QUALITATIVE BEHAVIOR OF SOLUTIONS OF DYNAMIC EQUATIONS ON TIME SCALES

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submitted by **RAZİYE MERT** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Mathematics Department, Middle East Technical University** by,

Prof. Dr. Canan Özgen _____
Dean, Graduate School of **Natural and Applied Sciences**

Prof. Dr. Zafer Nurlu _____
Head of Department, **Mathematics**

Prof. Dr. Ağacık Zafer _____
Supervisor, **Mathematics Department, METU**

Examining Committee Members:

Prof. Dr. Aydın Tiryaki
Department of Mathematics and Computer Science,
İzmir University _____

Prof. Dr. Ağacık Zafer
Mathematics Department, METU _____

Prof. Dr. Hasan Taşeli
Mathematics Department, METU _____

Prof. Dr. Dumitru Baleanu
Department of Mathematics and Computer Science,
Çankaya University _____

Assist. Prof. Dr. Ömür Uğur
Institute of Applied Mathematics, METU _____

Date: _____

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: Raziye Mert

Signature :

ABSTRACT

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Mert, Raziye
Ph.D., Department of Mathematics
Supervisor : Prof. Dr. Ağacık Zafer

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In this thesis, the asymptotic behavior and oscillation of solutions of dynamic equations on time scales are studied. In the first part of the thesis, asymptotic equivalence and asymptotic equilibrium of dynamic systems are investigated. Sufficient conditions are established for the asymptotic equivalence of linear systems and linear and quasilinear systems, respectively, and for the asymptotic equilibrium of quasilinear systems by unifying and extending some known results for differential systems and difference systems to dynamic systems on arbitrary time scales. In particular, for the asymptotic equivalence of differential systems, the well-known theorems of Levinson and Yakubovich are improved and the well-known theorem of Wintner for the asymptotic equilibrium of linear differential systems is generalized to arbitrary time scales. Some of our results for asymptotic equilibrium are new even for difference systems. In the second part, the oscillation of solutions of a particular class of second order nonlinear delay dynamic equations and, more generally, two-dimensional nonlinear dynamic systems, including delay-dynamic systems, are discussed. Necessary and sufficient conditions are derived for the oscillation of solutions of nonlinear delay dynamic equations by extending some continuous results. Specifically, the classical theorems of Atkinson and Belohorec are generalized. Sufficient conditions are

established for the oscillation of solutions of nonlinear dynamic systems by unifying and extending the corresponding continuous and discrete results. Particularly, the oscillation criteria of Atkinson, Belohorec, Waltman, and Hooker and Patula are generalized.

Keywords: Time scale, Dynamic equation, Asymptotic equivalence, Asymptotic equilibrium, Oscillation

ÖZ

ZAMAN SKALALARI ÜZERİNDE DİNAMİK DENKLEMLERİN ÇÖZÜMLERİNİN KALİTATİF DAVRANIŞI

Mert, Raziye
Doktora, Matematik Bölümü
Tez Yöneticisi : Prof. Dr. Ağacık Zafer

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Bu tezde, zaman skalaları üzerinde dinamik denklemlerin çözümlerinin asimptotik davranışı ve salınımı araştırılmıştır. İlk olarak, dinamik sistemlerin asimptotik denklikleri ve asimptotik dengeleri incelenmiştir. Diferensiyel sistemler ve fark sistemler için bilinen bazı sonuçlar birleştirilip keyfi zaman skalaları üzerindeki dinamik sistemlere genişletilerek, sırasıyla, lineer sistemlerin ve lineer ve quasi lineer sistemlerin asimptotik denklikleri için ve quasi lineer sistemlerin asimptotik dengeleri için yeter koşullar kanıtlanmıştır. Özellikle, Levinson ve Yakubovich'in iyi bilinen teoremleri diferensiyel sistemlerin asimptotik denklikleri için geliştirilmiş ve Wintner'in lineer diferensiyel sistemlerin asimptotik dengeleri için iyi bilinen teoremi keyfi zaman skalalarına genelleştirilmiştir. Ayrıca, asimptotik denge için elde edilmiş olan sonuçların bir kısmı fark denklemler için de yenidir. Daha sonra, belirli sınıftan ikinci dereceden lineer olmayan gecikmeli dinamik denklemlerin ve daha genel olarak gecikmeli dinamik sistemleri de kapsayan iki boyutlu lineer olmayan dinamik sistemlerin çözümlerinin salınımı ele alınmıştır. Sürekli durumdaki bazı sonuçlar genişletilerek, lineer olmayan gecikmeli dinamik denklemlerin çözümlerinin salınımı için gerek ve yeter koşullar elde edilmiştir. Özellikle, Atkinson ve Belohorec'in klasik teoremleri genelleştirilmiştir. Sürekli ve kesikli durumdaki bazı

sonuçlar birleştirilip genişletilerek, lineer olmayan dinamik sistemlerin çözümlerinin salınımı için yeter koşullar kanıtlanmıştır. Atkinson, Belohorec, Waltman ve Hooker ve Patula' nın salınım kriterleri genelleştirilmiştir.

Anahtar Kelimeler: Zaman skalası, Dinamik denklem, Asimptotik denklik, Asimptotik denge, Salınım

To my parents, Behçet and Azize,
and my brother, Caner

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My dear father you are not dead, you are still living in our hearts...

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LIST OF SYMBOLS

$\mathbb{T}$	:	time scale
$\mathbb{R}$	:	the set of all real numbers
$\mathbb{R}^+$	:	the set of all non-negative real numbers
$\mathbb{Z}$	:	the set of all integers
$\mathbb{N}$	:	the set of all natural numbers
$\mathbb{N}_0$	:	the set of all non-negative integers
$\mathbb{R}^n$	:	$n-$dimensional real Euclidean space, $n \in \mathbb{N}$
$\|.\|$	:	any convenient vector norm
$\|\|.\|\|$	:	any convenient matrix norm
$\oplus$	:	circle plus addition
$\ominus$	:	circle minus subtraction
$u(t) = o(v(t))$	:	$u(t)$ is little-oh of $v(t)$ as $t \to \infty$
$u(t) = O(v(t))$	:	$u(t)$ is big-oh of $v(t)$ as $t \to \infty$
$\mathbb{C}(A, B)$	:	The set of all continuous functions from A to B
$\mathbb{C}_{rd}(A, B)$	:	The set of all rd-continuous functions from A to B
$\mathbb{C}^1_{rd}(A, B)$	:	The set of all rd-continuously delta differentiable functions from A to B
$\mathcal{R}(A, B)$	:	The set of all regressive and rd-continuous functions from A to B
$\mathcal{R}^+(A, B)$	:	The set of all positively regressive rd-continuous functions from A to B

CHAPTER 1

INTRODUCTION

The theory of time scales was introduced by Stefan Hilger in his Ph.D. thesis [47] in 1988 in order to unify continuous and discrete analysis. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The so-called dynamic equations on time scales build bridges between continuous and discrete mathematics and allow a simultaneous treatment of differential and difference equations. The idea is to prove a general result for a dynamic equation where the domain of the unknown function is a time scale. This general result yields a result for an ordinary differential equation when the time scale is the set of real numbers, and it yields a result for a difference equation when the time scale is the set of integers, and it also extend these results to other dynamic equations, e.g., q-difference equations. In summary, **Unification** and **Extension** are the two main features of the time scale calculus.

The calculus of time scales has an enormous potential for applications, e.g., in the study of insect population models, neural networks, heat transfer, epidemic models, and economic models.

1.1 Structure of the Thesis

The thesis is organized as follows:

In the present chapter, we provide a related literature review for asymptotic behavior and oscillation of solutions of differential and difference equations and state the problems under consideration. In Chapter 2, we introduce some basic tools concerning calculus on time scales, extracted from the monographs by Bohner and Peterson [20, 21]. For additional references, the interested reader may also see the monograph

by Kaymakçalan, Lakshmikantham, and Sivasundaram [53]. Chapter 3 deals with asymptotic behavior of solutions of dynamic equations on time scales. In particular, the problem of asymptotic equivalence and asymptotic equilibrium are investigated. In Chapter 4, we deal with oscillation of solutions of dynamic equations on time scales. We consider a class of second order nonlinear delay dynamic equations and a system of two first order nonlinear dynamic equations. Finally, in Chapter 5, we summarize our findings in this thesis.

1.2 Literature Review

1.2.1 Asymptotic Behavior of Solutions

Two systems are said to be asymptotically equivalent if there is a homeomorphism between the solutions x and y of the systems such that the asymptotic relation

$$x(t) = y(t) + o(1) \quad \text{as } t \to \infty$$

holds. The asymptotic equivalence of two systems shows an asymptotic relationship between these systems. If two systems are asymptotically equivalent, and if the asymptotic behavior of the solutions of one of them is known, then information can be obtained about the asymptotic behavior of the solutions of the other system.
To the best of our knowledge, the first result on asymptotic equivalence is due to Levinson [59], where he considered the linear differential system

$$x' = A(t)x, \quad t \in \mathbb{R}^+ = [0, \infty) \tag{1.1}$$

and its perturbed system

$$y' = [A(t) + B(t)]y, \quad t \in \mathbb{R}^+, \tag{1.2}$$

where $x, y \in \mathbb{R}^n$, and $A, B \in \mathbb{C}(\mathbb{R}^+, \mathbb{R}^{n\times n})$, and proved the following theorem.

Theorem 1.2.1 (Levinson, [59]) *If $A(t) \equiv A$, all solutions of (1.1) are bounded, and*

$$\int^{\infty} \|B(t)\|\, dt < \infty, \tag{1.3}$$

then (1.1) and (1.2) are asymptotically equivalent.

Later, Wintner [80] showed that the same conclusion remains valid when A is a non-constant matrix.

Theorem 1.2.2 (Wintner, [80]) *Assume that all solutions of (1.1) are bounded, (1.3) is satisfied, and*

$$\liminf_{t\to\infty} \int_0^t Trace[A(s)]\, ds > -\infty.$$

Then (1.1) and (1.2) are asymptotically equivalent.

The asymptotic equivalence of the quasilinear system

$$y' = Cy + f(t, y), \quad t \in \mathbb{R}^+ \tag{1.4}$$

and the corresponding linear homogeneous system

$$x' = Cx, \quad t \in \mathbb{R}^+, \tag{1.5}$$

where $x, y \in \mathbb{R}^n$, $C \in \mathbb{R}^{n\times n}$, and $f(t, x) \in \mathbb{C}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ such that $f(t, 0) = 0$ for all $t \in \mathbb{R}^+$, was studied by Yakubovich [64, 82].

Theorem 1.2.3 (Yakubovich, [82]) *Assume that there exists $\eta \in \mathbb{C}(\mathbb{R}^+, \mathbb{R})$ such that*

$$|f(t, x_1) - f(t, x_2)| \le \eta(t)|x_1 - x_2|$$

for all $(t, x_1), (t, x_2) \in \mathbb{R}^+ \times \mathbb{R}^n$. If

$$\lim_{t\to\infty} \int_t^\infty s^{m_\beta + m_\alpha - 2} e^{\beta s} \eta(s)\, ds = 0,$$

where α and β are the minimum and maximum of the real parts of the eigenvalues of C, and m_α and m_β denote the multiplicities of these eigenvalues in the minimal polynomial of C, then (1.5) and (1.4) are asymptotically equivalent.

After the pioneering work of Levinson, Wintner, and Yakubovich, the problem of asymptotic equivalence of differential systems including linear, nonlinear, and functional differential equations was investigated by many authors; see e.g. [9, 17, 29, 30, 39, 43, 64, 67, 68, 71, 74, 75] and the references cited therein. Recently, in [9], Levinson's and Yakubovich's results have been improved by replacing their conditions with some mild ones.

The problem of asymptotic equivalence of difference systems has been initiated by H. Poincaré (1885) and O. Perron (1921). For some related works for difference systems, we may refer to [31, 33, 34, 60, 61, 62, 83] and the references cited therein. The results given in [83] are the discrete analogous of the corresponding ones given in [9]. With regard to asymptotic equivalence of systems, in almost all the articles, it is usually assumed that either the solutions of unperturbed systems are bounded or have special forms or the matrix $A(t)$ is independent of t. Such assumptions are excluded in [9, 83].

The asymptotic equilibrium of differential and difference equations has also been the subject of intensive investigations. In almost all works, the systems under consideration are of the form

$$x' = A(t)x, \quad t \in \mathbb{R}^+ \tag{1.6}$$

or

$$x' = A(t)x + f(t, x), \quad t \in \mathbb{R}^+. \tag{1.7}$$

The system (1.6) (or (1.7)) is said to have linear asymptotic equilibrium (or asymptotic equilibrium) if for each constant vector $c \in \mathbb{R}^n$, there is a solution x of (1.6) (or (1.7)) such that

$$\lim_{t\to\infty} x(t) = c.$$

The first result on asymptotic equilibrium of solutions of differential equations was obtained in 1929 by Perron [65]. Later, some extensions were given by Bellman [18], and by Hartman and Wintner [46], see also some related works by Wintner [79], Brauer [26, 27], and Atkinson [15]. The notion of asymptotic equilibrium can be used to investigate the asymptotic equivalence of two systems. Ladas et al. demonstrated this approach in [57] by studying the asymptotic manifold of solutions of nonlinear systems generated by the solutions of the corresponding linear system. They improved the results in [28] and [44]. In [76], Trench obtained sufficient conditions for the asymptotic equilibrium of perturbed linear systems.

The following two theorems motivate our study. The first one appeared in the literature more than a century ago and was attributed to Bôcher by Wintner [79].

Theorem 1.2.4 (Bôcher) *Let $A \in \mathbb{C}(\mathbb{R}^+, \mathbb{R}^{n\times n})$. If*

$$\int^{\infty} ||A(t)||\, dt < \infty,$$

then system (1.6) has linear asymptotic equilibrium.

Theorem 1.2.5 (Wintner, [79]) *Let $A \in \mathbb{C}(\mathbb{R}^+, \mathbb{R}^{n\times n})$ and suppose the integrals*

$$A_j(t) = \int_t^{\infty} A_{j-1}(s)A(s)\, ds, \quad 1 \le j \le k \quad (A_0 = I)$$

converge, and

$$\int^{\infty} ||A_k(t)A(t)||\, dt < \infty.$$

Then system (1.6) has linear asymptotic equilibrium.

Some extensions and improvements of Theorem 1.2.5 can be found in [76], where Trench studied the asymptotic behavior of solutions of (1.7) and gave sufficient conditions under which (1.7) has asymptotic equilibrium. The results in [77] are partly the discrete analogous of the corresponding ones in [76] although their proofs are somewhat different.

1.2.2 Oscillation of Solutions

The problem of oscillation and nonoscillation of solutions of nonlinear differential and difference equations is of ongoing interest and has been the subject of numerous investigations over many years. For a sampling of the work done, see the monographs by Agarwal [1], Agarwal, Grace, and O'Regan [2, 3], and Ladde, Lakshmikantham, and Zhang [58].

In the study of oscillation and nonoscillation of solutions of differential and difference equations, attention is restricted to those solutions x which exist on some ray $[T_x, \infty)$ and $\mathbb{N}(n_0) = \{n_0, n_0 + 1, \ldots\}$ for some $n_0 \in \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of non-negative integers, respectively, and nontrivial in the neighborhood of infinity. Such a solution is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Equation itself is said to be oscillatory if all its solutions are oscillatory.

Let us consider the forced second order delay differential equation

$$x''(t) + p(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) = g(t), \quad t \in \mathbb{R}^+, \tag{1.8}$$

where $\alpha > 0$ is a constant, p, g, $\tau \in \mathbb{C}(\mathbb{R}^+, \mathbb{R})$ with τ increasing, $\tau(t) \leq t$, and $\lim_{t\to\infty} \tau(t) = \infty$. When $g(t) \equiv 0$ and $\tau(t) \equiv t$, Eq. (1.8) takes the form

$$x''(t) + p(t)|x(t)|^{\alpha-1}x(t) = 0, \quad t \in \mathbb{R}^+. \tag{1.9}$$

In the case of $\alpha > 1$, Eq. (1.8) is the prototype of a class of nonlinear differential equations called Emden-Fowler superlinear differential equations, and if $0 < \alpha < 1$, it is the prototype of differential equations called Emden-Fowler sublinear differential equations.

Eq. (1.8) has been studied extensively by Nasr [63], where the author extends the well-known theorems of Atkinson [14] and Belohorec [19] from the unforced equation (1.9) to the forced one (1.8) in the superlinear and sublinear cases, respectively. The following theorems of Atkinson [14], Belohorec [19], and Nasr [63] are of particular importance for our purpose.

Theorem 1.2.6 (Atkinson, [14]) *Let $\alpha > 1$. If*

$$p(t) \geq 0 \text{ and } p(t) \not\equiv 0 \text{ on } [T, \infty) \text{ for any } T \geq 0, \tag{1.10}$$

then a necessary and sufficient condition for oscillation of Eq. (1.9) is that

$$\int^{\infty} tp(t)\,dt = \infty. \tag{1.11}$$

Theorem 1.2.7 (Belohorec, [19]) *Let $0 < \alpha < 1$ and (1.10) holds. Then a necessary and sufficient condition for oscillation of Eq. (1.9) is that*

$$\int^{\infty} t^{\alpha} p(t)\,dt = \infty. \tag{1.12}$$

Theorem 1.2.8 (Nasr, [63]) *Let $\alpha > 1$ and (1.10) holds. Assume that*

(1) $g = h''$ for some function $h \in \mathbb{C}^2(\mathbb{R}^+, \mathbb{R})$;

(2) *h is oscillatory and there exist two sequences $\{s_n\}$, $\{\bar{s}_n\}$ tending to infinity such that for all n,*

$$h(s_n) = \inf\{h(t) : t \geq s_n\},$$

$$h(\bar{s}_n) = \sup\{h(t) : t \geq \bar{s}_n\};$$

(3) *τ is differentiable and there exists a positive constant β such that $\tau'(t) \geq \beta$ for all $t \in \mathbb{R}^+$.*

Then condition (1.11) is a necessary and sufficient condition for Eq. (1.8) to be oscillatory.

Theorem 1.2.9 (Nasr, [63]) *Let $0 < \alpha < 1$ and (1.10) holds. Under the assumptions (1)-(3), condition (1.12) is a necessary and sufficient condition for Eq. (1.8) to be oscillatory.*

The sign condition on $p(t)$ was removed by Waltman [78].

Theorem 1.2.10 (Waltman, [78]) *Let $\alpha > 1$. If*

$$\int^{\infty} p(t)\,dt = \infty,$$

then Eq. (1.9) oscillates.

We shall also be interested in the second order Emden-Fowler difference equation

$$\Delta^2 x(n-1) + p(n)|x(n)|^{\alpha-1}x(n) = 0, \quad n \in \mathbb{N}_0, \tag{1.13}$$

where $\alpha > 0$ is a constant and $\{p(n)\}$ is a real sequence such that $p(n) \geq 0$ and $p(n) \not\equiv 0$ for infinitely many values of n, and the following two theorems.

Theorem 1.2.11 (Hooker, Patula, [49]) *If $\alpha > 1$, then a necessary and sufficient condition for oscillation of Eq. (1.13) is that*

$$\sum_{n=1}^{\infty} np(n) = \infty.$$

Theorem 1.2.12 (Hooker, Patula, [49]) *If* $0 < \alpha < 1$, *then a necessary and sufficient condition for oscillation of Eq. (1.13) is that*

$$\sum_{n=1}^{\infty} n^{\alpha} p(n) = \infty.$$

Now consider

$$\begin{aligned} x'(t) &= a_1(t) f_1(y(t)) \\ y'(t) &= -a_2(t) f_2(x(t)), \quad t \in \mathbb{R}^+ \end{aligned} \tag{1.14}$$

and

$$\begin{aligned} \Delta x(n) &= p_1(n) g_1(y(n)) \\ \Delta y(n) &= -p_2(n) g_2(x(n+1)), \quad n \in \mathbb{N}_0, \end{aligned} \tag{1.15}$$

where a_1, $a_2 \in \mathbb{C}(\mathbb{R}^+, \mathbb{R})$, $\{p_1(n)\}$, $\{p_2(n)\}$ are real sequences, and f_i, $g_i \in \mathbb{C}(\mathbb{R}, \mathbb{R})$, $i = 1, 2$, with the sign property that $u f_i(u) > 0$, $u g_i(u) > 0$ for all $u \neq 0$.

A continuable nontrivial solution (x, y) of the system (1.14) (or (1.15)) is said to be oscillatory if x and y are both oscillatory (i.e., neither eventually positive nor eventually negative). Otherwise it is said to be nonoscillatory. The system (1.14) (or (1.15)) is called oscillatory if all its solutions are oscillatory.

Eq. (1.14) has been studied by many authors, see for example [55, 56] and the references cited therein. And the oscillatory property of the difference system (1.15) has also received a great deal of attention. We refer the reader to the papers [41, 50, 51, 52] and the references cited therein. The systems (1.14) and (1.15) includes Eqs. (1.9) and (1.13), respectively, as special cases.

Of our particular interest will be the papers [41] and [56]. In [41], Graef and Thandapani have generalized the results of Hooker and Patula [49] to system (1.15). They have also established the discrete generalization of Waltman's [78] theorem. Similarly, in [56], Kwong and Wong have extended the results of Atkinson [14] and Belohorec [19] to system (1.14). They also have generalized Waltman's [78] theorem.

1.3 Statement of the Problems

The purpose of this section is to state the problems in this thesis. The calculus of time scales is based on the so-called delta derivative. The delta derivative of a function f

defined on $\mathbb{T}$ is denoted by f^Δ, and it turns out that $f^\Delta = f'$ is the usual derivative if $\mathbb{T} = \mathbb{R}$ and $f^\Delta = \Delta f$ is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$. For further details on calculus of time scales, we refer the reader to Chapter 2.

In this thesis, we consider dynamic equations of the form

$$x^\Delta = A(t)x, \tag{1.16}$$

$$y^\Delta = [A(t) + B(t)]\, y, \tag{1.17}$$

$$x^\Delta = Cx, \tag{1.18}$$

$$y^\Delta = Cy + f(t, y), \tag{1.19}$$

$$x^\Delta = A(t)x + f(t, x), \tag{1.20}$$

$$x^{\Delta\Delta}(t) + p(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) = g(t), \tag{1.21}$$

$$\begin{aligned} x^\Delta(t) &= a_1(t)\, f_1(y(t)) \\ y^\Delta(t) &= -a_2(t)\, f_2(x(\sigma(t))), \end{aligned} \tag{1.22}$$

$$\begin{aligned} x^\Delta(t) &= a_1(t)\, f_1(y(t)) \\ y^\Delta(t) &= -a_2(t)\, f_2(x(\tau(t))). \end{aligned} \tag{1.23}$$

We first study the asymptotic equivalence of linear systems (1.16) and (1.17), and linear and quasilinear systems (1.18) and (1.19), respectively, and establish sufficient conditions for their asymptotic equivalence. Next, we investigate the asymptotic equilibrium of quasilinear system (1.20) and provide sufficient conditions which guarantee that (1.20) has asymptotic equilibrium. We discuss the oscillatory behavior of solutions of second order delay dynamic equation (1.21) in the superlinear and sublinear cases, and the oscillation of two-dimensional nonlinear dynamic system (1.22). By considering the systems obtained from (1.22) by allowing inequality in the second equation, we also study the oscillation of delay-dynamic system (1.23). We present necessary and sufficient conditions for oscillation of (1.21) and some sufficient conditions for oscillation of (1.22) and (1.23).

Definitions of asymptotic equivalence, asymptotic equilibrium, and oscillation are adapted from $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ to any arbitrary time scale $\mathbb{T}$. We will give such definitions later in appropriate sections.

In Bohner and Lutz [22], some results on the asymptotic equivalence as well as asymptotic behavior of solutions of dynamic equations on time scales are given. Although the arguments have some common features, our method and results differ from the ones obtained in [22]. We impose no boundedness condition for solutions of (1.16) and (1.18), cf. [22, Theorem 3.1]. To the best of our knowledge, there is no result regarding the asymptotic equilibrium of dynamic equations on time scales.

The oscillation theory of second order nonlinear dynamic equations on time scales has been developed intensively after the discovery of time scale calculus. For some work, we refer in particular to the papers [4, 5, 6, 7, 8, 12, 13, 25, 35, 38, 45, 72, 73] and the references cited therein. In [45], Han et al. considered (1.21) when $g(t) \equiv 0$ and established sufficient conditions for the oscillation of (1.21) in the superlinear, linear, and sublinear cases. Their results extend the results of Agarwal et al. [4]. System (1.22) includes, in particular, second order nonlinear dynamic equations of the form

$$(\frac{1}{a_1(t)} x^{\Delta}(t))^{\Delta} + a_2(t) f_2(x(\sigma(t))) = 0, \tag{1.24}$$

$$x^{\Delta\Delta}(t) + a_2(t) f_2(x(\sigma(t))) = 0, \tag{1.25}$$

$$x^{\Delta\Delta}(t) + a_2(t)|x(\sigma(t))|^{\alpha-1} x(\sigma(t)) = 0. \tag{1.26}$$

Numerous oscillation and nonoscillation criteria have been established for solutions of dynamic equations (1.24) – (1.26), see for example [10, 11, 24, 36, 37] and the references cited therein. However, to the best of our knowledge, except [81], there is no result regarding the oscillation of solutions of two-dimensional dynamic systems on time scales.

CHAPTER 2

PRELIMINARIES

2.1 Time Scale Calculus

In this section, we state some basic definitions and theorems concerning the calculus on time scales, extracted from the monographs by Bohner and Peterson [20, 21], which we need for our purpose in this thesis.

2.1.1 Introduction

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, the natural numbers $\mathbb{N}$, the nonnegative integers $\mathbb{N}_0$, the h-numbers $h\mathbb{Z} = \{hn : n \in \mathbb{Z}\}$, where $h > 0$ is a fixed real number, and the q-numbers $\mathbb{K}_q = q^{\mathbb{Z}} \cup \{0\} = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$, where $q > 1$ is a fixed real number are examples of time scales, as are $[0, 1] \cup [2, 3]$, $[0, 1] \cup \mathbb{N}$, and the Cantor set. It is assumed throughout that a time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology.

On any time scale $\mathbb{T}$, the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. A point $t \in \mathbb{T}$ is said to be left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, and right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$. Points that are left-dense and right-dense at the same time are called dense. Also, if $\rho(t) < t$, we say that t is left-scattered, while if $\sigma(t) > t$, we say that t is right-scattered. Points that are left-scattered and right-scattered at the same time are called isolated. The graininess

function μ for a time scale $\mathbb{T}$ is defined by

$$\mu(t) = \sigma(t) - t.$$

If $a, b \in \mathbb{T}$ with $a \leq b$, we define the closed interval $[a, b]_{\mathbb{T}}$ in $\mathbb{T}$ by

$$[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Open intervals and half-open intervals etc. are defined accordingly.

Let us identify σ, ρ, and μ by considering special time scales.

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf(t, \infty) = t$$

and similarly

$$\rho(t) = \sup\{s \in \mathbb{R} : s < t\} = \sup(-\infty, t) = t.$$

Hence every point $t \in \mathbb{R}$ is dense. The graininess function μ turns out to be $\mu(t) \equiv 0$ for all $t \in \mathbb{R}$.

(ii) If $\mathbb{T} = h\mathbb{Z}$, then

$$\sigma(t) = \inf\{s \in h\mathbb{Z} : s > t\} = \inf\{t + nh : n \in \mathbb{N}\} = t + h$$

and

$$\rho(t) = \sup\{s \in h\mathbb{Z} : s < t\} = \sup\{t - nh : n \in \mathbb{N}\} = t - h.$$

Hence every point $t \in h\mathbb{Z}$ is isolated. The graininess function μ turns out to be $\mu(t) \equiv h$ for all $t \in h\mathbb{Z}$.

