ON THE PROBLEM OF LIFTING FIBRATIONS ON ALGEBRAIC SURFACES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

CELALETTİN KAYA

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

JUNE 2010

Approval of the thesis:

ON THE PROBLEM OF LIFTING FIBRATIONS ON ALGEBRAIC SURFACES

submitted by **CELALETTİN KAYA** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Mathematics Department, Middle East Technical University** by,

Prof. Dr. Canan Özgen
Dean, Graduate School of **Natural and Applied Sciences**

Prof. Dr. Zafer Nurlu
Head of Department, **Mathematics**

Prof. Dr. Mustafa Hurşit Önsiper
Supervisor, **Mathematics Department, METU**

Examining Committee Members:

Prof. Dr. Cem Tezer
Mathematics Department, METU

Prof. Dr. Mustafa Hurşit Önsiper
Mathematics Department, METU

Assoc. Prof. Dr. Sefa Feza Arslan
Mathematics Department, METU

Prof. Dr. Yıldıray Ozan
Mathematics Department, METU

Prof. Dr. Ali Sinan Sertöz
Mathematics Department, Bilkent University

Date:

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: CELALETTİN KAYA

Signature :

ABSTRACT

ON THE PROBLEM OF LIFTING FIBRATIONS ON ALGEBRAIC SURFACES

Kaya, Celalettin
Ph. D., Department of Mathematics
Supervisor : Prof. Dr. Mustafa Hurşit Önsiper

June 2010, 33 pages

In this thesis, we first summarize the known results about lifting algebraic surfaces in characteristic $p > 0$ to characteristic zero, and then we study lifting fibrations on these surfaces to characteristic zero.

We prove that fibrations on ruled surfaces, the natural fibration on Enriques surfaces of classical type, the induced fibration on $K3-$surfaces covering these types of Enriques surfaces, and fibrations on certain hyperelliptic and quasi-hyperelliptic surfaces lift. We also obtain some fragmentary results concerning the smooth isotrivial fibrations and the fibrations on surfaces of Kodaira dimension 1.

Keywords: Liftings, Fibrations.

ÖZ

CEBİRSEL YÜZEYLER ÜZERİNDEKİ LİFLENMELERİ KALDIRMA PROBLEMİ HAKKINDA

Kaya, Celalettin

Doktora, Matematik Bölümü

Tez Yöneticisi : Prof. Dr. Mustafa Hurşit Önsiper

Haziran 2010, 33 sayfa

Bu tezde, önce karakteristiği sıfırdan farklı olan cisimler üzerindeki yüzeylerin karakteristik sıfıra kaldırma problemiyle ilgili bilinen sonuçlar özetlenmiş, sonra da bu tip yüzeyler üzerindeki liflenmelerin karakteristik sıfıra kaldırılması çalışılmıştır.

Regle yüzeyler üzerindeki liflenmelerin, klasik tipteki Enriques yüzeyler üzerindeki doğal liflenmenin, bu tipteki Enriques yüzeylerini örten $K3$ yüzeyleri üzerinde elde edilen liflenmenin ve bazı hipereliptik ve kuasi-hipereliptik yüzeyler üzerindeki liflenmelerin kaldırldığı ispat edilmiştir. Ayrıca, pürüzsüz izotrivial liflenmelerle ve Kodaira boyutu 1 olan yüzeyler üzerindeki liflenmelerle ilgili bazı kısmi sonuçlar elde edilmiştir.

Anahtar Kelimeler: Kaldırmalar, Lif Uzayları.

To Hurşit Hocam

ACKNOWLEDGMENTS

In addition to this doctorate thesis, a long and stressful period is coming to an end; ”was it worth it?” is a question which comes to mind. Did I at any stage think about giving up? No. Because I knew I had the support of my supervisor, dear Hurşit Hocam who always was there for me during those hard periods.

Just to get to know Hurşit Hocam is a privilege, without him I would not have had the strength to endure the hard road to the end; certainly this thesis would have failed. His support and wisdom was not limited to the academic issues but extended to my personal life; as a father figure. I can not imagine to the life of me, how can I ever repay him; that is simply impossible.

It is customary to mention more names in such acknowledgement writings, but I find it hard to compare any one else to my supervisor whom I mentioned above as his input was exponentially greater than the numerous contributors to this effort. Also to avoid forgetting any one; I will simply thank the groups and institutions that provided support:

To my professors and lecturers, for learning so much from you, for knowing you;

To the staff in the department, for being ever helpful, for always smiling;

To my friends, within and without the mathematics circle, for being such a joyous crowd;

Thank you, I wish we never depart.

I could not end this without mentioning TÜBİTAK BİDEB for its support of this program and sponsoring many research programs; thanks for making all of this possible.

Last but certainly not least, thanks to my family for their never yielding believe in me, I love each one of you.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

In this thesis, we are concerned with the problem of lifting fibrations on surfaces in characteristic $p > 0$ to characteristic zero. More precisely, we let X be a smooth projective surface over an algebraically closed field k of characteristic $p > 0$, which admits a fibration

$$\phi : X \to Y$$

(with geometrically connected fibers) over a curve Y and we ask the following questions :

- **Existence of a lifting :** Do we have surface $\mathcal{X}$ and a curve $\mathcal{Y}$ over a complete discrete valuation ring R of mixed characteristic with residue field k, and a fibration

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Phi} & \mathcal{Y} \\ & \searrow \quad \swarrow & \\ & S = Spec(R) & \end{array}$$

  such that over the closed point $s : Spec(k) \hookrightarrow S$

$$\Phi_s : \mathcal{X}_s \cong X \to \mathcal{Y}_s \cong Y$$

  is the original fibration ϕ ?

- **Moduli of liftings :** What is the moduli of these liftings? That is, intuitively, in "how many" different ways can we lift $X \to Y$?

We recall in relation to the moduli question that two liftings

$$\Phi_1 : \mathcal{X}_1 \to \mathcal{Y}_1, \ \ \Phi_2 : \mathcal{X}_2 \to \mathcal{Y}_2$$

are identified if we have a commutative diagram given by S-isomorphisms α, β

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{\alpha} & \mathcal{X}_2 \\ \Phi_1 \downarrow & & \downarrow \Phi_2 \\ \mathcal{Y}_1 & \xrightarrow{\beta} & \mathcal{Y}_2 \end{array}$$

such that $\alpha_s = id_X, \beta_s = id_Y$.

Throughout the thesis, we will keep in focus the subtler question of whether "a lifting in the strong sense" (that is a lifting over the Witt ring $W(k)$ - "no ramification" case) exists. The importance of this question can not be over emphasized; it suffices to recall the remarkable consequences of the existence of a lifting of a variety to $W_2(k)$ ([7]).

Obviously, the existence part of the problem has two aspects :

1. **Problem 1 :** Lift the pair (X, Y) to a pair $(\mathcal{X}, \mathcal{Y})$ (over a suitable R).
2. **Problem 2 :** Lift ϕ to a morphism $\Phi : \mathcal{X} \to \mathcal{Y}$.

We start with an overview of Problem 1 which has been a subject of considerable interest over a long period of time. Since any smooth projective curve lifts without any restriction, we consider solely the lifting problem for surfaces.

We will adopt the following notation :

- k is a field (algebraically closed unless otherwise stated) of characteristic $p > 0$.
- R is a complete discrete valuation ring (dvr) of characteristic zero with residue field k.
- $W(k)$ is the ring of Witt vectors over k and $W_n(k) = W(k)/m^n$ is the ring of Witt vectors of length n.
- X is a projective smooth minimal surface over k.
- c_1, c_2 denote the first and the second Chern classes.
- A lifting of X means a projective smooth scheme $\mathcal{X}$ over $S = Spec(R)$ with special fiber $\mathcal{X} \times_S Spec(k) \cong X$.
- $\mu_p = \mathrm{Spec}(k[t]/(t^p - 1))$, $\alpha_p = \mathrm{Spec}(k[t]/(t^p))$ are the standard infinitesimal group schemes.

- For a variety X, Pic_X denotes the Picard scheme of X. Pic^0_X is the component of Pic_X containing the identity.

- For an abelian variety X, $X^\vee$ is the dual abelian variety, i.e., $X^\vee = Pic^0_X$.

We refer to a statement in the text by quoting the number of the chapter and the number assigned to that statement in that chapter.

An Overview of Problem 1 :

We recall the basic lifting results concerning smooth projective surfaces. The results are listed according to Enriques-Kodaira classification (it is well-known that the Kodaira dimension is invariant under lifting ([12], Theorem 9.1)) and we assume that all surfaces are minimal.

I) $\boldsymbol{\kappa = -1}$: Since $\mathbb{P}^2$ does not admit a fibration, it suffices to consider ruled surfaces.

It is easy to see that the lifting problem for ruled surfaces is unobstructed. In fact, any lifting of a ruled surface is a $\mathbb{P}^1$-bundle over a suitable curve (II, Proposition 1).

