the action of $\overline{S}A_q$ on V, we see that $C_V(A_q) \neq 0$ and $\operatorname{Ker}(C_{\overline{D}}(A_q) \text{ on } C_V(A_q)) = \operatorname{Ker}(C_{\overline{D}}(A_q) \text{ on } V)$ where $D = [S, A_q]^{q-1}$. This supplies that $\operatorname{Ker}([C_{\overline{D}}(A_q), A_p] \text{ on } C_V(A_q)) = \operatorname{Ker}([C_{\overline{D}}(A_q), A_p]$ on V) also. If $[C_{\overline{D}}(A_q), A_p] \neq 1$, then we apply Lemma 3.0.3 to the action of $[C_{\overline{D}}(A_q), A_p]A_p$ on $C_V(A_q)$ and get a contradiction. Hence $[C_{\overline{D}}(A_q), A_p] = 1$ forcing that $[\overline{D}, A_p] = 1$ by Thompson's $A \times B$ Lemma 2.1.15. But then $\overline{D} \leq \Phi(\overline{S}) = \overline{\Phi(S)}$, which is not the case. □

The following lemma is a sligth modification of Theorem 1 in [5].

Lemma 3.0.10. *Let $S\langle\alpha\rangle$ be a group such that $S \lhd S\langle\alpha\rangle$, S is an s-group with $\Phi(\Phi(S)) = 1$, $\Phi(S) \leq Z(S)$ and $\langle\alpha\rangle$ is cyclic of order p for primes s and p. Assume either $s = p \geq 5$ or $s \neq p$, p is odd and S is abelian whenever $s = 2$. Let V be a $kS\langle\alpha\rangle$-module where k is a field of characteristic not dividing ps such that $[S, \alpha]^{p-1}$ acts nontrivially on each irreducible submodule of $V|_S$. Let Ω be an $S\langle\alpha\rangle$-stable subset of V^* which linearly spans V^* and set $V_0 = \{Kerf \mid f \in \Omega - C_\Omega(\alpha)\}$. Then $C_V(\alpha) \nsubseteq V_0$ and Ker($C_D(\alpha)$ on $C_V(\alpha)/C_{V_0}(\alpha)$)= Ker($C_D(\alpha)$ on V) where*

$$D = \begin{cases} [S, \alpha]^{p-1} & \textit{when} \quad s = p \\ S & \textit{otherwise} \end{cases}$$

Proof. Assume that the lemma is false and consider a counterexample with $\dim V + |S\langle\alpha\rangle|$ minimal. Set $X = C_V(\alpha)/C_{V_0}(\alpha)$ and $C = C_D(\alpha)$.

Claim 1. We may assume that $S\langle\alpha\rangle$ acts irreducibly on V and k is the splitting field for all subgroups of $S\langle\alpha\rangle$.

Since char$k \nmid ps$, V is completely reducible as an $S\langle\alpha\rangle$-module and so we have a collection $\{V_1, \ldots, V_l\}$ of irreducible $S\langle\alpha\rangle$-submodules of V such that $V = \bigoplus_{i=1}^{l} V_i$. Now $[S, \alpha]^{p-1}$ acts nontrivially on each irreducible constituent of $V_i|_S$ and hence $[S, \alpha]^{p-1}$ acts nontrivially on each V_i for $i = 1, \ldots, l$. Fix $j \in \{1, \ldots, l\}$. Set $V_j = U$ and $\Omega_U = \{f|_U \mid : f \in \Omega\} \subseteq U^*$. Now we show that $\langle \Omega|_U \rangle = U^*$:

Consider $\gamma : V^* \to U^*$ with $\gamma(g) = g|_U$. γ is onto; to see this let $U = \langle e_1, \ldots, e_n \rangle$ and $V = \langle e_1, \ldots, e_n, \ldots, e_m \rangle$ and $V^* = \{f_1, \ldots, f_m\}$ where $f_i : V \to k$ with $\sum_{i=1,\ldots,m} a_i e_i \mapsto a_j$, for $j = 1, \ldots, m$. Let $f'_j(\sum_{i=1,\ldots,n} a_i e_i) = a_j$, $j = 1, \ldots, n$. Then $f'_j = f_j|_U$ for $j = 1, \ldots, n$ and $f_j|_U = 0$ for $j = n+1, \ldots, m$. Also $\langle f'_j \rangle = U^*$ and $\langle f_j \rangle = V^*$. Take $f \in U^*$ then $f = \sum_{i=1,\ldots,n} g_i f'_i$, let $F = \sum_{i=1,\ldots,n} g_i f_i$ then $\gamma(F) = f$. Second, Ω_U spans U^*. Let $\Omega = \{f_1, \ldots, f_m\}$ and $\Omega_U = \{f_1|_U, \ldots, f_m|_U\}$. Let $h \in U^*$. Then there exists $g \in V^*$ such that $g|_U = h$. As Ω spans V^*, $g = c_1 f_1 + \ldots + c_m f_m$ and $h = g|_U = c_1 f_1|_U + \ldots + c_m f_m|_U$. So, Ω_U spans U^*. Let $f|_U \in \Omega_U$ and $x \in S\langle\alpha\rangle$. Now $g = xf \in \Omega$ since Ω is $S\langle\alpha\rangle$-invariant, and hence $(xf)(v) = f(x^{-1}v) = g(v)$ for all $v \in V$ implying $(xf|_U)(v) = g|_U(v)$. So Ω_U is $S\langle\alpha\rangle$-invariant. Thus $\Omega|_{V_i}$ is an $S\langle\alpha\rangle$-stable subset of V_i^* and $\langle \Omega|_{V_i} \rangle = V_i^*$ for each $i = 1, \ldots, l$.

If V is not irreducible as an $S\langle\alpha\rangle$-module, we apply induction to the action of $S\langle\alpha\rangle$ on V_i for each i and get $C_{V_i}(\alpha) \nsubseteq (V_i)_0$ and Ker(C on $C_{V_i}(\alpha)/C_{(V_i)_0}(\alpha)$)=Ker($C$ on V_i). Set $X_i = C_{V_i}(\alpha)/C_{V_i \cap V_0}(\alpha)$. Now Ker($C$ on X_i)=Ker(C on V_i) since $(V_i)_0 = \cap\{\mathrm{Ker} g \mid g \in \Omega_i - C_{\Omega_i}(\alpha)\} \supseteq V_i \cap V_0$. As $V = \bigoplus_{i=1}^{l} V_i$ and $X \simeq \bigoplus_{i=1}^{l} X_i$, it follows that Ker($C$ on X)=Ker(C on V). Therefore we can regard V as an irreducible $S\langle\alpha\rangle$-module.

Since $S\langle\alpha\rangle$ has only a finite number of subgroups, we may choose a finite algebraic extension field K of k so that K is a splitting field for all subgroups of $S\langle\alpha\rangle$. Let $U = V \otimes_k K$. If one could show that $C_U(\alpha) \nsubseteq U_0 = V_0 \otimes_k K$ and Ker($C_D(\alpha)$ on $C_U(\alpha)/C_{U_0}(\alpha)$)= Ker($C_D(\alpha)$ on U), as $C_U(\alpha) = C_V(\alpha) \otimes_k K$ by Proposition 2.2.17. It would follow that $C_V(\alpha) \nsubseteq V_0$ and Ker($C_D(\alpha)$ on $C_V(\alpha)/C_{V_0}(\alpha)$)= Ker($C_D(\alpha)$ on V). Thus we may assume that k is a splitting field for all subgroups of $S\langle\alpha\rangle$.

Let $\overline{S} = S/\mathrm{Ker}(S \text{ on } V)$ and $\overline{C} = C_{\overline{D}}(\alpha)$.

Claim 2. $[Z(\overline{S}), \alpha, \alpha] = 1$.