(iii) If $\mathbb{T} = \mathbb{K}_q$, then

$$\sigma(t) = \inf\{s \in \mathbb{K}_q : s > t\} = \inf\{q^n t : n \in \mathbb{N}\} = qt$$

and

$$\rho(t) = \sup\{s \in \mathbb{K}_q : s < t\} = \sup\{q^{-n} t : n \in \mathbb{N}\} = \frac{t}{q}$$

for all $t \neq 0$. Obviously, $\sigma(0) = \rho(0) = 0$. Consequently, $\mu(t) = \sigma(t) - t = (q-1)t$ for all $t \in \mathbb{K}_q$. Hence 0 is a right-dense minimum and every other point in $\mathbb{K}_q$ is isolated.

The following induction principle is a useful tool.

Theorem 2.1.1 *(Induction Principle) Let $t_0 \in \mathbb{T}$ and assume that*

$$\{S(t) : t \in [t_0, \infty)_{\mathbb{T}}\}$$

is a family of statements satisfying:

(I) The statement $S(t_0)$ is true.

(II) If $t \in [t_0, \infty)_{\mathbb{T}}$ is right-scattered and $S(t)$ is true, then $S(\sigma(t))$ is also true.

(III) If $t \in [t_0, \infty)_{\mathbb{T}}$ is right-dense and $S(t)$ is true, then there is a neighborhood U of t such that $S(s)$ is true for all $s \in U \cap (t, \infty)_{\mathbb{T}}$.

(IV) If $t \in (t_0, \infty)_{\mathbb{T}}$ is left-dense and $S(s)$ is true for all $s \in [t_0, t)_{\mathbb{T}}$, then $S(t)$ is true.

Then $S(t)$ is true for all $t \in [t_0, \infty)_{\mathbb{T}}$.

2.1.2 Differentiation

Let $f : \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}^\kappa$, where $\mathbb{T}^\kappa$ is the set derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T}\backslash\{m\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. Then the delta (or Hilger) derivative of f at the point t is defined to be the number $f^\Delta(t)$ (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)\,(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

Moreover, we say that f is delta differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is then called the (delta) derivative of f on $\mathbb{T}^\kappa$. Similarly we define higher order derivatives $f^{\Delta^n} : \mathbb{T}^{\kappa^n} \to \mathbb{R}$, where $f^{\Delta^n} = (f^{\Delta^{n-1}})^\Delta$, $\mathbb{T}^{\kappa^n} = (\mathbb{T}^{\kappa^{n-1}})^\kappa$.

Some easy and useful relationships concerning the delta derivative are given in the following theorem.

Theorem 2.1.2 *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$.*

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If t is right-dense, then f is differentiable at t iff the limit

$$\lim_{s \to t} \frac{f(s) - f(t)}{s - t}$$

exists as a finite number. In this case

$$f^\Delta(t) = \lim_{s \to t} \frac{f(s) - f(t)}{s - t}.$$

(iv) If f is differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t). \tag{2.1}$$

As an example, we again consider the cases $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = h\mathbb{Z}$, and $\mathbb{T} = \mathbb{K}_q$.

(i) If $\mathbb{T} = \mathbb{R}$, then Theorem 2.1.2 (iii) yields that $f^\Delta(t) = f'(t)$, the ordinary derivative of f at t.

(ii) If $\mathbb{T} = h\mathbb{Z}$, then Theorem 2.1.2 (ii) yields that

$$f^\Delta(t) = \frac{f(t+h) - f(t)}{h}.$$

Note that when $h = 1$, i.e., $\mathbb{T} = \mathbb{Z}$, we have $f^\Delta(t) = f(t+1) - f(t) = \Delta f(t)$, where Δ is the usual forward difference operator defined by the last equation above.

(iii) If $\mathbb{T} = \mathbb{K}_q$, then Theorem 2.1.2 (ii) and (iii), respectively, yields that

$$f^\Delta(t) = \frac{f(qt) - f(t)}{(q-1)t}$$

for all $t \neq 0$, and

$$f^\Delta(0) = \lim_{s \to 0} \frac{f(s) - f(0)}{s}$$

provided that the limit exists.

The rules for derivatives as well as the mean value theorem are given in the following two theorems.

Theorem 2.1.3 *Let $f, g : \mathbb{T} \to \mathbb{R}$ be differentiable at $t \in \mathbb{T}^\kappa$. Then the following statements are valid.*

(i) The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t).$$

(ii) For any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(\alpha f)^\Delta(t) = \alpha f^\Delta(t).$$

(iii) The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(fg)^\Delta(t) = f^\Delta(t)g(\sigma(t)) + f(t)g^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t).$$

(iv) If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is differentiable at t with

$$\left(\frac{1}{f}\right)^\Delta(t) = -\frac{f^\Delta(t)}{f(t)f(\sigma(t))}.$$

(v) If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is differentiable at t with

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.$$

Theorem 2.1.4 *(Mean Value Theorem) Let f be a continuous function on $[a, b]_\mathbb{T}$ that is differentiable on $[a, b)_\mathbb{T}$. Then there exit $\xi, \tau \in [a, b)_\mathbb{T}$ such that*

$$f^\Delta(\xi) \leq \frac{f(b) - f(a)}{b - a} \leq f^\Delta(\tau).$$

The usual chain rule from calculus is no longer valid on an arbitrary time scale. One form of the extended chain rule that we need in this thesis involves integral. Therefore statement of this rule will be given in the integration section.

2.1.3 Integration

The Cauchy Integral

In order to describe a class of functions that are (delta) integrable, the concept of rd-continuous functions should be introduced. For further details on integration, we refer the reader to [42].

Definition 2.1.5 *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.*

The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted in this thesis by

$$\mathbb{C}_{rd} = \mathbb{C}_{rd}(\mathbb{T}) = \mathbb{C}_{rd}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$\mathbb{C}^1_{rd} = \mathbb{C}^1_{rd}(\mathbb{T}) = \mathbb{C}^1_{rd}(\mathbb{T}, \mathbb{R}).$$

Theorem 2.1.6 *Let $f : \mathbb{T} \to \mathbb{R}$.*

(i) If f is continuous, then f is rd-continuous.

(ii) The jump operator σ is rd-continuous.

(iii) If f is rd-continuous, then so is $f \circ \sigma$.

(iv) Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. If f is rd-continuous, then so is $g \circ f$.

Definition 2.1.7 *A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided*

$$F^{\Delta}(t) = f(t) \quad \text{holds for all} \quad t \in \mathbb{T}^{\kappa}.$$

Theorem 2.1.8 *(Existence of Antiderivatives) Every rd-continuous function f has an antiderivative. In particular if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) = \int_{t_0}^{t} f(s)\,\Delta s \quad \text{for} \quad t \in \mathbb{T}$$

is an antiderivative of f.

Theorem 2.1.9 *If F is an antiderivative of an integrable function f, then*

$$\int_a^b f(t)\,\Delta t = F(b) - F(a) \quad \text{for all} \quad a, b \in \mathbb{T}.$$

The following three theorems give a list of properties of the integral.

Theorem 2.1.10 *If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in \mathbb{C}_{rd}(\mathbb{T}, \mathbb{R})$, then*

(i) $\int_a^b [f(t) + g(t)]\,\Delta t = \int_a^b f(t)\,\Delta t + \int_a^b g(t)\,\Delta t$;

(ii) $\int_a^b (\alpha f)(t)\,\Delta t = \alpha \int_a^b f(t)\,\Delta t$;

(iii) $\int_a^b f(t)\,\Delta t = -\int_b^a f(t)\,\Delta t$;

(iv) $\int_a^b f(t)\,\Delta t = \int_a^c f(t)\,\Delta t + \int_c^b f(t)\,\Delta t$;

(v) $\int_a^a f(t)\,\Delta t = 0$;

(vi) *if $f(t) \geq 0$ on $[a, b)_{\mathbb{T}}$, then* $\int_a^b f(t)\,\Delta t \geq 0$;

(vii) *if $|f(t)| \leq g(t)$ on $[a, b)_{\mathbb{T}}$, then*

$$\left| \int_a^b f(t)\,\Delta t \right| \leq \int_a^b g(t)\,\Delta t.$$

Theorem 2.1.11 *If $a, b \in \mathbb{T}$, and $f, g \in \mathbb{C}^1_{rd}(\mathbb{T}, \mathbb{R})$, then*

(i) $\int_a^b f(\sigma(t)) g^{\Delta}(t)\,\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t) g(t)\,\Delta t$;

(ii) $\int_a^b f(t) g^{\Delta}(t)\,\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t) g(\sigma(t))\,\Delta t$.

Theorem 2.1.12 *If $f : \mathbb{T} \to \mathbb{R}$ is an arbitrary function and $t \in \mathbb{T}$, then*

$$\int_t^{\sigma(t)} f(s)\,\Delta s = \mu(t) f(t).$$

One can easily prove that the following theorem is valid.

Theorem 2.1.13 *Let $a, b \in \mathbb{T}$ and $f \in \mathbb{C}_{rd}(\mathbb{T}, \mathbb{R})$.*

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\,\Delta t = \int_a^b f(t)\,dt,$$

where the integral on the right is the usual Riemann integral from calculus.

(ii) If $[a, b]_{\mathbb{T}}$ consists of only isolated points, then

$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{t\in[a,b)_{\mathbb{T}}} \mu(t) f(t) & \text{if } a < b \\ 0 & \text{if } a = b \\ -\sum_{t\in[b,a)_{\mathbb{T}}} \mu(t) f(t) & \text{if } a > b. \end{cases}$$

(iii) If $\mathbb{T} = h\mathbb{Z}$, then

$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} h f(kh) & \text{if } a < b \\ 0 & \text{if } a = b \\ -\sum_{k=\frac{b}{h}}^{\frac{a}{h}-1} h f(kh) & \text{if } a > b. \end{cases}$$

(iv) If $\mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t) & \text{if } a < b \\ 0 & \text{if } a = b \\ -\sum_{t=b}^{a-1} f(t) & \text{if } a > b. \end{cases}$$

Improper Integral

We now define the improper integral $\int_a^\infty f(t)\,\Delta t$ as one would expect. We refer the reader to [23] for further details.

Definition 2.1.14 *If $a \in \mathbb{T}$, $\sup \mathbb{T} = \infty$, and $f \in \mathbb{C}_{rd}([a, \infty)_{\mathbb{T}}, \mathbb{R})$, then*

$$\int_a^\infty f(t)\,\Delta t = \lim_{b\to\infty} \int_a^b f(t)\,\Delta t$$

provided the limit exists. In this case the improper integral is called convergent. Otherwise, the improper integral is said to be divergent.

A Chain Rule

One form of the following chain rule, due to S. Keller [54] and generalized to measure chains by C. Pötzche [66], is as follows.

Theorem 2.1.15 *(Chain Rule) Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and $g : \mathbb{T} \to \mathbb{R}$ delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable with*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))\, dh \right\} g^{\Delta}(t).$$

The Substitution Rule for Integrals

Theorem 2.1.16 *(Substitution) Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} = \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)\nu^{\Delta}(t)\, \Delta t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\, \widetilde{\Delta} s.$$

2.2 Existence of Solutions and Matrix Exponential

Let A be an $m \times n$-matrix-valued function on $\mathbb{T}$. We say that A is rd-continuous on $\mathbb{T}$ if each entry of A is rd-continuous on $\mathbb{T}$, and the class of all such rd-continuous $m \times n$-matrix-valued functions on $\mathbb{T}$ is denoted, similar to the scalar case, by

$$\mathbb{C}_{rd} = \mathbb{C}_{rd}(\mathbb{T}) = \mathbb{C}_{rd}(\mathbb{T}, \mathbb{R}^{m\times n}).$$

We say that A is differentiable on $\mathbb{T}$ provided each entry of A is differentiable on $\mathbb{T}$, and in this case we put

$$A^{\Delta} = ({a_{ij}}^{\Delta})_{1\le i\le m, 1\le j\le n}, \quad \text{where} \quad A = (a_{ij})_{1\le i\le m, 1\le j\le n}.$$

Theorem 2.2.1 *If A is differentiable at $t \in \mathbb{T}^{\kappa}$, then*

$$A(\sigma(t)) = A(t) + \mu(t)A^{\Delta}(t).$$

Theorem 2.2.2 *Suppose that A and B are differentiable $n \times n$-matrix-valued functions. Then*

(i) $(A+B)^{\Delta}(t) = A^{\Delta}(t) + B^{\Delta}(t)$;

(ii) $(\alpha A)^{\Delta}(t) = \alpha A^{\Delta}(t)$ *if α is constant;*

(iii) $(AB)^{\Delta}(t) = A^{\Delta}(t)B(\sigma(t)) + A(t)B^{\Delta}(t) = A(\sigma(t))B^{\Delta}(t) + A^{\Delta}(t)B(t)$;

(iv) $(A^{-1})^{\Delta}(t) = -A^{-1}(\sigma(t))A^{\Delta}(t)A^{-1}(t) = -A^{-1}(t)A^{\Delta}(t)A^{-1}(\sigma(t))$ *if $A(t)A(\sigma(t))$ is invertible;*

(v) $(AB^{-1})^{\Delta}(t) = (A^{\Delta}(t) - A(t)B^{-1}(t)B^{\Delta}(t))B^{-1}(\sigma(t)) = (A^{\Delta}(t) - (AB^{-1})(\sigma(t))B^{\Delta}(t))B^{-1}(t)$ *if $B(t)B(\sigma(t))$ is invertible.*

Definition 2.2.3 *(Regressivity) An $n \times n$-matrix-valued function A on a time scale $\mathbb{T}$ is called regressive provided*

$$I + \mu(t)A(t) \quad \text{is invertible for all} \quad t \in \mathbb{T}^{\kappa},$$

where I denotes as usual the $n \times n$-identity matrix, and the class of all such regressive and rd-continuous functions is denoted by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n}).$$

Lemma 2.2.4 *An $n \times n$-matrix-valued function A is regressive iff the eigenvalues $\lambda_i(t)$ of $A(t)$ are regressive for all $1 \leq i \leq n$.*

Definition 2.2.5 *Let $\mathbb{T}$ be a time scale and $\mathbb{X}$ a Banach space. A function $f : \mathbb{T} \times \mathbb{X} \to \mathbb{X}$ is called*

(i) rd-continuous, if g defined by $g(t) = f(t, x(t))$ is rd-continuous for any continuous function $x : \mathbb{T} \to \mathbb{X}$;

(ii) regressive at $t \in \mathbb{T}^{\kappa}$, if the mapping

$$id + \mu(t)f(t, .) : \mathbb{X} \to \mathbb{X} \quad \text{is invertible}$$

(where id is the identity function), and f is called regressive on $\mathbb{T}^{\kappa}$, if f is regressive at each $t \in \mathbb{T}^{\kappa}$.

Let $\mathbb{T}$ be a time scale, $t_0 \in \mathbb{T}$, $x_0 \in \mathbb{R}^n$, $a > 0$ with $\inf \mathbb{T} \leq t_0 - a$ and $\sup \mathbb{T} \geq t_0 + a$, and let

$$I_a = [t_0 - a, t_0 + a]_{\mathbb{T}} \quad \text{and} \quad U_b = \{x \in \mathbb{R}^n : |x - x_0| \leq b\}.$$

Theorem 2.2.6 *(Global Existence and Uniqueness Theorem) Suppose that $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is rd-continuous and regressive, and that for each $(t, x) \in \mathbb{T} \times \mathbb{R}^n$, there exists a set $I_a \times U_b$ such that the Lipschitz condition*

$$|f(t, x_1) - f(t, x_2)| \leq L(t, x)|x_1 - x_2| \quad \textit{for all } (t, x_1), (t, x_2) \in I_a \times U_b$$

holds, where $L(t, x) > 0$. Then the IVP

$$x^{\Delta} = f(t, x), \quad x(t_0) = x_0$$

has a unique solution defined on $\mathbb{T}$.

In case the system is linear, the existence and uniqueness theorem reads as follows.

Theorem 2.2.7 *Let $A \in \mathcal{R}$ be an $n \times n$-matrix-valued function on $\mathbb{T}$ and suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$. Then the IVP*

$$x^{\Delta} = A(t)x + f(t), \quad x(t_0) = x_0$$

has a unique solution $x : \mathbb{T} \to \mathbb{R}^n$.

The uniqueness of solutions is proved by using the Gronwall's Inequality.

Theorem 2.2.8 *(Gronwall's Inequality) Let $y \in \mathbb{C}_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, $p \in \mathcal{R}^+([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ (i.e., $p \in \mathbb{C}_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and $1 + \mu(t)p(t) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$), $p(t) \geq 0$, and $\alpha \in \mathbb{R}$. Then*

$$y(t) \leq \alpha + \int_{t_0}^{t} y(s)p(s)\,\Delta s \quad \textit{for all} \quad t \geq t_0$$

implies

$$y(t) \leq \alpha e_p(t, t_0) \quad \textit{for all} \quad t \geq t_0.$$

It follows from Theorem 2.2.7 that the matrix IVP

$$X^{\Delta} = A(t)X, \quad X(t_0) = X_0,$$

where X_0 is a constant $n \times n$-matrix, has a unique (matrix-valued) solution X.

Definition 2.2.9 *(Matrix Exponential Function) Let $t_0 \in \mathbb{T}$ and assume that $A \in \mathcal{R}$ is an $n \times n$-matrix-valued function. The unique matrix-valued solution of the IVP*

$$X^{\Delta} = A(t)X, \quad X(t_0) = I \tag{2.2}$$

is called the matrix exponential function (at t_0), and it is denoted by $e_A(\cdot, t_0)$.

Remark 2.2.10 *If $n = 1$, (2.2) becomes*

$$x^{\Delta} = a(t)x, \quad x(t_0) = 1$$

and the matrix exponential function $e_A(t, t_0)$ becomes the generalized exponential function $e_a(t, t_0)$ which is also given explicitly by

$$e_a(t, t_0) = \exp\left(\int_{t_0}^{t} \xi_{\mu(s)}(a(s))\,\Delta s\right) \quad \text{for} \quad t \in \mathbb{T},$$

where

$$\xi_{\mu(t)}(a(t)) = \begin{cases} \frac{1}{\mu(t)}\mathrm{L}og(1 + \mu(t)a(t)), & \mu(t) > 0 \\ a(t), & \mu(t) = 0 \end{cases},$$

where L*og is the principal logarithm function.*

Before we give some properties of the exponential matrix, we need to introduce the circle plus addition $\oplus$ and the circle minus subtraction $\ominus$.

Definition 2.2.11 *Assume A and B are regressive $n \times n$-matrix-valued functions on $\mathbb{T}$. Then we define $A \oplus B$ by*

$$(A \oplus B)(t) = A(t) + B(t) + \mu(t)A(t)B(t) \quad \text{for all} \quad t \in \mathbb{T}^{\kappa},$$

and we define $\ominus A$ by

$$(\ominus A)(t) = -[I + \mu(t)A(t)]^{-1}A(t) \quad \text{for all} \quad t \in \mathbb{T}^{\kappa}.$$

Definition 2.2.12 *If the matrix-valued functions A and B are regressive on $\mathbb{T}$, then we define $A \ominus B$ by*

$$(A \ominus B)(t) = (A \oplus (\ominus B))(t) \quad \text{for all} \quad t \in \mathbb{T}^{\kappa}.$$

Lemma 2.2.13 *$(\mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n}), \oplus)$ is a group.*

For a matrix A, we let A^* denote its conjugate transpose.

Theorem 2.2.14 *If $A, B \in \mathcal{R}$ are $n \times n$-matrix-valued functions on $\mathbb{T}$, then*

(i) $e_0(t, s) \equiv I$ *and* $e_A(t, t) \equiv I$;

(ii) $e_A(\sigma(t), s) = (I + \mu(t)A(t))e_A(t, s)$;

(iii) $e_A^{-1}(t, s) = e^*_{\ominus A^*}(t, s)$;

(iv) $e_A(t, s) = e_A^{-1}(s, t) = e^*_{\ominus A^*}(s, t)$;

(v) $e_A(t, s)e_A(s, r) = e_A(t, r)$;

(vi) $e_A(t, s)e_B(t, s) = e_{A\oplus B}(t, s)$ *if* $e_A(t, s)$ *and* $B(t)$ *commute.*

We remark that in case A is a constant matrix, $e_A(t, t_0) = e^{A(t-t_0)}$ for $\mathbb{T} = \mathbb{R}$, while if $\mathbb{T} = \mathbb{Z}$ and $I + A$ is invertible, then $e_A(t, t_0) = (I + A)^{t-t_0}$.

Theorem 2.2.15 *If $A \in \mathcal{R}$ is an $n \times n$-matrix-valued function on $\mathbb{T}$, and $a, b, c \in \mathbb{T}$, then*

$$[e_A(c, .)]^{\Delta} = -[e_A(c, .)]^{\sigma} A$$

and

$$\int_a^b [e_A(c, \sigma(t))]A(t)\,\Delta t = e_A(c, a) - e_A(c, b).$$

Now we may state the variation of parameters formula.

Theorem 2.2.16 *Let $A \in \mathcal{R}$ be an $n\times n$-matrix-valued function on $\mathbb{T}$ and suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$. Then the IVP*

$$x^{\Delta} = A(t)x + f(t), \quad x(t_0) = x_0$$

has a unique solution $x : \mathbb{T} \to \mathbb{R}^n$. *Moreover, this solution is given by*

$$x(t) = e_A(t, t_0)x_0 + \int_{t_0}^{t} e_A(t, \sigma(s))f(s)\,\Delta s.$$

2.3 Linear Constant Coefficient Equations

In this section, we consider the dynamic equation

$$x^{\Delta} = Ax, \tag{2.3}$$

where $A \in \mathcal{R}$ is a real constant $n \times n$-matrix.

Theorem 2.3.1 *If* λ_0, ξ *is an eigenpair for the constant* $n \times n$*-matrix* A, *then* $x(t) = e_{\lambda_0}(t, t_0)\xi$ *is a solution of (2.3) on* $\mathbb{T}$.

Whenever a real constant $n \times n$-matrix A has n linearly independent eigenvectors (for example every symmetric matrix has n linearly independent eigenvectors), Theorem 2.3.1 can be used to compute $e_A(t, t_0)$. However not every constant $n \times n$-matrix A has n linearly independent eigenvectors. The Putzer algorithm for dynamic equations is quite useful for constructing $e_A(t, t_0)$. A more elementary method is given in [84] to calculate $e_A(t, t_0)$.

Theorem 2.3.2 *(Putzer Algorithm) Let* $A \in \mathcal{R}$ *be a constant* $n \times n$*-matrix. Suppose* $t_0 \in \mathbb{T}$. *If* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are the eigenvalues of* A, *then*

$$e_A(t, t_0) = \sum_{i=0}^{n-1} r_{i+1}(t)P_i,$$

where $r(t) = (r_1(t), r_2(t), \ldots, r_n(t))^T$ *is the solution of the IVP*

$$r^{\Delta} = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & 0 & \ddots & \vdots \\ 0 & 1 & \lambda_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \lambda_n \end{pmatrix} r, \quad r(t_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and the P-matrices $P_0, P_1, \ldots, P_n$ *are recursively defined by*

$$P_0 = I, \qquad P_k = (A - \lambda_k I)P_{k-1}, \quad k = 1, 2, \ldots, n.$$

2.4 Some Definitions and Theorems from Analysis

Finally, we state some definitions and theorems which we rely on proving certain results in this thesis.

Definition 2.4.1 *A metric (or a distance function) on a set* $\mathbb{M}$ *is a real-valued function* d *defined on* $\mathbb{M} \times \mathbb{M}$ *which satisfies the following properties:*

(i) $d(x, y) \geq 0$ *for all* $x, y \in \mathbb{M}$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in \mathbb{M}$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in \mathbb{M}$.

We call the pair $(\mathbb{M}, d)$ *a metric space.*

Definition 2.4.2 *A sequence* (x_n) *in* $\mathbb{M}$ *is said to converge to an element* x *of* $\mathbb{M}$ *if for every* $\varepsilon > 0$ *we can find a positive integer* N *so that* $d(x_n, x) < \varepsilon$ *for all* $n \geq N$. *A sequence* (x_n) *is called a Cauchy sequence if for every* $\varepsilon > 0$ *there is a positive integer* N *such that* $d(x_n, x_m) < \varepsilon$ *for all* $n, m \geq N$. *A metric space* $(\mathbb{M}, d)$ *is called complete if every Cauchy sequence in* $\mathbb{M}$ *converges to an element of* $\mathbb{M}$.

Definition 2.4.3 *Let* $(\mathbb{M}, d)$ *be a metric space and* Y *a subset of* $\mathbb{M}$. *A point* $x \in \mathbb{M}$ *is said to be a limit point of* Y *if there exists an eventually non-constant sequence* (x_n) *in* Y *which converges to* x. *We say a subset* Y *is closed if* Y *contains all of its limit points.*

Definition 2.4.4 *Let* $(\mathbb{M}, d)$ *be a metric space. A subset* Y *of* $\mathbb{M}$ *is relatively compact if every sequence in* Y *contains a subsequence which converges to a point in* $\mathbb{M}$.

Definition 2.4.5 *Let* $T : \mathbb{M}_1 \to \mathbb{M}_2$ *be a function from a metric space* $(\mathbb{M}_1, d)$ *into a metric space* $(\mathbb{M}_2, \rho)$ *and* $x_0 \in \mathbb{M}_1$. *We say* T *is continuous at* x_0 *if for every* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that* $\rho(Tx, Tx_0) < \varepsilon$ *whenever* $d(x, x_0) < \delta$. *Further,* T *is continuous on* $\mathbb{M}_1$, *or simply continuous, if it is continuous at all points of* $\mathbb{M}_1$.

Definition 2.4.6 *A vector space $\mathbb{V}$ equipped with a norm is called a normed vector space.*

Any normed vector space is a metric space. If this metric space is complete, then the normed space is called a Banach space.

Definition 2.4.7 *A subset Y of a normed vector space $\mathbb{V}$ is called convex if for any x, $y \in Y$, $\lambda x + (1-\lambda)y \in Y$ for all $\lambda \in [0,1]$.*

Definition 2.4.8 *Let $(\mathbb{M}, d)$ be a metric space. A function $T : \mathbb{M} \to \mathbb{M}$ is called a contraction mapping if there is a number $\lambda \in [0,1)$ such that for all x, $y \in \mathbb{M}$, we have*

$$d(Tx, Ty) \leq \lambda d(x, y).$$

Theorem 2.4.9 *(Banach fixed point theorem) Let $(\mathbb{M}, d)$ be a complete metric space and $T : \mathbb{M} \to \mathbb{M}$ a contraction mapping. Then T has a unique fixed point in $\mathbb{M}$, i.e., there exists a unique $x \in \mathbb{M}$ such that $Tx = x$.*

Theorem 2.4.10 *(Schauder's fixed point theorem) Let Y be a nonempty, closed, convex subset of a Banach space $\mathbb{X}$. Let $T : Y \to Y$ be a continuous mapping such that TY is a relatively compact subset of $\mathbb{X}$, then T has a fixed point in Y.*

The following theorem is the time scale version of the well-known Arzela-Ascoli theorem given by Řehák in [70].