II) $\boldsymbol{\kappa = 0}$: In this case X belongs to one of the classes listed below.

A. *K*3-Surfaces :

Deligne proved that all $K3$-surfaces lift to characteristic zero with no ramification if $char(k) > 2$ and to $W(k)[\sqrt{p}]$ in all cases ([6]).

B. Enriques Surfaces :

We have the following types of Enriques surfaces as explained in [5] :

- **Classical Enriques Surfaces :**

  Definition 1. An Enriques surface X is said to be *classical*, if the canonical divisor class $K_X \nsim 0$, i.e., K_X is not linearly equivalent to 0.

  It is known that every Enriques surface is classical if $char(k) \neq 2$ ([5], Theorem 1.1.3), and in this case they can be lifted to characteristic zero ([5], Corollary 1.4.1).

  In $char(k) = 2$, a classical Enriques surface X can be lifted to characteristic 0 if we have a regular 1-form with only isolated singularities on X ([5], Corollary 1.4.1).

- **Non-classical Enriques Surfaces :**

  Definition 2. A non-classical Enriques surface X is called a $\boldsymbol{\mu_2}$*-surface* (respectively an $\boldsymbol{\alpha_2}$*-surface*) if $Pic^{\tau}_{X/k} \cong \boldsymbol{\mu_2}$ (respectively $\boldsymbol{\alpha_2}$). In analogy with abelian varieties, a $\boldsymbol{\mu_2}$-surface (respectively an $\boldsymbol{\alpha_2}$-surface) is called *ordinary* (respectively *supersingular*).

  – **Ordinary Enriques Surfaces (μ_2-Surfaces) :**
    It is known that every $\boldsymbol{\mu_2}$-surface lifts to characteristic zero ([5], Corollary 1.4.1).

  – **Supersingular Enriques Surfaces (α_2-Surfaces) :**
    If X is a $\boldsymbol{\alpha_2}$-surface, then it can not be lifted even to $W_2(k)$. However, there are examples where such a surface can be lifted to a ramified extension of the Witt vectors. But it is not known whether every $\boldsymbol{\alpha_2}$-surface can be lifted to characteristic zero.

C. Abelian Surfaces :

The problem of lifting abelian varieties (not just abelian surfaces) to characteristic zero was solved completely. Before summarizing the results about this problem, we give some basic definitions.

Definition 3.

a. Let $char(k) = p > 0$. We say that an abelian variety X over k is *ordinary* if $\alpha_p \not\hookrightarrow X$. Equivalently, an abelian variety X of dimension n is said to be *ordinary* if the set of elements of order p, $X[p^n]$ has precisely $(\mathbb{Z}/p\mathbb{Z})^n$ elements. An equivalent formulation is that the kernel of the geometric Frobenius is μ_p^n.

b. A *polarization divisor* on an abelian variety X is an effective divisor D which is *ample*, i.e., for some positive integer N, the multiple ND of D is a hyperplane section $H_{|X}$ of $X \subset \mathbb{P}^m$ for some $m \geq 1$. The pair (X, D) is said to be a *polarized abelian variety*.

c. Let (X, D) be an n-dimensional polarized abelian variety. Then the degree of D is given by

$$D^n = (H^n_{|X})/N^n,$$

($(H^n_{|X})$ is the degree of the variety $X \subset \mathbb{P}^m$). D is said to be a *principally polarized divisor* if $D^n = n!$.

d. Let L be a line bundle on the abelian variety X and let

$$\mu : X \times X \to X \ \text{ (respectively } p_i : X \times X \to X,\ i = 1, 2)$$

be the multiplication (respectively the projection maps). We define $K(L)$ to be the maximal subscheme of X such that the line bundle

$$\mu^*(L) \otimes p_1^*(L)^{-1} \otimes p_2^*(L)^{-1}$$

is trivial on $K(L) \times X$ ([16], p. 123).

One knows that

1. $K(L)$ is the kernel of the homomorphism $\phi_L : X \to X^\vee = Pic^0_X$ defined set theoretically by

   $$\phi_L(x) = \text{the isomorphism class of } T_x^* L \otimes L^{-1}$$

   where $T_x : X \to X,\ T_x(y) = x + y$ is “the translation by x” ([16], Corollary 5, p. 131).

2. $K(L)$ is a finite subgroup scheme (equivalently, ϕ_L is an isogeny) if and only if L is ample.

Definition 4. A polarized abelian variety (X, L) is said to be *separably polarized* if the isogeny $\phi_L : X \to X^\vee$ is separable.

Now, we list the main results concerning the problem of lifting an abelian variety X to characteristic zero.

1. If $dim(X) \leq 2$, then we can find a polarization λ on X such that X together with λ lifts to $W(k)$:

Theorem 1 ([20], Proposition 11.1). *If X is an abelian variety,* $\dim(X) \leq 2$*, then there exists a polarization λ on X such that (X, λ) lifts to $W(k)$.*

2. If $\dim(X) \geq 3$, then we have examples X for which there exists no polarization λ such that the pair (X, λ) lifts to $W(k)$ ([20], p. 186-189).

3. The most general result in this direction is the following theorem of Mumford :

Theorem 2 ([20], Theorem (Mumford)). *Any polarized abelian variety can be lifted to characteristic zero (possibly with ramification!).*

The index of ramification needed is determined in an article by P. Norman :

Theorem 3 ([18], Main Theorem). *Let k be a field of characteristic p, $p \neq 0, 2$; let (X, λ) be a polarized abelian variety over k. Let W be a local, p-adically complete and separated ring of characteristic zero such that $W/pW \cong k$. Let A be a local W-algebra of characteristic zero that is also p-adically complete and separated; let e denote the ramification index of p in A. Assume either*

i. k is perfect and $1 < e \leq p - 1$, or

ii. $1 < e < p - 1$.

Then (X, λ) lifts to A.

Clearly, this result implies that any abelian variety X lifts in the weak sense, that is, it lifts over an integral domain R of characteristic zero which admits a surjective homomorphism $R \to k$.

However, if X has a separable polarization λ, then (X, λ) lifts to $W(k)$:

Theorem 4 ([19], Corollary 2.4.2). *Any abelian variety which admits a separable polarization (in particular, any abelian variety which admits a principal polarization) can be lifted to characteristic zero (with no ramification).*

Thus we see that for the example of (2) in ([20]), X does not admit a separable polarization.

4. The problem of lifting abelian varieties is related to the well-known fact that if p is not a square in R, then the group scheme α_p does not lift to $Spec(R)$. The following result shows that the main obstruction to lifting an abelian variety without ramification is the existence of infinitesimal unipotent subgroup schemes (i.e., non-ordinariness of the given abelian variety).

Theorem 5 ([20], Theorem (Serre and Tate)). *Let X be an ordinary abelian variety over a perfect field k of $char(k) = p > 0$. Then there exists an abelian scheme $\mathcal{X} \to Spec(W(k))$ such that every endomorphism of X lifts to $\mathcal{X}$:*

$$\mathrm{End}_W(\mathcal{X}) \tilde{\to} \mathrm{End}_k(X),$$

and every polarization of X lifts to $\mathcal{X}$ (and $\mathcal{X}/W(k)$ is called the canonical lifting of X/k).

D. Hyperelliptic and Quasi-hyperelliptic Surfaces :

We consider smooth projective surfaces X with invariants $\kappa(X) = 0 = \chi(\mathcal{O}_X) = K_X^2$ and $\dim(Alb_X) = 1$.

Definition 5. X is said to be *hyperelliptic* (respectively *quasi-hyperelliptic*) if the fibers of the albanese mapping $X \to E$ are elliptic curves (respectively rational curves with one cusp).

We know that a hyperelliptic surface X is of the form $X = (E_1 \times E_2)/G$ for a group G of automorphisms whose type and action on the elliptic curves E_1 and E_2 were worked out in ([2]). If there is no wild ramification in the action of G on E_2, then X lifts to characteristic zero trivially by taking $\mathcal{X} = \mathcal{E}_1 \times \mathcal{E}_2/G$ where $\mathcal{E}_1$ (respectively $\mathcal{E}_2$) is the lifting of E_1 (respectively E_2) with G-action.

Question. Does X lift if there is wild ramification in the action of G on E_2?

Remark 1. One knows that a pair (X, α) where X is a smooth projective curve and $\alpha \in \mathrm{Aut}(X)$ lifts if $p^2 \nmid ord(\alpha)$ ([20], p. 172). Still working with finite groups acting on curves, Green-Matignon obtained a more general result relating the ramification in the group action to the lifting problem (cf. II, p. 24).

Without employing the explicit construction given above, one can deduce the existence of lifting for certain hyperelliptic surfaces from the following vanishing result for cohomology.

