CONVERGENCE OF LOTZ-RÄBIGER NETS ON BANACH SPACES

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ABSTRACT

CONVERGENCE OF LOTZ-RÄBIGER NETS ON BANACH SPACES

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The concept of *LR*-nets was introduced and investigated firstly by H.P. Lotz in [27] and by F. Räbiger in [30]. Therefore we call such nets Lotz-Räbiger nets, shortly *LR*-nets. In this thesis we treat two problems on asymptotic behavior of these operator nets.

First problem is to generalize well known theorems for Cesàro averages of a single operator to LR-nets, namely to generalize the Eberlein and Sine theorems. The second problem is related to the strong convergence of Markov LR-nets on L^1 -spaces. We prove that the existence of a lower-bound functions is necessary and sufficient for asymptotic stability of LR-nets of Markov operators.

Keywords: Banach space, operator net, *LR*-net, Markov operator, strong convergence, mean ergodicity, asymptotic stability, lower bound function, attractor.

BANACH UZAYLARI ÜZERİNDEKİ LOTZ-RÄBİGER NETLERİNİN YAKINSAKLIĞI

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LR-ağ kavramı ilk olarak H. P. Lotz [27] ve F. Räbiger [30] tarafından tanımlanarak incelenmiştir. Bu nedenle biz bu özel ağları Lotz-Räbiger ağları ya da kısaca *LR*-ağları olarak adlandıracağız. Bu tezde *LR*-ağları olarak adlandırılan özel operatör ağların asimtotik özellikleri ile ilgili iki problem ele alınmaktadır.

İlk problemimiz bazı özel koşulları sağlayan bir operatörün Cesàro ortalaması olarak bilinen operatör ağları için verilmiş olan teoremlerin genişletilmesidir. Diğer bir deyişle Eberlein ve Sine teoremleri LR-ağlarına genişletilmiştir. İkinci problem L^1 -uzayı üzerinde Markov operatörlerinden oluşan LR-ağlarının kuvvetli yakınsaması ile ilgilidir. Daha sonra alt sınır fonksiyonunun varlığının bu özel ağların asimtotik kararlılığı için yeterli ve gerekli olduğu ispatlanmıştır.

Anahtar Kelimeler: Banach uzayı, operatör netleri, *LR*-net, Markov operatörü, kuvvetli yakınsaklık, ortalama ergodiklik, asimptotik kararlı-lık, alt sınır fonksiyonu, çekici. To My Family

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CHAPTER 1

INTRODUCTION

The notion of Cesaro averages of one-parameter semigroups is one of the oldest and extensively studied topic in the theory of operators in Banach spaces. Especially, the asymptotic behavior is one of the most important concepts in many disciplines. For example in PDE and in dynamical system theories they have many useful and remarkable applications. In the literature, there are dozens of textbooks and papers directly related to asymptotic behavior of them. Of course it is not possible to cover all references. In the thesis we consider the asymptotic behavior of new operator nets which is an *LR*-net. Those nets are generalization of one-parameter semigroups of Cesàro averages. The scope of the study is presented in the first chapter.

Second chapter present the general background which is needed in the thesis. In the first section, we deal with operator nets, operator semigroups on a Banach space X and operator net convergence. Mean ergodicity of the semigroup of operators is mentioned in the second section. Some of examples and some important theorems such as Mean Ergodic Theorem and Eberlein Theorem are given. The backbone of the thesis is based on Markov operator. We show some examples and results about Markov operator in the third section. The other significant issue is Lasota lower-bound criterion. Therefore we introduce the definition of a lower-bound function and asymptotic stability in the fourth section. This section is completed with Lasota criterion and its proof. In the remainder of this chapter, we present an extension of Sine's counterexample. On arbitrary Banach space if T^m is mean ergodic for some $m \in \mathbb{N}$ then T is mean ergodic. The converse is not true in general. But for reflexive Banach space, statement is that T^m is mean ergodic for some $m \in \mathbb{N}$ if and only if T is mean ergodic. R. Sine has constructed a positive isometry in a C(K)-space where K is compact Hausdorff space

such that T is mean ergodic but T^2 is not. We extend Sine's construction to p^{th} power of T where p is prime. This result is published in [14]

The main notion of the thesis is a special operator net, Lotz-Räbiger net or briefly LR-net. Therefore we mention the definition and basic result about LR-nets in the first section of **Chapter 3**. We give several examples which are related with different areas of mathematics in the second part of **Chapter 3**

Fourth chapter is focused on the generalization of Eberlein and Sine Theorems for *LR*-nets. We extend the Eberlein theorem which is known for more than sixty years for Cesaro averages of single operators and \mathcal{T} -ergodic nets to *LR*-nets. For further application of the convergence, the concept of attractor is important. So the last part of this section deals with this concept. At the end of the chapter, Sine theorem is extended to *LR*-nets. The results of Chapter 4 are published in [13]

Fifth chapter is the main part of the thesis. We discuss the strong convergence for Markov operator LR-nets on L^1 -spaces. The main result of the chapter stated in the first section given by Emelyanov and Wolff for single Markov operator on L^1 -spaces [11]. The extension of this theorem gives a relation between asymptotic stability and strong convergence of LR-nets with finite dimensional fixed space. In the second section, we give the asymptotic stability and lower-bound function definitions which are motivated be classical definitions to LR-nets. Classical definitions are discussed in the fourth section of Chapter 2. The rest of the section, the relation between lower-bound function and asymptotic stability of Markov LR-nets on L^1 -space are discussed and the section ends with the proposition giving an example of Markov LR-nets which need not be \mathcal{T} -ergodic nets. In the third section Lasota criterion for abelian Markov semigroups are studied. The results are published in [15].

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CHAPTER 2

PRELIMINARIES

For the convenience of the reader, we present in this chapter the general background needed in the thesis and we give some basic structural properties.

2.1 Operator nets and convergence

Let X be a Banach space. We denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators in X. In addition to the norm topology, we also consider the strong and weak operator topologies on $\mathcal{L}(X)$. Because the subject of this thesis is *LR*-nets, which depend on operator nets, firstly we mention operator nets and operator semigroups on X.

Definition 1 A family $\Theta = (T_{\lambda})_{\lambda \in \Lambda} \subseteq \mathcal{L}(X)$ indexed by a directed set $\Lambda = (\Lambda, \prec)$ is called an operator net where the directed set Λ is a nonempty set together with a reflexive and transitive binary relation with the additional property that every pair of elements has an upper bound.

In the following Λ always represents a directed set.

Definition 2 A nonempty subset $\mathcal{A} \subseteq \mathcal{L}(X)$ is called a semigroup if

$$T, S \in \mathcal{A} \Rightarrow T \circ S \in \mathcal{A} \ (\forall T, S \in \mathcal{A}).$$

A semigroup *A* is called abelian if

$$T \circ S = S \circ T \quad (\forall T, S \in \mathcal{A}).$$

If a semigroup has an a unit and every element is invertible, the semigroup is a group. Generally operator semigroups are indexed by non-negative integers or non-negative reals and are called one-parameter semigroups. Obviously, any one-parameter semigroup is abelian.

The following definition is about the norm-convergence of nets in Banach spaces.

Definition 3 Let X be a Banach space. A net $(z_{\lambda})_{\lambda}$ in X converges to $z_0 \in X$ in the norm if for every $\epsilon > 0$ there exists $\lambda_0 \in \Lambda$ such that $\forall \lambda \ge \lambda_0$ we get $||z_{\lambda} - z_0|| < \epsilon$. We use for the convergence in the norm $\lim_{\lambda \to \infty} ||z_{\lambda} - z_0|| = 0$ or simply the following symbol $||\cdot -|| \lim_{\lambda \to \infty} z_{\lambda} = z_0$.

The following two definitions are about special conditions on semigroups.

Definition 4 A one-parameter semigroup \mathcal{T} indexed by \mathbb{R}_+ is called strongly continuous if

$$\lim_{t \to 0} ||T_{s+t}x - T_sx|| = 0 \ (\forall s \ge 0, \ x \in X).$$

Hereafter we use the notation $\mathcal{T} = (T_t)_{t \in J}$ where $J = \mathbb{R}_+$ for a one-parameter semigroup in the continuous-parameter case and $(T^n)_{n=1}^{\infty}$ for the discrete semigroup, generated by a single operator T.

Definition 5 If a continuous-parameter semigroup \mathcal{T} satisfies $T_0 = I$ where I is the identity operator in X, it is called it a C_0 -semigroup.

The following definition is due to Eberlein [6], who firstly gave the definition under the name a system of almost invariant integrals for semigroup \mathcal{T} in 1949.

Definition 6 Let \mathcal{T} be a semigroup of $(\mathcal{L}(X), \circ)$. A uniformly bounded net $C = (C_{\lambda})_{\lambda \in *}$ of operators in X is called \mathcal{T} -ergodic net for the semigroup \mathcal{T} if

- (E1) $C_{\lambda}x$ is in the closed convex hull of $\mathcal{T}(x)$ for every $x \in X$ and every $\lambda \in \Lambda$,
- (E2) $\lim_{\lambda} ||C_{\lambda}(I-T)x|| = 0$ for all $x \in X$ and all $T \in \mathcal{T}$
- (E3) $\lim_{\lambda} ||(I T)C_{\lambda}x|| = 0$ for all $x \in X$ and all $T \in \mathcal{T}$

We refer to [18] for basic examples of \mathcal{T} -ergodic nets.

Example 7 Every uniformly bounded Abelian operator semigroup \mathcal{T} admits a \mathcal{T} -ergodic net. Take $\Lambda = co\mathcal{T}$ where $co\mathcal{T}$ is the convex hull of \mathcal{T} and $A_{\lambda} = \lambda$. Define a partial order in Λ by $T \ge S$ if and only if there exists an $R \in \Lambda$ with T = RS. It follows from RS = SR that Λ is a directed set and \mathcal{T} is \mathcal{T} ergodic net.

Proof: Firstly, we prove that $\Lambda = coT$ and $A_{\lambda} = \lambda$ is a directed set. Since the operator net is abelian, reflexivity is obvious. If $T \ge S$ and $S \ge K$ then there exists R_1 such that $T = R_1S$ and there exists R_2 such that $S = R_2K$. Therefore with $R = R_1R_2$, we obtain T = RK. Thus by the definition of partial order we get $T \ge K$ so transitivity holds. Finally given S and R are in Λ , consider the operator T = RS. By commutativity $T \ge S$ and $T \ge R$.

In the second step, we show $A_{\lambda} = \lambda \in \Lambda$ is a \mathcal{T} -ergodic net. (E1) is directly satisfied by definition of $A_{\lambda} = \lambda$. Consider A_{λ} and $A_n(T) = \frac{1}{n} \sum_{k=0}^{n-1} T^k$. For large enough λ the inequality $A_{\lambda} \ge A_n(T)$ holds. So there exists R with $A_{\lambda} = RA_n(T)$. As a consequence, we obtain

$$A_{\lambda} - A_{\lambda}T = RA_{n}(T) - RA_{n}(T)T$$
$$= R\frac{(I - T^{n})}{n}$$

and $\lim_{\lambda \to \infty} \|(A_{\lambda} - A_{\lambda}T)x\| = \lim_{n \to \infty} \left\| R \frac{(I-T^n)}{n}x \right\| = 0$ by uniform boundedness of the semigroup \mathcal{T} .

This clearly yields (E3) by commutativity. Accordingly (A_{λ}) is \mathcal{T} -ergodic net.

Example 8 Let T_1, T_2, \dots, T_d be commuting power bounded operators in X. Put $\Lambda = \mathbb{N}^d$. S is a semigroup generated by T_1, T_2, \dots, T_d . For $\lambda = (\lambda_1, \dots, \lambda_d)$ we take

$$A_{\lambda} = (\prod_{\nu=1}^{d} \lambda_{\nu})^{-1} \sum_{i_{1}=0}^{\lambda_{1}-1} \cdots \sum_{i_{d}=0}^{\lambda_{d}-1} T_{1}^{i_{1}} \cdots T_{d}^{i_{d}}$$

Then A_{λ} is S-ergodic net.

2.2 Mean Ergodic Theorem

The mean ergodic theorem mentioned above deals with the convergence of averages which bring along the orbits of elements of the space under a family of operators. In this section, we characterize the strong convergence concept needed to define mean ergodicity of operator nets and semigroups. In the meantime to understand mean ergodicity we need some definitions about Cesaro averages of operators.

Definition 9 Let \mathcal{T} be a one-parameter semigroup in $\mathcal{L}(X)$. We denote the Cesàro averages (means) of \mathcal{T} by

$$\mathcal{A}_t^{\mathcal{T}} = \mathcal{A}_n^T := \frac{1}{n} \sum_{k=0}^{n-1} T^k \quad (whenever \ \mathcal{T} = (T^n)_{n=0}^{\infty})$$

and

$$\mathcal{A}_t^{\mathcal{T}} := \frac{1}{t} \int_0^t T_s \, ds \, (whenever \, \mathcal{T} = (T_t)_{t \ge 0})$$

The integral above is taken with respect to the strong topology on $\mathcal{L}(X)$. A one-parameter semigroup \mathcal{T} in $\mathcal{L}(X)$ is called Cesàro bounded if

$$\sup_{t} \left\| \mathcal{A}_{t}^{\mathcal{T}} \right\| < \infty.$$

Then it is possible to mention about mean ergodicity of an operator semigroup.

Definition 10 A one-parameter semigroup \mathcal{T} is called mean ergodic if the norm limit $\lim_{\mathcal{T}\to\infty}\mathcal{A}_{\mathcal{T}}^{\mathcal{T}}$ exists for all $x \in X$. We call T mean ergodic whenever the semigroup $\mathcal{T} = (T^n)_{n=1}^{\infty}$ is mean ergodic.