Let $\mathbb{T}$ be a time scale unbounded from above, i.e., $\sup \mathbb{T} = \infty$, and $a \in \mathbb{T}$. Let $\mathbb{C}_0([a,\infty)_{\mathbb{T}})$ denote the vector space of all continuous bounded n-vector functions f (with real or complex components) defined on $[a,\infty)_{\mathbb{T}}$. Define the norm of a function $f \in \mathbb{C}_0([a,\infty)_{\mathbb{T}})$ as $\|f\|_0 = \sup_{t\in[a,\infty)_{\mathbb{T}}} |f(t)|$.

Theorem 2.4.11 *Let Y be a subset of $\mathbb{C}_0([a,\infty)_{\mathbb{T}})$ having the following properties:*

(a) Y is bounded;

(b) *on every compact subinterval J of $[a, \infty)_{\mathbb{T}}$, there exists, for any $\varepsilon > 0$, $\delta > 0$ such that $t_1, t_2 \in J$, $|t_1 - t_2| < \delta$ implies $|f(t_1) - f(t_2)| < \varepsilon$ for all $f \in Y$;*

(c) *for every $\varepsilon > 0$, there exists $b \in [a, \infty)_{\mathbb{T}}$ such that $t_1, t_2 \in [b, \infty)_{\mathbb{T}}$ implies $|f(t_1) - f(t_2)| < \varepsilon$ for all $f \in Y$.*

Then Y is relatively compact.

CHAPTER 3

ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF DYNAMIC EQUATIONS ON TIME SCALES

3.1 Asymptotic Equivalence

3.1.1 Introduction

Let $\mathbb{T}$ be an arbitrary time scale which is unbounded from above, i.e., $\sup \mathbb{T} = \infty$. Fix $a \in \mathbb{T}$, $a \geq 0$, and consider the following dynamic equations:

$$x^{\Delta} = A(t)x, \quad t \in [a, \infty)_{\mathbb{T}}, \tag{3.1}$$

$$y^{\Delta} = [A(t) + B(t)]\, y, \quad t \in [a, \infty)_{\mathbb{T}}, \tag{3.2}$$

$$x^{\Delta} = Cx, \quad t \in [a, \infty)_{\mathbb{T}}, \tag{3.3}$$

$$y^{\Delta} = Cy + f(t, y), \quad t \in [a, \infty)_{\mathbb{T}}, \tag{3.4}$$

where A and B are $n \times n$-matrix-valued functions, C is a constant matrix of size $n \times n$, and $f(t, x)$ is an n-vector-valued function. It is also assumed that A, $C \in \mathcal{R}([a, \infty)_{\mathbb{T}}, \mathbb{R}^{n\times n})$, $B \in \mathbb{C}_{rd}([a, \infty)_{\mathbb{T}}, \mathbb{R}^{n\times n})$, and $f \in \mathbb{C}_{rd}([a, \infty)_{\mathbb{T}} \times \mathbb{R}^n, \mathbb{R}^n)$ satisfies $f(t, 0) = 0$ for all $t \in [a, \infty)_{\mathbb{T}}$.

In this section, we will investigate the asymptotic equivalence of dynamic equations (3.1) and (3.2), and (3.3) and (3.4). By an asymptotic equivalence between two systems, we mean the following, adapted from [29, 64].

Definition 3.1.1 *Systems (3.1) and (3.2) are called asymptotically equivalent if there is a homeomorphism between solutions x of (3.1) and y of (3.2) such that the asymptotic relation*

$$x(t) = y(t) + o(1) \quad as\, t \to \infty$$

holds. Asymptotic equivalence of (3.3) and (3.4) is defined similarly.

Our purpose here is to unify the main results in [9, 83] and extend these results to dynamic equations on time scales.

3.1.2 Main Results

We first establish sufficient conditions under which (3.1) and (3.2) are asymptotically equivalent. Following Ráb [67, 68], as in [9, 83], we make use of an analogous change of dependent variable

$$y(t) = e_A(t, a)u(t)$$

which transforms (3.2) into

$$u^{\Delta} = P(t)u, \tag{3.5}$$

where

$$P(t) = e_A(a, \sigma(t))B(t)e_A(t, a).$$

Lemma 3.1.2 *If*

$$\int_a^{\infty} \|P(t)\|\, \Delta t < \infty, \tag{3.6}$$

then the matrix dynamic equation

$$\Psi^{\Delta} = P(t)\Psi + P(t)$$

has a solution Ψ *which satisfies* $\Psi(t) \to 0$ *as* $t \to \infty$.

Proof. The lemma can be accomplished similarly by using the arguments carried out in [9, 67, 68, 83]. For completeness, we sketch the proof.

Define

$$\Psi_0(t) = I, \qquad \Psi_k(t) = -\int_t^{\infty} P(s)\Psi_{k-1}(s)\, \Delta s, \quad k = 1, 2, \ldots.$$

In view of (3.6), it can be shown that the series $\sum_{k=1}^{\infty} \Psi_k(t)$ converges uniformly for $t \in [t_1, \infty)_{\mathbb{T}}$ for some $t_1 \geq a$. Furthermore, $\Psi(t) = \sum_{k=1}^{\infty} \Psi_k(t)$, $t \in [t_1, \infty)_{\mathbb{T}}$, satisfies $\Psi(t) \to 0$ as $t \to \infty$, and

$$\Psi(t) = -\int_t^{\infty} P(s)[I + \Psi(s)]\,\Delta s, \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

■

Theorem 3.1.3 *Assume that (3.6) is satisfied. If*

$$\lim_{t\to\infty} e_A(t, a)\Psi(t) = 0, \tag{3.7}$$

then (3.1) and (3.2) are asymptotically equivalent.

Proof. We use the arguments developed in [9]. Let t be sufficiently large, $t \geq t_1$ say. Then, in view of Lemma 3.1.2, we see that the function $u(t) = [I + \Psi(t)]c$, $c \in \mathbb{R}^n$, is a solution of (3.5) defined on $[t_1, \infty)_{\mathbb{T}}$ and hence

$$y(t) = e_A(t, a)[I + \Psi(t)]c$$

is a solution of (3.2).

Since $\Psi(t) \to 0$ as $t \to \infty$, there exists a $t_2 > t_1$ such that $I + \Psi(t_2)$ is nonsingular. Let $x_0 = e_A(t_2, a)c$ and $y_0 = e_A(t_2, a)(I + \Psi(t_2))c$. Denote by $x(t) = x(t, t_2, x_0)$ and $y(t) = y(t, t_2, y_0)$, the solutions of (3.1) and (3.2) satisfying $x(t_2) = x_0$ and $y(t_2) = y_0$, respectively. In view of the existence and uniqueness theorem for linear system of dynamic equations, Theorem 2.2.7, and the fact that $I + \Psi(t_2)$ is nonsingular, the relation

$$y_0 = e_A(t_2, a)[I + \Psi(t_2)]e_A(a, t_2)x_0$$

defines a one-to-one correspondence between solutions x of (3.1) and y of (3.2) such that

$$y(t) = x(t) + e_A(t, a)\Psi(t)c, \quad t \in [t_2, \infty)_{\mathbb{T}}.$$

This, in view of (3.7), completes the proof. ■

Example 3.1.4 *Let* $\mathbb{T} = \mathbb{R}$, *and consider the linear systems*

$$x^{\Delta} = \begin{bmatrix} 0 & 1 \\ 2t^{-2} & 0 \end{bmatrix} x, \quad t \in [1, \infty) \tag{3.8}$$

and

$$y^{\Delta} = \begin{bmatrix} 0 & 1 \\ 2t^{-2} & 0 \end{bmatrix} y + \begin{bmatrix} 0 & 0 \\ e^{-t} & 0 \end{bmatrix} y, \quad t \in [1, \infty). \tag{3.9}$$

It is easy to verify that, cf. [9, Example 2.1], all the conditions of Theorem 3.1.3 are satisfied ($\|P(t)\| \leq Ke^{-t/2}$ and $\|\Psi(t)\| \leq e^{2Ke^{-t/2}} - 1$ for all $t \geq 1$ and for some $K > 0$). Therefore, (3.8) and (3.9) are asymptotically equivalent. Note that $x_1 = (t^2, 2t)^T$ and $x_2 = (t^{-1}, -t^{-2})^T$ are solutions of (3.8). Since x_1 is not bounded, Wintner's theorem is not applicable here.

Next, we consider linear system (3.3) and the quasilinear system (3.4) and establish Yakubovich type conditions for their asymptotic equivalence. Along the lines, we also derive a formula for asymptotic representation of solutions of (3.4).

Let there exist $\eta \in \mathbb{C}_{rd}([a, \infty)_{\mathbb{T}}, \mathbb{R})$ such that

$$|f(t, x_1) - f(t, x_2)| \leq \eta(t)|x_1 - x_2| \tag{3.10}$$

for all $(t, x_1), (t, x_2) \in [a, \infty)_{\mathbb{T}} \times \mathbb{R}^n$. Assume that

$$\int^{\infty} \|e_C(a, \sigma(t))\| \, \|e_C(t, a)\| \, \eta(t) \, \Delta t < \infty. \tag{3.11}$$

The following lemmas are required, see [9] when $\mathbb{T} = \mathbb{R}$ and [83] when $\mathbb{T} = \mathbb{Z}$.

Lemma 3.1.5 *If (3.11) is valid, then every solution of*

$$u^{\Delta} = e_C(a, \sigma(t)) f(t, e_C(t, a)u) \tag{3.12}$$

is bounded on $[a, \infty)_{\mathbb{T}}$. Moreover, for each solution u of (3.12), there exists a constant vector $c \in \mathbb{R}^n$ such that $u(t) \to c$ as $t \to \infty$.

Proof. Let $u(t) = u(t, a, u_0)$ denote a solution of (3.12) satisfying the initial condition $u(a) = u_0$. In view of $f(t, 0) = 0$ and (3.10), we have

$$|u(t)| \leq |u(a)| + \int_a^t M(s)|u(s)| \, \Delta s, \quad t \in [a, \infty)_{\mathbb{T}},$$

where

$$M(t) = \|e_C(a, \sigma(t))\| \, \|e_C(t, a)\| \, \eta(t).$$

By using (3.11) and the fact that M is positively regressive (i.e., $1 + \mu(t)M(t) > 0$), and applying Gronwall's inequality (i.e., Theorem 2.2.8), we see that there exists M_0 such that $|u(t)| \leq M_0$ for all $t \in [a, \infty)_{\mathbb{T}}$.

To prove the second part of the lemma, we first note that

$$\left| \int_a^t e_C(a, \sigma(s)) f(s, e_C(s, a)u(s)) \, \Delta s \right| \leq M_0 \int_a^\infty M(s) \, \Delta s < \infty.$$

Hence, we may define

$$c = u_0 + \int_a^\infty e_C(a, \sigma(s)) f(s, e_C(s, a)u(s)) \, \Delta s.$$

Now the result follows from observing that

$$u(t) = c - \int_t^\infty e_C(a, \sigma(s)) f(s, e_C(s, a)u(s)) \, \Delta s.$$

■

Lemma 3.1.6 *If y is a solution of (3.4), then there is a solution u of (3.12) such that*

$$y(t) = e_C(t, a)u(t). \tag{3.13}$$

Conversely, if u is a solution of (3.12), then y given by (3.13) is a solution of (3.4).

Proof. The proof can be accomplished by substitution as in [9]. ■

In view of Lemma 3.1.5 and Lemma 3.1.6, we easily get an asymptotic representation of solutions of (3.4).

Theorem 3.1.7 *If (3.11) is satisfied, then every solution y of (3.4) possesses an asymptotic representation as $t \to \infty$, of the form*

$$y(t) = e_C(t, a)[c + o(1)],$$

where $c \in \mathbb{R}^n$ is a constant vector. Moreover,

$$o(1) = -\int_t^\infty e_C(a, \sigma(s)) f(s, e_C(s, a)u(s)) \, \Delta s,$$

where u is a solution of (3.12).

The main result is the following theorem, whose proof is also adapted from [9].

Theorem 3.1.8 *Assume that (3.11) holds and*

$$\lim_{t\to\infty}\int_t^\infty \|e_C(t,\sigma(s))\|\,\|e_C(s,a)\|\,\eta(s)\,\Delta s = 0. \tag{3.14}$$

Then (3.3) and (3.4) are asymptotically equivalent.

Proof. We see that

$$\begin{aligned} y(t) &= e_C(t,a)\left[c - \int_t^\infty e_C(a,\sigma(s))f(s,e_C(s,a)u(s))\,\Delta s\right] \\ &= x(t) - \int_t^\infty e_C(t,\sigma(s))f(s,e_C(s,a)u(s))\,\Delta s, \end{aligned}$$

where $x(t) = e_C(t,a)c$ is a solution of (3.3) and $u(t) = u(t,a,u_0)$ is a solution of (3.12). It is clear that any given u_0 results in a one-to-one correspondence between the solutions x and y. In view of (3.14), we see that $x(t) - y(t) \to 0$ as $t \to \infty$, which completes the proof. ■

Remark 3.1.9 *By considering special time scales $\mathbb{T}$, one can easily replace (3.11) and (3.14) by more concrete conditions.*

Remark 3.1.10 *[9, Theorem 3.2] and [83, Theorem 3.2] can be deduced from Theorem 3.1.8 when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, respectively. Moreover, Theorem 3.1.8 remains valid if the matrix C is replaced by a nonconstant matrix $C(t)$.*

Example 3.1.11 *Let $\mathbb{T} = 2^{\mathbb{N}_0} = \{2^k : k \in \mathbb{N}_0\}$, and consider the systems*

$$x^\Delta = \begin{bmatrix} 4 & 1 \\ 0 & 5 \end{bmatrix} x, \quad t \in [1,\infty)_{\mathbb{T}} \tag{3.15}$$

and

$$y^\Delta = \begin{bmatrix} 4 & 1 \\ 0 & 5 \end{bmatrix} y + \begin{bmatrix} (1-\cos y_1)\prod_{s\in[1,\sigma(t))_{\mathbb{T}}} \frac{1}{(1+5s)} \\ \sin y_2 \prod_{s\in[1,\sigma(t))_{\mathbb{T}}} \frac{1}{(1+5s)} \end{bmatrix}, \quad t \in [1,\infty)_{\mathbb{T}}. \tag{3.16}$$

We see that

$$e_C(t,1) = \begin{bmatrix} \prod_{s\in[1,t)_{\mathbb{T}}}(1+4s) & \prod_{s\in[1,t)_{\mathbb{T}}}(1+5s) - \prod_{s\in[1,t)_{\mathbb{T}}}(1+4s) \\ 0 & \prod_{s\in[1,t)_{\mathbb{T}}}(1+5s) \end{bmatrix}, \quad t \in [1,\infty)_{\mathbb{T}}.$$

Let $\eta(t) = \prod_{s\in[1,\sigma(t))_{\mathbb{T}}} \frac{1}{(1+5s)}$, *and consider the matrix infinity norm. It is easy to verify that (3.11) holds. Indeed,*

$$\int_1^\infty \|e_C(1,\sigma(t))\|\,\|e_C(t,1)\|\,\eta(t)\,\Delta t \le \sum_{k=0}^{\infty} \frac{2^{k+1}}{(1+5(2^k))\prod_{i=0}^{k}(1+2^{i+2})},$$

where the series on the right is convergent by the ratio test. Since

$$\int_t^\infty \|e_C(t,\sigma(s))\|\,\|e_C(s,1)\|\,\eta(s)\,\Delta s \le \sum_{m=n}^{\infty} \frac{2^{m+1}}{(1+5\,(2^m))\prod_{i=n}^{m}(1+2^{i+2})},$$

where $t = 2^n$, $s = 2^m$, $m \ge n$, *we see that (3.14) is also satisfied. Therefore, we may conclude from Theorem 3.1.8 that (3.15) and (3.16) are asymptotically equivalent.*

Example 3.1.12 *Let* $\mathbb{T} = P_{1,1} = \bigcup_{k=0}^{\infty}[2k, 2k+1]$, *and consider the systems*

$$x^\Delta = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} x, \quad t \in [0,\infty)_{\mathbb{T}} \tag{3.17}$$

and

$$y^\Delta = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} y + \begin{bmatrix} e^{-8t}(1-\cos y_1) \\ e^{-9t}\sin y_2 \end{bmatrix}, \quad t \in [0,\infty)_{\mathbb{T}}. \tag{3.18}$$

In this case, for $t \in [2k, 2k+1]$,

$$e_C(t,0) = \begin{bmatrix} \frac{1}{2}(3^k e^{2(t-k)} + (-1)^k e^{-2(t-k)}) & \frac{1}{4}(3^k e^{2(t-k)} - (-1)^k e^{-2(t-k)}) \\ 3^k e^{2(t-k)} - (-1)^k e^{-2(t-k)} & \frac{1}{2}(3^k e^{2(t-k)} + (-1)^k e^{-2(t-k)}) \end{bmatrix}.$$

Letting $\eta(t) = e^{-8t}$, *one can easily check that (3.11) is satisfied. Indeed,*

$$\begin{aligned} \int_0^\infty \|e_C(0,\sigma(t))\|\,\|e_C(t,0)\|\,\eta(t)\,\Delta t &\le 4e^2 \int_0^\infty e^{-4t}\,\Delta t \\ &= 4\sum_{k=0}^{\infty}\int_{2k}^{2k+1} e^{-4t+2}\,dt + 4\sum_{k=0}^{\infty} e^{-8k-2} \\ &= \frac{e^2+3e^{-2}}{1-e^{-8}}. \end{aligned}$$

And since

$$\int_t^\infty \|e_C(t,\sigma(s))\|\,\|e_C(s,0)\|\,\eta(s)\,\Delta s \le 8e^{2t+2}\int_t^\infty e^{-4s}\,\Delta s,$$

condition (3.14) holds as well. Therefore, (3.17) and (3.18) are asymptotically equivalent by Theorem 3.1.8.

3.1.3 Further Results on Asymptotic Equivalence

Note that conditions (3.11) and (3.14) may not be so easy to verify as they require the computation of matrix exponential $e_C(t, s)$. It is therefore plausible to derive conditions which are easier to verify than (3.11) and (3.14). In this subsection, we aim at providing such sufficient conditions.

The following result [69, Theorem 2.1], which is a generalization of a result obtained in [40], is useful.

Lemma 3.1.13 *Let $p \in \mathcal{R}$ and $a \in \mathbb{T}$. Define*

$$\mathcal{B}_p(t) = \begin{cases} \frac{1}{\mu(t)} \log|1 + \mu(t)p(t)|, & \mu(t) > 0 \\ \Re\, p(t), & \mu(t) = 0 \end{cases}, \quad t \in [a, \infty)_{\mathbb{T}},$$

where $\Re\, p(t)$ denotes the real part of $p(t)$.

Suppose that there exist real numbers r and q such that $r \le \mathcal{B}_p(t) \le q$. Then

$$e^{r(t-a)} \le |e_p(t, a)| \le e^{q(t-a)}, \quad t \in [a, \infty)_{\mathbb{T}}.$$

Let $\lambda_j \in \mathcal{R}$, $j = 1, 2, \ldots, l$, be the eigenvalues of C with multiplicity m_j in the minimal polynomial. It follows from Theorem 2.2.16 with $n = 1$ and Putzer Algorithm (i.e., Theorem 2.3.2), see also [84], that the elements of matrix exponential $e_C(t, a)$ consist of linear combination of the following functions:

$$e_{\lambda_j}(t, a), k_1^{\lambda_j}(t, a)e_{\lambda_j}(t, a), \ldots, k_{m_j-1}^{\lambda_j}(t, a)e_{\lambda_j}(t, a), \quad j = 1, 2 \ldots, l, \tag{3.19}$$

where

$$k_0^{\lambda_j}(t, a) \equiv 1, \qquad k_i^{\lambda_j}(t, a) = \int_a^t \frac{k_{i-1}^{\lambda_j}(\tau, a)}{1 + \lambda_j \mu(\tau)} \Delta\tau, \quad i = 1, 2, \ldots, m_j - 1. \tag{3.20}$$

We assume that there exist real numbers $M_0 > 0$ and β such that

$$\frac{1}{|1 + \lambda_j \mu(t)|} \le M_0 \tag{3.21}$$

and

$$\mathcal{B}_{\lambda_j}(t) \leq \beta \tag{3.22}$$

for all $t \in [a, \infty)_{\mathbb{T}}$ and $j = 1, 2, \ldots l$.

Condition (3.21) means that λ_j's are uniformly regressive. Under this condition, we easily obtain from (3.20) that

$$|k_i^{\lambda_j}(t, a)| \leq M_0^i (t - a)^i \tag{3.23}$$

for all $t \in [a, \infty)_{\mathbb{T}}$, $i = 1, 2, \ldots m_j - 1$, and $j = 1, 2 \ldots, l$. Making use of (3.22) in Lemma 3.1.13, we also obtain that

$$|e_{\lambda_j}(t, a)| \leq e^{\beta(t-a)} \tag{3.24}$$

for all $t \in [a, \infty)_{\mathbb{T}}$ and $j = 1, 2 \ldots, l$.

Let $m_0 = \max\{m_1, m_2, \ldots, m_l\}$. Using (3.19), (3.23), and (3.24), we see that there exist real numbers K and $t_1 \in [a, \infty)_{\mathbb{T}}$ such that

$$\|e_C(t, a)\| \leq K t^{m_0 - 1} e^{\beta t}, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.25}$$

Unfortunately, a similar result for $\|e_C(a, t)\|$ is not so obvious since it involves $\ominus C^*$, which is not a constant matrix. However, we may get an estimate for $\|e_C(a, t)\|$ (not as sharp as the previous one) by using the fact that if $S(t)$ is an $n \times n$-invertible-matrix, then

$$\|S^{-1}(t)\| \leq \frac{\|S(t)\|^{n-1}}{|\det S(t)|}.$$

Indeed, if we replace $S(t)$ by $e_C(t, a)$ in the above inequality, then

$$\|e_C(a, t)\| \leq \frac{\|e_C(t, a)\|^{n-1}}{|\det e_C(t, a)|}$$

from which on using Liouville's formula, see [32],

$$\det e_C(t, a) = e_{\lambda_1 \oplus \lambda_2 \oplus \ldots \oplus \lambda_l}(t, a),$$

we have

$$\|e_C(a, t)\| \leq \frac{[K t^{m_0 - 1} e^{\beta t}]^{n-1}}{|e_{\lambda_1 \oplus \lambda_2 \oplus \ldots \oplus \lambda_l}(t, a)|}, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.26}$$

Theorem 3.1.14 *Let $\lambda_j \in \mathcal{R}$, $j = 1, 2, \ldots, l$, be the eigenvalues of C with multiplicity m_j in the minimal polynomial of C, and let $m_0 = \max\{m_1, m_2, \ldots, m_l\}$. Suppose that (3.21) and (3.22) are satisfied. If*

$$\int^{\infty} \frac{[t(\sigma(t))^{n-1}]^{m_0-1} e^{\beta(n\sigma(t)-\mu(t))}}{|e_{\lambda_1 \oplus \lambda_2 \oplus \ldots \oplus \lambda_l}(\sigma(t), a)|} \eta(t)\, \Delta t < \infty \tag{3.27}$$

and

$$\lim_{t\to\infty} t^{m_0-1} e^{\beta t} \int_t^{\infty} \frac{[s(\sigma(s))^{n-1}]^{m_0-1} e^{\beta(n\sigma(s)-\mu(s))}}{|e_{\lambda_1 \oplus \lambda_2 \oplus \ldots \oplus \lambda_l}(\sigma(s), a)|} \eta(s)\, \Delta s = 0, \tag{3.28}$$

then (3.3) and (3.4) are asymptotically equivalent.

Proof. Clearly, (3.25) and (3.26) hold. Then (3.27) and (3.28) implies (3.11) and (3.14), respectively. By Theorem 3.1.8, the conclusion follows. ■

Example 3.1.15 *Let $\mathbb{T}$ be an arbitrary time scale such that $1 \in \mathbb{T}$. Consider the systems*

$$x^{\Delta} = Cx, \quad [1, \infty)_{\mathbb{T}} \tag{3.29}$$

and

$$y^{\Delta} = Cy + e^2_{\ominus 3}(t, 1) g(t, y), \quad [1, \infty)_{\mathbb{T}}, \tag{3.30}$$

where

$$C = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

and $g \in \mathbb{C}_{rd}([1, \infty)_{\mathbb{T}} \times \mathbb{R}^3)$ satisfies $g(t, 0) = 0$ and

$$|g(t, y_1) - g(t, y_2)| \le L|y_1 - y_2|$$

for some $L > 0$ and for all $(t, y_1), (t, y_2) \in [1, \infty)_{\mathbb{T}} \times \mathbb{R}^3$.

Eigenvalues of C are $\lambda_1 = 2$ and $\lambda_2 = 3$ with multiplicities $m_1 = 2$ and $m_2 = 1$, respectively. We take $a = 1$, $m_0 = 2$, and $\eta(t) = Le^2_{\ominus 3}(t, 1)$, and see that (3.21) and (3.22) are satisfied for $\mathbb{T} = \mathbb{N}$ ($M_0 = 1/3, \beta = \ln 4$) and for $\mathbb{T} = \mathbb{R}$ ($M_0 = 1, \beta = 3$). It is easy to check that (3.27) and (3.28) are satisfied as well. Note that if $\mathbb{T} = \mathbb{N}$, then

$$\int_t^{\infty} \frac{[s(\sigma(s))^{n-1}]^{m_0-1} e^{\beta(n\sigma(s)-\mu(s))}}{|e_{\lambda_1 \oplus \lambda_2 \oplus \lambda_3}(\sigma(s), a)|} \eta(s)\, \Delta s = 2^8 L \sum_{s=t}^{\infty} \frac{s(s+1)^2}{3^{2s}},$$

while if $\mathbb{T} = \mathbb{R}$, *then*

$$\int_t^\infty \frac{[s(\sigma(s))^{n-1}]^{m_0-1} e^{\beta(n\sigma(s)-\mu(s))}}{|e_{\lambda_1\oplus\lambda_2\oplus\lambda_3}(\sigma(s),a)|}\eta(s)\,\Delta s = L\int_t^\infty s^3 e^{-4s+13}\,ds.$$

By applying Theorem 3.1.14, we see that (3.29) and (3.30) are asymptotically equivalent systems on $\mathbb{T} = \mathbb{N}$ *and* $\mathbb{T} = \mathbb{R}$.

3.2 Asymptotic Equilibrium

3.2.1 Introduction

In this section, we study the asymptotic behavior of solutions of the nonlinear dynamic system

$$x^\Delta = A(t)x + f(t,x), \quad t \in [a,\infty)_{\mathbb{T}}, \tag{3.31}$$

where $A \in \mathbb{C}_{rd}([a,\infty)_{\mathbb{T}}, \mathbb{R}^{n\times n})$, $f : [a,\infty)_{\mathbb{T}}\times\mathbb{R}^n \to \mathbb{R}^n$ is a function such that $f(.,z(.)) \in \mathbb{C}_{rd}([a,\infty)_{\mathbb{T}}, \mathbb{R}^n)$ for any function $z \in \mathbb{C}_{rd}([a,\infty)_{\mathbb{T}}, \mathbb{R}^n)$, and continuous in x for each fixed t.

The following definition is adapted from $\mathbb{T} = \mathbb{R}$ to any arbitrary time scale $\mathbb{T}$.

Definition 3.2.1 *The system (3.31) is said to have asymptotic equilibrium if for each constant vector* $c \in \mathbb{R}^n$, *there is a solution x of (3.31) such that*

$$\lim_{t\to\infty} x(t) = c.$$

Our aim is to establish sufficient conditions for asymptotic equilibrium of (3.31) by extending some continuous results from differential equations to dynamic equations on arbitrary time scales. Our arguments are closely related to the ones given in [76, 77]. In particular, we obtain the time scale extension of Wintner's well-known theorem (i.e., Theorem 1.2.5) by considering the linear system

$$x^\Delta = A(t)x, \quad t \in [a,\infty)_{\mathbb{T}}. \tag{3.32}$$

To the best of our knowledge some of the results obtained in this section are new even for the special case $\mathbb{T} = \mathbb{Z}$.

