PRESERVICE SECONDARY MATHEMATICS TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE OF COMPOSITE AND INVERSE FUNCTIONS

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ABSTRACT

PRESERVICE SECONDARY MATHEMATICS TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE OF COMPOSITE AND INVERSE FUNCTIONS

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The main purpose of the study was to understand preservice secondary mathematics teachers' pedagogical content knowledge of composite and inverse functions.

The study was conducted with three preservice secondary mathematics teachers in Graduate School of Education at Bilkent University. The instruments of the study were qualitative in nature and in four different types of data forms: observations, interviews, documents, and audiovisual materials. Observation data came from fieldnotes by conducting an observation of lessons participants taught at Private Bilkent High School. Interview data came from the transcriptions of semi-structured interviews. Document data came from survey of function knowledge, journal writings, vignettes, and lesson plans. Audiovisual data came from the examination of the videotape of the lessons participants taught.

The findings reveal that preservice secondary mathematics teachers' knowledge levels in components of pedagogical content knowledge were not at the desired levels and also they experienced difficulty while integrating that knowledge. The results of the study indicate that teacher education should provide courses that cover the content relevant to students in order to assure both depth and breadth in subject matter knowledge of the preservice teachers. Moreover, the activities which mimics the classroom cases and assures the integration of knowledge components like vignettes would be used in teacher education programs. Results can inform educational practices, and reforms in Turkey, and provide a basis for further research, with increased pedagogical content knowledge as the ultimate goal.

Keywords: Pedagogical Content Knowledge, Preservice Secondary Mathematics Teachers, Composite Functions, Inverse Functions, Teacher Education

ÖZ

ORTAÖĞRETİM MATEMATİK ÖĞRETMEN ADAYLARININ BİLEŞKE VE TERS FONKSİYON HAKKINDAKİ PEDAGOJİK ALAN BİLGİLERİ

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Bu çalışmanın amacı ortaöğretim matematik öğretmen adaylarının bileşke ve ters fonksiyonlar hakkındaki pedagojik alan bilgilerinin araştırılmasıdır.

Çalışma Bilkent Üniversitesi eğitim bilimleri enstitüsünde ikinci sınıf öğrencisi olan üç öğretmen adayı ile yürütülmüştür. Çalışmada kullanılan araçlar nitel özelik taşımaktadır. Dört çeşit ölçme aracı kullanılmıştır; gözlem, görüşme, dökümanlar ve işitsel veriler. Gözlem verileri öğretmen adaylarının Özel Bilkent Lisesinde yaptıkları okul deneyimleri sırasında çekilen videolar ve bu gözlemler sırasında alınan notlardan oluşmaktadır. Görüşme verileri yarı yapılandırılmış görüşme kayıtlarının çözümlemelerinden oluşmaktadır. Dökümanlar ise fonksiyon bilgisi testi, irdeleme yazıları, vignette'ler ve ders planlarından gelmektedir. İşitsel veriler ise görüşme kayıtları ve ders kayıtlarından oluşmaktadır.

Sonuçlar ortaöğretim matematik öğretmen adaylarının bilgi düzeylerinin pedagojik alan bilgisini oluşturan boyutlarda istenen düzeyde olmadığını ve bu bilgileri istendiğinde gerekli şekilde entegre edemediği veya entegre etmekte güçlük çektiğini göstermiştir. Bu sonuçlar doğrultusunda, öğretmen eğitimi programları öğrencilerin öğrendikleri bilgileri göz önünde bulunduran alan eğimi dersleri

sağlamalı ve aynı zamanda bu derslerde konular derinlemesine verilmelidir. Ayrıca, ders içi durumları taklit eden ve pedagojik alan bilgisinin ortaya çıkmasına sebep veren vignette benzeri aktivitelere öğretmen eğitiminde yer verilmelidir. Bu çalışmanın sonuçları, öğretmen eğitimi programlarındaki uygulamalara ve Türkiye'de ki öğretmen eğitimi hakkındaki reformlara bilgi sağlayacaktır. Ayrıca, pedagojik alan bilgisinin artırılmasını temel amaç edinecek gelecek çalışmalara ışık tutacaktır.

Anahtar Kelimeler: Pedagojik alan bilgisi, Ortaöğretim Matematik Öğretmen Adayları, Bileşke Fonksiyon, Ters Fonksiyon, Öğretmen Eğitimi

To my daughter Bengisu

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Hence, this thesis is dedicated to my family, representing my appreciation.

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CHAPTER 1

INTRODUCTION

The complex nature of teaching mathematics effectively is emphasized by many researchers (Shulman, 1986; Thompson, 1992). Therefore, one needs to have effective teachers that should be able to provide students a rich learning environment to overcome this difficulty. However, there are no easy ways to equip teachers with such a content knowledge (Mason & Spence, 1999). For this purpose, teachers' content knowledge should be well developed to provide students with tasks, which should be able to model real world situations with different kinds of functions, represent and analyze functions in various representational forms, and develop an understanding of operations on functions, composite and inverse functions, and the general behavior of classes of functions (NCTM, 1991). In addition, to improve instruction, more emphasize should be given to the understanding of the concepts, problem solving, applications, and communication of ideas and a decreased emphasis on computation and facts and procedural questions (NCTM, 1989).

In relation to teach mathematics effectively, debates about the testing teachers' competence in subject matter and pedagogy are ideas dating back to last century (Shulman, 1986). In 1870's, the pedagogy was essentially ignored and teacher candidates were tested for their competence in subject matter. However, in 1980's the situation was vice versa where teachers' competence was only tested through pedagogical tests ignoring the subject matter. In order to balance this pendulum between the content and pedagogy in 1986 Shulman and her colleagues started a project called "Knowledge Growth in Teaching". Through the project they tried to bring to front unasked questions of teacher education like "Where do teachers explanations came from? How do teachers decide what to teach? … What are the sources of knowledge ?" (Shulman, 1986, p.8).

After Shulman and his colleagues' project, questions like how teachers' knowledge organized and what are the critical components of the teachers' knowledge? have been under discussion among scholars inside and outside the mathematics education community (Fennema & Franke, 1992; Gess-Newsome, 1999). Eventually, there is no consensus on what to count as a component for teachers' knowledge (Fennema & Franke, 1992; Gess-Newsome, 1999; Thompson, 1992). This is because of the nature of teachers' knowledge which is "a large, integrated, and functioning system where its components are difficult to isolate" (Fennema & Franke, 1992, p. 148).

In 1986, Shulman proposed three subcategories for the teachers' content knowledge by posing the question "How might we think about the knowledge that grows in the minds of teachers, with special emphasis on content?" (p. 9). The subcategories were subject matter knowledge, pedagogical content knowledge, and curricular knowledge. What was acknowledged in this work was a unique type of knowledge, called pedagogical content knowledge, specific to profession of teachers. Simply, pedagogical content knowledge (PCK) is a teacher's understanding of how to help students to understand subject matter and teacher's understanding is a result of transformations from subject matter knowledge, and curricular knowledge. Since then several other researchers have investigated pedagogical content knowledge, even though the terminology used was changed most of components used by the researchers overlaps the components of Shulman (1986).

Combining the research conducted about PCK, Gess-Newsome (1999) introduced the two types of model for teachers' knowledge: transformative model, and integrative model. In transformative models, like Shulman's model, pedagogical content knowledge exists as a distinct category, however in integrative models pedagogical content knowledge lies at the intersection of the categories. In transformative model, teaching was defined as the use of only one of the category of the teachers' knowledge which is the PCK. Because researchers using transformative models believe that although other knowledge categories exist, they are useful only when transformed into PCK; and an effective teacher is defined as the teacher having as many transformed knowledge, PCK, as possible. In integrative model, teaching was defined as the act of integrating knowledge across these categories; and a good teacher was defined as the one who has well-organized individual knowledge categories that are easily accessed, integrated, and flexibly used during teaching.

In the present study, PCK model used was between these two extremes, but close to integrative model. Because, as reported by Gess-Newsome (1999), educating teachers with transformative models would result in teachers having tricks for every topic in the subject. This means that teacher does not need to internalize the subject and make decisions, instead they would know only the tricks for teaching. However, as stated by Mason and Spence (1999) in order to use a knowledge it is priori to know about it and knowing about a subject for teachers does not necessarily mean knowing just bags of tricks.

Most of the studies that focusing on increasing the effectiveness of instruction emphasized the importance of teacher knowledge on related content areas. Even (1990) mentioned that mathematics teachers who have deficiencies in subject matter knowledge are likely to pass their misconceptions and misunderstandings to their students. In contrast, a teacher with well developed mathematical knowledge for teaching is more capable of helping his/her students achieve a meaningful understanding of the content.

Based on these ideas, again we could question pre-service teachers' pedagogical content knowledge of specific mathematics topics. What teachers know or how they teach could be primary questions that should be answered before trying to improve the mathematics instruction. It is reasonable that the teacher who has a lack of understanding about specific topic will be unable to transfer the correct knowledge to the students, so it's important to pay attention to the teacher's knowledge specific to the mathematics topics.

1.1 The Statement of the Problem

Students' difficulties in understanding mathematics were stated by many researchers. Related to the concept of functions as being an important and unifying concept in modern mathematics (Selden & Selden, 1992), students also faced these difficulties (Akkoç, 2006; Bakar & Tall, 1991; Dubinsky, 1991; Eisenberg, 1991; Gray & Tall, 1994; Leinhardt, Zaslavsky, & Stein, 1990; Sierpinska, 1992; Tall & Vinner, 1981; Thompson, 1994; Vinner, 1983; 1991; Vinner & Dreyfus, 1989). Strong conception of function is an indispensable part of the background of any student who hopes to learn something about calculus (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & Harel, 1992). Beyond its use in calculus and analysis, functions are widely used as elements of abstract mathematical structures such as vector spaces, rings and groups, algebraic operations on numbers, geometric transformations, and operations on sets. Examples of functions, like the relationship between speed and distance, the relationship between time and growth of a plant, and the relationship between interest and principal, can be found in the real world situations. The importance of function concept is emphasized by different institutions (MAA, 1991; NCTM, 1989; NCTM, 1991). When the national mathematics curriculum in Turkey (MEB, 2005) is analyzed, it shows that functions appear and reappear throughout the curriculum. Besides being a central topic, the study of functions is at the critical time in the study of mathematics where the transformations between different forms of representations lead to powerful learning (Ebert, 1993; Leinhardt, 1990). Although it's important, the function concept is often misunderstood by students (Ubuz, 1996; Vinner & Dreyfus, 1989) and even by teachers (Even, 1989; Howald, 1998; Wilson, 1990).

Considering the central role of the function concept in mathematics, teachers should be able to provide students a rich learning environment which takes into account the above aspects of the function concept (Howald, 1998). For this purpose, teachers' content knowledge should be well developed to provide students with rich learning environments. These environments should provide students with tasks, which should be able to model real world situations with different kinds of functions, represent and analyze functions in various representational forms, and develop an understanding of operations on functions, composite and inverse functions, and the general behavior of classes of functions (NCTM, 1991).

In the light of the previous discussion, this study intended to provide information to teacher educators that will help them understand preservice secondary mathematics teachers' pedagogical content knowledge of composite and inverse functions. Namely, this study aims to answer the following question:

What is the pedagogical content knowledge of preservice secondary mathematics teachers about the composite and inverse functions?

a. What is the extent and organization of preservice secondary mathematics teachers' subject matter knowledge of composite and inverse functions?
b. What is the nature of preservice secondary mathematics teachers' general pedagogical knowledge?
c. What is the preservice secondary mathematics teachers' awareness about the value of teaching composite and inverse functions?
d. What is the nature of preservice secondary mathematics teachers' knowledge of context?
e. What is the nature of preservice secondary mathematics teachers' knowledge about learners' conception of composite and inverse functions?

1.2 Definition of Important Terms

The research question consists of several terms that needs to be defined.

Pedagogical Content Knowledge

The existence of multiple PCK definitions addressed in the literature (e.g. Abd Rahman & Scaife, 2005; Ebert, 1994; Marks, 1990; Shulman, 1987) make it a necessary step to employ a definition for the concept of PCK in this study in line with the integrative model. PCK definition used in the study was inspired from definitions of Cochran, King, and DeRuiter (1993) and Abd Rahman and Scaife (2005).

The working PCK definition in this study can be summarized as follows: in order to have PCK preservice secondary mathematics teachers should have a good understanding of each category (subject matter knowledge, general pedagogical knowledge, knowledge of learners, value of teaching a mathematical concept, and knowledge of context) and then integrate these when needed. This definition allows for the framing of the categories of PCK and their relationships. It also allows researcher to narrow down the teachers' knowledge literature mostly to the studies that used similar PCK categories, hence, forming a more consistent set of previous studies in order to support the results. Categories investigated are consistent with the current views about the teachers' knowledge bases which defined three of those as critical and interrelated: knowledge of content, knowledge of learners and knowledge of general pedagogy (Cohen & Ball, 1999; Harel & Lim, 2004).

Subject Matter Knowledge

Subject matter knowledge was defined as the facts, concepts, principles, procedures, and syntax that are typically taught in secondary school mathematics curriculum in Turkey. For this study, by using the term facts, concepts, principles, procedures and syntax, I referred specific relations related to composite and inverse functions typically taught in the 9th grade national curriculum in Turkey.

Knowledge of Learners

Knowledge of learners defined as knowledge of common areas of students' conceptions, misconceptions, and difficulties about a topic (Abd Rahman & Scaife, 2005; Cochran, King, & DeRuiter, 1993; Ebert, 1994; Grossman, 1990; Magnusson, Krajcik, & Borko, 1999; Marks, 1990; Morine-Dershimer & Kent, 1999; Pitts, 2003; Shulman, 1987; Smith & Neale, 1989; Veal & MaKinster, 1999).

Knowledge of Context

Knowledge of context defined as the preservice teachers' understanding of (Abd Rahman & Scaife, 2005; Cochran, King, & DeRuiter, 1993) physical facilities and setting, types of students, parents, school and community characteristics, resource availability, classroom climate, school climate, degree of support provided by others, expectations, effects of standardized assessments, demands made on the teacher, and departmental guidelines (Grossman, 1990).

General Pedagogical Knowledge

General pedagogical knowledge defined as skills related to teaching and instruction (Abd Rahman & Scaife, 2005; Ebert, 1994; Grossman, 1990; Shulman, 1987) and skills related to classroom management (Abd Rahman & Scaife, 2005; Grossman, 1990; Shulman, 1987)

Value of Teaching Composite and Inverse Functions

Value of teaching composite and inverse functions was defined as how preservice teachers value the importance of teaching functions through three value types: intrinsic value, pedagogical value and excitement and beauty value.

1.3 Significance of the Study

Most of the previous studies related to teachers' PCK on functions have generally investigated definition of function, different representations of functions, examples and non-examples of functions (Cha, 1999; Critchfield, 2001; Duah-Agyeman, 1999; Ebert, 1994; Even, 1989; Gilbert, 2003; Klanderman, 1996; Lloyd, 1996; McGehee, 1990; Pitts, 2003; Sherin, 1996; Wick, 1998; Winsor, 2003; Wyberg, 2002; Zbiek, 1992).

A few studies investigated teachers' subject matter and pedagogical content knowledge of composition of functions (Even, 1989; Lucus, 2005) and inverse function of functions (Even, 1989; Howald, 1998; Lucus, 2005). These studies represented different methodological approaches. Even (1989) and Howald (1998) investigated the existence and properties of composite and inverse functions through conditional questions in a survey. The participants in the Even's (1989) study were preservice teachers whereas in Howald's (1998) study participants were experienced teachers. Different from Even (1989), Howald (1998) also made observations. On the other hand, Lucus (2005) investigated teachers' knowledge about composite and inverse functions through interviews in which teachers were asked to list prerequisites for composite and inverse functions define them, give properties and associated examples for them. Moreover, the teachers in the study wrote lesson plans for teaching composite and inverse functions by choosing their own order for teaching. In this study, in order to get a rich data both Even's (1989) and Lucus' (2005) assessment styles were used. First, subject matter knowledge about composite and inverse functions were investigated concerning different kinds of knowledge types namely, declarative, conditional and procedural knowledge. Second, preservice teachers were required to write lesson plans for teaching composite and inverse functions by choosing their own order for teaching. Furthermore, as suggested by Lucus (2005) for future research, preservice secondary mathematics teachers were observed while teaching their own lesson plans. So, this study contributes to mathematics teacher education research literature on functions by providing a full array of data on preservice secondary mathematics teachers' knowledge about composite and inverse functions theoretically and practically.

Although preservice teachers' pedagogical content knowledge of composition and inverse functions seems investigated, data only comes from United States (Even, 1989; Howald, 1998) and Canada (Lucus, 2005) and they are few studies on the topic when compared to the studies about definition of function, different representations of functions, examples and non-examples of functions. Moreover, the dramatic differences between the high school curricula make U.S. and Canada research findings about pedagogical content knowledge on composition and inverse functions difficult to generalize to the Turkish context. By this study, Turkish preservice secondary mathematics teachers' pedagogical content knowledge about composite and inverse functions were identified.

Besides, there is an increase in research about pedagogical content knowledge after Shulman's (1986) work, educational research in Turkey has not focused on it. There is research conducted only on preservice elementary mathematics teachers' pedagogical content knowledge (Bütün, 2005; Işıksal, 2006). Hence, conducting a research on pedagogical content knowledge of preservice secondary mathematics teachers on composition and inverse functions seems necessary and important. The importance of studying PCK is also stated by Murray and Porter (1996) as follows: "discussions of pedagogical content knowledge are at the heart of the teacher educator's work and cannot be avoided" (p. 163).

In addition, preservice secondary mathematics teachers were selected since it is believed that findings of the study in terms of knowledge structures will help to draw valuable implications to teacher educators and policy makers in terms of designing content of the courses in teacher education programs.

To summarize, I tried to characterize the pedagogical content knowledge of the preservice secondary mathematics teachers about composite and inverse functions. The primary outcomes of the study will be the current PCK of preservice secondary mathematics teachers, graduating from a two-year master program without thesis in teaching mathematics, about composite and inverse functions in relation with the components subject matter knowledge, general pedagogical knowledge, knowledge of learners, knowledge of context, and value of teaching. The teacher education program the study was conducted uses integrative model because each knowledge category is given in separate courses and preservice secondary mathematics teachers are required to combine and integrate these knowledge. Therefore, results of the study will reveal to what extend preservice secondary mathematics teachers were able to integrate each knowledge component (course) and will suggest ways of improving preservice secondary mathematics teachers' integration among courses. Suggestions for the future practices of secondary mathematics teacher education in Turkey as well as future research paths that will explore the field will be the additional outcome of this study.