Assume the contrary. Set $S_1 = Z(\overline{S})\overline{C}$. Then S_1 is an $\langle\alpha\rangle$-invariant subgroup of $\overline{S}$ and $V|_{S_1\langle\alpha\rangle}$ is completely reducible as char$k \nmid |S_1\langle\alpha\rangle|$. Note that $\overline{C} \lhd S_1\langle\alpha\rangle$. Let V_i be an irreducible $S_1\langle\alpha\rangle$-submodule of V and W be a homogeneous component of $V_i|_{\overline{C}}$.

Now $Z(\overline{S})\langle\alpha\rangle \leq C_{S_1\langle\alpha\rangle}(\overline{C}) \leq N_{\overline{S}\langle\alpha\rangle}(W)$ by Clifford's theorem 2.2.7. This yields that $V_i|_{\overline{C}}$ is homogeneous. We also observe that Ker($Z(\overline{S})$ on V_i)=Ker($Z(\overline{S})$ on V)=1 by applying

Lemma 3.0.5 to the action of $\overline{S}\langle\alpha\rangle$ on V.

We shall prove that $C_{Z(\overline{S})}(f) = 1$ for each $0 \neq f \in C_\Omega(\alpha)$: Consider $\langle f\rangle = \{cf \mid c \in k\}$, a $C_{Z(\overline{S})}(f)$-submodule of V^*. Appealing to Lemma 3.0.5 together with $\langle f\rangle$ and $C_{Z(\overline{S})}(f)$, we get $C_{Z(\overline{S})}(f)$ =Ker($C_{Z(\overline{S})}(f)$ on $V^*) = 1$, as desired.

Since $[Z(\overline{S}), \alpha] \neq 1$, $[Z(S_1), \alpha]$ is nontrivial on V_i. Applying Lemma 3.0.8 to the action of $S_1\langle\alpha\rangle$ on V_i, we obtain $C_{V_i}(\alpha) \neq 0$. If $C_{V_i}(\alpha) \nsubseteq V_0$, it follows that Ker($\overline{C}$ on $C_{V_i}(\alpha)/C_{V_i \cap V_0}(\alpha)$)=Ker($\overline{C}$ on V_i) as $V_i|_{\overline{C}}$ is homogeneous. Hence Ker(C on $C_{V_i}(\alpha)/C_{V_i\cap V_0}(\alpha)$) =Ker($C$ on V_i). This forces that there is an irreducible $S_1\langle\alpha\rangle$-submodule V_i of the completely reducible module $V_{S_1\langle\alpha\rangle}$ such that $C_{V_i}(\alpha) \subseteq V_0$. Since $0 \neq C_{V_i}(\alpha)$, we have $V_i \cap V_0 \neq 0$. Set $\Omega_i = \Omega|_{V_i}$. Now Ω_i is an $S_1\langle\alpha\rangle$-stable subset of V_i^*, and $(V_i)_0 = \cap\{\mathrm{Ker}h \mid h \in \Omega_i - C_{\Omega_i}(\alpha)\} \neq 0$ as $V_i \cap V_0 \subseteq (V_i)_0$. Let $f \in \Omega$ be such that $f((V_i)_0) \neq 0$. Then $f_i = f|_{V_i} \in C_{\Omega_i}(\alpha)$. Consider $\langle f_i\rangle = \{cf_i \mid c \in k\}$, a $C_{Z(\overline{S})}(f_i)\langle\alpha\rangle$-submodule of V_i^*. Appealing Lemma 3.0.5 together with $\langle f_i\rangle$ and $C_{Z(\overline{S})}(f_i)$, we get $C_{Z(\overline{S})}(f_i)$ =Ker($C_{Z(\overline{S})}(f_i)$ on $V_i^*) = 1$. On the other hand, there exists $x \in Z(\overline{S})$ such that $[x, \alpha, \alpha] \neq 1$, as $[Z(\overline{S}), \alpha, \alpha] \neq 1$. It follows that $[x, a, \alpha] \neq 1$ for any $1 \neq a \in \langle\alpha\rangle$, that is $[x, a, \alpha] \notin C_{S_1}(f_i)$, for any $1 \neq a \in \langle\alpha\rangle$. Now Lemma 3.0.6 applied to the action of $S_1\langle\alpha\rangle$ on V_i, together with f_i and Ω_i, gives that $C_{V_i}(\alpha) \nsubseteq (V_i)_0$. This is a contradiction as $V_i \cap V_0 \subseteq (V_i)_0$ and $C_{V_i}(\alpha) \subseteq V_0$. Thus we have the claim.

Now assume that $s = p \geq 5$. Since $[\overline{S}, \alpha]^{p-1} \neq 1$, $[\overline{S}, \alpha]^{p-3} \neq 1$. Set $S_1 = [\overline{S}, \alpha]^{p-3}$. We can prove that $[S_1, [\overline{S}, \alpha]^{p-1}] \leq [\Phi(\overline{S}), \alpha]^{p-3} = 1$ (see ([3],5.37)). Hence $[\overline{S}, \alpha]^{p-1} \leq Z(S_1)$.

We have a collection $\{V_1, \ldots, V_l\}$ of irreducible $S_1\langle\alpha\rangle$-modules such that $V = \bigoplus_{i=1}^{l} V_i$. Fix $i \in \{1, \ldots, l\}$. We notice that $\overline{C} = C_{[\overline{S},\alpha]^{p-1}}(\alpha) \lhd S_1\langle\alpha\rangle$ implying $V|_{\overline{C}}$ is completely reducible. In particular, $\overline{C} \leq Z(S_1\langle\alpha\rangle)$ and so $V_i|_{\overline{C}}$ is homogeneous.

Set $X_i = C_{V_i}(\alpha)/C_{V_i\cap V_0}(\alpha)$ and assume that Ker($\overline{C}$ on X_i) $\neq$ Ker($\overline{C}$ on $C_{V_i}(\alpha)$). If $[\overline{S}, \alpha]^{p-1}$ is trivial on V_i, then $\overline{C}$ acts trivially on V_i, and this is a contradiction. Hence $[\overline{S}, \alpha]^{p-1}$ is not trivial on V_i. If $V_i \cap V_0 = 0$, then Ker($\overline{C}$ on X_i)=Ker($\overline{C}$ on $C_{V_i}(\alpha)$), and again we have a contradiction. Hence, $V_i \cap V_0 \neq 0$, and there exists some $f \in \Omega$ such that $f(V_i \cap V_0) \neq 0$. Now $f \in C_\Omega(\alpha)$. Set $f|_{V_i} = f_i$. Now $\langle f_i\rangle = \{cf_i \mid c \in k\}$ is a $C_{[\overline{S},\alpha]^{p-1}}(f_i)\langle\alpha\rangle$-submodule of V_i^*. Appealing Lemma 3.0.5, we get $C_{Z(S_1)}(f_i)$ =Ker($C_{Z(S_1)}(f_i)$ on V_i^*). We also have $C_{[\overline{S},\alpha]^{p-1}}(f_i) \leq C_{Z(S_1)}(f_i)$. Thus $C_{[\overline{S},\alpha]^{p-1}}(f_i)$ is properly contained in $[\overline{S}, \alpha]^{p-1}$, that is, there is $1 \neq y \in [\overline{S}, \alpha]^{p-1} - C_{[\overline{S},\alpha]^{p-1}}(f_i)$, and $x \in [\overline{S}, \alpha]^{p-3}$ such that $y = [x, \alpha, \alpha]$. It follows that $1 \neq [x, a, \alpha] \notin C_{[\overline{S},\alpha]^{p-1}}(f_i)$ for any $1 \neq a \in \langle\alpha\rangle$. Now we can apply Lemma 3.0.6 to the action of $S_1\langle\alpha\rangle$ on V_i together with $\Omega_i = \Omega|_{V_i}$ and f_i and obtain that $C_{V_i}(\alpha) \nsubseteq V_0$. As $V_i|_{\overline{C}}$ is homogeneous, we already have Ker($\overline{C}$ on X_i)=Ker($\overline{C}$ on $C_{V_i}(\alpha)$).