To prove the main results, we will use the well-known Schauder's fixed point theorem (i.e., Theorem 2.4.10) and the time scale version of the well-known Arzela-Ascoli theorem (i.e., Theorem 2.4.11).

3.2.2 Preparatory Lemmas

Let w be a real-valued, rd-continuous, nonincreasing function defined on $[a,\infty)_{\mathbb{T}}$ such that $0 < w(t) \leq 1$, and either $\lim_{t\to\infty} w(t) = 0$ or $w(t) \equiv 1$. For $c \in \mathbb{R}^n$ and $M > 0$ a real number, define as in [76] the set

$$S = \{(t,x) \in [a,\infty)_{\mathbb{T}} \times \mathbb{R}^n : \mid x - c \mid \leq Mw(t)\}.$$

We list the following conditions for convenience.

(H1) For any given $c \in \mathbb{R}^n$, there is a constant $M > 0$ such that

$$\mid f(t,x) - f(t,c) \mid \leq R(t, \mid x - c \mid) \quad \text{for all } (t,x) \in S,$$

where $R(t,\lambda)$ is a real-valued function defined on the set

$$\{(t,\lambda) : \ t \in [a,\infty)_{\mathbb{T}}, \ \ 0 \leq \lambda \leq Mw(t)\}$$

such that $R(.,z(.)) \in \mathbb{C}_{rd}([a,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ for any function $z \in \mathbb{C}_{rd}([a,\infty)_{\mathbb{T}}, \mathbb{R})$ with $0 \leq z(t) \leq Mw(t)$, and nondecreasing in λ for each fixed t.

(H2) The integrals defined recursively by

$$A_j(t) = \int_t^\infty A_{j-1}(s)A(s)\,\Delta s, \quad 1 \leq j \leq k, \quad t \in [a,\infty)_{\mathbb{T}} \quad (A_0 = I)$$

are convergent.

(H3) The integral

$$\int^\infty \Big[A_k(s)A(s)c + B_k(s)f(s,c) - \mu(s)B_{k-1}(s)A(s)[A(s)c + f(s,c)]\Big]\,\Delta s$$

converges for $k \geq 0$ and for every constant vector $c \in \mathbb{R}^n$, where

$$B_k(t) = A_0(t) + A_1(t) + \cdots + A_k(t), \quad t \in [a,\infty)_{\mathbb{T}}.$$

Since $\lim_{t\to\infty} B_k(t) = I$, B_k is invertible for large t, and $\lim_{t\to\infty} B_k^{-1}(t) = I$.

Let $t_0 \geq a$ be sufficiently large so that B_k^{-1} exists on $[t_0, \infty)_{\mathbb{T}}$, and let V_{t_0} denote a closed convex subset of $\mathbb{C}_0([t_0, \infty)_{\mathbb{T}})$ defined by

$$V_{t_0} = \{x \in \mathbb{C}_0([t_0, \infty)_{\mathbb{T}}) : \mid x(t) - c \mid \leq Mw(t),\ t \in [t_0, \infty)_{\mathbb{T}}\}. \tag{3.33}$$

Lemma 3.2.2 *Let (H1), (H2), and (H3) be satisfied. If*

$$\int^{\infty} \left[(1 + \mu(s)\|A(s)\|) \frac{R(s, Mw(s))}{M} + (\|A_k(s)A(s)\| + \mu(s)\|A^2(s)\|)w(s) \right] \Delta s \tag{3.34}$$

converges, then any fixed point of

$$\begin{aligned} (\mathcal{T}x)(t) = B_k^{-1}(t)c - B_k^{-1}(t) \int_t^{\infty} \Big\{ & A_k(s)A(s)x(s) + B_k(s)f(s, x(s)) \\ & - \mu(s)B_{k-1}(s)A(s)[A(s)x(s) + f(s, x(s))] \Big\} \Delta s \end{aligned} \tag{3.35}$$

in V_{t_0} is a solution of (3.31).

Proof. The integral in (3.35) converges for $x \in V_{t_0}$ by the hypothesis (H1), (H3), and (3.34). Let $x \in V_{t_0}$ be a fixed point of $\mathcal{T}$. It follows that

$$\begin{aligned} x^{\Delta}(t) &= (\mathcal{T}x)^{\Delta}(t) \\ &= B_k^{-1}(\sigma(t))[A_k(t)A(t)x(t) + B_k(t)f(t, x(t)) - \mu(t)B_{k-1}(t)A^2(t)x(t) \\ &\quad - \mu(t)B_{k-1}(t)A(t)f(t, x(t))] + B_k^{-1}(\sigma(t))B_{k-1}(t)A(t)(\mathcal{T}x)(t) \\ &= B_k^{-1}(\sigma(t))[A_k(t)A(t)x(t) + B_k(\sigma(t))f(t, x(t)) \\ &\quad - \mu(t)B_{k-1}(t)A^2(t)x(t)] + B_k^{-1}(\sigma(t))B_{k-1}(t)A(t)x(t) \\ &= B_k^{-1}(\sigma(t))[A_k(t)A(t)x(t) + B_k(\sigma(t))f(t, x(t)) + B_k(\sigma(t))A(t)x(t) \\ &\quad - B_k(t)A(t)x(t)] + B_k^{-1}(\sigma(t))B_{k-1}(t)A(t)x(t) \\ &= A(t)x(t) + f(t, x(t)), \end{aligned}$$

where we have used the product formula for differentiation and the relationships

$$B_k^{\Delta} = -B_{k-1}A, \quad B_{k-1} + A_k = B_k, \quad k \geq 0 \quad (B_{-1} = 0)$$

and the identities

$$z(\sigma(t)) = z(t) + \mu(t)z^{\Delta}(t), \quad (B_k^{-1})^{\Delta}(t) = -B_k^{-1}(\sigma(t))B_k^{\Delta}(t)B_k^{-1}(t).$$

■

Lemma 3.2.3 *Let (H1), (H2), and (H3) be satisfied. Suppose that*

$$\limsup_{t\to\infty}\frac{1}{w(t)}\int_t^\infty\Big[(1+\mu(s)\|A(s)\|)\frac{R(s,Mw(s))}{M} + (\|A_k(s)A(s)\|+\mu(s)\|A^2(s)\|)w(s)\Big]\Delta s=\theta<1 \tag{3.36}$$

and

$$\limsup_{t\to\infty}\frac{1}{w(t)}\Big|(B_k^{-1}(t)-I)c-B_k^{-1}(t)\int_t^\infty\Big\{A_k(s)A(s)c+B_k(s)f(s,c) - \mu(s)B_{k-1}(s)A(s)[A(s)c+f(s,c)]\Big\}\Delta s\Big|=\alpha. \tag{3.37}$$

If

$$\alpha<M(1-\theta), \tag{3.38}$$

then the operator $\mathcal{T}$ given by (3.35) has a fixed point in V_{t_0} for t_0 sufficiently large.

Proof. We may rewrite the operator $\mathcal{T}$ as follows:

$$\begin{aligned}(\mathcal{T}x)(t)=c+h(t)-B_k^{-1}(t)\int_t^\infty\Big[&A_k(s)A(s)(x(s)-c)\\&+B_k(s)(f(s,x(s))-f(s,c))-\mu(s)B_{k-1}(s)A^2(s)(x(s)-c)\\&-\mu(s)B_{k-1}(s)A(s)(f(s,x(s))-f(s,c))\Big]\Delta s,\end{aligned} \tag{3.39}$$

where

$$h(t)=(B_k^{-1}(t)-I)c-B_k^{-1}(t)\int_t^\infty\Big[A_k(s)A(s)c+B_k(s)f(s,c) - \mu(s)B_{k-1}(s)A(s)\big[A(s)c+f(s,c)\big]\Big]\Delta s. \tag{3.40}$$

Because of (3.33), (3.36), and (H1), the integral in (3.39) is convergent. We apply the Schauder's fixed point theorem.

(i) $\mathcal{T}V_{t_0}\subseteq V_{t_0}$.

Let $x\in V_{t_0}$. It is clear that

$$|(\mathcal{T}x)(t)-c|\le|h(t)|+M\tau_k(t)\int_t^\infty\Big[(1+\mu(s)\|A(s)\|)\frac{R(s,Mw(s))}{M} + (\|A_k(s)A(s)\|+\mu(s)\|A^2(s)\|)w(s)\Big]\Delta s, \tag{3.41}$$

where

$$\tau_k(t)=\sup_{s\ge t}\|B_k^{-1}(s)\|\max\Big\{\sup_{s\ge t}\|B_k(s)\|,\ \sup_{s\ge t}\|B_{k-1}(s)\|\Big\}. \tag{3.42}$$

Using (3.36), (3.37), (3.38), and $\tau_k(t) = 1 + o(1)$ as $t \to \infty$, we see that the right side of (3.41) is less than or equal to $Mw(t)$ for $t \geq t_0$ provided that t_0 is sufficiently large.

(ii) $\mathcal{T}$ is continuous on V_{t_0}.

Let $\{x_j\}$ be a sequence in V_{t_0} which converges to a limit x in V_{t_0}. We see that as long as $t \geq t_0$,

$$|(\mathcal{T}x_j)(t) - (\mathcal{T}x)(t)| \leq \tau_k(t_0) \int_{t_0}^{\infty} \Big[(\|A_k(s)A(s)\| + \mu(s)\|A^2(s)\|)|x_j(s) - x(s)| \\ + (1 + \mu(s)\|A(s)\|)|f(s, x_j(s)) - f(s, x(s))| \Big] \Delta s.$$

The integrand is dominated by the function $2Mw(s)(\|A_k(s)A(s)\|+\mu(s)\|A^2(s)\|)+ 2R(s, Mw(s))(1 + \mu(s)\|A(s)\|)$. In view of the Lebesgue dominated convergence theorem on time scales, see [16], and (3.36), it follows that $\{\mathcal{T}x_j\}$ converges to $\mathcal{T}x$ uniformly on $[t_0, \infty)_{\mathbb{T}}$, and hence $\mathcal{T}$ is continuous.

(iii) $\mathcal{T}V_{t_0}$ is relatively compact.

We verify the conditions of Theorem 2.4.11 with $Y = \mathcal{T}V_{t_0}$ and $a = t_0$. The condition (a) trivially holds by the definition of V_{t_0}. Let us show that (b) holds. Indeed, let J be a compact subinterval of $[t_0, \infty)_{\mathbb{T}}$, $\varepsilon > 0$ be arbitrary, and $t_1, t_2 \in J$ such that $t_2 \geq t_1$. From (3.39), we have

$$|(\mathcal{T}x)(t_1) - (\mathcal{T}x)(t_2)| \leq |h(t_1) - h(t_2)| + \|B_k^{-1}(t_1)\| \int_{t_1}^{t_2} |G(s)| \Delta s \\ + \|B_k^{-1}(t_2) - B_k^{-1}(t_1)\| \int_{t_2}^{\infty} |G(s)| \Delta s,$$

where

$$G(t) = [A_k(t)A(t) - \mu(t)B_{k-1}(t)A^2(t)](x(t) - c) \\ + [B_k(t) - \mu(t)B_{k-1}(t)A(t)](f(t, x(t)) - f(t, c)).$$

Further, we get

$$|(\mathcal{T}x)(t_1) - (\mathcal{T}x)(t_2)| \leq |h(t_1) - h(t_2)| + M\tau_k(t_0) \int_{t_1}^{t_2} H(s) \Delta s \\ + \left\{ M \frac{\tau_k(t_0)}{\sup_{t \geq t_0} \|B_k^{-1}(t)\|} \int_{t_0}^{\infty} H(s) \Delta s \right\} \|B_k^{-1}(t_2) - B_k^{-1}(t_1)\|,$$

where

$$H(t) = (1 + \mu(t)\|A(t)\|) \frac{R(t, Mw(t))}{M} + (\|A_k(t)A(t)\| + \mu(t)\|A^2(t)\|)w(t).$$

Now, since h, B_k^{-1} are continuous and H is rd-continuous (for the integral, note that H is rd-continuous and hence has an antiderivative Q, and thus the integral is equal to $Q(t_2) - Q(t_1)$ and Q is continuous), it follows that if $|t_1 - t_2| < \delta$ with δ sufficiently small, then

$$|(\mathcal{T}x)(t_1) - (\mathcal{T}x)(t_2)| < \varepsilon$$

for all $x \in V_{t_0}$, i.e., (b) holds. To see (c) holds, let $\varepsilon > 0$ be given. Choose t_0^* sufficiently large so that

$$|h(t_1) - h(t_2)| < \frac{\varepsilon}{3} \quad \text{for all } t_1, t_2 \geq t_0^*, \tag{3.43}$$

$$\tau_k(t) \leq 2 \quad \text{for all } t \geq t_0^*, \tag{3.44}$$

and

$$\int_{t_0^*}^{\infty} H(s)\,\Delta s < \frac{\varepsilon}{6M}, \tag{3.45}$$

where we have used the fact that $\lim_{t\to\infty} h(t) = 0$, $\lim_{t\to\infty} \tau_k(t) = 1$, and the integral $\int^{\infty} H(t)\,\Delta t$ is convergent. We write

$$\begin{aligned}(\mathcal{T}x)(t_1) - (\mathcal{T}x)(t_2) = h(t_1) - h(t_2) - B_k^{-1}(t_1)\int_{t_1}^{\infty} G(s)\,\Delta s \\ + B_k^{-1}(t_2)\int_{t_2}^{\infty} G(s)\,\Delta s.\end{aligned}$$

It follows that

$$\begin{aligned}|(\mathcal{T}x)(t_1) - (\mathcal{T}x)(t_2)| \leq |h(t_1) - h(t_2)| + M\tau_k(t_1)\int_{t_0^*}^{\infty} H(s)\,\Delta s \\ + M\tau_k(t_2)\int_{t_0^*}^{\infty} H(s)\,\Delta s.\end{aligned}$$

Employing (3.43), (3.44), and (3.45), we obtain

$$|(\mathcal{T}x)(t_1) - (\mathcal{T}x)(t_2)| < \varepsilon$$

for all $x \in V_{t_0}$, i.e., (c) holds. Thus, $\mathcal{T}V_{t_0}$ is relatively compact.

In view of (i), (ii), and (iii), we may conclude by using Theorem 2.4.10 that the operator $\mathcal{T}$ has a fixed point in V_{t_0}.

■

3.2.3 Main Results

Theorem 3.2.4 *Suppose that (H1), (H2), (H3), (3.36), (3.37), and (3.38) are satisfied. Then there is a solution x of (3.31) such that*

$$|x(t) - c| \le Mw(t), \quad t \in [t_0, \infty)_{\mathbb{T}}; \tag{3.46}$$

$$\limsup_{t\to\infty} \frac{1}{w(t)}|x(t) - c| \le \alpha + M\theta; \tag{3.47}$$

$$\limsup_{t\to\infty} \frac{1}{w(t)}|x(t) - c - h(t)| \le M\theta. \tag{3.48}$$

Proof. By Lemma 3.2.3, the operator $\mathcal{T}$ has a fixed point $x \in V_{t_0}$. Since (3.36) implies (3.34), the fixed point is a solution of (3.31) by Lemma 3.2.2.

Now, from the definition of V_{t_0}, we see that (3.46) holds. The estimate (3.47) follows from (3.41), while (3.48) can be easily obtained from (3.39). ■

Corollary 3.2.5 *Under the conditions of Theorem 3.2.4, system (3.31) has asymptotic equilibrium.*

Proof. Let $w(t) \equiv 1$, then we necessarily have $\alpha = \theta = 0$ and so

$$\lim_{t\to\infty} x(t) = c.$$

If $\lim_{t\to\infty} w(t) = 0$, the limit is obvious. ■

A better estimate is possible for (3.47) provided that the function R has a special monotonicity property.

Theorem 3.2.6 *Let the assumptions of Theorem 3.2.4 hold. In addition, assume that $R(t, \lambda)/\lambda$ is nondecreasing in λ for each fixed t and $R(t, 0) = 0$ for all $t \in [a, \infty)_{\mathbb{T}}$. Then*

$$\limsup_{t\to\infty} \frac{1}{w(t)}|x(t) - c| \le \alpha/(1 - \theta). \tag{3.49}$$

Proof. Let $u(t) = |x(t) - c|$ and $L = \limsup_{t\to\infty} u(t)/w(t)$. Setting $\mathcal{T}x = x$ in (3.39) and estimating the integrals results in

$$\begin{aligned} u(t) \le |h(t)| + \tau_k(t) \int_t^\infty \Big[& (1 + \mu(s)\|A(s)\|)R(s, u(s)) \\ & + (\|A_k(s)A(s)\| + \mu(s)\|A^2(s)\|)u(s) \Big] \Delta s, \end{aligned} \tag{3.50}$$

where h and τ_k are defined by (3.40) and (3.42), respectively. Using the nondecreasing nature of $R(t,\lambda)/\lambda$ together with $R(t,0) = 0$ yields

$$\frac{u(t)}{w(t)} \le \frac{|h(t)|}{w(t)} + \frac{\tau_k(t)}{w(t)} \sup_{s\ge t} \frac{u(s)}{w(s)} \int_t^\infty \Big[(1+\mu(s)\|A(s)\|) \frac{R(s, Mw(s))}{M} + (\|A_k(s)A(s)\| + \mu(s)\|A^2(s)\|) w(s) \Big] \Delta s,$$

from which, we get $L \le \alpha + \theta L$. This proves (3.49). ■

In case $w(t) \equiv 1$, Theorem 3.2.4 can be simplified as follows.

Theorem 3.2.7 *Let (H1), (H2), and (H3) be satisfied with $w(t) \equiv 1$. Suppose that*

$$\int^\infty R(t,M)\,\Delta t < \infty, \quad \int^\infty \|A_k(t)A(t)\|\,\Delta t < \infty,$$
$$\int^\infty \mu(t)\|A^2(t)\|\,\Delta t < \infty, \quad \int^\infty \mu(t)\|A(t)\|R(t,M)\,\Delta t < \infty.$$

Then system (3.31) has asymptotic equilibrium.

The set S results in a restriction on the choices of c. Replacing (H1) by (H1*) below, we may get rid of such a restriction.

(H1*) Let f satisfy

$$|f(t,x_1) - f(t,x_2)| \le R(t, |x_1 - x_2|) \quad \text{for all } (t,x_1),(t,x_2) \in [a,\infty)_{\mathbb{T}} \times \mathbb{R}^n,$$

where $R(t,\lambda)$ is a real-valued function defined on $[a,\infty)_{\mathbb{T}} \times \mathbb{R}^+$ such that $R(.,z(.)) \in \mathbb{C}_{rd}([a,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ for any function $z \in \mathbb{C}_{rd}([a,\infty)_{\mathbb{T}}, \mathbb{R}^+)$, and nondecreasing in λ for each fixed t.

Theorem 3.2.8 *Let (H1*) and (H2) hold. Suppose that for some $k \ge 0$ and for every constant vector $c \in \mathbb{R}^n$,*

$$\lim_{t\to\infty} \frac{1}{w(t)} \int_t^\infty B_k(s) f(s,c)\,\Delta s = 0, \tag{3.51}$$
$$\lim_{t\to\infty} \frac{1}{w(t)} \int_t^\infty A_{j-1}(s)A(s)\,\Delta s = 0,\ 1 \le j \le k+1, \tag{3.52}$$
$$\lim_{t\to\infty} \frac{1}{w(t)} \int_t^\infty \mu(s) B_{k-1}(s) A^2(s)\,\Delta s = 0, \tag{3.53}$$
$$\lim_{t\to\infty} \frac{1}{w(t)} \int_t^\infty \mu(s) B_{k-1}(s) A(s) f(s,c)\,\Delta s = 0. \tag{3.54}$$

If for all $M > 0$,

$$\lim_{t\to\infty} \frac{1}{w(t)} \int_t^\infty \Big[(1 + \mu(s)\|A(s)\|) \frac{R(s, Mw(s))}{M} + (\|A_k(s)A(s)\| + \mu(s)\|A^2(s)\|)w(s) \Big] \Delta s = 0, \tag{3.55}$$

then system (3.31) has asymptotic equilibrium.

Proof. In view of (3.51), (3.52), (3.53), and (3.54), we see that (3.37) holds with $\alpha = 0$. The proof is complete by applying Theorem 3.2.4. ■

Finally, let us consider

$$x^\Delta = [A(t) + B(t)]x + g(t), \quad t \in [a, \infty)_{\mathbb{T}}, \tag{3.56}$$

where $B \in \mathbb{C}_{rd}([a, \infty)_{\mathbb{T}}, \mathbb{R}^{n\times n})$, $g \in \mathbb{C}_{rd}([a, \infty)_{\mathbb{T}}, \mathbb{R}^n)$. Observe that since $f(t, x) = B(t)x + g(t)$, we have $R(t, \lambda) = \|B(t)\|\lambda$. From Theorem 3.2.8, we have the following corollaries.

Corollary 3.2.9 *Suppose that (H2) holds and for some $k \geq 0$,*

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{w(t)} \int_t^\infty B_k(s)B(s)\,\Delta s = 0,\\
&\lim_{t\to\infty} \frac{1}{w(t)} \int_t^\infty B_k(s)g(s)\,\Delta s = 0,\\
&\lim_{t\to\infty} \frac{1}{w(t)} \int_t^\infty A_{j-1}(s)A(s)\,\Delta s = 0,\ 1 \leq j \leq k+1,\\
&\lim_{t\to\infty} \frac{1}{w(t)} \int_t^\infty \mu(s)B_{k-1}(s)A(s)B(s)\,\Delta s = 0,\\
&\lim_{t\to\infty} \frac{1}{w(t)} \int_t^\infty \mu(s)B_{k-1}(s)A(s)g(s)\,\Delta s = 0,\\
&\lim_{t\to\infty} \frac{1}{w(t)} \int_t^\infty \mu(s)B_{k-1}(s)A^2(s)\,\Delta s = 0.
\end{aligned}$$

If

$$\lim_{t\to\infty} \frac{1}{w(t)} \int_t^\infty \Big[(1 + \mu(s)\|A(s)\|)\,\|B(s)\|w(s) + (\|A_k(s)A(s)\| + \mu(s)\|A^2(s)\|)w(s) \Big] \Delta s = 0,$$

then system (3.56) has linear asymptotic equilibrium.

Corollary 3.2.10 *Suppose that (H2) holds and for some $k \geq 0$,*

$$\lim_{t\to\infty} \frac{1}{w(t)} \int_t^\infty A_{j-1}(s)A(s)\,\Delta s = 0,\ 1 \leq j \leq k+1,$$
$$\lim_{t\to\infty} \frac{1}{w(t)} \int_t^\infty \mu(s)B_{k-1}(s)A^2(s)\,\Delta s = 0.$$

If

$$\lim_{t\to\infty} \frac{1}{w(t)} \int_t^\infty \Big[\|A_k(s)A(s)\| + \mu(s)\|A^2(s)\|\Big]w(s)\,\Delta s = 0,$$

then system (3.32) has linear asymptotic equilibrium.

Remark 3.2.11 *If we set $\mathbb{T} = \mathbb{R}$ ($\mu(t) \equiv 0$), then our results coincide with the ones obtained by Trench in [76]. For $\mathbb{T} = \mathbb{Z}$ ($\mu(t) \equiv 1$), we recover the result [77, Theorem 1] and its corollaries in the special case $w(t) \equiv 1$.*

Remark 3.2.12 *When $\mathbb{T} = \mathbb{R}$, Corollary 3.2.10 reduces to Theorem 1.2.5 in the particular case $w(t) \equiv 1$, and thus gives an extension of Theorem 1.2.5 to the linear system (3.32) on time scales.*

3.2.4 Applications

In this section, we state only Theorem 3.2.8, Corollary 3.2.9, and Corollary 3.2.10 for the special cases $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = q^{\mathbb{N}}$ ($q > 1$). The other corresponding theorems can be given in a similar manner.

Difference equations

We consider the systems of difference equations

$$\Delta x(n) = A(n)x(n) + f(n, x(n)), \quad n \in \mathbb{N}, \tag{3.57}$$

$$\Delta x(n) = [A(n) + B(n)]x(n) + g(n), \quad n \in \mathbb{N}, \tag{3.58}$$

and

$$\Delta x(n) = A(n)x(n), \quad n \in \mathbb{N}, \tag{3.59}$$

where A, $B : \mathbb{N} \to \mathbb{R}^{n\times n}$, $g : \mathbb{N} \to \mathbb{R}^n$, $f : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$, $f(n, x)$ is continuous in x for each fixed n, and Δ is the forward difference operator defined by $\Delta x(n) = x(n + 1) - x(n)$, which in the time scale language is the derivative of x at n.

The conditions (H1*) and (H2) take the following forms.

(C1) Let f satisfy

$$|f(n, x_1) - f(n, x_2)| \le R(n, |x_1 - x_2|) \quad \text{for all } (n, x_1), (n, x_2) \in \mathbb{N} \times \mathbb{R}^n,$$

where $R(n, \lambda)$ is a real-valued function defined on $\mathbb{N}\times\mathbb{R}^+$ and nondecreasing in λ for each fixed n.

(C2) The series

$$A_j(n) = \sum_{s=n}^{\infty} A_{j-1}(s)A(s), \quad 1 \le j \le k, \quad n \in \mathbb{N} \quad (A_0 = I)$$

are convergent.

Let w be a real-valued nonincreasing function defined on $\mathbb{N}$ such that $0 < w(n) \le 1$, and either $\lim_{n\to\infty} w(n) = 0$ or $w(n) \equiv 1$, and as before

$$B_k(n) = A_0(n) + A_1(n) + \cdots + A_k(n), \quad n \in \mathbb{N}.$$

The results below generalize and improve the ones contained in [77], where the author considers the case $w(n) \equiv 1$.

Theorem 3.2.13 *Let (C1) and (C2) hold. Suppose that for some $k \ge 0$ and for every constant vector $c \in \mathbb{R}^n$,*

$$\lim_{n\to\infty} \frac{1}{w(n)} \sum_{s=n}^{\infty} B_k(s)f(s, c) = 0,$$

$$\lim_{n\to\infty} \frac{1}{w(n)} \sum_{s=n}^{\infty} A_{j-1}(s)A(s) = 0, \ 1 \le j \le k + 1,$$

$$\lim_{n\to\infty} \frac{1}{w(n)} \sum_{s=n}^{\infty} B_{k-1}(s)A^2(s) = 0,$$

$$\lim_{n\to\infty} \frac{1}{w(n)} \sum_{s=n}^{\infty} B_{k-1}(s)A(s)f(s, c) = 0.$$

If for all $M > 0$,

$$\lim_{n\to\infty} \frac{1}{w(n)} \sum_{s=n}^{\infty} \left[(1 + ||A(s)||) \frac{R(s, Mw(s))}{M} + (||A_k(s)A(s)|| + ||A^2(s)||)w(s) \right] = 0,$$

then system (3.57) has asymptotic equilibrium.

From Theorem 3.2.13, we have the following corollaries.