CHAPTER 2

REVIEW OF LITERATURE

The purpose of this study was to provide a picture of the preservice secondary mathematics teachers' pedagogical content knowledge of composition and inverse functions. Fennema and Franke (1992) provide a perspective on how teachers' integrated knowledge (pedagogical content knowledge) should be investigated.

> The transforming of knowledge is understandably complex. Little research is available that explains the relationships between components of knowledge as new knowledge develops in teaching nor is information available regarding the parameters of knowledge being transferred through teacher implementation. Here all aspects of teacher knowledge and beliefs come together and all must be considered to understand the whole. The challenge is to develop methodologies and systematic studies that will provide information to enlighten our thinking in this area. The future lies in the understanding the dynamic interaction between components of teacher knowledge and beliefs, the roles they play, and how the roles differ as teacher differ in the knowledge and beliefs they possess (p.163).

In light of this comment and the purpose of the study, through this chapter relevant research concerning the pedagogical content knowledge in terms of definitions, models, components, and previous researches, and teachers' knowledge of functions, composite and inverse functions were reviewed.

2.1 Students' Conceptions and Misconceptions about Functions

The concept of functions is without doubt one of the most important concepts in modern mathematics. It has many aspects and subcomponents which may account for some of the difficulties it seems to cause in school mathematics (Eisenberg, 1991). There is much research done in this area (Akkoç, 2006; Bakar & Tall, 1991; Dubinsky, 1991; Eisenberg, 1991; Gray & Tall, 1994; Leinhardt, Zaslavsky, & Stein, 1990; Sierpinska, 1992; Tall & Vinner, 1981; Thompson, 1994; Vinner, 1983; 1991; Vinner & Dreyfus, 1989).

The study conducted by Vinner (1983) differentiates from the others since the focus of the study was not to reveal misconceptions but to identify students' understandings of the concept of functions. The following understandings of functions were observed in students:

- A function can have one rule. If there are two rules, there are two functions.
- A function must have a smooth graph.
- Every function is one to one.
- Confusing the definition of functions with being one to one.
- Functions which cannot be written algebraically are not accepted as functions.
- For sign and greatest integer functions not considering the conditions of existence of functions.

Even though the purpose of the preceding study was not to reveal misconceptions, since students' understandings include misconceptions, like every function is one to one, so as the findings.

The results of the studies related the students' misconceptions related to the concept of functions showed similarities in terms of the misconceptions identified. The misconceptions students have can be summarized as follows: 1) *definitions* (Akkoç, 2006; Dubinsky, 1991; Eisenberg, 1991; Leinhardt, Zaslavsky, & Stein, 1990; Vinner & Dreyfus, 1989); 2) *representational difficulties* (Bakar & Tall, 1991; Eisenberg, 1991; Leinhardt, Zaslavsky, & Stein, 1990; Tall & Vinner, 1981; Thompson, 1994; Vinner, 1983; 1991); 3) *concept of variable* (Eisenberg, 1991; Leinhardt, Zaslavsky, & Stein, 1990; Vinner & Dreyfus, 1989); 4) *difficulties related to constant functions* (Bakar & Tall, 1991; Clement, 2001; Montiel, Vidakovic, & Kabael, 2008); 5) *notational difficulties* (Eisenberg, 1991; Gray & Tall, 1994; Leinhardt, Zaslavsky, & Stein, 1990; Sierpinska, 1992; Tall & Vinner, 1981; Thompson, 1994; Vinner, 1983; 1991; Vinner & Dreyfus, 1989); 6) *graph* *and visualization related difficulties* (Leinhardt, Zaslavsky, & Stein, 1990; Vinner & Dreyfus, 1989).

The students' definitions of the concept of functions influence their understanding so Vinner and Dreyfus (1989) grouped students' definitions into six different categories namely: correspondence, dependence relation, rule, operation, formula, representation. Even though students define functions in any of these forms and aware of the formal definition of functions studies conducted by Akkoç (2006) and Leinhardt, Zaslavsky, and Stein (1990) showed that when students were asked to decide the existence of functions given in different representations, students experienced difficulties when the functions were given in algebraic form or Venn diagram.

Difficulties related to representations of functions are generally faced with when students were asked to decide whether functions given in different representations are function or not (Akkoç, 2006; Leinhardt, Zaslavsky, & Stein, 1990). Since functions have many representational forms like ordered pairs, equations, graphs, tables, and verbal descriptions, translations among them necessarily became a topic of research. Deciding whether functions in these different representations are functions students ignore the definition of the concept and decide intuitively depending on the familiarity of the given function (Tall & Vinner, 1981; Bakar & Tall, 1991). Difficulties related to representations of functions are also faced when students are asked to make translations among different representations. However, most of the studies dealt with translations from equations to graphs not vice versa. The study conducted by Leinhardt, Zaslavsky, and Stein (1990) asked both of them and the results revealed that students experienced difficulty moving from a graph to an equation than the reverse. Because, the latter one has series of steps to follow.

Change of representation needs students to define appropriate variables and relationships among them. However, as stated earlier students faced problems with understanding the concept of variable (Eisenberg, 1991). Furthermore, Leinhardt, Zaslavsky, and Stein (1990) reported that changing the name of the variable used in the equation was taken as a different equation by students.

In relation to concept of variables, when students were provided a graph of a constant function and asked whether it is a function or not, most of the students answered as not a function since the students were thinking that if there was not any variable there cannot be any function (Bakar &Tall, 1991). Similar to this finding, many other studies reported students' not thinking constant functions as functions since there is nothing changing or varying (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Clement, 2001; Montiel, Vidakovic, & Kabael, 2008). The same confusion was reported by Lovell (1971 cited in Leinhardt, Zaslavsky, and Stein (1990)) between one-to-many and many-to-one correspondences given in set notation. He suggested a possible reason for this confusion as counting arrows rather than elements in the set.

Another misconception mentioned was the notation. The notation includes both graphical and algebraic symbols. Leinhardt, Zaslavsky, and Stein (1990) mentioned only problems with the Cartesian plane. The misconceptions mentioned were misunderstanding of the interval scale, scaling issues, and construction of axes and points. On the other hand, Gray and Tall (1994) mentioned the students misunderstandings regarding the similarity and the difference between the notations f and $f(x)$, where $f(x)$ denotes both the function and the values the function take, and f denotes the function itself. Notation often hinders students' understandings regardless of the difficulty of the topic (Eisenberg, 1991). The possible reason for this was stated by Eisenberg (1991) as flexibly using and understanding notations requires some level of abstraction.

Last misconception mentioned was graphs and visualization related difficulties. Students think graphs independent from functions (Vinner & Dreyfus, 1989). As a result, they incorrectly relate the data in graphs on functions (Eisenberg, 1991) and tendency to focus on individual points on the graphs even if the graph is continuous (Leinhardt, Zaslavsky, & Stein, 1990). Moreover, while interpreting graphs instead of giving correct intervals they prefer to choose a correct point. Therefore, it was seen that students tied to processing information given in graphs either pointwise or analytically not visually (Kleiner, 1988).

2.2 Epistemology of Teachers' Knowledge

Testing teachers' competence in subject matter and pedagogy are ideas dating back to last century (Shulman, 1986). In 1870's, the pedagogy was essentially ignored and teacher candidates were tested for their competence in subject matter. However, in 1980s the situation was vice versa where teachers competence was only tested through pedagogical tests ignoring the subject matter. In order to balance this pendulum between the content and pedagogy in 1986 Shulman and her colleagues started a project called "Knowledge Growth in Teaching". Through the project they tried to bring to front unasked questions of teacher education like "Where do teachers explanations came from? How do teachers decide what to teach? … What are the sources of knowledge ?" (Shulman, 1986, p.8).

After Shulman and his colleagues' project, questions like how teachers' knowledge organized and what are the critical components of the teachers' knowledge? have been under discussion among scholars inside and outside the mathematics education community (Fennema & Franke, 1992; Gess-Newsome, 1999). Eventually, there is no consensus on what to count as a component for teachers' knowledge (Fennema & Franke, 1992; Gess-Newsome, 1999; Thompson, 1992). This is because of the nature of teachers knowledge which is "a large, integrated, and functioning system where its components are difficult to isolate" (Fennema & Franke, 1992, p. 148).

By posing the question "How might we think about the knowledge that grows in the minds of teachers, with special emphasis on content?" (p. 9), Shulman (1986) proposed three subcategories for the teachers content knowledge which are subject matter knowledge, pedagogical content knowledge, and curricular knowledge. Here, subject matter knowledge refers to the knowledge of the subject the teachers supposed to teach, curricular knowledge refers to the knowledge of the programs specific to subjects for each grade. What was acknowledged in this work was a unique type of knowledge, called pedagogical content knowledge (PCK), specific to profession of teachers. The first definition of PCK was as follows:

> A second kind of content knowledge is pedagogical knowledge, which goes beyond the knowledge of subject matter per se to the dimension of subject matter for teaching. The category of pedagogical content knowledge includes the most regularly taught topics in one's subject area, the most useful forms

> of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, am demonstration-in a word, ways of representing and formulating the subject that makes it comprehensible to others. … Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult; the conceptions and preconceptions that students of different ages and backgrounds bring with them to learning of those most frequently taught topics and lessons (Shulman, 1986, p. 9)

Simply, PCK refers not only to the knowledge of subject but also knowing how to teach the subject.

The concept of PCK was developed by Shulman and his colleagues in the Knowledge Growth in Teaching Project (Shulman, 1987). Firstly, a model about the conceptualization of the teachers' knowledge domain was proposed. This model has seven categories: 1) subject matter knowledge, 2) general pedagogical knowledge, 3) curriculum knowledge, 4) pedagogical content knowledge, 5) an understanding of the learners and their characteristics, 6) knowledge of educational ends, purposes, and values, and 7) teachers' philosophical and historical grounds. PCK has been described as "the blending of content and pedagogy into an understanding of how particular topics, problems , or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (Shulman, 1987, p. 8). When functions are considered, pedagogical content knowledge refers not only to knowledge about functions, but also to knowledge about the teaching and learning of functions. Furthermore, Shulman (1987) emphasized the importance of PCK by showing it as discriminator between the pedagogue and the subject teacher. The discriminating power of PCK was mentioned by Geddis (1993) as follows:

> The outstanding teacher is not simply a 'teacher', but rather a 'history teacher', a 'chemistry teacher', or an 'English teacher'. While in some sense there are generic teaching skills, many of the pedagogical skills of the outstanding teacher are content-specific. Beginning teachers need to learn not just 'how to teach', but rather 'how to teach electricity', 'how to teach world history', or 'how to teach fractions'. (p. 675)

One might add 'how to teach functions' or 'how to teach limits', etc. Apparently, the requirements of teaching functions and limit are different. This difference comes from the PCK of the teacher. In line with this, the difference between an expert mathematician (subject matter expert) and a mathematics teacher (subject matter teacher) was put by Mason (1998):

> The mark of an expert mathematician is that they make problem solving and proof look easy: they articulate with technical terms, they make the choice and use of techniques look easy, and they are aware of connections between apparently disparate topics. The mark of an expert teacher is that they make exposition; explanation, task-design, and relating to students look easy. (p. 243)

Secondly, Shulman (1987; 1991) created a model of pedagogical reasoning which is composed of a cycle of activities for teaching including comprehension, transformation, instruction, evaluation, reflection, and new comprehension. This model was created under the assumption that teaching occurs when teachers "presented with the challenge of taking what he or she already understands and making it ready for instruction" (Shulman, 1987, p. 14) by using any teaching material. Although presented, these six processes may not occur in the sequence given at all or may occur in different order (Shulman, 1987) because, these are not the rigid steps to follow. When the model of pedagogical reasoning analyzed it can be seen that the concept of PCK redefined implicitly as transformation of subject matter knowledge by the help of other categories in the teachers' knowledge domain in order to create a reasonable teaching sequence with appropriate activities.

When the literature about PCK was reviewed, it was seen that scholars elaborated on Shulman's model of knowledge base for teaching put some of the categories separate in Shulman's model under PCK. As Shulman (1987) admitted PCK defined several times in his articles, however there was not "great cross-article consistency" (p. 8). Furthermore, Gess-Newsome and Lederman (1999), Hashew (2005), Marks, (1990), and Magnusson, Krajcik, and Borko (1999) stated that the concept of PCK has fuzzy boundaries and there are no clear distinctions between PCK and other knowledge domains. The reason for this is put forward by Magnusson, Krajcik, and Borko (1999): "This due to the fact that PCK represents an integrated knowledge system, but equally important is the recognition that the distinctions between domains are necessarily arbitrary and ambiguous" (p.117). Like Shulman, scholars in the teacher education do not have great consistency in defining boundaries for PCK. In their book called “Examining Pedagogical Content Knowledge” one of the editors Gess-Newsome (1999), collected the studies of many scholars, about the studies of PCK. Although PCK does not have clear boundaries, her review of research results in two distinct models of teachers’ knowledge: the transformative model, and the integrative model. In the following section these models were discussed.

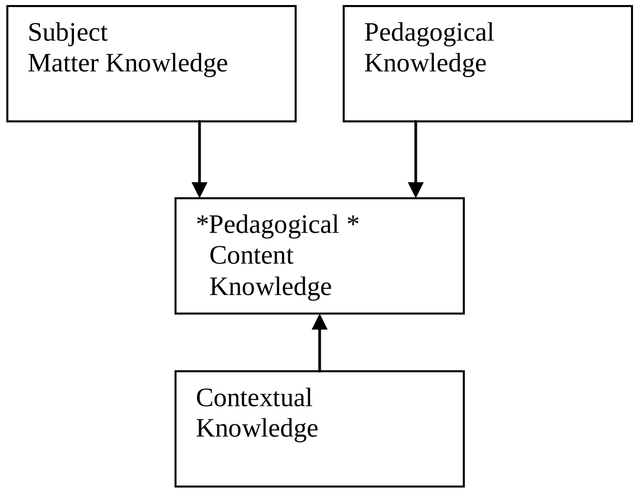
The transformative model, and the integrative model were described by Gess-Newsome (1999) by using an analogy from Chemistry.

> When two materials mixed together, they can form a mixture or a compound. In a mixture, the original elements remain chemically distinct, though their visual impact may imply a total integration. Regardless of the level of apparent combination, the parent ingredients in a mixture can be separated through relatively unsophisticated, physical means. In contrast, compounds are created by the addition or release of energy. Parent ingredients can no longer be easily separated and their initial properties can no longer be detected. A compound is a new substance distinct from its original ingredients, with chemical and physical properties that distinguish it from all other materials (p. 11).

In this analogy, the integrative model and the transformative model were matched with a mixture, and a compound respectively.

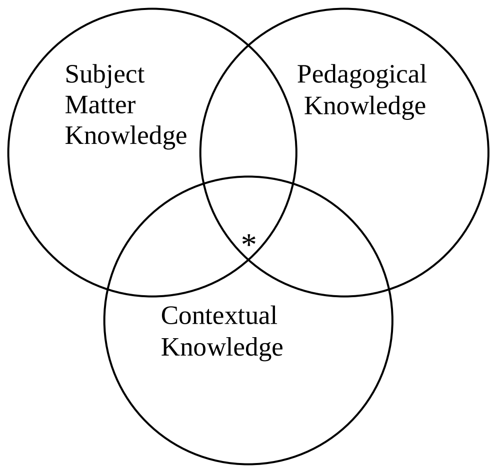
In the transformative model, subject matter, pedagogy, and context are synthesized into a new category of knowledge called PCK (See Figure 2.1). PCK is the only knowledge used during instruction. Although other knowledge categories exist, they are useful only when transformed into PCK. An expert teacher in the transformative model should have a well formed PCK for all topics commonly taught (Gess-Newsome, 1999).

In the integrative model, teachers’ knowledge is at the intersection of knowledge from subject matter, pedagogy, and context (See Figure 2.2). PCK does not exist as a distinct knowledge category (Gess-Newsome, 1999). According to this model, teaching was defined as the act of integrating knowledge across these categories; and a good teacher was defined as one who has well-organized individual knowledge categories that are easily accessed and flexibly used during teaching.



*=knowledge needed for classroom teaching

Figure 2.1: The Transformative model (Gess-Newsome, 1999, p.12)



*=knowledge needed for classroom teaching

Figure 2.2: The Integrative model (Gess-Newsome, 1999, p.12)

Gess-Newsome (1999) compared and contrasted two models in terms of knowledge domains, teaching expertise, implications for teacher preparation, and implications for research in Table 2.1.

Table 2.1: Overview of Integrative and Transformative models of teacher cognition (Gess-Newsome, 1999, p.13)

	Integrative Model	Transformative Model
Knowledge Domains	Knowledge of subject matter, pedagogy, and context are developed separately and integrated in the act of teaching. Each knowledge base must be well structured and easily accessible.	Knowledge of subject matter, pedagogy, and context, whether developed separately or integratively, are transformed into PCK, the knowledge base used for teaching. PCK must be well structured and easily accessible.
Teaching Expertise	Teachers are fluid in the active integration of knowledge bases for each topic taught.	Teachers possess PCK for all topics taught.
Implications for Teacher Preparation	Knowledge bases can be taught separately or integrated. Integration skills must be fostered. Teaching experience and reflection reinforces the development, selection, integration, and use of knowledge bases.	Knowledge bases are best taught in an integrated fashion. Teaching experience reinforces the development, selection, and use of PCK.
Implications for Research	Identify teacher preparation programs that are effective. How can transfer and integration of knowledge best be fostered?	Identify exemplars of PCK and their conditions for use. How can these examples and selection criteria best be taught?

When the two models compared, it can be seen that they are at the extremes of a continuum. The word extreme is used in order to emphasize that in transformative model PCK exists as a distinct category, and in the integrative one it does not exist as a distinct category. In the literature, examples of both models exist.