Therefore we conclude that Ker($\overline{C}$ on $C_V(\alpha)/C_{V_0}(\alpha)$)=Ker($\overline{C}$ on $C_V(\alpha)$). Appealing

Lemma 3.0.8 together with V and $\overline{S}\langle\alpha\rangle$, we also see that $C_V(\alpha) \neq 0$ and Ker($\overline{C}$ on $C_V(\alpha)$)=Ker($\overline{C}$ on V)=1 hold. Thus Ker($\overline{C}$ on $C_V(\alpha)/C_{V_0}(\alpha)$)=Ker($\overline{C}$ on V)=1. Thus Ker(C on $C_V(\alpha)/C_{V_0}(\alpha)$)=Ker($C$ on V). Since $[\overline{S},\alpha]^{p-1} \neq 1$ and $s = p$, $C \neq 1$. Hence $\overline{C}$ is nontrivial on V and so is on $C_V(\alpha)/C_{V_0}(\alpha)$. This supplies that $C_V(\alpha) \nsubseteq V_0$, a contradiction. Thus $s \neq p$, p is odd and S is abelian whenever $s = 2$.

Now $[\Phi(\overline{S}),\alpha] = 1$. Then $\Phi(\overline{S}) \leq Z(\overline{S}\langle\alpha\rangle)$. Since $[\overline{S},\alpha] \cap C_{\overline{S}}(\alpha) \leq \Phi(\overline{S})$ and $[[\overline{S},\alpha], C_{\overline{S}}(\alpha)] = 1$, $\overline{S}$ is a central product of $[\overline{S},\alpha]$ and $C_{\overline{S}}(\alpha)$. As $\overline{C} = C_{\overline{S}}(\alpha) \lhd \overline{S}\langle\alpha\rangle$, $V|_{\overline{C}}$ is completely reducible. In fact $V|_{\overline{C}}$ is homogeneous, because any homogeneous component is stabilized by $\overline{S}\langle\alpha\rangle$ as $\overline{C}$ is centralized by $[\overline{S},\alpha]\langle\alpha\rangle$. It follows that Ker($\overline{C}$ on $C_V(\alpha)/C_{V_0}(\alpha)$)=Ker($\overline{C}$ on V)=1 if $C_V(\alpha) \nsubseteq V_0$, that is Ker($C$ on $C_V(\alpha)/C_{V_0}(\alpha)$)=Ker($C$ on V) if $C_V(\alpha) \nsubseteq V_0$. Hence $C_V(\alpha) \subseteq V_0$. Note that $C_V(\alpha) \neq 0$: Assume that $C_V(\alpha) = 0$. Since $[S,\alpha]$ acts nontrivially on V we can apply Theorem 2.3.2 (Gagola) to the action of $[S,\alpha]\langle\alpha\rangle$ on V. Then we get $s = 2$. By hypothesis, $\overline{S}$ is abelian and by claim 2, $[\overline{S},\alpha] = [\overline{S},\alpha,\alpha] = 1$, that is $[S,\alpha]$ acts trivially on V, a contradiction. Then there exists $0 \neq f \in C_\Omega(\alpha)$ with $f(V_0) \neq 0$. Now $C_{Z(\overline{S})}(f)$=Ker($C_{Z(\overline{S})}(f)$ on V^*) $= 1$ by Lemma 3.0.5. It follows that $C_{Z([\overline{S},\alpha])}(f) = 1$, as $[C_{\overline{S}}(\alpha), [\overline{S},\alpha]] = 1$. Then $C_{[\overline{S},\alpha]}(f)$ is properly contained in $[\overline{S},\alpha]$. Let M be a maximal α-invariant subgroup of $[\overline{S},\alpha]$ containing $C_{[\overline{S},\alpha]}(f)$. The abelian group $[\overline{S},\alpha]/M = \widetilde{[\overline{S},\alpha]}$ forms an irreducible $\langle\alpha\rangle$-module on which $\langle\alpha\rangle$ acts fixed point freely. Thus we have $[\tilde{x}, a] \neq 0$ for any $0 \neq \tilde{x} \in \widetilde{[\overline{S},\alpha]}$. It follows that $[\tilde{x}, a, \alpha] \neq 0$ for each $1 \neq a \in \langle\alpha\rangle$. Put $\tilde{x} = xM$ for $x \in [\overline{S},\alpha]$. Then $[x,a,\alpha] \notin M$. In particular, $[x,a,\alpha] \notin C_{[\overline{S},\alpha]}(f)$ for each $1 \neq a \in \langle\alpha\rangle$. Recall that $V|_{\overline{C}}$ is homogeneous. Then Lemma 3.0.6 applied to the action of $\overline{S}\langle\alpha\rangle$ on V gives that $C_V(\alpha) \nsubseteq V_0$. This contradiction completes the proof. □

Before the next result we need to give the following definitions due to Dade [3].

Definition 3.0.2. *Suppose a group K acts on a finite solvable group G denoted by $(K$ on $G)$. Then each K-composition factor A/B of G is an elementary abelian p-group, for some prime p. So it can be viewed as an irreducible module which is called an irreducible component of $(K$ on $G)$. If K also acts on another finite solvable group H, then $(K$ on $G)$ and $(K$ on $H)$ are weakly equivalent if each nontrivial irreducible component of $(K$ on $G)$ is K-isomorphic to an irreducible component of $(K$ on $H)$ and vice versa.*

Let a group K act on a group G and another group L act on both K and G. We say that $(K$ on $G)$ is L-invariant if $(\sigma^\tau) = (\sigma^\rho)^{\tau\rho}$, for all $\sigma \in G$, $\tau \in K$, $\rho \in L$. In that case we may form the triple semi direct product LKG.

If K acts on G and L acts on K, then $(K$ on $G)$ is weakly L-invariant if the actions $(K$

on G) and $(K$ on $G)^\sigma$, the latter given by

$$\tau \to (K \text{ on } G)(\tau^{\sigma^{-1}})\text{for } \tau \in K,$$

are weakly equivalent for all $\sigma \in L$.

Definition 3.0.3. *A Fitting chain consists of groups $A_1, \dots, A_t$ and actions A_i on A_{i+1} for $i = 1, \dots, t-1$ satisfying:*

1) Each A_i is a nontrivial p_i-group for some prime p_i, for $i = 1, \dots, t$.

2) $\Phi(A_i) \leq Z(A_i)$, for $i = 1, \dots, t$.

3) $\Phi(\Phi(A_i)) = 1$, for $i = 1, \dots, t$.

4) If p_i is odd, then A_i has exponent p_i, for $i = 1, \dots, t$.

5) $p_i \neq p_{i+1}$, for $i = 1, \dots, t-1$.

6) $[\Phi(A_{i+1}), A_i] = 1$, for $i = 1, \dots, t-1$.

7) Ker(A_i on $A_{i+1}) = 1$, for $i = 1, \dots, t-1$.

8) (A_{i+1} on $A_{i+2}/\Phi(A_{i+2})$ is weakly A_i-invariant, for $i = 1, \dots, t-2$.

