ENTANGLEMENT TRANSFORMATIONS

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ABSTRACT

ENTANGLEMENT TRANSFORMATIONS

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Entanglement is a special correlation of the quantum states of two or more particles. It is also a useful resource enabling us to complete tasks that cannot be done by classical means. As a result, the transformation of entangled states of distant particles by local means arose as an important problem in quantum information theory. In this thesis, we first review some of the studies done on the entanglement transformations. We also develop the necessary and sufficient conditions for the deterministic transformation of *W*-type states.

Keywords: Entanglement, Entanglement Transformations, Multipartite *W*-Type States

ÖZ

DOLANIKLIK DÖNÜŞÜMLERİ

Kıntaş, Seçkin
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Dolanıklık iki veya daha fazla parçacığın kuvantum durumları arasında var olan özel bir tür korelâsyondur. Ayrıca dolanıklık, klasik yöntemlerle yapamayacağımız birçok işi halletmemizi sağlayan bir kaynak olma özelliğine de sahiptir. İşte bu nedenle, birbirlerinden uzaktaki parçacıklar arasındaki dolanık durumun, sadece yerel imkânlar kullanılarak dönüştürülmesi, kuantum bilgi kuramı içinde önemli bir problem olarak ortaya çıkmıştır. Bu tezde, öncelikle dolanıklık dönüşümleri üzerine şimdiye kadar yapılmış bazı çalışmaların bir derlemesi yapılmıştır. Ayrıca, *W*-tipi durumların belirlenimci dönüşümlerini sağlayacak gerekli ve yeterli koşullar geliştirilmiştir.

Anahtar kelimeler: Dolanıklık, Dolanıklık Dönüşümleri, Çok taraflı *W*-tipi Durumlar

To my family

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TABLE OF CONTENTS

ABSTRACT iv

ÖZ v

ACKNOWLEDGMENTS vii

TABLE OF CONTENTS viii

CHAPTERS

1. INTRODUCTION 1

2. THEORY 3
   - 2.1 Generalized Measurement 3
   - 2.2 Density Matrix 4
   - 2.3 Reduced Density Matrix 7
   - 2.4 Schmidt Decomposition 8
   - 2.5 Entanglement 10
   - 2.6 Majorization 11
   - 2.7 Doubly Stochastic Matrices 12
   - 2.8 Entanglement Monotones 15

3. ENTANGLEMENT TRANSFORMATIONS OF BIPARTITE PURE STATES 17
   - 3.1 Deterministic Entanglement Transformation 17
   - 3.2 Jensen – Schack Algorithm 20
   - 3.3 Entanglement Catalysis 24
   - 3.4 Probabilistic Entanglement Transformations of Bipartite Pure States 25

4. TRANSFORMATION OF W-TYPE STATES 29
   - 4.1 W-Type States 29
   - 4.2 Local Operations by One Party 33
   - 4.3 Deterministic W-Type State Transformations 37

5. CONCLUSION 41

REFERENCES 43

CHAPTER 1

INTRODUCTION

Although most of the formal novelties of non-relativistic quantum mechanics had been completed by von Neumann; Einstein, Podolsky and Rosen (EPR) [1] and Schrödinger [2] discovered the most non-classical feature of the quantum world; this phenomenon which is called as “entanglement”. It is originally named as “Verschrankung” by Schrödinger. In the beginning, entanglement was thought as a sort of a statistical relation between subsystems of a composite system. However, nowadays it is recognized as a fundamental feature of a quantum system and it lies at the heart of quantum information theory and is used for many applications.

In quantum information theory, entanglement is a useful resource for accomplishing tasks which are (seemed to be) impossible classically such as teleportation [3] and dense coding [4]. Quantum channels which, can be described as communication channels, are used for sending quantum information as well as classical information. Distribution of entanglement is also performed by the quantum communication channels. But these channels are noisy because the distribution of entanglement yields in mixed state that cannot be useful in for mentioned applications. So, the parties sharing the entangled state are faced with the problem of transforming the state to one that can be used in applications. This problem is called “Entanglement Transformation”

The parties that share the entangled state work in corporation with each other when transforming a given state to a target state. This cooperation has to be done by local quantum operations (including local unitaries). But they are also allowed to communicate by sending the measurement results to each other (classical communication). Such operations are called LOCC (local operation assisted with classical communication) operations. The LOCC operations can be considered as special measurement operations that can be carried out by local actions of all parties.

Different aspects of the entanglement transformation problem are investigated by many people such as, Nielsen [5], Vidal [6], Jonathan and Plenio [7], [8]. The methods and approaches for solving the problem differ depending on the nature of the given and desired states. While approaching the problem, the given state can be bipartite, multipartite, pure or mixed. Furthermore the transformation can be done deterministically or probabilistically. Some of the problems in entanglement transformation have not been solved yet (for example the transformation of mixed states) but, great deals of problems have been solved successfully.

In this study, the main approach to solving the problem of entanglement transformation is investigated. In Chapter 2 some basic mathematical tools and theories which are used for solving the problems are described (density matrix, majorization, etc). In Chapter 3, the brief review of entanglement transformation conditions of bipartite pure states is discussed. Firstly, the transformation of a single copy of bipartite pure state is discussed. The definite reference for this chapter is Bhatia`s book [9] and Nielsen`s theorem [5]. Secondly, the definite solution of probabilistic transformation is given. The necessary and sufficient conditions are given in terms of basic majorization relations. In Chapter 4, as an original work, the necessary and sufficient conditions for the deterministic transformations of *W*-type states are investigated. SLOCC (stochastic local operations assisted with classical communication) classification and LU (local unitary) equivalence is introduced. In Chapter 5, a summary of the study is given and the main results are discussed.

CHAPTER 2

THEORY

2.1 Generalized Measurement

Generalized measurement is described by a set of operators $\{M_i\}$ which are not need to be self-adjoint satisfying the following condition,

$$\sum_i M_i^\dagger M_i = I \tag{2.1}$$

which is called the completeness relation. Before the measurement, if the system is the state $|\psi\rangle$, then the probability of obtaining outcome *i* is,

$$p_i = \langle \psi | M_i^\dagger M_i | \psi \rangle \tag{2.2}$$

and the state collapses to,

$$|\psi_i\rangle = \frac{1}{\sqrt{p_i}} M_i |\psi\rangle \tag{2.3}$$

The completeness relation given in Eqn. 2.1 satisfies the rule which says that the total probability is equal to unity. That is,

$$\sum_i p_i = \sum_i \langle \psi | M_i^\dagger M_i | \psi \rangle = \langle \psi | \psi \rangle = 1 \tag{2.4}$$

Projective measurements are special types of generalized measurements. They correspond to the case where the equation $M_i^\dagger = M_i$ is satisfied, then the measurement operators are orthogonal projectors.

$$M_i M_j = \delta_{ij} M_i \tag{2.5}$$

$$\sum_i M_i = I \tag{2.6}$$

Generalized measurement operators can be converted to unitary operations as follows.

$$U|\psi\rangle \otimes |0\rangle = \sum_i M_i\, |\psi\rangle \otimes |i\rangle \tag{2.7}$$

The completeness relation satisfies that U is unitary. If $|\psi\rangle$ and $|\phi\rangle$ are two states then, U preserves the inner product.

$$\langle 0| \otimes \langle\phi|U^{\dagger}U|\psi\rangle \otimes |0\rangle = \sum_{ij}\langle j| \otimes \langle\phi|M_j^{\dagger}M_i\, |\psi\rangle \otimes |i\rangle \tag{2.8}$$

$$= \sum_{ij}\langle\phi|M_j^{\dagger}M_i|\psi\rangle\langle j|\, i\rangle \tag{2.9}$$

$$= \sum_i\langle\phi|M_i^{\dagger}M_i|\psi\rangle \tag{2.10}$$

$$= \langle\phi|\psi\rangle \tag{2.11}$$

We see that the inner product is preserved. This is the feature of a unitary transformation.

2.2 Density Matrix

In quantum mechanics, the physical system is represented by a pure the state vector $|\psi\rangle$ which contains the whole information about the physical system. However, this information can be incomplete. For example, consider electrons emitted by a hot source where the state of the electron is unknown. We only know that the state of the electron is taken from the ensemble $\{|\psi_i\rangle\}$ with corresponding probabilities $\{|p_i\}$ satisfying the equation $\sum_i p_i = 1$. In that case we say that our physical system is a statistical mixture of pure states (mixed state). But single states $|\psi_k\rangle$ are pure states. Here one can introduce a density matrix (density operator) to describe the physical system.

The density operator (density matrix) is defined as follows [10];

$$\rho = \sum_{i=1}^{n} p_i\, |\psi_i\rangle\langle\psi_i| \tag{2.12}$$

The measurement of any observable A on a mixed state with density operator ρ can be described as follows

$$\langle A\rangle = \sum_{k=1}^{n} \lambda_k p(k) \tag{2.13}$$

where λ_k is the kth eigenvalue of the operator A and $p(k)$ denotes the probability of outcome λ_k when A is measured. This probability can be computed as

$$p(k) = \sum_{m=1}^{l} p_m \langle\psi_m|P_k|\psi_m\rangle = tr(\rho P_k) \tag{2.14}$$

where P_k is the projector on to the eigensubspace corresponding the eigenvalue λ_k of A. For the case of a non degenerate eigenvalue, it is defined as,

$$P_k = |k\rangle\langle k| \tag{2.15}$$

where, $|k\rangle$ is the corresponding eigenvector.

If the Eqn.s 2.12 and 2.14 are combined,

$$\langle A\rangle = \sum_{k=1}^{n}\sum_{m=1}^{l} \lambda_k\, p_m \langle\psi_m|P_k|\psi_m\rangle \tag{2.16}$$

$$= \sum_{k=1}^{n}\sum_{m=1}^{l} \lambda_k\, p_m \langle\psi_m|k\rangle\langle k|\psi_m\rangle \tag{2.17}$$

$$= \sum_{k=1}^{n}\sum_{m=1}^{l} p_m\, \langle k|\psi_m\rangle\langle\psi_m|A|k\rangle \tag{2.18}$$

$$= \sum_{k=1}^{n} \langle k|\rho A|k\rangle = tr(\rho A) \tag{2.19}$$

Now, let us describe the collapse due to measurement for a mixed state. In the case where the state is a pure state $|\psi_k\rangle$ and if the measurement yields λ_k, then the final collapsed state

$$|\psi'_{lk}\rangle = \frac{1}{\sqrt{\langle\psi_l|P_k|\psi_l\rangle}} P_k|\psi_l\rangle \tag{2.20}$$

For the case of a mixed state, the probability distribution of the pure state ensemble and the information gain in the measurement has to be taken into account. In such a case, it can be found that, when measurement yields λ_k, the final density matrix is,

$$\rho' = P_k \rho P_k \frac{1}{tr(\rho P_k)} \tag{2.21}$$

Now let us look at the time evolution equation of a mixed state. The time evolution of any state $|\psi_m\rangle$ in the ensemble is described by the Schrödinger equation.

$$i\hbar \frac{d}{dt}|\psi_m(t)\rangle = \hat{\mathcal{H}}|\psi_m(t)\rangle. \tag{2.22}$$

hence,

$$-i\hbar \frac{d}{dt}\langle\psi_m(t)| = \langle\psi_m(t)|\hat{\mathcal{H}} \tag{2.23}$$

and therefore,

$$\frac{d}{dt}\rho(t) = \sum_{m=1}^{n} p_m \left[\frac{d}{dt}|\psi_m(t)\rangle\langle\psi_m(t)| + |\psi_m(t)\rangle\frac{d}{dt}\langle\psi_m(t)|\right] \tag{2.24}$$

$$= \sum_{m=1}^{n} p_m \left[-\frac{i}{\hbar}\widehat{\mathcal{H}}|\psi_m(t)\rangle\langle\psi_m(t)| + \frac{i}{\hbar}|\psi_m(t)\rangle\langle\psi_m(t)|\widehat{\mathcal{H}}\right] \tag{2.25}$$

$$= \sum_{m=1}^{n} p_m \frac{-i}{\hbar}\left[\widehat{\mathcal{H}}|\psi_m(t)\rangle\langle\psi_m(t)| - |\psi_m(t)\rangle\langle\psi_m(t)|\widehat{\mathcal{H}}\right] \tag{2.26}$$

$$\frac{d}{dt}\rho(t) = \frac{1}{i\hbar}\left[\widehat{\mathcal{H}}, \rho(t)\right] \tag{2.27}$$

This equation is known as von Neumann equation, which describes the time evolution of the density operator. A formal solution of this equation can be expressed as,

$$\rho(t) = U(t, t_0)\rho(t_0)U^{\dagger}(t, t_0) \tag{2.28}$$

where $U(t, t_0)$ is the time evolution operator.