Theorem 6 ([15], Theorem 4.9). *If X is a hyperelliptic surface over a field k of characteristic $\neq 2$ with $ord(K_X) = 3, 4, 6$, then*

$$H^2(X, \Theta_X) = 0, \qquad H^2(X, \mathcal{O}_X) = 0.$$

Corollary 1. *If X is a hyperelliptic surface in characteristic $\neq 2$ with $ord(K_X) = 3, 4, 6$, then X lifts to $S = Spec(R)$, for any complete discrete valuation ring R with residue field k.*

Remark 2. Comparing with the following list of possible values for K_X ([2], p. 37)

- $ord(K_X) = 2, 3, 4, 6$ if $char(k) \neq 2, 3$,
- $ord(K_X) = 1, 3$ if $char(k) = 2$,
- $ord(K_X) = 1, 2, 4$ if $char(k) = 3$,

we see that there are hyperelliptic surfaces which lift ($ord(K_X) = 3$ in characteristic 2, $ord(K_X) = 2$ in characteristic 3) but are not covered by the Corollary 1.

Quasi-hyperelliptic surfaces exist only in characteristics 2 and 3. A quasi-hyperelliptic surface X is of the form $X = (E_1 \times C_0)/G$ where E_1, (respectively C_0) is an elliptic curve (respectively a cuspidal rational curve), G is a finite subgroup scheme of E_1 and the action of G is given by $g.(u, v) = (u + g, \alpha(g)v)$ for some injective homomorphism $\alpha : G \to Aut(C_0)$; X admits a natural fibration $X \to E = E_1/G$ with cuspidal fibers ([3]).

For a quasi-hyperelliptic surface X, the possible values of $ord(K_X)$ are as follows ([3], p. 214):

- $ord(K_X) = 1, 2, 3, 4, 6$ if $char(k) = 2$,
- $ord(K_X) = 1, 2, 3, 6$ if $char(k) = 3$.

Recall. A fibration with a section is called a *Jacobian fibration*.

Theorem 7 ([15], Theorem 4.2 and Theorem 4.3). *If X is a Jacobian quasi-hyperelliptic surface with $ord(K_X) = 3, 6$ or a non-Jacobian quasi-hyperelliptic surface with $ord(K_X) = 2, 3, 6$ over a field k of characteristic 3, then*

$$H^2(X, \Theta_X) = 0, \qquad H^2(X, \mathcal{O}_X) = 0.$$

Thus we see that the obstruction to lifting such X vanishes and we obtain the following corollary.

Corollary 2. *A surface X of one of the types listed in Theorem 7 lifts to $S = Spec(R)$, for any complete discrete valuation ring R with residue field k.*

III) $\kappa = 1$: In this case X is elliptic or quasi-elliptic.

- **Elliptic Surfaces :**

  Definition 6. A smooth projective surface X with $\kappa(X) = 1$ is said to be *elliptic* if there exists a morphism $f : X \to C$, where C is a smooth curve, such that the general fiber is a smooth curve of arithmetic genus $g = 1$.

  Theorem 8 ([12], Theorem 5.2). *The elliptic fibration $X \to C$ arises from the n-th canonical map for every $n \geq 14$.*

  There is a class of elliptic surfaces obtained as the étale quotient of a product $C \times E$ by a group-scheme of the form $\mathbb{Z}/p^2\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z} \times \mu_n$, $(p, n) = 1$ ([12], Section 8). Since such actions on curves lift to characteristic zero, the elliptic surfaces of this type lift. It is not known whether a general elliptic surface lifts.

- **Quasi-elliptic Surfaces :**

  Definition 7. A smooth projective surface X of Kodaira dimension $\kappa(X) = 1$ is said to be *quasi-elliptic* if there is a morphism $f : X \to C$, where C is a smooth curve, such that the general fiber is a singular curve of arithmetic genus $g = 1$.

  Quasi-elliptic surfaces exist only in characteristic 2 and 3.

  The general fiber of a quasi-elliptic fibration has one ordinary cusp ([15]).

  It is also not known whether a general quasi-elliptic surface lifts.

- A surface X may be both elliptic and quasi-elliptic.

IV) $\kappa = 2$:

There are surfaces of general type which do not lift to characteristic zero. We will recall an example due to Serre. In Chapter II (p. 18), we will discuss Szpiro's example of a family of non-liftable surfaces of general type admitting smooth fibrations.

Example 1. (Serre's Example)

This example is due to Serre (in $dim \geq 3$) and it was modified by Mumford to get an example of a non-liftable surface.

Serre constructs a smooth projective scheme X_0 which does not lift over any integral, complete, local, noetherian ring A with residue field k and field of fractions K of characteristic zero. To do this, he first constructs a non-liftable homomorphism ρ_0, and then shows that the constructed X_0 is not liftable by using the non-liftability of this ρ_0. The details of Serre's arguments are given in ([8], p. 228-231), and we summarize this construction in the sequel.

Let k be an algebraically closed field of characteristic $p > 0$, r and n be integers with $2 \leq r < n$, and $p > n+1$. And let $G = \mathbb{F}_p^s$, with $s \geq n+1$.
Choose an injective homomorphism $h : G \to k$, where k is considered as an additive group.
Let $N = (u_{ij})$ be the nilpotent matrix of order $n+1$ defined by $u_{ij} = 1$ if $j = n+1$ and $u_{ij} = 0$ otherwise.
For $g \in G$, let

$$\tilde{\rho_0}(g) = \exp(h(g)N) \in GL_{n+1}(k)$$

(which makes sense since $p \geq n+1$), and let $\rho_0(g)$ be the image of $\tilde{\rho_0}(g)$ in $PGL_{n+1}(k)$ $(= GL_{n+1}(k)/k^*)$. We thus get a representation

$$\rho_0 : G \to PGL_{n+1}(k)$$

which is faithful, because h is injective.

First, by the following theorem, ρ_0 is not liftable to any integral local ring A with residue field k and field of fractions K of characteristic zero.

Theorem 9 ([8], Proposition 8.6.6). *Assume that $p > n+1$. Let A be an integral local ring with residue field k and field of fractions K of characteristic zero. Then there exists no homomorphism*

$$\rho : G \to PGL_{n+1}(A)(= GL_{n+1}(A)/A^*)$$

lifting ρ_0.

Now, since the group of k-automorphisms of $P_0 = \mathbb{P}_k^n$ is $PGL_{n+1}(k)$, ρ_0 defines a (right) action of G on P_0.

For $g \in G$, denote by $\mathrm{Fix}(g)$ the closed subscheme of fixed points of g (intersection of the graph of g and the diagonal in $P_0 \times_k P_0$).
Let $Q_0 \subseteq P_0$ be the union of the $\mathrm{Fix}(g)$'s for $g \neq e$.
In our case, for any $g \in G$, $g \neq e$, $\mathrm{Fix}(g)$ consists of the single rational point [1:0:...:0] of P_0.
In particular, $\dim(Q_0) = 0$.
Then, since $r + \dim(Q_0) < n$, we can find a smooth, projective, complete intersection Y_0 as stated explicitly in the following theorem.

Theorem 10 ([8], Proposition 8.6.2). *Assume that*

$$r + \dim(Q_0) < n.$$

Then there exists an integer $d_0 \geq 1$ such that, for any integer d divisible by d_0, one can find a smooth, projective, complete intersection $Y_0 = V(h_1, ..., h_{n-r})$ of dimension r in P_0, with $\deg(h_i) = d$ for $1 \leq i \leq n - r$, which is stable under the action of G on P_0 defined by the representation ρ_0, and on which G acts freely.

Finally, let

$$X_0 = Y_0/G$$

be the quotient of Y_0 by G. Then, since G acts freely on Y_0, X_0/k is a smooth, projective scheme of dimension r. And it is non-liftable as stated explicitly in the following theorem.

Theorem 11 ([8], Corollary 8.6.7). *Let r, n be integers such that $2 \leq r < n$ and $p > n + 1$. Let $G = \mathbb{F}_p^s$, with $s \geq n + 1$. There exists a smooth, projective, complete intersection Y_0 of dimension r in P_0, stable under the action of G on P_0 defined by the representation ρ_0 constructed above, and on which G acts freely, and such that the smooth, projective scheme*

$$X_0 = Y_0/G$$

has the following property. Let A be an integral, complete, local noetherian ring with residue field k and field of fractions K of characteristic zero. Then there exists no formal scheme $\mathcal{X}$, flat over A, lifting X_0. $\square$

Now we return to our main problem; lifting fibrations on surfaces.

Once the first question (Problem 1) is answered affirmatively, the second question (which is the relative version of the problem of lifting curves) is a problem in the general theory of deformation of maps. Both of these problems can be treated, in principle, by the infinitesimal deformation theory combined with the techniques of algebraization in formal geometry. This theory developed by Grothendieck ([10]) particularly for smooth morphisms, was later extended by Illusie ([8]) to cover non-smooth cases too.