Lemma 3.0.11. *Let $G \lhd GA$ and $\langle a \rangle \trianglelefteq A$ of prime order p. Assume that $P_1, \dots, P_t$ is an A-Fitting chain of G such that $[P_1, a] \neq 1$, P_i is a p_i-group for a prime $p_i$and $t \geq 3$. We also assume that P_i is abelian whenever $p_i = 2$ and $p \geq 5$ whenever $p_i = p$ for some $i \in \{1, \dots, t\}$. Then there are sections $D_{i_0}, \dots, D_t$ of $P_{i_0}, \dots, P_t$ respectively, forming an A-Fitting chain of G such that a centralizes each D_j for $j = i_0, \dots, t$ where* $i_0 = \begin{cases} 2 & \textit{if} \quad p_1 \neq p \\ 3 & \textit{if} \quad p_1 = p \end{cases}$

Proof. It will be sufficient to demonstrate Claim 1 and Claim 2 appearing in the proof of [5, Theorem 3] with the hypothesis as revised above. One can observe that these claims can be restated and proven as by-products of Lemma 3.0.10 and Lemma 3.0.4 which are slightly altered versions of Theorem 1 and Theorem 2 in [5] by an analogous reasoning. □

CHAPTER 4

MAIN RESULT

In this chapter we shall state and prove our main result. A contradiction will be deduced over a series of steps from the assumption of the existence of a counterexample. This chapter also contains a corollary as an immediate consequence of the main result.

Main Theorem. *Let G be finite group admitting a fixed point free automorphism group $\langle\alpha\rangle$ whose order is a product of three primes which are coprime to* 6. *If the Sylow* 2*-subgroups of G are abelian, then G has Fitting length at most* 3.

Proof of the theorem. If $\langle\alpha\rangle$ is a p-group for a prime p, because of fixed point free action $|G|$ is not divisible by p. Since $\langle\alpha\rangle$ is abelian of order coprime to $|G|$, $\langle\alpha\rangle$ acts with regular orbits on G by a remark given in introduction. Then the result follows by Turull's work in [26]. Also if $|\langle\alpha\rangle|$ is a product of three distinct primes, a theorem due to Ercan and Güloğlu [4] (see also [5]) gives the result. Thus we may assume that $|\langle\alpha\rangle| = p^2q$ for two distinct primes p and q. Set $\langle\alpha\rangle = \langle\alpha_p\rangle \times \langle\alpha_q\rangle$ where $|\alpha_p| = p^2$ and $|\alpha_q| = q$ and $z = \alpha_p^p$. Let G be a minimal counterexample to the theorem. Then we may assume that $F_4(G) = G$. As $C_G(\alpha) = 1$, for any prime dividing $|G|$ we have a unique $\langle\alpha\rangle$-invariant Sylow subgroup of G by Proposition 2.1.11.

(1) There is an irreducible $\langle\alpha\rangle$-tower (C_i), $i = 1, 2, 3, 4$ in the sense of [25] satisfying the following:

(i) $\pi(C_i) = \{p_i\}$ consists of a single prime for $i = 1, 2, 3, 4$ and $p_i \neq p_{i+1}$ for $i = 1, 2, 3$.

(ii) C_i is $\langle\alpha\rangle$-invariant for $i = 1, 2, 3, 4$ and C_i is normalized by C_j for $j > i$ and $i = 1, 2, 3$.

(iii) $\overline{C_i} = C_i/D_i$ is a special group on the Frattini factor group of which $(\prod_{j>i} C_j)\langle\alpha\rangle$ acts irreducibly for $i = 1, 2, 3$ where $D_i = C_{C_i}(C_{i-1}/D_{i-1})$ for $i > 1$ and $D_1 = 1$;

(iv) $[C_i, C_{i+1}] = C_i$ for $i = 1, 2, 3$.

Let M be a minimal normal subgroup of $G\langle\alpha\rangle$ contained in G. Since $\langle\alpha\rangle$ acts fixed point free on G by the classification of finite simple groups, G is solvable. Thus M is nontrivial. Now M is an elementary abelian p-group for some prime p and it is the unique minimal normal subgroup of $G\langle\alpha\rangle$ contained in G. Because if M_1 and M_2 are two distinct minimal normal subgroups of $G\langle\alpha\rangle$ contained in G, then G can be embedded into $G/M_1 \times G/M_2$ and so by induction applied to both G/M_1 and G/M_2 with $\langle\alpha\rangle$ we have $f(G) \leq 3$, a contradiction. Since M is unique, $F(G) = O_p(G)$ because otherwise $M \leq O_{p'}(F(G))$, impossible as M is a p-group. Now if $\Phi(F(G)) \neq 1$, then $f(G) \leq 3$ by induction applied to $G/\Phi(G)$, a contradiction. Thus $F(G)$ is an elementary abelian p-group. Now if $M < F(G)$, then by Proposition 2.1.7, we can write

$$1 \neq F(G)/M < F(G/M) = O_p(F(G/M)) \times O_{p'}(F(G/M)) = F(G)/M \times TM/M$$

where T is a Hall p'-subgroup of the inverse image of $F(G/M)$ in G.

Since $[F(G), T] = [F(G), TM] \leq M$, we have that $[F(G), T] \leq M$. As M is the unique minimal normal subgroup of $G\langle\alpha\rangle$ contained in G and $[F(G), TM] \triangleleft G\langle\alpha\rangle$, $[F(G), TM]$ is 1 or M. If $[F(G), TM] = 1$, then $[F(G), T] = 1$ and $T \leq C_G(F(G)) \leq F(G)$. And this implies that $T = 1$. But then $F(G/M) = F(G)/M$ and $f(G) \leq 3$, which is not the case. Thus $[F(G), T] = M$. Since $(|F(G)|, |T|) = 1$, $F(G) = MC_{F(G)}(T)$. As T is a Hall p'-subgroup of the inverse image of $F(G/M)$ in G, we use Frattini's argument (see Proposition 2.1.1 a)) to write $G = F(G)N_G(T)$. Thus $C_{F(G)}(T)$ is normal in $G\langle\alpha\rangle$. Since M is the unique minimal normal subgroup of $G\langle\alpha\rangle$ contained in G, $C_{F(G)}(T)$ is 1 or M. Thus $F(G) = M$ in any case.

Let $K/F_3(G)$ be a proper minimal normal $\langle\alpha\rangle$-invariant subgroup of the nilpotent group $G/F_3(G)$. Since $\langle\alpha\rangle$ acts fixed point freely on $G/F_3(G)$, $K/F_3(G)$ is non-trivial because we know that there exists a unique $\langle\alpha\rangle$-invariant Sylow subgroup of $G/F_3(G)$ of order dividing $|G/F_3(G)|$ by Proposition 2.1.11. Since $G/F_3(G)$ is nilpotent, $K/F_3(G)$ is an elementary abelian t-group for some prime t. Thus $K/F_3(G)$ can be regarded as a vector space over a field of characteristic t. If K is a proper subgroup of G, then the minimality of G implies that $f(K) \leq 3$. However since $F_3(G) \lneqq K = F_3(K) \leq G$, $f(K) > 3$, a contradiction. Thus $G/F_3(G)$ is a nontrivial irreducible $\langle\alpha\rangle$-module as being an elementary abelian t-group for some prime t.

Let T be an $\langle\alpha\rangle$-invariant Sylow t-subgroup of G. Then $TF_3(G) = G$ and so $T \nsubseteq F_3(G)$.

If T acts trivially on $O_{t'}(F_3(G)/F_2(G))$, then $TF_2(G)/F_2(G) \times O_{t'}(F_3(G)/F_2(G))$ is nilpotent. Now $G/F_2(G) = TF_3(G)/F_2(G) = (TF_2(G)/F_2(G))(F_3(G)/F_2(G))$ or equivalently $G/F_2(G) = TF_2(G)/F_2(G)[O_t(F_3(G)/F_2(G)) \times O_{t'}(F_3(G)/F_2(G))]$. As $O_t(F_3(G)/F_2(G)) = O_t(G/F_2(G))$ is contained in Sylow t-subgroup $TF_2(G)/F_2(G)$, we conclude that $G/F_2(G)$

is in $TF_2(G)/F_2(G) \times O_{t'}(F_3(G)/F_2(G))$. Thus the quotient group $G/F_2(G)$ is nilpotent (or equivalently, $T \subseteq F_3(G)$), a contradiction. Therefore there exists a prime s, which is different from t, such that T acts non-trivially on $O_s(F_3(G)/F_2(G))$.