The density matrices have the following properties.

i. They are hermitian

ii. They are positive semi definite $\rho \geq 0$

iii. $tr(\rho) = 1$

Proof:

i. From the Eqn. 2.1 given above

$$\rho^{\dagger} = \sum_{m}^{n} p_m^{*}|\psi_m\rangle\langle\psi_m| = \rho \tag{2.29}$$

ii. For any vector $|u\rangle$ we have,

$$\langle u|\rho|u\rangle = \langle u|\sum_{m=1}^{n} p_m |\psi_m\rangle\langle\psi_m|u\rangle \tag{2.30}$$

$$= \sum_{m=1}^{n} p_m |\langle u|\psi_m\rangle|^2 \geq 0 \tag{2.31}$$

iii.

$$tr(\rho) = \sum_i \langle i|\rho|i\rangle \tag{2.32}$$

$$= \sum_i \langle i|\sum_{m=1}^{n} p_m |\psi_m\rangle\langle\psi_m|i\rangle \tag{2.33}$$

$$= \sum_i |\langle i|\psi_m\rangle|^2 (\sum_{m=1}^{n} p_m) = \sum_i p_i (\sum_{m=1}^{n} p_m) = 1 \tag{2.34}$$

Density matrices are also used to determine whether a given state is mixed or pure; because one gets $tr\rho^2 = 1$ for mixed states and $tr\rho^2 \leq 1$ for pure states. To prove these, let us define a state $\{|\psi_k\rangle\}$ by means of orthonormal bases $\{|\phi_j\rangle\}$.

$$|\psi_k\rangle = \sum_j a_j|\phi_j\rangle \tag{2.35}$$

where, $\sum_j|a_j|^2 = 1$

Then the density matrix is defined as,

$$\rho = \sum_{i=1}^{n} \sum_j \sum_m p_i a_j a_m^* \, |\phi_j\rangle\langle\phi_m| \tag{2.36}$$

hence,

$$\rho^2 = \sum_{i=1}^{n} \sum_j \sum_m \sum_t \sum_l {p_i}^2 a_j a_m^* a_t a_l^* \, |\phi_j\rangle\langle\phi_m|\phi_t\rangle\langle\phi_l| \tag{2.37}$$

where $\langle\phi_m|\phi_t\rangle = \delta_{mt}$

$$\rho^2 = \sum_{i=1}^{n} \sum_j \sum_m \sum_l {p_i}^2 a_j \, |a_m|^2 a_l^* |\phi_j\rangle\langle\phi_l| \tag{2.38}$$

$$tr(\rho^2) = \sum_{i=1}^{n} \sum_j \sum_m \sum_l \sum_\sigma {p_i}^2 a_j \, |a_m|^2 a_l^* \langle\sigma|\phi_j\rangle\langle\phi_l|\,\sigma\rangle \tag{2.39}$$

$$= \sum_{i=1}^{n} \sum_m \sum_j |a_m|^2 |a_j|^2 \, |p_i|^2 = \sum_i |p_i|^2 \tag{2.40}$$

Since $\sum_i p_i = 1$, $\sum_i |p_i|^2 = 1$ if and only if, $p_i = 1$. The density matrix for pure state is defined as,

$$\rho = (|\psi_k\rangle\langle\psi_k|) \tag{2.41}$$

On the other hand, for a mixed state, $\sum_i |p_i|^2 < 1$. The density matrix for mixed state is defined as,

$$\rho = \sum_{k=1}^{n} p_k \, |\psi_k\rangle\langle\psi_k| \tag{2.42}$$

2.3 Reduced Density Matrix

Now consider a bipartite pure state $|\psi\rangle = \sum_{ij} c_{ij} |i\rangle_A |j\rangle_B$ where the total Hilbert space is the tensor product of those of the associated subsystems A and B.

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \tag{2.43}$$

Where the orthonormal bases $\{|i\rangle_A\}$ and $\{|i\rangle_B\}$ are in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. The density operator is defined as,

$$\rho = \sum_{ij} \sum_{km} c_{ij} c_{km}^* \, |i\rangle_A |j\rangle_B \, {}_A\langle k|_B\langle m|, \tag{2.44}$$

where, $c_{ij} c_{km}^* = {}_A\langle i|_B\langle j|\rho|k\rangle_A|m\rangle_B$

The total system is described by ρ and our aim is to calculate the expectation value of the operator M_A acting on the system A. It is possible to extend the operator M_A to the entire Hilbert space by defining the operator,

$$\widetilde{M} = M_A \otimes I_B \tag{2.45}$$

The expectation value of the operator is,

$$\langle M_A \rangle = tr(\rho\widetilde{M}) = \sum_{\alpha\beta A}\langle\alpha|_B\langle\beta|\rho\,\widetilde{M}|\alpha\rangle_A|\beta\rangle_B, \tag{2.46}$$

$$= \sum_{\alpha\beta A}\langle\alpha|_B\langle\beta|\left(\sum_{ij}\sum_{km} c_{ij}c_{km}^{*}\,|i\rangle_A|j\rangle_{B\,A}\langle k|_B\langle m|\right) M_A \otimes I_B|\alpha\rangle_A|\beta\rangle_B, \tag{2.47}$$

$$= \sum_{ij}\sum_{km}\sum_{\alpha\beta} c_{ij}c_{km\;A}^{*}\langle\alpha|\,i\rangle_{A\,B}\langle\beta|j\rangle_{B\,A}\langle k|M_A|\alpha\rangle_{A\,B}\langle m|\beta\rangle_{B,} \tag{2.48}$$

where ${}_A\langle\alpha|i\rangle_A = \delta_{\alpha i}$, ${}_B\langle\beta|j\rangle_B = \delta_{\beta j}$ and ${}_B\langle m|\beta\rangle_B = \delta_{m\beta}$. The equation becomes,

$$tr(\rho\widetilde{M}) = \sum_i\sum_k\sum_{j\,A}\langle i|_B\langle j|\rho|k\rangle_A|m\rangle_{B\,A}\,\langle k|M_A|\alpha\rangle_A, \tag{2.49}$$

$$= \sum_j\sum_{k\,A}\langle k|M_A|k\rangle_{A\;B}\langle j|\rho|j\rangle_B. \tag{2.50}$$

Let us introduce a matrix such that,

$$\rho_A = \sum_{j\,B}\langle j|\rho|j\rangle_B = tr_A(\rho), \tag{2.51}$$

where $tr_A(\rho)$ denotes the partial trace over subsystem B. In that case, the expectation value of the operator M_A is defined as,

$$\langle M_A \rangle = tr(\rho_A M_A) \tag{2.52}$$

It is easy to see that; it is possible to compute the expectation value of an operator acting on the only subsystem A as if the system A is isolated and is described by the reduced density matrix ρ_A.

2.4 Schmidt Decomposition

A useful expansion of states exists for composite systems formed by two more subsystems. This is called the Schmidt decomposition. It can be exposed as follows [10].

For any given bipartite pure state $|\psi\rangle$ in $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, there exists orthonormal states $\{|i\rangle_1\}$ in $\mathcal{H}_1$ and $\{|i\rangle_2\}$ in $\mathcal{H}_2$, so that

$$|\psi\rangle = \sum_{i=1}^{k} \sqrt{\lambda_i}|i\rangle_1|i\rangle_2 \tag{2.53}$$

where λ_i (Schmidt coefficients) are real numbers satisfying the condition $\sum_{i=1}^{k} \lambda_i = 1$. The orthonormal sets $\{|i\rangle_1\}$ and $\{|i\rangle_2\}$ depend on the state $|\psi\rangle$.

Proof: $|\psi\rangle \in \mathcal{H}$ can be expressed as,

$$|\psi\rangle = \sum_{i,m} c_{im}\, |i\rangle_1|m\rangle_2 \tag{2.54}$$

which $\{|i\rangle_1\} \in \mathcal{H}_1$ and $\{|m\rangle_2\} \in \mathcal{H}_2$. In that case we can write the decomposition of the state $|\psi\rangle$ as

$$|\psi\rangle = \sum_i |i\rangle_1|i^{'}\rangle_2 \tag{2.55}$$

where we defined $\{|i'\rangle_2\} \in \mathcal{H}_2$ as

$$|i^{'}\rangle_2 = \sum_m c_{im}|m\rangle_2 \tag{2.56}$$

Note that $\{|i'\rangle_2\}$ bases are neither orthogonal nor normalized. These vectors also depend on $\{|i\rangle_1\}$ which is an arbitrary orthonormal basis in $\mathcal{H}_1$. In order to prove the statement, we must choose $|i\rangle_1$ so that the reduced density matrix ρ_1 is diagonalized. The reduced density matrix is defined as follows.

$$\rho_1 = tr_2(|\psi\rangle\langle\psi|) = tr_2(\textstyle\sum_{ij}|i\rangle_1\, |i'\rangle_{21}\langle j|_2\langle j'|) \tag{2.57}$$

$$= \textstyle\sum_{\alpha 2}\langle\alpha\big|(\sum_{i,j}|i\rangle_1|i'\rangle_{21}\langle j|_2\langle j|)\big|\alpha\rangle_2 \tag{2.58}$$

$$= \textstyle\sum_{i,j}|i\rangle_{11}\langle j|\sum_{\alpha 2}\langle\alpha|i'\rangle_{2\,2}\langle j'|\alpha\rangle_2 \tag{2.59}$$

$$= \textstyle\sum_{i,j}|i\rangle_{11}\langle j|\sum_{\alpha 2}\langle j'|\alpha\rangle_{2\,2}\langle\alpha|i'\rangle_2 \tag{2.60}$$

$$= \textstyle\sum_{ij}|i\rangle_{11}\langle j|\,{}_2\langle j'|i'\rangle_2 \tag{2.61}$$

where $\{|\alpha\rangle_2\}$ is an orthonormal bases in $\mathcal{H}_2$ and satisfies the completeness relation.

$$\textstyle\sum_\alpha |\alpha\rangle_{22}\langle\alpha| = I \tag{2.62}$$

If $\rho_1 = \sum_i p_i|i\rangle_{11}\langle i|$ then, ${}_1\langle i'|j'\rangle_1 = p_i\delta_{ij}$ must be satisfied. For this reason $|i'\rangle_2$ vectors are orthogonal to each other and normalized as,

$$\langle i'|i'\rangle = p_i \tag{2.63}$$

Hence, $\{|i_2\rangle\}$ defined by,

$$|i\rangle_2 = \frac{1}{\sqrt{p_i}} |i'\rangle_2 \tag{2.64}$$

is an orthonormal set.

After normalizing the base vector $|i'\rangle_2$ substituting into the Eqn. 2.55 we get the Eqn. 2.64. In Schmidt decomposition, the number of product elements is called Schmidt rank. If the Schmidt rank is greater than 1, then we call the state entangled.

2.5 Entanglement

The term entanglement has crucial importance in quantum information technologies, where quantum nature of systems is used for processing information in new ways. It has been well understood by now that the entanglement of a quantum state is one of the most important properties in quantum information science.