Any mean ergodic one-parameter semigroup is *Cesàro* bounded and satisfies the condition $\lim_{t\to\infty} ||t^{-1}T_tx|| = 0$. The first is a consequence of the Uniform Boundedness Principle. The statement is that let X and Y be Banach spaces and $M \subset \mathcal{L}(X, Y)$, then M is uniformly bounded if and only if for each $x \in X$ the set $\{Tx : T \in M\}$ is a bounded subset of Y. The second basically for single operator follows from the identity

$$\frac{n+1}{n}A_{n+1} = \frac{1}{n}T^n + A_n \quad (n \in \mathbb{N}).$$

The important conditions for the mean ergodicity is firstly given by Eberlein in 1949 for \mathcal{T} ergodic nets. In 1938, F. Riesz proved it for L_p spaces and independently K. Yosida in 1938
and S. Kakutani in 1938 gave the proof for general Banach spaces. The next theorem is a
special case of results of Eberlein (cf. Krengel [18, Thm.1.1]), stated later in Theorem 14.

Theorem 11 (Mean Ergodic Theorem) Let T be a Cesàro bounded operator in a Banach space X. Then for any $x \in X$ satisfying

$$\lim_{n \to \infty} \left\| n^{-1} T_n x \right\| = 0$$

and for any $y \in X$, the following conditions are equivalent

(i) Ty = y and $y \in \overline{co}\{x, T_x, T^2x, \dots\}$; where $\overline{co}\{x, T_x, T^2x, \dots\}$ is the closed convex hull of the sequence $\{T^nx, n \in \mathbb{N}\}$

- (*ii*) $y = \lim_{n \to \infty} A_n x$;
- (*iii*) $y = w \lim_{n \to \infty} A_n x$;
- (iv) y is a weak cluster point of the sequence $(A_n x)$.

Our next purpose is to state a mean ergodic theorem for X_{me} spaces given by Yosida for single operator *T* in a Banach space *X*. Before Yosida theorem statement, we need the notations:

$$X_{me}(T) := \left\{ x \in X : \exists \lim_{n \to \infty} \mathcal{A}_n^T x \right\} \quad N(T) := (I - T)X$$

Clearly if T is Cesàro bounded, then X_{me} is a closed linear subspace of X.

Theorem 12 (Yosida) Let T be a Cesàro bounded operator in a Banach space X which satisfies $\lim_{t\to\infty} ||t^{-1}T_tx|| = 0$ for all $x \in X$. Then $X_{me}(T) = Fix(T) \oplus \overline{N(T)}$, and the operator $P: X_{me}(T) \to X$, which is defined by

$$Px := \lim_{n \to \infty} \mathcal{A}_n^T x \ (x \in X_{me}(T))$$

is a projection onto Fix(T) satisfying $P = T \circ P = P \circ T$.

Corollary 13 Let T be a Cesàro bounded operator in a Banach space X which satisfies $\lim_{t\to\infty} ||t^{-1}T_tx|| = 0$ for all $x \in X$. Then T is mean ergodic if and only if $X = Fix(T) \oplus \overline{N(T)}$

Theorem 14 (Eberlein(1949)) If \mathcal{T} is a semigroup of continuous linear operators in a Banach space X and admits a \mathcal{T} -ergodic net $\{A_{\lambda} : \lambda \in \Lambda\}$ then for any $x, y \in X$ the following conditions are equivalent:

- (*i*) Ty = y for all $T \in \mathcal{T}$ and $y \in \overline{co}\mathcal{T}(x)$;
- (*ii*) $y = \lim_{\lambda} A_{\lambda} x$;
- (*iii*) $y = w \lim_{\lambda} A_{\lambda} x$;
- (iv) *y* is a weak cluster point of $\{A_{\lambda}x : \lambda \in \Lambda\}$.

2.3 Markov Operators

Chapter 4 is based on Markov operators so in this section we introduce Markov operators on the Banach space L^1 . The theory of Markov operators is very rich. Many authors have been interested in this subject for many years. In the next section, we investigate asymptotic behavior of Markov operators. Therefore in the section, the concept of Markov operator and some of its properties is studied.

Let (Ω, Σ, μ) be a σ -finite measure space, and let $L^1 = L^1(\Omega, \Sigma, \mu)$ be the space of all real valued Lebesgue-integrable functions on (Ω, Σ, μ) equipped with the integral norm $\|\cdot\| := \|\cdot\|_1$. By $\mathcal{D} = \mathcal{D}(\Omega, \Sigma, \mu)$ we denote the set of all *densities* on Ω , that is

$$\mathcal{D} = \{ f \in L^1 : f \ge 0, \|f\| = 1 \},\$$

and denote $L_0^1 := \{f \in L^1 : ||f_+|| = ||f_-||\}$. A linear operator $T : L^1 \to L^1$ is called a *Markov operator* if $T(\mathcal{D}) \subseteq \mathcal{D}$.

Now we give some examples of Markov operators and Markov semigroups [31].

Example 15 Markov Operators

1.Frobenius-Perron operator. Let (X, Σ, m) be a σ -finite measure space and let S be a transformation of X. If a measure μ describes the distribution of points in the space X, then the measure ν given by $\nu(A) = \mu(S^{-1}(A))$ describes the distribution of points after S.

Assume that S is non-singular, that is, if m(A) = 0 then $m(S^{-1}(A)) = 0$. If μ is absolutely continuous with respect to m, then v is also absolutely continuous. μ is absolutely continuous with respect to m, ($\mu \ll m$) means if $\mu(E) = 0$ for every $E \in \Sigma$ for which m(E) = 0. Therefore $\nu(E) = \mu(S^{-1}(A)) = 0$.

If f is the density of μ and if g is the density of ν then we define the operator P_S by $P_S f = g$. This operator can be extended to a linear operator $P_S : L^1 \to L^1$. Hence we obtain a Markov operator which is called the Frobenius-Perron operator for the transformation S.

2.Iterated function system. Let S_1, \dots, S_n be non-singular transformations of the space X. Let P_1, \dots, P_n be the Frobenius-Perron operators corresponding to the transformations S_1, \dots, S_n . Let $p_1(x), \dots, p_n(x)$ be non-negative measurable functions defined on X such that $p_1(x) + \dots + p_n(x) = 1$ for all $x \in X$. Let x be a point in X. We choose a transformation S_i with probability $p_i(x)$ and $S_i(x)$ describes the position of x after the action of the system.

The evolution of densities of the distribution is described by the Markov operator

$$Pf = \sum_{i=1}^{n} P_i(p_i f).$$

3. *Integral operator.* If $k : X \otimes X \rightarrow [0, \infty)$ is a measurable function such that

$$\int_X k(x, y)m(dx) = 1$$

for each $y \in X$, then

$$Pf(x) = \int_X k(x, y)f(y)m(dy)$$

is a Markov operator.

Example 16 Markov Semigroups

1. Fokker-Planck equation. In the d-dimensional space \mathbb{R}^d the Fokker-Planck equation has the form

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{d} \frac{\partial^2 (a_{ij}(x)u)}{\partial x_i \partial x_j} - \sum_{i=1}^{d} \frac{\partial (b_i(x)u)}{\partial x_i}, \quad u(x,0) = v(x).$$

We assume that the functions a_{ij} and b_i are sufficiently smooth and

$$\sum_{i,j=1}^{d} a_{ij}(x)\lambda_i\lambda_j \ge \alpha \, |\lambda|^2$$

for some $\alpha > 0$ and every $\lambda \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$. The solution of this equation describes the distribution of a diffusion process. This equation generates a Markov semigroup given by P(t)v(x) = u(x, t), where v(x) = u(x, 0).

2. Liouville equation. If we assume that $a_{ij} \equiv 0$ in above example, then we obtain the Liouville equation

$$\frac{\partial u}{\partial t} = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i} (b_i(x)u)$$

As in the previous example, this equation generates a Markov semigroup given by P(t)v(x) = u(x, t), where v(x) = u(x, 0).

Markov operators have several properties that we will use in the following sections. First, if $f, g \in L^1$, then for any Markov operator T, we get

$$Tf(x) \ge Tg(x)$$
 whenever $f(x) \ge g(x)$ (1)

An operator T satisfying (1) is said to be monotonic. Thus Markov operators are monotonic.

Proposition 17 Let (Ω, Σ, μ) be a σ -finite measure space and T be a Markov operator. Then, for every $f \in L^1$,

- (i) $(Tf)^+ \leq T(f^+);$
- (*ii*) $(Tf)^{-} \leq T(f^{-});$
- (*iii*) $|Tf| \le T |f|$;

(*iv*)
$$||Tf|| \le ||f||$$
.

Markov operators satisfy the following inequalities, needed in Chapter 4. For the proofs, see [22]. If any operator T satisfies it then it is called a contraction. Since Markov operator is a contraction, it has the stability property of iterates. For any $f \in L^1$

$$||T^n f|| = ||T(T^{n-1}f)|| \le ||T^{n-1}f||$$

and thus for any two different elements f_1 , $f_2 \in L^1$ and $f_1 \neq f_2$, we obtain

$$||T^n f_1 - T^n f_2|| = ||T^n (f_1 - f_2)|| \le ||T^{n-1} (f_1 - f_2)|| = ||T^{n-1} f_1 - T^{n-1} f_2||$$

Inequality simply states that during the process of iteration of two individual functions the distance between them can decrease but never increase.

The support of the function g is the set of all x such that $g(x) \neq 0$, that is,

$$supp g = \{x : g(x) \neq 0\}.$$

Proposition 18 ||Tf|| = ||f|| if and only if Tf^+ and Tf^- have disjoint supports.

Having developed some of the more important elementary properties of Markov operators, we can introduce a fixed point of T as in section 3.1. A function f satisfying Tf = f for a Markov operator T is called a fixed element and if f is a density and fixed point then we call it stationary density of T. From Proposition 17, we can easily prove the following.

Proposition 19 If Tf = f then $Tf^+ = f^+$ and $Tf^- = f^-$.

2.4 Lasota's Criterion

In this section we introduce the concept of asymptotic stability for Markov operators, which is the generalization of exactness for Frobenius-Perron operators. Then a lower-bound function definition is introduced. Lastly, the relation between these two notions are shown and called the Lasota's criterion.

At first, we mention the exactness of Frobenius-Perron operator because the asymptotic stability is its generalization.

Definition 20 Let (Ω, Σ, μ) be a normalized measure space and $S : X \to X$ a measure preserving transformation such that $S(A) \in \Sigma$ for each $A \in \Sigma$. If

$$\lim_{n \to \infty} \mu(S^n(A)) = 1 \text{ for every } A \in \Sigma, \ \mu(A) > 0$$

then S is exact.

The following theorem gives the relation between exactness of Frobenius-Perron operator and its strong convergence.

Theorem 21 Let (Ω, Σ, μ) be normalized measure space, $S : X \to X$ a measure preserving transformation and P the Frobenius-Perron operator corresponding to S. Then S is exact if and only if $(P^n f)$ is strongly convergent to 1 for all $f \in \mathcal{D}$.

Since *P* is linear, convergence of $(P^n f)$ to 1 for every $f \in \mathcal{D}$ is equivalent to the convergence of $(P^n f)$ to $\langle f, 1 \rangle$ for every $f \in L^1$.

Theorem 22 *S* is exact if and only if $\lim_{n\to\infty} |P^n f - \langle f, 1 \rangle| = 0$ for $f \in L^1$.

For the proof of the preceding theorem, see [22].

The notion of exactness for Frobenius-Perron operators associated with a transformation is generalized for Markov semigroups on arbitrary measure space. A normalized measure space is not required.

Definition 23 Let $\Theta = (T_{\lambda})_{\lambda \in \Lambda}$ be a Markov semigroup in $L^{1}(\Omega, \Sigma, \mu)$. It is called asymptotically stable whenever there exists a density u such that

$$\lim_{\lambda \to \infty} \|T_{\lambda}f - u\| = 0 \qquad (\forall f \in \mathcal{D}).$$

To state the Lasota criterion we need the definition of lower-bound function for Markov semigroups.

Definition 24 A function $h \in L^1_+$ is called a lower-bound function for Θ if

$$\lim_{\lambda \to \infty} \|(h - T_{\lambda}f)_{+}\| = 0 \qquad (\forall f \in \mathcal{D})$$

We say that h is nontrivial if $h \neq 0$.

For a single operator, lower-bound function figuratively mean successive iterates of for every density f by T are finally almost everywhere above h. Any nonpositive function of course can be a lower-bound function but since $f \in \mathcal{D}$ so $T^n f \in \mathcal{D}$ and all of them are positive, so a negative lower-bound function is not interesting. Therefore a nontrivial function $h \ge 0$ is taken.

Now we state the following theorem of A. Lasota and give its proof [11] accordingly to [12].

Theorem 25 Let $\Theta = (T_t)_{t \in \mathcal{J}}$ be a (not necessarily continuous if $\mathcal{J} = \mathbb{R}_+$) one-parameter Markov semigroup in $E := L^1(\Omega, \Sigma, \mu)$. Then the following assertions are equivalent:

(i) Θ is asymptotically stable;

(ii) There is $0 \neq h \in L^1_+$ such that, for any density $f \in L^1$ and for any $t \in \mathcal{J}$, there exists $f_t \in L^1_+$ with $\lim_{t\to\infty} ||f_t|| = 0$ and $T_t f + f_t \ge h$ for all $t \in \mathcal{J}$;

(iii) There exists a nontrivial lower-bound function for Θ .

Proof:

 $(i) \Rightarrow (ii)$: Let a density $u \in L^1$ satisfy $\lim_{t\to\infty} ||T_t f - u|| = 0$ for any density f, then u is a nontrivial lower-bound function for Θ .

(*iii*) \Leftrightarrow (*ii*): Let $0 \neq h \in L^1_+$ be a nontrivial lower-bound function for Θ . Then for any density f, the condition (*ii*) is satisfied with $f_t := (T_t f - h)_-$ for all $t \in J$.