Let S be an $\langle\alpha\rangle$-invariant Sylow s-subgroup of $F_3(G)$. Since $F_3(G)/F_2(G)$ is nilpotent, $SF_2(G)/F_2(G) = O_s(F_3(G)/F_2(G))$ where $[T, SF_2(G)/F_2(G)] \neq 1$. $SF_2(G) \triangleleft G$ because $SF_2(G)/F_2(G) \triangleleft G/F_2(G)$. Now $[SF_2(G)/F_2(G), T] = [S,T]F_2(G)/F_2(G)$. Set $K = [S,T]F_2(G)$. Obviously, $[SF_2(G)/F_2(G), O_{s'}(F(G/F_2(G)))] = 1$. Moreover since $[SF_2(G)/F_2(G), T]$ is normal in $STF_2(G)/F_2(G)$, we conclude that

$$K/F_2(G) \triangleleft [F_3(G)/F_2(G)][TF_2(G)/F_2(G)] = G/F_2(G).$$

Since $K = F_2(G)[S,T] \triangleleft G$, $F_2(K) \leq F_2(G) \lneqq K$. If $KT < G$, then the minimality of G implies that $f(KT) \leq 3$. Now K is normal in KT because $[[SF_2(G)/F_2(G), T], T] = [SF_2(G)/F_2(G), T]$. Then $F_2(K) \leq F_2(KT)$.

As $F(K)$ is a characteristic subgroup of K, which is normal in G, $F(K)$ is normal in G. Also since $F(G)$ is in K, we have $F(K) = F(G)$. Now since $F_2(K)/F(K) = F_2(K)/F(G) \leq F(G/F(G)) = F_2(G)/F(G) \leq K/F(G) = K/F(K)$, we have $F_2(K) = F_2(G)$. Then as the Fitting length of KT is at most 3, $KT/F_2(KT)$ is nilpotent and so we have $F_2(K) = F_2(KT)$. This leads to a contradiction because T acts non-trivially on $SF_2(G)/F_2(G)$. Thus $G = KT$, that is $G = F_2(G)[S,T]T$. Assume that $[S,T]$ acts trivially on $O_{s'}(F_2(G)/F(G))$. Then the normal subgroup $[[S,T]F(G)/F(G)][O_{s'}(F_2(G)/F(G))][O_s(F_2(G)/F(G))]$ in $G/F(G)$ is nilpotent. This forces that $f(G/F(G))$ is at most 2, leading to a contradiction because the Fitting length of $G/F(G)$ is equal to 3 by the assumption.

Hence we can find a prime r, which is different from s, such that S acts non-trivially on $O_r(F_2(G)/F(G))$. Let R be an α-invariant Sylow r-subgroup of $F_2(G)$. Then $RF(G)/F(G) = O_r(F_2(G)/F(G))$. Note that T also acts non-trivially on $RF(G)/F(G)$ because otherwise we can apply the three subgroup lemma (see Proposition 2.1.1 b)) with S, and we get that $[S,T]$ acts trivially on $RF(G)/F(G)$, a contradiction. $RF(G) \triangleleft G$ because $RF(G)/F(G)$ is normal in $G/F(G)$. Write $\overline{G} = G/F(G)$ and put $K/F(G) = \overline{K} = [\overline{R}, [S,T]]$. Since $[O_{r'}(F(\overline{G})), \overline{K}] \leq [O_{r'}(F(\overline{G})), O_r(F(\overline{G}))] = 1$, $[\overline{K}, O_{r'}(F(\overline{G}))] = 1$. Due to the coprimeness, $[\overline{K}, [S,T]] = \overline{K}$. Also $[\overline{K}, T] \leq \overline{K}$. Thus $\overline{K}$ is normal in $\overline{G}$. If $H = K[S,T]T < G$, then by induction the Fitting length of $K[S,T]T$ is at most three. As $F(G) \leq K$ and K is normal in G, $F(K) = F(G)$. Now $\overline{H}/F(\overline{H})$ is nilpotent, then $\overline{[S,T]} \leq F(\overline{H})$. More precisely, $[S,T]F(G)/F(G) \leq F_2(H)/F(G)$. Also $\overline{K} = [\overline{K}, \overline{[S,T]}] \leq F(\overline{H})$. Thus both $[S,T]F(G)/F(G)$ and $[R,[S,T]]F(G)/F(G)$ are contained in $F(H/F(H))$ which is a nilpotent group. Coprimeness implies that $[S, T]$ acts trivially on $[R, [S, T]]F(G)/F(G)$, a contradiction. Therefore

$$G = F(G)[R, [S, T]][S, T]T$$

It should be noted that $p \neq r$ because otherwise $F(G)[R, [S, T]]$ is a normal p-subgroup of G implying that $[R, [S, T]]$ is contained in the unique minimal normal subgroup $F(G)$. This leads to a contradiction because if $[R, [S, T]] \leq F(G)$, then $f(G) \leq 3$.

Write $C_1 = F(G)$, $C_2 = [R, [S, T]]$, $C_3 = [S, T]$ and $C_4 = T$. Then we may assume that $G = C_1C_2C_3C_4$ where C_i $(i = 1, 2, 3, 4)$ satisfy the following:

(i) $\pi(C_i) = \{p_i\}$ for $i = 1, 2, 3, 4$ and $p_i \neq p_{i+1}$ for $i = 1, 2, 3$;

(ii) C_i is $\langle\alpha\rangle$-invariant for $i = 1, 2, 3, 4$ and C_i is normalized by C_j for $i < j$ and $i = 1, 2, 3$;

(iii) $[C_i, C_{i+1}] = C_i$ for $i = 1, 2, 3$.

Moreover we will show that C_i/D_i is a special group on the Frattini factor group of which $(\prod_{j>i} C_j)\langle\alpha\rangle$ acts irreducibly for $i = 1, 2, 3$ where $D_i = C_{C_i}(C_{i-1}/D_{i-1}))$ for $i > 1$ and $D_1 = 1$;

$C_1 = F(G)$ is an elementary abelian p_1-group being the minimal normal subgroup of $G\langle\alpha\rangle$ contained in G with $D_1 = 1$. Then we may consider C_1 as a vector space over a field of characteristic p_1 and $\phi(C_1) = 1$. Thus C_1 is a special group and an irreducible $C_2C_3C_4\langle\alpha\rangle$-module. As G is solvable $D_2 = C_{C_2}(C_1) \leq C_1$. As $p_1 \neq p_2$, we have $D_2 = 1$.

Let M be the minimal element of the following set

$$\{A | A \leq C_2, \text{A is } C_3C_4\langle\alpha\rangle\text{-invariant and } [A, C_3] \neq 1\}$$

Then $M/\phi(M)$ is an irreducible $C_3C_4\langle\alpha\rangle$-module with $[M, C_3] = M$, $[\phi(M), C_3] = 1$ and M is special by Proposition 2.2.9. Our aim is to show that $M = C_2$.

If $H = C_1MC_3C_4 < C_1C_2C_3C_4$, then $f(H) \leq 3$ by induction and so $H/F_2(H)$ is nilpotent. Then $C_3 = [C_3, C_4] \leq F_2(H)$ and $M = [M, C_3] \leq F_2(H)$. Thus both C_3 and M are in $F_2(H)$. Since $F_2(H)/F(H)$ is nilpotent, $M = [M, C_3] \leq F(H)$. Since $F(H) char H \triangleleft G$ and $F(G) = C_1 \leq H$, $F(H) = C_1$. Then $M = [M, C_3] \leq F(G)$. Thus $M \leq F(G)$ which is impossible. Therefore $M = C_2$.