The entanglement concept emerged from a 1935 paper written by Einstein Podolsky and Rosen [1] which analyzed a problem that is called "EPR paradox" in later years. In their paper they developed a thought experiment to demonstrate what they felt about the lack of completeness in quantum mechanics. The idea of the EPR paradox was the following. Consider two entangled particles,

$$|\phi^+\rangle = \frac{1}{\sqrt{2}} (|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B) \tag{2.65}$$

where the distance of the particles is not important. If Alice makes measurement on the first particle and she obtains the outcome in which state collapses to $|0\rangle$ state, then Bob`s particle collapses to $|0\rangle$ too. If she obtains the outcome in which state collapses to $|1\rangle$ state, then Bob`s particle collapses to $|1\rangle$. In that case, the measurement on the first particle seems to affect the second particle. But according to the theory of relativity, nothing (at least no message) can travel faster than light. Einstein called this instantaneous collapse "spooky action at a distance" and believed that, the quantum theory is incomplete.

It is thought that in order to overcome those problem hidden variable theories must be used. In 1964, Bell [11] showed that the hidden variable theories desired by Einstein and quantum theory have different predictions. He then proposed a class of experiments which will enables us to distinguish between two types of theories. He found several inequalities and showed that quantum mechanics violated these inequalities.

The mathematical definition of entanglement is defined as follows; consider two states represented as,

$$|\psi\rangle_A = \sum_i c_i{}^A\, |i\rangle_A, \tag{2.66}$$

$$|\phi\rangle_B = \sum_j m_j{}^B\, |j\rangle_B, \tag{2.67}$$

where, $\{|i\rangle_A\}$ is a basis in $\mathcal{H}_A$ and $\{|j\rangle_B\}$ is in $\mathcal{H}_B$. The state in $\mathcal{H}_A \otimes \mathcal{H}_B$ is represented as,

$$|\psi\rangle_{AB} = \sum_{ij} c_{ij}\, |i\rangle_A |j\rangle_B. \tag{2.68}$$

Then, the state is separable if $c_{ij} = c_i{}^A m_j{}^B$. It is not seperable if $c_{ij} \neq c_i{}^A m_j{}^B$. The in separable states are entangled states. Look at the following state,

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B). \tag{2.69}$$

The state $|\psi\rangle_{AB}$ is an entangled state since it can not be written in product form. The system can be either in A or in B. If observer in A performs a measurement on his system and finds the result 0 then the state collapses to $|0\rangle_A \otimes |1\rangle_B$. Similarly if he finds the result 1 then the state collapses to $|1\rangle_A \otimes |0\rangle_B$.

2.6 Majorization

Majorization is a partial order relation between vectors [9]. Consider two n-dimensional vectors $x = (x_1, x_2 \ldots\ldots, x_n)$ and $y = (y_1, y_2 \ldots\ldots, y_n)$. Their components are real numbers. The components of vectors can be arranged into increasing or decreasing order. The symbol $x^{\downarrow} = (x_1^{\downarrow}, x_2^{\downarrow}, \ldots\ldots, x_n^{\downarrow})$ indicates that the components of vector x arranged into decreasing order.

$$x_1^{\downarrow} \geq x_2^{\downarrow} \geq \ldots\ldots \geq x_n^{\downarrow}. \tag{2.70}$$

We say that x is majorized by y ($x \prec y$), if,

$$x_1^{\downarrow} \leq y_1^{\downarrow}, \tag{2.71}$$

$$x_1^{\downarrow} + x_2^{\downarrow} \leq y_1^{\downarrow} + y_2^{\downarrow}, \tag{2.72}$$

and

$$x_1^{\downarrow} + x_2^{\downarrow} + \dots\dots\dots + x_{n-1}^{\downarrow} \leq y_1^{\downarrow} + y_2^{\downarrow} + \dots\dots\dots + y_{n-1}^{\downarrow}, \tag{2.73}$$

and the sum of the components are equal.

$$x_1^{\downarrow} + x_2^{\downarrow} + \dots\dots\dots + x_n^{\downarrow} = y_1^{\downarrow} + y_2^{\downarrow} + \dots\dots\dots + y_n^{\downarrow}. \tag{2.74}$$

The majorization relation $x \prec y$ can also be expressed by using $x^{\uparrow}$ which is a vector formed by the components of x arranged into increasing order. In that case, the following inequality must be satisfied.

$$\sum_{j=1}^{k} x_j^{\uparrow} \geq \sum_{j=1}^{k} y_j^{\uparrow}, \tag{2.75}$$

for $(k = 1,2,3 \dots, n)$ and,

$$S = \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j. \tag{2.76}$$

This can be proved by using the obvious relations

$$\sum_{j=1}^{k} x_j^{\uparrow} = S - \sum_{j=1}^{n-k} x_j^{\downarrow}, \tag{2.77}$$

$$\sum_{j=1}^{k} y_j^{\uparrow} = S - \sum_{j=1}^{n-k} y_j^{\downarrow}. \tag{2.78}$$

In that case,

$$\sum_{j=1}^{k} x_j^{\uparrow} \geq \sum_{j=1}^{k} y_j^{\uparrow}, \tag{2.79}$$

if and only if

$$\sum_{j=1}^{k} x_j^{\downarrow} \leq \sum_{j=1}^{k} y_j^{\downarrow}. \tag{2.80}$$

These two inequalities form two alternative ways of defining the condition $x \prec y$.

2.7 Doubly Stochastic Matrices

There is a close relation between majorization and doubly stochasticity. Majorization relation between two vectors can be described by doubly stochastic matrices.

Definition: Any $n \times n$ square matrix a is called doubly stochastic matrix if,

$$a_{ij} \geq 0, \tag{2.81}$$

$$\sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} = 1. \tag{2.82}$$

Majorization given in Eqn. 2.75 can be described with the help of doubly stochastic matrices.

<u>*Theorem 2.1: Horn`s theorem*</u> Let a be a doubly stochastic matrix. If $x = ay$, then, $x \prec y$

Horn`s theorem is very useful; it gave rise to many applications in linear algebra.

<u>*Theorem 2.2: Ky Fan`s maximum principle*</u> For any hermitian matrix A,

$$\sum_{j=1}^{k} \lambda_j^{\downarrow}(A) = \max\,[tr(AP)] \tag{2.83}$$

where maximization is taken over k-dimensional projection operators P and $\lambda_j^{\downarrow}$ denotes the j^{th} largest eigenvalue [12].

Proof: It is enough to show that

$$\sum_{j=1}^{k} \lambda_j(A) \geq tr(AP) \tag{2.84}$$

for all k-dimensional projection operators P

Let $\{|i_1\rangle, |i_2\rangle, \ldots |i_k\rangle\}$ be orthonormal basis for k-dimensional projector P and $\{|m_1\rangle, |m_2\rangle, \ldots |m_k\rangle\}$ be the orthonormal basis of eigenvectors of A. The projector P can be expressed as

$$P = \sum_{\alpha=1}^{k} |i_\alpha\rangle\langle i_\alpha| \tag{2.85}$$

Every hermitian matrix has the following decomposition.

$$A = \sum_{l=1}^{n} \lambda_l |m_l\rangle\langle m_l| \tag{2.86}$$

$$\langle i_j|A|i_j\rangle = \sum_{j=1}^{n} \left|u_{lj}\right|^2 \lambda_l \tag{2.87}$$

where $u_{lj} = \langle m_l|j\rangle$ is a unitary matrix and $\left|u_{lj}\right|^2 = a_{lj}$ is doubly stochastic matrix.

According to theorem 2.1 [9]

$$\langle i_j|A|i_j\rangle \prec \lambda(A) \tag{2.88}$$

From the basic majorization relations it is easy to see that

$$\sum_{j=1}^{k} \langle i_j | A | i_j \rangle \leq \sum_{j=1}^{k} \lambda_j^{\downarrow}(A). \tag{2.89}$$

And,

$$tr(AP) = \sum_{j=1}^{k} \langle j | A \sum_{n=1}^{k} | i_n \rangle \langle i_n | j \rangle \tag{2.90}$$

$$tr(AP) = \sum_{j=1}^{k} \langle i_j | A | i_j \rangle. \tag{2.91}$$

Eqns. 2.89 and 2.91 proves the statement

$$tr(AP) \leq \sum_{j=1}^{k} \lambda_j^{\downarrow}(A) \tag{2.92}$$

Ky Fan`s theorem is used for the majorization relation between hermitian matrices.

<u>*Theorem 2.3:*</u> For two Hermitian matrices $And\ B$

$$\lambda(A+B) \prec \lambda^{\downarrow}(A) + \lambda^{\downarrow}(B) \tag{2.93}$$

Proof: Let projector P projects to the k-dimensional subspace formed by k eigenvectors corresponding to k largest eigenvalues.

$$\lambda_1^{\downarrow}(A+B), \lambda_2^{\downarrow}(A+B), \ldots \lambda_k^{\downarrow}(A+B) \tag{2.94}$$

$$\sum_{j=1}^{k} \lambda_j^{\downarrow}(A+B) = tr((A+B)P) = tr(AP) + tr(BP) \tag{2.95}$$

From the inequality 2.92 we get

$$\sum_{j=1}^{k} \lambda_j^{\downarrow}(A+B) \leq \sum_{j=1}^{k} \lambda^{\downarrow}(A) + \sum_{j=1}^{k} \lambda^{\downarrow}(B) \tag{2.96}$$

which completes the proof.

<u>*Theorem 2.4: Birkhoff`s Theorem*</u> *Any* $n \times n$ doubly stochastic matrix can be written as a convex combination of permutation matrices; that is,

$$\mathcal{D} = \sum_{i=1}^{k} c_i P_i \tag{2.97}$$

where P_i denotes permutation matrices and $0 \leq c_i \leq 1$ and $\sum_{i=1}^{k} c_i = 1$

Let us sketch the proof:

Let P_1 be a permutation matrix such that all entries $D_{ij} \geq 0$ whenever $(P_1)_{ij} = 1$ [13]. Let c_1 be a real number such that $(\mathcal{D} - c_1 P_1)$ has non-negative elements. The matrix $\mathcal{D}$ can be written as,

$$\mathcal{D} = c_1 P_1 + (1 - c_1)\left\{\frac{1}{1-c_1}(\mathcal{D} - c_1 P_1)\right\} \tag{2.98}$$

It is possible to choose a matrix

$$M = \left\{ \frac{1}{1-c_1} (\mathcal{D} - c_1 P_1) \right\} \tag{2.99}$$

M is doubly stochastic and one zero element. It can be written as,

$$M = c_2 P_2 + (1 - c_2) N \tag{2.100}$$

Repeating the argument on M k times it is clear to see that doubly stochastic matrix $\mathcal{D}$ can be written as convex combination of permutation matrices. Johnson, Dulmage and Mendelsohn showed that it is not possible to write less than $n^2 - 2n + 2$ permutation matrices [14].

For example any general 2×2 doubly stochastic matrix can be represented by combination of permutation matrices as,

$$\begin{pmatrix} t & 1-t \\ 1-t & t \end{pmatrix} = t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (1-t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.101}$$

where $0 \leq t \leq 1$.

2.8 Entanglement Monotones

The monotonic behavior of entanglement transformations under local operations and classical communications is called entanglement monotones [15]. ($E(\rho)$) where ρ is the density operator of two or more systems. The main properties of entanglement monotones are as follows,

i. $E(\rho) \geq 0$

ii. If ρ is separable then, $E(\rho) = 0$

iii. Local unitary operations does not change $E(\rho)$

$$E(\rho) = E(U_1 \otimes U_2 \otimes \dots . \otimes U_n)\, \rho (U_1 \otimes U_2 \otimes \dots . \otimes U_n)^{\dagger}) \tag{2.102}$$

iv. $E(\rho)$ does not increase on average by local operations.

$$E(\rho) \geq \sum_i p_i\, E(\rho_i) \tag{2.103}$$

where ρ_i is a possible final state appearing with probability p_i

v. $E(\rho)$ is a convex function under discarding information.

$$E\left(\sum_i p_i \rho_i\right) \leq \sum_i p_i\, E(\rho_i) \tag{2.104}$$

Consider bipartite pure state in Schmidt form,

$$|\psi\rangle = \sum_{j=1}^{N} \sqrt{\lambda_i} |j\rangle_A |j\rangle_B \tag{2.105}$$

where $\lambda_1^{\downarrow} \geq \lambda_2^{\downarrow} \geq \cdots \geq \lambda_n^{\downarrow}$. The following functions are shown to be entanglement monotones [16].

$$E_l(|\psi\rangle_{AB}) = \sum_{i=l}^{n} \lambda_i^{\downarrow} \tag{2.106}$$

The entropy of entanglement which is defined as $S(\psi) = -\sum_{i=1}^{n} \lambda_i \ln(\lambda_i)$ is also an entanglement monotones.