 $(ii) \Rightarrow (i)$:

Case I : Assume $\Theta = (T^n)_{n=0}^{\infty}$ is discrete. Let $0 \neq h \in L^1_+$ be a nontrivial lower-bound function for *T* and denote,

$$L_0^1 := \{ f \in L^1 : ||f_+|| = ||f_-|| \}$$

Since h is a nontrivial lower-bound function, we obtain

$$\lim \sup_{n \to \infty} \left\| (\mathcal{A}_n^T - h)_+ \right\| \le 1 - \|h\|$$

< 1 (\forall f \in \mathcal{D})

and so *T* is mean ergodic. Then there exists *T*-invariant density, say *u*. Since $L^1 = L_0^1 \oplus \mathbb{R} \cdot u$, it suffices to show that

$$\lim_{n \to \infty} \left\| T^n f \right\| = 0 \quad (\forall f \in L_0^1).$$
⁽²⁾

Notice that $(||T^n f||)_{n=1}^{\infty}$ is a monotone sequence since T is a contraction. Hence

$$\|f\| \geq \lim_{n \to \infty} \|T^n f\|$$

= $\inf_n \|T^n f\|$ ($\forall f \in L^1$)

Now suppose that there exists $f \in L_0^1$ with $2\alpha := \lim_{n \to \infty} ||T^n f|| > 0$. Then because *h* is a lower-bound function and $||f_+|| = ||f_-|| \ge \alpha$ holds;

$$2\alpha = \lim_{n \to \infty} \left\| T^n f \right\|$$

=
$$\lim_{n \to \infty} \left\| T^n (f_+ - f_-) \right\|$$

=
$$\lim_{n \to \infty} \left\| (T^n f_+ - \alpha h)_+ - (T^n f_- - \alpha h)_+ \right\|$$

$$\leq \lim_{n \to \infty} \left(\left\| (T^n f_+ - \alpha h)_+ \right\| + \left\| (T^n f_- - \alpha h)_+ \right\| \right)$$

=
$$2\alpha (1 - \|h\|)$$

which is impossible. Consequently the condition (2) holds.

Case II : Now assume that $\Theta = (T_t)_{t\geq 0}$ is a semigroup of Markov operators, which is not necessarily continuous. We shall prove the implication $(ii) \Rightarrow (i)$ in this case. In this way, we reproduce the elegant arguments in [10]. Take any $t_0 > 0$ and define $T = T_{t_0}$. Then *h* is a nontrivial lower-bound function for $(T^n)_{n=1}^{\infty}$. The first part of the proof implies that there exists a unique *T*-invariant density *u* such that

$$\lim_{n\to\infty}T^nf=u\quad (\forall f\in\mathcal{D})$$

Having shown that $T_t u = u$ for $t \in kt_{0k=1}^{\infty}$, we now demonstrate that $T_t u = u$ for all $t \in \mathbb{R}_+$. Pick t' > 0, set $f' = T'_t u$, and note that

$$u=T^n u=T_{nt_0}u.$$

Therefore

$$\begin{aligned} \|T'_t u - u\| &= \lim_{n \to \infty} \|T'_t u - u\| \\ &= \lim_{n \to \infty} \|T'_t (T_{nt_0} u) - u\| \\ &= \lim_{n \to \infty} \|T_{nt_0} (T'_t u) - u\| \\ &= \lim_{n \to \infty} \|T^n (T'_t u) - u\| \\ &= \lim_{n \to \infty} \|T^n f - u\| \\ &= 0 \end{aligned}$$

Since t' is arbitrary, we have that u is T-invariant.

Finally to show that Θ is asymptotically stable. Take a density f. Then

$$t \to ||T_t f - u|| = ||T_t f - T_t u||$$

is a non-increasing function. Take a subsequence $t_n := nt_0$. It is known that $\lim_{n\to\infty} ||T_{t_n}f - u|| = 0$, then $\lim_{t\to\infty} ||T_tf - u|| = 0$.

2.5 An extension of Sine's counterexample

In this section, we generalize Sine's counterexample of a positive contraction in a C(K)-space which is mean ergodic but its square is not [36]. For this purpose, we need some basic

definition as lemmata which are important not only for stating the mean ergodic theorem but also for other discussions throughout the thesis.

Theorem 26 *S* is mean ergodic if S^m is mean ergodic for some $m \in \mathbb{N}$

The converse is true for positive operators in ideally ordered Banach spaces where any Banach space is ideally ordered Banach space if X_+ is strongly normal and all order intervals in X are weakly compact. However, the converse is not true in general, even for positive contractions. The first example of such an operator is due to R. Sine, who had constructed a positive isometry T in a C(K)-space, such that T is mean ergodic, but T^2 is not. Note that, if we omit the positivity assumption on the operator, such an example can be constructed much more easily even in C[0, 1].

We extend Sine's construction and present a positive mean ergodic isometry in a C(K)-space, such that its q-th power is not mean ergodic for an arbitrary fixed $q \in \mathbb{N}$, $q \neq 1$. Consider the operator $T \in \mathcal{L}(\ell^{\infty}(\mathbb{Z}))$ generated by the left shift transformation:

$$T((a_n)_{n\in\mathbb{Z}}):=(a_{n+1})_{n\in\mathbb{Z}}.$$

Then *T* is a positive invertible isometry in $\ell^{\infty}(\mathbb{Z})$. Let $1 be a prime. Define a bilateral sequence <math>\mathbf{c_p} = (c_n^p)_{n \in \mathbb{Z}}$ by the formula

$$c_n^p = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } p^m \le n < p^{m+1} \text{ and } n = m + kp \\ 0 & \text{if } p^m \le n < p^{m+1} \text{ and } n \ne m + kp \\ c_{-n}^p & \text{if } n < 0 \end{cases}$$

Let $C_{\mathbf{p}}$ be the closed subalgebra of $\ell^{\infty}(\mathbb{Z})$ generated by the sequences $T^{s}(\mathbf{c_{p}})$ for all $s \in \mathbb{Z}$ and by the constant sequence $(1)_{n \in \mathbb{Z}}$. Obviously, $C_{\mathbf{p}}$ is a Banach lattice algebra and T^{q} is a positive invertible isometry on $C_{\mathbf{p}}$ for all $q \in \mathbb{Z}$. A routine computation shows that T^{q} is mean ergodic on $C_{\mathbf{p}}$ if and only if $q \neq p \cdot r$ for all $r \in \mathbb{Z}$, $r \neq 0$.

Let $\Delta = \{p_1, ..., p_j\}$ be a finite set of primes. Take the direct sum $C = \bigoplus_{i=1}^{j} C_{\mathbf{p}_i}$ of Banach lattice algebras $C_{\mathbf{p}_i}$ (here we assume that they do not possess common elements, otherwise we replace them by isometrically isomorphic copies). Since a finite direct sum $\ell^{\infty} \oplus ... \oplus \ell^{\infty}$ is isometrically isomorphic to ℓ^{∞} , we may consider *C* as a Banach lattice subalgebra of $\ell^{\infty}(\mathbb{Z})$. Define $R \in \mathcal{L}(C)$ as follows:

$$R((x_1,...,x_j)) := (T(x_1),...,T(x_j)) \quad (x_1 \in C_{\mathbf{p}_1},...,x_j \in C_{\mathbf{p}_j})$$

Obviously, an operator R^l in the Banach lattice algebra *C* is mean ergodic if and only if p_i is not a divisor of *l* for all i = 1, ..., j.

Note that the Banach lattice algebra *C* constructed above is commutative and possesses a strong unit. Therefore, by Gelfand's theorem, *C* can be identified with C(K), where *K* is a Hausdorff compact space. Moreover, *K* is an appropriate quotient space of the Čech – Stone compactification $\beta(\mathbb{Z})$ of integers. Note that *K* can be obtained from $\beta(\mathbb{Z})$ by identification of points which cannot be distinguished by elements of $C \subset C(\beta(\mathbb{Z}))$. The main disadvantage of the construction above is that it cannot be extended to an infinite set of primes. If we do this by defining in a similar way an algebra *G* and an operator $R \in \mathcal{L}(G)$ for an infinite set Δ of primes, the operator *R* is no longer mean ergodic.

However, it is still possible to construct, in a commutative Banach lattice algebra without a strong unit, a positive mean ergodic operator U such that all nontrivial powers U^q are not mean ergodic. For this purpose, we take for every prime p an operator T on C_p such that T is mean ergodic, but T^p is not. Denote by Λ the set of all primes p > 1. Consider the c_0 -direct sum of the set $(C_p)_{p \in \Lambda}$ of Banach lattice algebras

$$\mathcal{G} = \bigoplus_{\mathbf{p} \in \Delta} C_{\mathbf{p}}$$

Then \mathcal{G} is a commutative Banach lattice algebra without a strong unit. We define an operator U in \mathcal{G} setting $U|_{C_p} := T$. Then U is mean ergodic, but, for any $n \neq 0, \pm 1, n$ has a prime divisor p, say $n = k \cdot p$. Hence U^n is not mean ergodic for all $n \neq 0, \pm 1$, since $U|_{C_p}^n := T^{k \cdot p}$ is not mean ergodic. Note that, in our construction, the operators U^{-1} and U are mean ergodic. The results of Section (2.5) was published in [14].

CHAPTER 3

LR-NETS

This chapter is devoted to LR-nets which are the main theme of the thesis. First of all, the definition of an LR-net and some elementary results are given. Then we discuss some examples which are needed in the next chapters.

3.1 Elementary Results

For the following definition we prefer to call Lotz-Räbiger net, briefly LR-net. First of all, it is introduced at the beginning of 80th by Heinrich Lotz under the name M-sequence and published in 1984. The main aim of Lotz was to find a unified approach to various Tauberian theorems for operators in Banach spaces. In 1993 Frank Räbiger introduced the following notion and called it M-net.

Definition 27 A net $\Theta = (T_{\lambda})_{\lambda \in \Lambda} \subseteq \mathcal{L}(X)$ is called a Lotz-Räbiger net (= LR-net) if

*LR*1 : Θ *is uniformly bounded;*

$$LR2 : \lim_{\lambda \to \infty} ||T_{\lambda} \circ (T_{\mu} - I)x|| = 0 \text{ for every } \mu \in \Lambda \text{ and for every } x \in X;$$

LR3 : $\lim_{\lambda \to \infty} \left\| (T_{\mu} - I) \circ T_{\lambda} x \right\| = 0$ for every $\mu \in \Lambda$ and for every $x \in X$.

We may suppose that an *LR*-net contains the identity operator. In fact, for a given *LR*-net $\Theta = (T_{\lambda})_{\lambda} \in \Lambda$, denote by $\widehat{\Lambda}$ the set $\Lambda \cup \lambda_0$ for $\lambda_0 \notin \Lambda$ and extend the partial order from Λ to $\widehat{\Lambda}$ setting $\lambda_0 < \lambda$ for all $\lambda \in \Lambda$. Put $T_{\lambda_0} := I_X$. The family $\widehat{\Theta} = (T_{\lambda})_{\lambda \in \widehat{\Lambda}}$ is an *LR*-net containing the identity operator. We always suppose that an *LR*-net contains the identity

operator because the only interesting questions concerning *LR*-nets are those of asymptotic nature (e.g. whenever $\lambda \to \infty$).

The next definition indicates a fixed vector of an operator net Θ .

Definition 28 Let $\Theta = (T_{\lambda})_{\lambda \in \Lambda}$ be a net in a Banach space X. A vector x is called fixed under Θ if $T_{\lambda}x = x$ for every $T_{\lambda} \in \Lambda$. Denote by $Fix(\Theta)$ the set of all fixed vectors of Θ .

It is easy to see that $Fix(\Theta)$ is a closed subspace in *X*.

The proof of the next proposition is straightforward. It says us that for proving a vector in X is a fixed vector, it is enough to show that it is eventually fixed. This proposition can not be extended to an arbitrary uniformly bounded operator nets.

Proposition 29 Let $\Theta = (T_{\lambda})_{\lambda \in \Lambda}$ be an LR-net in X and x is any vector in X. Then $x \in Fix(\Theta)$ if and only if there exists $\lambda(x) \in \Lambda$ satisfying $T_{\lambda}(x) = x$ for all $\lambda \ge \lambda(x)$.

Proof: The necessity is obvious.

For sufficiency part, take an element $x \in X$ satisfying $T_{\lambda}(x) = x$ for all $\lambda \ge \lambda(x)$. Let μ be an arbitrary element of Λ . Because of the condition (*LR* 3) and the continuity of ($T_{\mu} - I$) we obtain

$$0 = \lim_{\mu \to \infty} (T_{\mu} - I) \circ T_{\lambda} x = (T_{\mu} - I) \lim_{\mu \to \infty} T_{\lambda} x.$$

By the assumption, the limit equality $\lim_{\lambda \to \infty} T_{\lambda} x = x$ holds. Therefore $T_{\mu} x = x$ for arbitrary $\mu \in \Lambda$. Hence $x \in Fix(\Phi)$.

Next definition will form the backbone of the thesis.

Definition 30 The net Θ is called strongly convergent if the norm limit $\lim_{\lambda \to \infty} T_{\lambda} x$ exists for each $x \in X$.

The following elementary result explains the relationship between the strong convergence and the fixed space of an LR-net. Note that Proposition 31 cannot be extended to an arbitrary uniformly bounded operator net. In Chapter 1, we mentioned the splitting theorem for operator

semigroups under the name mean ergodicity of operator semigroups. The next theorem is the splitting theorem for *LR*-nets. Originally the next theorem statement consists of the direct sum of Fix(Θ) and span $\bigcup_{\lambda \in \Lambda} (I - T_{\lambda})X$, stated by Räbiger without proof in [30].

We state the following elementary fact.