Let N be the minimal element of

$$\{A | A \leq C_3/D_3, \text{ A is } C_4\langle\alpha\rangle\text{-invariant and} [A, C_4] \neq 1\}$$

Then $N/\phi(N)$ is an irreducible $C_4\langle\alpha\rangle$-module with $[N, C_4] = N$, $[\phi(N), C_4] = 1$ and N is special by Proposition 2.2.9. Let S be the inverse image of N in C_3. If $H = C_1C_2SC_4 <$

$C_1C_2C_3C_4$, then by induction the Fitting length of H is at most 3. Then $S = [S, C_4] \leq F_2(H)$.

Note that $F(H) = F(G) = C_1$, C_1C_2/C_1 is a normal and p_2-subgroup of $H/F(H)$ and so it is in $F(H/F(H))$. Then $C_1C_2 \leq F_2(H)$. As $F_2(H)/F(G)$ is nilpotent, $[S, C_2] \leq F(G) = C_1$, this leads to a contradiction because $[S, C_2] \leq [C_3, C_2] = C_2$ and $[S, C_2] \neq 1$. Thus $S = C_3$.

Let $\Omega_{p_4}(C_4/D_4) = T/D_4$. Then T/D_4 is a nontrivial elementary abelian p_4-subgroup of C_4/D_4. Assume that $T \neq C_4$ and set $H = C_1C_2C_3T$. By induction the Fitting length of H is at most 3. Then $F_2(H)/F(H) = F_2(H)/F(G)$ is nilpotent and so $[C_2, [C_3, T]] \leq F(G) = C_1$. Since C_1 is abelian, $[C_1, [C_2, [C_3, T]]] \leq [C_1, C_1] = 1$. But this implies that $[C_3, T] \leq D_2 = 1$ or equivalently, $T \leq D_4$, which is impossible. Thus $C_4 = T$ that is, C_4/D_4 is an elementary abelian p_4-group. Hence (1) follows. $\square$

(2) Set $H = C_2C_3C_4$. We may assume that $(|C_1|, |H\langle\alpha\rangle|) = 1$. Moreover, there exists a proper subgroup B of $\langle\alpha\rangle$ such that $(|H|, |B|) = 1$, $C_{C_1}(B) = 1$ and $|B|$ is either pq or a divisor of p^2.

Now $C_H(C_1) = 1$. As C_1 is an irreducible $H\langle\alpha\rangle$-module, by the Fong-Swan Theorem 2.2.8, we may assume that $(|C_1|, |H\langle\alpha\rangle|) = 1$.

Let W be a homogeneous H-component of C_1 on which C_2 acts nontrivially. Put $B = N_{\langle\alpha\rangle}(W)$ and $\overline{H} = H/\mathrm{Ker(H\ on\ W)}$. Then W is a homogeneous and faithful $\overline{H}$-module. We have also $C_W(B) = 0$ as $C_{C_1}(\alpha) = 1$ by Proposition 2.2.12. Therefore $B \neq 1$ and $C_{C_1}(B) = 0$.

If $(|H|, |B|) = 1$, we see that $C_W(C_B(supp_B(\overline{H}))) = 0$ by Proposition 2.2.16. Then $C_B(supp_B(\overline{H})) \neq 1$ and by the definition of support subgroup we have also $1 \neq \overline{C_2} \leq supp_B(\overline{H})$. It follows that $C_B(supp_B(\overline{H})) \leq C_B(\overline{C_2})$. Now assume that $C_B(\overline{C_2}) \neq 1$. As $[C_B(\overline{C_2}), \overline{C_2}] = 1$, we have by the three subgroup lemma 2.1.1 that $[[C_B(\overline{C_2}), C_3], \overline{C_2}] = 1$, that is $[[C_B(\overline{C_2}), C_3], C_2] \leq$Ker($C_2$ on W). It follows that $[[C_B(\overline{C_2}), C_3], C_2]$ is contained in the kernel of each homogeneous component of C_1 because they are all $\langle\alpha\rangle$-congruent to each other. Then $[C_B(\overline{C_2}), C_3] = 1$. By the three subgroup lemma 2.1.1 again, we get $[C_B(\overline{C_2}), C_4] = 1$ and so by the coprimeness condition on $|H|$ and $|B|$, $[\overline{H}, C_B(\overline{C_2})] = 1$. On the other hand, the centralizer of a Sylow subgroup of $\langle\alpha\rangle$ has Fitting length at most 2 by [4, Corollary]. This leads to a contradiction because $f(\overline{H}) = 3$. Thus we have $(|H|, |B|) \neq 1$.

Now $B = \langle\alpha\rangle$, then C_1 is a homogeneous H-module and so irreducible by Proposition 2.2.10. Now we apply Theorem 2.2.13 and get $C_H(\alpha) \neq 1$, a contradiction. Thus $1 \neq B < \langle\alpha\rangle$. First assume that $B = \langle\alpha_q\rangle$. Then $Y = C_1C_H(\alpha_p)$ is nilpotent since B is fixed point free on Y. This forces that $C_H(\alpha_p) = 1$ because $C_H(C_1) = 1$ and so H has Fitting length at most 2, a contradiction. Therefore we have $|B|$ is pq or a divisor of p^2. $\square$

(3) $\tilde{C}_2 = C_2/\Phi(C_2)$ *is centralized by neither* z *nor* α_q.

Assume that $[\tilde{C}_2, z] = 1$. Then both C_3/D_3 and C_4/D_4 are centralized by z. If C_4 is a p'-group, then we may assume that $[C_4, z] = 1$. Now $\tilde{C}_2(C_3/D_3)C_4$, as a group on which $\langle\alpha\rangle/\langle z\rangle$ acts fixed point freely, has Fitting length at most two, a contradiction. Hence C_4 is a p-group. If C_3 is a q-group, then we may consider the action of $(C_3/D_3)C_4\langle\alpha_q\rangle$ on $\tilde{C}_2$, and get $[C_{\tilde{C}_2}(\alpha_q), C_{C_3/D_3}(\alpha_q)] \neq 1$ by Lemma 3.0.2. This contradicts the fact that $C_{\tilde{C}_2(C_3/D_3)}(\alpha_q)$ is a group on which $\langle\alpha\rangle/\langle z\alpha_q\rangle$ acts fixed point freely. Thus C_3 is an s-group for some prime s different from both p and q. If C_2 is a q'-group, then $C_{C_1}(\alpha_q)C_{C_2}(\alpha_q)C_{C_3}(\alpha_q)$ is the only tower inside $C_G(\alpha_q)$ by Theorem 2.1.14, contradicting the fact that $C_G(\alpha_q)$ has Fitting length at most two, as a group on which $\langle\alpha_p\rangle$ acts fixed point freely. This supplies that C_2 is a q-group. Notice that $[C_3/D_3, \alpha_p] = C_3/D_3$ because otherwise $[C_3/D_3, \alpha_p] = 1$ and so α_q acts fixed point freely on $(C_3/D_3)C_4$ which is impossible. Suppose that C_3/D_3 is abelian. Then $C_{C_3/D_3}(\alpha_p) = 1$. Now $\tilde{C}_2(C_3/D_3)$ must be nilpotent as a group on which $\langle\alpha_p\rangle/\langle z\rangle$ acts fixed point freely, a contradiction. It follows that C_3/D_3 is a nonabelian special group. Let W be a homogeneous component of $\tilde{C}_2|_{\Phi(C_3/D_3)}$. Now $(C_3/D_3)C_4\langle z\rangle \leq N = N_{(C_3/D_3)C_4\langle\alpha\rangle}(W)$ since $\Phi(C_3/D_3)$ lies in the center of $(C_3/D_3)C_4\langle z\rangle$. Notice that N is the stabilizer of each such homogeneous component. Put $A = N \cap \langle\alpha\rangle$. We have $C_W(A) = 0$ as $C_{\tilde{C}_2}(\alpha) = 0$ by Proposition 2.2.12. This yields that $C_{\tilde{C}_2}(A) = C_{\tilde{C}_2}(A/\langle z\rangle) = 0$.