Concurrence is also an entanglement monotone. It is defined for mixed states of two qubits [17] as.

$$C(P) = \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}), \tag{2.107}$$

where, $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}, \sqrt{\lambda_4}$ are arranged into decreasing order and eigenvalues of,

$$\rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y) \tag{2.108}$$

where σ_y is a Pauli spin matrix defined as,

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{2.109}$$

CHAPTER 3

ENTANGLEMENT TRANSFORMATIONS OF BIPARTITE PURE STATES

3.1 Deterministic Entanglement Transformation

How can one transform a bipartite pure state to another deterministically using local operations and classical communication (LOCC)? As an example consider the following bipartite pure state shared by Alice and Bob [18].

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{3.1}$$

Their aim is to transform that state to

$$|\phi\rangle = \cos\theta|01\rangle + sin\theta|10\rangle \tag{3.2}$$

They can do this as follows. First, Alice performs a generalized measurement on her system. The measurement operators are described as,

$$M_{1=}\begin{pmatrix} cos\theta & 0 \\ 0 & sin\theta \end{pmatrix} \tag{3.3}$$

$$M_{2=}\begin{pmatrix} sin\theta & 0 \\ 0 & cos\theta \end{pmatrix}. \tag{3.4}$$

The unnormalized final states are,

$$|\psi_1\rangle = (M_1 \otimes I_B)\,\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{3.5}$$

$$|\psi_2\rangle = (M_2 \otimes I_B)\,\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{3.6}$$

After the measurement the state is either

$$\sqrt{2}|\psi_1\rangle = \cos\theta|01\rangle + sin\theta|10\rangle \tag{3.7}$$

or

$$\sqrt{2}|\psi_2\rangle = \sin\theta|01\rangle + cos\theta|10\rangle \tag{3.8}$$

If Alice gets $|\phi\rangle = \cos\theta|01\rangle + sin\theta|10\rangle$ then, she does nothing on her system and sends result to Bob. He does not do anything on his system as they got the desired state. However if the measurement results in $(\sin\theta|01\rangle + cos\theta|10\rangle)$ then she performs unitary operation on her system.

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.9}$$

and then the state becomes,

$$(U_A \otimes I_B)(\sin\theta|01\rangle + cos\theta|10\rangle) = \sin\theta|11\rangle + \cos\theta|00\rangle \tag{3.10}$$

After that, she communicates with Bob. Then he performs the same operation on his system. The final state is the desired state.

$$|\phi\rangle = (I_A \otimes U_B)(\sin\theta|11\rangle + \cos\theta|00\rangle) = \cos\theta|01\rangle + sin\theta|10\rangle \tag{3.11}$$

Hence they can obtain the final state $|\phi\rangle$ with certainty.

Nielsen was able to find the complete set of rules for deciding whether such a transformation protocol exists for arbitrary bipartite pure states [5]. He described necessary and sufficient conditions for transforming bipartite pure state $|\psi\rangle$ to a desired state $|\phi\rangle$. This transformation can be done with one way classical communication only [19].

Theorem 3.1:Nielsen`s theorem The state $|\psi\rangle$ can be transformed to $|\varphi\rangle$ using local operations and classical communication if and only if the Schmidt coefficients of $|\psi\rangle$ is majorized by the Schmidt coefficients of $|\varphi\rangle$,

$$\lambda(\psi) \prec \lambda(\phi), \tag{3.12}$$

where, $\lambda(\psi)$ denotes the vector of Schmidt coefficients of $|\psi\rangle$ and similarly for $|\phi\rangle$.

Suppose Alice and Bob share bipartite entangled pure state $|\psi\rangle$ and their aim is the transform $|\psi\rangle$ to desired state $|\phi\rangle$. Lo and Popescu [19] have shown that for any protocol converting $|\psi\rangle$ to $|\phi\rangle$, the same job can be done by, Alice performing a measurement, a one way classical communication from Alice to Bob basically conveying the outcome of the measurement and a final arbitrary transformation by

Bob. They have proved this by showing that, any measurement performed by Bob, can also be performed by Alice and she can get the same result.

Now, let us sketch the proof of necessity of Nielsen`s theorem. Let the states $|\psi\rangle$ and $|\phi\rangle$ have the following decompositions,

$$|\psi\rangle = \sum_{i=1}^{N} \sqrt{\alpha_i}\, |i\rangle_A \mid i\rangle_B \tag{3.13}$$

$$|\phi\rangle = \sum_{j=1}^{N} \sqrt{\beta_j} | j\rangle_{A|} | j\rangle_B \tag{3.14}$$

Suppose that Alice performs a generalized measurement on her side with measurement M_t where *t* are the outcomes. After that Alice will tell Bob the value of *t* and then Bob a unitary transformation. Let,

$$(M_t \otimes I_B)\, |\psi\rangle = \sqrt{p_t} |\psi_t\rangle \tag{3.15}$$

$$(M_t \otimes I_B) \sum_{i=1}^{N} \sqrt{\alpha_i}\, |i\rangle_A |i\rangle_B = \sqrt{p_t} |\psi_t\rangle \tag{3.16}$$

Here, $|\psi_t\rangle$ should be convertible to $|\phi\rangle$ by local unitaries. Hence $\lambda(\psi_t) = \lambda(\phi)$. The reduced density matrix of Bob`s particle is;

$$\rho_B = tr_A(|\psi\rangle\langle\psi|) \tag{3.17}$$

$$= \sum_{kA}\langle k| \left(\sum_{i=1}^{N} \sqrt{\alpha_i}\, |i\rangle_A |i\rangle_B\right)\left(\sum_{j=1\,A}^{N}\langle j|_B\langle j|\sqrt{\alpha_j}\right)|k\rangle_A \tag{3.18}$$

$$= \sum_{i=1}^{N} \alpha_i\, |i\rangle_{B\,B}\langle i| \tag{3.19}$$

If we take the partial trace of the Eqn. 3.15 we get,

$$tr_A(M_t \otimes I_B \sum_{i=1}^{N} \sqrt{\alpha_i}\, |i\rangle_A |i\rangle_B \sum_{j=1}^{N} \sqrt{\alpha_j}\ {}_A\langle j|_B\langle j| M_t^{\dagger} \otimes I_B) = tr_A(p_t|\psi_t\rangle\langle\psi_t|) \tag{3.20}$$

$$= tr_A\left(\sum_{i=1}^{N} \sum_{j=1}^{N} M_t|i\rangle_A|i\rangle_B\ {}_A\langle j|_B\langle j|M_t^{\dagger}\right)\sqrt{\alpha_i}\sqrt{\alpha_j} = p_t\rho_B^t \tag{3.21}$$

$$= \sum_k \sum_{j=1}^{N} \sum_{i=1\,A}^{N}\langle k|M_t|i\rangle_A|i\rangle_{B\,A}\langle j|_B\langle j|M_t^{\dagger}\, |k\rangle_A\sqrt{\alpha_i}\, \sqrt{\alpha_j} \tag{3.22}$$

$$= \sum_k \sum_{j=1}^{N} \sum_{i=1\,B}^{N}\langle j|M_t^{\dagger}|k\rangle_{A\,A}\langle k|M_t|i\rangle_A|i\rangle_{B\,B}\langle j|\sqrt{\alpha_i}\, \sqrt{\alpha_j}. \tag{3.23}$$

According to completeness relation, $\sum_k |k\rangle_{A\,A}\langle k| = I$ and we get the following.

$$\sum_{i=1}^{N} \sum_{j=1\,A}^{N}\langle j|M_t^{\dagger}M_t|i\rangle_A|i\rangle_{B\,B}\langle j|\sqrt{\alpha_i}\sqrt{\alpha_j} = p_t\rho_B^t \tag{3.24}$$

$$= \sum_{t=1} \sum_{i=1}^{N} \sum_{j=1\,A}^{N}\langle j|M_t^{\dagger}M_t|i\rangle_A|i\rangle_{B\,B}\langle j|\sqrt{\alpha_i}\, \sqrt{\alpha_j} \tag{3.25}$$

$\sum_t M_t^\dagger M_t = I$ and ${}_A\langle j|j\rangle_A = \delta_{ij}$

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\delta_{ij}|i\rangle_{B\,B}\langle j|\sqrt{\alpha_i}\sqrt{\alpha_j} = \sum_i p_t\rho_B^t \tag{3.26}$$

$$\sum_{i=1}^{N}|i\rangle_{B\,B}\langle i|\alpha_i = \sum_t p_t\rho_B^t. \tag{3.27}$$

As a result,

$$\rho^B = \sum_t p_t\rho_B^t, \tag{3.28}$$

where ρ_B^t is the reduced density matrix if outcome is t and $\sum_t p_t\rho_B^t$ is the density matrix if Bob does not know the outcome. The relation given in Eqn. 3.28 is called no communication theorem. The main idea of that theorem is any measurement performed by Alice does not affect the statistics of Bob`s measurement.

According to theorem 2.3 then,

$$\lambda(\rho^B) \prec \sum_i p_i\lambda_i^{\downarrow}(\rho_i^B) \tag{3.29}$$

which implies the relation given in theorem 3.1. This proves the necessity of the majorization relation. The sufficiency of the relation can be shown by using a particular algorithm.

3.2 Jensen – Schack Algorithm

Nielsen [5] has found the necessary and sufficient conditions for converting a bipartite pure state $|\psi\rangle$ to another bipartite pure state $|\phi\rangle$ by means of LOCC. Jensen and Schack [20] discovered a simple algorithm for transforming a bipartite pure state to another one. This algorithm takes into account a measurement performed on one party and a local unitary operation which are essentially as permutations of a given orthonormal basis. The algorithm is constructed as follows;

Suppose that the bipartite pure state $|\psi\rangle$ is needed to be transformed to another bipartite pure state $|\phi\rangle$ by means of LOCC. Suppose that the Schmidt coefficients of $|\psi\rangle$ is majorized by the Schmidt coefficients of $|\phi\rangle$. In other words if,

$$|\psi\rangle_{AB} = \sum_i \sqrt{\alpha_i}\,|i\rangle_A\,|i\rangle_B \tag{3.30}$$

$$|\phi\rangle_{AB} = \sum_j \sqrt{\beta_j}\,|j\rangle_A\,|j\rangle_B \tag{3.31}$$

then,

$$\alpha \prec \beta \tag{3.32}$$

We will now show that we can create a protocol where $|\psi\rangle$ is transformed into $|\phi\rangle$.

Find doubly stochastic matrix Remember that by theorem by theorem 2.1 we can find doubly stochastic matrix such that,

$$\alpha = D\beta \tag{3.33}$$

where $\alpha \prec \beta$

Decompose doubly stochastic matrix:

Every doubly stochastic matrix can be written as a convex combination of permutation matrices by Birkhoff`s theorem (theorem 2.2). Hence,

$$\mathcal{D} = \sum_\sigma p_\sigma P_\sigma \tag{3.34}$$

where P_σ are some permutation matrices.