Proposition 31 Let $\Theta = (T_{\lambda})_{\lambda \in \Lambda}$ be an LR-net in X. Then the set

$$\bigcup_{\lambda \in \Lambda} (I - T_{\lambda}) X$$

is a linear space. Thus we may replace $\overline{\text{span}} \bigcup_{\lambda \in \Lambda} (I - T_{\lambda}) X$ with $\overline{\bigcup_{\lambda \in \Lambda} (I - T_{\lambda}) X}$.

Proof: Let $x = (I - T_{\lambda_1})u$ and $y = (I - T_{\lambda_2})v$ be in $\bigcup_{\lambda \in \Lambda} (I - T_{\lambda})X$. Since for every $\mu \in \lambda$ and for each $x \in X$, $\lim_{\lambda \to \infty} \left\| (I - T_{\mu}) \circ T_{\lambda}x \right\| = 0$ by (*LR* 2), we obtain

$$\begin{split} \lim_{\lambda \to \infty} \|T_{\lambda} x\| &= \lim_{\lambda \to \infty} \|T_{\lambda} \circ (I - T_{\lambda_{1}}) u\| = 0 \,, \\ \lim_{\lambda \to \infty} \|T_{\lambda} y\| &= \lim_{\lambda \to \infty} \|T_{\lambda} \circ (I - T_{\lambda_{2}} v)\| = 0 \,. \end{split}$$

Therefore $\lim_{\lambda \to \infty} T_{\lambda}(x + y) = 0$ and $x + y \in \bigcup_{\lambda \in \Lambda} (I - T_{\lambda})X$. The same argument is true for arbitrary $x, y \in \overline{\bigcup_{\lambda \in \Lambda} (I - T_{\lambda})X}$. Consequently, $\overline{\bigcup_{\lambda \in \Lambda} (I - T_{\lambda})X}$ is a linear subspace of X.

Theorem 32 Let $\Theta = (T_{\lambda})_{\lambda \in \Lambda}$	be an LR-net in X.	Then Θ is strongly	convergent if and	only if
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$$X = \operatorname{Fix}(\Theta) \oplus \overline{\bigcup_{\lambda \in \Lambda} (I - T_{\lambda}) X}.$$
 (1)

Moreover, in this case, the strong limit P of Θ is a projection onto $Fix(\Theta)$.

Proof: For the first implication, assume $X = \text{Fix}(\Theta) \oplus \overline{\bigcup_{\lambda \in \Lambda} (I - T_{\lambda}) X}$. It is enough to prove that the norm limit exists for all $x \in \overline{\bigcup_{\lambda \in \Lambda} (I - T_{\lambda}) X}$, because $T_{\lambda} x = x$ for all $x \in \text{Fix}(\Theta)$. For fixed $\mu \in \Lambda$ and $x \in (I - T_{\mu}) X$, there exists some $v \in X$ such that $x = (I - T_{\mu}) v$. The norm convergence of the net $(T_{\lambda} x)_{\lambda \in \Lambda}$ is provided by (LR 2)

$$\lim_{\lambda \to \infty} \|T_{\lambda} x\| = \lim_{\lambda \to \infty} \left\| T_{\lambda} \circ (I - T_{\mu}) x \right\| = 0$$

So for every $x \in \overline{\bigcup_{\lambda \in \Lambda} (I - T_{\lambda}) X}$ the norm limit of the net $(T_{\lambda} x)_{\lambda \in \Lambda}$ is equal to zero.

For the other implication, assume Θ is strongly convergent. Hence the strong limit *P* of Θ is a continuous operator. Consider any point $x \in X$ we have

$$T_{\mu}(Px) - Px = (T_{\mu} - I) \lim_{\lambda \to \infty} T_{\lambda}x = \lim_{\lambda \to \infty} (T_{\mu} - I) \circ T_{\lambda}x = 0$$

for any $\mu \in \Lambda$ because of the condition (*LR* 3). Hence $T_{\mu}Px = Px$, $\forall \mu \in \Lambda$ and $Px \in Fix(\Theta)$ Moreover $P^2x = P(Px) = Px$ and by arbitrariness of x, P is the continuous projection onto Fix(Θ). Henceforth $X = P(X) \oplus \ker P$ where $P(X) = Fix(\Theta)$. The last step of the proof is showing that $\overline{\bigcup_{\lambda \in \Lambda} (I - T_{\lambda})X} = \ker P$..

Let *x* be an arbitrary element of $\overline{\bigcup_{\lambda \in \Lambda} (I - T_{\lambda}) X}$. By (*LR* 2), $\lim_{\lambda \to \infty} T_{\lambda} x = \lim_{\lambda \to \infty} T_{\lambda} \circ (I - T_{\mu}) x = 0$ so $x \in \ker P$. For the other inclusion assume $x \in \ker P$ then we have $\lim_{\lambda \to \infty} T_{\lambda} x = Px = 0$ and $x = \lim_{\lambda \to \infty} (I - T_{\lambda}) x \in \overline{\bigcup_{\lambda \in \Lambda} (I - T_{\lambda}) X}$

3.2 Examples of *LR*-nets

3.2.1 Single operators

Example 33 (Lotz) Let $T \in \mathcal{L}(X)$ be a contraction (i.e. $||T|| \le 1$). Then the sequence $(\mathcal{A}_n^T)_{n_1}^{\infty}$ of Cesàro means $\mathcal{A}_n^T := \frac{1}{n} \sum_{k=0}^{n-1} T^k$ of the operator T is an LR-net.

Proof. Consider

$$\begin{aligned} \mathcal{A}_{n}^{T} \circ (T-I)x &= \frac{1}{n} \sum_{k=0}^{n-1} T^{k} \circ (T-I)x \\ &= \frac{1}{n} \left(\sum_{k=0}^{n-1} T^{k+1}x - \sum_{k=0}^{n-1} T^{k}x \right) \\ &= \frac{1}{n} (T^{n} + T^{n-1} + \dots T - (T^{n-1} + \dots + T + I))x \\ &= \frac{1}{n} (T^{n} - I)x \end{aligned}$$

Since *T* is a contraction the limit value

$$\lim_{n \to \infty} \left\| \mathcal{A}_n^T \circ (T - I)x \right\| = \lim_{n \to \infty} \frac{\|T^n x - x\|}{n}$$
$$\leq \lim_{n \to \infty} \left(\frac{\|T^n x\|}{n} + \frac{\|x\|}{n} \right)$$
$$\leq \lim_{n \to \infty} \left(\frac{\|T\|^n}{n} \|x\| + \frac{\|x\|}{n} \right)$$
$$= 0$$

Now for any s > n

.

$$\begin{aligned} \mathcal{A}_{n}^{T} \circ (T^{s} - I)x &= \frac{1}{n} \sum_{k=0}^{n-1} T^{k} \circ (T^{s} - I)x \\ &= \frac{1}{n} \left(\sum_{k=0}^{n-1} T^{k+s} x - \sum_{k=0}^{n-1} T^{k} x \right) \\ &= \frac{1}{n} (T^{s+n-1} + T^{s+n-2} + \dots T^{s} - (T^{n-1} + \dots + T + I))x \\ &= \frac{1}{n} (T^{s+n-1} + \dots T^{n} - (T^{s-1} + \dots + I))x \end{aligned}$$

Since *T* is a contraction the limit value of $\mathcal{A}_n^T \circ (T^s - I)x$ in norm is zero as n goes to infinity. Then,

$$\begin{aligned} \mathcal{A}_n^T \circ (\mathcal{A}_m^T - I)x &= \mathcal{A}_n^T \circ (\frac{1}{m} \sum_{k=0}^{m-1} T^k - I)x \\ &= \left(\mathcal{A}_n^T \circ (\frac{1}{m} \sum_{k=0}^{m-1} T^k - \sum_{k=0}^{m-1} I^k)x \right) \\ &= \left(\mathcal{A}_n^T \circ (\frac{1}{m} \sum_{k=0}^{m-1} (T^k - I^k))x \right) \\ &= \frac{1}{m} \sum_{k=0}^{m-1} \mathcal{A}_n^T (T^k - I)x \end{aligned}$$

Since above is true for every k, $\mathcal{A}_n^T(T^k - I)$ converges to zero, sum of these terms goes to zero. Therefore the *LR*-net definition conditions are satisfied and \mathcal{A}_n^T net is an *LR*-net.

Example 34 If $T \in \mathcal{L}(X)$ is an operator in X with uniformly bounded Cesàro averages $\mathcal{A}_n^T := \frac{1}{n} \sum_{k=0}^{n-1} T^k$, $n \in \mathbb{N}$ and if $n^{-1}T^n \to 0$ strongly, then $(\mathcal{A}_n^T)_{n=1}^{\infty}$ is an LR-net.

Proof. Consider the proof of Example 33, $\mathcal{A}_n^T \circ (T - I)x = \frac{1}{n}(T^n - I)x$. Since $\frac{T^n}{n}$ converges 0 strongly, $\mathcal{A}_n^T \circ (T - I)x$ converges to zero as $n \to \infty$. Now, for any s > n, we know that $\mathcal{A}_n^T \circ (T^s - I)x = \frac{1}{n}(T^{s+n-1} + \cdots T^n - (T^{s-1} + \cdots + I))x$. Again since $\frac{T^n}{n}$ converges 0 strongly and $(T^{s-1} + \cdots + I)x$ is an any element of X and n goes to ∞ , $\mathcal{A}_n^T \circ (T^s - I)x$ converges to zero as $n \to \infty$. Then from equation $\mathcal{A}_n^T \circ (\mathcal{A}_m^T - I)x = \frac{1}{m} \sum_{k=0}^{m-1} \mathcal{A}_n^T (T^k - I)x$ and for every k, $\mathcal{A}_n^T (T^k - I)$ converges to zero, sum of these terms goes to zero. Therefore the *LR*-net definition conditions are satisfied and $(\mathcal{A}_n^T)_n$ is an *LR*-net.

3.2.2 Operator semigroups

Example 35 (Lotz) Let $\Theta = (T(t))_{t>0}$ be a strongly continuous one-parameter semigroup of operators in X. Let $C_t := t^{-1} \int_0^t T(s) ds$ t > 0, be the Cesaro means of Θ , where the integral (if it exists) is defined strongly. If there exists a > 0 such that $(C_t)_{0 < t \le a}$ is uniformly bounded, then $(I - C_t)_{0 < t \le a}$ is an LR-net (for $t \to 0$). If there exists b > 0 such that $(C_t)_{t \ge b}$ is uniformly bounded and $t^{-1}T(t)$ tends strongly to zero as $t \to \infty$, then $(C_t)_{t \ge b}$ is an LR-net for $t \to \infty$.

Example 36 Let $\mathcal{G} \subseteq \mathcal{L}(X)$ be a uniformly bounded Abelian operator semigroup. Then \mathcal{G} is a directed set with respect to the natural partial order \prec defined by $S \prec T$ if there exists an $R \in \mathcal{G}$ with $T = R \circ S$.

If the operator net $(T_T)_{T \in (\mathcal{G}, \prec)}$, where $T_T := T$ for all $T \in \mathcal{G}$, converges strongly as $T \to \infty$, then $(T_T)_{T \in (\mathcal{G}, \prec)}$ is an LR-net.

In the case, when the strong limit satisfies the condition

$$\lim_{T \in (\mathcal{G}, \prec)} S \circ T = S$$

for every $S \in \mathcal{G}$, the net $(I - T_T)_{T \in (\mathcal{G}, \prec)}$, where $T_T := T$ for all $T \in \mathcal{G}$, is an LR-net.

3.2.3 \mathcal{T} -ergodic nets

Example 37 Every \mathcal{T} -ergodic net for a given operator semigroup $\mathcal{T} \subseteq \mathcal{L}(X)$ is an LR-net.

Proof.

$$C_{\lambda}x = \overline{\operatorname{co}(\mathcal{T}x)} = \left\{ \sum_{j=1}^{n} \lambda_j T_j x : \lambda_j \in \mathbb{R}, \ T_j \in \mathcal{T} and \sum_{j=1}^{n} \lambda_j = 1 \right\}.$$

For some fix λ_0 , $C_{\lambda_0} x = \lim_{\lambda'} K_{\lambda'} x$ where $K_{\lambda'} x = \sum_{j=1}^n \lambda'_j T_j x$.

Since \mathcal{T} -ergodic net is uniformly bounded, then the first condition of *LR*-net is directly satisfied. Then

$$\begin{aligned} \left\| C_{\lambda}(C_{\mu} - I)x \right\| &= \left\| C_{\lambda}(C_{\mu} - I)x + C_{\lambda}(I - T)x - C_{\lambda}(I - T)x \right\| \\ &\leq \left\| C_{\lambda}(C_{\mu} - I + I - T)x \right\| + \left\| C_{\lambda}(I - T)x \right\| \\ &\leq \left\| C_{\lambda}(C_{\mu} - T)x \right\| + \frac{\epsilon}{2} \\ &\leq \left\| C_{\lambda} \right\| \left\| C_{\mu} - T \right\| \left\| x \right\| + \frac{\epsilon}{2} \end{aligned}$$

Since C_{μ} is in the closed convex hull of *T*;

$$\leq \frac{\epsilon}{2 \left\| C_{\lambda} \right\| \left\| x \right\|} + \frac{\epsilon}{2} \leq \epsilon.$$

Therefore the second condition of LR-net is satisfied.

$$\begin{aligned} \left\| (C_{\mu} - I)C_{\lambda}x \right\| &= \left\| (C_{\mu} - I)C_{\lambda}x + (I - T)C_{\lambda}x - (I - T)C_{\lambda}x \right\| \\ &\leq \left\| (C_{\mu} - I + I - T)C_{\lambda}x \right\| + \left\| (I - T)C_{\lambda}x \right\| \\ &\leq \left\| (C_{\mu} - T)C_{\lambda}x \right\| + \frac{\epsilon}{2} \\ &\leq \left\| C_{\lambda} \right\| \left\| C_{\mu} - T \right\| \left\| x \right\| + \frac{\epsilon}{2} \end{aligned}$$

Again C_{μ} is in the closed convex hull of *T*;

$$\leq \frac{\epsilon}{2 \|C_{\lambda}\| \|x\|} + \frac{\epsilon}{2} \leq \epsilon$$

and the third condition is satisfied. Hence every \mathcal{T} -ergodic net is an LR-net.