We have $[\Phi(C_3/D_3), A] = 1$, as $\Phi(C_3/D_3)$ acts by scalars. Hence A is a nontrivial proper subgroup of $\langle\alpha\rangle$ containing $\langle z\rangle$. Since $C_{\tilde{C}_2}(\alpha_q) = C_{\tilde{C}_2}(z\alpha_q) \neq 0$, we see that $A \nleq \langle z\alpha_q\rangle$ and so $A = \langle\alpha_p\rangle$. Now, $C_{\tilde{C}_2}(\langle\alpha_p\rangle/\langle z\rangle) = 0$ and $[C_3/D_3, \langle\alpha_p\rangle/\langle z\rangle] = C_3/D_3$. Now we may apply Theorem 2.3.2 to the action of $(C_3/D_3)\langle\alpha_p\rangle/\langle z\rangle$ on $\tilde{C}_2$. Then $\pi(C_3) = \{2\}$ and so C_3 is abelian, a contradiction. Therefore $[\tilde{C}_2, z] \neq 1$.

We next assume that $[\tilde{C}_2, \alpha_q] = 1$. Then $C_{\tilde{C}_2}(\alpha_p) = 1$. So C_2 is not a p-group. We also have $(C_3/D_3)(C_4/D_4)$ is centralized by α_q. Now C_3 should be a p'-group because otherwise we would have $C_{C_3/D_3}(\alpha) \neq 1$, a contradiction. If $[C_3/D_3, z] = 1$, then $[C_4/D_4, z] = 1$ and so $\langle\alpha\rangle/\langle z\alpha_q\rangle$ acts fixed point freely on $(C_3/D_3)(C_4/D_4)$, which is impossible. Thus $[C_3/D_3, z] \neq 1$. Since $\tilde{C}_2$ is completely reducible as a $C_3/D_3\langle\alpha_p\rangle$-module, we can write $\tilde{C}_2 = X_1 \oplus \ldots \oplus X_m$ where X_i are irreducible $C_3/D_3\langle\alpha_p\rangle$-module, for $i = 1, \ldots, m$. We may assume that $[C_3/D_3, z]$ acts nontrivially on X_1. By Gross' theorem 2.3.3, we get a contradiction because $C_{\tilde{C}_2}(\alpha_p) = 1 = C_{C_3/D_3}(\alpha_p)$. Thus $[\tilde{C}_2, \alpha_q] \neq 1$.

$\square$

(4) The theorem follows when $|B|$ divides p^2.

Assume that $|B|$ divides p^2. Then $C_{C_1}(\alpha_p) = 0$. Assume that C_2 is a p'-group. Since C_1 is completely reducible as a $C_2\langle\alpha_p\rangle$-module by Maschke's theorem (see Proposition 2.2.6), we may consider an irreducible $C_2\langle\alpha_p\rangle$-submodule X of C_1. Now we apply Lemma 3.0.3 to the action of $C_2\langle\alpha_p\rangle$ on X and get $[C_2, z] = 1$ which is impossible by (3). Hence C_2 is a p-group. Then C_3 is a p'-group. Now applying Lemma 3.0.3 to the action of $C_3\langle\alpha_p\rangle$ on an irreducible $C_3\langle\alpha_p\rangle$-submodule of C_1 gives that $[C_3, z] = 1$. It follows that $[C_4, z] \leq D_4$. If C_4 is a p-group, then $D_4 = C_{C_4}(\tilde{C}_2)$ and so $[C_4, z] \leq D_4 = C_{C_4}(\tilde{C}_2)$. If C_4 is a p'-group, then we may assume that $[C_4, z] = 1$. Thus as $[C_3/D_3, z] = 1$, in any case $[\tilde{C}_2, z]$ is 1 or $\tilde{C}_2$ by the irreducibility of $\tilde{C}_2$ as a $(C_3/D_3)C_4\langle\alpha\rangle$-module. As $p_2 = p$, we should have $[\tilde{C}_2, z] = 1$ which is impossible by (3). □

From now on we assume that $B = \langle z\alpha_q\rangle$.

(5) C_2 is not a p-group.

Assume that C_2 is a p-group. Then $C_{C_2}(\alpha_q) = 1$. We first suppose that $[C_4/D_4, \alpha_q] = C_4/D_4$. Then $p_4 \neq q$ and also we may assume that $[C_4, \alpha_q] = C_4$. If $p_3 = q$, then Lemma 3.0.2 applied to the action of $(C_3/D_3)C_4\langle\alpha_q\rangle$ on $\tilde{C}_2$ gives that $C_{\tilde{C}_2}(\alpha_q) \neq 1$, a contradiction. Now G is a q'-group. If $[C_3/D_3, \alpha_q] = 1$, then $[C_4/D_4, \alpha_q] = 1$, a contradiction. Thus $[C_3/D_3, \alpha_q] \neq 1$. Since $\tilde{C}_2$ is completely reducible $[C_3/D_3, \alpha_q]\langle\alpha_q\rangle$-module by Maschke's theorem 2.2.6, we choose an irreducible $[C_3/D_3, \alpha_q]\langle\alpha_q\rangle$-submodule X of $\tilde{C}_2$. Now we may apply Gross' theorem 2.3.3 to the action of $[C_3/D_3, \alpha_q]\langle\alpha_q\rangle$ on X, and get $C_{\overline{[C_3/D_3,\alpha_q]}}(\alpha_q) \neq 1$ where $\overline{[C_3/D_3, \alpha_q]\langle\alpha_q\rangle} = [C_3/D_3, \alpha_q]\langle\alpha_q\rangle/\mathrm{Ker}([C_3/D_3, \alpha_q]\langle\alpha_q\rangle \text{ on } X)$. Since $(|C_3|, |\alpha_q|) = 1$, $C_{\overline{[C_3/D_3,\alpha_q]}}(\alpha_q) = \overline{C_{[C_3/D_3,\alpha_q]}(\alpha_q)}$ and so $C_{[C_3/D_3,\alpha_q]}(\alpha_q) \neq 1$, a contradiction.

Thus $[C_4/D_4, \alpha_q] = 1$. If $p_4 = p$, then $[C_4/D_4, \alpha] = 1$, a contradiction. Hence $p_4 \neq p$. If $[C_3/D_3, z] = 1$, then $[C_4/D_4, z] = 1$. Since $p_4 \neq p$, we have $[C_4, z] = 1$ and so $[\tilde{C}_2, z] = 1$ or $\tilde{C}_2$. As $p_2 = p$ we have $[\tilde{C}_2, z] = 1$ which is impossible by (3). Thus $[C_3/D_3, z] \neq 1$. Now, applying Lemma 3.0.4 to the action of $(C_3/D_3)\langle z\rangle$ on $\tilde{C}_2$ we get $[\tilde{C}_2, z]^{p-1} \neq 0$. That is, $[C_2, z]^{p-1} \nleq \Phi(C_2)$. Also $[C_2, \alpha_q] = C_2$, since, $C_{C_2}(\alpha_q) = 1$. Then we apply Lemma 3.0.9 to the action of C_2B on C_1 and get $C_{C_1}(B) \neq 1$, a contradiction. □

(6) Either $[C_4/D_4, z] \neq 1$ or $[C_4/D_4, \alpha_q] \neq 1$.

Assume that C_4/D_4 is centralized by both z and α_q. It follows that $p_4 \neq p$, because otherwise $C_{C_4/D_4}(\alpha) \neq 1$, which is impossible. Thus we may assume that $[C_4, z] = 1$.