Construct the measurement operator:

For each σ, define M_σ as,

$$M_\sigma = \sum_{i=1}^{N} \sqrt{p_\sigma}\sqrt{\frac{\beta_\sigma^i}{\alpha_i}}\ |i\rangle_{AA}\langle i| \tag{3.35}$$

where the index σ is an index in the decomposition of D and $\beta_\sigma^i = (P_\sigma\beta)_i$. Note that β_σ is a vector whose components are same as those of β but ordered in a different way. In vector notation $\beta_\sigma = P_\sigma\beta$

The measurement operators satisfy the relation,

$$\sum_\sigma M_\sigma^\dagger M_\sigma = \sum_i |i\rangle\langle i| \left(\frac{\sum_\sigma P_\sigma \beta_\sigma^i}{\alpha_i}\right) = \sum_i |i\rangle\langle i| = I \tag{3.36}$$

Measurement:

Alice performs a measurement on her system described by the generalized measurement operators. M_σ above. So, when the outcome σ is obtained, the unnormalized collapsed state is

$|\psi\rangle_{AB}$ is shared by Alice and Bob and Alice performs a measurement on her side

$$(M_\sigma \otimes I_B)\left(\sum_i^N \sqrt{\alpha_i}|i\rangle_A|i\rangle_B\right) = \sqrt{p_\sigma}\sum_j^N \sqrt{\beta_\sigma^i}\,|i\rangle_A|i\rangle_B \tag{3.37}$$

Note that the outcome σ occurs with probability p_σ.

If the measurement results in target state she communicates to Bob and he does nothing on his system

If the measurement result is different than the target state she performs local unitary operation (permutation) on her system and sends the result to Bob. Bob performs the same operation on his system and gets the desired system.

As an example, consider two bipartite pure states.

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|11\rangle + |22\rangle + |33\rangle) \tag{3.38}$$

$$|\phi\rangle = \frac{1}{\sqrt{6}}(|11\rangle + \sqrt{2}|22\rangle + \sqrt{3}|33\rangle) \tag{3.39}$$

It is obvious that Schmidt coefficients of $|\psi\rangle$ are majorized by the coefficients of $|\phi\rangle$.

$$\vec{\alpha} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} \tag{3.40}$$

$$\vec{\beta} = \begin{pmatrix} 1/6 \\ 1/3 \\ 1/2 \end{pmatrix} \tag{3.41}$$

We first find a doubly stochastic matrix *D* such that,

$$\begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} \mathcal{D}_{11} & \mathcal{D}_{12} & \mathcal{D}_{13} \\ \mathcal{D}_{21} & \mathcal{D}_{22} & \mathcal{D}_{23} \\ \mathcal{D}_{31} & \mathcal{D}_{32} & \mathcal{D}_{33} \end{pmatrix} \begin{pmatrix} 1/6 \\ 1/3 \\ 1/2 \end{pmatrix} \tag{3.42}$$

Considering the matrix Eqn. 3.42 and 2.82 a possible solution of doubly stochastic matrix is;

$$\mathcal{D} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \tag{3.43}$$

The doubly stochastic matrix $\mathcal{D}$ is the combination of permutation matrices.

$$\begin{pmatrix} {}^{1}/_{3} & {}^{1}/_{3} & {}^{1}/_{3} \\ {}^{1}/_{3} & {}^{1}/_{3} & {}^{1}/_{3} \\ {}^{1}/_{3} & {}^{1}/_{3} & {}^{1}/_{3} \end{pmatrix} = {}^{1}/_{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + {}^{1}/_{3}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$+ {}^{1}/_{3}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.44}$$

Therefore, the measurement operators are,

$$M_1 = \frac{1}{\sqrt{3}}\left[\frac{1}{\sqrt{2}}|1\rangle\langle 1| + |2\rangle\langle 2| + \sqrt{\frac{3}{2}}|3\rangle\langle 3|\right] \tag{3.45}$$

$$M_2 = \frac{1}{\sqrt{3}}\left[|1\rangle\langle 1| + \sqrt{\frac{3}{2}}|2\rangle\langle 2| + \sqrt{\frac{1}{2}}|3\rangle\langle 3|\right] \tag{3.46}$$

$$M_3 = \frac{1}{\sqrt{3}}\left[\sqrt{\frac{3}{2}}|1\rangle\langle 1| + \sqrt{\frac{1}{2}}|2\rangle\langle 2| + |3\rangle\langle 3|\right] \tag{3.47}$$

After Alice performs M_1 to her system the system collapses to,

$$|\psi'\rangle = \frac{M_1|\psi\ \rangle}{\sqrt{\langle\psi|M^\dagger\ M|\psi\rangle}} \tag{3.48}$$

$$|\psi'\rangle = \frac{1}{\sqrt{{}^{1}/_{3}}}\left[\frac{1}{3\sqrt{2}}\ |11\rangle + \frac{1}{3}\ |22\rangle + \frac{1}{\sqrt{6}}\ |33\rangle\right] \tag{3.49}$$

$$|\psi'\rangle = \frac{1}{\sqrt{6}}\left[|11\rangle + \sqrt{2}|22\rangle + \sqrt{3}|33\rangle\right] \tag{3.50}$$

$$|\psi'\rangle = |\phi\rangle \tag{3.51}$$

The target state and the collapsed states are same. So, Alice does not need to perform unitary on her side. She communicates with Bob and tells him about the result. (The local unitary matrix is the identity matrix). Second measurement operator is

$$M_2 = \frac{1}{\sqrt{3}}\left[|1\rangle\langle 1| + \sqrt{\frac{3}{2}}|2\rangle\langle 2| + \sqrt{\frac{1}{2}}|3\rangle\langle 3|\right] \tag{3.52}$$

If second outcome is obtained in Alice`s measurement, the collapsed state is,

$$|\psi''\rangle = \frac{1}{\frac{1}{\sqrt{3}}}\left[\frac{1}{3}|11\rangle + \frac{1}{\sqrt{6}}|22\rangle + \frac{1}{\sqrt{18}}|33\rangle\right] \tag{3.53}$$

$$|\psi''\rangle = \left[\frac{1}{\sqrt{3}}|11\rangle + \frac{1}{\sqrt{2}}|22\rangle + \frac{1}{\sqrt{6}}|33\rangle\right] \tag{3.54}$$

The collapsed state is not the target state. Therefore Alice performs a local unitary operation on her system (the inverse of the second permutation matrix is local unitary operator).

$$\left|\phi'\right\rangle = (P_2^{-1} \otimes I_B)\left|\psi'\right\rangle = \left[\frac{1}{\sqrt{3}}|21\rangle + \frac{1}{\sqrt{2}}|32\rangle + \frac{1}{\sqrt{6}}|13\rangle\right] \tag{3.55}$$

Alice sends this result to Bob, who also performs local unitary operation to his system,

$$\left|\phi''\right\rangle = (I_A \otimes P_2^{-1})\left|\phi'\right\rangle = \left[\frac{1}{\sqrt{3}}|22\rangle + \frac{1}{\sqrt{2}}|33\rangle + \frac{1}{\sqrt{6}}|11\rangle\right] \tag{3.56}$$

$$|\phi''\rangle = |\phi\rangle \tag{3.57}$$

Performing local operations the state $|\psi\rangle$ is transformed to desired state $|\phi\rangle$. If Alice has obtained the third outcome, the procedure will be the same.

3.3 Entanglement Catalysis

There are situations where, according to Nielsen`s theorem [5] a state $|\psi\rangle$ cannot be transformed into $|\phi\rangle$, but it is possible to find an appropriate entangled state $|\chi\rangle$ such that it is possible to transform $|\psi\rangle\otimes|X\rangle$ to $|\phi\rangle\otimes|X\rangle$ deterministically [7]. At the end of the protocol, the state of the catalyst particles $|X\rangle$ does not change. This is the reason why they are called catalysts. They can be used over and over again to make transformation possible. As an example consider the following.

$$\left|\psi\right\rangle = \sqrt{0{,}4}\left|00\right\rangle + \sqrt{0{,}4}|11\rangle + \sqrt{0{,}1}|22\rangle + \sqrt{0{,}1}|33\rangle \tag{3.58}$$

$$|\phi\rangle = \sqrt{0{,}5}|00\rangle + \sqrt{0{,}25}|11\rangle + \sqrt{0{,}25}|22\rangle \tag{3.59}$$

The Schmidt coefficients vectors are

$$\vec{\alpha} = \begin{pmatrix} 0{,}4 \\ 0{,}4 \\ 0{,}1 \\ 0{,}1 \end{pmatrix} \text{ and } \vec{\beta} = \begin{pmatrix} 0{,}5 \\ 0{,}25 \\ 0{,}25 \\ 0 \end{pmatrix} \tag{3.60}$$

It is easy to see that the Schmidt coefficients of $\vec{\alpha}$ is not majorized by the coefficients of $\vec{\beta}$ $(\alpha \nprec \vec{\beta})$. Therefore, it is not possible to transform $|\psi\rangle$ to $|\phi\rangle$ by LOCC. Consider the entangled state,

$$|X\rangle = \sqrt{0,6}|00\rangle + \sqrt{0,4}|11\rangle \tag{3.61}$$

The Schmidt coefficients of states $|\psi\rangle \otimes |X\rangle$ and $|\phi\rangle \otimes |X\rangle$ are,

$$\overrightarrow{k} = (0.24, 0.24, 0.06, 0.06, 0.16, 0.16, 0.04, 0.04,) \tag{3.62}$$

$$\vec{m} = (0.3, 0.15, 0.15, 0, 0.2, 0.1, 0.1, 0) \tag{3.63}$$

It is easy to see that $\overrightarrow{k} \prec \vec{m}$. This is a simple example in which $|\psi\rangle \otimes |X\rangle$ can be converted into $|\phi\rangle \otimes |X\rangle$.

3.4 Probabilistic Entanglement Transformations of Bipartite Pure States

Suppose that there is a bipartite pure state and several desired states $\{|\phi_i\rangle\}$ (i=1,2,....,m) and p_i be the corresponding probability for the state $|\varphi_i\rangle$ $(p_1 + p_2 + .. + p_m = 1)$. Is it possible to transform $|\psi\rangle$ to $|\phi_i\rangle$ with probability p_i? Nielsen [4] in his theorem described necessary and sufficient conditions for the single target state with probability one. The solution of probabilistic transformation of the bipartite pure state is obtained by Jonathan and Plenio [8]. The maximum probability of obtaining a particular final state has been found by Vidal [6]. The necessary and sufficient conditions can be expressed in terms of majorization relation.

Theorem 2.2: The probabilistic transformation $|\psi\rangle \to \{p_i, |\phi_i\rangle\}$ can be carried out by means of local operations and classical communications if and only if;

$$\lambda(\psi) \prec \sum_{j=1}^{N} p_i \lambda^{\downarrow}(\phi_i) \tag{3.64}$$

Let us sketch the proof. First we show that the Eqn. 3.64 is necessary. To show this we make use of the result of Lo and Popescu [19]: Any transformation protocol can be carried out by local measurements by Alice, classical communications from Alice to Bob and Bob doing a local unitary operation. Let $|\psi\rangle \longrightarrow \{p_i, |\phi_i\rangle\}$ transformation be achieved by such a protocol. Let $M_{i\alpha}$ be a local measurement operator performed by Alice such that,

$$(M_{i\alpha} \otimes I)|\psi\rangle = \sqrt{p_{i\alpha}}|\phi_{i\alpha}\rangle \tag{3.65}$$

where $|\phi_{i1}\rangle, |\phi_{i2}\rangle, \dots |\phi_{in}\rangle$ are all LU equivalent to $|\phi_i\rangle$ If Alice does not report the result to Bob yet, then Bob`s density matrix is $\sum_{i\alpha} p_{i\alpha}\rho_B^{(i\alpha)}$

where $\rho_B^{(i\alpha)} = tr_A|\phi_{i\alpha}\rangle\langle\phi_{i\alpha}|$ which is equal to $\rho_B = tr_A|\psi\rangle\langle\psi|$ by no communication theorem.

$$\rho_B = \sum_{i\alpha} p_{i\alpha}\rho_B^{(i\alpha)} \tag{3.66}$$

By the theorem 2.3

$$\lambda(\rho_B) \prec \sum_{i\alpha} p_{i\alpha}\lambda^{\downarrow}(\rho_B^{(i\alpha)}) \tag{3.67}$$