3.2.4 Pseudoresolvents

Example 38 (Räbiger) Let Λ be a nonempty subset of \mathbb{C} and let $(R_{\lambda})_{\lambda \in \Lambda} \subseteq \mathcal{L}(X)$ be a pseudoresolvent, (*i.e. satisfies the Hilbert identity* $R_{\lambda} - R_{\mu} = (\mu - \lambda)R_{\lambda} \circ R_{\mu}$ for all $\lambda, \mu \in \Lambda$). Then

 $(\lambda R_{\lambda})_{\lambda \in \Lambda}$ is called Abel means relative to $(R_{\lambda})_{\lambda \in \Lambda}$. Let α_{λ} be a net in Λ such that $(\alpha_{\lambda} R_{\alpha_{\lambda}})$ is uniformly bounded.

- (a) If $\lim_{\lambda \to \infty} \alpha_{\lambda} = a \in \mathbb{C}$ then $((\alpha_{\lambda} a)R_{\alpha_{\lambda}})_{\lambda \in \Lambda}$ is an LR-net.
- (b) If $\lim_{\lambda \to \infty} |\alpha_{\lambda}| = \infty$ then $(I \alpha_{\lambda} R_{\alpha_{\lambda}})_{\lambda \in \Lambda}$ is an LR-net.

Proof.

Since $(\alpha_{\lambda}R_{\alpha_{\lambda}})$ is equi-continuous, $(I - \alpha_{\lambda}R_{\alpha_{\lambda}})$ is uniformly bounded. Consider

$$(I - \alpha_{\lambda}R_{\alpha_{\lambda}})(I - (I - \alpha_{\mu}R_{\alpha_{\mu}}))x$$

$$= (I - \alpha_{\lambda}R_{\alpha_{\lambda}})\alpha_{\mu}R_{\alpha_{\mu}}x$$

$$= \alpha_{\mu}R_{\alpha_{\mu}}x - \alpha_{\lambda}\alpha_{\mu}R_{\alpha_{\lambda}}R_{\alpha_{\mu}}x$$

$$= \alpha_{\mu}R_{\alpha_{\mu}}x - \frac{\alpha_{\lambda}\alpha_{\mu}}{\alpha_{\lambda} - \alpha_{\mu}}(R_{\alpha_{\lambda}} - R_{\alpha_{\mu}})x$$

$$= \alpha_{\mu}R_{\alpha_{\mu}}x - \frac{\alpha_{\lambda}\alpha_{\mu}}{\alpha_{\lambda} - \alpha_{\mu}}R_{\alpha_{\lambda}}x + \frac{\alpha_{\mu}\alpha_{\lambda}}{\alpha_{\lambda} - \alpha_{\mu}}R_{\alpha_{\mu}}x$$

If the limit is taken, it is enough to check the limit value as λ goes to ∞ .

$$\begin{split} &\lim_{\lambda \to \infty} (I - \alpha_{\lambda} R_{\alpha_{\lambda}}) (I - (I - \alpha_{\mu} R_{\alpha_{\mu}})) x \\ &= \lim_{\lambda \to \infty} \left(\alpha_{\mu} R_{\alpha_{\mu}} x - \frac{\alpha_{\lambda} \alpha_{\mu}}{\alpha_{\lambda} - \alpha_{\mu}} R_{\alpha_{\lambda}} x + \frac{\alpha_{\mu} \alpha_{\lambda}}{\alpha_{\lambda} - \alpha_{\mu}} R_{\alpha_{\mu}} x \right) \\ &= \alpha_{\mu} R_{\alpha_{\mu}} x - \lim_{\lambda \to \infty} \frac{\alpha_{\lambda} \alpha_{\mu}}{\alpha_{\lambda} - \alpha_{\mu}} R_{\alpha_{\lambda}} x + \lim_{\lambda \to \infty} \frac{\alpha_{\mu} \alpha_{\lambda}}{\alpha_{\lambda} - \alpha_{\mu}} R_{\alpha_{\mu}} x \\ &= \alpha_{\mu} R_{\alpha_{\mu}} x - \lim_{\lambda \to \infty} \frac{\alpha_{\mu} (\alpha_{\lambda} R_{\alpha_{\lambda}} x)}{\alpha_{\lambda} - \alpha_{\mu}} + \lim_{\lambda \to \infty} \frac{\alpha_{\mu} \alpha_{\lambda}}{\alpha_{\lambda} (1 - \frac{\alpha_{\mu}}{\alpha_{\lambda}})} R_{\alpha_{\mu}} x \\ &= -\lim_{\lambda \to \infty} \frac{\alpha_{\mu} (\alpha_{\lambda} R_{\alpha_{\lambda}} x)}{\alpha_{\lambda} - \alpha_{\mu}} \\ &= 0 \end{split}$$

Because $\alpha_{\lambda}R_{\alpha_{\lambda}}$ is uniformly bounded and $\lim_{\lambda \to \infty} |\alpha_{\lambda}| = \infty$. Therefore $(I - \alpha_{\lambda}R_{\alpha_{\lambda}})$ is an *LR*-net. The proof of the first part is the same.

Example 39 (Lotz) Let (Ω, Σ, μ) be a probability space, let $1 \le p \le \infty$, and let $X = L^p(\Omega, \Sigma, \mu)$. Let $(\Sigma_n)_{n=1}^{\infty}$ be an increasing sequence of σ -subalgebras of Σ . For every $f \in X$, let $C_n f \in X$ be the conditional expectation of f with respect to Σ_n . Then the sequence $(I - C_n)_{n=1}^{\infty}$ is an commutative LR-net.

Proof. For any $(\Sigma_n)_{n=1}^{\infty} \sigma$ -subalgebra (Σ_n) of Σ , conditional expectation is a contractive projection from $L^p(\Omega, \Sigma, \mu)$ to $L^p(\Omega, \Sigma_n, \mu)$. Since X is a probability space, $(I - C_n)_n$ is uniformly bounded. The condition *LR*2 is satisfied because conditional expectation has properties such that assume Σ_n and Σ_m is an σ -subalgebras of Σ , then the relation between conditional expectation of subalgebras is $C_n \circ C_m = C_{\min(n,m)}$. So

$$(I - C_n)(I - (I - C_m))x = ((I - C_n)C_m)x$$
(3.1)

$$= (C_m - C_n C_m) x \tag{3.2}$$

$$= (C_m - C_{\min(n,m)})x$$
(3.3)

Since $(\Sigma_n)_{n=1}^{\infty}$ is an increasing sequence and consider the strong limit, then for n > m,

$$(I - C_n)(I - (I - C_m))x = (C_m - C_{\min(n,m)})x$$
(3.4)

$$= (C_m - C_{\min(n,m)})x$$
 (3.5)

$$= (C_m - C_m)x = 0 (3.6)$$

Hence LR2 and because of the commutativity of conditional expectation, LR3 are satisfied.

Example 40 Let X be a Banach space and let $(P_n)_{n=1}^{\infty} \subseteq \mathcal{L}(X)$ be a uniformly bounded sequence of projections with $P_n P_m = P_{\min(n,m)}$. Then the sequence $(I - P_n)_{n=1}^{\infty}$ is an LR-net.

Proof of Example 40 is the same like Example 39.

3.2.6 Ordinal numbers

Example 41 Let $\ell^2(\kappa)$ be the standard Hilbert space (see [5]), where κ is a limit ordinal $(\ell^2(\kappa) \text{ is separable if and only if } \kappa \leq \omega)$. The vectors $e_i(j) = \delta_{i,j}$, for $\iota < \kappa$, form a complete orthonormal basis for $\ell^2(\kappa)$. Let Λ be the set of all finite subsets (including the empty set) of κ with a natural ordering $\lambda_1 \leq \lambda_2$ if and only if $\lambda_1 \subseteq \lambda_2$. Given $\lambda = \{\xi_1, \dots, \xi_n\} \in \Lambda$ define T_{λ} to be the orthogonal projection on $\lim\{e_{\xi_1}, \dots, e_{\xi_n}\}^{\perp}$. Obviously T_{λ} form an (LR) net. We notice that for each $\eta \in \ell^2(\kappa)$ we have $\lim_{\lambda \to \infty} ||T_{\lambda}\eta||_2 = 0$. Moreover, given η there exists an increasing sequence $\lambda_1 \leq \lambda_2 \leq \cdots$ (depending on η) such that the ordinary limit $\lim_{n\to\infty} \left\|T_{\lambda_n}\right\|_2 = 0$. However, if κ is uncountable then there exists no subsequence $\lambda_1 \leq \lambda_2 \leq \cdots$ which is universally good for all $\eta \in \ell^2(\kappa)$. Clearly the net under consideration may be uplifted to the Banach lattice C(X) of all (real valued) continuous functions, where X stands for the unit (closed) ball of $\ell^2(\kappa)$ endowed with *weak=weak topology. Namely, define $T_{\lambda}f(\eta) = f(T_{\lambda}\eta)$. It follows that $\lim_{\lambda \to \infty} T_{\lambda}f(\eta) = f(0)$ pointwise (a close look on the structure of weakly continuous functions gives that the convergence $\lim_{\lambda \to \infty} T_{\lambda}f = f(0)I$ is uniform on X i.e. in the sup norm).

Proof. An ordinal number is defined as the order type of a well-ordered set. The motivation is to define an ordinal number as the set of all ordinals less than itself. Any nonzero ordinal has the minimum element zero. It may or may not have a maximum element. If an ordinal number α , then it is the next ordinal after α is called **successor ordinal**, namely the successor of α written $\alpha + 1$. A nonzero ordinal which is not a successor is called **a limit ordinal**. Another important definition about ordinal number is the following. If α is a limit ordinal and *X* is a set, an α -index sequence of elements of *X* is a function from α to *X*. This notion is a generalization of a sequence. An ordinary sequence is the case $\alpha = \omega$. Now, if *X* is a topological space, we say that an α -indexed elements of *X* converges to a limit *x* when it converges as a net. That is to say, for every neighbourhood *U* of *x*, there exists an ordinal $\beta < \alpha$ such that $x_t \in U$ for all $t \ge \beta$. Ordinary-indexed sequences are more powerful than ordinary sequences to determine limits in topology.

Other background for such example is a definition of Hilbert space for arbitrary set. Let *I* be any set. $\ell_2(I)$ to be the set of all functions $x : I \to \mathbb{F}$ such that $x(i) \neq 0$ for at most a countable number of *i* and $||x||^2 = \sum_{i \in I} |x(i)|^2 < \infty$. Then $\ell^2(I)$ is a **Hilbert space** with respect to inner

product $\langle x, y \rangle = \sum_{i \in I} x(i) \overline{y(i)}$. Moreover the vectors $e_{\iota}(j) = \delta_{\iota,j}$, for $\iota \prec \kappa$, form a complete orthonormal basis (Conway basis) for $\ell^2(\kappa)$.

Let Λ be the set of all finite subsets (including the empty set) of κ with a natural ordering $\lambda_1 \leq \lambda_2$ if and only if $\lambda_1 \subseteq \lambda_2$. For $\lambda = \{\xi_1, \dots, \xi_n\} \in \Lambda$ define T_{λ} to be the orthogonal projection on $lin\{e_{\xi_1}, \dots, e_{\xi_n}\}^{\perp}$. If λ is large enough and $\mu \leq \lambda$, we obtain $lin\{e_{\lambda_1}, \dots, e_{\lambda_n}\}^{\perp} \subset lin\{e_{\mu_1}, \dots, e_{\mu_m}\}^{\perp}$, additionally $T_{\lambda} \circ (I - T_{\mu})x = T_{\lambda}y$ where *y* is an element of $lin\{e_{\mu_1}, \dots, e_{\mu_m}\}$. Finally, we obtain $||T_{\lambda}y|| = 0$ for enough large λ . Therefore (*LR2*) condition is satisfied. For the last condition of *LR*-nets, $(I - T_{\mu}) \circ T_{\lambda}x = (I - T_{\mu})y$ where $y \in lin\{e_{\lambda_1}, \dots, e_{\lambda_n}\}^{\perp}$. Since $\mu \leq \lambda$, $lin\{e_{\mu_1}, \dots, e_{\mu_m}\} \subseteq lin\{e_{\lambda_1}, \dots, e_{\lambda_n}\}$, we obtain $(I - T_{\mu})y = 0$. Hence we obtain (T_{λ}) is an *LR*-net on *X*.

Example 42 Let us consider the order interval $X = \{\xi : 0 \le \xi \le \omega_1\}$, where ω_1 is the first uncountable ordinal. Clearly (cf. [16]) X equipped with the topology generated by the basis $\{(\iota_1, \iota_2] : 0 \le \iota_1 \le \iota_2 \le \omega_1\}$ is a compact Hausdorff space. As usual $\mathfrak{X} = C(X)$ denotes the Banach space of all functions on X with sup norm. Let us consider again Λ to be the family of all finite (nonempty) subsets of $[0, \omega) = \mathbb{N}$ and define

(•)
$$T_{\lambda}f(\iota) = \begin{cases} \frac{1}{card(\lambda)} \sum_{j \in \lambda} f(\iota+j) & \text{if } \iota \text{ is not a limit ordinal} \\ f(\iota) & \text{if } \iota \text{ is a limit ordinal.} \end{cases}$$

Clearly T_{λ} are positive linear contractions on C(X), $T_{\lambda}I = I$ and form an (LR) net. For each $f \in X$ we have $\lim_{\lambda \to \infty} T_{\lambda}f = \overline{f}$ exists in the sup norm and for non limit ordinals $\overline{f}(\iota) = f(\xi_{\iota})$, where $\xi_{\iota} = \min\{\xi > \iota : \xi$ is a limit ordinal}. Modifying; we introduce

$$(\diamond) \quad S_{\lambda}f(\iota) = \frac{1}{card(\lambda)}\sum_{j\in\lambda}f(\iota-j),$$

(if $\iota = \eta_{\iota} + k_{\iota}$, where $k_{\iota} \in \mathbb{N}$, η_{ι} is a limit ordinal supporting ι from below and $k_{\iota} < j$ then we set $\iota - j = \eta_{\iota}$). Clearly, $\lim_{\lambda \to \infty} S_{\lambda} f(\iota) = \overline{f}(\iota) = f(\eta_{\iota})$ exists pointwise, but the limit function is not continuous in general.