Corollary 3.2.14 *Suppose that (C2) holds and for some $k \geq 0$,*

$$\lim_{n\to\infty} \frac{1}{w(n)} \sum_{s=n}^{\infty} B_k(s)B(s) = 0,$$

$$\lim_{n\to\infty} \frac{1}{w(n)} \sum_{s=n}^{\infty} B_k(s)g(s) = 0,$$

$$\lim_{n\to\infty} \frac{1}{w(n)} \sum_{s=n}^{\infty} A_{j-1}(s)A(s) = 0, \ 1 \leq j \leq k+1,$$

$$\lim_{n\to\infty} \frac{1}{w(n)} \sum_{s=n}^{\infty} B_{k-1}(s)A(s)B(s) = 0,$$

$$\lim_{n\to\infty} \frac{1}{w(n)} \sum_{s=n}^{\infty} B_{k-1}(s)A(s)g(s) = 0,$$

$$\lim_{n\to\infty} \frac{1}{w(n)} \sum_{s=n}^{\infty} B_{k-1}(s)A^2(s) = 0.$$

If

$$\lim_{n\to\infty} \frac{1}{w(n)} \sum_{s=n}^{\infty} \left[(1 + ||A(s)||)||B(s)||w(s) + (||A_k(s)A(s)|| + ||A^2(s)||)w(s) \right] = 0,$$

then system (3.58) has linear asymptotic equilibrium.

Corollary 3.2.15 *Suppose that (C2) holds and for some $k \geq 0$,*

$$\lim_{n\to\infty} \frac{1}{w(n)} \sum_{s=n}^{\infty} A_{j-1}(s)A(s) = 0, \ 1 \leq j \leq k+1,$$

$$\lim_{n\to\infty} \frac{1}{w(n)} \sum_{s=n}^{\infty} B_{k-1}(s)A^2(s) = 0.$$

If

$$\lim_{n\to\infty} \frac{1}{w(n)} \sum_{s=n}^{\infty} \left[||A_k(s)A(s)|| + ||A^2(s)|| \right] w(s) = 0,$$

then system (3.59) has linear asymptotic equilibrium.

Taking $w(n) \equiv 1$ in the last corollary, we obtain the discrete analogous of Wintner's theorem.

Corollary 3.2.16 *Suppose that for some $k \geq 0$, the following series are convergent:*

$$\sum_{s=1}^{\infty} A_{j-1}(s)A(s),\ 1 \leq j \leq k+1, \quad \sum_{s=1}^{\infty} B_{k-1}(s)A^2(s),$$

$$\sum_{s=1}^{\infty} \Big[\|A_k(s)A(s)\| + \|A^2(s)\| \Big].$$

Then system (3.59) has linear asymptotic equilibrium.

The discrete analogous of Theorem 1.2.4 is obtained by taking $k = 0$ and using the convention made earlier.

Corollary 3.2.17 *If the series*

$$\sum_{s=1}^{\infty} \|A(s)\|$$

converges, then system (3.59) has linear asymptotic equilibrium.

q-Difference equations

We consider the systems of q-difference equations

$$D_q x(t) = A(t)x(t) + f(t, x(t)), \quad t \in q^{\mathbb{N}}, \tag{3.60}$$

$$D_q x(t) = [A(t) + B(t)]x(t) + g(t), \quad t \in q^{\mathbb{N}}, \tag{3.61}$$

and

$$D_q x(t) = A(t)x(t), \quad t \in q^{\mathbb{N}}, \tag{3.62}$$

where $A, B : q^{\mathbb{N}} \to \mathbb{R}^{n\times n}$, $g : q^{\mathbb{N}} \to \mathbb{R}^n$, $f : q^{\mathbb{N}} \times \mathbb{R}^n \to \mathbb{R}^n$, $f(t, x)$ is continuous in x for each fixed t, $q^{\mathbb{N}} = \{q^n : n \in \mathbb{N}, q > 1\}$, and D_q is the operator defined by $D_q x(t) = [x(qt) - x(t)]/(q-1)t$, which is called the q-derivative of x at t. In this case, we have $\sigma(t) = qt$, and $\mu(t) = (q-1)t$, and $x^{\Delta}(t) = D_q x(t)$.

The conditions (H1*) and (H2) take the following forms.

(C1*) Let f satisfy

$$|f(t,x_1)-f(t,x_2)|\le R(t,|x_1-x_2|) \quad \text{for all } (t,x_1),(t,x_2)\in q^{\mathbb{N}}\times\mathbb{R}^n,$$

where $R(t,\lambda)$ is a real-valued function defined on $q^{\mathbb{N}}\times\mathbb{R}^+$ and nondecreasing in λ for each fixed t.

(C2*) The series

$$A_j(t)=(q-1)t\sum_{n=0}^{\infty}q^nA_{j-1}(q^nt)A(q^nt),\quad 1\le j\le k,\quad t\in q^{\mathbb{N}}\quad (A_0=I)$$

are convergent.

Let w be a real-valued nonincreasing function defined on $q^{\mathbb{N}}$ such that $0<w(t)\le 1$, and either $\lim_{t\to\infty}w(t)=0$ or $w(t)\equiv 1$, and as before

$$B_k(t)=A_0(t)+A_1(t)+\cdots+A_k(t),\quad t\in q^{\mathbb{N}}.$$

Theorem 3.2.18 *Let (C1*) and (C2*) hold. Suppose that for some $k\ge 0$ and for every constant vector $c\in\mathbb{R}^n$,*

$$\lim_{t\to\infty}\frac{t}{w(t)}\sum_{n=0}^{\infty}q^nB_k(q^nt)f(q^nt,c)=0,$$
$$\lim_{t\to\infty}\frac{t}{w(t)}\sum_{n=0}^{\infty}q^nA_{j-1}(q^nt)A(q^nt)=0,\ 1\le j\le k+1,$$
$$\lim_{t\to\infty}\frac{t^2}{w(t)}\sum_{n=0}^{\infty}q^{2n}B_{k-1}(q^nt)A^2(q^nt)=0,$$
$$\lim_{t\to\infty}\frac{t^2}{w(t)}\sum_{n=0}^{\infty}q^{2n}B_{k-1}(q^nt)A(q^nt)f(q^nt,c)=0.$$

If for all $M>0$,

$$\lim_{t\to\infty}\frac{t}{w(t)}\sum_{n=0}^{\infty}q^n\Big[(1+(q^{n+1}-q^n)t\|A(q^nt)\|)\frac{R(q^nt,Mw(q^nt))}{M}$$
$$+(\|A_k(q^nt)A(q^nt)\|+(q^{n+1}-q^n)t\|A^2(q^nt)\|)w(q^nt)\Big]=0,$$

then system (3.60) has asymptotic equilibrium.

From Theorem 3.2.18, we have the following corollaries.

Corollary 3.2.19 *Suppose that (C2*) holds and for some $k \geq 0$,*

$$\lim_{t\to\infty} \frac{t}{w(t)} \sum_{n=0}^{\infty} q^n B_k(q^n t) B(q^n t) = 0,$$

$$\lim_{t\to\infty} \frac{t}{w(t)} \sum_{n=0}^{\infty} q^n B_k(q^n t) g(q^n t) = 0,$$

$$\lim_{t\to\infty} \frac{t}{w(t)} \sum_{n=0}^{\infty} q^n A_{j-1}(q^n t) A(q^n t) = 0,\ 1 \leq j \leq k+1,$$

$$\lim_{t\to\infty} \frac{t^2}{w(t)} \sum_{n=0}^{\infty} q^{2n} B_{k-1}(q^n t) A(q^n t) B(q^n t) = 0,$$

$$\lim_{t\to\infty} \frac{t^2}{w(t)} \sum_{n=0}^{\infty} q^{2n} B_{k-1}(q^n t) A(q^n t) g(q^n t) = 0,$$

$$\lim_{t\to\infty} \frac{t^2}{w(t)} \sum_{n=0}^{\infty} q^{2n} B_{k-1}(q^n t) A^2(q^n t) = 0.$$

If

$$\lim_{t\to\infty} \frac{t}{w(t)} \sum_{n=0}^{\infty} q^n \Big[(1 + (q^{n+1} - q^n) t \|A(q^n t)\|) \|B(q^n t)\| w(q^n t) + (\|A_k(q^n t) A(q^n t)\| + (q^{n+1} - q^n) t \|A^2(q^n t)\|) w(q^n t) \Big] = 0,$$

then system (3.61) has linear asymptotic equilibrium.

Corollary 3.2.20 *Suppose that (C2*) holds and for some $k \geq 0$,*

$$\lim_{t\to\infty} \frac{t}{w(t)} \sum_{n=0}^{\infty} q^n A_{j-1}(q^n t) A(q^n t) = 0,\ 1 \leq j \leq k+1,$$

$$\lim_{t\to\infty} \frac{t^2}{w(t)} \sum_{n=0}^{\infty} q^{2n} B_{k-1}(q^n t) A^2(q^n t) = 0.$$

If

$$\lim_{t\to\infty} \frac{t}{w(t)} \sum_{n=0}^{\infty} q^n \Big[(\|A_k(q^n t) A(q^n t)\| + (q^{n+1} - q^n) t \|A^2(q^n t)\|) w(q^n t) \Big] = 0,$$

then system (3.62) has linear asymptotic equilibrium.

Taking $w(t) \equiv 1$ in the last corollary, we obtain the q-difference analogous of Wintner's theorem.

Corollary 3.2.21 *Suppose that for some $k \geq 0$, the following series are convergent:*

$$\sum_{n=1}^{\infty} q^n A_{j-1}(q^n)A(q^n),\ 1 \leq j \leq k+1, \quad \sum_{n=1}^{\infty} q^{2n} B_{k-1}(q^n)A^2(q^n),$$

$$\sum_{n=1}^{\infty} q^n \Big[\|A_k(q^n)A(q^n)\| + (q^{n+1} - q^n)\|A^2(q^n)\| \Big].$$

Then system (3.62) has linear asymptotic equilibrium.

The q-difference analogous of Theorem 1.2.4 is obtained by taking $k = 0$ and using the convention made earlier.

Corollary 3.2.22 *If the series*

$$\sum_{n=1}^{\infty} q^n \|A(q^n)\|$$

converges, then system (3.62) has linear asymptotic equilibrium.

CHAPTER 4

OSCILLATION OF SOLUTIONS OF DYNAMIC EQUATIONS ON TIME SCALES

4.1 Oscillation of Second Order Nonlinear Equations

4.1.1 Introduction

Let $\mathbb{T}$ be a time scale unbounded from above and $a \in \mathbb{T}$, $a \geq 0$, be a fixed point. In this section, we consider the second order delay dynamic equations of the form

$$x^{\Delta\Delta}(t) + p(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) = g(t), \quad t \in [a, \infty)_{\mathbb{T}}, \tag{4.1}$$

where $\alpha > 0$ is a constant, $p, g \in \mathbb{C}_{rd}([a, \infty)_{\mathbb{T}}, \mathbb{R})$, $\tau \in \mathbb{C}_{rd}(\mathbb{T}, \mathbb{T})$, τ is increasing, $\tau(t) \leq t$, and $\lim_{t\to\infty} \tau(t) = \infty$. The following conditions will be assumed to hold without further mention.

(i) $p(t) \geq 0$ and $p(t) \not\equiv 0$ on $[T, \infty)_{\mathbb{T}}$ for any $T \geq a$;

(ii) $g = h^{\Delta\Delta}$ for some bounded function $h \in \mathbb{C}^2_{rd}([a, \infty)_{\mathbb{T}}, \mathbb{R})$.

In the case of $0 < \alpha < 1$, Eq. (4.1) is the prototype of a class of nonlinear dynamic equations called Emden-Fowler sublinear dynamic equations. When $\alpha > 1$, it is the prototype of dynamic equations called Emden-Fowler superlinear dynamic equations. It is interesting to study Eq. (4.1) since its continuous version has several physical applications, see e.g., [48] and its discrete version is also important in applications.

Our attention is restricted to those solutions x of Eq. (4.1) which exist on some half-line $[T, \infty)_{\mathbb{T}}$, where $T \geq a$ may depend on the particular solution, and satisfy

$\sup\{|x(t)| : t \geq t_0\} > 0$ for any $t_0 \geq T$. Such a solution is said to be oscillatory if it is neither eventually positive nor eventually negative; i.e., if for any given $t_1 \in [T, \infty)_{\mathbb{T}}$, there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $x(t_2)x(\sigma(t_2)) \leq 0$. Otherwise, it is called nonoscillatory. Eq. (4.1) is said to be oscillatory if all its solutions are oscillatory.

We study the oscillatory behavior of solutions of Eq. (4.1) in the sublinear and superlinear cases. For the sublinear case ($0 < \alpha < 1$), we find a more appropriate extension from $\mathbb{T} = \mathbb{R}$ to an arbitrary time scale $\mathbb{T}$ of a necessary condition in [63] and remove a superfluous condition in the sufficiency part of a theorem in [45]. In addition, by combining two theorems, we give a necessary and sufficient condition for oscillation of Eq. (4.1) which is the complete extension of Belohorec's [19] theorem. And for the superlinear case ($\alpha > 1$), we extend a necessary condition in [63] from $\mathbb{T} = \mathbb{R}$ to an arbitrary time scale $\mathbb{T}$ and give conditions under which Eq. (4.1) is oscillatory. As a combination of two theorems, we give a necessary and sufficient condition for oscillation of Eq. (4.1) which is an extension of Atkinson's [14] theorem.

4.1.2 Sublinear case

We begin with a necessary condition for oscillation of Eq. (4.1). The proof is based on Banach fixed point theorem (i.e., Theorem 2.4.9).

Theorem 4.1.1 *Let* $0 < \alpha < 1$. *If*

$$\int^{\infty} (\tau(s))^{\alpha} p(s)\,\Delta s < \infty, \tag{4.2}$$

then Eq. (4.1) has an eventually positive solution.

Proof. Let $M = \sup\{|h(t)| : t \in [a, \infty)_{\mathbb{T}}\}$. In view of (4.2), we may choose a sufficiently large $T \in \mathbb{T}$ so that $T > 8M$ and

$$\int_{T}^{\infty} (\tau(s))^{\alpha} p(s)\,\Delta s \leq \frac{1}{4}. \tag{4.3}$$

Consider the complete metric space

$$\mathbb{M} = \left\{ x : x \in \mathbb{C}_{rd}([T, \infty)_{\mathbb{T}}, \mathbb{R}),\ \frac{t}{2} \leq x(t) \leq t \right\}$$

endowed with the metric

$$d(x,y) = \sup_{t\in[T,\infty)_{\mathbb{T}}} \frac{1}{t}\,|x(t)-y(t)|\,.$$

Pick a point $T^* \in \mathbb{T}$ in such a way that $T^* > T$ and $T^* \in \tau([T,\infty)_{\mathbb{T}})$. Define a map $\mathcal{T}$ on $\mathbb{M}$ by

$$(\mathcal{T}x)(t) = \begin{cases} \frac{t}{2} + M + h(t) + t\displaystyle\int_{\tau^{-1}(T^*)}^{\infty} p(s)x^{\alpha}(\tau(s))\,\Delta s, & t \in [T,\tau^{-1}(T^*))_{\mathbb{T}}; \\ \frac{t}{2} + M + h(t) + \displaystyle\int_{\tau^{-1}(T^*)}^{t} \sigma(s)p(s)x^{\alpha}(\tau(s))\,\Delta s \\ \qquad\qquad + t\displaystyle\int_{t}^{\infty} p(s)x^{\alpha}(\tau(s))\,\Delta s, & t \in [\tau^{-1}(T^*),\infty)_{\mathbb{T}}. \end{cases}$$

$\mathcal{T}$ maps $\mathbb{M}$ into itself. Indeed, let $x \in \mathbb{M}$. Then $\mathcal{T}x$ is rd-continuous (in fact continuous) and

$$(\mathcal{T}x)(t) \geq \frac{t}{2}.$$

Moreover, from (4.3), it follows that

$$\begin{aligned}(\mathcal{T}x)(t) &\leq \frac{t}{2} + M + h(t) + t\int_{\tau^{-1}(T^*)}^{\infty} p(s)x^{\alpha}(\tau(s))\,\Delta s \\ &\leq \frac{t}{2} + 2M + t\int_{T}^{\infty} (\tau(s))^{\alpha}p(s)\,\Delta s \\ &\leq \frac{t}{2} + \frac{t}{4} + \frac{t}{4} = t.\end{aligned}$$

$\mathcal{T}$ is a contraction mapping. For this, let $x, y \in \mathbb{M}$. Then we have the estimate

$$|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)| \leq t\int_{\tau^{-1}(T^*)}^{\infty} p(s)\,|x^{\alpha}(\tau(s)) - y^{\alpha}(\tau(s))|\;\Delta s.$$

Applying the continuous version of the mean value theorem, we see that

$$\begin{aligned}&|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)| \leq \\ &\qquad \alpha t\int_{\tau^{-1}(T^*)}^{\infty} p(s)(z(\tau(s)))^{\alpha-1}\,|x(\tau(s)) - y(\tau(s))|\;\Delta s, \end{aligned} \tag{4.4}$$

where $z(\tau(s))$ lies between $x(\tau(s))$ and $y(\tau(s))$. Since $0 < \alpha < 1$, we have

$$(z(\tau(s)))^{\alpha-1} \leq 2^{1-\alpha}(\tau(s))^{\alpha-1}. \tag{4.5}$$

By using inequalities (4.4) and (4.5), we see that

$$d(\mathcal{T}x,\mathcal{T}y) \leq \alpha 2^{1-\alpha} d(x,y)\int_{T}^{\infty} (\tau(s))^{\alpha}p(s)\,\Delta s$$

and hence by (4.3),

$$d(\mathcal{T}x, \mathcal{T}y) \leq \alpha 2^{-(\alpha+1)} d(x, y).$$

Thus, by Banach fixed point theorem, $\mathcal{T}$ has a unique fixed point $x \in \mathbb{M}$. It is easy to verify that this fixed point is a solution of Eq. (4.1) for $t > \tau^{-1}(T^*)$. ■

Remark 4.1.2 *The conclusion of Theorem 4.1.1 remains true if condition (4.2) is replaced by*

$$\int^{\infty} s^{\alpha} p(s)\, \Delta s < \infty.$$

In this case, when $\mathbb{T} = \mathbb{R}$, our result coincides with [63, Theorem 3].

Next, we give a sufficient condition for oscillation of Eq. (4.1).

Theorem 4.1.3 *Let $0 < \alpha < 1$. Assume that the function h is oscillatory and $\lim_{t\to\infty} h(t) = 0$. If*

$$\int^{\infty} (\tau(s))^{\alpha} p(s)\, \Delta s = \infty, \tag{4.6}$$

then Eq. (4.1) is oscillatory.

Proof. Our arguments have common features with that of [10, Theorem 5.5]. Assume to the contrary that there exists a nonoscillatory solution x of Eq. (4.1). Without loss of generality, we may assume that

$$x(t), \quad x(\tau(t)) > 0, \quad t \geq t_0$$

for some sufficiently large $t_0 \geq a$. We shall consider only this case, since the substitution $y = -x$ transforms Eq. (4.1) into an equation of the same form. Define

$$x(t) = y(t) + h(t), \quad t \geq t_0. \tag{4.7}$$

From Eq. (4.1), it follows that

$$y^{\Delta\Delta}(t) + p(t)x^{\alpha}(\tau(t)) = 0, \quad t \geq t_0. \tag{4.8}$$

Consequently, there is a $t_1 \geq t_0$ such that

$$y(t) > 0, \quad y^{\Delta}(t) > 0, \quad y^{\Delta\Delta}(t) \leq 0 \tag{4.9}$$

for all $t \geq t_1$. In view of (4.7), (4.9), and the assumption that

$$\lim_{t\to\infty} h(t) = 0,$$

it follows that there exist a $0 < \lambda < 1$ and a $t_2 \geq 2t_1$ such that

$$x(t) \geq \lambda y(t), \quad t \geq t_2. \tag{4.10}$$

Moreover

$$\begin{aligned} y(t) = y(t_1) + \int_{t_1}^{t} y^{\Delta}(s)\,\Delta s &\geq y^{\Delta}(t)(t - t_1) \\ &\geq \frac{t}{2} y^{\Delta}(t) \end{aligned} \tag{4.11}$$

for all $t \geq t_2$. By the chain rule on time scales, we have

$$\begin{aligned} ((y^{\Delta})^{1-\alpha})^{\Delta}(t) &= (1-\alpha)\left\{\int_0^1 \left[hy^{\Delta}(\sigma(t)) + (1-h)y^{\Delta}(t)\right]^{-\alpha}\, dh\right\} y^{\Delta\Delta}(t) \\ &\leq (1-\alpha)(y^{\Delta}(\tau(t)))^{-\alpha} y^{\Delta\Delta}(t), \quad t \geq t_2 \end{aligned} \tag{4.12}$$

for sufficiently large t_2. Dividing both sides of (4.8) by $(y^{\Delta}(\tau(t)))^{\alpha}$ and using (4.10), (4.11), and (4.12), we get

$$\begin{aligned} 0 &= \frac{y^{\Delta\Delta}(t) + p(t)x^{\alpha}(\tau(t))}{(y^{\Delta}(\tau(t)))^{\alpha}} \\ &= y^{\Delta\Delta}(t)(y^{\Delta}(\tau(t)))^{-\alpha} + p(t)\left(\frac{x(\tau(t))}{y^{\Delta}(\tau(t))}\right)^{\alpha} \\ &\geq \frac{((y^{\Delta})^{1-\alpha})^{\Delta}(t)}{1-\alpha} + \left(\frac{\lambda}{2}\right)^{\alpha} p(t)(\tau(t))^{\alpha}, \quad t \geq t_3 \end{aligned}$$

for sufficiently large $t_3 > t_2$. Integration from t_3 to t yields

$$\frac{(2/\lambda)^{\alpha}}{\alpha - 1} \int_{t_3}^{t} ((y^{\Delta})^{1-\alpha})^{\Delta}(s)\,\Delta s \geq \int_{t_3}^{t} p(s)(\tau(s))^{\alpha}\,\Delta s.$$

Thus

$$\frac{(2/\lambda)^{\alpha}}{\alpha - 1}(y^{\Delta})^{1-\alpha}(t) - \frac{(2/\lambda)^{\alpha}}{\alpha - 1}(y^{\Delta})^{1-\alpha}(t_3) \geq \int_{t_3}^{t} p(s)(\tau(s))^{\alpha}\,\Delta s$$

and hence

$$\frac{(2/\lambda)^{\alpha}}{1-\alpha}(y^{\Delta})^{1-\alpha}(t_3) \geq \int_{t_3}^{t} p(s)(\tau(s))^{\alpha}\,\Delta s.$$

Letting $t \to \infty$, we get a contradiction. ■

Remark 4.1.4 *In [45, Theorem 3.5], the authors require that*

$$\int^{\infty} \sigma(s)p(s)\,\Delta s = \infty$$

be satisfied in addition to (4.6). Therefore, Theorem 4.1.3 substantially improves their result.

In view of Theorems 4.1.1 and 4.1.3, we may state the following necessary and sufficient condition for the oscillation of solutions of Eq. (4.1).

Theorem 4.1.5 *Let* $0 < \alpha < 1$. *Assume that the function* h *is oscillatory and* $\lim_{t\to\infty} h(t) = 0$. *Then Eq. (4.1) is oscillatory if and only if*

$$\int^{\infty} (\tau(s))^{\alpha} p(s)\,\Delta s = \infty.$$

Special cases

Note that if $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = q^{\mathbb{N}}$ $(q > 1)$, then Eq. (4.1) takes, respectively, the following forms

$$x''(t) + p(t)\,|x(\tau(t))|^{\alpha-1}\,x(\tau(t)) = g(t), \quad t \in [a,\infty),\ a \in \mathbb{R}^{+}, \tag{4.13}$$

$$\Delta^2 x(n) + p(n)\,|x(\tau(n))|^{\alpha-1}\,x(\tau(n)) = g(n), \quad n \in \{n_0, n_0+1, \ldots\},\ n_0 \in \mathbb{N}_0, \tag{4.14}$$

and

$$D_q^2 x(t) + p(t)\,|x(\tau(t))|^{\alpha-1}\,x(\tau(t)) = g(t), \quad t \in \{q^{n_0}, q^{n_0+1}, \ldots\},\ n_0 \in \mathbb{N}, \tag{4.15}$$

where

$$D_q x(t) = x^{\Delta}(t) = \frac{x(qt) - x(t)}{(q-1)t}.$$

Corollary 4.1.6 *Let* $0 < \alpha < 1$. *Assume that the function* h *is oscillatory and* $\lim_{t\to\infty} h(t) = 0$.

(a) Every solution of Eq. (4.13) is oscillatory if and only if

$$\int^{\infty} (\tau(s))^{\alpha} p(s)\,ds = \infty.$$

(b) Every solution of Eq. (4.14) is oscillatory if and only if

$$\sum^{\infty}(\tau(n))^{\alpha}p(n) = \infty.$$

(c) Every solution of Eq. (4.15) is oscillatory if and only if

$$\sum^{\infty} q^{n}(\tau(q^{n}))^{\alpha}p(q^{n}) = \infty.$$

An Example

Example 4.1.7 *Now we give an example to illustrate the main result of this section. We first note that if $\mathbb{T}$ is one of the time scales in Corollary 4.1.6, then*

$$\int_{a}^{\infty}\frac{\Delta t}{t^{p}} = \infty \Leftrightarrow 0 \leq p \leq 1.$$

For more details, see [23].

Let $\frac{\tau(t)}{t} \to k$ as $t \to \infty$ for some $k > 0$. We consider

$$x^{\Delta\Delta}(t) + \frac{1}{t^{\beta}}|x(\tau(t))|^{\alpha-1}x(\tau(t)) = g(t), \quad t \in [a,\infty)_{\mathbb{T}}, \tag{4.16}$$

where $0 < \alpha < 1$, g is a function that satisfies the condition of Theorem 4.1.5. It follows from Theorem 4.1.5 that Eq. (4.16) is oscillatory if and only if $\alpha \leq \beta \leq \alpha + 1$.

To give a more specific example, let $\mathbb{T} = q^{\mathbb{N}}$ $(q > 1)$, and consider the q-difference equation

$$D_{q}^{2}x(t) + \frac{1}{t^{\beta}}\,|x(t/q)|^{\alpha-1}\,x(t/q) = \frac{(1+q)(1+q^{2})}{q^{3}(1-q)^{2}}\frac{(-1)^{n}}{t^{3}}, \tag{4.17}$$

where $t \in \{q^{n_0}, q^{n_0+1}, \ldots\}$ with $n_0 \geq 2$ an integer and $n = \ln t/\ln q$. It is easy to verify that $h(t) = \frac{(-1)^{n}}{t}$. We may conclude that Eq. (4.17) is oscillatory if and only if $\alpha \leq \beta \leq \alpha + 1$.

4.1.3 Superlinear Case

Our first theorem is concerned with the existence of an eventually positive solution of Eq. (4.1).

Theorem 4.1.8 *Let $\alpha > 1$. If*

$$\int^{\infty} \sigma(s)p(s)\,\Delta s < \infty, \tag{4.18}$$

then Eq. (4.1) has an eventually positive solution.

Proof. In view of (4.18), let $T \in \mathbb{T}$ be sufficiently large so that

$$\int_T^{\infty} \sigma(s)p(s)\,\Delta s \le \frac{1}{2}\min\left\{\frac{1}{(2M+1)^{\alpha}}, \frac{1/\alpha}{(2M+1)^{\alpha-1}}\right\}, \tag{4.19}$$

where $M = \sup\{|h(t)| : t \in [a,\infty)_{\mathbb{T}}\}$.