First transformative model in the literature was Shulman's (1986) model about the conceptualization of the teachers' knowledge domain consisting of three categories: subject matter knowledge, pedagogical content knowledge, and curricular knowledge. Building upon the Shulman (1986) work, PCK defined as the result of transformation of the teachers' existing knowledge domains by many researchers (Grossman, 1990; Magnusson, Krajcik, & Borko, 1999; Morine-Dershimer & Kent, 1999; Shulman, 1987; Smith & Neale, 1989; Wilson, Shulman, & Richert, 1988). The only difference between these models were the terms that were used to define knowledge components of teachers but most of these new terms overlap with the components defined by Shulman (1986). The study conducted by Magnusson, Krajcik, and Borko (1999) had a belief component which was not taken as a component by any of the transformative models before.

In a similar vein, the integrative models in the literature (Abd Rahman & Scaife, 2005; Cochran, King, & DeRuiter, 1993; Marks, 1990) was build their components on Shulman's (1986, 1987) studies and again most of the components overlap with his components. Only the study conducted by Abd Rahman and Scaife (2005) included the component knowledge of self, beliefs, which was not a category in Shulman's models (1986, 1987) and in other integrative models (Cochran, King, & DeRuiter, 1993; Marks, 1990). Therefore, the only difference between the transformative and integrative models comes from how they define PCK. In the integrative models PCK was defined as teacher's integrated understanding of knowledge components. These models did not put a separate category called PCK instead they stated the general structure which constituted the PCK, which showed evidence for the integrated nature of these models.

When the Transformative and Integrative models analyzed, some similarities and differences were realized with respect to components included. Different scholars' conceptualization of Shulman's model of knowledge base for teaching is summarized in Table 2.2. It can be inferred from the table that "...there is no universally accepted conceptualization of PCK. Between scholars, differences occur with respect to the elements they include or integrate in PCK, and to specific labels or descriptions of these elements" (Van Drieal, Verloop, & De Vos, 1998, p. 677). However, one common thing which was stated by all scholars regardless of they used a transformative or integrative model was that the importance of having a substantive content knowledge.

Table 2.2: Knowledge components in different models of PCK

		Knowledge of						
Scholars	**Transformative/ Integrative Model**	**Subject Matter**	**General Pedagogy**	**Students Learning & Conceptions**	**Self**	**Context**	**Curriculum**	**Purposes**
Shulman (1986)	T	d	d	n.e.	n.e.	d	d	n.e.
Shulman (1987)	T	d	d	PCK	n.e.	d	d	d
Smith & Neale (1989)	T	PCK	PCK	PCK	n.e.	n.e.	n.e.	n.e.
Grossman (1990)	T	d	d	PCK	n.e.	d	PCK	PCK
Magnusson, Krajcik, & Borko (1999)	T	d	d	PCK	n.e.	d	PCK	PCK
Morine-Dershimer & Kent (1999)	T	PCK	PCK	PCK	d	PCK	PCK	PCK
Rowan, Schilling, Ball, & Miller (2001)	T	PCK	PCK	PCK	n.e.	n.e.	n.e.	n.e.
Ebert (1994)	T	d	d	d	d	n.e.	n.e.	n.e.
Marks (1990)	I	PCK	n.e.	PCK	n.e.	n.e.	PCK	n.e.
Cochran, King, & DeRuiter (1993)	T&I	PCKg	PCKg	PCKg	n.e.	PCKg	PCKg	PCKg
Abd Rahman & Scaife (2005)	T&I	PCK	PCK	PCK	PCK	PCK	PCK	n.e

T : The Transformative Model
I : The Integrative Model
d : Distinct category in the model
n.e. : Not discussed explicitly

Apart from leading to two distinct models of PCK, Shulman's (1986) work also lead to a creation of a taxonomy for PCK by Veal and MaKinster (1999). The taxonomy was defined in the educational context by Krathwohl et al. (1964) as follows:

> A true taxonomy is a set of classifications which is ordered and arranged on the basis of a single principle or on the basis of a consistent set of principles. Such a true taxonomy may be tested by determining whether it is in agreement with empirical evidence and whether the way in which the classifications are ordered corresponds to a real order among the relevant phenomena. The taxonomy must also be consistent with sound theoretical views available in the field...finally; a true taxonomy should be of value in pointing to phenomena yet to be discovered. (Krathwohl, et al., 1964, p. 11).

Previous discussions and models of PCK in education have not been classified as taxonomies and even they did not mention about a hierarchical relationship (Cochran, King, & DeRuiter, 1993; Magnusson, Krajcik, & Borko, 1999; Morine-Dershimer & Kent, 1999; Abd Rahman & Scaife, 2005; Shulman 1987; Smith & Neale, 1989; Tamir, 1987). Typically the attributes of these PCK models are represented so that the overlap or relatedness of all the attributes determines the amount or development of PCK. "However, these lists of attributes are similar to taxonomies because of the relationships and connections among the attributes" (Cited from Veal & MaKinster, 1999; Tamir, 1998). By considering these relationships between attributes, a taxonomy was constructed by Veal & MaKinster (1999) as Taxonomy of PCK Attributes (components) (See Figure 2.7).

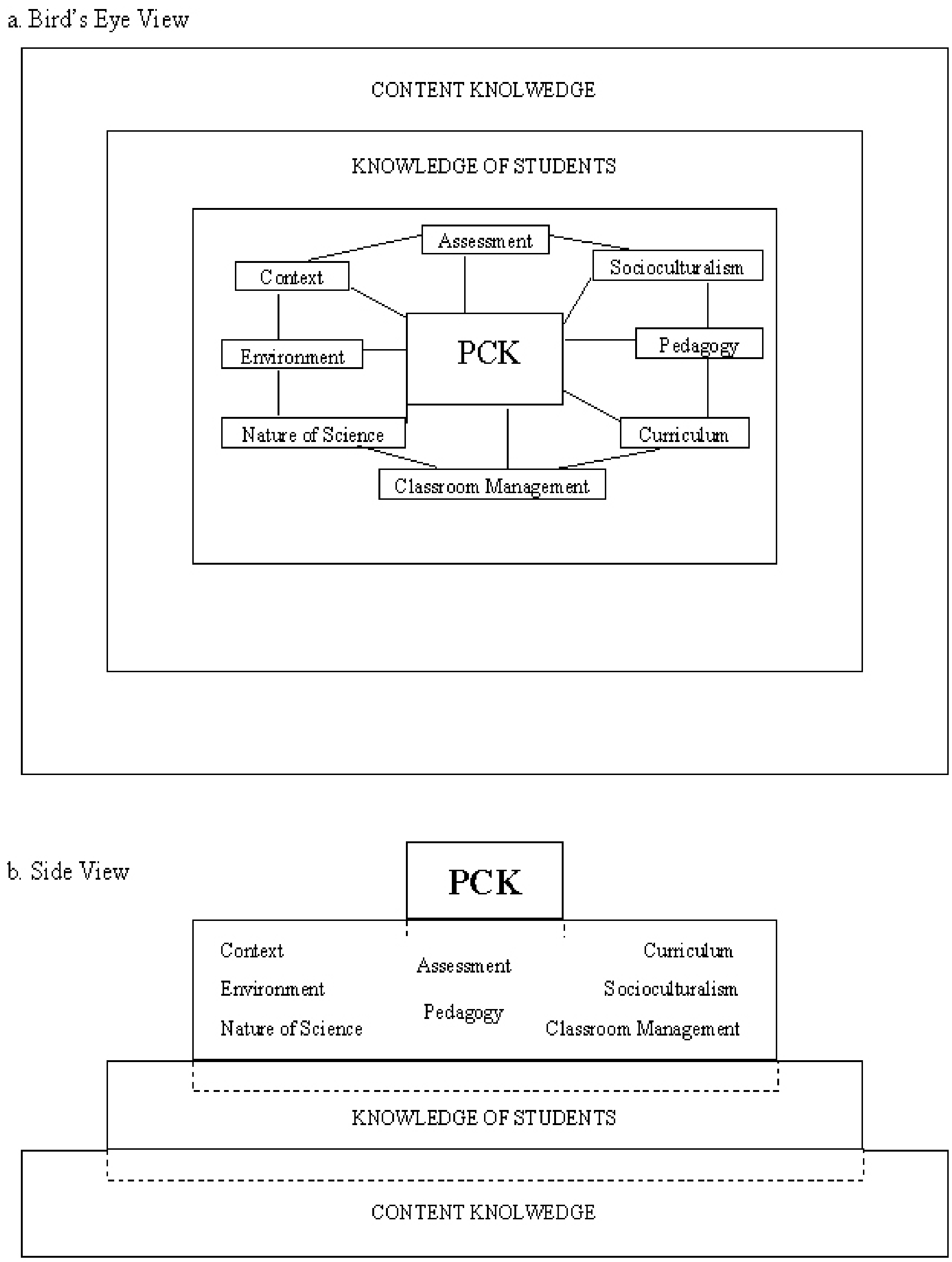


Figure 2.3: Taxonomy of PCK Attributes (Veal & MaKinster, 1999, p.12)

When the taxonomies were analyzed, it was seen that they were prepared for science teaching; however, it can be applied to any field of education. In order to measure and identify a term, from actions and/or operations one needs to have an operational definition. Veal and MaKinster (1999) made an operational definition of

PCK after creating the taxonomies which is “the ability to translate subject matter to a diverse group of students using multiple strategies and methods of instruction and assessment while understanding the contextual, cultural, and social limitations within the learning environment” (p. 14). This definition gave the clue that even though they created a taxonomy their approach close to the transformative model and like all the other studies SMK was taken as an indispensible part of teachers’ knowledge as being a base in the taxonomy.

Shulman’s ideas not only affected researchers but also affected teacher assessment practices and licensing examinations in education (Rowan, Schilling, Ball, & Miller, 2001). As a result, widely used teacher assessment series, called Praxis, have been revised to measure both preservice teachers’ subject matter knowledge and pedagogical content knowledge. In line with this, survey researchers in the field of education moved their attention to teachers’ knowledge of subject matter and pedagogy. However, Rowan, Schilling, Ball, and Miller (2001) analyzed the large scale surveys in the literature and found that these studies have tried to assess the effect of teachers’ subject matter knowledge on students’ achievement. Therefore, measuring PCK in specific content and examining the effect of PCK on students’ achievement through large-scale surveys is a missing paradigm. In order to fill this gap, as a part of the Teaching and Learning to Teach (TELT) project, a set of questionnaire items was developed for assessing PCK in mathematics and reading/language arts (Rowan et al., 2001). In the study, for measuring PCK, three attributes were used: content knowledge, knowledge of students’ thinking, and knowledge of pedagogical strategies. The questionnaire items used for measuring PCK consists of vignettes; that is, scenarios of classroom situations.

2. 3 Teachers’ Knowledge on Functions

Teachers PCK on functions and their knowledge of students’ conceptions have been the focus of many studies after Shulman (1986) presented the idea of PCK. In this part of the review of literature studies on teachers’ PCK of functions were explained.

Every research studying PCK of functions also investigated the subject matter knowledge. The areas of research on functions basically run through the following key themes of SMK: definition of function, different representations of functions and moving between them, connectedness of function knowledge, applications of the rules related to the function concept (Cha , 1999; Duah-Agyeman, 1999; Ebert, 1994; Even, 1989; Klanderman, 1996; McGehee, 1990; Pitts, 2003; Wick, 1998; Winsor, 2003; Wyberg, 2002). However, for complete understanding of functions teachers need to know composite and inverse functions. Only a few studies investigated the teachers pedagogical content knowledge on these subtopics (Ebert, 1994; Even , 1989; Lucus, 2005, 2006). Most of the research studies related to concept of functions are unpublished doctoral dissertations or master thesis.

The preservice teachers use of definitions about the concept of function were investigated by most of the studies (Cha , 1999; Duah-Agyeman, 1999; Ebert, 1994; Even, 1989; Klanderman, 1996; McGehee, 1990; Pitts, 2003; Wick, 1998; Winsor, 2003; Wyberg, 2002). These studies investigated the use of definitions through surveys and/or interviews via questions asking directly the definitions. Results of the studies were close to each other in terms of use of definitions, preservice teachers were able to describe the functions but the definitions they gave lacks the univallence property. Even though knowledge of definitions were assessed through a question, use of just this kind of a question is not enough to decide whether that knowledge exists. Since as stated by Mason and Spence (1999) knowing about a subject requires having three iterative stages: first declarative knowledge (knowing that) forms the ground and forthcoming actions depends on this stage, second conditional knowledge (knowing why) gives a sense of direction and provides an overview for the actions, third procedural knowledge (knowing how) provides actions and applications of knowledge (Smith & Ragan, 1993, as cited in Yıldırım, Özden, & Aksu, 2001).Therefore, use of questions addressing three knowledge types would be beneficial to describe a knowledge about a topic.

Use of different representations and translations among them another theme emerged from the studies of preservice teachers (Even, 1989; McGehee, 1990; Wick, 1998). While all of them used survey questions to see to what degree preservice teachers were able to translate among different representations, McGehee (1990) used an additional instrument which asks for classifying functions given in different representations. Results of the studies were not consistent. Results of Even (1989) and McGehee (1990) revealed that preservice teachers experienced difficulty when questions were given in graph format, whereas the results of the study conducted by Wick (1998) revealed that the participants were successful in translation among the different representational forms. This difference might be because of the participants involved in the studies. In the former studies the participants were preservice teachers taking the regular methods course, whereas in the latter one participant were voluntary preservice teachers attending to one week summer course. Of course, as stated earlier for the use of definitions, questions simply asking transformations do not guarantee complete understanding.

Preservice teachers' answers to procedural questions which require application of the rules were also investigated by the researchers through surveys and/ or interviews (Even, 1989; Klanderman, 1996; McGehee, 1990; Pitts, 2003; Winsor, 2003; Wyberg, 2002).The results revealed that most of the preservice teachers were able to apply rules about the concept of functions. However, this result does not guarantee deep and connected understanding of the concept of functions. The study conducted by Schroder, Schaffer, Reisch, and Donavan (2002) used non-routine problems (problems which are not very similar to ones they solved before but require combination of known facts or principals) in order to see whether preservice teachers with a certain knowledge about the function concept can or cannot solve these questions. The results indicated that preservice teachers have difficulty in solving non-routine problems which requires use of different knowledge domains at the same time. So, they suggested the use of non-routine questions both in the teacher education programs and assessment of teachers' content knowledge so that teachers use of connections in mathematics will increase.

Organization and connectedness of mathematical knowledge one of the important themes emerged from the analysis of the related literature about presevice teachers knowledge of functions. Connectedness of knowledge was investigated through concept maps and card sorting activities through the research studies (Duah-Agyeman, 1999; Ebert, 1994; Howald, 1998; McGehee, 1990). The combined use of concept maps, essays and interview was to assess the connectedness of preservice teachers' knowledge of functions (Bolte, 1999). Results revealed through the concept maps and/or card sorting activities were consistent throughout the studies. Preservice teachers mostly unable to classify functions and construct meaningful groups of subtopics related to concept of functions.

The last theme discussed in the literature was the composite and inverse functions (Ebert, 1994; Even , 1989; Lucus, 2005, 2006). The studies collected the data in different ways. Even (1989) used questionnaire and results revealed that preservice teachers had a limited understanding of composite and inverse functions. Lucus (2005) collected the data through clinical interviews and lesson plans written by preservice and inservice teachers. The results revealed that for both composite and inverse functions teachers showed procedural approach in treating the topic of composite and inverse functions, and a poor conceptual knowledge of the topics. She suggested the use of both lesson plans and teaching practices for further studies since being able to observe the teaching will complete the cycle of assessing pedagogical content knowledge of teachers. For assessing pedagogical content knowledge use of observations and lesson plans were also suggested by Baxter and Lederman (1999) and Winsor (2003) as an effective tool since it requires the use of all components of PCK. The study conducted by Ebert (1994) similar to previous studies in terms of findings but differentiates from the others since it used variety of assessment tasks and instructional practices (subject matter knowledge test, vignettes, card sort tasks, unit plan, and interviews). The most important instrument she used was the vignettes (short story presenting an issue, such as descriptions of students' misunderstandings in a math class, in a context) since the research has shown that vignettes would appear to have a more realistic data compared to surveys for getting information about the teachers' knowledge (Ruiz-Primo & Li, 2002; Stecher, Le, Hamilton, Ryan, Robyn, & Lockwood, 2006). Moreover, the study conducted by Stecher, Le, Hamilton, Ryan, Robyn, and Lockwood, (2006) provides a partial evidence that teachers' responses to vignettes shows similarities with their instruction.

Also, Ebert (1994) found vignettes as a consistent assessment tool for teachers' knowledge components, especially for knowledge of learners since vignettes were mostly constructed upon students' misconceptions or misunderstandings and through their answers teachers have to show understanding of those. Because to generate appropriate explanations and representations, teachers must have some knowledge about the students' current state about the topic and the things that are likely to be puzzling for the students (Grossman, 1990). Due to its importance, knowledge of learners was taken as a component of pedagogical content knowledge regardless of the model used in the literature (Abd Rahman & Scaife, 2005; Cochran, King, & DeRuiter, 1993; Cohen & Ball, 1999; Ebert, 1994; Fennema & Franke, 1992; Grossman, 1990; Harel & Lim, 2004; Magnusson, Krajcik, & Borko, 1999; Marks, 1990; Morine-Dershimer & Kent 1999; Rowan, Schilling, Ball, & Miller, 2001; Shulman, 1987; Smith & Neale, 1989; Veal & MaKinster, 1999).

In addition to knowledge of learners, if a teacher does not have necessary pedagogical skills to provide explanations to students in the context of teaching, teaching cannot occur. Therefore, having general pedagogical knowledge (Abd Rahman & Scaife, 2005; Cochran, King, & DeRuiter, 1993; Ebert, 1994; Grossman, 1990; Shulman, 1987) and knowledge of context (Abd Rahman & Scaife; 2005; Cochran, King, & DeRuiter, 1993; Grossman, 1990; Magnusson, Krajcik, & Borko, 1999; Marks, 1990; Morine-Dershimer & Kent, 1999; Shulman, 1987; Veal & MaKinster, 1999) are seen as indispensible parts of teachers' knowledge. Through instruction, teachers made choices depending on their beliefs and Bishop (2001) stated that they found three general types of values that teachers wanted to transmit to their students: the general educational values, the mathematical values, and the mathematics educational values. By restricting the domain, Cha (1999) investigated how preservice teachers value the importance of teaching functions and found that most of the preservice teachers aware of the practical aspects of teaching mathematics (intrinsic value), and half of them know about the intrinsic value for functions. Even though most of the preservice teachers mentioned the excitement and beauty in mathematics, they could not give examples for functions. Lastly a few of them mentioned the importance of functions to other mathematics topic (pedagogical value).