3.2.7 Heat Equations

Example 43 (Emelyanov) Let $(H_t)_{t\geq 0}$ be a C_0 -semigroup of kernel operators acting on the space $L^1(\mathbb{R})$ as follows:

$$(H_t f)[x] = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} \exp[\frac{-(x-y)^2}{4t}] f(y) dy, \quad (\forall x \in X, t \in \mathbb{R}_+).$$

These operators deliver the solution $u(t, x) = (H_t f)[x]$ of the **heat equation** on the real line $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2}$ with the initial condition u(0, x) = f(x). Then $(H_t)_{t\geq 0}$ is an LR-net.

Proof.

This semigroup elements are Markov operator and completely mixing. So more generally we should prove that any one-parameter semigroup of **completely mixing** Markov operators on L^1 -space (cf. [22]) is an *LR*-net.

Let us remind definition of completely mixing for semigroups.

Definition 44 Let $(T_t)_{t\geq 0}$ be a one-parameter semigroup on L^1 -space. It is called **completely** *mixing* if $\lim_{t\to\infty} ||T_tx|| = 0$ for all $f \in L^1_0$.

Consider the (*LR* 2) condition. $\lim_{\lambda \to \infty} ||T_{\lambda} \circ (I - T_{\mu})f|| = \lim_{\lambda \to \infty} ||T_{\lambda}z|| = 0$ because $(I - T_{\mu})f = z \in L_0^1$. Therefore (*LR* 2) condition is satisfied. Since *T* is a Markov operator and Θ is completely mixing, condition (*LR* 3) is satisfied directly.

3.2.8 Approximate Identities

Example 45 (Emelyanov) Let $A = (A, \bullet, ||\cdot||)$ be a Banach algebra. We embed A to L(A) isometrically as follows;

$$\pi(a)(x) := a \bullet x \ (\forall x \in A)$$

Then for any approximate identity $(e_{\lambda})_{\lambda \in \Lambda}$ in A, the operator net $(I - \pi(e_{\lambda}))_{\lambda \in \Lambda} \subseteq L(A)$ is a *LR*-net.

Proof. An approximate identity in a Banach algebra *A* is a net $(e_{\lambda})_{\lambda \in \Lambda}$ such that for every element $a \in A$ the net $(a \bullet e_{\lambda})_{\lambda \in \Lambda}$ and $(e_{\lambda} \bullet a)_{\lambda \in \Lambda}$ have limit *a*. Therefore, we obtain the following conditions.

$$\begin{split} \lim_{\lambda \to \infty} \left\| (I - \pi(e_{\lambda})) \circ (I - (I - \pi(e_{\mu})))a \right\| &= \lim_{\lambda \to \infty} \left\| (I - \pi(e_{\lambda})) \circ \pi(e_{\mu})a \right\| \\ &= \lim_{\lambda \to \infty} \left\| (I - \pi(e_{\lambda})) \circ (e_{\mu} \bullet a) \right\| \\ &= \lim_{\lambda \to \infty} \left\| (e_{\mu} \bullet a - e_{\lambda} \bullet e_{\mu} \bullet a) \right\| \\ &= \left\| e_{\mu} \bullet a - e_{\mu} \bullet a \right\| = 0 \end{split}$$

and

$$\begin{split} \lim_{\lambda \to \infty} \left\| (I - (I - \pi(e_{\mu}))) \circ (I - \pi(e_{\lambda}))a \right\| &= \lim_{\lambda \to \infty} \left\| \pi(e_{\mu}) \circ (I - \pi(e_{\lambda}))a \right\| \\ &= \lim_{\lambda \to \infty} \left\| \pi(e_{\mu}) \circ (a - e_{\lambda} \bullet a) \right\| \\ &= \lim_{\lambda \to \infty} \left\| (e_{\mu} \bullet a - e_{\mu} \bullet e_{\lambda} \bullet a) \right\| \\ &= \left\| e_{\mu} \bullet a - e_{\mu} \bullet a \right\| = 0 \end{split}$$

Additionally, since A is a Banach algebra $||T_{\lambda}|| < 1$, the net is uniformly bounded. Therefore it is an *LR*-net.

CHAPTER 4

CONVERGENCE OF LR-NETS

4.1 Extension of Eberlein's theorem to *LR*-net

4.1.1 Theorem

A lot of results about concrete *LR*-nets belongs to the classical ergodic theory. However, only very few facts about general *LR*-nets, like Proposition 29 and Theorem 32, are known. For instance, Theorem 47 below is well known more than sixty years for *LR*-nets of Cesàro averages as the *Mean Ergodic Theorem*, and for \mathcal{T} -ergodic nets as the *Eberlein Theorem*. Theorem 47 had been proved for *M*-sequences by Lotz in [27, Thm. 3]. The general form of Theorem 47 had been stated, without a proof, by Räbiger in [30, Prop. 2.3]. In this section, we present a complete proof (see [13]). We remind the following theorem needed as a tool to prove the Eberlein Theorem for *LR*-nets.

Theorem 46 (Mazur) *Let X be a Banach space and let* $A \subset X$ *. Then the norm closure of A coincides with its weak closure.*

Theorem 47 (Räbiger) Let $\Theta = (T_{\lambda})_{\lambda \in \Lambda}$ be an LR-net in X. Then the following assertions are equivalent:

- (i) Θ is strongly convergent.
- (ii) The net $(T_{\lambda}x)_{\lambda \in \Lambda}$ has a weak cluster point for every $x \in X$.

Proof: The proof of the first implication is straightforward. For the converse, let $x \in X$ be an arbitrary element and *y* be a weak cluster point of the net $(T_{\lambda}x)_{\lambda \in \Lambda}$. By the Mazur theorem,

$$y \in \overline{\operatorname{co}(T_{\lambda}x)_{\lambda \in \Lambda}} \,. \tag{1}$$

To begin with, we need to show that $y \in Fix(\Theta)$. Since y is a weak cluster point of the *LR*-net $(T_{\lambda})_{\lambda \in \Lambda}$ for the point x, for given ϵ , $(T_{\lambda}x)_{\lambda \in \Lambda}$ satisfies the following

$$|\langle T_{\lambda}x,h\rangle-\langle y,h\rangle|\leq \frac{\epsilon}{3}$$

By using the condition (*LR*2), we obtain for fix $\mu \in \Lambda$ and $\epsilon > 0$ there exists $\zeta \in \Lambda$ satisfying the following three formulas:

$$|\langle T_{\mu} y, h \rangle - \langle T_{\mu} \circ T_{\zeta} x, h \rangle| \le \frac{\epsilon}{3}, \qquad (2)$$

$$|\langle T_{\mu} \circ T_{\zeta} x, h \rangle - \langle T_{\zeta} x, h \rangle| \le \frac{\epsilon}{3}, \qquad (3)$$

$$|\langle T_{\zeta} x, h \rangle - \langle y, h \rangle| \le \frac{\epsilon}{3} \,. \tag{4}$$

Consequently, the summation of (2), (3), (4) we get;

$$\begin{split} |\langle T_{\lambda}x,h\rangle - \langle y,h\rangle| &= |\langle T_{\mu}y,h\rangle - \langle T_{\mu}\circ T_{\zeta}x,h\rangle + \langle T_{\mu}\circ T_{\zeta}x,h\rangle - \langle T_{\zeta}x,h\rangle + \langle T_{\zeta}x,h\rangle - \langle y,h\rangle| \\ &\leq |\langle T_{\mu}y,h\rangle - \langle T_{\mu}\circ T_{\zeta}x,h\rangle| + |\langle T_{\mu}\circ T_{\zeta}x,h\rangle - \langle T_{\zeta}x,h\rangle| \\ &+ |\langle T_{\zeta}x,h\rangle - \langle y,h\rangle| < \epsilon \end{split}$$

As a result $y \in Fix(\Theta)$.

Secondly, we prove that the net $(T_{\lambda}x)_{\lambda \in \Lambda}$ converges to *y* in the norm topology. On account of the uniform boundedness of *LR*-nets, the supremum $M := \sup_{\lambda \in \Lambda} ||T_{\lambda}x||$ is finite. Given $\epsilon > 0$, there exists an $S \in \overline{\operatorname{co}(T_{\lambda})_{\lambda \in \Lambda}}$ satisfying

$$\|y - Sx\| \le \epsilon \,. \tag{5}$$

By (*LR* 2), there exists $\lambda_0 \in \Lambda$ such that

$$\|T_{\lambda} \circ S \ x - T_{\lambda} \ x\| \le \epsilon \quad (\forall \lambda \ge \lambda_0) \,. \tag{6}$$

Combining of these conditions with $y \in Fix(\Theta)$, we have

$$\begin{aligned} ||y - T_{\lambda} x|| &= ||y - T_{\lambda} \circ S \ x + T_{\lambda} \circ S \ x - T_{\lambda} x|| \\ &\leq ||T_{\lambda} (y - S \ x)|| + ||T_{\lambda} \circ S \ x - T_{\lambda} x|| \\ &\leq M \epsilon + \epsilon \quad (\forall \lambda \ge \lambda_0) \,. \end{aligned}$$

Since $x \in X$ and $\epsilon > 0$ were chosen arbitrary, the formula $||y - T_{\lambda}x|| < \epsilon$ implies that the net Θ converges strongly

4.1.2 Application to attractors

The following definition is needed for more application of the convergence theorem. The notion of attractor or constrictor was invented for a one-parameter discrete Markov semigroup by A. Lasota, T. Y. Li and J. A. Yorke. Later, for an abelian linear operator semigroup on a Banach space, many authors investigated the notion of attractors (see [12, 22]).

Definition 48 Let $\Theta = (T_{\lambda})_{\lambda \in \Lambda}$ be an operator net on a Banach space X. A subset A is called *an* attractor for Θ if

$$\lim_{\lambda \to \infty} \operatorname{dist}_{\|\cdot\|}(T_{\lambda}x, A) = 0 \quad (\forall x \in B_X).$$

where B_X is the closed unit ball of X. The family of all attractors of Θ is denoted by $Att(\Theta)$.

The first application of attractor is the following proposition.

Proposition 49 Every LR-net possessing a weakly compact attractor is strongly convergent.

Proof: Let *A* be an weakly compact attractor for an *LR*-net Θ . Given an element $x \in X$ and a sequence $(\lambda_n)_{n=0}^{\infty}$ in Λ that converges to ∞ . Take $a_{\lambda_n} \in A$ for any λ_n such that $||a_{\lambda_n} - T_{\lambda_n}x|| \rightarrow 0$. Since *A* is weakly compact, the sequence (a_{λ_n}) has a weakly convergent subsequence $(a_{\lambda_{n_m}})$. Then the subsequence $(T_{\lambda_{n_m}}x)_{m=0}^{\infty}$ of $(T_{\lambda_n}x)_{n=0}^{\infty}$ is weakly convergent to the same limit.

Theorem 50 (*Emel'yanov*) Every LR-net containing a weakly compact operator is strongly convergent.

Proof:

Let $\Theta = (T_{\lambda})_{\lambda \in \Lambda}$ be an *LR*-net on a Banach space *X* and let T_{λ_0} be a weakly compact. Take $x \in B_X$ because of (*LR* 3) and uniform boundedness of *LR*-net, ie, $T_{\lambda}x \in MB_X$ for all λ we get

$$\lim_{\lambda\to\infty}\operatorname{dist}_{\parallel\cdot\parallel}(T_{\lambda}x,T_{\lambda_0}(MB_X))=0.$$

Since the norm-closure of $T_{\lambda_0}(MB_X)$ is a weakly compact and $\lim_{\lambda \to \infty} \text{dist}_{\|\cdot\|}(T_{\lambda}x, T_{\lambda_0}(MB_X)) = 0$ then $T_{\lambda_0}(MB_X)$ is weakly compact attractor of Θ . By Proposition 49 Θ is strongly convergent.

4.2 Extension of Sine's theorem to LR-nets

In this section, we present an extension of Sine's ergodic theorem to *LR*-nets. The theorem was discovered by R. Sine [35] in the special case when an *LR*-net is a net of Cesàro averages of a single operator (cf. Krengel's book [18, Thm. 2.1.4]). It had been extended to arbitrary \mathcal{T} -ergodic nets, by J.J. Koliha, R. Nagel, and R. Sato (cf. [18, Thm. 2.1.9]). For "small" *LR*-net (=*M*-sequence) it is due to H.P. Lotz [27, Thm. 3].

Theorem 51 An LR-net $\Theta = (T_{\lambda})_{\lambda \in \Lambda}$ in X is strongly convergent if and only if its fixed space Fix(Θ) separates the fixed space Fix(Θ^*) of the adjoint operator net $\Theta^* = (T^*_{\lambda})_{\lambda \in \Lambda}$ in X^{*}.

Proof: Assume that $Fix(\Theta)$ separates $Fix(\Theta^*)$. In view of Theorem 32, to show the strong convergence of Θ , it suffices to prove (1). If (1) is failed then, by the Hahn – Banach theorem, there exists an $h \in X^*$, $h \neq 0$, with $\langle x, h \rangle = 0$ for all

$$x \in \operatorname{Fix}(\Theta) \oplus \overline{\bigcup_{\lambda \in \Lambda} (I - T_{\lambda}) X}$$
.