Note that

$$\lambda^{\downarrow}(\rho_B^{(i1)}) = \lambda^{\downarrow}(\rho_B^{(i2)}) = \cdots = \lambda^{\downarrow}(\phi_i) \tag{3.68}$$

hence,

$$\lambda(\rho_B) \prec \sum_i (p_{i1} + p_{i2} + \cdots)\lambda^{\downarrow}(\phi_i) = \sum_i p_i\lambda^{\downarrow}(\phi_i) \tag{3.69}$$

This completes the proof of necessity of the Eqn. 3.64

Now let us prove the sufficiency of the Eqn. 3.64. In other words suppose that Eqn. 3.64 is satisfied. We will show that $|\psi\rangle \to \{p_i, |\phi_i\rangle\}$ transformation can be carried out. First define a vector $\vec{\beta}$ by,

$$\vec{\beta} = \sum_i p_i\vec{\lambda}^{\downarrow}(\phi_i) \tag{3.70}$$

Note that $\sum_k \beta_k = 1$ and $\beta_k \geq 0$. Let $|\phi\rangle$ be state with Schmidt coefficient vector equal to $\vec{\beta}$. As a result $|\psi\rangle$ can be converted to $|\phi\rangle$. Since the Schmidt coefficients of $|\psi\rangle$ is majorized by the Schmidt coefficients of $|\phi\rangle$ given in Eqn. 3.64. Suppose that Alice and Bob do this transformation. After obtaining $|\phi\rangle$, second local measurement is carried out by them. The local measurement carried out by Alice is,

$$M_i \equiv \sum_j \sqrt{\frac{\mu_{ij}}{\beta_j}}|j\rangle\langle j| \tag{3.71}$$

$$|\phi_i\rangle \cong_{LU} \sum_j \mu_{ij} \, |j\rangle\langle j| \tag{3.72}$$

As an example, consider a bipartite pure state shared by Alice and Bob, which is defined as,

$$|\psi\rangle = \sqrt{\tfrac{2}{5}}|11\rangle + \sqrt{\tfrac{2}{5}}|22\rangle + \sqrt{\tfrac{1}{5}}|33\rangle \tag{3.73}$$

Is it possible to transform $|\psi\rangle$ to target states with corresponding probabilities?

$$|\psi\rangle \longrightarrow \left\{p_1 = \tfrac{1}{2}, |\psi\rangle_1;\ p_2 = \tfrac{1}{3}, |\psi\rangle_2; p_3 = \tfrac{1}{6}, |\psi\rangle_6 \right\} \tag{3.74}$$

where,

$$|\psi\rangle_1 = \tfrac{1}{\sqrt{3}}(|11\rangle + |22\rangle + |33\rangle) \tag{3.75}$$

$$|\psi\rangle_1 = \tfrac{1}{\sqrt{2}}(|11\rangle + |22\rangle) \tag{3.76}$$

$$|\psi\rangle_3 = |33\rangle \tag{3.77}$$

Now look at the following

$$\lambda(\psi) = \begin{pmatrix} 2/5 \\ 2/5 \\ 1/5 \end{pmatrix} \tag{3.78}$$

$$\sum_{i=1}^{3} p_i \lambda^{\downarrow}(\psi_i) = \begin{pmatrix} 1/2 \\ 1/3 \\ 1/6 \end{pmatrix} \tag{3.79}$$

it is easy to see that

$$\lambda(\psi) \prec \sum_{i=1}^{3} p_i \lambda^{\downarrow}(\psi_i). \tag{3.80}$$

Note that, it is possible to choose a state which has the same Schmidt coefficients with the term $\sum_{i=1}^{3} p_i \lambda^{\downarrow}(\psi_i)$.

$$\vec{\beta} = \sum_i p_i \vec{\lambda}^{\downarrow}(\psi_i). \tag{3.81}$$

So, the desired state can be defined as,

$$\tfrac{1}{\sqrt{6}}(\sqrt{3}|11\rangle + \sqrt{2}|22\rangle + |33\rangle). \tag{3.82}$$

The measurement operators are,

$$M_1 \equiv \sqrt{\frac{2}{3}}|1\rangle\langle 1| + |2\rangle\langle 2| + \sqrt{\frac{1}{2}}|3\rangle\langle 3| \tag{3.83}$$

$$M_2 \equiv |1\rangle\langle 1| + \sqrt{\frac{3}{2}}|2\rangle\langle 2| \tag{3.84}$$

$$M_3 \equiv \sqrt{6}|3\rangle\langle 3|. \tag{3.85}$$

Applying these measurement operators to given state in Eqn. 3.82 we get the target states with corresponding probabilities.

CHAPTER 4

TRANSFORMATION OF W-TYPE STATES

4.1 W-Type States

Consider the entangled states between p parties where $p \geq 3$. The generalized W state is,

$$|W\rangle = \frac{1}{\sqrt{p}}(|100\ldots0\rangle + |010\ldots0\rangle + \cdots + |000\ldots1\rangle). \tag{4.1}$$

We call a state $|\psi\rangle$ a W-type state if it is stochastically reducible from $|W\rangle$, i.e., if there are local operators $A_1, A_2, \ldots A_p$ such that $(A_1 \otimes A_2 \otimes \ldots \otimes A_p)\,|W\rangle = |\psi\rangle$. The state $|\psi\rangle$ is called W class or it is SLOCC (stochastic local operations assisted with classical communications) equivalent to $|W\rangle$ if it is possible to choose A_i to be invertible. We will be interested in general W-type states. Note that W-type states do not need to be multipartite; bipartite and even product states can be of W-type.

The transformations of multipartite entangled states have not been studied and not much is known about them. The main reason for this is that these states do not have a suitable representation such as the Schmidt decomposition. However there are some special subsets of multipartite states which have a suitable representation. We will study the entanglement transformations of these types of states in this chapter.

Any W-type state can be expressed as

$$|\psi\rangle = |b_1 \otimes a_2 \otimes \ldots \otimes a_p\rangle + |a_1 \otimes b_2 \otimes \ldots \otimes a_p\rangle + \ldots.. + |a_1 \otimes a_2 \otimes \ldots \otimes b_p\rangle \tag{4.2}$$

where $|a_k\rangle$ and $|b_k\rangle$ are some vectors in the Hilbert space that do not need to be orthogonal or normalized. This representation is not unique. For the purpose of finding a unique representation it is possible to choose orthonormal bases $\{|\alpha_k\rangle, |\beta_k\rangle\}$ in the Hilbert space of the *kth* party`s particle by using,

$$|\alpha_k\rangle = \frac{|a_k\rangle}{\||a_k\rangle\|} \tag{4.3}$$

$$|b_k\rangle \ = \ c_k|\alpha_k\rangle + d_k|\beta_k\rangle \tag{4.4}$$

where c_k is a complex number and d_k is a real number. In this representation, $\{|\alpha_k\rangle, |\beta_k\rangle\}$ forms an orthonormal bases in the kth particle`s state space. The following expression can be obtained from the Eqn. 4.2

$$|\psi\rangle = \ \sqrt{x_0}\left|\alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_p\right\rangle + \sqrt{x_1}\left|\beta_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_p\right\rangle + \sqrt{x_2}\left|\alpha_1 \otimes \beta_2 \otimes \ldots \otimes \alpha_p\right\rangle$$
$$+ \cdots + \sqrt{x_p}\left|\alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \beta_p\right\rangle \tag{4.5}$$

where $x_i \geq 0$ $(i = 0,1,2 \ldots., p)$and by the normalization condition, we have

$$x_0 + \sum_{i=1}^{p} x_i = 1 \tag{4.6}$$

This is a possible representation of the *W*-type states. But this form may not be unique. Consider the following state

$$\left|\psi\right\rangle = c_0\left|\alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_p\right\rangle + \sum_{i=1}^{p} c_i\left|\alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \beta_i \otimes \ldots \otimes \alpha_p\right\rangle \tag{4.7}$$

where c_i are complex numbers and $\{|\alpha_k\rangle, |\beta_k\rangle\}$ are orthonormal bases. In that case, it is possible to adapt the phases of the vectors $\{|\alpha_k\rangle, |\beta_k\rangle\}$ so that the standard form which is given in Eqn. 4.5 can be obtained with $x_i = |c_i|^2$

For the given state in Eqn. 4.5, it is possible to consider a subset of these p parties, $\{i_1, i_2, \ldots, i_m\}$ as a single party labeled by the letter *P*. It is easy to see that the same state is a W-type state between *P* and rest of the other parties. In that case the parameter *X* for *P* can be found as

$$x_p = x_{i_1} + x_{i_2} + \cdots + x_{i_m} \tag{4.8}$$

If the state given in Eqn.4.5 is considered as an entangled state between *P*, *Q*, *R*….,*Z* (all of each are subsets of *p* parties), it is easy to find the representation of the state with corresponding parameters and vectors. If *P* = $\{i_1, i_2, \ldots, i_m\}$ is the subset of *P* then,

$$X_P = x_{i_1} + x_{i_2} + \cdots + x_{i_m} \tag{4.9}$$

$$|A_P\rangle = \left|\alpha_{i_1} \otimes \alpha_{i_2} \otimes \alpha_{i_3} \ldots \otimes \alpha_{i_m}\right\rangle \tag{4.10}$$

$$|B_P\rangle = \frac{1}{\sqrt{X_p}}\left(\sqrt{x_{i_1}}\left|\beta_{i_1}\otimes\alpha_{i_2}\otimes \ldots \otimes\alpha_{i_m}\right\rangle + \cdots + \sqrt{x_{i_m}}\left|\alpha_{i_1}\otimes\alpha_{i_2}\otimes \ldots \otimes\beta_{i_m}\right\rangle\right) \quad (4.11)$$

In that case,

$$|\psi\rangle = \sqrt{x_0}\left|A_P\otimes A_Q\otimes \ldots \otimes A_Z\right\rangle + \sqrt{X_P}\left|B_P\otimes A_Q\otimes \ldots \otimes A_Z\right\rangle + \cdots$$
$$+\sqrt{X_Z}\left|A_P\otimes A_Q\otimes \ldots \otimes B_Z\right\rangle \quad (4.12)$$

which is also a *W*-type state.

Now, let us define an entanglement measure that helps us to determine whether a given party is entangled with the rest of the parties. Let us recall that the concurrence is defined as a measure of bipartite entanglement of two qubits [17]. For the state given in Eqn. 4.5, the concurrence for the entanglement of the party k with the rest is defined as,

$$C_k = 2\sqrt{det\rho^{(k)}} \quad (4.13)$$

where $\rho^{(k)}$ is the reduced density matrix for the party k which is calculated as follows

$$\rho^{(k)} = tr_k(|\psi\rangle\langle\psi|) \quad (4.14)$$

$$\rho^{(k)} = \sum_{\sigma_1\sigma_2\ldots\sigma_p}{}_1\langle\sigma_1|\otimes_2\langle\sigma_1|\otimes \ldots \otimes_p\langle\sigma_p|\left(\sqrt{x_0}\left|\alpha_1\otimes\alpha_2\otimes \ldots \otimes\alpha_p\right\rangle\right.$$
$$+\sum_i \sqrt{x_i}\left|\alpha_1\otimes\alpha_2\otimes \ldots \otimes\beta_i\otimes \ldots \otimes\alpha_p\right\rangle \sqrt{x_0}\left\langle\alpha_1\otimes\alpha_2\otimes \ldots \otimes\alpha_p\right|$$
$$+\sum_j \sqrt{x_j}\left\langle\alpha_1\otimes\alpha_2\otimes \ldots \otimes\beta_j\otimes \ldots \otimes\alpha_p\right|\left|\sigma_1\rangle\otimes_1|\sigma_1\rangle_2\otimes \ldots \otimes|\sigma_p\rangle_p \quad (4.15)$$

$$= x_0|\alpha_k\rangle\langle\alpha_k| + \sqrt{x_o x_k}|\alpha_k\rangle\langle\beta_k| + \sqrt{x_o x_k}|\beta_k\rangle\langle\alpha_k| + x_k|\beta_k\rangle\langle\beta_k|$$
$$+\sum_{i\neq k} x_i|\alpha_k\rangle\langle\alpha_k| \quad (4.16)$$

The matrix representation of $\rho^{(k)}$ in the basis $\{|\alpha_k\rangle, |\beta_k\rangle\}$ is

$$\rho^{(k)} = \begin{pmatrix} 1-x_k & \sqrt{x_o x_k} \\ \sqrt{x_o x_k} & x_k \end{pmatrix} \quad (4.17)$$

And therefore the concurrence is

$$C_k = 2\sqrt{x_k(1-x_0-x_k)} \quad (4.18)$$

The definition of the concurrence can be generalized for a subset P of all parties. It is easy to see that the reduced density matrix $\rho^{(P)}$ has the matrix rank 2. This means that the bipartite entanglement between P and the rest has Schmidt rank 2. The concurrence for P and the rest is then,

$$C_P = 2\sqrt{x_P(1 - x_P - x_0)} \tag{4.19}$$

$$C_P = 2\sqrt{\left(x_{i_1} + x_{i_2} + \cdots + x_{i_m}\right)(1 - x_0 - x_{i_1} - x_{i_2} - \cdots - x_{i_m})} \tag{4.20}$$

Now consider the following expression for two different parties

$$D_{kl} = \tfrac{1}{8}(C_k^2 + C_l^2 - C_{kl}^2) \tag{4.21}$$

It is then easy to see that $D_{kl} = x_k x_l$. These quantities, therefore, enable us to compute the x_k parameters from the concurrences.