First we shall observe that $p_2 = q$:

Assume that C_2 is an s-group where s is a prime different from both p and q. Now we consider an irreducible $[C_2, z]B$-submodule Y of C_1 on which $[C_2, z]$ acts nontrivially.

It follows by Lemma 3.0.3 that $[C_2, z, \alpha_q]$ is trivial on Y. In fact $[C_2, z, \alpha_q]$ is trivial on each irreducible $[C_2, z]B$-constituent of C_1 and hence it is trivial on C_1. It follows that $[C_2, z, \alpha_q] = 1$ by the faithful action of C_2 on C_1. Then $[C_2, z] \neq C_2$ by (3). Hence we must have $[C_3/D_3, z] \neq 1$ because otherwise by the irreducibility of $\tilde{C_2}$ as a $(C_3/D_3)C_4\langle\alpha\rangle$-module we would have either $[\tilde{C_2}, z] = 1$ or $[\tilde{C_2}, z] = \tilde{C_2}$, both are impossible.

It follows that $p_3 \neq p$ because otherwise $[\widetilde{C_3/D_3}, z] = 1$ by the irreducibility of $\widetilde{C_3/D_3} = C_3/D_3/\Phi(C_3/D_3)$ as a $(C_4/D_4)\langle\alpha\rangle$-module. Now we can apply Lemma 3.0.7 to the action of $C_2[C_3, z]C_4\langle\alpha_p\rangle$ on C_1 and get $[C_{C_1}(\alpha_p), C_{C_2}(\alpha_p)] \neq 1$, contradicting the fact that α_q acts fixed point freely on $C_{C_1C_2}(\alpha_p)$ and hence it is nilpotent. This shows that $p_2 = q$.

Next we shall observe that $[C_3/D_3, \alpha_q] \neq 1$:

Assume that $[C_3/D_3, \alpha_q] = 1$. If $p_4 \neq q$, then we may assume that $[C_4, \alpha_q] = 1$. So we have $[\tilde{C_2}, \alpha_q] = 1$, as $\tilde{C_2}$ is irreducible as a $(C_3/D_3)C_4\langle\alpha\rangle$-module. This is not the case by (3) and so $p_4 = q$. Now $D_4 = C_{C_4}(\tilde{C_2})$ and so $\tilde{C_2}$ is an irreducible $(C_3/D_3)(C_4/D_4)\langle\alpha\rangle$-module. It follows that $[\tilde{C_2}, \alpha_q] = 1$ again, as $p_2 = q$, which is impossible. Hence $[C_3/D_3, \alpha_q] \neq 1$.

Applying Lemma 3.0.4 to the action of $(C_3/D_3)\langle\alpha_q\rangle$ on $\tilde{C_2}$, we see that $[C_2, \alpha_q]^{q-1} \nleq \Phi(C_2)$. If $[C_2, z] = C_2$, then Lemma 3.0.9 applies to the action of C_2B on an irreducible C_2B-submodule C_1 on which $[C_2, \alpha_q]^{q-1}$ acts nontrivially, and gives that $C_{C_1}(B) \neq 1$, a contradiction. Hence $[C_2, z] \neq C_2$. This forces that $p_3 \neq p$ and $[C_3/D_3, z] = C_3/D_3$, as $[C_4, z] = 1$, by the fact that $\tilde{C_2}$ is an irreducible $(C_3/D_3)C_4\langle\alpha\rangle$-module. Now Lemma 3.0.7 applied to the action of $C_2C_3\langle\alpha_p\rangle$ on C_1 supplies that $C_{C_1C_2}(\alpha_p)$ is not nilpotent, contradicting the fact that α_q acts fixed point freely on this group. $\square$

(7) *The theorem follows.*

We may assume that $[C_4/D_4, a] \neq 1$, where $a = z$ or $a = \alpha_q$. Now $p_4 \neq |a|$ where $|a|$ denotes the order of a. By Lemma 3.0.11 there are $\langle\alpha\rangle$-invariant sections U_1, U_2, U_3 of C_1, C_2, C_3/D_3 respectively such that U_i normalizes U_{i-1}, $\mathrm{Ker}(U_i \text{ on } U_{i-1}) = 1$ for $i = 2, 3$ and each U_i is centralized by a for $i = 1, 2, 3$.

Now $\langle\alpha\rangle/\langle a\rangle$ acts fixed point freely on each U_i for $i = 1, 2, 3$. $|\langle\alpha\rangle/\langle a\rangle|$ is either pq or p^2. We may assume that U_3 is an elementary abelian group on which $A = \langle\alpha\rangle/\langle a\rangle$ acts irreducibly. If every element $b \in A$ of prime order centralizes U_3, then A must be cyclic of order p^2: For if $|A| = pq$, then $\langle z\rangle$, $\langle\alpha_p\rangle/\langle z\rangle$ and $\langle\alpha_q\rangle$ all centralize U_3 and so $\langle\alpha\rangle$ is not fixed point free on U_3, which is impossible. Hence we may assume that $A = \langle\alpha_p\rangle$ and $a = \alpha_q$. Now $[U_2, z] \neq 1$, because otherwise $[U_2U_3, z\alpha_q] = 1$ and so $A/\langle z\rangle$ acts fixed point freely on U_2U_3. This is impossible as $A/\langle z\rangle$ is of prime order and $[U_2, U_3] \neq 1$. Since $\langle\alpha_q\rangle$ centralizes U_1 and $C_{U_1}(z\alpha_q) = 1$, $C_{U_1}(z) = 1$. We can apply Gagola's theorem 2.3.2 to the action of

$[U_2, z]\langle z\rangle$ on U_1 and get U_2 is a 2-group. It follows that U_2 is abelian and so $C_{[U_2,z]}(z) = 1$. This forces that $[[U_2, z], U_1] = 1$ and so $[U_2, z] = 1$, a contradiction.

Thus $|A| = pq$ and there exists $x \in A$ of prime order such that $[U_3, x] \neq 1$. Now $[U_3, x] = U_3$ by the irreducibility of U_3 as an irreducible A-module and so $p_3 \neq |x|$. We first consider the case $p_2 = |x|$. Then Lemma 3.0.4 applied to the action of $U_3\langle x\rangle$ on U_2 gives that $[U_2, x]^{|x|-1} \neq 1$. This enables us to apply Lemma 3.0.8 to the action of $U_2\langle x\rangle$ on U_1. It follows that $[C_{U_1}(x), C_{[U_2,x]^{|x|-1}}(x)] \neq 1$, which is impossible as $A/\langle x\rangle$ is fixed point free on $C_{U_1U_2}(x)$. Hence $p_2 \neq |x|$. Notice that $C_{U_2}(x) \neq 1$ because otherwise $\langle x\rangle$ acts fixed point freely on U_2U_3, a contradiction. Applying Lemma 3.0.10 to the action of $U_2\langle x\rangle$ on U_1 respectively, we see that $[C_{U_1}(x), C_{U_2}(x)] \neq 1$, which is not the case because $A/\langle x\rangle$ acts fixed point freely on $C_{U_1U_2}(x)$. □

As an immediate consequence of our main result we state the following.

Corollary. *Let A be a finite abelian group whose order is a product of three primes coprime to 6. Assume that A acts fixed point freely on a finite group G of odd order. Then $f(G) \leq 3$.*

Proof of the corollary. There are four possible cases for A: Either (i) A is of order p^3 for some prime p, or (ii) A is a cyclic group whose order is a product of three distinct primes, or (iii) A is of order p^2q for two distinct primes p and q and its Sylow p-subgroups are elementary abelian, or (iv) A is cyclic of order p^2q for distinct primes p and q. In cases (ii) and (iv), A is cyclic and so the result follows by the main theorem. When (i) holds, A acts with regular orbits on G because $(|G|, |A|) = 1$ and $C_G(A) = 1$. So we have the result by Turull's work in [26]. Finally the result follows by Ercan and Güloğlu [5] in case (iii). □

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