Show that $h \in Fix(\Theta^*)$. Since

$$(y - T_{\mu}y) \in \overline{\bigcup_{\lambda \in \Lambda} (I - T_{\lambda})X} \quad (\forall y \in X, \forall \mu \in \Lambda),$$

we have

$$\langle y,h\rangle = \langle T_{\mu}y,h\rangle = \langle y,T_{\mu}^{*}h\rangle = 0 \quad (\forall y \in X,\forall \mu \in \Lambda).$$
(7)

It follows from (7) that $T_{\mu}^*h = h$ for all $\mu \in \Lambda$ and therefore $h \in Fix(\Theta^*)$. Thus, h is a nonzero fixed point of Θ^* such that $\langle x, h \rangle = 0$ for all $x \in Fix(\Theta)$. This contradicts the assumption.

Assume the net Θ converges strongly, and denote its limit in the strong operator topology by *P*. Take an $h \in \text{Fix}(\Theta^*)$, $h \neq 0$. In view of $h \neq 0$, there exists an $x \in X$ with $\langle x, h \rangle \neq 0$. Consequently

$$\langle Px,h\rangle = \lim_{\lambda \to \infty} \langle T_{\lambda}x,h\rangle = \lim_{\lambda \to \infty} \langle x,T_{\lambda}^*h\rangle = \langle x,h\rangle \neq 0.$$
(8)

Since $Px \in Fix(\Theta)$, the formula (8) shows that $Fix(\Theta)$ separates $Fix(\Theta^*)$.

CHAPTER 5

LR-NETS OF MARKOV OPERATORS IN *L*¹-SPACES

In this chapter, we investigate LR-nets of Markov operators in L^1 -spaces. In the first section, we give the conditions for strong convergence of LR-nets of Markov operators. Then the asymptotic stability and Lasota criteria are discussed.

5.1 The main result

The main goal of the section is to study the strong convergence of *LR*-net of Markov operator in L^1 -spaces. The result has first given by Emelyanov and Wolff [11] for single Markov operator *T* on L^1 space. Then Alpay, Binhadjah, Emelyanov and Ercan [2] generalized the result for positive power bounded operators in KB-spaces.

The following theorems are necessary for the proof of the Theorem 54 in this section. The first theorem present property of positive projections defined on Banach lattices.

Theorem 52 [32, Prop III,11.5] Let P be a positive projection in L(E) where E is any Banach lattice. The range P(E) is a Riesz space under the order induced by E and a Banach lattice under a norm equivalent to the norm induced by E. If P is strictly positive, then P(E) is a Riesz subspace of E.

Therefore $Fix(\Theta)$ is a Banach lattice under a norm equivalent to the norm induced by $E := L^1$. Additionally, by Kakutani theorem which is the statement that a Banach lattice is an AL-space if and only if it is Riesz isomorphic to $L^1(\mu)$. Because of the additivity of the norm on $Fix(\Theta)$, it is itself an L^1 -space. Moreover the following theorem gives a condition under which an Archimedean Riesz space is finite dimensional [1].

Theorem 53 (Judin) If every subset of pairwise disjoint elements in an Archimedean Riesz space *E* is finite then *E* is Riesz isomorphic to some \mathbb{R}^n .

In the mean time, we can state our main theorem in this chapter. Note that the preceding of the following theorem [11, Thm 1] is generalized in [2] to Cesàro averages of a positive power bounded operator in arbitrary *KB*-space (see [2, Thm 1, Thm 2]).

Theorem 54 Let $\Theta = (T_{\lambda})_{\lambda \in \Lambda}$ be a LR-net of Markov operators in $E := L^1(\Omega, \Sigma, \mu)$. Then the following assertions are equivalent:

(*i*) there exist a function $g \in L^1_+$ and a real η , $0 \le \eta < 1$, such that

$$\overline{\lim_{\lambda \to \infty}} \left\| (T_{\lambda} f - g)_+ \right\| \le \eta \quad (\forall f \in \mathcal{D});$$
(1)

(*ii*) the net Θ is strongly convergent and dim Fix(Θ) < ∞ .

Proof: By the remark after the definition of *LR*-nets, we may assume that the Markov *LR*-net Θ contains the identity operator.

 $(ii) \Rightarrow (i)$: For the first assertion, we show that Θ converges strongly in the first step. By Theorem 51, it is enough to check that $Fix(\Theta)$ separates $Fix(\Theta^*)$, more precisely for every $0 \neq \psi \in Fix(\Theta^*)$, there exists a vector $0 \neq w \in Fix(\Theta)$ which satisfies $\langle \psi, w \rangle \neq 0$.

Let $0 \neq \psi \in E^* = L^{\infty}(\Omega, \Sigma, \mu)$, $T^*_{\lambda}\psi = \psi$ for all $\lambda \in \Lambda$. We may assume $||\psi_+|| = ||\psi|| = 1$. Take some $f \in E$ which satisfies ||f|| = 1 and $\langle \psi_+, f \rangle \ge 1 - \epsilon$ for $\epsilon := (1 - \eta)/3$. We have |||f||| = ||f|| = 1 and

$$1 \ge \langle |\psi|, |f| \rangle \ge \langle \psi_+, |f| \rangle \ge \langle \psi_+, f \rangle \ge 1 - \epsilon.$$

Consequently

$$\langle \psi, |f| \rangle = \langle 2\psi_+, |f| \rangle - \langle |\psi|, |f| \rangle \ge 2(1-\varepsilon) - 1 = 1 - 2\epsilon.$$

Let $f'' \in E^{**}$ be a w^* -cluster point of the net $(T_{\lambda}(|f|))_{\lambda \in \Lambda}$. This cluster point exists because of Banach-Alaoglu Theorem. Then $\langle T_{\lambda}|f|, x \rangle \rightarrow \langle f'', x \rangle$, $\langle (I - T_{\mu}) \circ T_{\lambda}|f|, x \rangle \rightarrow \langle (I - T_{\mu})f'', x \rangle$ and by (*LR* 3), $\langle (I - T_{\mu}) \circ T_{\lambda} | f |, x \rangle \to 0$. Therefore f'' satisfies $T_{\mu} f'' = f''$ for arbitrary $\mu \in \Lambda$. By assumption, we obtain

$$\overline{\lim_{\lambda \to \infty}} \mathrm{dist}(T_{\lambda}(|f|), [0, g]) \leq \eta$$

and order interval [0, g] is weakly^{*} compact in E. Hence

$$f'' \in [0,g] + \eta B_{E^{**}} \subseteq E + \eta B_{E^{**}},$$

where $B_{E^{**}}$ denotes the unit ball of E^{**} . Since the dual to any normed lattice is Dedekind complete, any band in a Dedekind complete vector lattice is a projection band and any *AL*space is *KB*-space, *E* is a projection band in E^{**} (see [1, Thm. 14.12]). Denote by *P* the band projection $P : E^{**} \to E$. Then $(I - P)f'' \in \eta B_{E^{**}}$, and

$$\langle \psi, Pf'' \rangle = \langle \psi_+, Pf'' \rangle - \langle \psi_-, Pf'' \rangle =$$

$$\langle f'', \psi_+ \rangle - \langle (I - P)f'', \psi_+ \rangle - \langle \psi_-, Pf'' \rangle \ge$$

$$\langle f'', \psi \rangle - \eta = \langle \psi, |f| \rangle - \eta \ge 1 - 2\epsilon - \eta = \epsilon > 0.$$
(2)

It follows from (2) that $Pf'' \neq 0$. Since f'' is a *w*^{*}-cluster point of the net $(T_{\lambda}(|f|))_{\lambda \in \Lambda}$, Pf'' > 0. For any $\zeta, \mu \in \Lambda$, we have

$$T_{\zeta} \circ P f'' = T_{\zeta} \circ P \circ T_{\mu}^{**} f'' \ge T_{\zeta} \circ P \circ T_{\mu}^{**} \circ P f'' =$$
$$T_{\zeta} \circ P \circ T_{\mu} \circ P f'' = T_{\zeta} \circ T_{\mu} \circ P f'' .$$

In particular, for $T_{\zeta} = I_E$, we obtain

$$Pf'' \ge T_{\mu} \circ Pf'' \quad (\forall \mu \in \Lambda).$$
(3)

Since T_{μ} is a Markov operator and $P f'' \ge 0$, it follows from (3) that

$$T_{\mu}(Pf'') = Pf'' \quad (\forall \mu \in \Lambda).$$
(4)

Clearly $\langle \psi, Pf'' \rangle > 0$, therefore there exists a vector $w \in Fix(\Theta)$, namely w = Pf'', which satisfies $\langle \psi, w \rangle \neq 0$. By Theorem 51, the net Θ converges strongly.

The space $Fix(\Theta)$ is an L^1 -space as the range of a Markov projection. By the condition (1), the inequality

$$\|(y-g)_+\| = \overline{\lim_{\lambda \to \infty}} \left\| (T_\lambda y - g)_+ \right\| \le \eta \quad (\forall y \in \mathcal{D} \cap \operatorname{Fix}(\Theta))$$

holds. Since $||x|| = ||(x - z)_+|| + ||x \wedge z||$ for all $x, z \in L^1_+$, we obtain

$$||y \wedge g|| \ge 1 - \eta > 0$$
 $(\forall y \in \mathcal{D} \cap Fix(\Theta)).$

Hence there exist at most finitely many pairwise disjoint densities in $Fix(\Theta)$. Thus dim $Fix(\Theta) < \infty$.

 $(ii) \Rightarrow (i)$: If dim Fix(Θ) < ∞ there exists a family of pairwise disjoint densities $u_1, u_2, ..., u_p$ such that

$$\operatorname{Fix}(\Theta) = \operatorname{span}\{u_1, u_2, ..., u_p\}.$$

Denote $g := u_1 + u_2 + \dots + u_p$. Take a density f, then $Pf := \overline{\lim_{\lambda \to \infty} T_{\lambda} f}$ is a linear combination of u_1, u_2, \dots, u_p since $Pf \in Fix(\Theta)$. In view of pairwise disjointness of densities u_1, u_2, \dots, u_p , we obtain

$$Pf = \sum_{i=1}^{p} \alpha_i u_i \le \sum_{i=1}^{p} u_i = g.$$

Thus

$$\overline{\lim_{\lambda \to \infty}} \left\| (T_{\lambda} f - g)_+ \right\| = \left\| (Pf - g)_+ \right\| = 0 \quad (\forall f \in \mathcal{D}),$$

which completes the proof.

5.2	Asymptotic stability of Mark	ov nets

The section is reserved for asymptotic stability in terms of lower bounds. The first theorem is well known as Lasota's lower bound criterion of asymptotic stability of Markov semigroups. In addition we discuss a theorem of Komornik and Lasota [19] for *LR*-nets. The following definitions are motivated by the terminology used in the Lasota-Mackey [22].

Definition 55 Let $\Theta = (T_{\lambda})_{\lambda \in \Lambda}$ be a Markov net in $L^{1}(\Omega, \Sigma, \mu)$. We call Θ is asymptotically stable whenever there exists a density u such that

$$\lim_{\lambda \to \infty} \|T_{\lambda}f - u\| = 0 \qquad (\forall f \in \mathcal{D}).$$
(5)

Definition 56 A function $h \in L^1_+$ is called a lower-bound function for Θ if

$$\lim_{\lambda \to \infty} \|(h - T_{\lambda}f)_{+}\| = 0 \qquad (\forall f \in \mathcal{D}).$$
(6)

We say that h is nontrivial if $h \neq 0$. Note that any lower-bound function has the norm at most one.

The main result of this section is the following theorem which generalizes the main result of the paper [11] on Cesàro averages of Markov semigroups (see results on Markov semigroups [23, Thm. 2.], [20, Thm. 1.1.], [21, Cor. 1.4.]) to an arbitrary Markov *LR*-net.

Theorem 57 Let $\Theta = (T_{\lambda})_{\lambda \in \Lambda}$ be an LR-net of Markov operators in $E := L^1(\Omega, \Sigma, \mu)$. Then the following assertions are equivalent:

- (i) Θ is asymptotically stable.
- (ii) There exists a nontrivial lower-bound function for Θ .

Proof: (*i*) \Rightarrow (*ii*): Since Θ is asymptotically stable, there exists a density *u* satisfying $\lim_{\lambda \to \infty} ||T_{\lambda}f - u|| = 0$ for every $f \in \mathcal{D}$. Obviously, *u* is a non-trivial lower-bound function for Θ .

 $(ii) \Rightarrow (i)$: Let $0 \neq h \in E_+$ be a lower-bound function for Θ . Then

$$\overline{\lim_{\lambda\to\infty}}\,\,\|(T_\lambda f-h)_+\|\leq\eta\ \, (\forall f\in\mathcal{D})\,,$$

with $\eta := 1 - ||h||$. In view of Theorem 54, the net Θ converges strongly and dim Fix(Θ) := $p < \infty$. Theorem 51 implies that

$$E = \operatorname{Fix}(\Theta) \oplus \overline{\bigcup_{\lambda \in \Lambda} (I - T_{\lambda}) L^1}.$$

The subspace Fix(Θ) of *E* is an *L*¹-space as the range of a Markov projection, and hence it possesses a linear basis $\{u_i\}_{i=1}^p$ which consists of pairwise disjoint densities. Since $T_{\lambda}u_i = u_i$ for all $\lambda \in \Lambda$,

$$||(h - u_i)_+|| = \lim_{\lambda \to \infty} ||(h - T_\lambda u_i)_+|| = 0$$

Therefore

$$u_i \ge h > 0 \quad (i = 1, ..., p).$$
 (7)

Since the family $\{u_i\}_{i=1}^p$ consists of pairwise disjoint densities, the condition (7) ensures dim Fix(Θ) = 1.