This approach consists of two steps :

1. One constructs a formal lifting of the structure under consideration. In our problem, we want to obtain a commutative diagram

$$\begin{tikzcd} \hat{X} \arrow[rr] \arrow[dr] & & \hat{Y} \arrow[dl] \\ & \hat{S} & \end{tikzcd}$$

of formal schemes.

2. Then one proves that the formal solution in step (1) is in fact the completion of an algebraic lifting. In our problem, we need to show that the digram in step (1) arises by completion from an algebraic solution over S :

$$\begin{tikzcd} \mathcal{X} \arrow[rr] \arrow[dr] & & \mathcal{Y} \arrow[dl] \\ & S & \end{tikzcd}$$

In both steps of this program, as in the case with any deformation theoretic approach, it is hard to determine whether the relevant obstructions vanish. The theory is efficient only for the cases where the cohomology groups, in which the obstructions survive, vanish. As expected such cases are quite rare.

In this thesis, on one hand we work out the obstruction theory in favourable cases (i.e., cases in which vanishing of the obstruction(s) can be verified readily). On the other hand, we give explicit constructions whenever possible by exploiting the geometry of the given fibration.

$\kappa = 0$ case is more or less covered in our M. Sc. Thesis ([13]). Here, we elaborate on the techniques used in that thesis to give a more systematic and uniform treatment. These results comprise part of Chapter II of this thesis. The rest of Chapter II contains fragmentary results concerning isotrivial fibrations and the fibrations on surfaces of Kodaira dimension $\kappa = 1$.

We do not have considerable progress related to the most interesting case, namely the fibrations on surfaces of general type. We did not include the obvious applications of our basic results (on isotrivial fibrations, canonical fibrations and albanese fibrations) to surfaces of general type. We quot in Chapter II a fundamental "uniqueness" result (due to Szpiro) concerning the lifting of families of curves of genus $g \geq 2$, again without elaborating on obvious applications to surfaces of general type.

As expected, the problem of lifting fibrations on surfaces is related to other interesting problems in geometry in positive characteristic. We will observe connections with

- the non-smoothness of the Picard scheme Pic_X,
- the ordinariness of the fibration $X \to Y$,
- the existence of non-closed regular differential forms on X.

As a brief indication of these relations, we note that

1. The obstruction to projectivization of $\hat{X}/\hat{S}$ vanishes if Pic_X is smooth.

2. It is known that semi-stable (relatively) ordinary fibrations are isotrivial ([23]). This fact will be used in Chapter II to prove an elementary lifting result for such fibrations.

3. The existence of non-closed differential forms is an obstruction to lifting X to $W_2(k)$ (as follows from the work of Deligne-Illusie in [7]). In fact, exploiting this property, W. Lang constructed examples of hyperelliptic surfaces which do not lift to $W(k)$ but lift to $W(k)(\sqrt{p})$ ([14]).

RESULTS :

We state a sample of results. For the details and for some more fragmentary results, we refer to Chapter II.

Proposition 1. *Let $\pi : X = \mathbb{P}(E) \to C$ be a ruled surface over a smooth projective curve C and let R be a complete discrete valuation ring with residue field k. Then :*

- *π lifts to a $\mathbb{P}^1$-bundle over any curve $\mathcal{C}/S$ lifting C.*
- *Any lifting $\mathcal{X}$ of X over $S = Spec(R)$, is a $\mathbb{P}^1$-bundle over a suitable lifting $\mathcal{C}$ of C.*

Proposition 2. *Assume that $p > 2$ and let X be an Enriques surface with the double étale covering $\varphi : Y \to X$. Then :*

- *The natural fibration $X \xrightarrow{\pi} \mathbb{P}^1$ lifts to a fibration $\mathcal{X} \to \mathbb{P}^1_S$.*
- *The induced elliptic fibration $\pi \circ \varphi : Y \to \mathbb{P}^1$ on the $K3$-surface Y lifts to a fibration $\mathcal{Y} \to \mathbb{P}^1_S$.*

Proposition 3. *Let $X \to Y$ be a smooth fibration of the form*

$$(Y' \times F)/G,$$

where F is the fiber and $Y' \to Y$ is a Galois cover with group G which is of the form described in the Theorem of Green-Matignon (II, Theorem 16). Then $X \to Y$ lifts to a fibration

$$\mathcal{X} \to \mathcal{Y}$$

over $Spec(W(k)[\zeta_{(2)}])$.

Corollary. *Let X be a hyperelliptic surface. And assume that there is no wild ramification in the action of G on E_2. Then the natural fibrations*

$$X \to E_i/G,$$

$i = 1, 2$ lift over $W(k)$ to give

$$\mathcal{X} = (\mathcal{E}_1 \times_S \mathcal{E}_2)/G \to \mathcal{E}_i/G.$$

CHAPTER 2

MAIN RESULTS

We first recall the basic results of deformation theory and the main examples which can be worked out by standard cohomological techniques. For the deformation theory and formal geometry, we follow the treatment given in ([8]).

A. The Smooth Fibration Case

Theorem 12 ([8], Theorem 8.5.9). a. *Let X and Y be schemes over a scheme S, with Y smooth over S, and let $j : X_0 \to X$ be a closed subscheme defined by an ideal J of square zero. Let $g : X_0 \to Y$ be an S-morphism. There is an obstruction*

$$o(g, j) \in H^1(X_0, J \otimes_{O_{X_0}} g^*\Theta_{Y/S})$$

whose vanishing is necessary and sufficient for the existence of an S-morphism $h : X \to Y$ extending g, i.e., such that $hj = g$. When $o(g, j) = 0$, the set of extensions h of g is an affine space under $H^0(X_0, J \otimes_{O_{X_0}} g^*\Theta_{Y/S})$.
b. *Let $i : S_0 \to S$ be a thickening of order one defined by an ideal I of square zero, and let X_0 be a smooth S_0-scheme. There is an obstruction*

$$o(X_0, i) \in H^2(X_0, f_0^* I \otimes \Theta_{X_0/S_0})$$

(where $f_0 : X_0 \to S_0$ is the structural morphism) whose vanishing is necessary and sufficient for the existence of a deformation X of X_0 over S (8.5.7). When $o(X_0, i) = 0$, the set of isomorphism classes of such deformations is an affine space under $H^1(X_0, f_0^* I \otimes \Theta_{X_0/S_0})$, and the group of automorphism of a fixed deformation is isomorphic to $H^0(X_0, f_0^* I \otimes \Theta_{X_0/S_0})$. In particular, if X_0 is étale over S_0, there exists a deformation X of X_0 over S, which is unique up to a unique isomorphism.

This theorem applied in case of $I = \mathbf{m}^n/\mathbf{m}^{n+1}$, where $\mathbf{m}$ is the maximal ideal of R, together with

Theorem 13 ([8], Corollary 8.5.6). *Let $\mathcal{X}$ be a proper, flat adic locally noetherian formal scheme over $\hat{S}$. Then :*

a. *If X/S is a proper scheme such that $\mathcal{X} = \hat{X}$, X is flat over S. Moreover, if $H^2(X_0, \mathcal{O}_{X_0}) = 0$, any line bundle L_0 on X_0 can be lifted to a line bundle L on X, which is unique (up to an isomorphism) if $H^1(X_0, \mathcal{O}_{X_0}) = 0$.*

b. *If X_0 is projective and an ample line bundle L_0 on X_0 can be lifted to a line bundle $\mathcal{L}$ on $\mathcal{X}$, there exists a projective and flat scheme X/S such that $\mathcal{X} = \hat{X}$ and an ample line bundle L on X such that $\hat{L} = \mathcal{L}$.*

leads to the following basic result

Theorem 14 ([8], Theorem 8.5.19 [SGA1, III 7.3]). *Let A be a complete local noetherian ring, with residue field k. Let $S = \mathrm{Spec} A$, $s = \mathrm{Spec} k$, and let X_0 be a projective and smooth scheme over s satisfying*

$$(i) \qquad H^2(X_0, \Theta_{X_0/s}) = 0.$$

Then there exists a proper and smooth formal scheme (8.5.8) $\mathcal{X}$ over $\hat{S}$ lifting X_0. If, in addition to (i), X_0 satisfies

$$(ii) \qquad H^2(X_0, \mathcal{O}_{X_0}) = 0,$$

then there exists a projective and smooth scheme X over S such that $X_s = X_0$.

Example 2. (Ruled Surfaces)

The case of ruled surfaces, by the virtue of having a very simple geometry, is practically the only class where we can answer completely all the questions posed in the Introduction. We will need the following results.