Consider the Banach space $\mathbb{X}$ of all bounded rd-continuous real functions x defined on $[T,\infty)_{\mathbb{T}}$, endowed with the norm

$$\|x\| = \sup_{t\in[T,\infty)_{\mathbb{T}}} |x(t)|\,.$$

Let

$$\mathbb{M} = \left\{x \in \mathbb{X} : \frac{1}{2} \le x(t) \le 2M+1\right\}.$$

Clearly, $\mathbb{M}$ is a closed bounded subset of $\mathbb{X}$. Pick a point $T^* \in \mathbb{T}$ in such a way that $T^* > T$ and $T^* \in \tau([T,\infty)_{\mathbb{T}})$. Define a map $\mathcal{T} : \mathbb{M} \to \mathbb{X}$ by

$$(\mathcal{T}x)(t) = \begin{cases} (M+1) + h(t) - \int_{\tau^{-1}(T^*)}^{\infty} (\sigma(s)-t)p(s)x^{\alpha}(\tau(s))\,\Delta s, & t \in [T, \tau^{-1}(T^*))_{\mathbb{T}}; \\ \\ (M+1) + h(t) - \int_{t}^{\infty} (\sigma(s)-t)p(s)x^{\alpha}(\tau(s))\,\Delta s, & t \in [\tau^{-1}(T^*),\infty)_{\mathbb{T}}. \end{cases}$$

$\mathcal{T}$ maps $\mathbb{M}$ into itself. Let $x \in \mathbb{M}$. It is clear that $\mathcal{T}x$ is rd-continuous (in fact continuous) and $(\mathcal{T}x)(t) \le 2M+1$. Moreover, in view of (4.19), we get

$$\begin{aligned}(\mathcal{T}x)(t) &\ge 1 - \int_{\tau^{-1}(T^*)}^{\infty} \sigma(s)p(s)x^{\alpha}(\tau(s))\,\Delta s \\ &\ge 1 - (2M+1)^{\alpha}\int_T^{\infty} \sigma(s)p(s)\,\Delta s \\ &\ge \frac{1}{2}.\end{aligned}$$

$\mathcal{T}$ is a contraction mapping. For this, let $x, y \in \mathbb{M}$. Then

$$\begin{aligned}|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)| &\le \int_{\tau^{-1}(T^*)}^{\infty} (\sigma(s)-t)p(s)\,|x^{\alpha}(\tau(s)) - y^{\alpha}(\tau(s))|\,\Delta s \\ &\le \int_{\tau^{-1}(T^*)}^{\infty} \sigma(s)p(s)\,|x^{\alpha}(\tau(s)) - y^{\alpha}(\tau(s))|\,\Delta s.\end{aligned}$$

Applying the continuous version of the mean value theorem, we see that

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \alpha \|x - y\| \int_{\tau^{-1}(T^*)}^{\infty} \sigma(s)p(s)(z(\tau(s)))^{\alpha-1}\,\Delta s,$$

where $z(\tau(s))$ lies between $x(\tau(s))$ and $y(\tau(s))$. It follows that

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \alpha \|x - y\| (2M+1)^{\alpha-1} \int_{T}^{\infty} \sigma(s)p(s)\,\Delta s,$$

and hence by (4.19),

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \frac{1}{2}\|x - y\|.$$

Employing Banach fixed point theorem, we conclude that $\mathcal{T}$ has a unique fixed point $x \in \mathbb{M}$. It is easy to verify that this fixed point is a solution of Eq. (4.1) for $t > \tau^{-1}(T^*)$.
. ■

Next, we give conditions under which Eq. (4.1) is oscillatory.

Theorem 4.1.9 *Let $\alpha > 1$. Assume that $g(t) \equiv 0$ and $\sigma(t) = O(\tau(t))$ as $t \to \infty$. If*

$$\int^{\infty} \sigma(s)p(s)\,\Delta s = \infty, \tag{4.20}$$

then Eq. (4.1) is oscillatory.

Proof. Suppose to the contrary that there is a nonoscillatory solution x of Eq. (4.1). Without loss of generality, we may assume that x is eventually positive. If x is eventually negative, we write $y = -x$ and get back to the previous case.

Choose t_0 sufficiently large so that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geq t_0 \geq a$. From Eq. (4.1), we see that $x^{\Delta\Delta}(t) \leq 0$ for $t \geq t_0$. Consequently, for some $t_1 \geq t_0$, we have

$$x(t) > 0, \quad x^{\Delta}(t) > 0, \quad x^{\Delta\Delta}(t) \leq 0, \quad t \geq t_1.$$

Define the function r by the Riccati substitution

$$r(t) = -t\frac{x^{\Delta}(t)}{x^{\alpha}(t)}, \quad t \geq t_1.$$

Then r satisfies

$$r^{\Delta}(t) = -\frac{x^{\Delta}(t)}{x^{\alpha}(t)} - \sigma(t)\left[\frac{x^{\Delta}(t)}{x^{\alpha}(t)}\right]^{\Delta}, \quad t \geq t_1.$$

In view of Eq. (4.1), we have

$$r^{\Delta}(t) = -\frac{x^{\Delta}(t)}{x^{\alpha}(t)} + \sigma(t)p(t)\frac{x^{\alpha}(\tau(t))}{x^{\alpha}(\sigma(t))} + \sigma(t)\frac{x^{\Delta}(t)(x^{\alpha})^{\Delta}(t)}{x^{\alpha}(t)x^{\alpha}(\sigma(t))}, \quad t \geq t_1. \tag{4.21}$$

On the other hand, by the chain rule, we have

$$\begin{aligned}(x^{\alpha})^{\Delta}(t) &= \alpha\left\{\int_0^1 (x(t) + h\mu(t)x^{\Delta}(t))^{\alpha-1}\,dh\right\}x^{\Delta}(t)\\ &\geq \alpha x^{\alpha-1}(t)x^{\Delta}(t), \quad t \geq t_1,\end{aligned} \tag{4.22}$$

where we have used $x(t) > 0$ and $x^{\Delta}(t) > 0$.

Using (4.22) in (4.21), we obtain

$$\begin{aligned}r^{\Delta}(t) &\geq -\frac{x^{\Delta}(t)}{x^{\alpha}(t)} + \sigma(t)p(t)\frac{x^{\alpha}(\tau(t))}{x^{\alpha}(\sigma(t))} + \frac{\alpha\sigma(t)}{t^2}\frac{x^{2\alpha-1}(t)}{x^{\alpha}(\sigma(t))}r^2(t)\\ &\geq -\frac{x^{\Delta}(t)}{x^{\alpha}(t)} + \sigma(t)p(t)\frac{x^{\alpha}(\tau(t))}{x^{\alpha}(\sigma(t))} + \frac{c}{t}\frac{x^{\alpha}(t)}{x^{\alpha}(\sigma(t))}r^2(t), \quad t \geq t_1,\end{aligned} \tag{4.23}$$

where $c = \alpha x^{\alpha-1}(t_1)$.

Let $s \in \mathbb{T}$, $s \geq t_1$ be a fixed point. By the mean value theorem applied to x on $[s,t]_{\mathbb{T}}$, $t > s$, we have

$$\frac{x^{\Delta}(t)}{x(t)} < \frac{1}{t-s} \tag{4.24}$$

since $x(t) > 0$, $x^{\Delta\Delta}(t) \leq 0$. From (4.24), it follows that

$$x(t) - (t-s)x^{\Delta}(t) > 0, \quad t \geq s. \tag{4.25}$$

Dividing both sides of (4.25) by $x(t)x(\sigma(t))$, we see that

$$\left(\frac{t-s}{x(t)}\right)^{\Delta} > 0, \quad t \geq s. \tag{4.26}$$

Integrating (4.26) from $\tau(t)$ to $\sigma(t)$, for sufficiently large t, we get

$$\frac{x(\tau(t))}{x(\sigma(t))} \geq \frac{\tau(t)-s}{\sigma(t)-s}$$

and hence

$$\frac{x(\tau(t))}{x(\sigma(t))} \geq K, \quad t \geq t_2 \tag{4.27}$$

for sufficiently large $t_2 > t_1$ and some constant $K > 0$ since $\sigma(t) = O(\tau(t))$ as $t \to \infty$. Similar computations yield that for sufficiently large t_2,

$$\frac{x(t)}{x(\sigma(t))} \geq L, \quad t \geq t_2 \tag{4.28}$$

for some constant $L > 0$.

Using (4.27) and (4.28) in (4.23), we see that

$$r^{\Delta}(t) \geq -\frac{x^{\Delta}(t)}{x^{\alpha}(t)} + \beta\sigma(t)p(t) + \gamma\frac{r^2(t)}{t}, \quad t \geq t_2$$

for some constants $\beta, \gamma > 0$. Integrating from t_2 to t, we obtain

$$r(t) \geq r(t_2) - \int_{t_2}^{t} \frac{x^{\Delta}(s)}{x^{\alpha}(s)}\,\Delta s + \beta\int_{t_2}^{t} \sigma(s)p(s)\,\Delta s + \gamma\int_{t_2}^{t} \frac{r^2(s)}{s}\,\Delta s. \tag{4.29}$$

To prove the convergence of the integral

$$\int^{\infty} \frac{x^{\Delta}(s)}{x^{\alpha}(s)}\,\Delta s,$$

we consider the following two cases separately.

Case 1: $\lim_{t\to\infty} x(t) < \infty$.

Case 2: $\lim_{t\to\infty} x(t) = \infty$.

First consider Case 1. In this case, it follows from

$$0 < \frac{x^{\Delta}(t)}{x^{\alpha}(t)} \leq \frac{x^{\Delta}(t)}{x^{\alpha}(t_2)}, \quad t \geq t_2$$

and the comparison test.

Next consider Case 2. Clearly, $x([t_2,\infty)_{\mathbb{T}})$ is also a time scale unbounded from above. Let $\widetilde{\mathbb{T}} = x([t_2,\infty)_{\mathbb{T}})$ and $\widetilde{\sigma}$ denote the forward jump operator on $\widetilde{\mathbb{T}}$. Then by the substitution rule for integrals, we have

$$\int_{t_2}^{\infty} \frac{x^{\Delta}(s)}{x^{\alpha}(s)}\,\Delta s = \int_{x(t_2)}^{\infty} \frac{\widetilde{\Delta}u}{u^{\alpha}},$$

where the integral on the right side is convergent. To prove this, we will show that $\widetilde{\sigma}(x(t)) = O(x(t))$ as $t \to \infty$ (for details see [23, Corollary 5.14]). We have two possibilities to consider: (i) $x(t)$ is right-dense and (ii) $x(t)$ is right-scattered. In case (i), $\widetilde{\sigma}(x(t)) = x(t)$ by definition. Let (ii) hold. We see that t is also right-scattered and hence $\widetilde{\sigma}(x(t)) = x(\sigma(t))$. Then using (2.1), (4.24), and the fact that $\sigma(t) = O(t)$ as $t \to \infty$, it is easy to see that $x(\sigma(t)) = O(x(t))$ as $t \to \infty$. Thus we get the desired result.

From (4.29), we see that there exists $t_3 > t_2$ such that

$$r(t) \geq \gamma \int_{t_2}^{t} \frac{r^2(s)}{s}\,\Delta s, \quad t \geq t_3.$$

Setting

$$z(t) = \gamma \int_{t_2}^{t} \frac{r^2(s)}{s}\,\Delta s,$$

we get

$$\frac{z^{\Delta}(t)}{z^2(t)} \geq \frac{\gamma}{t}, \quad \text{where} \quad z^{\Delta}(t) = \gamma \frac{r^2(t)}{t}.$$

Integrating from t_3 to t, we obtain

$$\int_{t_3}^{t} \frac{z^{\Delta}(s)}{z^2(s)}\,\Delta s \geq \gamma \int_{t_3}^{t} \frac{\Delta s}{s},$$

which leads to a contradiction as $t \to \infty$. Since we already know that

$$\int^{\infty} \frac{\Delta s}{s} = \infty$$

(see [23, Theorem 5.11]), to get a contradiction we will show that

$$\int^{\infty} \frac{z^{\Delta}(s)}{z^2(s)}\,\Delta s < \infty.$$

For this, we repeat the previous steps for $z(t)$ instead of $x(t)$. As in the previous case, it can easily be shown that the integral is convergent if $\lim_{t\to\infty} z(t) < \infty$. And when $\lim_{t\to\infty} z(t) = \infty$, we have $z([t_3, \infty)_{\mathbb{T}})$ is a time scale unbounded from above. Let $\widehat{\mathbb{T}} = z([t_3, \infty)_{\mathbb{T}})$ and $\widehat{\sigma}$ denote the forward jump operator on $\widehat{\mathbb{T}}$. To prove the convergence, we will show that $\widehat{\sigma}(z(t)) = O(z^{\nu}(t))$ as $t \to \infty$ is satisfied for any $\nu \in [1, 2)$ (see [23, Corollary 5.14]). If $z(t)$ is right-dense, then $\widehat{\sigma}(z(t)) = z(t)$. Also $\widehat{\sigma}(z(t)) = z(\sigma(t))$ when $z(t)$ is right-scattered. In this case, from (4.24), we have

$$\begin{aligned}
\widehat{\sigma}(z(t)) = z(\sigma(t)) &= z(t) + \gamma\mu(t)\frac{r^2(t)}{t} \\
&= z(t) + \gamma t\mu(t)\left(\frac{x^{\Delta}(t)}{x^{\alpha}(t)}\right)^2 \\
&\leq z(t) + \gamma t\mu(t)\frac{1}{(t-s)^2 x^{2\alpha-2}(t)},
\end{aligned}$$

where the last expression is $O(z^{\nu}(t))$ as $t \to \infty$. This completes the proof. ■

The following theorem is a direct consequence of the above results.

Theorem 4.1.10 *Let $\alpha > 1$. Assume that $g(t) \equiv 0$ and $\sigma(t) = O(\tau(t))$ as $t \to \infty$. Then Eq. (4.1) is oscillatory if and only if*

$$\int^{\infty} \sigma(s)p(s)\,\Delta s = \infty.$$

Special Cases

By considering special time scales $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = q^{\mathbb{N}}$ $(q > 1)$, we easily obtain the following oscillation criteria.

Corollary 4.1.11 *Let $\alpha > 1$. Then every solution of the delay differential equation*

$$x''(t) + p(t)\,|x(t-r)|^{\alpha-1}\,x(t-r) = 0, \quad t \in [a, \infty),$$

where $a \geq 0$, $r > 0$ are constants, is oscillatory if and only if

$$\int^{\infty} sp(s)\,ds = \infty.$$

Corollary 4.1.12 *Let $\alpha > 1$. Then every solution of the delay difference equation*

$$\Delta^2 x(n) + p(n)\,|x(n-k)|^{\alpha-1}\,x(n-k) = 0, \quad n \in \{n_0, n_0+1, \ldots\},$$

where $n_0 \in \mathbb{N}_0$, $k \in \mathbb{N}$, is oscillatory if and only if

$$\sum^{\infty} (n+1)p(n) = \infty.$$

Corollary 4.1.13 *Let $\alpha > 1$. Then every solution of the delay q-difference equation*

$$D_q^2 x(t) + p(t)\,|x(t/q)|^{\alpha-1}\,x(t/q) = 0, \quad t \in \{q^{n_0}, q^{n_0+1}, \ldots\},$$

where $n_0 \geq 2$ an integer, is oscillatory if and only if

$$\sum^{\infty} q^{2n} p(q^n) = \infty.$$

4.2 Oscillation of Two-Dimensional Nonlinear Dynamic Systems

4.2.1 Introduction

In this section, we consider the two-dimensional nonlinear dynamic systems of the form

$$\begin{aligned} x^{\Delta}(t) &= a_1(t)f_1(y(t)) \\ y^{\Delta}(t) &= -a_2(t)f_2(x(\sigma(t))), \quad t \in [a,\infty)_{\mathbb{T}}, \end{aligned} \tag{4.30}$$

$$\begin{aligned} x^{\Delta}(t) &= a_1(t)f_1(y(t)) \\ y^{\Delta}(t) &\leq -a_2(t)f_2(x(\sigma(t))), \quad t \in [a,\infty)_{\mathbb{T}}, \end{aligned} \tag{4.31}$$

$$\begin{aligned} x^{\Delta}(t) &= a_1(t)f_1(y(t)) \\ y^{\Delta}(t) &\geq -a_2(t)f_2(x(\sigma(t))), \quad t \in [a,\infty)_{\mathbb{T}}, \end{aligned} \tag{4.32}$$

$$\begin{aligned} x^{\Delta}(t) &= a_1(t)f_1(y(t)) \\ y^{\Delta}(t) &= -a_2(t)f_2(x(\tau(t))), \quad t \in [a,\infty)_{\mathbb{T}}, \end{aligned} \tag{4.33}$$

where $a_1,\ a_2 \in \mathbb{C}_{rd}([a,\infty)_{\mathbb{T}},\mathbb{R})$, $\tau \in \mathbb{C}_{rd}(\mathbb{T},\mathbb{T})$, $\tau(t) \leq t$, $\lim_{t\to\infty}\tau(t) = \infty$, and $f_i \in \mathbb{C}(\mathbb{R},\mathbb{R})$, $i = 1, 2$, with the sign property

$$uf_i(u) > 0 \quad \text{for all} \quad u \neq 0. \tag{4.34}$$

We will restrict our attention only those solutions (x, y) of system (4.30) (or (4.33)) which exist on some half-line $[T,\infty)_{\mathbb{T}}$, where $T \geq a$ may depend on the particular solution, and satisfy $\sup\{|x(t)|+|y(t)| : t \geq t_0\} > 0$ for any $t_0 \in [T,\infty)_{\mathbb{T}}$. Such a solution is called oscillatory if x, y are both oscillatory; otherwise it is called nonoscillatory. A component x (or y) of a solution (x, y) is said to be oscillatory if and only if x (or y) is neither eventually positive nor eventually negative; i.e., if and only if given any $t_0 \in [T,\infty)_{\mathbb{T}}$, there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t_1)x(\sigma(t_1)) \leq 0$ (or $y(t_1)y(\sigma(t_1)) \leq 0$). Otherwise it is called nonoscillatory. System (4.30) (or (4.33)) is said to be oscillatory if all of its solutions are oscillatory. We note that if $a_1(t) \geq 0$ and is not eventually identically zero, it follows that oscillation of x implies oscillation of y. Thus, if (x, y) is a nonoscillatory solution of system (4.30) (or (4.33)), then x is always nonoscillatory. Furthermore, if $a_2(t) \geq 0$ and is not eventually identically zero, then from the second equation in (4.30) (or (4.33)), y is eventually of one sign.

We study the oscillation of solutions of system (4.30) and establish sufficient conditions for all solutions to be oscillatory. We consider the following two cases separately:

Case 1: $a_1(t) \geq 0$ on $[a, \infty)_{\mathbb{T}}$ and $a_1(t) \not\equiv 0$ eventually.

Case 2: $a_1(t) > 0$ on $[a, \infty)_{\mathbb{T}}$.

In the first case, we also consider the systems (4.31) and (4.32) obtained from (4.30) by allowing inequality in the second equation. The primary motivation for this consideration is to obtain oscillation criteria for the related delay-dynamic system (4.33). Our results, in this case, not only extend the results of Kwong and Wong [56] for the differential system (1.14) and that of Graef and Thandapani [41] for the difference system (1.15), but also unify them. In particular, we generalize the well-known results of Atkinson [14], Belohorec [19], Hooker and Patula [49], and Waltman [78]. In the second case, our results generalize the results of Atkinson [14] and Bohner, Erbe, and Peterson [24].

We first state the following conditions (see [24]).

A function $g : \mathbb{T} \to \mathbb{R}$ satisfies **condition (A)** if

$$\liminf_{t\to\infty} \int_T^t g(s)\,\Delta s \geq 0 \quad \text{and} \quad \not\equiv 0 \tag{4.35}$$

for all large T. It was shown that (4.35) implies either $\int_a^\infty g(s)\,\Delta s = \infty$ or that

$$\int_T^\infty g(s)\,\Delta s = \lim_{t\to\infty} \int_T^t g(s)\,\Delta s$$

exists and satisfies $\int_T^\infty g(s)\,\Delta s \geq 0$ for all large T.

We say that a function $g : \mathbb{R}\backslash\{0\} \to (0, \infty)$ satisfies **condition (B)** if for each $k > 0$, there exists $m > 0$ such that $g(x) \geq m$ provided $|x| \geq k$. This condition says that $g(x)$ is bounded away from 0 if x is.

We list the following conditions for convenience.

$$a_1(t) \geq 0 \text{ on } [a, \infty)_{\mathbb{T}} \text{ and } a_1(t) \not\equiv 0 \text{ eventually} \tag{4.36}$$

$$a_2(t) \geq 0 \text{ on } [a, \infty)_{\mathbb{T}} \text{ and } a_2(t) \not\equiv 0 \text{ eventually} \tag{4.37}$$

$$a_1(t) > 0 \text{ on } [a, \infty)_{\mathbb{T}}. \tag{4.38}$$

4.2.2 Preparatory Lemmas

The following lemmas are essential in proving the main results. We first prove the following two lemmas which will allow us to compare the solution of (4.33) to a system of one of the forms (4.31) and (4.32).

Lemma 4.2.1 *Assume that (4.36) and (4.37) hold. Assume further that*

$$f_1 \text{ is nondecreasing on } \mathbb{R}. \tag{4.39}$$

If (x, y) *is a nonoscillatory solution of (4.33) with* $x(t) > 0$ *for* $t \geq t_0 \geq a$ *for some* $t_0 \in \mathbb{T}$, *then for* $t_1 > t_0$ *sufficiently large, we have*

$$x(\tau(t)) \geq F(t)x(\sigma(t)), \quad t \geq t_1,$$

where

$$F(t) = \frac{\int_{t_0}^{\tau(t)} a_1(s)\,\Delta s}{\int_{t_0}^{\sigma(t)} a_1(s)\,\Delta s}.$$

Proof. Since $x(t) > 0$ for $t \geq t_0$ and $\tau(t) \to \infty$ as $t \to \infty$, we may assume also that $x(\tau(t)) > 0$ for $t \geq t_0$. Let $t_1 > t_0$ so that $\tau(t) > t_0$ for $t \geq t_1$. We have, from the monotonicity of f_1,

$$\begin{aligned} x(\sigma(t)) &= x(\tau(t)) + \int_{\tau(t)}^{\sigma(t)} x^{\Delta}(s)\,\Delta s \\ &= x(\tau(t)) + \int_{\tau(t)}^{\sigma(t)} a_1(s) f_1(y(s))\,\Delta s \\ &\leq x(\tau(t)) + f_1(y(\tau(t))) \int_{\tau(t)}^{\sigma(t)} a_1(s)\,\Delta s, \quad t \geq t_1 \end{aligned}$$

since $y^{\Delta}(t) \leq 0$ for $t \geq t_0$. Therefore

$$\frac{x(\sigma(t))}{x(\tau(t))} \leq 1 + \frac{f_1(y(\tau(t)))}{x(\tau(t))} \int_{\tau(t)}^{\sigma(t)} a_1(s)\,\Delta s, \quad t \geq t_1. \tag{4.40}$$

Similarly

$$\begin{aligned} x(\tau(t)) &= x(t_0) + \int_{t_0}^{\tau(t)} x^{\Delta}(s)\,\Delta s \\ &= x(t_0) + \int_{t_0}^{\tau(t)} a_1(s) f_1(y(s))\,\Delta s \\ &\geq f_1(y(\tau(t))) \int_{t_0}^{\tau(t)} a_1(s)\,\Delta s, \quad t \geq t_1, \end{aligned}$$

from which, it follows that

$$\frac{f_1(y(\tau(t)))}{x(\tau(t))} \leq \frac{1}{\int_{t_0}^{\tau(t)} a_1(s)\,\Delta s}, \quad t \geq t_1. \tag{4.41}$$

Then, from (4.40) and (4.41), we have

$$\frac{x(\sigma(t))}{x(\tau(t))} \leq \frac{\int_{t_0}^{\sigma(t)} a_1(s)\,\Delta s}{\int_{t_0}^{\tau(t)} a_1(s)\,\Delta s}, \quad t \geq t_1,$$

which proves the lemma. ■

Lemma 4.2.2 *Assume that (4.36), (4.37), and (4.39) hold. If (x, y) is a nonoscillatory solution of (4.33) with $x(t) < 0$ for $t \geq t_0 \geq a$ for some $t_0 \in \mathbb{T}$, then for $t_1 > t_0$ sufficiently large, we have*

$$x(\tau(t)) \leq F(t)x(\sigma(t)), \quad t \geq t_1,$$

where F is the same as in Lemma 4.2.1.

Proof. The proof is similar to the proof of Lemma 4.2.1. ■

The following lemma gives another simple consequence of condition (A) (see [24]).

Lemma 4.2.3 *Suppose that g satisfies condition (A). Then given any T_0, there exists $T_1 \geq T_0$ so that*

$$\int_{T_1}^{t} g(s)\,\Delta s \geq 0 \quad \text{for all} \quad t \geq T_1.$$

Lemma 4.2.4 *Assume that (4.36) and (4.39) hold and a_2 satisfies condition (A). Assume further that*

$$f_2 \in \mathbb{C}^1(\mathbb{R}\backslash\{0\}, \mathbb{R}) \text{ and } f_2'(u) \geq 0 \text{ for all } u \neq 0 \tag{4.42}$$

and

$$\int^{\infty} a_1(t)\,\Delta t = \infty. \tag{4.43}$$

If (x, y) is a nonoscillatory solution of (4.30) defined on $[T, \infty)_{\mathbb{T}}$, then there exists $T_1 \geq T$ such that

$$x(t)y(t) > 0 \quad \text{for} \quad t \geq T_1.$$

Proof. Suppose that (x, y) is a nonoscillatory solution of system (4.30) defined on $[T, \infty)_{\mathbb{T}}$ and without loss of generality, assume $x(t) > 0$ for $t \geq T_0 \geq T$. Since a_2 satisfies condition (A), we may assume by Lemma 4.2.3 that $T_1 \geq T_0$ is sufficiently large so that

$$\int_{T_1}^{t} a_2(s)\,\Delta s \geq 0 \quad \text{for all} \quad t \geq T_1.$$

Let us assume, for the sake of contradiction, that $y(t)$ is not strictly positive for all large t. First consider the case when $y(t) < 0$ for all large t. Then without loss of generality, $y(t) < 0$ for all $t \geq T_1$. An integration of the second equation in (4.30) for $t > T_1$ gives

$$y(t) + \int_{T_1}^{t} a_2(s) f_2(x(\sigma(s)))\,\Delta s = y(T_1) < 0. \tag{4.44}$$

Now by the integration by parts formula, we have

$$\begin{aligned} \int_{T_1}^{t} a_2(s) f_2(x(\sigma(s)))\,\Delta s = f_2(x(t)) \int_{T_1}^{t} a_2(s)\,\Delta s \\ - \int_{T_1}^{t} (f_2 \circ x)^{\Delta}(s) \int_{T_1}^{s} a_2(u)\,\Delta u\,\Delta s. \end{aligned} \tag{4.45}$$

By the chain rule, we have

$$(f_2 \circ x)^{\Delta}(t) = \left\{ \int_0^1 f_2'(x_h(t))\,dh \right\} x^{\Delta}(t) \leq 0,$$

where

$$x_h(t) = x(t) + h\mu(t)x^{\Delta}(t)$$

since $f_2'(u) \geq 0$ for all $u \neq 0$ and $x^{\Delta}(t) \leq 0$ for all $t \geq T_1$. Hence, it follows that

$$\int_{T_1}^{t} (f_2 \circ x)^{\Delta}(s) \int_{T_1}^{s} a_2(u)\,\Delta u\,\Delta s \leq 0 \tag{4.46}$$

and so from (4.45), we have

$$\int_{T_1}^{t} a_2(s) f_2(x(\sigma(s)))\,\Delta s \geq f_2(x(t)) \int_{T_1}^{t} a_2(s)\,\Delta s \geq 0. \tag{4.47}$$

Consequently, from (4.44), we have

$$y(t) \leq y(T_1) < 0, \quad t \geq T_1. \tag{4.48}$$

Now from the monotonicity of f_1 and (4.48), it follows that

$$f_1(y(t)) \leq f_1(y(T_1)) < 0, \quad t \geq T_1.$$

Integration of the first equation in (4.30) yields

$$\begin{aligned} x(t) &= x(T_1) + \int_{T_1}^{t} a_1(s) f_1(y(s))\,\Delta s \\ &\leq x(T_1) + f_1(y(T_1)) \int_{T_1}^{t} a_1(s)\,\Delta s \to -\infty \end{aligned}$$

as $t \to \infty$, which is a contradiction. Hence $y(t)$ is not negative for all large t and since we are assuming $y(t)$ is not positive for all large t, it follows that $y(t)$ must change sign infinitely often. Make the "Riccati–like" substitution

$$w(t) = -\frac{y(t)}{f_2(x(t))}, \quad t \geq T_0.$$

We may suppose that $T_1 > T_0$ is sufficiently large so that

$$\liminf_{t\to\infty} \int_{T_1}^{t} a_2(s)\,\Delta s \geq 0$$

holds and is such that $w(T_1) > 0$ (i.e., $y(T_1) < 0$). Differentiating w and using the chain rule gives

$$\begin{aligned} w^{\Delta}(t) &= a_2(t) + \frac{a_1(t) y(t) f_1(y(t))}{f_2(x(t)) f_2(x(\sigma(t)))} \left\{ \int_0^1 f_2'(x_h(t))\,dh \right\} \\ &\geq a_2(t), \quad t \geq T_1 \end{aligned}$$

and this yields

$$w(t) \geq w(T_1) + \int_{T_1}^{t} a_2(s)\,\Delta s, \quad t \geq T_1. \tag{4.49}$$

Now taking the lim inf of both sides of (4.49), we see that

$$\liminf_{t\to\infty} w(t) \geq w(T_1) > 0,$$

which implies that $y(t) < 0$ for all large t, which is a contradiction to the assumption that $y(t)$ changes sign infinitely often. ■

The following lemma may be found in [10] (Lemma 3.1).