Now, $E = \mathbb{R} \cdot u_1 \oplus \overline{\text{span} \bigcup_{\lambda \in \Lambda} (I - T_{\lambda}) L^1}$ and hence

$$\lim_{\lambda\to\infty}T_\lambda f=u_1$$

for every density f, since Θ is a Markov *LR*-net.

The next simple proposition provides us with examples of Markov *LR*-nets which need not to be \mathcal{T} -ergodic nets. We remark that the assertion below is trivial, whenever the Markov net is an abelian operator semigroup considered with natural ordering (see Example 36).

Proposition 58 Every asymptotically stable Markov net is an LR-net.

Proof: Let $\Theta = (T_{\lambda})_{\lambda \in \Lambda}$ be a Markov net in L^1 such that there exists a density *u* such that

$$\lim_{\lambda \to \infty} \|T_{\lambda}f - u\| = 0 \qquad (\forall f \in \mathcal{D}).$$
(8)

Since any Markov net is equi-continuous, we need to check only conditions (*LR2*) and (*LR3*) of Definition 27. Obviously, it is enough to prove these conditions for an arbitrary density x. Fix a $\mu \in \Lambda$ and take an $x \in \mathcal{D}$. Denote $z := (I - T_{\mu})x$, then $||z_{+}|| = ||z_{-}||$ since T_{μ} is a Markov operator. Thus we obtain

$$\lim_{\lambda \to \infty} T_{\lambda} \circ (I - T_{\mu}) x = \lim_{\lambda \to \infty} T_{\lambda}(z_{+} - z_{-}) =$$
$$\lim_{\lambda \to \infty} T_{\lambda}(z_{+}) - \lim_{\lambda \to \infty} T_{\lambda}(z_{-}) = ||z_{+}|| \cdot u - ||z_{-}|| \cdot u = 0,$$

which gives (LR2). Moreover,

$$\lim_{\lambda \to \infty} (I - T_{\mu}) \circ T_{\lambda} x = (I - T_{\mu}) \lim_{\lambda \to \infty} T_{\lambda} x = (I - T_{\mu}) u = 0.$$

So (LR3) is also satisfied.

5.3 The Extension of Lasota's stability criterion to an arbitrary abelian Markov semigroup

The Lasota's famous criterion of asymptotic stability says that a one-parameter Markov semigroup is asymptotically stable if and only if there is a nontrivial lower-bound function for this semigroup (see, for example [20, Thm.1.1.], [22, Thm.5.6.2 and Thm.7.4.1] and [12, Thm.3.2.1]). It is worthy of note that this criterion, in the case of Frobenius-Perron operators, goes back to the work of A. Lasota and J. A. Yorke [23, Thm. 1, Thm. 2] We prove Theorem 46 which generalizes Lasota's lower-bound criteria [20, Thm.1.1.] to abelian Markov semigroups. In this section, we always assume that an abelian Markov semigroup is an operator net with respect to the natural ordering mentioned in Example 36.

Theorem 59 Let $\Theta = (T_{\lambda})_{\lambda \in \Lambda}$ be an abelian Markov semigroup in L^1 . Then the following assertions are equivalent:

- (i) Θ is asymptotically stable.
- (ii) There exists a nontrivial lower-bound function for Θ .

The proof of this theorem is postponed to the end of the section. What may happened in the case of arbitrary non-abelian Markov semigroup is not known.

The following lemma is an important step in the proof of Theorem 59.

Lemma 60 Let $\Theta = (T_{\lambda})_{\lambda \in \Lambda}$ be an abelian Markov semigroup in L^1 possessing a nontrivial lower-bound function then Θ is an LR-net.

Proof: Let $0 \neq h \in L^1_+$ be a nontrivial lower bound function for Θ , then obviously $||h|| \leq 1$. As in the proof of Proposition 58, it is enough to check conditions (*LR2*) and (*LR3*), moreover, since Θ is abelian, it suffices to prove (*LR2*) only. Thus we have to prove the following formula

$$\lim_{\lambda \to \infty} \|T_{\lambda} \circ (I - T_{\mu})f\| = 0 \quad (\forall \mu \in \lambda, f \in L^{1})$$

Obviously, for any $f \in L^1$, the vector $(I - T_\mu) f$ belongs to L_0^1 , therefore it is enough to prove

$$\lim_{\lambda \to \infty} \|T_{\lambda} z\| = 0 \quad (\forall z \in L_0^1).$$
⁽⁹⁾

Take an arbitrary $z \in L_0^1$. Write z in the form

$$z = 2^{-1} ||z|| (y_1 - y_2),$$

where $y_1 = 2 \cdot ||z||^{-1} \cdot z_+$, $y_2 = 2 \cdot ||z||^{-1} \cdot z_-$. Therefore y_1 and y_2 are densities. Since *h* is a lower bound function for the Markov semigroup Θ , there exists an $\lambda_1 \in \Lambda$ such that, the following

$$\|(h - T_{\lambda}y_1)_+\| \le 4^{-1} \|h\| \& \|(h - T_{\lambda}y_2)_+\| \le 4^{-1} \|h\|$$
(10)

holds for every $\lambda > \lambda_1$. The formula (10) can be rewritten as

$$||T_{\lambda}y_{1} \wedge h|| \ge \frac{3}{4} ||h|| \& ||T_{\lambda}y_{1} \wedge h|| \ge \frac{3}{4} ||h|| \quad (\forall \lambda > \lambda_{1}).$$
(11)

It follows from (11) that $||T_{\lambda}y_1 \wedge T_{\lambda}y_2|| \ge 1/2$ for all $\lambda > \lambda_1$. Hence

$$||T_{\lambda}y_1 - T_{\lambda}y_2|| \le 2 - 2^{-1} ||h|| \quad (\forall \lambda > \lambda_1),$$

and

$$||T_{\lambda}z|| \le (1 - 4^{-1}||h||)||z|| \quad (\forall \lambda > \lambda_1).$$

Replacing z with $T_{\lambda_1}z \in L_0^1$ and repeating the above arguments gives an element $\lambda_2 \in \Lambda$ satisfying

$$\|T_{\lambda} \circ T_{\lambda_1} z\| \le (1 - 4^{-1} \|h\|) \|T_{\lambda_1} z\| \quad (\forall \lambda > \lambda_2) \,.$$

By induction, we obtain a sequence $(\lambda_n)_{n=1}^{\infty} \subseteq \Lambda$ satisfying

$$||T_{\lambda} \circ T_{\lambda_{n-1}}z|| \le (1 - 4^{-1}||h||)||T_{\lambda_{n-1}}z|| \quad (\forall \lambda > \lambda_n)$$

$$\tag{12}$$

for every n. It follows from (12) that

$$||T_{\lambda}z|| \le ||T_{\lambda_{n}} \circ T_{\lambda_{n-1}} \dots \circ T_{\lambda_{1}}z|| \le (1 - 4^{-1}||h||)^{n} \cdot ||z|| \quad (\forall \lambda > \lambda_{1} + \lambda_{2} + \dots + \lambda_{n}).$$
(13)

Since ||h|| > 0, we obtain $\lim_{\lambda \to \infty} ||T_{\lambda z}|| = 0$, which is exactly the formula (9). The proof is completed.

Now we are able to prove the main result of the section.

Proof of Theorem 46: $(i) \Rightarrow (ii)$: If the Markov semigroup Θ is asymptotically stable then it is an *LR*-net by Proposition 58. The existence of nontrivial lower-bound function for Θ follows now from Theorem 57.

 $(ii) \Rightarrow (i)$: Suppose that there exists a nontrivial lower-bound function for Θ . Then Θ is an *LR*-net by Lemma 60, and the asymptotic stability of Θ follows from Theorem 57.

REFERENCES

- Aliprantis C. D.; Burkinshaw O., *Positive Operators*, Academic Press, Orlando, London, (1985)
- [2] Alpay Ş; Binhadjah A.; Emel'yanov E. Yu.; Ercan Z., Mean Ergodicity of Positive Operators in KB-spaces, J. Math. Anal. Appl. 323 (2006), 371-378
- [3] Bartoszek W.; Erkurşun N., On Quasi Compact Markov Nets, (to appear in Ergodic Theory and Dynamical Systems)
- [4] Bermúdez T.; González M.; Mbekhta M., Operators with an ergodic power, Studia Math. 141 (2000), no.3, 201–208.
- [5] Bessaga C. and Pelczynski A., Selected Topics in Infinite dimensional Topology Mathematical Monographs PWN 58 Warsawa 1975.
- [6] Eberlein W. F., Abstract ergodic theorems and weak almost periodic functions, Trans. Amer. Math. Soc. 67 (1949), 217–240.
- [7] Emel'yanov E. Yu., Asymptotic behavior of Lotz-R\"abiger nets and of martingale nets, (to appear in Sibirsk. Mat. Zh.)
- [8] Emel'yanov E. Yu., Banach lattices on which every power-bounded operator is mean ergodic, Positivity 1 (1997), no.4, 291-296
- [9] Emel'yanov E. Yu.; Wolff M. P. H., Mean ergodicity on Banach lattices and Banach spaces, Arch. Math. (Basel) 72 (1999), no.3, 214-218

- [10] Emel'yanov E. Yu., Conditions for a C₀-semigroup to be asymptotically finitedimensional, Siberian Math. J. 44 (2003), no. 5, 793-796
- [11] Emel'yanov E. Yu.; Wolff M. P. H., *Mean lower bounds for Markov operators*, Ann.Pol. Math. 83 (2004), no.1, 11–19.
- [12] Emel'yanov E. Yu., Non-spectral asymptotic analysis of one-parameter operator semigroups Operator Theory: Advances and Applications, 173. Birkhäuser Verlag, Basel, 2007. viii+174 pp.
- [13] Emel'yanov E. Yu.; Erkurşun N., Generalization of Eberlein's and Sine's Ergodic Theorems to LR-nets, Vladikavkaz. Mat. Zh. 9 (2007), no.3, 22–26.
- [14] Emel'yanov E. Yu.; Erkurşun N., An extension of Sine's counterexample, Positivity 13 (2009), no.1, 125–127.
- [15] Emel'yanov E. Yu.; Erkurşun N., Lotz-Räbiger nets of Markov operators in L₁-spaces,
 (to appear in J. Math. Anal. Appl.)
- [16] Engelking R., General Topology, Heldermann Verlag, Berlin 1985
- [17] Hille E.; Phillips R. S., Functional Analysis and semigroups. Rev. Ed. Amer. Soc. Colloq. Publ., vol.31 Amer. Math. Soc., New York, (1957)
- [18] Krengel U., Ergodic Theorems, De Gruyter, Berlin New York (1985).
- [19] Komornik J.; Lasota A., Asymptotic decomposition of Markov operators, Bull. Polish Acad. Sci. Math. 35 (1987), no.5-6, 321-327
- [20] Lasota A., Statistical stability of deterministic systems, Equadiff 82 (Wurzburg, 1982),
 386–419, Lecture Notes in Math., 1017, Springer, Berlin, (1983).

- [21] Lasota A.; Li T. Y.; Yorke J. A., Asymptotic periodicity of the iterates of Markov operators, Trans. Amer. Math. Soc. 286 (1984), no.2, 751–764.
- [22] Lasota A. and Mackey M. C., Chaos, fractals, and noise.Stochastic aspects of dynamics, Second edition. Applied Mathematical Sciences, 97. Springer-Verlag, New York,(1994). xiv+472 pp.
- [23] Lasota A.; Yorke J. A., *Exact dynamical systems and the Frobenius-Perron operator*, Trans. Amer. Math. Soc. 273 (1982), no.1, 375–384
- [24] Lasota A.; Yorke J. A., Lower bound technique for Markov operators and iterated function systems, Random Comput. Dynam. 2 (1994), no.1, 41–77.
- [25] Lotz H. P., Uniform ergodic theorems for Markov operators on C(X), Math. Z. 178 (1981), 145-156
- [26] Lotz H. P., Tauberian theorems for operators on Banach spaces, Semesterbericht Functionalanalysis, Tübingen, WS 1983/84, 1–15.
- [27] Lotz H. P., Tauberian theorems for operators on L_∞ and similar spaces, In: Biersted,
 K.-D., Fuchssteiner, B. (ed.) Functional analysis: Surveys and recent results, pp. 117-133, Amsterdam: North Holland 1984.
- [28] Lotz H. P., Uniform convergence of operators on L_{∞} and similar spaces, Math. Z. **190** (1985), no.2, 207–220.
- [29] Oxtoby J., Ergodic sets, Bull. Amer. Math. Soc. 58 (1952), 116-136
- [30] Räbiger F., Stability and ergodicity of dominated semigroups: II. The strong case, Math. Ann. 297 (1993), 103–116.

- [31] Rudnicki R., Markov operators: Applications to diffusion processes and population dynamics, Appl. Math. (Warsaw) 27 (2000), no.1, 67–79.
- [32] Schaefer H. H., *Banach Lattices and Positive Operators*, Die Grundlehren der mathematischen Wissenschaften, Band 215, Springer-Verlag, New York- Heidelberg, (1974)
- [33] Shaw S. Y., Abstract ergodic theorems, Proceedings of the Third International Conference on Functional Analysis and Approximation Theory, Vol. I (Acquafredda di Maratea, 1996). Rend. Circ. Mat. Palermo (2) Suppl. No.52, Vol.I (1998), 141-155
- [34] Sine R., Geometric theory of a single Markov operator, Pacific J. Math. 27 (1968), 155-166
- [35] Sine R., A mean ergodic theorem, Proc. Amer. Math. Soc. 24 (1970), no.2, 438–439.
- [36] Sine R., A note on the ergodic properties of homeomorphisms, Proc. Amer. Math. Soc.57 (1976), no.1, 169–172.
- [37] Yosida K., *Functional Analysis*, Second edition. Classics in Mathematics. Springer-Verlag, Berlin (1968)

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3) E. Yu. Emel'yanov; N. Erkurşun, LR-nets of Markov Operators, (To be appeared in Journal of Mathematical Analysis and Applications)

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