Theorem 15 ([8], Theorem 8.5.3). *Suppose a smooth projective scheme X lifts to $\mathcal{X}$ over $SpecR$ and let $\mathcal{E}$ be a locally free sheaf on X. We have :*

- ***i.*** *If $H^2(X, End(\mathcal{E})) = 0$, then $\mathcal{E}$ lifts to a locally free sheaf on $\mathcal{X}$.*
- ***ii.*** *If furthermore $H^1(X, End(\mathcal{E})) = 0$, then this lifting is unique.*

In particular,

Any locally free sheaf $\mathcal{E}$ on a smooth proper curve C/k, lifts to a locally free sheaf on any given lifting $\mathcal{C}$ of C.

Lemma 1 ([13], Lemma 2). *Let $\mathcal{X}$ be a lifting of X over a complete discrete valuation ring and assume that the fibration $\pi : X \to C$ satisfies one of the following conditions :*

a. *π is the albanese fibration and Pic^0_X is smooth or lifts with ramification index $e < p - 1$.*

b. *π is the n-th canonical fibration.*

Then π lifts to a fibration $\mathcal{X} \to \mathcal{C}$ for a suitable lifting $\mathcal{C}$ of C.

Proof. a. We consider the dual of the reduced component of the Picard scheme $Pic_{\mathcal{X}/S}$ containing the identity. Under the given hypothesis this is an abelian scheme and is the relative albanese scheme $Alb_{\mathcal{X}/S}$ of $\mathcal{X}/S$. As the base scheme is Henselian, the point in $X(k)$ used in defining the albanese map of the special fiber, lifts to a section in $\mathcal{X}(S)$. Thus the relative albanese map $\mathcal{X} \to Alb_{\mathcal{X}/S}$ is defined over S and gives the required curve $\mathcal{C}$.

b. If the given fibration corresponds to the n-th canonical map, then clearly the image of the map $\mathcal{X} \to \mathbb{P}(\pi_{S*}(w^{\otimes n}_{\mathcal{X}/S}))$ is a curve $\mathcal{C}$; the result follows. □

Proposition 1 ([13], Lemma 4). *Let $\pi : X = \mathbb{P}(E) \to C$ be a ruled surface over a smooth projective curve C and let R be a complete discrete valuation ring with residue field k. Then :*

a. *π lifts to a $\mathbb{P}^1$-bundle over any curve $\mathcal{C}/S$ lifting C.*

b. *Any lifting $\mathcal{X}$ of X over $S = Spec(R)$, is a $\mathbb{P}^1$-bundle over a suitable lifting $\mathcal{C}$ of C.*

Proof. a. Let $X = \mathbb{P}(\mathcal{E})$ for a vector bundle $\mathcal{E}$ of rank 2 on C. For any lifting

$$\mathcal{C} \to Spec(R)$$

of C, the obstruction to lifting $\mathcal{E}$ to some $\tilde{\mathcal{E}}$ on $\tilde{\mathcal{C}}$ vanishes. We let $\tilde{\mathcal{E}}$ be a lifting of $\mathcal{E}$ to $\mathcal{C}$ and we take

$$\mathcal{X} = \mathbb{P}(\tilde{\mathcal{E}}) \to \mathcal{C}.$$

b. This follows from Lemma 1(a), because $X \to C$ is the Albanese fibration and $Pic^0_X = J_C$ is smooth. □

Other than this general obstruction theory, one has obstructions arising from the special geometry we are working in.

It is well-known in characteristic zero that if X is a surface of general type, then $c_1^2(X) \leq c_2(X)$. However, there are examples which show that this inequality does not hold in characteristic $p > 0$. On the other hand, one has the following fact ([13], Lemma 1) :

Fact 1. *If a surface X lifts to characteristic zero, then the Bogomolov inequality holds for X.*

Thus, by constructing fibered surfaces violating the Bogomolov-Miyaoka-Yau inequality, one can give examples of non-liftable surfaces. One such example was given by Szpiro in ([23], p. 195). Starting with a non-isotrivial fibration $f : X \to C$ with fiber genus $g \geq 2$, Szpiro constructs a family of surfaces of general type with fixed positive second Chern class c_2 and with c_1^2 unbounded in the family. His construction is as follows.

Example 3. (Szpiro's Example)

Let C be a curve of genus $q \geq 2$, $f : X \to C$ be a smooth non-isotrivial fibration with fiber genus $g \geq 2$. (Note : Such fibrations exist ([23], 3.1).)
Let $F^n : C \to C$ be the n-th iteration of Frobenius on C, and let $X^{(p^n)} \to C$ be the corresponding pull back of $X \to C$.
Let $d = deg(f_* \Omega_{X/C})$. (Note : It is known that d is positive.)
Then one proves that

- $c_2(X^{(p^n)}) = 4(g-1)(q-1)$ for each n,
- $c_1^2(X^{(p^n)}) = p^n d + 8(g-1)(q-1)$,

from which one concludes that $c_1^2 \to \infty$, but c_2 is a fixed positive integer. □

Fact 2. *If a surface X lifts to $W(k)$ (even to $W_2(k)$), then X is free from certain pathologies, e.g. Hodge-deRham spectral sequence degenerates. In particular, all regular forms on X are closed.*

W. Lang constructed examples of hyperelliptic surfaces which do not lift to $W(k)$ by verifying that the corresponding Hodge-deRham spectral sequence does not degenerate ([14]). These surfaces lift to characteristic zero if one permits (minimal) ramification, i.e., $e = 2$.

In cases where the lifting of the surface is possible only over ramified extensions of $W(k)$, it is a challenging problem to show that the obstruction in

$$H^2(X, \Theta_X) \otimes \frac{\mathbf{m}}{\mathbf{m}^2},$$

where $\mathbf{m}$ is the maximal ideal of $W(k)$, is non-zero, but is zero in

$$H^2(X, \Theta_X) \otimes \frac{\tilde{\mathbf{m}}}{\tilde{\mathbf{m}}^2},$$

where $\tilde{\mathbf{m}}$ is the maximal ideal of $W(k)[\pi]$. (Note : The equation satisfied by π over $W(k)$ determines e.)

Deformation theoretic approach to lifting fibrations on a surface X which lifts as a surface, is reduced to the problem of deforming the morphism $X \to Y$ inducing the fibration. In the (rare) case of smooth fibrations $g : X \to Y$ the obstruction to a formal lifting is in

$$H^1(X, g^*\Theta_Y) \otimes \frac{\mathbf{m}^n}{\mathbf{m}^{n+1}}.$$

In this case too, as is illustrated in the following example, it is hard to deduce the existence of liftings of fibrations by computing the obstructions.

Example 4. Let $g : X \to Y$ be a smooth geometrically connected fibration over an elliptic curve. Then $g^*\Theta_Y = g^*\mathcal{O}_Y = \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) \neq 0$;

$$dim(H^1(X, \mathcal{O}_X)) = dim(Lie(PicX)) \geq dim((PicX)_{red}) = dim(Alb_X) \geq 1$$

since the base Y is an elliptic curve. Therefore, computing the obstruction to lifting g is not trivial even for ruled surfaces (by Proposition 1, we know that it vanishes). □

Example 5. One can write examples of abelian surfaces admitting fibrations over elliptic curves, which lift as an abelian surface over $W(k)$, but yet the fibration does not extend over this lifting. To demonstrate this situation, we quot the following example given in our M. Sc. Thesis ([13]).

Consider the affine "plane" curve C given by

$$y^2 = x(x-1)(x-2)(x-5)(x-6) \ \text{ over } S = Spec(W(\overline{\mathbb{F}}_7)).$$

The complete nonsingular model is a curve of genus 2. We take the jacobian scheme $\mathcal{J}_{C/S}$. The generic fiber is a geometrically simple abelian surface (cf. [4], p. 159), but the special fiber is the jacobian of the curve birational to the plane curve

$$y^2 = x(x-1)(x-2)(x+2)(x+1) \ \ (\text{since } 6 \equiv -1, 5 \equiv -2 \pmod 7)$$

which admits an elliptic fibration ([4], Thm. 14.1.1(iii)) over an elliptic curve E. □

This result, clearly is in conformity with the obstruction theory for liftings. The obstruction to *infinitesimal* lifting of the fibration is in the cohomology group $H^2(J, \Theta_{J/E}) \otimes \pi^*(I)$ where J is the special fiber of $\mathcal{J}_{C/S}$ and $\Theta_{J/E}$ is the relative tangent bundle. Since $\Theta_{J/E} \cong \mathcal{O}_J$ we have $H^2(J, \Theta_{J/E}) = \overline{\mathbb{F}}_7$. Therefore, it is not surprising to find out that the obstruction does not vanish.

Question. Can we write a <u>non-isotrivial</u> smooth fibration $f : X \to C$ of fiber genus $g \geq 2$ such that X lifts, but f does not?

B. Non-smooth Fibrations

As a motivating example, we consider $K3$-surfaces admitting fibrations; here we come across an example of Problem 2 (in the Introduction) for non-smooth fibrations.