Lemma 4.2.5 *If z and x are differentiable on $\mathbb{T}$ and $x(t) \neq 0$ on $\mathbb{T}$, then*

$$x^{\Delta}\left(\frac{z^2}{x}\right)^{\Delta} = \left(z^{\Delta}\right)^2 - x x^{\sigma}\left[\left(\frac{z}{x}\right)^{\Delta}\right]^2.$$

Lemma 4.2.6 *Assume that (4.38) holds and*

$$f_2 \in \mathbb{C}^1(\mathbb{R}\backslash\{0\}, \mathbb{R}) \text{ and } f_2'(u) > 0 \text{ for all } u \neq 0. \tag{4.50}$$

Let (x, y) be a nonoscillatory solution of (4.30) with either $x(t)y(t) > 0$ or $x(t)y(t) < 0$ for all $t \geq T_0 \geq a$ and z be a differentiable function on $\mathbb{T}$. If we define

$$w = \frac{z^2 y}{f_2 \circ x},$$

then w satisfies

$$-w^\Delta = a_2(z^2)^\sigma - \frac{y(z^\Delta)^2}{(f_2 \circ x)^\Delta} + \frac{y(f_2 \circ x)(f_2 \circ x)^\sigma}{(f_2 \circ x)^\Delta}\left[\left(\frac{z}{f_2 \circ x}\right)^\Delta\right]^2.$$

Proof. Let (x, y), z, and w be as in the statement of this lemma. Then we have

$$\begin{aligned}
-w^\Delta &= -\left(\frac{z^2 y}{f_2 \circ x}\right)^\Delta \\
&= -\left(\frac{z^2}{f_2 \circ x}\right)^\sigma y^\Delta - \left(\frac{z^2}{f_2 \circ x}\right)^\Delta y \\
&= a_2(z^2)^\sigma - \left(\frac{z^2}{f_2 \circ x}\right)^\Delta y \\
&= a_2(z^2)^\sigma - \frac{y}{(f_2 \circ x)^\Delta}\left[(f_2 \circ x)^\Delta \left(\frac{z^2}{f_2 \circ x}\right)^\Delta\right].
\end{aligned}$$

Now, using Lemma 4.2.5 with x replaced by $f_2 \circ x$, we get

$$(f_2 \circ x)^\Delta \left(\frac{z^2}{f_2 \circ x}\right)^\Delta = (z^\Delta)^2 - (f_2 \circ x)(f_2 \circ x)^\sigma \left[\left(\frac{z}{f_2 \circ x}\right)^\Delta\right]^2. \tag{4.51}$$

Therefore, using (4.51), we have

$$-w^\Delta = a_2(z^2)^\sigma - \frac{y(z^\Delta)^2}{(f_2 \circ x)^\Delta} + \frac{y(f_2 \circ x)(f_2 \circ x)^\sigma}{(f_2 \circ x)^\Delta}\left[\left(\frac{z}{f_2 \circ x}\right)^\Delta\right]^2,$$

which proves the lemma. ■

4.2.3 Main Results

Case 1: $a_1(t) \geq 0$ on $[a, \infty)_\mathbb{T}$ and $a_1(t) \not\equiv 0$ eventually.

We first restrict our attention to the case $a_2(t) \geq 0$ on $[a, \infty)_\mathbb{T}$ and $a_2(t) \not\equiv 0$ eventually and establish oscillation criteria for the systems (4.30) and (4.33).

Theorem 4.2.7 *Assume that (4.36), (4.37), (4.42), and (4.43) hold. Assume further that*

$$f_1 \in \mathbb{C}^1(\mathbb{R}, \mathbb{R}) \text{ and } f_1'(u) > 0 \text{ for all } u, \tag{4.52}$$

the superlinearity condition

$$\int_{\pm 1}^{\pm\infty} \frac{du}{f_2(u)} < \infty \tag{4.53}$$

holds, and

$$\int^{\infty} A(t)a_2(t)\,\Delta t = \infty, \tag{4.54}$$

where

$$A(t) = \int_a^t a_1(s)\,\Delta s. \tag{4.55}$$

Then system (4.30) is oscillatory.

Proof. Assume to the contrary that system (4.30) has a nonoscillatory solution (x, y). Since $a_1(t)$, $a_2(t) \geq 0$, and neither of them vanishes identically, as pointed out earlier, x and y are eventually of one sign. Without loss of generality, assume that $x(t) > 0$ for $t \geq t_0 \geq a$ for some $t_0 \in \mathbb{T}$. Then, from the second equation in (4.30), it follows that either $y(t) < 0$ or $y(t) > 0$ for $t \geq t_0$ for a sufficiently large t_0. Suppose first that $y(t) < 0$. Since $y^{\Delta}(t) \leq 0$ for $t \geq t_0$, assumption (4.52) together with (4.43) implies that

$$\int^{\infty} a_1(t)f_1(y(t))\,\Delta t = -\infty.$$

Integrating the first equation in (4.30) from t_0 to t, we get

$$x(t) = x(t_0) + \int_{t_0}^t a_1(s)f_1(y(s))\,\Delta s.$$

But now, if we take the limit of both sides as $t \to \infty$, we see that the right-hand side diverges to $-\infty$, which is a contradiction to the assumption that $x(t) > 0$ for $t \geq t_0$. To handle the second case $y(t) > 0$, we define

$$w(t) = \frac{(f_1 \circ y)(t)}{(f_2 \circ x)(t)}, \quad t \geq t_0. \tag{4.56}$$

Then by the quotient rule,

$$w^{\Delta}(t) = \frac{(f_1 \circ y)^{\Delta}(t)}{(f_2 \circ x)(\sigma(t))} - \frac{(f_1 \circ y)(t)(f_2 \circ x)^{\Delta}(t)}{(f_2 \circ x)(t)(f_2 \circ x)(\sigma(t))}, \quad t \geq t_0. \tag{4.57}$$

From the assumption (4.52) and compactness, it follows that there exists a positive constant c such that

$$f_1'(u) \geq c > 0 \quad \text{for all} \quad y(t_0) \geq u \geq 0.$$

Now, from the chain rule and (4.30), we have

$$\begin{aligned}(f_1 \circ y)^{\Delta}(t) &= \left(\int_0^1 f_1'(y_h(t))\, dh\right) y^{\Delta}(t) \\ &= -\left(\int_0^1 f_1'(y_h(t))\, dh\right) a_2(t)(f_2 \circ x)(\sigma(t)),\end{aligned}$$

where

$$0 < y_h(t) = y(t) + h\mu(t)y^{\Delta}(t) \leq y(t_0), \quad t \geq t_0$$

since $y^{\Delta}(t) \leq 0$ for $t \geq t_0$. Therefore, we have

$$\frac{(f_1 \circ y)^{\Delta}(t)}{(f_2 \circ x)(\sigma(t))} \leq -ca_2(t), \quad t \geq t_0. \tag{4.58}$$

Similarly

$$\begin{aligned}(f_2 \circ x)^{\Delta}(t) &= \left(\int_0^1 f_2'(x_h(t))\, dh\right) x^{\Delta}(t) \\ &= \left(\int_0^1 f_2'(x_h(t))\, dh\right) a_1(t)(f_1 \circ y)(t) \geq 0, \quad t \geq t_0 \end{aligned} \tag{4.59}$$

since $f_2'(u) \geq 0$ for all $u \neq 0$ and $x^{\Delta}(t) \geq 0$ for $t \geq t_0$. Consequently, from (4.56) – (4.59), it follows that

$$w^{\Delta}(t) \leq -ca_2(t) - a_1(t)\left(\int_0^1 f_2'(x_h(t))\, dh\right)(w(\sigma(t)))^2, \quad t \geq t_0,$$

from which, we have

$$w^{\Delta}(t) \leq -ca_2(t), \quad t \geq t_0. \tag{4.60}$$

Multiplying both sides of (4.60) by $A(t)$, then integrating from t_0 to t, and using integration by parts on the left-hand side, we get

$$A(t)w(t) - \int_{t_0}^{t} a_1(s)w(\sigma(s))\,\Delta s \leq A(t_0)w(t_0) - c\int_{t_0}^{t} A(s)a_2(s)\,\Delta s, \quad t \geq t_0.$$

We note that the right-hand side tends to $-\infty$ as $t \to \infty$ by (4.54), and the first term on the left-hand side is nonnegative. Thus, if we can prove that the second term on the left-hand side is bounded, then we obtain the desired contradiction and our theorem will be proved.

Now, from the first equation in (4.30) and definition of w, we have

$$\int_{t_0}^{t} a_1(s)w(\sigma(s))\,\Delta s = \int_{t_0}^{t} \frac{x^{\Delta}(s)(f_1 \circ y)(\sigma(s))}{(f_1 \circ y)(s)(f_2 \circ x)(\sigma(s))}\,\Delta s \leq \int_{t_0}^{t} \frac{x^{\Delta}(s)}{(f_2 \circ x)(\sigma(s))}\,\Delta s$$

since

$$0 < \frac{(f_1 \circ y)(\sigma(t))}{(f_1 \circ y)(t)} \leq 1, \quad t \geq t_0.$$

Let

$$G(u) = \int_{u_0}^{u} \frac{1}{f_2(s)}\,ds, \quad u \geq u_0 \ (u_0 = x(t_0) > 0)$$

so that $G'(u) = 1/f_2(u)$. By the chain rule

$$(G \circ x)^{\Delta}(t) = \left(\int_0^1 \frac{1}{f_2(x_h(t))}\,dh\right) x^{\Delta}(t) \geq \left(\int_0^1 \frac{1}{f_2(x(\sigma(t)))}\,dh\right) x^{\Delta}(t), \quad t \geq t_0$$

since $x_h(t) \leq x(\sigma(t))$ for $t \geq t_0$, and therefore

$$\frac{1}{f_2(x_h(t))} \geq \frac{1}{f_2(x(\sigma(t)))}, \quad t \geq t_0.$$

Hence, we have

$$(G \circ x)^{\Delta}(t) \geq \frac{x^{\Delta}(t)}{(f_2 \circ x)(\sigma(t))}, \quad t \geq t_0. \tag{4.61}$$

Now, since $x^{\Delta}(t) \geq 0$ for $t \geq t_0$, we get

$$0 < \lim_{t\to\infty} x(t) = M \leq \infty.$$

Therefore, from (4.53), we obtain

$$\lim_{t\to\infty}(G \circ x)(t) = \lim_{t\to\infty} \int_{u_0}^{x(t)} \frac{1}{f_2(s)}\,ds < \infty. \tag{4.62}$$

On the other hand, we also have

$$\int_{t_0}^{t} (G \circ x)^{\Delta}(s)\,\Delta s = (G \circ x)(t) - (G \circ x)(t_0) = (G \circ x)(t). \tag{4.63}$$

Hence, the result follows from (4.61), (4.62), and (4.63). The proof is similar for the case $x(t) < 0$ eventually. (Note that, in this case, we also have $y(t) < 0$ eventually.) ■

Remark 4.2.8 *In the case, $a_1(t) \equiv 1$, $f_1(u) \equiv u$, $f_2(u) \equiv |u|^{\alpha-1}u$, $\alpha > 1$, criterion (4.54) reduces to the oscillation criterion of Atkinson [14] for the superlinear Emden-Fowler differential equation (1.9) and that of Hooker and Patula [49] for the superlinear Emden-Fowler difference equation (1.13).*

Theorem 4.2.9 *Assume that (4.36), (4.37), (4.39), and (4.43) hold. Assume further that*

$$f_2 \text{ is nondecreasing on } \mathbb{R} \tag{4.64}$$

and satisfies the following inequalities

$$f_2(uv) \geq f_2(u)f_2(v) \quad \text{for all } u > 0,\ v \geq \delta, \tag{4.65}$$

$$f_2(uv) \leq f_2(u)f_2(v) \quad \text{for all } u < 0,\ v \geq \delta, \tag{4.66}$$

for some constant $\delta > 0$, the sublinearity condition

$$\int_0^{\pm 1} \frac{du}{(f_2 \circ f_1)(u)} < \infty \tag{4.67}$$

holds, and

$$\int^{\infty} f_2(A(t))a_2(t)\,\Delta t = \infty, \tag{4.68}$$

where function A is defined by (4.55). Then system (4.30) is oscillatory.

Proof. We will first prove that if $t_0 > a$ is fixed, then

$$\int^{\infty} f_2(A(t) - A(t_0))a_2(t)\,\Delta t = \infty. \tag{4.69}$$

Since $A(t) \to \infty$ as $t \to \infty$, there exists a point $T > t_0$ so that

$$A(t) \geq \max\{\delta, 2A(t_0)\}, \quad t \geq T.$$

Therefore

$$A(t) - A(t_0) \geq \frac{1}{2}A(t), \quad t \geq T.$$

From the monotonicity of f_2 and (4.65), we have

$$\begin{aligned} f_2(A(t) - A(t_0)) &\geq f_2(\frac{1}{2}A(t)) \\ &\geq f_2(\frac{1}{2})f_2(A(t)), \quad t \geq T, \end{aligned}$$

and this yields

$$\int_T^t f_2(A(s) - A(t_0))a_2(s)\,\Delta s \geq f_2(\frac{1}{2}) \int_T^t f_2(A(s))a_2(s)\,\Delta s,$$

where the right-hand side tends to ∞ as $t \to \infty$ by (4.68).

Now assume that system (4.30) has a nonoscillatory solution (x, y). As stated earlier, x and y are of one sign for $t \geq t_0 \geq a$ for some $t_0 \in \mathbb{T}$. We may assume, without loss of

generality, that $x(t) > 0$ for $t \geq t_0$. Then, in a manner similar to the previous case, we have either $y(t) < 0$ or $y(t) > 0$ for $t \geq t_0$. The case $y(t) < 0$ can be handled similarly as earlier. So we now assume that $y(t) > 0$ for $t \geq t_0$. Integrating the first equation in (4.30) from t_0 to t yields

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} a_1(s) f_1(y(s))\,\Delta s \\ &> f_1(y(t)) \int_{t_0}^{t} a_1(s)\,\Delta s \end{aligned} \tag{4.70}$$

since f_1 is nondecreasing on $\mathbb{R}$ and y is nonincreasing on $[t_0, \infty)_{\mathbb{T}}$.

Let

$$B(t) = \int_{t_0}^{t} a_1(s)\,\Delta s,$$

and note that $B(t) \to \infty$ as $t \to \infty$ by (4.43). From (4.65) and (4.70), for large enough t, say $t \geq t_1 > t_0$ such that $B(t) \geq \delta$ for $t \geq t_1$, we have

$$\begin{aligned} f_2(x(\sigma(t))) \geq f_2(x(t)) &\geq f_2(B(t) f_1(y(t))) \\ &\geq f_2(B(t)) f_2(f_1(y(t))) \end{aligned} \tag{4.71}$$

since f_2 is nondecreasing on $\mathbb{R}$ and x is nondecreasing on $[t_0, \infty)_{\mathbb{T}}$.

Let us denote

$$g(u) = (f_2 \circ f_1)(u).$$

Then, from (4.30) and (4.71), it follows that

$$y^{\Delta}(t) \leq -a_2(t) f_2(B(t)) g(y(t)), \quad t \geq t_1$$

or

$$-\frac{y^{\Delta}(t)}{g(y(t))} \geq a_2(t) f_2(B(t)), \quad t \geq t_1.$$

Integration from t_1 to t yields

$$-\int_{t_1}^{t} \frac{y^{\Delta}(s)}{g(y(s))}\,\Delta s \quad \geq \quad \int_{t_1}^{t} a_2(s) f_2(B(s))\,\Delta s, \quad t \geq t_1,$$

where the right-hand side tends to ∞ as $t \to \infty$ by (4.69). Thus, to get a contradiction, it is sufficient to show that the left-hand side is bounded for large values of t. To prove this, we define

$$H(u) = \int_{u}^{u_0} \frac{1}{g(s)}\,ds, \quad 0 < u \leq u_0 \ (u_0 = y(t_1) > 0).$$

By the chain rule

$$(H \circ y)^{\Delta}(t) = -\left(\int_0^1 \frac{1}{g(y_h(t))}\, dh\right) y^{\Delta}(t) \geq -\left(\int_0^1 \frac{1}{g(y(t))}\, dh\right) y^{\Delta}(t), \quad t \geq t_1$$

since $y_h(t) \leq y(t)$ for $t \geq t_1$, and therefore

$$\frac{1}{g(y_h(t))} \geq \frac{1}{g(y(t))}, \quad t \geq t_1.$$

Consequently, we have

$$(H \circ y)^{\Delta}(t) \geq -\frac{y^{\Delta}(t)}{g(y(t))}, \quad t \geq t_1. \tag{4.72}$$

Now since $y^{\Delta}(t) \leq 0$ for $t \geq t_0$, we have

$$0 \leq \lim_{t\to\infty} y(t) = L < \infty.$$

Therefore, from (4.67), we have

$$\lim_{t\to\infty}(H \circ y)(t) = \lim_{t\to\infty} \int_{y(t)}^{u_0} \frac{1}{g(s)}\, ds < \infty. \tag{4.73}$$

Similar to the previous case, we also have

$$\int_{t_1}^{t} (H \circ y)^{\Delta}(s)\, \Delta s = (H \circ y)(t) - (H \circ y)(t_1) = (H \circ y)(t). \tag{4.74}$$

Hence, the result follows from (4.72), (4.73), and (4.74). The proof is similar for the case $x(t) < 0$ eventually (and hence $y(t) < 0$ eventually). We also use the assumption (4.66) in this case. ■

There are many functions satisfying (4.65) and (4.66). For instance, one may take $f_2(u) = |u|^{\alpha-1}u$, $\alpha > 0$, $\delta > 0$, and $f_2(u) = 1 - e^{-u}$, $\delta \geq 1$.

Remark 4.2.10 *If $a_1(t) \equiv 1$, $f_1(u) \equiv u$, $f_2(u) \equiv |u|^{\alpha-1}u$, $0 < \alpha < 1$, then criterion (4.68) reduces to the oscillation criterion of Belohorec [19] for the sublinear Emden-Fowler differential equation (1.9) and that of Hooker and Patula [49] for the sublinear Emden-Fowler difference equation (1.13).*

We next consider the systems (4.31) and (4.32). The following corollaries follow easily from Theorems 4.2.7 and 4.2.9.

Corollary 4.2.11 *Assume that conditions of Theorem 4.2.7 hold. Then system (4.31) cannot have a solution* (x, y) *with* $x(t)$, $y(t) > 0$ *eventually. Similarly, system (4.32) cannot have a solution* (x, y) *with* $x(t)$, $y(t) < 0$ *eventually.*

Corollary 4.2.12 *Assume that conditions of Theorem 4.2.9 hold. Then system (4.31) cannot have a solution* (x, y) *with* $x(t)$, $y(t) > 0$ *eventually. Similarly, system (4.32) cannot have a solution* (x, y) *with* $x(t)$, $y(t) < 0$ *eventually.*

We next apply the above results to investigate the oscillatory behavior of the delay-dynamic system (4.33).

Theorem 4.2.13 *Assume that conditions of Theorem 4.2.7 hold with* $a_2(t)$ *replaced by* $a_2(t)f_2(F(t))$*, where* F *is defined as in Lemma 4.2.1. Assume further that* f_2 *satisfies the following inequalities*

$$f_2(uv) \geq f_2(u)f_2(v) \quad \text{for all } u > 0,\ v > 0, \tag{4.75}$$

$$f_2(uv) \leq f_2(u)f_2(v) \quad \text{for all } u < 0,\ v > 0. \tag{4.76}$$

Then system (4.33) is oscillatory.

Proof. Assume to the contrary that system (4.33) has a nonoscillatory solution (x, y). As in the previous cases, since $a_1(t)$, $a_2(t) \geq 0$, and neither of them vanishes identically, x and y are eventually of one sign. Without loss of generality, assume that $x(t) > 0$ for $t \geq t_0 \geq a$ for some $t_0 \in \mathbb{T}$. We may also assume that $x(\tau(t)) > 0$ for $t \geq t_0$. Similar to the previous cases, we have either $y(t) < 0$ or $y(t) > 0$ for $t \geq t_0$ for a sufficiently large t_0. The case $y(t) < 0$ can be handled as earlier. So consider the case $y(t) > 0$. Then, from Lemma 4.2.1 and (4.75), it follows that

$$\begin{aligned} f_2(x(\tau(t))) &\geq f_2(F(t)x(\sigma(t))) \\ &\geq f_2(F(t))f_2(x(\sigma(t))), \quad t \geq t_1 \end{aligned} \tag{4.77}$$

sufficiently large. But now, using (4.77), we see from (4.33) that (x, y) with $x(t)$, $y(t) > 0$, $t \geq t_1$, is a solution of a system of type (4.31) with $a_2(t)$ replaced by $a_2(t)f_2(F(t))$, which is a contradiction by Corollary 4.2.11. The proof is similar when $x(t) < 0$ eventually (and hence $y(t) < 0$ eventually). In this case, the result follows from (4.76), Lemma 4.2.2, and Corollary 4.2.11. ■

Theorem 4.2.14 *Assume that conditions of Theorem 4.2.9 hold with (4.65) and (4.66) replaced by (4.75) and (4.76), respectively, and $a_2(t)$ replaced by $a_2(t)f_2(F(t))$, where F is defined as in Lemma 4.2.1. Then system (4.33) is oscillatory.*

Proof. The proof is similar to the proof of Theorem 4.2.13. In this case, we use Corollary 4.2.12, instead of Corollary 4.2.11, to obtain a contradiction. ■

We will now establish oscillation criteria for system (4.30) with the fact that a_2 can change sign on $[a, \infty)_{\mathbb{T}}$.

Theorem 4.2.15 *Assume that (4.36), (4.42), and (4.43) hold. Assume further that*

$$\liminf_{u \to \pm\infty} |f_1(u)| \neq 0 \tag{4.78}$$

and

$$\int^{\infty} a_2(t)\,\Delta t = \infty. \tag{4.79}$$

Then system (4.30) is oscillatory.

Proof. Assume to the contrary that system (4.30) has a nonoscillatory solution (x, y). Without loss of generality, assume that $x(t) > 0$ for $t \geq t_0 \geq a$ for some $t_0 \in \mathbb{T}$. Dividing both sides of the second equation in (4.30) by $f_2(x(\sigma(t)))$ and integrating from t_0 to t, we get

$$-\int_{t_0}^{t} \frac{y^{\Delta}(s)}{f_2(x(\sigma(s)))}\,\Delta s = \int_{t_0}^{t} a_2(s)\,\Delta s.$$

Using integration by parts on the first term, we see that

$$-\frac{y(t)}{f_2(x(t))} = -\frac{y(t_0)}{f_2(x(t_0))} + \int_{t_0}^{t} a_2(s)\,\Delta s + \int_{t_0}^{t} \frac{y(s)(f_2 \circ x)^{\Delta}(s)}{f_2(x(s))f_2(x(\sigma(s)))}\,\Delta s. \tag{4.80}$$

By the chain rule, we have

$$\begin{aligned}(f_2 \circ x)^{\Delta}(t) &= \left\{\int_0^1 f_2'(x_h(t))\,dh\right\} x^{\Delta}(t) \\ &= \alpha(t)x^{\Delta}(t),\end{aligned}$$

where

$$\alpha(t) = \int_0^1 f_2'(x_h(t))\,dh \geq 0, \quad t \geq t_0$$

since $f_2'(u) \geq 0$ for all $u \neq 0$. Note that the last term in (4.80) is nonnegative since

$$y(t)(f_2 \circ x)^{\Delta}(t) = \alpha(t)a_1(t)f_1(y(t))y(t) \geq 0$$

from $uf_1(u) > 0$ for all $u \neq 0$ and $a_1(t), \alpha(t) \geq 0$ for $t \geq t_0$. From (4.79), it follows that, there exists a point $t_1 > t_0$ so that the sum of the first two terms on the right-hand side of (4.80) is greater than or equal to 1 for $t \geq t_1$. We thus have the integral inequality

$$-\frac{y(t)}{f_2(x(t))} \geq 1 + \int_{t_1}^{t} \left[-\frac{\alpha(s)x^{\Delta}(s)}{f_2(x(s))f_2(x(\sigma(s)))} \right] (-y(s))\,\Delta s, \quad t \geq t_1.$$

In particular, we infer that

$$y(t) < 0, \quad t \geq t_1.$$

By comparing $y(t)$ with the solution of the corresponding integral equation

$$\frac{z(t)}{f_2(x(t))} = 1 + \int_{t_1}^{t} \left[-\frac{\alpha(s)x^{\Delta}(s)}{f_2(x(s))f_2(x(\sigma(s)))} \right] z(s)\,\Delta s, \quad t \geq t_1, \tag{4.81}$$

we conclude that

$$-y(t) \geq z(t), \quad t \geq t_1.$$

To see this, let $\varepsilon > 0$ be small, and let $z(t, \varepsilon)$ be the solution of the integral equation

$$z(t, \varepsilon) = f_2(x(t)) - \varepsilon + \int_{t_1}^{t} f_2(x(t)) \left[-\frac{\alpha(s)x^{\Delta}(s)}{f_2(x(s))f_2(x(\sigma(s)))} \right] z(s, \varepsilon)\,\Delta s, \quad t \geq t_1.$$

Then, we claim that

$$-y(t) > z(t, \varepsilon), \quad t \geq t_1. \tag{4.82}$$

We have

$$-y(t) \geq f_2(x(t)) + \int_{t_1}^{t} G(t, s, -y(s))\,\Delta s, \quad t \geq t_1,$$
$$z(t, \varepsilon) < f_2(x(t)) + \int_{t_1}^{t} G(t, s, z(s, \varepsilon))\,\Delta s, \quad t \geq t_1,$$

where

$$G(t, s, v) = f_2(x(t)) \left[-\frac{\alpha(s)x^{\Delta}(s)}{f_2(x(s))f_2(x(\sigma(s)))} \right] v,$$

which is nondecreasing in v for each fixed t, s.