2.4 Summary

The knowledge required for teaching mathematics is complex and unfortunately many teachers do not possess this knowledge which will help students to acquire the mathematical proficiency as described in the national curricula. Although the scholars stated that the content domain of functions is broader and important for school mathematics, it was less defined than the other areas of research on content domains such as; addition and subtraction or fractions with young children or teachers. With respect to functions, it was reported that preservice teachers hold several misconceptions that can affect the way they teach. The areas of research on functions basically run through the following key themes: different representations of functions and moving between them, linear functions, definition of function with emphasis to univalence property, graphical representations. However, for complete understanding of functions teachers need to know composite and inverse functions. Only a few studies investigated the teachers pedagogical content knowledge on these where only one of those gave special emphasis to composite and inverse functions and these studies had a limitation since teachers were not observed while teaching. Therefore, there is a need for further examination of what preservice secondary mathematics teachers' pedagogical content knowledge of composite and inverse functions are. As we understand the pedagogical content knowledge of preservice teachers about composite and inverse functions, accordingly we get better understanding about the nature of pedagogical content knowledge of functions, so we stand to gain a better understanding of the teacher education programs and ways to improve them.

The next chapter describes the methods used in order to elicit the pedagogical content knowledge of preservice secondary mathematics teachers about composite and inverse functions.

CHAPTER 3

METHODOLOGY

This chapter involves a full account of research design and implementation. Within this perspective, it gives details of research questions, design of the study, context, participants, research procedure, instruments, data analysis procedures, trustworthiness, and researcher's role, background and biases.

3.1 Research Questions

In this study, preservice secondary mathematics teachers' pedagogical content knowledge of composite and inverse functions were investigated through the following research question with subsidiary questions:

What is the nature of pedagogical content knowledge of preservice secondary mathematics teachers about the composition and inverse of functions?

a. What is the extent and organization of preservice secondary mathematics teachers' subject matter knowledge of composite and inverse functions?

f. What is the nature of preservice secondary mathematics teachers' general pedagogical knowledge?

g. What is the preservice secondary mathematics teachers' awareness about the value of teaching composite and inverse functions?

h. What is the nature of preservice secondary mathematics teachers' knowledge of context?

b. What is the nature of preservice secondary mathematics teachers' knowledge about learners' conception of composite and inverse functions?

3.2 Participants

The participants in this study were three female preservice secondary mathematics teachers who were in the second year of the two-year non-thesis master program in the Graduate School of Education at Bilkent University.

The study was conducted in the fall term of 2006-2007 academic year. At the time of the study, there were eight preservice secondary mathematics teachers enrolled in the program, and they were taking the following courses: Planning and Assessment in Teaching, and Teaching Practice in Mathematics.

For the Teaching Practice in Mathematics course, eight preservice secondary mathematics teachers were divided into three different schools for a six-week practice. Three of the preservice secondary mathematics teachers took the teaching practice course in Private Bilkent High School (PBH). Since the researcher is also a mathematics teacher in PBH, these students were conveniently selected as the participants for the study. The three preservice mathematics teachers' demographic data (See Table 3.1) were gathered from university records. The names given in the table are pseudonyms.

Table 3.1: Preservice Secondary Mathematics Teachers Demographic Data

Preservice Mathematics Teacher	High School Graduated	University	CGPA	CGPA in Master Program	LES* Score	English Exam Score
Deniz	TED Karadeniz Ereğli Private High School	Middle East Technical University	2.70/4.00	3.54/4.00	67.27	TOEFL CBT 237
Yeliz	Çankaya Super High School, Ankara	Ankara University	81.44/100	3.38/4.00	65.97	COPE C
Gizem	Yunus Emre Anatolian High School, Izmir	Middle East Technical University	3.05/4.00	3.59/4.00	63.63	TOEFL CBT 207

*LES: Lisansüstü Eğitim Sınavı (Graduate Education Examination)

3.3. Context of the Study

The study was conducted in the Bilkent University Graduate School of Education Mathematics Teacher Education Program, in Ankara, Turkey. This program is a two-year non-thesis master program and accepts mathematics majors and trains them for the certification of secondary mathematics teachers. Since it provides preservice mathematics teachers a full scholarship, they are chosen with respect to their cumulative grade point average (CGPA), Graduate Education Examination (abbreviated LES in Turkish), English proficiency, and a personal interview (See Table 3.1). For CGPA minimum expectation is 2.5 out of 4 or 70 out of 100. LES score is required for all Turkish graduate applicants. This exam is conducted by Öğrenci Seçme ve Yerleştirme Merkezi (ÖSYM) and includes multiple choice questions in two domains: Turkish literature and mathematics. LES requirement is waived for applicants with GRE scores of a combined minimum verbal and quantitative total of 950 and 3.5 in analytical writing. Minimum verbal LES score accepted is 50. For English proficiency, Governmental Personal Language Examination (abbreviated KPDS in Turkish), Foreign Language Exam for Academic purposes (abbreviated ÜDS in Turkish), Certification of Proficiency in English (COPE), TOEFL, and IELTS were accepted. KPDS is an English proficiency exam and is required for taking a degree in governmental jobs and was also used for application to graduate programs. ÜDS is also an English proficiency exam that is required from all applications of graduate programs. Both KPDS and ÜDS are conducted by ÖSYM. Certification of Proficiency in English (COPE) is an English proficiency exam conducted by Bilkent University. Minimum accepted scores from the tests were as follows KPDS 70 out of 100, ÜDS 70 out of 100, TOEFL (193 CBT / 69 iBT), IELTS 6, COPE C.

The program requires the Graduate School of Education in Bilkent University to offer their graduate students four different teaching practice courses spread over to all semesters (See details of the courses in Appendix A). These experiences differ from semester to semester. These school experiences include observation, teaching practices, departmental activities and school activities. All of them are held in different leading private schools of Turkey such as, Bilkent Schools (Private Bilkent High School and Primary School), Bilkent Labratory and International School-BLIS (previous name was BUPS), TED Ankara College, METU Schools, HEV Schools, Robert College and American Collegiate İzmir (ACI). Besides seeing different leading private schools in Turkey, student teachers have the chance to go to the U.S. and have another valuable teaching practice there.

All of the schools, except Robert College, are K-12 schools. ACI and Private Bilkent High School are only respective high school. Except Bilkent Schools and BLIS, the other schools have a five year high school program, where the first year of the program is a prepatory grade. An intensive English program is given to students during this year. On the other hand, Private Bilkent High School and BLIS have a four year program.

ACI and TED Ankara College offer their students an IB Diploma Programme in addition to the national curriculum. BLIS applies only IB Diploma Programme. Robert College and HEV Schools applies AP Program. Bilkent Schools and METU Schools only applied national curriculum at the time of the study.

Another difference among the schools visited was in terms of the schools' population. The most populated school among these is TED Ankara College. TED Ankara College has more than 5,500 students, 530 teachers, 230 employees and 26,000 alumni. Robert College has approximately 1000 students. On the other hand, Private Bilkent High School had around 200 students and 25 teachers at the time of the study. Therefore, preservice secondary mathematics teachers have a chance to see schools and classes in different sizes.

Except Robert College, after 8^{th} grade, students have a right to continue their high school without an entrance examination. Moreover, all of these schools accept students according to SBS grades (Seviye Belirleme Sınavı is an exam designed for assessing primary school graduates' level of Turkish, Mathematics, Science, and Social Sciences. The exam consists of 100 questions equally distributed to each subject and exam results are used for entrance to high schools in Turkey). According to the choice of students who has higher SBS grades, the leading school in Turkey is Robert College.

Preservice secondary mathematics teachers are first assigned to collaborating schools, and then to mentor teachers in those schools. The first teaching practice course called School Experience I is based mostly on observation of the classroom and teaching context, and lives of the students and teachers. Preservice secondary mathematics teachers spend a day in the Bilkent schools and BUPS for fourteen weeks. During this time, they observe different classes and different teachers in order to examine different teaching methods and classroom management techniques. They have their first teaching experiences in classrooms during the last two weeks of the course. They have to complete six hours of teaching either individually or collaboratively.

In School Experience II, preservice secondary mathematics teachers spend a day at TED Ankara College for 11 weeks. Moreover, they are sent to either Istanbul or Izmir for school experience only for two weeks in the middle of the semester. They visit either Robert College or Hisar Eğitim Vakfı Schools in Istanbul and İzmir American College in İzmir. This is different from other school experiences as they spend an entire two weeks at schools which provides them with seeing the continuity of the courses, and school environment.

In the first week, they just observe the classes and try to get some information about the school in every aspects. In the second week, they teach and are observed by their supervisors from Bilkent University. After Istanbul and İzmir experiences, they continue to go to TED. They teach in TED for a total of at least 10 hours. They continuously get feedback both from their mentors and supervisors. During these periods, preservice secondary mathematics teachers reflect on their teaching experiences together with their supervisors in weekly meetings.

In the first term of the second year, preservice secondary mathematics teachers have the teaching practice course. They are spread over 3 different private high schools in Ankara (METU Schools, Bilkent Schools, and BUPS) during the first six weeks of the semester. This again is different from school experience courses, because preservice secondary mathematics teachers work at school during the whole school working time, every day of the week. They have more chance to observe different classes, continuum of the courses, assessment and evaluation techniques, department work, and school environment. The most important thing that distinguishes this course from others is that they have more opportunity to teach in classes. In the first week, they just observe the classes and try to get to know students. At the end of the teaching practice, they have to complete at least 30 hours of teaching. They are also involved in the departmental studies and try to learn as much as they can in the school environment.

Having seen quite a good number of leading private schools in Turkey, preservice secondary mathematics teachers have the chance to go to the United States as part of their program. It is the last school experience of the program. Bilkent University Graduate School of Education has an agreement with Full Bright regarding the Turkish Student Internship Project which allows them to be involved in the project. They went to the US (city of Ames in Iowa) for 2 months and they are distributed to different high school in Ames. They have an opportunity to compare different curriculums and different educational applications in American high schools. During this project, they also visit some high schools which are particular in that area. Moreover, they observe classes, help their mentor teachers, and teach collaboratively or individually in the classes. Teaching to native speakers is also an important experience for these student teachers. They teach at least 25 hours during these two months.

The context of the study described above does not represent the all Turkish teacher education programs in Turkey, it is unique to the Graduate School of Education at the Bilkent University. Thus, the participants of the study did not constitute a representative sample for the preservice teachers in Turkey.

3.3 Design of the Study

In order to examine the preservice secondary mathematics teachers' pedagogical content knowledge of composite and inverse functions qualitative research methodologies were used in the study.

Qualitative research was defined by Denzin and Lincoln (2005) as follows:

> Qualitative research is a situated activity that locates the observer in the world. It consists of a set of interpretive, material practices that make the world visible. These practices transform the world. They turn the world into series of representations, including fieldnotes, interviews, conversations, photographs, recordings and memos to the self. At this level, qualitative research involves an interpretive, naturalistic approach to the world. This means that qualitative research study things in their natural settings, attempting to make sense of, or interpret, phenomena in terms of the meanings people bring to them (p.3).

Bogdan and Biklen (1998) defined qualitative research as an umbrella term referring to specific research strategies, researchers' role, and data collection and analysis strategies.

Different researchers' categorized qualitative research methodologies in education under different headings (Creswell, 2007; Merriam, 1998; Miles & Huberman, 1994; Yin, 2003). Creswell (2007) stated five different approaches for the qualitative research, which are narrative research, phenomenological research, grounded theory research, ethnographic research, and case study research. Similar to Creswell, Merriam (1998) categorized qualitative research methodologies under five headings: basic or generic qualitative study, ethnography, phenomenology, grounded theory, and case study. Both researchers mentioned that these five approaches are not purely distinct and they work in conjunction with each other. In this study, case study was used as the qualitative research methodology.

When the qualitative case study definitions in the literature were examined, it was seen that there are slightly different definitions. Creswell (2007) defined it as an approach in which researcher explores the bounded system, or a case, through multiple, rich, and in-depth data collection and reports them in detail depending on the case. Similarly, Merriam (1998) defined qualitative study as "intensive holistic description and analysis of a single instance, phenomenon, or a social unit" (p.21). Different from the other two, Yin (2003) defined the case study more technically in two phases. First, she defined case studies when the context and the phenomenon can be easily distinguishable as "an empirical inquiry that investigates a contemporary phenomenon within its real life context, especially when the boundaries between the phenomenon and the context are not clearly evident" (p.13). Second, she defined case studies when phenomenon and context are not always distinguishable as " an inquiry copes with technically distinctive situation in which there will be many more variables of interest than the data points, and as one result relies on multiple source of evidence, with data needing to converge in a triangulating fashion, and as another result benefits from the prior development of theoretical propositions to guide data collection and analysis" (pp. 13-14).

From the definitions, it can be concluded that the most important characteristic of the case study is the object of the study, case, and its relation with its context (Creswell, 2007; Merriam, 1998; Yin, 2003). Therefore, researchers must carefully define the case and its context. This is because, a case is a bounded system where some components of the study are within the case and some are outside. Merriam (1998) stated that bounds of the case are very important in defining the case. So, as a result of case studies, researchers describe and interpret the case within its bounds and context, they don't represent the world (Yin, 2003).

This study was characterized through Creswell, Merriam, and Yin's definitions of the case study. The aim was to "gain in-depth understanding of the situation and meaning for those who are involved" (Merriam, 1998, p. 19) and the study was particularly interested in analyzing the nature of pedagogical content knowledge of preservice secondary mathematics teachers about composite and inverse functions.

The importance of having research design is stated by many researchers (Creswell, 2007; Merriam, 1998; Stake, 2005; Yin, 2003; Yin, 2009), because, having a research design can provide a strong guidance in deciding which data to collect.

Types of case studies are distinguished according to the size of the bounded case and the intent case of the case (Creswell, 2007). Creswell (2007) categorized the case studies according to their intent as the single instrumental case study, the multiple case study, intrinsic case study and he influenced from the categorization of Stake (1995) which are intrinsic case study, instrumental case study and collective case study. The term the single instrumental case study used where researcher focuses on an issue or concern, and then selects one bounded case to illustrate this issue (Creswell, 2007). He used the term multiple case study when the researchers selects multiple cases to illustrate the same issue. In this type, generalization was made from one case to the other one. Finally, he defined intrinsic case study as the study where the case is the focus of interest because of an unusual or unique situation case representing. Similarly, Merriam (1998) categorized case study into three categories according to its overall intent of the study. First category is the descriptive case studies in this type basic information about the phenomenon under study is presented. Second category is interpretive case studies which contain rich and thick descriptions. Unlike the descriptive case study, in interpretive case studies descriptive data are used to develop a typology, a continuum, or conceptual categories. Last category is the evaluative case studies which involve description, evaluation, and judgment. For example, presenting judgments about worth of applying a certain program is under this category. Apart from these categories, Merriam (1998) mentioned the multiple case studies under a different heading and she stated that having multiple cases is a strategy for increasing the external validity and generalizability of the findings. And also, she mentioned that there can be purely descriptive, interpretive or evaluative case studies but generally case studies involve combination of those.

Thus, this study was multiple case study which involved both description and interpretation since the purpose was to provide an insight and get rich and thick description of the pedagogical content knowledge of preservice secondary mathematics teachers about composition and inverse functions. Although the participants of the study seems close to each other in terms of the tabulated values in Table 3.1. It was also seen from the how multiple Also, selecting preservice secondary mathematics teachers at PBH as participants provided a controlled environment for the researcher in which all of the participants could complete the instruments under the same circumstances. Besides, this provided researcher to observe them in actual classroom setting as they transform their knowledge into instructional practices.

In this study, the context of my research was the teacher education program, the participants involved and preservice secondary mathematics teachers attending teaching practice course at Private Bilkent High School (PBH) all constituted the case of the study.

3.5 The Research Procedure

Data collection for the study was conducted from October 2006 to December 2006. Data for this study were collected in two phases. The first phase included all preservice secondary mathematics teachers in the program whereas the second phase included only the three preservice secondary mathematics teachers who were taking the Teaching Practice course at PBH. The reason for including eight preservice secondary mathematics teachers in the first phase was that it was not known which preservice secondary mathematics teachers would be teaching in the PBH. A timeline of data collection used in both phases is given in Table 3.2.

Table 3.2: Timeline of Data Collection

Phase	Week	Date	Data Collection Activity
I	1	*October 18*	Survey of Function Knowledge
	2	*November 2*	Knowledge of Context Focus Group Interview
II	3	*November 6* *November 7* *November 9* *November 10*	Sample Vignette -d- (distributed) Sample Vignette-c- (collected) and discussion Concept Map Terminology discussion Concept Map Activity Definition of Functions, Inverse Functions and Composite Functions Activity
	4	*November 13* *November 14* *November 15* *November 16* *November 17*	Concept Map Essay-c- Survey of Function Knowledge Follow-up Interview with Deniz Lesson Plan Format-d- Survey of Function Knowledge Follow-up Interview with Yeliz & Gamze 3 Vignettes-d-
	5	*November 21* *November 22* *November 23*	Non-routine Problems Interview with Deniz Non-routine Problems Interview with Yeliz & Gamze 3 Vignettes-c-
	6	*November 29* *November 30* *December 1*	Lesson Plans-c- Value of Teaching Functions, Inverse functions, Composite Functions Focus Group Interview 5 Vignettes-d- Preservice teachers' teachings started
	7	*December 6* *December 8*	5 Vignettes-c- 5 Vignettes-d-
	8	*December 13* *December 14* *December 15*	5 Vignettes-c- Evaluation Interview with Deniz, Yeliz & Gamze Concept Map Activity Follow up Focus Group Interview Preservice teachers' teachings ended

3.6 Instruments

This section describes in detail the instruments and how they were implemented. The instruments of the study resulted in four different types of data forms: observations, interviews, documents, and audiovisual materials as categorized by Creswell (2007). Observation data came from gathered fieldnotes by conducting an observation of participants' teachings at PBH. Interview data came from transcriptions of the semi-structured interviews. Document data came from the survey of function knowledge, journal writings, vignettes, and lesson plans.

Audiovisual data came from the examination of the videotape of the lessons in which the participants taught.