Example 6. ($K3$-Surfaces)

First step : Lifting X as a surface.
It is well known that $H^2(X, \Theta_X) = 0$.
Therefore, X lifts as a formal scheme ([8], Theorem 8.5.19(a)). But since $H^2(X, \mathcal{O}_X) \cong H^0(X, \mathcal{O}_X) \cong k$, one can not deduce that the obstruction to lifting very ample bundles vanishes. However, it is true that X lifts with no ramification if $char(k) > 2$ and to $W(k)[\sqrt{p}]$ in all cases ([6]).

Suppose the given $K3$-surface has a fibration

$$\phi : X \to Y$$

and let $\mathcal{X} \to Spec(R)$ be a lifting of X. Does ϕ lift?

We first recall that any such fibration is elliptic with rational base curve :

Lemma 2 ([13], Lemma 5). *A generically smooth fibration on a $K3$-surface X is necessarily elliptic with base $\mathbb{P}^1$.*

Proof. Since $H^0(X, \Omega_X) = 0$ in all characteristics, the base is $\mathbb{P}^1$. If F is the generic fiber, then $2g(F) - 2 = F.(K_X + F) = 0$ since $K_X = 0$. Thus F is an elliptic curve. □

Restricting ourselves to $K3$-surfaces which cover Enriques surfaces, we have the following complete solution (given in our M. Sc. Thesis) to the lifting problem without resorting to deformation theory.

Proposition 2. *Assume that $p > 2$ and let X be an Enriques surface with the double étale covering $\varphi : Y \to X$. Then :*

- *The natural fibration $X \xrightarrow{\pi} \mathbb{P}^1$ lifts to a fibration $\mathcal{X} \to \mathbb{P}^1_S$.*
- *The induced elliptic fibration $\pi \circ \varphi : Y \to \mathbb{P}^1$ on the $K3$-surface Y lifts to a fibration $\mathcal{Y} \to \mathbb{P}^1_S$.*

To prove Proposition 2, we will need the following fact describing the induced fibration on the $K3$-surface Y.

Lemma 3. *The fibration $\pi \circ \varphi : Y \to \mathbb{P}^1$ is not connected. After a suitable "Stein factorization" we obtain a connected fibration $Y \to \mathbb{P}^1$.*

Proof. We first verify that $\pi \circ \varphi$ is not connected, where $\varphi : Y \to X$ is the "universal covering map" and $\pi : X \to \mathbb{P}^1$ is the natural fibration :

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ & & \downarrow \pi \\ & & \mathbb{P}^1 \end{array}$$

If $\pi o \varphi$ is connected, then we obtain an elliptic fibration $\pi o \varphi : Y \to \mathbb{P}^1$ on the $K3-$surface Y, with precisely two double fibers, say over p_1 and p_2. Then by the canonical bundle formula for elliptic fibrations, we get $w_Y = (\pi o \varphi)^*(\mathcal{L}) \otimes \mathcal{O}_Y(F_1^{'} + F_2^{'})$ where $\mathcal{L}$ is a line bundle on $\mathbb{P}^1$ of degree, $deg(\mathcal{L}) = \chi(\mathcal{O}_Y) - \chi(\mathcal{O}_{\mathbb{P}^1}) = 0$, since $\chi(\mathcal{O}_Y) = 2$. Therefore, $w_Y \cong \mathcal{O}_Y(F_1^{'} + F_2^{'}) \neq \mathcal{O}_Y$; contradiction since $K_Y = 0$. □

Therefore, we need "Stein factorization" obtained from the double cover $\mathbb{P}^1 \to \mathbb{P}^1$, to get a connected fibration $Y \to \mathbb{P}^1$ (cf. [1], p.274, Remarks). (We note that this covering $\mathbb{P}^1 \to \mathbb{P}^1$ corresponds to the line bundle $\mathcal{O}(p_1 + p_2) \cong \mathcal{O}(2)$ on $\mathbb{P}^1$).

Proof of Proposition 2.

First of all, it is known that an Enriques surface X of classical type lifts to characteristic zero if $p > 2$ ([5]), and Y is the degree 2 étale covering of X.

Now, let $\mathcal{X}$ be a lifting of X over a Henselian ring (for instance $W(k)$). To prove that the fibration $\pi : X \to \mathbb{P}^1$ lifts to a fibration $\mathcal{X} \to \mathbb{P}^1_S$, we construct $\mathbb{P}(\mathcal{E})$ over S for a suitable rank 2 locally free sheaf $\mathcal{E}$ and the map $\mathcal{X} \to \mathbb{P}(\mathcal{E})$ lifting π.

The map $X \to \mathbb{P}^1$ corresponds to the linear system determined by the line bundle $L = \pi^*(\mathcal{O}_{\mathbb{P}^1}(1)) \cong \mathcal{O}(2F_i^{'})$, where $2F_i^{'}$ is one of the double fibers of π lying over $p_1, p_2 \in \mathbb{P}^1$. Since $End(L) \cong \mathcal{O}_X$, $H^2(X, End(L)) \cong H^2(X, \mathcal{O}_X) = 0$ and $H^1(X, End(L)) \cong H^1(X, \mathcal{O}_X) = 0$, because X is an Enriques surface. Therefore by ([8], Thm.8.5.3) L lifts to a unique line bundle $\mathcal{L}$ on $\mathcal{X}$. Taking $\mathcal{E} = \varphi_*(\mathcal{L})$ we obtain $\mathcal{X} \to \mathbb{P}(\mathcal{E})$ (corresponding to the natural map $\varphi^*(\varphi_*(\mathcal{L})) \to \mathcal{L} \to 0$) which lifts $\pi : X \to \mathbb{P}^1$.

Then one checks that the induced fibration on the generic fiber $\mathcal{X}_\eta$ is connected and has precisely two double fibers; in fact these double fibers lie over the generic points of the sections $s_i : S \to \mathbb{P}(\mathcal{E})$ which lift the points $p_1, p_2 \in \mathbb{P}^1(k)$ in the special fiber (Henselian base!).

And to prove that the induced elliptic fibration $Y \to \mathbb{P}^1$ on the $K3$-surface Y lifts to a fibration $\mathcal{Y} \to \mathbb{P}^1_S$, we first note that the covering $Y \to X$ lifts to give $\mathcal{Y} \to \mathcal{X}$, because the base scheme is Henselian. Then the composite map $\mathcal{Y} \to \mathcal{X} \to \mathbb{P}(\mathcal{E})$ induces an elliptic fibration which we proved is not connected. The "Stein factorization" $\mathcal{Y} \to \mathbb{P}(\mathcal{E})$ obtained from the double cover $\mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E})$ which ramifies precisely over $s_1 \cup s_2$, lifts the elliptic fibration on the $K3$-surface Y. $\square$

Next, still working with the fibrations on $K3$-surfaces, we take X to be a generalized Kummer surface which is the minimal desingularization of $E_1 \times E_2/G$, the quotient by a finite group of the product of the elliptic curves E_1, E_2. Katsura proves in ([11], Theorem 3.7) that if $char(k) \neq 2, 3, 5$, then G is isomorphic to one of the following groups :

$$G = \begin{cases} \text{cyclic group of order } 2, 3, 4, 5, 6, 8, 10, 12, \\ \text{binary dihedral group } < 2, 2, n >, \; n = 2, 3, 4, 5, 6, \\ \text{binary tetrahedral group } < 2, 3, 3 >, \\ \text{binary octahedral group } < 2, 3, 4 >, \\ \text{binary icosahedral group } < 2, 3, 5 >. \end{cases}$$

Now, X admits a natural fibration

$$X \to E_1/G = \mathbb{P}^1.$$

And the corresponding lifting problem is solved if we can lift the action of G on $E_1 \times E_2$ to an action on $\mathcal{E}_1 \times \mathcal{E}_2$ and then resolve the singularities of $(\mathcal{E}_1 \times \mathcal{E}_2)/G$. Thus the problem is related to the following equivariant lifting problem.

Problem G. Does the pair (Z, G) where Z is a variety and G is a group of automorphisms of Z lift ?

Remark 3. Comparing the list given by Katsura with the list of groups G for (generalized) Kummer surfaces in $char(k) = 0$, and applying Deligne's result on lifting $K3$-surfaces, one can compile a list of examples for lifting Kummer surfaces to obtain *(relative) Kummer surfaces*.

Example 7. (Hyperelliptic and Quasi-hyperelliptic Surfaces)

Fibrations on hyperelliptic surfaces and on quasi-hyperelliptic surfaces furnish the other two examples of the equivariant lifting problem. The results concerning the hyperelliptic surfaces is in fact a special case of the problem of lifting isotrivial fibrations. We first recall the definition of an isotrivial fibration.

Definition 8. A fibration $X \to Y$ is *isotrivial* if after a (finite) étale extension $Y' \to Y$, the surface $X' = X \times_Y Y'$ is birational to a trivial fibration $Y' \times F$.