We apply the induction principle of Theorem 2.1.1. It is obvious that (4.82) holds at $t = t_1$. If t_1 is right-scattered, then we have

$$\begin{aligned}
-y(\sigma(t_1)) &\geq f_2(x(\sigma(t_1))) + \int_{t_1}^{\sigma(t_1)} G(\sigma(t_1), s, -y(s))\,\Delta s \\
&= f_2(x(\sigma(t_1))) + \mu(t_1) G(\sigma(t_1), t_1, -y(t_1)) \\
&\geq f_2(x(\sigma(t_1))) + \mu(t_1) G(\sigma(t_1), t_1, z(t_1, \varepsilon)) \\
&= f_2(x(\sigma(t_1))) + \int_{t_1}^{\sigma(t_1)} G(\sigma(t_1), s, z(s, \varepsilon))\,\Delta s \\
&> z(\sigma(t_1), \varepsilon).
\end{aligned}$$

If t_1 is right-dense, then by the continuity of $y(t)$ and $z(t, \varepsilon)$, it follows that there exists a neighborhood U of t_1 so that

$$-y(r) > z(r, \varepsilon) \quad \text{for all} \quad r \in [t_1, \infty)_{\mathbb{T}} \cap U.$$

Let $t > t_1$ so that

$$-y(r) > z(r, \varepsilon) \quad \text{for all} \quad r \in [t_1, t)_{\mathbb{T}}.$$

If t is left-dense, then

$$-y(t) - z(t, \varepsilon) > \int_{t_1}^{t} G(t, s, -y(s)) - G(t, s, z(s, \varepsilon))\,\Delta s \geq 0.$$

If t is left-scattered, then

$$\begin{aligned}
-y(t) - z(t, \varepsilon) &> \int_{t_1}^{t} G(t, s, -y(s)) - G(t, s, z(s, \varepsilon))\,\Delta s \\
&= \mu(\rho(t)) \left(G(t, \rho(t), -y(\rho(t))) - G(t, \rho(t), z(\rho(t), \varepsilon))\right) \\
&\quad + \int_{t_1}^{\rho(t)} G(t, s, -y(s)) - G(t, s, z(s, \varepsilon))\,\Delta s \geq 0.
\end{aligned}$$

Now, if we take the limit of both sides of (4.82) as $\varepsilon \to 0^+$, we see that

$$-y(t) \geq z(t), \quad t \geq t_1. \tag{4.83}$$

Differentiation of (4.81) gives

$$z^{\Delta}(t) = 0, \quad t \geq t_1,$$

from which, it follows that

$$z(t) = f_2(x(t_1)), \quad t \geq t_1.$$

Thus, (4.83) reduces to

$$y(t) \leq -f_2(x(t_1)), \quad t \geq t_1.$$

From (4.78), we see that

$$f_1(y(t)) \leq \sup_{u \leq -f_2(x(t_1))} f_1(u) = k < 0, \quad t \geq t_1. \tag{4.84}$$

Thus, from (4.43) and (4.84), we get

$$\int_{t_1}^{t} a_1(s) f_1(y(s))\,\Delta s \leq k \int_{t_1}^{t} a_1(s)\,\Delta s \to -\infty \quad \text{as } t \to \infty.$$

Integrating the first equation in (4.30) from t_1 to t, we obtain

$$x(t) = x(t_1) + \int_{t_1}^{t} a_1(s) f_1(y(s))\,\Delta s.$$

The fact that, the right-hand side tends to $-\infty$ as $t \to \infty$, is a contradiction to the fact that $x(t) > 0$ eventually. The proof is similar for the case $x(t) < 0$ eventually. ■

Case 2: $a_1(t) > 0$ **on** $[a, \infty)_{\mathbb{T}}$.

We will now again consider system (4.30) with no restriction imposed on the sign of the coefficient a_2.

Theorem 4.2.16 *Assume that (4.38), (4.39), (4.42), and (4.53) hold, a_2 satisfies condition (A), and*

$$\limsup_{t \to \infty} \int_{a}^{t} \sigma(s) a_2(s)\,\Delta s = \infty. \tag{4.85}$$

In addition, let there exist constants $k_0, \delta_0 > 0$ such that

$$\frac{f_1(u)}{u} \geq k_0 \quad \textit{for all } u \neq 0 \tag{4.86}$$

and

$$a_1(t) \geq \delta_0 \quad \textit{for all large } t. \tag{4.87}$$

Then system (4.30) is oscillatory.

Proof. If not, by Lemma 4.2.4, we may assume without loss of generality that (x, y) is a solution of (4.30) with $x(t), y(t) > 0$ for all $t \geq T_0 \geq a$. We define

$$H(t) = \frac{t}{f_2(x(t))}, \quad t \geq T_0,$$

then multiplying the first term of the second equation in (4.30) by $H(\sigma(t))$ and integrating by parts gives

$$\begin{aligned}\int_{T_0}^{t} H(\sigma(s))y^{\Delta}(s)\,\Delta s &= H(t)y(t) - H(T_0)y(T_0) - \int_{T_0}^{t} y(s)H^{\Delta}(s)\,\Delta s \\ &= H(t)y(t) - H(T_0)y(T_0) \\ &\quad - \int_{T_0}^{t} y(s)\left(\frac{f_2(x(s)) - s(f_2\circ x)^{\Delta}(s)}{f_2(x(s))f_2(x(\sigma(s)))}\right)\Delta s \\ &= H(t)y(t) - H(T_0)y(T_0) \\ &\quad - \int_{T_0}^{t} \frac{y(s)}{f_2(x(\sigma(s)))}\,\Delta s + \int_{T_0}^{t} \frac{H(s)y(s)(f_2\circ x)^{\Delta}(s)}{f_2(x(\sigma(s)))}\,\Delta s.\end{aligned}$$

Now from the chain rule,

$$\int_{T_0}^{t} \frac{H(s)y(s)(f_2\circ x)^{\Delta}(s)}{f_2(x(\sigma(s)))}\,\Delta s = \int_{T_0}^{t} \frac{H(s)y(s)}{f_2(x(\sigma(s)))}\left(\int_0^1 f_2'(x_h(s))\,dh\right)x^{\Delta}(s)\,\Delta s \geq 0$$

since $x^{\Delta}(t) > 0$ for $t \geq T_0$. Therefore, we get

$$\int_{T_0}^{t} H(\sigma(s))y^{\Delta}(s)\,\Delta s \geq H(t)y(t) - H(T_0)y(T_0) - \int_{T_0}^{t} \frac{y(s)}{f_2(x(\sigma(s)))}\,\Delta s. \quad (4.88)$$

From the second equation in (4.30), we have after a multiplication by $H(\sigma(t))$ and an integration

$$\int_{T_0}^{t} H(\sigma(s))y^{\Delta}(s)\,\Delta s + \int_{T_0}^{t} \sigma(s)a_2(s)\,\Delta s = 0$$

and so by (4.88), we have after rearranging

$$H(t)y(t) + \int_{T_0}^{t} \sigma(s)a_2(s)\,\Delta s \leq H(T_0)y(T_0) + \int_{T_0}^{t} \frac{y(s)}{f_2(x(\sigma(s)))}\,\Delta s. \quad (4.89)$$

Now we would like to show that

$$\int_{T_0}^{\infty} \frac{y(s)}{f_2(x(\sigma(s)))}\,\Delta s < \infty. \quad (4.90)$$

We already know that

$$\int_{T_0}^{\infty} \frac{x^{\Delta}(s)}{f_2(x(\sigma(s)))}\,\Delta s = \int_{T_0}^{\infty} \frac{a_1(s)f_1(y(s))}{f_2(x(\sigma(s)))}\,\Delta s < \infty \quad (4.91)$$

(See proof of Theorem 3.3 in [24], and define $G(u) = \int_{u_0}^{u} 1/f_2(s)\,ds$ $(u_0 = x(T_0) > 0)$). From (4.86) and (4.87), it follows that

$$\delta_0 \frac{y(t)}{f_2(x(\sigma(t)))} \leq \frac{1}{k_0}\frac{a_1(t)f_1(y(t))}{f_2(x(\sigma(t)))} \quad \text{for all large } t. \quad (4.92)$$

Therefore (4.90) follows from (4.91) and (4.92). But now if we take the lim sup on both sides of (4.89), we see that the left side is not bounded above and this is a contradiction. A similar argument will be valid if $x(t)$, $y(t) < 0$ eventually. ■

Remark 4.2.17 *Theorem 4.2.16 extends the result of Atkinson [14] for the superlinear Emden-Fowler differential equation (1.9).*

We can replace the nonlinearity assumption (4.53) by assuming that f_2' satisfies condition (B). A simple example of such a function is $f_2(u) = cu^\alpha + du^\beta$, where $c, d > 0$ and $0 < \alpha \leq 1 \leq \beta$, with α, β quotients of odd positive integers. We also note that one can also give an example of a function f_2 which satisfies the nonlinearity condition (4.53) but is such that f_2' does not satisfy condition (B) (see [24]).

We have therefore the following result.

Theorem 4.2.18 *Assume that (4.38), (4.39), and (4.50) hold, f_2' satisfies condition (B), and a_2 satisfies condition (A). Assume further that (4.86) and (4.87) hold and that there exists an rd-continuously delta differentiable function z such that for all $K > 0$,*

$$\limsup_{t\to\infty} \int_a^t [a_2(s)(z(\sigma(s)))^2 - K(z^\Delta(s))^2]\,\Delta s = \infty. \tag{4.93}$$

Then system (4.30) is oscillatory.

Proof. If not, by Lemma 4.2.4, we may assume without loss of generality that, there is a solution (x, y) of (4.30) such that $x(t)$, $y(t) > 0$ for $t \geq T_0$ for some sufficiently large T_0. As in Lemma 4.2.6, let

$$w = \frac{z^2 y}{f_2 \circ x}, \quad t \geq T_0$$

and note that $w(t) \geq 0$, and from Lemma 4.2.6,

$$\begin{aligned} -w^\Delta &= a_2(z^\sigma)^2 - \frac{y(z^\Delta)^2}{(f_2 \circ x)^\Delta} + \frac{y(f_2 \circ x)(f_2 \circ x)^\sigma}{(f_2 \circ x)^\Delta}\left[\left(\frac{z}{f_2 \circ x}\right)^\Delta\right]^2 \\ &\geq a_2(z^\sigma)^2 - \frac{y(z^\Delta)^2}{(f_2 \circ x)^\Delta} \end{aligned}$$

since $x^{\Delta}(t) > 0$ for $t \geq T_0$. Therefore

$$-\int_{T_0}^{t} w^{\Delta}(s)\,\Delta s = -w(t) + w(T_0)$$
$$\geq \int_{T_0}^{t} a_2(s)(z(\sigma(s)))^2\,\Delta s - \int_{T_0}^{t} \frac{y(s)(z^{\Delta}(s))^2}{(f_2 \circ x)^{\Delta}(s)}\,\Delta s. \tag{4.94}$$

Now from the chain rule,

$$(f_2 \circ x)^{\Delta}(t) = \left(\int_0^1 f_2'(x_h(t))\,dh\right) x^{\Delta}(t)$$
$$= \left(\int_0^1 f_2'(x_h(t))\,dh\right) a_1(t) f_1(y(t)),$$

where

$$x_h(t) = x(t) + h\mu(t)x^{\Delta}(t) \geq x(t) \geq x(T_0), \quad t \geq T_0$$

since $x^{\Delta}(t) > 0$ for $t \geq T_0$. Since f_2' satisfies condition (B) corresponding to $k = x(T_0) > 0$, there is an $m > 0$ such that

$$f_2'(x_h(t)) \geq m, \quad t \geq T_0.$$

Therefore

$$0 < \frac{y(t)}{(f_2 \circ x)^{\Delta}(t)} \leq \frac{1}{mk_0\delta_0}, \quad t \geq T_0$$

for sufficiently large T_0. It follows that

$$-\int_{T_0}^{t} \frac{y(s)(z^{\Delta}(s))^2}{(f_2 \circ x)^{\Delta}(s)}\,\Delta s \geq -\frac{1}{mk_0\delta_0}\int_{T_0}^{t} (z^{\Delta}(s))^2\,\Delta s$$

and so we have from (4.94) that

$$w(T_0) \geq -w(t) + w(T_0) = -\int_{T_0}^{t} w^{\Delta}(s)\,\Delta s$$
$$\geq \int_{T_0}^{t} a_2(s)(z(\sigma(s)))^2\,\Delta s - \frac{1}{mk_0\delta_0}\int_{T_0}^{t} (z^{\Delta}(s))^2\,\Delta s$$
$$= \int_{T_0}^{t} \left[a_2(s)(z(\sigma(s)))^2 - K(z^{\Delta}(s))^2\right]\Delta s \quad \left(K = \frac{1}{mk_0\delta_0}\right),$$

which leads using (4.93) to a contradiction. A similar argument will be valid if $x(t)$, $y(t) < 0$ eventually. ■

Remark 4.2.19 *Note that if $f_2(u) = u$, then f_2' satisfies condition (B) so Theorem 4.2.18 applies to the linear case as well, in contrast to Theorem 4.2.16.*

If we do not assume that f_2' satisfies condition (B) in the previous theorem, then we can conclude that all bounded solutions are oscillatory. That is, we have

Corollary 4.2.20 *Assume that all conditions of Theorem 4.2.18 hold except f_2' satisfies condition (B). Then all bounded solutions of (4.30) are oscillatory.*

4.2.4 Examples

Example 4.2.21 *Let $\mathbb{T} = \mathbb{R}$. And consider the delay-dynamic system*

$$\begin{aligned} x'(t) &= t^{\alpha_1} y(t) \\ y'(t) &= -t^{\alpha_2} |x(t^\beta)|^{\alpha-1} x(t^\beta), \quad t \in [1, \infty),\ 0 < \beta \le 1,\ 1 < \alpha, \end{aligned}$$

where $a_1(t) = t^{\alpha_1}$, $a_2(t) = t^{\alpha_2}$, $f_1(u) = u$, $f_2(u) = |u|^{\alpha-1}u$, and $\tau(t) = t^\beta$. It is easy to check that conditions (4.34), (4.36), (4.37), (4.42), (4.52), (4.53), (4.75), and (4.76) of Theorem 4.2.13 hold. And in order for the condition (4.43) to hold, we need to choose $\alpha_1 \ge -1$. For simplicity, we only consider the case $\alpha_1 > -1$ since we can treat the case $\alpha_1 = -1$ similarly. Noting that

$$A(t) \sim t^{\alpha_1+1} \quad \text{and} \quad F(t) \sim t^{(\beta-1)(\alpha_1+1)},$$

we see that

$$a_2(t) f_2(F(t)) A(t) \sim t^{\alpha_2+(\alpha_1+1)(1+\alpha(\beta-1))}.$$

Therefore, condition (4.54) holds if

$$\alpha_2 + (\alpha_1 + 1)(1 + \alpha(\beta - 1)) \ge -1.$$

This gives an example of a delay system and, in particular, a range of values of the delay term t^β for which oscillation holds. In the case of no delay ($\beta = 1$), the criterion reduces to Atkinson's [14] criterion for the case $\alpha_1 = 0$.

Example 4.2.22 *Now consider the previous example with $0 < \alpha < 1$. It is easy to check that conditions (4.34), (4.36), (4.37), (4.39), (4.64), (4.67), (4.75), and (4.76) of Theorem 4.2.14 hold. Again in order for the condition (4.43) to hold, we need to choose $\alpha_1 \ge -1$. For simplicity, we only consider the case $\alpha_1 > -1$. We have*

$$a_2(t) f_2(F(t)) f_2(A(t)) \sim t^{\alpha_2+\alpha(\alpha_1+1)(1+(\beta-1))}.$$

Therefore, condition (4.68) holds if

$$\alpha_2 + \alpha(\alpha_1 + 1)(1 + (\beta - 1)) \geq -1.$$

Example 4.2.23 *Let* $\mathbb{T} = \mathbb{Z}$, $N \in \mathbb{N}$, *and* $\alpha > 1$ *be fixed. Consider the system*

$$\begin{aligned} \Delta x(n) &= ny(n) \\ \Delta y(n) &= -n^{\alpha_3}|x(n-N)|^{\alpha-1}x(n-N), \quad n \in \{1, 2, \ldots\}. \end{aligned}$$

Here $a_1(n) = n$, $a_2(n) = n^{\alpha_3}$, $f_1(u) = u$, $f_2(u) = |u|^{\alpha-1}u$, *and* $\tau(n) = n - N$. *Again it is easy to check that conditions (4.34), (4.36), (4.37), (4.42), (4.43), (4.52), (4.53), (4.75), and (4.76) of Theorem 4.2.13 hold. Noting that*

$$A(n) = \sum_{k=1}^{n-1} k \sim n^2 \quad \text{and} \quad F(n) = \frac{\sum_{k=n_0}^{n-N-1} k}{\sum_{k=n_0}^{n} k} \sim 1,$$

we see that

$$a_2(n)f_2(F(n))A(n) \sim n^{\alpha_3+2}.$$

Therefore, condition (4.54) holds if

$$\alpha_3 \geq -3.$$

Example 4.2.24 *Now consider the previous example with* $0 < \alpha < 1$. *It is easy to check that conditions (4.34), (4.36), (4.37), (4.39), (4.43), (4.64), (4.67), (4.75), and (4.76) of Theorem 4.2.14 hold. Since*

$$a_2(n)f_2(F(n))f_2(A(n)) \sim n^{2\alpha+\alpha_3},$$

condition (4.68) holds if

$$2\alpha + \alpha_3 \geq -1.$$

In all the examples below, we assume that $f_i \in \mathbb{C}(\mathbb{R}, \mathbb{R})$, $i = 1, 2$, such that (4.34), (4.39), and (4.42) hold.

Example 4.2.25 *Let* $\mathbb{T} = \mathbb{N}$, *and consider the difference system*

$$\begin{aligned} \Delta x(n) &= a_1(n)f_1(y(n)) \\ \Delta y(n) &= -a_2(n)f_2(x(n+1)), \end{aligned} \tag{4.95}$$

where $a_1(n) > 0$ *and satisfies condition (4.87),*

$$a_2(n) = \frac{\lambda}{n^\alpha(n+1)} + \frac{\beta(-1)^n}{n^\alpha},$$

where $\beta, \lambda > 0, 0 < \alpha < 1$. *In [24, Example 4.2], it was shown that if* $\lambda/\alpha > \beta$, *then* a_2 *satisfies condition (A). Also*

$$-\infty = \liminf_{t\to\infty} \int_1^t \sigma(s)a_2(s)\,\Delta s < \limsup_{t\to\infty} \int_1^t \sigma(s)a_2(s)\,\Delta s = \infty$$

provided $\lambda < \beta(1-\alpha)$. *Hence, if*

$$\beta\alpha < \lambda < \beta(1-\alpha)$$

and if (4.53) and (4.86) hold, then system (4.95) is oscillatory by Theorem 4.2.16.

Example 4.2.26 *Let* $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, *and consider the q-difference system*

$$\begin{aligned} D_q x(t) &= a_1(t) f_1(y(t)) \\ D_q y(t) &= -a_2(t) f_2(x(qt)), \end{aligned} \tag{4.96}$$

where $a_1(t) > 0$ *and satisfies condition (4.87). Assume that* $1 < \alpha < 2$, *let*

$$m = \frac{q^{\alpha-1} - 1}{q^{\alpha-1} + 1},$$

and assume further that $\beta > 0$ *and* $0 < m\beta \le \lambda$. *Define*

$$a_2(t) = \frac{1}{t^\alpha}(\lambda + \beta(-1)^n), \quad t = q^n \in \mathbb{T}.$$

In [24, Example 4.3], it was shown that a_2 *satisfies condition (A) and (4.85) holds. Hence if we assume that (4.53) and (4.86) hold, then system (4.96) is oscillatory by Theorem 4.2.16. Note also that if* $0 < \lambda < \beta$, *we have*

$$-\infty = \liminf_{t\to\infty} \int_1^t \sigma(s)a_2(s)\,\Delta s < \limsup_{t\to\infty} \int_1^t \sigma(s)a_2(s)\,\Delta s = \infty.$$

Example 4.2.27 *Let* $\mathbb{T} = [1, \infty)$, *and consider the differential system*

$$\begin{aligned} x'(t) &= a_1(t) f_1(y(t)) \\ y'(t) &= -a_2(t) f_2(x(t)), \end{aligned} \tag{4.97}$$

where $a_1(t) > 0$ *and satisfies condition (4.87),*

$$a_2(t) = \frac{\lambda}{t^{1+\alpha}} + \frac{\beta \sin t}{t^\alpha},$$

where λ, α, β are all positive numbers and satisfy

$$\beta\alpha < \lambda, \quad 0 < \alpha < 1. \tag{4.98}$$

In [24, Example 4.4], it was shown that a_2 satisfies condition (A) and (4.85) holds. If, in addition, $\lambda < \beta(1-\alpha)$, then we also have

$$\liminf_{t\to\infty} \int_1^t s a_2(s)\, ds = -\infty.$$

Hence, if in addition to (4.98), we also assume that (4.53) and (4.86) hold, then system (4.97) is oscillatory by Theorem 4.2.16.

Remark 4.2.28 *One can also apply Theorem 4.2.18 to these examples if f_2' satisfies condition (B) instead of (4.53).*

CHAPTER 5

CONCLUSION

This thesis is devoted to the qualitative behavior of solutions of dynamic equations on time scales. The study of dynamic equations on time scales is a fairly new area of mathematics, and research in this area is rapidly growing. It has been created in order to unify the theories of differential and difference equations and to extend these theories to other dynamic equations, e.g., q-difference equations. In this thesis, we have illustrated this new understanding by extending and in most cases by also unifying some continuous and discrete results from differential and difference equations, respectively, to dynamic equations on arbitrary time scales.

In Chapter 3, we have studied the asymptotic behavior of solutions of dynamic equations on time scales, in particular, asymptotic equivalence and asymptotic equilibrium have been investigated. First, we have established sufficient conditions for the asymptotic equivalence of linear systems and linear and quasilinear systems, respectively, by unifying and extending the results in [9] and [83]. Particularly, for the asymptotic equivalence of differential systems, we have improved the well-known theorems of Levinson [59] and Yakubovich [82]. Next, we have presented sufficient conditions for the asymptotic equilibrium of quasilinear systems by unifying and extending the results in [76] and [77]. Specifically, we have obtained the time scale extension of Wintner's [79] well-known theorem. Moreover, some of our results are new even for the asymptotic equilibrium of difference systems.

In Chapter 4, we have discussed the oscillatory behavior of solutions of dynamic equations on time scales. First, we have provided necessary and sufficient conditions for the oscillation of solutions of a particular class of second order nonlinear delay dynamic equations by extending the results in [63]. Particularly, we have gen-

eralized the classical theorems of Atkinson [14] and Belohorec [19]. Next, we have established sufficient conditions for the oscillation of solutions of a two-dimensional nonlinear dynamic system by unifying and extending the results in [41] and [56], and by generalizing the results in [24]. In particular, we have generalized the oscillation criteria of Atkinson [14], Belohorec [19], Hooker and Patula [49], and Waltman [78]. Furthermore, we have considered the systems by allowing inequality in the second equation and established sufficient conditions for the oscillation of solutions of a related delay-dynamic system.

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VITA

PERSONAL INFORMATION

Surname, Name: Mert Raziye
Date and Place of Birth: 3 August 1978, Ankara
email: raziyemert@cankaya.edu.tr

EDUCATION

Degree	Institution	Year of Graduation
BS	METU, Mathematics	2002
High School	Batıkent High School	1996

WORK EXPERIENCE

Year	Place	Enrollment
2009-	Çankaya University, Department of Mathematics and Computer Science	Instructor
2002-2008	METU, Mathematics Department	Research Assistant

RESEARCH VISITS

1 September , 2008 - 28 February , 2009, University of Nebraska Lincoln, Department of Mathematics, Lincoln, Nebraska, USA, supervised by Prof. Dr. Allan Peterson and Prof. Dr. Lynn Erbe.

FOREIGN LANGUAGES

Advanced English

SCHOLARSHIPS

1. TUBITAK National Scholarship for MSc Students
2. TUBITAK International Research Fellowship

PUBLICATIONS

1. Mert, R. and Zafer, A. *Eventually positive solutions of second-order superlinear dynamic equations*, Further progress in analysis: Proceedings of the 6th International ISAAC Congress, 13-18 August, 2007, Ankara, Turkey, World Scientific, 535–544, (2009).

2. Erbe, L. and Mert, R. *Some new oscillation results for a nonlinear dynamic system on time scales*, Appl. Math. Comput., **215**, 2405–2412, (2009).

3. Kaymakçalan, B., Mert, R. and Zafer, A. *Asymptotic equivalence of dynamic systems on time scales*, Discrete Contin. Dyn. Syst., Supplement Volume, 558–567, (2007).

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6. Mert, R. and Zafer, A. *A necessary and sufficient condition for oscillation of second-order sublinear delay dynamic equations*, in progress.

REFEREEING

Mathematical and Computer Modelling

PARTICIPATION IN INTERNATIONAL SCIENTIFIC MEETINGS

1. 14th International Conference on Difference Equations and Applications, Bahçeşehir University, İstanbul, Turkey, 21-25 July, 2008.

2. 6th International ISAAC Congress, METU, Ankara, Turkey, 13-18 August, 2007.

3. AIMS Sixth International Conference on Dynamical Systems, Differential Equations and Applications, University of Poitiers, Poitiers, France, 25-28 June, 2006.

4. The First International Workshop on Dynamic Equations on Time Scales, Bahçeşehir University, İstanbul, Turkey, 27 June-1 July, 2005.

PARTICIPATION IN NATIONAL SCIENTIFIC MEETINGS

1. 9th Dynamic Systems Workshop, İzmir University, İzmir, Turkey, 18-19 June, 2009.

2. 4th Ankara Mathematics Days, METU, Ankara, Turkey, 4-5 June, 2009.

3. 8th Differential Equations Workshop, Yeditepe University, İstanbul, Turkey, 19-21 June, 2008.

4. Workshop on Differential Equations and Applications, METU, Ankara, Turkey, 8-10 February, 2007.

5. Ankara Differential Equations Seminars, Atılım University, Ankara, Turkey, 9 June, 2006.

PRESENTATIONS IN SCIENTIFIC MEETINGS

1. *Oscillation for a nonlinear dynamic system on time scales*, 9th Dynamic Systems Workshop, İzmir, Turkey, 2009.

2. *Time scale extensions of a theorem of Wintner on systems with asymptotic equilibrium*, 14th International Conference on Difference Equations and Applications, İstanbul, Turkey, 2008.

3. *Extensions of a theorem of Wintner on dynamic systems on time scales with asymptotically constant solutions*, 6th International ISAAC Congress, Ankara, Turkey, 2007.

4. *Asymptotic Equivalence of Dynamic Equations on Time Scales*, AIMS Sixth International Conference on Dynamical Systems, Differential Equations and Applications, Poitiers, France, 2006.

5. *Asymptotic Behavior of Dynamic Systems on Time Scales*, Ankara Differential Equations Seminars, Ankara, Turkey, 2006.

CONFERENCE ORGANIZATIONS

Member of the Organization Committee of the 3rd International Conference on Nonlinear Science and Complexity, Çankaya University, Ankara, Turkey, 28-31 July, 2010.