3.6.1 The Development of the Instruments

The complex nature of PCK is emphasized by many researchers (Abd Rahman & Scaife, 2005; Baxter & Lederman, 1999). Because of its complexity, it is also difficult to assess (Baxter & Lederman, 1999). A number of challenges were identified by Kagan (1990) while trying to study and assess teacher's knowledge. Her concerns about teachers' knowledge can be applied to PCK since it's a knowledge type unique to teachers (Shulman, 1987).

The first problem is that PCK is an internal construct; that is, it is teachers' understanding of subject specific examples, representations, analogies, and explanations. So while gathering information about it, relying only on observation or interview, problems may arise. Conducting only observation gives incomplete information since teachers might use just the limited portion of their representations, analogies, strategies, and methods. As a result, the observer would not get the full picture about the participant. Also, observer might not give meaning to selection of some of the representations, analogies, strategies, and methods, but not the others. Second, Kagan (1990) warns about the fact that teachers do not always possess the language to express their thoughts and beliefs or may refrain from expressing unpopular beliefs. This warning leads us to the fact that when using solely paper pencil instruments like questionnaires or short answer tests suffer from the same problems with observations and interviews. To conclude, it is appropriate to use an array of instruments to assess PCK.

To prepare and select instruments for this study, methodologies used in the previous researches were used as a guide. Ebert's (1994) criteria for selection of instruments inspired the researcher in the preparation of instruments. The criteria includes five steps: first use the PCK definition as a guide, second use variety of tasks since they may provide an evidence for PCK, third create tasks specific to topics in a unit, fourth use tasks which may provide evidence for teacher's knowledge and beliefs about learner and mathematics, and fifth use tasks of a qualitative nature which have proven to be effective as a means of describing teachers' knowledge and beliefs.

The use of combination of methods, data, or perspectives in a study is called triangulation (Denzin, & Lincoln, 1994; Maxell, 1996; Patton, 2002; Yin, 2003). By triangulation, the risk that the conclusions will reflect only systematic biases or limitations of a specific method will be reduced, and it allows you to gain a better assessment of validity by collecting information from a diverse range of individuals and settings (Maxell, 1996). So, triangulation of data was used in the study since the research theme of this study requires a considerable amount of description and interpretation

Apart from these, methodologies which have been used in previous researches were also included in the investigation of pedagogical content knowledge. While deciding the scope of the questions, national curriculum, textbooks, and researcher's own experience were taken into consideration. The following table (See Table 3.3) will serve as a guide for understanding which instruments were used for assessing the categories of PCK.

All the instruments in the study were checked by the researcher, two experienced mathematics teachers, one of whom also has a PhD in mathematics education, and the research supervisor, all of whom constituting a team, to determine the face and the content validity.

Table 3.3: Relationship between the categories of PCK and the instruments

	PCK				
Instruments	**SMK**	**KL**	**V**	**KC**	**GPK**
Survey of Function Knowledge	*				
Survey of Function Knowledge Follow-up Interview	*				
Concept Map Activity	*				
Non-routine Problems Interview	*				
Definition of Functions, Inverse Functions, and Composite Functions Activity	*				*
Vignettes	*	*	*	*	*
Knowledge of Context Focus Group Interview				*	
Value of Teaching Functions, Inverse Functions, and Composite Functions Focus Group Interview			*		
Lesson Planning Activity	*	*	*		*
Teaching Practice	*	*	*	*	*

SMK : Subject Matter Knowledge
KL : Knowledge of Learners
V : Value of Teaching
KC : Knowledge of Context
GPK : General Pedagogical Knowledge
* : PCK category measured by this instrument

3.6.2 Survey of Function Knowledge

The survey of function knowledge (See Appendix B) covering the content of the functions unit in the 9th grade national mathematics curriculum (MEB, 2006) in Turkey was developed to measure the preservice secondary mathematics teachers' subject matter knowledge on functions. Although the main concern of the study is composite and inverse functions, basic function knowledge (including definition, domain, range, and representations), properties of functions, and operations on functions were also assessed in the survey since they are prerequisite knowledge for composite and inverse functions. The survey consists of 19 open-ended questions, six of which having some subitems. The survey included 33 items altogether. The open ended questions were used in order to have insight on preservice secondary mathematics teachers' computation and process knowledge and deeper understanding of their conceptual understanding. In the survey five aspects were assessed: basic function knowledge, operations on functions, properties of functions, composite, and inverse functions. While selecting questions declarative, conditional, and procedural knowledge types were taken into consideration. Some of the questions in the survey of function knowledge were taken from the literature and some of them were developed by the researcher. The Table 3.4 includes information about question numbers with their associated aspect, knowledge type, and origin.

The survey of function knowledge was submitted to team along with a checklist including the following categories: 1) survey provides a relevant and adequate representative sample for the 9^{th} grade functions unit in national mathematics curriculum in Turkey, 2) questions are appropriate to grade level, 3) questions wording is understandable, 4) questions context is appropriate for the national mathematics curriculum in Turkey, 5) question contributes to relevant knowledge types.

They were also given the table of specifications and the knowledge type of each question (See Table 3.4). Before their examination, definitions about the three knowledge types were discussed with the team in order to clear the gaps and inconsistencies between the team and the researcher. The survey was found adequate for assessing the 9^{th} grade functions unit in national mathematics curriculum in Turkey and associated knowledge types of questions were also found relevant. Only some of the questions were revised in order to make the wordings clear and suitable for the knowledge type. The draft form of the final version was resubmitted to the team and they all commented on the clarity of questions, their face and content validity, and the correctness of their categorization into knowledge types. After the final comments, no more revisions were made on the test.

Table 3.4: Question numbers with their associated aspect, knowledge type, origin, and objectives

Question Number	**Aspect(s)**	**Knowledge Type**	**Origin**	**Objectives**
1	Basic function knowledge	Declarative	Karahasan (2002)	Define the concept of function
2	Basic function knowledge	Declarative	Karahasan (2002)	List different representations of functions
3(a)	Composite functions	Declarative	Researcher written	Define the concept of composition of functions
3(b)	Inverse functions	Declarative	Researcher written	Define the concept of inverse function
4	Basic function knowledge	Declarative	Researcher written	Decide whether the given relations are functions and explain the reasons
5	Basic function knowledge	Declarative	Researcher written	Define the concept of domain and range
6	Basic function knowledge	Procedural	Researcher written	Apply the properties of a domain of a function
7	Basic function knowledge	Procedural	Researcher written	Calculate the range of a given function
8	Operations on functions, Composite and inverse functions	Declarative	Researcher written	Read the graphs of functions and apply rules about the operations of functions
9	Operations on functions	Procedural	Researcher written	Apply operation of functions
10	Properties of functions	Procedural	Researcher written	Apply the properties of 1-1 and onto functions
11	Properties of functions	Conditional	Researcher written	Justify given statements about 1-1 and onto functions

Table 3.4: (continued)

12	Inverse functions, Properties of functions	Declarative	Karahasan (2002)	Decide whether the given functions have inverse function and explain the reasons
13	Composite and inverse functions	Procedural	Researcher written	Apply the properties of composite and inverse functions
14	Composite and inverse functions	Procedural	Researcher written	Apply the properties of composite and inverse functions
15	Operations on functions	Procedural	Researcher written	Apply the properties of operations of functions
16	Composite and inverse functions	Procedural	Researcher written	Apply the properties of composite and inverse functions
17	Composite and inverse functions	Conditional	Even (1989)	a) Explain and justify existence of composite function b) Explain and justify existence of inverse function
18	Composite functions	Conditional	Researcher written	a)Find out functions which satisfy the given composite function b) Decide and explain the existence of multiple functions satisfying the same composite functions
19	Basic function knowledge	Procedural	Researcher written	Apply the basic function knowledge

Basic function knowledge was the first aspect in the survey of function knowledge. This aspect includes questions 1, 2, 4, 5, 6, 7, and 19. These questions were designed to explore knowledge of definition of the function concept, representations of functions, examples and non-examples of functions, domain and range of a function.

The operations of functions aspect was addressed by questions 8, 9, and 15. These questions were designed to explore ability to use operations on functions both algebraically and on graphs.

The properties of functions aspect was addressed by questions 10, 11, and 12. These questions were designed to explore the understanding of one to one and onto properties of functions through algebraically, and on working inverse functions where one- to oneness was a prerequisite.

The composite functions aspect was addressed by questions 3, 8, 13, 14, 16, 17, and 18. These questions were designed to explore definition of composite functions, awareness of conditions for taking compositions, and operations about composite functions both algebraically, and graphically.

The inverse functions aspect was addressed by questions 3, 8, 12, 13, 14, 16, and 17. These questions were designed to explore the definition of inverse functions, awareness of conditions for being an inverse function, and operations about inverse functions both algebraically, and graphically.

The survey of function knowledge was administered to preservice secondary mathematics teachers before they started their internship in the PBH, so eight preservice secondary mathematics teachers participated in the survey. They were allowed 100 minutes for completion and were observed by the researcher. The language of the survey was English since the medium of instruction in Bilkent University is English. Preservice secondary mathematics teachers were told that if they had any questions about the language of the exam they were free to ask.

3.6.3 Survey of Function Knowledge Follow up Interview

The purpose of this interview was to gain an additional insight on the participants' knowledge of the content assessed in the Survey of Function Knowledge. Before the interview, the researcher analyzed the preservice secondary mathematics teachers' surveys and identified items with incorrect or partially correct or unclear responses and items which were left blank. During the interview each preservice secondary mathematics teacher was given an empty survey of function knowledge that was implemented and asked to solve items which were incorrect, partially correct, unclear, or left blank. Also, they were asked to explain their reasoning while they were solving. In addition, they were asked to evaluate the relative difficulty of the items. The interviews were conducted at PBH only with the participants and each took about 50 minutes. All the interviews were audiotaped.

3.8.4 Knowledge of Context Focus Group Interview Protocol

In order to gather data about preservice secondary mathematics teachers' knowledge of context, a focus group interview was conducted in order to reveal to what extend preservice secondary mathematics teachers were aware of the effect of opportunities provided by the school and the mathematics department, and students' mathematics level, SES, family while teaching the same subject through different level classes within the same school and through same level classes between different schools.

Knowledge of context was mentioned by many researchers (Abd Rahman & Scaife; 2005; Cochran, King, & DeRuiter, 1993; Grossman; 1990; Magnusson, Krajcik, & Borko, 1999; Marks, 1990; Morine-Dershimer & Kent, 1999; Shulman, 1987; Veal & MaKinster, 1999) but has never been assessed before. Since characteristics of knowledge of context are qualitative in nature, a focus group interview was conducted in order to reveal to what extent preservice secondary mathematics teachers were aware of the effect of knowledge of context while teaching the same subject through different level classes within same school and through same level classes between different schools. Focus group interview is an interview conducted on a small group of people (six to eight) on a specific topic (Patton, 1987). The need for focus group stated by Patton (1987) is as follows: "Focus group interviewing was developed in recognition that many of the consumer decisions that people make are made in a social context, and often growing out of discussions with other people" (p. 135). The focus group interview was conducted in order to get the advantage of discussions and for more elaboration. Discussion amongst the participants was expected to provide rich data compared to an individual interview. Considering these issues, a semi-structured focus group interview was conducted with the eight preservice secondary mathematics teachers before some of them started their internship at PBH. The interview was moderated by the researcher by following the interview protocol (See Appendix C). During the interview, the preservice secondary mathematics teachers were told to answer questions by considering schools in their School Experience I & II courses which were private high schools in Ankara, İstanbul, and İzmir, with more resources when compared to public schools where preservice secondary mathematics teachers had experience as a student.

As noted by Patton (1987), it was difficult to take notes during such an interview; due to this fact the interview was videotaped. It took 60 minutes.

3.6.5 Concept Map Activity

In this study, concept maps were used to see the organization of preservice mathematics teacher's subject matter knowledge on functions. Bolte (1999) used concept maps in combination with essays and interviews and found that this combination was effective in assessing the connectedness of preservice mathematics teachers' subject matter knowledge of functions. Hence, after constructing concept maps, participants were asked to write an essay and also an interview was conducted. Furthermore, before starting constructing concept maps, the researcher discussed with preservice secondary mathematics teachers what concept mapping is, how one constructs concept maps, and the kinds of concept maps –hierarchical, and web-like designs (See Appendix D). The importance of this step was mentioned by Özdemir as follows "In order to evaluate concept maps with scores, first of all your students should have learned to make concept maps sufficiently. When students learn to make concept maps, their maps can be evaluated by giving scores" (Özdemir, 2005, p.141). Although preservice secondary mathematics teachers previously constructed concept maps within their Educational Technology and Materials Development course via using a software, by this way they remembered the concept mapping terminology. This discussion was held two days before the concept mapping activity.

Secondly, preservice secondary mathematics teachers were required to construct two concept maps showing the organization of their knowledge about functions. For the first concept map, preservice secondary mathematics teachers were given the first activity sheet (See Appendix D) which included the instructions about how they should proceed while concept mapping. In this concept map, preservice secondary mathematics teachers were required to generate terms (such as, concepts, rules, definitions, examples) related with the 9^{th} grade functions topic and decide which ones to use. For the second concept map, everything was the same except for the fact that the researcher provided the preservice secondary mathematics teachers with some terms about functions in the second activity sheet. These terms were drawn from research (eg. Bolte, 1999) on functions and appropriate ones for the 9th grade national curriculum in Turkey were selected by the researcher (See Appendix D for list of items). Although the terms were provided, the participants were allowed to use extra terms or not to use the terms provided. The instructions in the activity sheets in both concept maps had a multi-step process suggesting a guide for constructing concept maps. In the instructions, preservice secondary mathematics teachers were first required to group the terms (self-generated or provided by the researcher) into clusters, then arrange the clusters, draw linking lines and label the linking lines, and lastly were asked to indicate the directional arrows between them.

After that, for the third part preservice secondary mathematics teachers were required to write an essay for describing thought processes while constructing both maps and comparing their two concept maps in terms of similarities and differences. They were asked to elaborate on their responses and gave any additional information that might be relevant.

Although not planned previously, during the evaluation interview it appeared that participants wanted to clarify their minds about concept mapping and would like to discuss it with their friends. Therefore, a follow up interview about concepts map was conducted. This interview was conducted in order to see consistencies and inconsistencies among the group. Before the interview, all preservice secondary mathematics teachers were given the other two teachers concept maps so that they could think about the other possible concept maps.

All the instruments prepared for the concept map activity were found adequate, appropriate, and valid by the team. Discussion on concept maps were made in the mathematics department at the PBH and it took about 30 minutes. The researcher took notes about this stage after the discussion. Two concept maps were completed at the PBH under the observation of researcher in 80 minutes. Essays were written at home. The follow up interview was also conducted at PBH and took about 40 minutes. It was a group interview, therefore the interview was videotaped in order to distinguish different voices.

3.6.6 Journals about Definition of Functions, Composite Functions, and Inverse Functions

In this activity several lists of true definitions about functions, inverse functions, and composite functions described by mathematicians or taken from textbooks were provided for the preservice secondary mathematics teachers in order to see whether they were familiar with all types of definitions and to see their preferences of definition(s) to be used in the class.

Investigation of the definition of function was a part of almost every research conducted about teachers' knowledge on functions (See Cha, 1999; Even, 1989; Vinner & Dreyfus, 1989). All of them used different categorizations for the analysis. This study differentiated from the other studies since it also covered definitions of composite and inverse functions which were not assessed in this way in the context of the mathematics teacher education before in the accessible studies. Therefore, a new and a simple categorization of definitions which reflected a general categorization for the definition of functions, composite and inverse functions were chosen.

In this categorization there are two types: formal definitions, and informal definitions. Formal definitions are defined as definitions which satisfy all the required conditions for that concept. Despite its importance, giving formal definitions to the students does not result in clear understanding of the concept's meaning (Schultze, 1939, as cited in Cha, 1999). Restatements of the part of formal definition for understanding of the students are called informal definitions. In the following table an example was provided for each type of definition (See Table 3.5).

Table 3.5: Examples for the definition types

Definition Type	**Example**
Formal definitions	A function is any correspondence between two sets which assigns to every element in the domain exactly one element in the range.
Informal definitions	In logic if a→b and b→c then a→c. Therefore, if there exists a function f which takes a to b and another function g which takes b to c, then one can talk about a third function, say h, which takes a to c. This new function is denoted by h=gof and called the composition of g and f.

Function definitions in the journals were taken from Cha (1999), Even (1989), and several textbooks (Adams, 2003; Aydın & Peken, 2000; Çavdar, Çaputlu, Arslan, Ayhan, & Yalçınkaya, 1997; Ellis & Gulick, 1991; Kaya & Salman, 1997; Larson, Hostetler, & Edwards, 2001; Silverman, 1990). Composition and inverse functions' definitions were taken from several textbooks. The journal regarding function definitions includes four formal, and twelve informal definitions. The journal regarding inverse function definition includes three formal, and ten informal definitions. The journal regarding composite functions definition includes four formal, and six informal definitions (See Appendix E). Whether the definitions fall into an associated category was controlled by the team. First, the researcher discussed with them two definition categories in order to clear the gaps and/or inconsistencies. The journals and the table (See Appendix E) which shows the associated category of definitions were submitted to them along with the definitions of each definition category. They also checked the face validity, content validity and wording of the definitions. Some rewording of the statements was suggested by them. Then, they checked the final version of the journals and they were found understandably worded, and each definition reflected the associated category.

In each journal, instead of just asking what your definition of function is, inverse function, and composite function, preservice secondary mathematics teachers were provided with several lists of definitions on a paper (See Appendix E). By this way, they were supposed to decide which definition was most appropriate for teaching students. In the journals, first, the preservice secondary mathematics teachers were asked to choose their favorite three definitions from the list and give reasons for their choices. Second, they were required to choose their least favorite definition. Third, they were asked which definition they would use if they were teaching functions, composite functions and inverse functions to 9^{th} grade and their underlying reasoning for choosing that definition. Last, they were required to respond to the case that they taught from the definitions they picked in the previous step, and some of the students did not understand it. Specifically, participants were asked what they would do to clear up the confusion.

Participants wrote in journals individually. Writing the journals took about an hour, and then they were collected. The journal writing activity was observed by the researcher.