In case $Y' \to Y$ is Galois with group G, the following result of Green-Matignon ([9]), which is a fundamental result in this direction, can be used to prove the existence of liftings of fibrations.

Theorem 16 ([9], Theorem 2). *Let C/k be a smooth integral proper curve of genus $g = g(C)$. Let G be a finite subgroup of $\mathrm{Aut}_k(C)$. Let $f : C \to C/G := D$ be a G-Galois cover of smooth integral proper curves over k. Assume that the inertia groups are $p^a e$-cyclic with $a \leq 2$ and $(e, p) = 1$. Then f can be lifted over $R = W(k)[\zeta_{(2)}]$ as a G-Galois cover of smooth integral proper R-curves, where $\zeta_{(2)}$ is a primitive p^2-root of unity.*

And applying this theorem, we obtain the following elementary result :

Proposition 3. *Let $X \to Y$ be a smooth fibration of the form*

$$(Y' \times F)/G,$$

where F is the fiber and $Y' \to Y$ is a Galois cover with group G which is of the form described in the above theorem. Then $X \to Y$ lifts to a fibration

$$\mathcal{X} \to \mathcal{Y}$$

over $Spec(W(k)[\zeta_{(2)}])$.

Proof. Action of G lifts to curves $\mathcal{Y}', \mathcal{F}$ over $Spec(W(k)[\zeta_{(2)}])$ which gives a lifting

$$\mathcal{X} = (\mathcal{Y}' \times \mathcal{F})/G \to \mathcal{Y}'/G$$

of $X \to Y$. $\square$

In particular, this result applies to certain hyperelliptic surfaces $X = (E_1 \times E_2)/G$ (see [2] for all possible types and the action of G on E_1 and E_2) and we obtain the following result concerning the lifting of an hyperelliptic surface :

Corollary. *Let X be a hyperelliptic surface. And assume that there is no wild ramification in the action of G on E_2. Then the natural fibrations*

$$X \to E_i/G,$$

$i = 1, 2$ lift over $W(k)$ to give

$$\mathcal{X} = (\mathcal{E}_1 \times_S \mathcal{E}_2)/G \to \mathcal{E}_i/G.$$

Proof. Since there is no ramification in the action of G on E_1, the pair $(E_1,\ G)$ lifts over $W(k)$ ([21]). The second pair $(E_2,\ G)$ lifts over $W(k)$ if there is no wild ramification in the action of G on E_2 ([21]). $\square$

Remark 4.

a. For the corollary, we may replace the Theorem of Green-Matignon by a weaker result given in ([20]).

b. In characteristics 2 and 3 we have quasi-hyperelliptic surfaces which we know to lift in certain cases (I, Corollary 2). On a quasi hyperelliptic surface we have two fibrations $X \to C_0/\alpha(G) \cong \mathbb{P}^1$ and $X \to E_1/G \cong E$ where $E = Alb(X)$. It follows from Lemma 1(a) that the second fibration lifts for surfaces indicated in (I, Corollary 2).

c. W. Lang's example ([14]) is a hyperelliptic surface in characteristic $p = 2$ and with $G \cong \mathbb{Z}_2$ which acts on the second component with wild ramification. Lang proves that X does not lift over $W(k)$, but lifts over an extension of degree 2 of $W(k)$. Thus, for lifting the fibrations in this example we obtain the same conclusion as in the preceding paragraph, only after we allow ramification of degree 2 (minimum possible !).

We note that this example of W. Lang is related to non-smoothness of Pic_X. $Pic_X = \alpha_2$ is a non-smooth group scheme which lifts over a discrete valuation ring R of residue characteristic 2 if and only if $2 \in m^2$, where m is the maximal ideal of R.

Next result on lifting smooth isotrivial fibrations, makes use of the concept of *ordinariness* for relative curves.

Let $X \to Y$ be a smooth family of curves over a field of characteristic p and consider $X^{(p)} = X \times_F Y$ where $F : Y \to Y$ is the (absolute) Frobenius.

Definition 9. $X \to Y$ is *ordinary (relative to Y)* if the Y-group scheme $N := \mathrm{Ker}(J_{X^{(p)}/Y} \to J_{X/Y})$ is locally isomorphic to μ_p^g, where g is the fiber genus of $X \to Y$.

Lemma 4. *Let $\varphi : X \to Y$ be a smooth family of ordinary curves of genus $g \geq 2$. Then φ lifts to $W(k)$.*

Proof. We recall the proof of ([22], Theorem 5).

$Y' = \underline{\mathrm{Isom}}_{Y-groups}(N, \mu_p^g \times Y)$ is a finite étale covering and $X \times_Y Y' \cong Y' \times F$.
Lifting Y' and F to $W(k)$, we obtain $\mathcal{Y}' \times_{W(k)} \mathcal{F}$ of $Y' \times F$ lifting the group action $Y' \to Y$. □

Remark 5. It follows from the isotriviality of smooth families of ordinary curves ([22], Theorem 5) that the fibrations on the surfaces in Szpiro's example ([23]) are non-ordinary.

Now we return to non-smooth fibrations.

In the case of non-smooth fibrations, one has to replace the tangent bundle by the tangent complex. In fact, one works with the cotangent complex and derives the following result.

Theorem 17 ([8], Theorem 8.5.31). a. *Let X and Y be schemes over a scheme S, and let $j : X_0 \to X$ be a closed subscheme defined by an ideal J of square zero. Let $g : X_0 \to Y$ be an S-morphism. There is an obstruction*

$$o(g, j) \in \mathrm{Ext}^1(g^* L_{Y/S}, J)$$

whose vanishing is necessary and sufficient for the existence of an S-morphism $h : X \to Y$ extending g, i.e., such that $hj = g$. When $o(g, j) = 0$, the set of extensions h of g is an affine space under $\mathrm{Ext}^0(g^* L_{Y/S}, J) = \mathrm{Hom}(g^* \Omega^1_{Y/S}, J)$.
b. *Let $i : S_0 \to S$ be a thickening of order 1 defined by an ideal I of square zero, and let X_0 be a flat S_0-scheme. There is an obstruction*

$$o(X_0, i) \in \mathrm{Ext}^2(L_{X_0/S_0}, f_0^* I)$$

(where $f_0 : X_0 \to S_0$ is the structural morphism) whose vanishing is necessary and sufficient for the existence of a deformation X of X_0 over S (8.5.7). When $o(X_0, i) = 0$, the set of isomorphism classes of such deformations is an affine space under $\mathrm{Ext}^1(L_{X_0/S_0}, f_0^* I)$*, and the group of automorphisms of a fixed deformation is isomorphic to* $\mathrm{Ext}^0(L_{X_0/S_0}, f_0^* I) = \mathrm{Hom}(\Omega^1_{X_0/S_0}, f_0^* I)$.

Illusie applies his theory to prove that singular curves which are complete intersections can be lifted ([8]). However, if the singularities are restricted suitably, one can prove the existence of liftings (to singular S-schemes) without applying the obstruction theory. We work out the following example.

Example 8. (Lifting by Rigidification)

Details and the relative version will appear elsewhere.

Let X be a smooth projective curve, and J_X be its Jacobian.
Let X_{sing} be the curve obtained by glueing $\{p_1, .., p_m\} \subset X$, and $J_{X_{sing}}$ be its Jacobian. Note that this set-up corresponds to a rigidification by $\mathcal{R} = \coprod_1^m Spec(k)_i$ and we have the following exact sequence

$$\begin{array}{ccccccccc} 0 & \to & G_L & \to & J_{X_{sing}} & \to & J_X & \to & 0 \quad (*) \\ & & & & \updownarrow & & & & \\ & & & & Pic^0(X, \mathcal{R}) & & & & \end{array}$$

Let $\mathcal{X}$ be a lifting of X to $W(k)$ and $\mathcal{J}_{\mathcal{X}/W(k)}$ be its Jacobian.

Note that

- **i.** Since $\mathcal{R} = \coprod Spec(k)_i$ is reduced, G_L is multiplicative, i.e., does not have unipotent part.
- **ii.** $\mathcal{R}$ has a trivial lifting $\tilde{\mathcal{R}} = \coprod Spec(W(k))_i$.

Now, let

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{f} & X \\ \searrow & & \swarrow \\ & Spec(k) & \end{array}$$

be the set of sections $f(Spec(k)_i) = p_i$.

Since the base scheme is Henselian and $\mathcal{X}$ is smooth over the base, f lifts to

$$\begin{array}{ccc} \tilde{\mathcal{R}} & \xrightarrow{\tilde{f}} & \mathcal{X} \\ \searrow & & \swarrow \\ & Spec(W(k)) & \end{array}$$

We glue $\mathcal{X}$ along $\tilde{f}(\tilde{R})$; the resulting non-smooth scheme $\mathcal{X}_{sing}$ is a lifting of X_{sing} and $Pic^0_{\mathcal{X}_{sing}}$ is the jacobian of the curve $\mathcal{X}_{sing}$ over $Spec(W(k))$. As a by-product one obtains a lifting

$$0 \to \mathcal{G}_L \to Pic^0_{\mathcal{X}_{sing}} \to \mathcal{J}_{\mathcal{X}} \to 0$$

of (*). $\square$

Now we return to the application of Illusie's theory to lifting non-smooth fibrations, as given in Szpiro's article ([23]).