Suppose that there are at least three parties with non-zero concurrences (C_r, C_s and C_t). It implies that x_r, x_s and x_t are also non-zero. Therefore for any party k with $k \neq r.s$ we have

$$x_k = \sqrt{\tfrac{D_{kr} D_{ks}}{D_{rs}}} \tag{4.22}$$

which implies that x_k is unique. In other words, for every W-type state $|\psi\rangle$, we can find only one set of values $x_0, x_1, \dots . x_p$

To summarize, let x denote the p-tuple $\boldsymbol{x} = \left(x_1, x_2, \dots, x_p\right)$ and $x_0(\boldsymbol{x})$ denote the function

$$x_0(\boldsymbol{x}) = 1 - \sum_{\mathrm{i}=1}^{\mathrm{p}} x_{\mathrm{i}} \tag{4.23}$$

Every W-type state can be represented by a point $\boldsymbol{x}$. This point is unique if at least three components of $\boldsymbol{x}$ is non-zero. In that case the state is truly multipartite. If at most two components of the point $\boldsymbol{x}$ is non-zero, then the point is not unique. That is there can be other points **y** such that, $\boldsymbol{x}$ and $\boldsymbol{y}$ represent the same state. We say that these points are equivalent if that happens.

Consider the case where only two components of $\boldsymbol{x}$ is non-zero (x_r and x_s). $\boldsymbol{x} \sim \mathbf{y}$ if and only if $y_k = 0$ for all $k \neq r, s$ and $2\sqrt{x_r x_s} = 2\sqrt{y_r y_s}$. In that case, parties *r* and *s* are bipartite entangled with concurrence being $C = 2\sqrt{x_r x_s}$

If only one component of $\boldsymbol{x}$ is non-zero, or if $\boldsymbol{x} = 0$ then, $\boldsymbol{x} \sim \mathbf{y}$ if and only if $\mathbf{y}$ has the same feature**.**

If three or more components of $\boldsymbol{x}$ is non-zero then $\boldsymbol{x} \sim \mathbf{y}$ if and only if $\boldsymbol{x} = \mathbf{y}$

LU equivalence of two states is very important in studying entanglement transformations. Two states are LU equivalent to each other if and only if there exists local unitary operators such that

$$|\psi\rangle = (U_1 \otimes U_2 \ldots \otimes U_p)|\phi\rangle \tag{4.24}$$

Bennett *et al.* [21] have shown that if $|\psi\rangle$ and $|\phi\rangle$ can be converted into each other by LOCC, then they are LU equivalent to each other. If these are W-type states represented by the points $\boldsymbol{x}$ and $\boldsymbol{y}$, then $|\psi\rangle$ and $|\phi\rangle$ are LU equivalent to each other if and only if $\boldsymbol{x} \sim \mathbf{y}$**.** Let us define $|\Phi(\boldsymbol{x})\rangle$ as,

$$|\Phi(\boldsymbol{x})\rangle = \sqrt{x_0(\boldsymbol{x})}|000 \ldots 0\rangle + \sqrt{x_1}|100 \ldots 0\rangle + \cdots \sqrt{x_p}|000 \ldots 1\rangle \tag{4.25}$$

This represents a W-type state with parameter $\mathbf{x}$.$|\Phi(\boldsymbol{x})\rangle$ and $|\Phi(\boldsymbol{y})\rangle$ are LU equivalent if and only if $\boldsymbol{x} \sim \mathbf{y}$. Also any state $|\psi\rangle$ with parameter $\boldsymbol{x}$ is LU equivalent to $|\Phi(\vec{\boldsymbol{x}})\rangle$.

4.2 Local Operations by One Party

Consider a *W*-type state as the one given in Eqn. 4.5. Suppose that the *kth* party has carried out a measurement on his particle. Let this measurement be described by the (generalized) measurement operators M_l where *l* indicates an outcome. Obviously, these operators satisfy $\sum_l M_l^\dagger M_l = 1$. Let us define the following quantities for each outcome *l*.

$$A_l = \| M_l |\alpha_k\rangle \| \tag{4.26}$$

$$|\alpha_k^{(l)}\rangle = \frac{1}{A_l} M_l |\alpha_k\rangle \tag{4.27}$$

$$D_l = \langle \alpha_k | M_l^\dagger M_l | \beta_k \rangle \tag{4.28}$$

$$E_l = \left\| M_l|\beta_k\rangle - D_l|\alpha_k^{(l)}\rangle \right\| \tag{4.29}$$

$$|\beta_k^{(l)}\rangle = \frac{1}{E_l}\left(M_l|\beta_k\rangle - D_l|\alpha_k^{(l)}\rangle\right) \tag{4.30}$$

where $|\alpha_k^{(l)}\rangle$ and $|\beta_k^{(l)}\rangle$ form a new orthonormal basis and A_l and E_l are real numbers. But in general D_l is a complex number. The measurement operators can be expressed as,

$$M_l|\alpha_k\rangle = A_l|\alpha_k^{(l)}\rangle \tag{4.31}$$

$$M_l|\beta_k\rangle = D_l|\alpha_k^{(l)}\rangle + E_l|\beta_k^{(l)}\rangle \tag{4.32}$$

In that case, the measurement operator is defined as ,

$$M_l = A_l\left|\alpha_k^{(l)}\right\rangle\langle\alpha_k| + \mathrm{D_l}\left|\alpha_k^{(l)}\right\rangle\langle\beta_k| + E_l|\beta_k^{(l)}\rangle\langle\beta_k| \tag{4.33}$$

The relationship between A_l, D_l and E_l can be found from the relation$\sum_l M_l^\dagger M_l = I$ satisfied by generalized measurement operators. They satisfy the following three relations.

$$\sum_l A_l^2 = 1 \tag{4.34}$$

$$\sum_l E_l^2 + |D_l|^2 = 1 \tag{4.35}$$

$$\sum_l A_l D_l = 0 \tag{4.36}$$

Suppose that the initial state has parameter point $\boldsymbol{x}$**.** Let $\boldsymbol{x}^{(l)}$ be the final state`s parameter vector when the outcome l is obtained. It is convenient to express the final states and their respective probabilities in terms of three sets of parameters s_l, s_l' and p_l such that

$$x_i^{(l)} = \begin{cases} s_l x_i & i \neq k \\ s_l' x_i & i = k \end{cases} \tag{4.37}$$

$$x_0^{(l)} = 1 - s_l(1 - x_0 - x_k) - s_l' x_k \geq 0 \tag{4.38}$$

where p_l is the probability outcome of l and defined as in Eqn. 2.2

$$p_l = A_l^2 x_0 + |D_l|^2 x_k + E_l^2 x_k + \sqrt{x_k x_0}(A_l D_l + A_l D_l^*) + A_l^2 \sum_{i\neq k} x_i \tag{4.39}$$

$$s_l = \frac{A_l^2}{p_l}, \tag{4.40}$$

$$s_l' = \frac{E_l^2}{p_l}. \tag{4.41}$$

It is easy to see that these three new parameters satisfy the following relations.

$$\sum_l s_l\, p_l = \sum_l \frac{A_l^2}{p_l} p_l = 1, \tag{4.42}$$

$$\sum_l p_l \sqrt{s_l x_0^{(l)}} \geq x_0, \tag{4.43}$$

The proof of Eqn. 4.43 is as follows: The measurement acting on a state given in Eqn. 4.5 results in,

$$\frac{1}{\sqrt{p_l}}(A_l\sqrt{x_0}\left|\alpha_1 \otimes \alpha_2 \ldots \otimes \alpha_p\right\rangle + A_l \sum_{i \neq k} \sqrt{x_i}\left|\alpha_1 \otimes \alpha_2 \ldots \beta_i \ldots \otimes \alpha_p\right\rangle$$

$$+ E_l\sqrt{x_0}\left|\alpha_1 \otimes \alpha_2 \ldots \beta_i \ldots \otimes \alpha_p\right\rangle + D_l\sqrt{x_k}\left|\alpha_1 \otimes \alpha_2 \ldots \otimes \alpha_p\right\rangle) \tag{4.44}$$

Thus, we obtain the following relation:

$$\sqrt{x_0^{(l)}} = \left|\frac{1}{\sqrt{p_l}}(A_l\sqrt{x_0} + D_l\sqrt{x_k})\right| \tag{4.45}$$

and hence,

$$\sqrt{x_0^{(l)}} \geq \frac{1}{\sqrt{p_l}}\left(A_l\sqrt{x_0} + Re(D_l)\sqrt{x_k}\right). \tag{4.46}$$

Therefore we get,

$$\sum_l p_l \sqrt{s_l x_0^{(l)}} \geq \sum_l A_l^2 \sqrt{x_0} + \sum_l A_l (Re) D_l \sqrt{x_k} = \sqrt{x_0} \tag{4.47}$$

Now, we will prove the opposite. Let s_l, s_l' and p_l be a set of non negative numbers satisfying the following conditions:

i. $$\sum_l p_l = 1 \tag{4.48}$$

ii. $$\sum_l p_l\, s_l = 1 \tag{4.49}$$

iii. $$\sum_l p_l\, s_l' \leq 1 \tag{4.50}$$

iv. $$\sum_l p_l \sqrt{s_l x_0^{(l)}} \geq \sqrt{x_0} \tag{4.51}$$

v. $$x_0^{(l)} = 1 - s_l(1 - x_0 - x_k) - s_l' x_k \geq 0 \tag{4.52}$$

Then, there is a local operation which can be done by the party k such that the final state $\boldsymbol{x}^{(l)}$ is obtained with the probability p_l.