Definition questions generally asked during the assessment of subject matter knowledge tests in the literature but since there was a time-limit for the survey of function knowledge, this activity was done separately. Moreover, writing in journals provided the researcher an opportunity to compare participants' definitions given in the survey of function knowledge with the definitions given in the journals. Apart from these, the participants' definitions were compared with the lesson plans and observations.

3.6.7 Vignettes

In order to assess the preservice secondary mathematics teachers' understanding of student conceptions and misconceptions about inverse functions, and composite functions, thirteen different vignettes (see Appendix F) were used. Vignettes are scenarios including student comments, questions, and/or solutions, and are generally used for searching PCK of preservice teachers (Ebert, 1994).

Thirteen vignettes were divided into three basic topics: six vignettes related to composite functions, five vignettes related with inverse functions, and two vignettes related with both composite and inverse functions. In Table 3.6, details about the vignettes were given.

Table 3.6: Vignette numbers with their associated topic, and conflicts (and/or problems) embedded in the vignettes

Vignette number	Topic	Conflicts (and/or Problems) Embedded in the Vignettes
1	Composition	Misunderstanding of the notation $h(x) = f(g(x))$ and mixing it with the ordinary multiplication $f(x).g(x)$.
2		Mixing order of operations when taking compositions of functions and mixing it with the ordinary multiplication $f(x).g(x)$.
3		Mixing composition with the ordinary multiplication when one of the functions is a constant function
4		Misunderstanding of the notation $h(x) = f(g(x))$ while working backwards in composition of function problems
5		Use (or misuse) of analogy for definition of composite functions
13		Use (or misuse)of analogy for understanding the idea of composition of functions
6	Inverse	Mixing the -1 in f^{-1} with the multiplicative/additive inverse
7		Importance of domain of a function when taking inverse of a function
8		Understanding the inverse functions as "undoing"
9		Use (or misuse) of analogy for definition of inverse functions
12		Use (or misuse) of analogy for understanding the idea of inverse functions
10	Composition & Inverse	Understanding of combined use of inverse and composition of functions in questions and ability to state the meaning behind the procedures used
11		Use of the fact $fof^{-1} = I$ while solving questions.

Vignette number 1 was taken from Ebert (1994) and vignette number 6 was inspired from Ebert's (1994) work. The other vignettes were prepared by the researcher. The researcher written vignettes were checked by the team regarding the purposes given in the table to determine the face and the content validity. Consensus was reached by the team. Only some of the vignettes were reworded after the control.

Before participants started to write in their vignettes, the researcher gave them a sample vignette (Even, 1989, see Appendix F) to complete at home. After they wrote their sample vignettes and handed them back, the researcher analyzed their responses and conducted a discussion at PBH with them about what a vignette is, and how they should write in their vignettes based on their work. During the discussion, the researcher talked about the characteristics of how to write a good vignette. For example, first of all the case or problem situation in the vignette should be carefully defined and then how the case or the problem situation would be solved by the teacher should be explained in detail. In addition, the researcher emphasized that the effectiveness and quality of the responses are more important than their length. The researcher took field notes about this discussion.

Each vignette including a case describing a part of a mathematics lesson related to either composite or inverse functions, and a confusion in the class was given to participants on a sheet of paper. First, for each vignette, they were asked to analyze the lesson excerpt and decide whether the thing that started the confusion in the class was correct or incorrect and to explain the reasons for their choice. Then, they were required to explain how they would respond to this case as a teacher and how they would clear up the confusion in the class.

Vignettes were given to the participants in three separate groups. Vignettes in each group were a mixture of composition and inverse functions because for the lesson planning activity they were required to choose their own order for teaching composite, and inverse functions. Moreover, giving only composition vignettes or inverse vignettes as a first group might have led participants to think that the researcher would prefer teaching composition during the lesson planning. The first group include vignettes numbers 1, 6, and 10. The second group include vignettes number 2, 7, 11, 12, and 13. The third group includes vignette numbers 3, 4, 5, 8, and 9. Vignettes were given to preservice secondary mathematics teachers on Fridays to be completed at home and after they had completed their written responses, they handed them back on the following Wednesdays. Participants were told to spend around 30 minutes on each vignette.

3.6.8 Interview Protocol about Non-routine Problems

Although participants' knowledge of functions, composite and inverse functions were assessed through the survey of function knowledge, non-routine problems were used in order to see the depth of their understanding about composite and inverse functions. Non-routine problems are defined as problems which are not very similar to the ones solved before but require combination of known facts or principals (Schroder, Schaffer, Reisch, & Donavan, 2002; Selden, Selden, Hauk, & Mason, 2000). When related to the knowledge types (declarative, conditional, and procedural knowledge), non-routine problems have the properties of conditional and procedural knowledge since they require students to recall related knowledge and use it in the appropriate conditions.

Non-routine problems were used in task-based interviews. The task based interview was defined by Davis (as cited in, Schroder, Schaffer, Reisch, and Donovan, 2002) as follows:

> Task based interviews vary along a number of dimensions, including the nature and amount of intervention by the interviewer, the extent to which participants are asked to verbalize their thoughts as they work at the task, the tools and materials available to them, and the equipment used to make records of the interview (p. 7).

In order to give preservice secondary mathematics teachers in the task based interviews a non-routine problems sheet (See Appendix G) and an interview protocol (See Appendix H) were prepared. The non-routine problems sheet included six problems. The first four problems in the task based interviews were taken from Schroder, Schaffer, Reisch, and Donovan, (2002) and they were related to composition of functions. They were non-routine since their solution requires combining knowledge of graphs, definition of functions and composition of functions, and domain of function. The fifth and sixth problems were related to inverse functions and they were inspired from Lucus (2005). Likewise from the first four problems, these two problems were called non-routine since their solution requires combining knowledge of graphs, definition of functions and inverse functions, and domain of function. The team was given the non-routine problems sheet (See Appendix G) to check the non-routinness of the problems with respect to mathematics education in Turkey and to determine the face and content validity. They also agreed that those questions were non-routine when we compare them with our national curriculum, and no problems were reported about face and content validity.

Task based interviews were conducted at PBH with each participant individually. All of them were audiotaped and each took about 40 minutes. During the task based interviews, participants were given time to work on the problem alone prior to talking with the researcher about what they did to complete the problem. After discussion about the problem, the researcher gave hints to participant in order to see whether they are able to solve the question after a hint. Then, the next problem was given. Meanwhile, each participant was asked to document as much of their thinking as possible and their written work on the non-routine problem sheet that was also collected.

3.6.9 Lesson Planning Activity

The lesson planning activity was chosen for this study because it was reported as one of the efficient ways for accessing preservice teachers' pedagogical content knowledge (Gess & Newsome, 1999) on composite and inverse functions (Ebert, 1994). Preservice secondary mathematics teachers were asked to prepare lesson plans for teaching composite and inverse functions considering the fact that after the properties of functions were presented to students in 9th grade national mathematics curriculum in Turkey, some books started to teach inverse functions, and some books started to teach composition of functions. However, participants were told that they were free to choose which one to teach first. They were also told that their lesson plans should be prepared for a minimum of eight 40 minutes lessons (2 weeks) and a maximum for twelve 40 minutes lessons (3 weeks). They were allowed to use any resource they wished as long as they cited them. In order to standardize the lesson plans, they were provided lesson planning activity instructions adapted from Winsor (2003) and a lesson plan format (See Appendix H). In the instructions part, preservice secondary mathematics teachers were told to be as detailed as possible when writing their lesson plans and to include examples to be solved in class, questions to be asked to students, homework that would be assigned, any handouts or overhead transparencies that would be used, and the prerequisite skills assumed. In the lesson plan format, the subtitles included were title of the lesson, name and surname of the teacher, grade level, prerequisite skills, materials/equipment, objectives, methods/techniques, procedure (introduction, development, closure), and evaluation/assessment/homework.

Participants were given two and a half weeks in order to complete the lesson planning activity. They were handed in their lesson plans before their teaching started.

3.6.10 Journal and Interview about Value of Teaching Functions, Inverse Functions, and Composite Functions

Journals and an interview about value of teaching functions, inverse functions and composite functions aimed to see to what degree the preservice secondary mathematics teachers were aware of the value of the topics they were expected to teach. For this purpose, three journals (see Appendix I) were provided to enable them to reflect on their own understanding of function, inverse function, and composite function and the value of teaching them. Each journal had statements based on the categories for the value of teaching a topic which were pedagogical value (how the concept of function is related with mathematics and other mathematics disciplines, like geometry), intrinsic value (modeling real world situations), and excitement and beauty value (showing beauties explained by mathematics in order to break the prejudices about mathematics) (Cha, 1999). These three categories reflect the mathematics' applied, and pure sides of mathematics which are all about mathematics. Statements given in the journals included teachers' perspectives on the value of teaching functions, composite and inverse functions. Statements about functions were taken from Cha (1999). Statements about composite and inverse functions were written by the researcher concerning these categories. Then the team and the researcher discussed definitions about three categories in order to clear the gaps and/or inconsistencies. The journals and the table (See Table 3.7) which shows the associated category of statements were submitted to them along with the definitions of the Cha's (1999) categories for the value of teaching. They were asked to check whether the given statements matched with the associated category and whether the statements' wording was understandable. Also, they checked the face and the content validity. Some rewording of the statements were suggested by the teachers and then they checked the final version of the journals and they were evaluated to be understandable, and reflected the associated category.

Table 3.7: The Associated Category of Statements in the Journals about Value of Teaching Functions, Composite Functions, and Inverse Functions

Journals about Value of Teaching					
Functions		**Composite Functions**		**Inverse Functions**	
Statement	**Category**	**Statement**	**Category**	**Statement**	**Category**
A	Pedagogical	G	Intrinsic	K	Pedagogical
B	Pedagogical	H	Pedagogical	L	Pedagogical
C	Intrinsic	I	Excitement & Beauty	M	Intrinsic
D	Pedagogical	J	Pedagogical	N	Intrinsic
E	Pedagogical			O	Excitement & Beauty
F	Excitement & Beauty				

The activity regarding the value of teaching functions, inverse functions and composite functions was conducted at PBH. All preservice secondary mathematics teachers were given three journal sheets. Each journal sheet included statements about functions, inverse functions and composite functions that expressed their teaching rationales on what they considered was important about teaching functions (six statements), inverse functions (five statements), and composite functions (four statements). Then the participants completed them individually by analyzing the statements. After analyzing the statements, the participants were required to allocate a total of 100 points for each statement (the total points of statements not to exceed 100, for each topic). Finally, a focus group interview was conducted to share their judgments and extend their ideas. The focus group interview took about 40 minutes and videotaped.

3.6.11 Teaching Practice

The teaching practices aimed to obtain the data regarding all the categories of PCK of preservice secondary mathematics teachers about composite and inverse functions. Although all of the components of PCK was assessed previously, composition and integration of these categories were observed through teaching practices. Moreover, data allowed the researcher to see and hear what concepts the participants thought important in teaching the concepts of composite and inverse functions, and to compare how much of their lesson plans really got into action.

After the participants gave their lesson plans on composite and inverse functions to the researcher, they taught the topics with the following teaching schedule (See Table 3.8). The participants knew that they would start teaching after the properties of functions was taught by the class teacher.

All lessons, except the lessons observed by the researcher, were videotaped by the class teacher. The researcher observed lessons were audiotaped and also the researcher took fieldnotes. Since all 9th grade lessons were at the same time, by changing place with the other class teacher the researcher was able to observe each participant at least once. The researcher observed lessons were marked with a * in the Table 3.8.

Table 3.8 : Preservice Secondary Mathematics Teachers' Teaching Practice Schedule

Preservice Teacher	Date	Minutes
Deniz	*December 1*	40
	December 4	80*
	December 7	80
	December 8	80
	December 15	80*
Yeliz	*December 1*	80*
	December 4	80
	December 8	80
	December 12	80
	December 13	40
	December15	80
Gizem	*December 5*	80
	December 8	80*
	December 12	40
	December 13	40
	December 14	80
	December 15	80

3.6.12 Evaluation Interview Protocol

Semi-structured evaluation interviews (see Appendix J) were conducted with each participant at the end of the study. The interview aimed to give participants an opportunity to share their feelings, impressions, and thoughts about the study. They were asked to evaluate each activity in the study. Also, the idea of this kind of study becoming a part of a teacher education program was discussed. Finally, they were asked to give an overall impression of the study including ways to improve the study. Each interview, took around 20 minutes, were audiotaped and conducted at the PBH.

3.7 Data Analysis Procedure

The data analysis was conducted in order to identify preservice secondary mathematics teachers' pedagogical content knowledge of composition and inverse functions.

The analysis of preservice secondary mathematics teachers' pedagogical content knowledge of composition and inverse functions were done by using Miles and Huberman (1994)'s view which include three con-current components: data reduction, data display, and conclusion drawing/verification. In this view, data reduction defined as a process of selecting, focusing, simplifying, and transforming data in the original data in the field notes, transcripts etc. It is a part of analysis but it does not mean that the data reduced quantitatively. The second component is data display which includes means to present data in an organized and compressed way, such as, matrices (tables), graphs, charts, and networks. Similarly, Yin (2003) suggested the use of word tables which include the summary of the results. The last component conclusion drawing and verification includes meaning emerging from the data in light of patterns, regularities, explanations, propositions noted during data collection, data reduction, data display and after.

In this study, in the light of the Miles and Huberman's (1994) data analysis view, the data was collected and categorized according to the components of pedagogical content knowledge. Then, data analysis was started by transcribing and coding data and then results were organized in tables in order to see the whole picture. Last, inferences were made depending on the tables made, the evidences found in the instruments about the components of PCK, and the nature of data. Moreover, comparisons were made for the inferences drawn with other relevant instruments.

The data was analyzed by the researcher and two second-coders in order to reduce bias in data analysis and to increase the reliability of the results. Since there are different kinds of data, coders with different characteristics were necessary. First group of data source was related with a direct account of mathematical knowledge through a test (survey of function knowledge) and a task-based interview (non-routine questions interview), whereas the other groups of data sources had more qualitative nature. The first second-coder (SC1) was a mathematics teacher in the PBH with a PhD in the Secondary Science and Mathematics Education Department at Middle East Technical University, and he was also offering two of the courses in the Bilkent University Graduate School of Education Mathematics teacher education program at the time of the study. Since he is an experienced secondary mathematics teacher, he was knowledgeable and competent in teaching 9th grade National Mathematics Curriculum in Turkey. Therefore, the survey of function knowledge and the interview about non-routine questions, which were directly asking mathematics knowledge, were coded with him. The second second-coder (SC2) was a researcher in the mathematics education field with focus on teacher education and qualitative research. The rest of the instruments were coded with the SC2.

Data analysis procedure for each instrument was given in the following sections. The recruitment and the training of the second coders were explained in each of the instrument's section. All data were prepared so that the coders would not see the names of the participants. The percentage of agreement between the coders were less than 15% for the analysis of the all instruments, which is less than the required percentage for reliability (Yıldırım & Şimşek, 2004). Therefore, the percentages for each instrument were not specifically given.

3.7.1 Survey of Function Knowledge

The survey of function knowledge was assessed through a focused holistic scoring scheme (See Appendix K) which was adapted from Lane (1993) and Aydın (2007). The scheme reflected the conceptual framework of declarative knowledge, conditional knowledge, and procedural knowledge. For each question of the survey, a five-score level (0-4) was assigned. The highest score of 4 was awarded for responses that the researchers regarded as being entirely correct and satisfactory, while the lowest score of 0 was reserved for a no answer.

For the analysis of survey of function knowledge, the researcher and SC1 worked on an adapted holistic criteria and agreed on what was meant by each criteria. Then, the researcher chose randomly one of the preservice secondary mathematics teacher who was not in PBH during the teaching practice course and her survey of function knowledge was scored by working independently. Afterwards, the scorings were compared between the coders and a few disagreements were found from all of them that related with the declarative questions. This is because, for example, it is hard to decide how well a definition is formed even though it is correct. Then ideas were shared and the scoring criteria was reviewed, and a full consensus was reached for the example scoring.

Lastly, the survey of function knowledge of the preservice secondary mathematics teachers at PBH were scored by two coders independently and the scorings were compared. As the number of disagreements were much less than the example scoring, and a full consensus was reached at the end.

3.7.2 Survey of Function Knowledge Follow up Interview

The survey of function knowledge follow up interview's transcripts were analyzed in order to explain the preservice secondary mathematics teachers thinking while answering the survey questions and why they responded as they indicated. Furthermore, these interview transcripts were used as a comparative instrument for the results of the instruments focusing on knowledge of functions. Evidence of consistencies as well as discrepancies between the survey of function knowledge and follow up interview were noted.

3.7.3 Knowledge of Context Focus Group Interview

Knowledge of context focus group interview video and transcripts were analyzed using the qualitative method of constant comparison by the researcher and SC2. This procedure was not chosen in order to guarantee the same results for different analysts working independently, but rather to allow for flexibility in the identification of patterns. Analysis of interview data progressed through several stages during which evidences of knowledge of context were explored in light of the umbrella categories emerged from the knowledge of context of the definition. These categories were physical facilities and setting, types of students, parents, school and community characteristics, resource availability, classroom climate, school climate, degree of support provided by others, expectations, effects of standardized assessments, demands made on the teacher, and departmental guidelines.To gain an overall feeling for responses to interview questions, first, the researcher and SC2 individually read the transcript while watching the videotape of the interview and insights for any indications of knowledge of context and emerging evidences for the existing dimensions were noted. During a second reading of the transcript, detailed notes on each preservice secondary mathematics teachers' responses were recorded and evidence of general patterns and indications of knowledge of context were searched. Then the researcher and SC2 compared the categories constructed and agreed on the category names and descriptions. Subsequent readings served to confirm evidence to those identified dimensions. Lastly, overall awareness of the preservice secondary mathematics teachers and the awareness of the participants were described by using examples from the transcripts. The data from interviews were then compared with the other assessment instruments like vignettes, lesson plans, and observations since they covered all PCK components.

3.7.4 Concept Map Activity

Two concept maps and essays were evaluated with a set of holistic scoring criteria taken from Bolte (1999).