Theorem 18 ([23], Theorem 2.3). *Let R be a complete discrete valuation ring, and let $\pi : C \to S = Spec(R)$ be a smooth projective curve. Let $D \subset C$ be a flat divisor over S. Let $f_s : X \to C_s$ be a non-isotrivial morphism from a smooth surface X over k, the residue field of R into the special fiber C_s of π.*
Suppose that the generic fiber of f_s is projective, smooth, and geometrically connected curve of genus $g \geq 2$. Suppose in addition that singular fibers of f_s are semi-stable, and lie over D_s (the special fiber of D).
Then there exist at most one smooth surface $\mathcal{X}$ over S, and a morphism $f : \mathcal{X} \to C$ such that

a. *on the special fiber over S, f is f_s,*

b. *f is smooth outside D.*

Theorem 19 ([23], Corollary 2, page 184). *A* <u>*non-isotrivial*</u> *smooth fibration $X \to C$ with fiber genus $g \geq 2$ has at most one lifting $\mathcal{X} \to \mathcal{C}$ over a given lifting $\mathcal{C}$ of C.*

Finally, we consider fibrations on properly elliptic surfaces X, i.e., surfaces with $\kappa(X) = 1$. It is known that the elliptic fibration on such X is unique and that it arises from the n-th canonical map, for every $n \geq 14$ (I, Theorem 8).

If X lifts to a surface $\mathcal{X}$ over a complete discrete valuation ring R, then the n-th canonical fibration lifts (Lemma 1(b)). This applies in particular to elliptic surfaces which are quotients by étale group actions on products $C \times E$ of curves (I). However, we do not know whether all elliptic surfaces lift to characteristic zero.

An elliptic surface X may admit a fibration (necessarily non-elliptic) which does not arise from the n-th canonical map. We give an elementary example of a family of elliptic surfaces (with varying base curve E and fiber F) with liftable non-elliptic fibration.

Example 9. We work in $char(k) \neq 2$, we take an elliptic curve E, and a smooth projective curve F with $g(F) \geq 2$ which is a double covering of an elliptic curve $E_2 \cong F/\mathbb{Z}_2$.

We consider a subgroup $G \subset E(k)$, $G \cong \mathbb{Z}_2$ and we let $X = (F \times E)/\mathbb{Z}_2$, where the action of $\mathbb{Z}_2$ is defined componentwise.

Then we have the following diagram :

$$\begin{array}{ccccc} F \times E & \longrightarrow & (F \times E)/\mathbb{Z}_2 = X & \xrightarrow{\pi_2} & F/\mathbb{Z}_2 = E_2 \\ & & \downarrow \pi_1 & & \\ & & E/G = E' & & \end{array}$$

In this diagram, $\pi_1 : X \to E'$ is a smooth genus 2 fibration and $\pi_2 : X \to E_2$ is an elliptic fibration with 2 double fibers F_1, F_2. Therefore, $\omega_X = \omega_{X/E_2} = \mathcal{O}_X(F_1 + F_2)$. And this implies that π_2 is the canonical map.

Now, F, E lift with $\mathbb{Z}_2$-action (since $p \neq 2$). Therefore,

$$\mathcal{X} = (\mathcal{F} \times \mathcal{E})/\mathbb{Z}_2 \to Spec(R)$$

lifts X, and

$$\begin{array}{ccc} \mathcal{X} & \to & \mathcal{F}/\mathbb{Z}_2 \\ \downarrow & & \\ \mathcal{E}/\mathbb{Z}_2 & & \end{array}$$

lift the fibrations π_1 and π_2. $\square$

REFERENCES

[1] W. Barth, C. Peters, A. Van de Ven, *Compact Complex Surfaces*, Springer-Verlag, Berlin-New York, 1984.

[2] E. Bombieri, D. Mumford, *Enriques classification of surfaces in characteristic p II*, Complex Analysis and Algebraic Geometry, pp. 23–42, Iwanami Shoten, Tokyo, 1977.

[3] E. Bombieri, D. Mumford, *Enriques classification of surfaces in characteristic p III*, Invent. Math. 35 (1976), 197–232.

[4] J. W. S. Cassels, E. V. Flynn, *Prolegomena to a Middlebrow Arithmetic of Curves of Genus* 2, London Mathematical Society Lecture Note Series, no. 230, Cambridge University Press, Cambridge, 1996.

[5] R. F. Cossec, I. V. Dolgachev, *Enriques Surfaces I*, Progress in Math. Vol. 76, Birkhäuser, Boston-Basel-Berlin, 1989.

[6] P. Deligne, *Relévement des surfaces* $K3$ *en caractéristique nulle*, Algebraic surfaces (Orsay, 1976–78), pp. 58–79, Lecture Notes in Math., Vol. 868, Springer-Verlag, Berlin-New York, 1981.

[7] P. Deligne, L. Illusie, *Relévements modulo* p^2 *et décomposition du complexe de de Rham*, Invent. Math. 89 (1987), no. 2, 247–270.

[8] B Fantechi, L. Göttsche, L. Illusie, S. L. Kleiman, N. Nitsure, A. Vistoli, *Fundamental Algebraic Geometry: Grothendieck's FGA Explained*, Mathematical Surveys and Monographs Vol. 123, American Mathematical Society, 2005.

[9] B. Green, M. Matignon, *Liftings of galois covers of smooth curves*, Compositio Math. 113 (1998), 237–272.

[10] A. Grothendieck, *Fondements de la Géométrie Algébrique (FGA)*, Extraits du S?minaire Bourbaki, 1957–1962, Secrétariat mathématique, Paris, 1962.

[11] T. Katsura, *Generalized Kummer surfaces and their unirationality in characteristic p*, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. 34 (1987), 1–41.

[12] T. Katsura, K. Ueno, *On elliptic surfaces in characteristic p*, Math. Ann. 272 (1985), 291–330.

[13] C. Kaya, *Lifting fibrations on algebraic surfaces to characteristic zero*, M. Sc. Thesis, METU (2005).

[14] W. E. Lang, *Examples of liftings of surfaces and a problem in de Rham cohomology*, Compositio Math. 97 (1995), no. 1–2, 157–160.

[15] W. E. Lang, *Quasi-elliptic surfaces in characteristic three*, Ann. Scient. Éc. Norm. Sup. 4^e série, t. 12, no. 4 (1979), 473–500.

[16] D. Mumford, *Abelian Varieties*, Oxford University Press, London, 1970.

[17] D. Mumford, *Enriques classification of surfaces in characteristic p I*, Global Analysis, pp. 325–339, Univ. Tokyo Press, Tokyo, 1969.

[18] P. Norman, *Lifting abelian varieties*, Invent. Math. 64, 431–443 (1981).

[19] F. Oort, *Finite group schemes, local moduli for abelian varieties, and lifting problems*, 5th Nordic Summer-School in Mathematics, Oslo, August 4-25, 1970, p. 223–254.

[20] F. Oort, *Lifting algebraic curves, abelian varieties and their endomorphisms to characteristic zero*, Proc. Sympos. Pure Math., Vol.46 (1987), Part 2, 165–195.

[21] T. Sekugichi, F. Oort, N. Suwa, *On the deformation of Artin-Schreier to Kummer*, Ann. Sci. École Norm. Sup., 4é serie (1989), no. 3, 345–375.

[22] L. Szpiro, *Proprietes numeriques du faisceaux dualisant relatif*, in Sem. sur les Pinceaux de Courbes de Genre au Moins Deux, Ast. 86, SMF (1981), 44-78.

[23] L. Szpiro, *Sur le téorème de rigidité de Parsin et Arakelov*, Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. II, volume 64 of Astérisque, pp. 169–202, Soc. Math. France, Paris, 1979.

CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name	Kaya, Celalettin
Nationality	Republic of Turkey
Date and Place of Birth	18.03.1972, Ankara
Phone Number	+90 536 776 88 16
E-mail Adress	celalettinkaya@gmail.com

EDUCATION

Degree	**Institution**	**Year of Graduation**
Ph. D.	METU, Mathematics Department	2010
M. Sc.	METU, Mathematics Department	2005
B. Sc.	METU, Mathematics Department	1999

WORK EXPERIENCE

Year	**Place**	**Enrollment**
2001-2008	METU, Mathematics Department	Research Assistant

FOREIGN LANGUAGE

English

FIELD OF STUDY

Algebraic Geometry