The proof can be carried out as follows: First, consider the inequality $\sum_l p_l \sqrt{s_l x_0^{(l)}} \geq x_0$. This is a generalized triangle inequality, hence, we can find the phase angles θ_l such that,

$$\sum_l p_l \sqrt{s_l x_0^{(l)}} e^{i\theta_l} = \sqrt{x_0} \tag{4.53}$$

Define:

$$A_l = \sqrt{p_l s_l} \tag{4.54}$$

$$E_l = \sqrt{p_l s_l'} \tag{4.55}$$

$$D_l = \frac{1}{\sqrt{x_k}} \left(\sqrt{p_l s_l x_0} - \sqrt{p_l x_0^{(l)}} e^{i\theta_l}\right) \tag{4.56}$$

Now, it is easy to check that,

$$\sum_l A_l^2 = \sum_l p_l\, s_l = 1 \tag{4.57}$$

$$\sum_l E_l^2 + |D_l|^2 = \sum_l p_l s_l' + \frac{1}{\sqrt{x_k}} \sum_l \left[p_l s_l x_0 + p_l x_0^{(l)} - 2 p_l \sqrt{s_l x_0^{(l)}} \sqrt{x_0} cos\theta_l\right] \tag{4.58}$$

$$= \sum_l p_l\, s_l' + \frac{1}{x_k} \left[x_0 + \sum_l p_l x_0^{(l)} - 2x_0\right] \tag{4.59}$$

$$= \frac{1}{x_k} \left[x_0 + (x_0 + x_k) - 2x_0\right] = 1 \tag{4.60}$$

$$\sum_l A_l D_l = \frac{1}{x_k} \sum_l \left[p_l s_l \sqrt{x_0} - p_l \sqrt{s_l x_0^{(l)}} e^{i\theta_l}\right] \tag{4.61}$$

$$= \frac{1}{x_k} \left[\sqrt{x_0} - \sqrt{x_0}\right] = 0 \tag{4.62}$$

Hence, $M_l = A_l |\alpha_k^{(l)}\rangle\langle\alpha_k| + \mathrm{D_l} |\alpha_k^{(l)}\rangle\langle\beta_k| + E_l |\beta_k^{(l)}\rangle\langle\beta_k|$ is a set of measurement operators satisfying $\sum_l M_l^\dagger M_l = 1$. If the kth party carries out this measurement, the probability of l^{th} outcome is

$$p(l) = A_l^2 x_0 + |D_l|^2 x_k + E_l^2 x_k + \sqrt{x_k x_0}(A_l D_l + A_l D_l^*) + (A_l^2) \sum_{i \neq k} x_i \quad (4.63)$$

Finally the parameters of the final state are $\boldsymbol{x}^{(l)}$where, $x_i^{(l)} = s_l x_i$ for $i \neq k$ and $x_i^{(l)} = s_l' x_i$ for $i = k$.

4.3 Deterministic W-Type State Transformations

Let $\boldsymbol{x}$ be the parameter vector for a multipartite, and $\boldsymbol{y}$ be the parameter vector for any non product *W*-type state. The following statement describes the necessary and sufficient conditions for deterministic LOCC transformation.

Theorem 4.1:

If $\boldsymbol{y}$ is also multipartite, then $\boldsymbol{x}$ can be LOCC converted to $\boldsymbol{y}$ if and only if ($x_i \geq y_i$ for all i =1,2, ... p).

Proof: We first start with proof of the necessity for the first measurement performed on the state. From the Eqn.s 4.49 and 4.50 the, averages of parameters can be written as,

$$\sum_l x_i s_l p_l = x_i \qquad \text{for } i \neq k \quad (4.64)$$

$$\sum_l x_i s_l' p_l \leq x_i \qquad \text{for } i = k \quad (4.65)$$

If both of the equations are combined together, one can write the following statement:

$$\sum_l x_i^{(l)} p_l \leq x_i \quad (4.66)$$

After the measurement, the parameter x_i is converted to $x_i^{(l)}$. For the second measurement,

$$\sum_t p(t|l)\, s_t^{(2,l)} = 1 \quad (4.67)$$

$$\sum_t p(t|l)\, s_t^{(2,l)'} \leq 1 \quad (4.68)$$

$p(t|l)$ means that the probability of finding *t* after the measurement results *l*. $s_t^{(2,l)}$ stands for the outcome of the second measurement.

$$\sum_t p(t|l)\, x_i^{(l)} s_t^{(2,l)} \le x_i^{(l)} \tag{4.69}$$

$$\sum_t p(t|l)\, x_i^{(lt)} \le x_i^{(l)} \tag{4.70}$$

Combining the Eqn.s 4.69 and 4.70, we get

$$\sum_l \sum_t p(t|l)\, p_l x_i^{(lt)} \le \sum_l x_i^{(l)} p_l \le x_i. \tag{4.71}$$

Performing repeated measurements on the system many times, one obtain

$$y_i = \sum_k \sum_m \dots \sum_l \sum_t p(k|m \dots tl) \dots p(t|l) p(l) x_i^{(lt\dots mk)} \le \dots \le \sum_l \sum_t p(t|l)\, p_l x_i^{(lt)}$$
$$\le \sum_l x_i^{(l)} p_l \le x_i \tag{4.72}$$

Where we have used the fact that $x_i^{(lt\dots mk)} = y_i$.

Therefore, $x_i \ge y_i$ is satisfied. This completes the proof of necessity. It is enough for us to show that only one component of vector $\vec{\boldsymbol{x}}$ can be decreased without changing the other parameters. In other words if $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{y}}$ are vectors such that, $x_1 \ge y_1$ and, $x_2 = y_2$, $x_3 = y_3, \dots$, $x_p = y_p$ then, we will show that $|\Phi(\vec{x})\rangle$ can be transformed to $|\Phi(\vec{y})\rangle$, by LOCC by a simple measurement by the first party.

Consider a two outcome measurement with l = 1,2. The parameters satisfy,

$$y_1 = s_l' x_1 \tag{4.73}$$

$$y_j = s_l x_j \tag{4.74}$$

Hence,

$$E_1 = \sqrt{p_1 \frac{y_1}{x_1}}, E_2 = \sqrt{p_2 \frac{y_1}{x_1}} \tag{4.75}$$

$$A_1 = \sqrt{p_1}, A_2 = \sqrt{p_2} \tag{4.76}$$

From the given Eqn.s 4.35 and 4.36,

$$D_1 = -\sqrt{\frac{p_2}{x_1}(x_1 - y_1)} \tag{4.77}$$

$$D_2 = \sqrt{\frac{p_1}{x_1}(x_1 - y_1)} \tag{4.78}$$

It is convenient to choose $D_l,$ as real numbers. The measurement operators are,

$$M_1 = \sqrt{p_1}|\alpha_k^{(l)}\rangle\langle\alpha_k| + \sqrt{p_1\frac{y_1}{x_1}}|\beta_k^{(l)}\rangle\langle\beta_k| - \sqrt{\frac{p_2}{x_1}(x_1 - y_1)}|\alpha_k^{(l)}\rangle\langle\beta_k| \quad (4.79)$$

$$M_2 = \sqrt{p_2}|\alpha_k^{(l)}\rangle\langle\alpha_k| + \sqrt{p_2\frac{y_1}{x_1}}|\beta_k^{(l)}\rangle\langle\beta_k| + \sqrt{\frac{p_1}{x_1}(x_1 - y_1)}|\alpha_k^{(l)}\rangle\langle\beta_k| \quad (4.80)$$

After some algebra, one can shows that

$$M_1^{\dagger}M_1 + M_2^{\dagger}M_2 = I \quad (4.81)$$

Moreover, it is also easy to see that both final states are LU equivalent to $|\Phi(\vec{y})\rangle$.

The condition given in theorem 4.1, stands for multipartite target state. If the target state is bipartite, the situation is more complicated. Because, the bipartite states do not have a unique representation. The theorem given below defines the necessary and sufficient conditions for transforming a multipartite state to a bipartite one.

Theorem 4.2:

If $\boldsymbol{y}$ is bipartite, then $\boldsymbol{x}$ can be LOCC converted to $\boldsymbol{y}$ if and only if there is a $\mathbf{z}\sim \boldsymbol{y}$ such that $x_i \geq z_i$ for all $i = 1,2 \ldots p$

Proof: We only need to show the necessity, because the proof of sufficiency proof also follows the same line of reasoning used in the proof of theorem 4.1. For necessity, suppose that, $|\Phi(\vec{y})\rangle$ is a bipartite entangled state between parties 1 and 2 with concurrence C. Hence we have,

$$y_3 = y_4 = \cdots y_p = 0 \quad (4.82)$$

and,

$$C = 2\sqrt{y_1 y_2} \quad (4.83)$$

Let $\vec{x}^{(L)}$ denote the *Lth* result when protocol is completed. L denotes the collection of all outcomes from all measurements done until the protocol is finalized. Since all of these are LU equivalent to $|\Phi(\vec{y})\rangle$ we have,

$$x_3^{(L)} = x_4^{(L)} = \cdots x_p^{(L)} = 0 \quad (4.84)$$

and,

$$C = 2\sqrt{x_1^{(L)} x_2^{(L)}} \quad (4.85)$$

Consider the averages,

$$\bar{x}_1 = \sum_L p_L x_1^{(L)} \leq x_1 \tag{4.86}$$

$$\bar{x}_2 = \sum_L p_L x_2^{(L)} \leq x_2 \tag{4.87}$$

where the inequalities that are used here, have been already proven above, in Eqn. 4.66. By Schwarz inequality,

$$\sqrt{\bar{x}_1 \bar{x}_2} = \sqrt{\sum_L p_L x_1^{(L)}} \sqrt{\sum_L p_L x_2^{(L)}} \geq \sum_L p_L \sqrt{x_1^{(L)} x_2^{(L)}} = \frac{1}{2} C \tag{4.88}$$

therefore, we have,

$$2\sqrt{\bar{x}_1 \bar{x}_2} \geq C \tag{4.89}$$

Let $z_1 = \bar{x}_1$, then z_2 be chosen such that,

$$C = 2\sqrt{z_1 z_2} \tag{4.90}$$

and,

$$z_3 = z_4 = \cdots z_p = 0 \tag{4.91}$$

Hence, we have the proof of theorem 4.2 is completed since, $z \sim y$

$$z_1 = \bar{x}_1 \leq x_1, \tag{4.92}$$

$$z_2 \leq \bar{x}_2 \leq x_2. \tag{4.93}$$

CHAPTER 5

CONCLUSION

Although, there have been achievements for transforming a given state to another, some of the problems in this field have not been solved yet. The main difficulty comes from the nature of given and desired states. To overcome these difficulties, different strategies have been carried out. In this thesis, we have reviewed the some of the results in the field. We have also presented the necessary and sufficient conditions for transforming W-type states deterministically. This is the original part of this thesis.

The problem of the transformation of bipartite pure states has been solved for both deterministic and probabilistic cases. These accomplishments are due to the fact that, bipartite pure states can be represented by Schmidt decomposition which has a very important role in these achievements. We presented the transformation rules for bipartite pure states. In Chapter 3, for deterministic transformations, the conditions can be expressed in terms of the linear algebraic majorization theory. Nielsen [5] has shown that under LOCC the given state can be transformed to the desired state, if and only if the Schmidt coefficients of the given state are majorized by the coefficients of the target state. To understand this theory better, some mathematical tools (majorization, Horn theorem, etc.) must be known. The deterministic transformation protocol is simplified by Jensen and Schack [20]. Their algorithm basically contains permutation matrices which are special type unitary operators that appear in the decomposition of the doubly stochastic matrix that appear in the majorization relation. These permutation matrices are used in constructing the measurement operators. In some cases a given bipartite pure states cannot be transformed deterministically into each other but, it is possible to find a suitable entangled state which is coupled to the given as well as another state such that the transformation can be accomplished [8]. These types of states are called catalysts, because the state of the particles of catalyst does not change during the transformation.

Another case of interest is the probabilistic transformations. This time there are several target states and our aim is to transform the given state to the target states with given probabilities. Jonathan and Plenio [7] have found the necessary and sufficient conditions for accomplishing this task. Vidal has computed the maximum probability [6]. Again the necessary and sufficient conditions can be expressed in terms of the majorization relations.

The main part of this thesis is to find the necessary and sufficient conditions for transforming a given W-type state to another deterministically by LOCC. As indicated above, this is the original part of this thesis. One of the difficulties in approaching that problem is that the W-type states do not have a suitable representation. In order to overcome this problem it is possible to choose new orthonormal bases and represent the state by them. This is called state parameterization. Every W-type state can be represented by a p-component vector $\vec{\boldsymbol{x}}$ where p is the number of parties. The point $\vec{\boldsymbol{x}}$ is unique if at least three components of its components are nonzero. The LU equivalence condition between these states can be expressed in a simple way in terms of the $\vec{\boldsymbol{x}}$ parameter.

The effect of the local measurements on the parameter vector is found and two theorems that describe the deterministic transformations of W-type states are proven in Chapter 4. The central result in these theorems is this: Any party can decrease its own parameter without changing the others. Moreover, any deterministic transformation can be achieved by this reason.

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