Bolte's holistic scoring criteria for concept maps (See Appendix L) focuses on organization and accuracy. Organization referred to creating meaningful clusters and efficiently using links and linking words which all showed the in-depth understanding. A seven-score level (0-6) was assigned for organization. The highest score of 6 was awarded for concept maps that shows in-depth understanding of the links among the terms by using exemplary linking words, while the lowest score of 0 was reserved for a no answer. Accuracy referred to identifying any inaccuracies and misconceptions. A five-score level (0-4) was assigned for accuracy. The highest score of 4 was awarded for concepts maps with no errors, while the lowest score of 0 was reserved for maps with many major conceptual errors. So the concept maps were scored on a scale from 0 to 10, with up to 6 points for organization and up to 4 points for accuracy.

Bolte's (1999) holistic scoring criteria for essays (See Appendix L) focuses on communication, organization, and mechanics (grammar and punctuation). Communication represented the ability to clarify understandings and express mathematical ideas through the essay. A seven-score level (0-6) was assigned for communication. The highest score of 6 was awarded for essays that shows interpretations and understandings in a clear, systematic, and organized manner, while the lowest score of 0 was reserved for a no answer. Organization represented how well the ideas were presented. A five-score level (0-4) was assigned for accuracy. The highest score of 4 was awarded for essays with excellent method of presentation, while the lowest score of 0 was reserved for no answers. Mechanics represented how well the grammar and punctuation of the essay was. A two-score level (0-1) was assigned for mechanics. The score of 1 was awarded for essays with few violations in grammar and punctuation, while the score of 0 was reserved for essays with grammar and/or punctuation errors interferring with the understanding of the essay. So the essays were scored on a scale from 0 to 10, with up to 6 points for communication, up to 3 points for organization, and up to 1 point for mechanics.

For the analysis of concept map activity the researcher and SC2 worked on Bolte's (1999) holistic criteria of concept maps and essays and agreed on what was meant on each criteria. Then they worked independently and scored the concept maps and essays. Afterwards, the scorings were compared between the coders. The concept maps were sorted according to the scores, and the concept maps with the same scores for organization were compared with the others with the same scores. A few disagreements about the communication and accuracy scores given were discussed, scoring criteria was reviewed and a full consensus was reached. Moreover, comparisons were made between each participants' first and second concept maps and a concluding statement was written from this comparison for each participant. These statements were then compared with each participant's concept map essays where they did their comparison of first and second concept maps.

3.7.5 Journals about Definition of Functions, Composite Functions, and Inverse Functions

Journals regarding definition of functions, composite functions, and inverse functions were analyzed with SC2. SC2 was given three journals with the definitions of formal, and informal definition types and asked to categorize the definitions in each journal. The definition types that the researcher assigned to a definition were compared with the SC2's and a full consensus was reached.

Within the journals, the first question asked for the favorite three definitions and the second question asked the least favorite definition. The third question asked for their choice of definition and reason behind their choice if they were a 9th grade mathematics teacher. The last question asked what they would do if the students in their class did not understand the definition they gave in the previous question. In order to summarize the results, first a table, including participants answers was constructed. In the table, each answers' related definition category was also provided.

The analysis procedure conducted for the journals were multi-dimensional. First, for each journal, each participant's choice of definition type was defined. Then, by considering all journals, participants' preferences of definition type was identified. Furthermore, consistency of their reasons for choosing a favorite and least favorite definition were investigated from the first and second questions in the journals. For the third question, their answers were checked as to whether they reflected the reasons the participants gave for their choice of least and favorite definitions. For the last question, it was checked whether their answers were consistent through the three journals and their choice of definition was categorized.

The researcher and SC2 worked on the data independently by having in mind the above analysis procedure. A few disagreements occurred were resolved by discussion and a full consensus was reached.

3.7.6 Vignettes

The data analyses of the responses to vignettes were made similar to the method used by Ebert (1993). First, for each vignette of each participant the kinds of responses related to the SMK, KL, and GPK components of PCK were evaluated by considering the levels of the combined framework (See section 3.10).

The analyses were conducted with the SC2. First, the levels of the combined framework were discussed with the SC2. Second, the sample vignette was coded independently by the researcher and the SC2. The only inconsistencies of the level assigned to the vignettes were about the SMK levels, and especially between the level 0 and level 1, as foreseen by Lindgren and Thompson. However, by specifying the reasons for the level choice and discussion on the levels, the disagreements were resolved and a full consensus was reached. Thirdly, both coders worked independently and coded all vignettes and compared their assigned levels. Very few disagreements were found and they were easily overcome with a full consensus.

Lastly, for each component, the participants' responses were described and comprehensive analysis was provided. Furthermore, vignettes was used as a comparative instrument for the results of the instruments directly focusing on each component of PCK.

3.7.7 Interview about Non-routine Problems

The scheme (See Appendix K) used in the survey of function knowledge was used in the analysis of the non-routine problems interview. Since non-routine problems were of the type conditional knowledge, only the part of the scheme focusing on the conditional knowledge was used. Since the researcher and SC1 worked on the scheme while analyzing the survey of function knowledge, they were both competent and there was no need to work on the criteria application again.

During the interview, participants were asked to solve each question first and then if a participant had difficulty, the researcher gave hints so as to complete the question. Therefore, participants' answers were assessed by the researcher and SC1 through this scheme, up to the point where the researcher started to give hints. A five-score level (0-4) was assigned for each question in the interview. The highest score of 4 was awarded for responses that the researchers regarded as being entirely correct and satisfactory, while the lowest score of 0 was reserved for no answer. Two coders worked simultaneously and after the analysis of each question, the coders compared the scores allocated according to scheme and their notes about the participants' answers, especially any misunderstandings.

In the analysis of questions, there were two main disagreements. The first one was regarding the score a certain work should get and the second was regarding what work and points of the participants' work should be noted as evidence for cross-analysis with the other instruments. The scoring criterion was reviewed and discussion was conducted as to which evidences should be used. For the scoring criteria, a full consensus was reached through reestablishing a common understanding of all questions at the end. From the evidences, it was decided that after evaluating each participant, coders shared the evidences they had written and it was seen that after the discussion nearly all of the evidences taken were the same.

3.7.8 Lesson Planning Activity

The lesson planning activity was evaluated similar to that of the vignettes since the instruments content also covers the same components of the PCK. The analysis was conducted with SC2 since the vignettes were also analyzed with her, so there was no need to go over the combined framework. Even though they were familiar with the combined framework, first they analyzed one of the lesson plans in order to check under which criteria a level was assigned to a lesson plan and which points were seen as important as evidence to PCK components. After this it was seen that the coders were consistent with the levels assigned. However, the important parts of the lesson plans as an evidence to PCK components had both common and uncommon selections. Therefore, it was decided that after every lesson plan both the levels assigned and evidences discussed and those that could be used would be chosen.

3.7.9 Journal and Interview about Value of Teaching Functions, Inverse Functions, and Composite Functions

The journals regarding value of teaching functions, inverse functions, and composite functions were analyzed through the following steps. First, the distribution of points given to statements in the journals was tabulated. Then, the reasons for participants' choices were read. Afterwards, the videotape of the focus group interview was watched and the consistency or inconsistency of participants' reasons behind the choices and any additional comments were noted. As a result, each participants' value choices were described by the researcher and SC2 individually. Then, these descriptions were compared and a full consensus was reached for the descriptions of the preservice secondary mathematics teachers value choices. Furthermore, the results were used as a comparative instrument for the results of the vignettes, lesson plans and observations.

3.7.10 Teaching Practice

The teaching practices were analyzed with SC2 by using the combined framework for all components of PCK. Since the framework was used by the SC2 and the researcher before, there was no need to review it again. Even though they were familiar with the combined framework, first they analyzed one of the observation videos in order to check the accuracy of the levels assigned. During the analysis, apart from assigning levels, the evidence for the assigned levels were noted and the consistency of the associated lesson plans were checked. They analyzed the sample video by stopping after every 10 minutes and checking the consistency of the evidences noted, and other additional comments. At the end of the video the coders decided the levels of each component individually and checked the consistency afterwards. It was seen that the coders were consistent both in levels assigned and their evidences and comments noted. In order to complete the analysis of the videos, coders individually watched every video till the end and assigned a level for each component of PCK, took notes about evidences and comments by comparing the video with the lesson plan. Then, after every video the coders compared their analyses and reached a full consensus. At the end, coders described the general characteristics of the teachings for each participant.

3.7.11 Evaluation Interviews

The interview transcripts were analyzed with SC2 to gain additional insight to participants view about the study. The coders worked on the transcripts individually and came out with some comments and shared their ideas by agreeing on which ones to use. Then, they analyzed the transcript once more regarding the agreed ideas. These ideas were used as supporting evidence where suitable.

3.8 Researcher's Background, Role, and Biases

In a qualitative study, the researcher is the primary instrument for gathering, analyzing and interpreting data (Merriam, 1998). Therefore, it is important to state the researcher's position in research (Goetz & LeCompte, 1982) in order to understand potential research bias that can affect the research results (Johnson, 1997). This part of the study will state the researcher's role and possible bias throughout the study.

The researcher got her B.S. and M.Sc. from the Secondary Science and Mathematics Education department at Middle East Technical University in Ankara. Her M.Sc. thesis is about the effect of using journal writing in mathematics classes and she used both qualitative and quantitative methods. After graduating from the B.S. program, she started to teach at PBH and she completed her master degree while teaching at the PBH.

After graduating from the Masters program, she started to work at Bilkent University Graduate School of Education as a part time instructor. She offered Mathematics Curriculum Review I and II jointly for 4 years and Mathematics Teaching Methods I and Instructional Technology and Materials Design jointly for a year. Moreover, she has been jointly mentoring preservice secondary mathematics teachers during their school experience and teaching practice courses since 2003.

During the study, the researcher was not only an insider of the research context but also was a co-mentor of the teaching practice course and an observer of the research context the whole time. She knew the participants of the study and had a pre-existing strong relationship since she was their instructor for the courses Mathematics Curriculum Review I and II. Knowing the participants turned out to be an advantage for the researcher because when she explained the purpose of the study to all of them, they reacted very positively. In order to comfort all participants throughout the study, the researcher explained to them that the work they will do for the study will not be taken into account as part of the grading teaching practice course and ensured them about the confidentiality of the data. In other words, she assures that her role in the department and PBH did not affect the way that participants completed the instruments of the study.

During the data collection, no communication problems were detected between the researcher and the participants. The researcher tried her best to be a good listener and observer in every step of the study. Based on her experience in mentoring preservice teachers and the demands of the program, she planned the order of implementation of instruments so as not to make the feel under too much pressure. She gave one or two days extensions for their completion of the written instruments when they needed. The participants' interview times were arranged in terms of timing so that their answers would not be rushed. For the interviews and observations, she took permission to either audiotape or videotape. During the interviews, it was emphasized that there were no correct answers for the questions. Moreover, after every interview researcher summarized the interview results and asked participants whether she understood their point of view correctly or not.

3.9 Quality of the Research

The accuracy of the findings and the correct interpretation of data was a major concern for qualitative research (Creswell, 2007). These questions are related with the concerns about the issues that are related with the quality of research. Miles and Huberman (1994) refer to these issues as the practical standards that help in judging the quality of the conclusions drawn from the research. When the qualitative research literature was examined, different views existed about how to decide the quality of a qualitative study (Creswell, 2007; Golafshani, 2003; Merriam, 1998; Miles & Huberman, 1994; Stake, 2005; Yin, 2003). Moreover, there are contrasting views about the applicability of the quantitative research terminology and methods, such as reliability and validity, to the qualitative research (Creswell, 2007; Golafshani, 2003). Therefore, with regard to the qualitative terminology instead of internal validity, external validity, reliability and objectivity, the terms, credibility, transferability, dependability, and confirmability are used by many researchers (Creswell, 2007). Moreover, reliability and validity are generally not discussed separately in qualitative research but rather terms such as “credibility” or “trustworthiness” are suggested in order to address both reliability and validity (Golafshani, 2003). To widen the spectrum of conceptualization of reliability and revealing the congruence of reliability and validity in qualitative research, Lincoln and Guba (1985) state that: "Since there can be no validity without reliability, a demonstration of the former [validity] is sufficient to establish the latter [reliability]" (p. 316). Patton (2001) with regards to the researcher's ability and skill in any qualitative research also states that reliability is a consequence of the validity in a study. Although some qualitative books discussed reliability and validity under different headings it was seen that the subcategories suggested for the analysis are the same or include each other (Merriam, 1998). Considering that terms such as reliability, and validity have several different approaches in the qualitative research paradigm, in this study the quality of the research was described under the term “credibility” to address them all.

3.9.1. Credibility

In this section, measures taken during data collection and analysis to increase the credibility of study will be explained. Moreover, convergence between multiple coders' accounts will be explained.

For ensuring validation of qualitative study, Creswell and Miller (2000) proposed nine different procedures which are triangulation, disconfirming evidence, researcher reflexivity, member checking, prolonged engagement in the setting, collaboration, audit trail, thick and rich description, and peer debriefing. Apart from these validation procedures, Creswell (2007) emphasized the importance of the multiple coders and their agreements through the data analysis. In the present study, some of the Creswell's validation procedures that were used for ensuring credibility and reliability were not discussed separately since having multiple coders is also a part of the triangulation process of the study.

Triangulation

One of the strengths of case studies is the possibility of gathering multiple sources of data, called triangulation (Yin, 2003). Four different types of triangulation exists in the qualitative research literature: (1) triangulation across data sources (i.e., participants), (2) triangulation of theories/perspectives, (3) triangulation of methods (i.e., interview, observations, documents), and (4) triangulation among different investigators (Creswell & Miller, 2000; Creswell, 2007; Patton, 2000). Triangulation in a study provides collaborating evidence collected through multiple methods, such as observations, interviews, and documents. This establishes validity since multiple sources of data provide multiple measures of the same phenomenon (Creswell & Miller, 2000; Yin, 2003).

In this study, data triangulation, investigator triangulation, and method triangulation were used. There were three different cases as a data source, and two second coders were used for the analysis of data. In addition, different types of data sources were used including surveys, interviews, observations, lesson plans, journals, concept maps, and vignettes.

As discussed in the data analysis procedure, there were two second coders in the study. The second coders were trained for the assessment procedures of each instrument as described in the data analysis sections. Having specific procedures for coding and analyzing data also increased the transferability of the findings. After the data analysis procedures were explained to the second coders, the researcher and the second coders analyzed all the instruments separately by following a data analysis procedure. Then, they came together and discussed if there exists any inconsistencies and reached full-consensus. Both coders analyzed the data with pseudonyms for the participants in order to eliminate the bias.

Researcher reflexivity

Researcher reflexivity is the process whereby researchers acknowledge and describe their entering beliefs and biases about the study (Creswell & Miller, 2000; Creswell, 2007; Yin, 2003). The researcher's role and biases in this study were explicitly stated in the previous section.

Prolonged engagement in the setting

The purpose of prolonged engagement in the setting is that the researchers build trust with their participants, and establish rapport so that the participants are comfortable with disclosing information (Creswell, 2007; Merriam, 1998; Yin, 2003). Also, prolonged engagement in the field has no fixed time duration (Creswell & Miller, 2000). In this study, the researcher was with the participants for six weeks during all working-days at the PBH. Moreover, as discussed earlier in the researcher's role and biases section, she had already established a good rapport with the participants.

Member Checks

An immediate member-check was made after collecting each type of written data from the participants and after each interview. For example, after implementing an instrument, the researcher read them all and talked to the participants and told them her understanding about their answers. If there were any conflicts between the understandings, the researcher noted them and sometimes gave the instrument back to the participant so as to rewrite it.

Thick and rich description

By writing thick and rich description, researchers provide as much detail as possible so that the readers will be able to understand whether the research is credible and they will be able to make decisions about the applicability of the findings to other settings or similar contexts (Creswell & Miller, 2000; Merriam, 1998).The researcher explicitly defined all stages of the research design and findings in detail in order to associate her findings with the readers in an efficient way.

Peer debriefing

A peer review or debriefing is defined as "the review of the data and research process by someone who is familiar with the research or the phenomenon being explored" (Creswell & Miller, 2000, p. 129). A peer reviewer provides support, plays "devil's advocate", challenges the researchers' assumptions, pushes the researchers, and asks difficult questions about their methods and interpretations (Creswell, 2007; Creswell & Miller, 2000). In this study, the researcher had a chance to get feedbacks from an academic person who is qualified in both qualitative research and PCK. Moreover, she welcomed any feedbacks coming from researchers having specialized in qualitative research and so she received continuous supervision.

Applying all the instruments in a timeline was a threat to the credibility of this study, since a particular event or inference might be resulted from some earlier occurrence, based on interview and documentary instruments (Yin, 2003). Yin suggested using analytical tactics such as explanation building, addressing rival explanations, and using logical models for the analysis of such case studies. In this study, a constant-comparative method was used for the analysis as described in data analysis procedure and this method was very similar to Yin's analytic tactic called explanation building.

Yin (2003) describes the process of explanation building as follows: first, the researcher makes an initial proposition about a phenomenon; second, comparing the initial findings with the forthcoming ones; third, comparing other details of the case against revision; last, repeating this process as many times as needed. In this way, this threat was tried to be reduced. Also, since the explanation building was done by two coders, the findings were more robust.

3.10 A Combined Framework for Categorization of PCK Components

In this section, the framework that was used to categorize the components of PCK that preservice secondary mathematics teachers have is described in detail. This framework integrates three similar models of teachers' conceptions of mathematics proposed by Thompson (1991), Lindgren (1996), and Ebert (1994).

Thompson (1991) and Lindgren (1996) used a similar framework in order to analyze conceptions related to mathematics, mathematics teaching and learning. However, conception of mathematics is a broader and general mental structure (Lloyd & Wilson, 1998; Thompson, 1992; Törner, 2002). Lloyd and Wilson (1998) defined conceptions of mathematics as a person's mental structure encompassing knowledge, beliefs, understandings, preferences, and views. In a similar vein, Thompson (1992) defined it as a mental structure encompassing beliefs, meanings, concepts, propositions, rules, mental images, preferences, and the like. The main difference between these two definitions is that Thompson (1991) did not explicitly state that conceptions included knowledge, however in the former definition she included terms concepts, rules, and propositions which are components of knowledge. In line with this discussion, Ebert (1994) used Thompson's (1991) framework, not only to analyze beliefs related to mathematics, mathematics teaching and learning but also to analyze subject matter knowledge, knowledge of learners and pedagogical knowledge.

Thompson's Framework

Thompson (1991) proposes a framework for investigating and analyzing the development of teachers' conceptions of mathematics teaching. This framework is developed using the results of her five-year work with seven preservice and five inservice teachers. Thompson (1991) claims that the framework documents that she has observed a "fairly consistent pattern of development of teachers' conceptions of mathematics teaching" (p.8). Since her framework is limited to the experiences and the existing conceptual schemes of the teachers she worked with, she asks other researchers to examine the viability of her framework.

In this framework, she categorizes the development of teachers' mathematics related conceptions in three developmental levels: *Level 0*, *Level 1*, and *Level 2*. She characterizes the levels depending on the conceptions of: (a) mathematics; (b) learning mathematics; (c) teaching mathematics; (d) roles of teachers and students; and (e) evidences of student knowledge and criteria for judging correctness, accuracy, or acceptability of mathematical results and conclusions. The characterizations of the levels are given as follows:

Level 0. Mathematics is conceptualized as using arithmetic skills in daily life. Hence teaching mathematics is focusing on the development of students' skills in arithmetic. This is performed through memorization of the mathematical knowledge which is composed of facts, rules, formulas, and procedures.

The teacher's role is limited to demonstrating the facts and procedures in the classroom and the student's role is imitating and practicing those procedures until they become a habit. The goal of the mathematics teaching and problem solving at this level is to implement the correct procedure or obtain the correct answer, usually in the ways demonstrated in the class. Mental processes are not considered during problem solving. The teacher or the book is generally considered as the authority for mathematical knowledge.

Level 1. At this level, mathematics is still considered as a collection of facts and rules, but the principles behind the rules are realized. This slight shift is considered to be a result of the use of instructional representations and manipulatives in teaching. However, this new pedagogical approach to teaching mathematics (such as use of manipulatives) is not considered as a way of improving conceptual understanding, but rather increasing the enjoyment of students in the mathematics classroom. Problem solving is seen as being isolated from the mathematical concepts and problems and are taught separately with almost no relation to the concepts. It is not seen as a way to teach mathematics.

The teacher has similar roles described in *Level 0*. The student's role is extended and it includes some understanding of the principles behind the procedures. Although there is a change in the way mathematics and mathematics teaching is considered, there is still an authority who decides on the correctness of mathematical ideas.

Level 2. Thompson does not specifically claim much about Level 2 conceptions within the nature of mathematics. She only claims that centrality of the mathematical ideas are realized at this level. Unlike the Level 1 teaching beliefs, using materials and different methods in mathematics teaching targets conceptual understanding. Mathematics teaching for understanding includes students' engagement. Thus, the teacher is considered as a guide in catalyzing students' thinking. The teacher allows students to express their ideas in order to have a better understanding of their learning process. The student's role is to understand the logical connection between the mathematical concepts and ideas. Students are expected to participate in the mathematics classroom by expressing their ideas and reasoning. Hence, proving and generalization are seen as a way of learning mathematics.

Thompson claims that the patterns of movement from one level to the other suggests a relatively easy move from Level 0 to Level 1 compared to that of from Level 1 to Level 2. She explains this difference by the nature of restructuring needed to achieve the level change. Moving from Level 0 conceptions to Level 1 conceptions requires no major structuring of conceptual schemes, but an expansion of or broadening in Level 0 conceptions. However, moving from Level 1 to Level 2 requires questioning deeply rooted ideas and unexamined assumptions of what it means to know, learn, and teach mathematics. Within this complicated process of restructuring, Thompson cautions that teachers' resistance to change their conceptions should not be underestimated.

Thompson's (1991) framework appears as a result of a qualitative study with few participants. In order to have a more accurate and stronger analysis tool, Lindgren's (1996) framework, which is a modification of Thompson's (1991) framework in Finnish context through both qualitative and quantitative methods, is additionally considered here.

Lindgren's Framework

Lindgren (1996) characterizes mathematical beliefs as implicit personal mathematical knowledge. For Lindgren, conscious beliefs form conceptions. In this perspective, Lindgren's (1996) study with preservice teachers in Finland seems to validate Thompson's (1991) framework. Her study includes the use of both quantitative (N = 163) and qualitative (N = 12) methods. She initially uses a Likert-type belief inventory and then conducts interviews with a selected group of participants. Her study results in a framework with three partly overlapping categories named Rules and Routines, Discussion and Games, and Open-Approach, which she claimed to correspond with Thompson's (1991) *Level 0*, *Level 1*, and *Level 2*, respectively. Lindgren's framework emphasizes the teaching and learning of mathematics, and roles of teachers and students. The categories and their characterizations are given as follows:

Rules and Routines - RR (*Level 0).* This category refers to an understanding of teaching mathematics based on routine procedures that should be demonstrated by the teacher and be memorized by the students.

Discussion and Games - DG (*Level 1*). This category characterizes teachers as having different approaches in teaching such as using games and promoting classroom discussions.

Open-Approach - OA (*Level 2).* This is the category where students have responsibility for their own learning and where teachers encourage and guide students. Mathematics is a way of thinking operationalized by problem solving.

Lindgren (1996) claims that there are sublevels in the Discussion and Games (Level 1) area where common sub-areas with the other two levels appear. Figure 3.1 illustrates this structure in Lindgren's (1996) study.

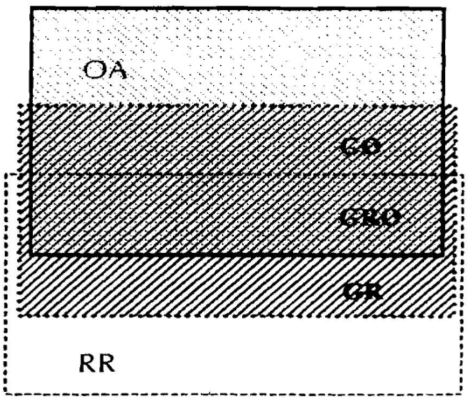


Figure 3.1: Lindgren's (1996) levels and sublevels of development of beliefs about teaching mathematics

In Figure 3.1, GR (Games and Rules), GRO (Games, Rules, and Openness), and GO (Games and Openness) are the intersection areas where teachers have beliefs from at least two different levels. For example, GR (Games and Rules) is the intersection of Rules and Routines (*Level 0*) and Discussion and Games (*Level 1*), where teachers might believe that facts and rules are the focus of mathematics but they might also promote class discussion. Lindgren's (1996) analysis yields a conjoint area of all three levels (GRO – Games, Rules, and Openness) where teachers simultaneously believe in issues from all three levels. The existence of conjoint areas in Lindgren's study suggests that Discussion and Games level (*Level 1*) is the area of development for preservice teachers' beliefs about teaching mathematics.

As can be seen from the characterization of categories, Lindgren's (1996) framework focuses more on teaching and learning. When combined with Thompson's (1991) framework, Lindgren's (1996) framework brings additional descriptions about teaching and learning, and makes Thompson's (1991) framework stronger in these areas. However, none of them categorized knowledge in terms of categories, although Thompson (1991) included it in the conception of mathematics definition. Ebert (1994) used Thompson's (1991) levels in order to categorize subject matter knowledge, beliefs about learners, beliefs about mathematics, and pedagogical knowledge. Hence, in order to supplement these two frameworks with a better description of knowledge levels, Ebert's (1994) description of each level was used.

Ebert's Level Descriptions

Ebert (1994) examined the PCK of preservice secondary mathematics teachers with respect to the content area of functions and graphs through an analysis of the transformation of knowledge and beliefs. She proposed that transformation takes place through developing explanations, planning lessons, simulating teaching, and reflecting on teaching. Therefore, she designed five vignettes which present a composite view of preservice secondary mathematics teachers PCK in the area of functions and graphs. Her study included 11 preservice secondary mathematics teachers enrolled in a secondary methods class. The data analysis of these vignettes was conducted by recording the kinds of responses in each of the following categories: subject matter knowledge; knowledge of learners and learning mathematics; beliefs about mathematics; pedagogical knowledge; and explanations. The initial analysis revealed that the strengths and weaknesses for each category seemed to fit well within the framework proposed by Thompson (1991). The categories and their characterizations were given as follows:

Subject Matter Knowledge

Inadequate (Level 0) subject matter knowledge is characterized by an inability to express definitions correctly, to use notation sensibly, to diagnose errors, and by presence of misconceptions. Good (Level 1) subject matter knowledge is characterized by expression of definitions correctly, interpreting graphical and other representations to obtain information, and suggesting possible real-world situations. However, at this level teachers still have a difficulty in diagnosing student errors, and even if they address the student error they focus on surface features of the misunderstanding. Strong (Level 2) subject matter knowledge is characterized by an ability to use definitions correctly, to diagnose all student errors, to express the distinctions between different representations, and to extend students' conceptions in one mathematics topic to future mathematics topics.

Pedagogical Knowledge

While giving categories for the pedagogical knowledge she described the teacher behaviors and preferences during teaching. Teachers with inadequate (Level 0) pedagogical knowledge are seen as knowledge providers and demonstrators for the students who are required to practice that knowledge until they do it perfectly. The importance of introducing procedures after concepts is also shown in thr characteristics of these teachers. Although this is a valuable tool, the impact may be lost since they are the sole source of the authority. Teachers view their role as one of advising, admonishing, and appraising the students so the flow of information is limited to the path between the teacher and student. Teachers with good (Level 1) pedagogical knowledge not only provide necessary rules and procedures but also help students to develop meaning and understanding. So, they value student understanding. Teachers still view their role as one of advising, admonishing, and appraising, and the flow of information is still from teacher to student. The role of teacher is to provide possible uses of representations for achieving cognitive objectives. Teachers with strong (Level 2) pedagogical knowledge facilitate and guide students rather than provide answers and explanations. They value student understanding and extending that understanding with questions that elicit further mathematical knowledge. They value students' input and often praise their intellectual comments. These teachers value student-to-student interactions and use methods like group work for increasing the number of such interactions. Apart from possessing strong subject matter knowledge, they also possess pedagogical tools to construct analogies, examples, non-examples, explanations and demonstrations.

Furthermore, they allow students to construct mathematical knowledge through authentic mathematical inquiry.

Knowledge about learners and learning mathematics

Teachers with inadequate (Level 0) knowledge of learners view responding to students' misconceptions as an opportunity to set the student straight by telling them the rule or the procedure. Teachers with good (Level 1) knowledge of learners appreciate the importance of discussion and solving similar numerical examples, practice problems for resolving conflicts. However, they also believe that students should be told what to do in certain mathematics topics. Teachers with strong (Level 2) knowledge of learners see themselves as guides or facilitators for the students rather than providing answers and explanations. They show awareness of difficulties inherent in mathematical topics that cause cognitive obstacles for students leading to misconceptions.

Similar to Thompson (1991) and Lindgren (1996), Ebert (1994) stated that there are midlevels in the conceptions of preservice teachers but did not name them. For example, if she could not decide whether it is level 0 or level 1, she defined that preservice teachers level as 0 or 1.

Combined Framework for the Components of PCK

The framework that was used to analyze the data (interview transcripts of pre- and inservice teachers) in this study is described in detail. This framework integrates three similar models of teachers' mathematics related beliefs proposed by Thompson (1991), Lindgren (1996), and Ebert (1994). Thompson's (1991) framework is an overall framework that draws a general developmental picture for categorization of components of PCK. It is used as the main analysis framework of the present study. The terminology (*Level 0*, *Level 1*, and *Level 2*) and the characterization of the levels are used as a starting point. In order to make the characterizations of the levels richer, Lindgren's (1996) and Ebert (1994) characterizations are also inserted into the main framework. For each component of PCK, the main characteristics of the levels were summarized in tables. Main characteristics of SMK, pedagogical knowledge, and knowledge of learners were taken from Ebert (1994) (See Table 3.9, 3.10, and 3.11), main characteristics of knowledge of context were written by the researcher in light of Thompson (1991) and Lindgren (1996) frameworks (See Table 3.12). Value of teaching functions is another component of PCK however since this component was already categorized by different labels it was not possible categorize in terms of Thompson's (1991) levels.

Table 3.9: Main Characteristics of the Subject Matter Knowledge

Levels	Main Characteristics
Level 0	Preservice secondary mathematics teachers • unable to express definitions correctly • unable to use appropriate notation sensibly • use only declarative and/or procedural questions • unable to interpret and use different representations easily • face difficulty when there is a need to see connections between different topics/subunits
Level 1	Preservice secondary mathematics teachers • express definitions correctly • use appropriate notation sensibly • still use declarative and/or procedural questions • interpret and use graphical and other representations • see connections between different topics/subunits
Level 2	Preservice secondary mathematics teachers • express definitions correctly • use appropriate notation sensibly • use all type of questions (declarative, procedural, and conditional) in an appropriate positions • interpret and use graphical and other representations sensibly • see connections between different topic/subunits and move among them smoothly

Table 3.10: Main Characteristics of the Knowledge of Learners

Levels	Main Characteristics
Level 0	Preservice secondary mathematics teachers • have difficulty in diagnosing errors of the students • view responding to students' misconceptions as an opportunity for them to tell the student the direct rule or procedure • have difficulty in realizing students' needs for understanding
Level 1	Preservice secondary mathematics teachers • diagnosing some of the student errors and even if they address the error they focus on the surface futures of the error • solve similar numerical examples, practice problems but also appreciate the importance of discussion • from time to time realize students' needs for understanding and prepare learning environments
Level 2	Preservice secondary mathematics teachers • easily diagnose student errors and address students difficulties • guide and facilitate students rather than providing answers and explanations • aware of students' needs for understanding and accordingly able to create rich learning environments

Table 3.11: Main Characteristics of the General Pedagogical Knowledge

Levels	Main Characteristics
Level 0	Preservice secondary mathematics teachers • are seen as knowledge providers and demonstrators for the students • introduce procedures after concepts • dominate the flow of information that is a path between the teacher and student • have problems sequencing the topics and problems during teaching/ lesson planning • have difficulty in controlling the class to have a democratic teaching environment
Level 1	Preservice secondary mathematics teachers • not only provide necessary rules and procedures but also help students to develop meaning and understanding • view their role as one of advising, appraising, and admonishing • still dominate the flow of information which is a path between teacher to the student • only have problems sequencing the problems during teaching/ lesson planning • sometimes controls the class to have a democratic teaching environment
Level 2	Preservice secondary mathematics teachers • facilitate and guide students rather than provide answers and explanations • value student understanding and extend that understanding by questioning further mathematical knowledge • value student-to-student interactions • allow and encourage students to construct mathematical knowledge through mathematical inquiry • sequence the topics and problems in an appropriate way • controls the class to have a democratic teaching environment

Table 3.12: Main Characteristics of the Knowledge of Context

Levels	Main Characteristics
Level 0	Preservice secondary mathematics teachers • rarely use school, student, and class related issues in the teaching environment
Level 1	Preservice secondary mathematics teachers • use school, student, and class related issues in the teaching environment and but have difficulty in adaptation
Level 2	Preservice secondary mathematics teachers • use school, student, and class related issues in the teaching environment and adopt those easily

Tables about main characteristics of components of PCK and Lindgren (1996) supports Thompson's (1991) claim about the differences among the nature of moving from one level to the other. The differences between the characterizations of *Level 0* and *Level 1* are not dramatic yet can be found in the combined framework, but still distinguishable whereas the differences between *Level 1* and *Level 2* characterizations are quite definite. Although the combined framework was the main tool to analyze preservice secondary mathematics teachers' PCK, this study was also an examination of the viability of the framework.

CHAPTER 4

PRESENTATION OF RESULTS

The following chapter documents the results of the study through excerpts from the participants' responses to instruments. The chapter's sections are organized in the order of research questions namely the components of PCK: subject matter knowledge, general pedagogical knowledge, value of teaching composite and inverse functions, and knowledge of learners. Since, this study is a multi-case study under these sections both a portrayal of each participant's knowledge and their comparisons were presented by using excerpts from the participants' instruments. This portrayal includes in fine detail the extend and organization of each component of pedagogical content knowledge. In these sections tables were used to report each participant's performances and to show differences and similarities among the group. In order to increase the readability of the results chapter instead of "preservice secondary mathematics teachers", "preservice teachers" was used.

4.1 Subject Matter Knowledge

This section summarizes the results obtained from eight instruments (survey of function knowledge, survey of function knowledge follow up interview, concept map activity, non-routine problems interview, vignettes, lesson planning activity, and teaching practices) administered to the participants in order to assess their subject matter knowledge as a part of their pedagogical content knowledge. This data provides a broad characterization of the extent, organization, and application of their subject matter knowledge of composite and inverse functions concerning the knowledge types declarative, conditional, and procedural.

Participants' scores on each item of the survey of function knowledge was given in Table 4.1. Furthermore, summary of the each participants total scores for the survey of function knowledge concerning the knowledge types were given in Table 4.2.

Table 4.1: Scores of the participants on each item of the survey of function knowledge

Question Number	**Objectives**	**Yeliz**	**Gizem**	**Deniz**
1	Define the concept of function	4	3	2
2	List different representations of functions	1	1	1
3a	Define the concept of composition of functions	2	2	3
3b	Define the concept of inverse function	2	1	2
4a	Decide whether the given relations are functions and explain the reasons	1	1	3
4b		1	3	3
4c		4	0	3
4d		3	3	3
4e		3	2	3
4f		1	1	4
5	Define the concept of domain and range	3	4	2
6	Apply the properties of a domain of a function	1	3	3
7	Calculate the range of a given function	1	3	3
8	Read the graphs of functions and apply rules about the operations of functions	4	4	4
9	Apply operations on functions	4	3	3
10	Apply the properties of 1-1 and onto functions	4	2	2
11a	Justify the given statements about 1-1 and onto functions	0	1	1
11b		0	1	1
12a	Decide whether the given functions have inverse functions by explaining reason	0	3	3
12b		0	3	4
13	Apply the properties of composite and inverse functions	4	4	4
14	Apply the properties of composite and inverse functions	4	4	4
15	Apply the properties of composite and inverse functions and operations on functions	0	4	4
16	Apply the properties of composite and inverse functions	4	3	4
17a	Explain and justify existence of composite function	3	3	4
17b	Explain and justify existence of inverse function	4	0	3
18a	Find out functions which satisfy the given composite function	4	4	4
18b	Decide and explain the existence of multiple functions satisfying the same composite functions	3	2	3
19	Apply the basic function knowledge	4	4	4

Table 4.2: The total scores of the participants on the Survey of Function Knowledge according to knowledge types

	Yeliz	Gizem	Deniz
Declarative	29 (51%)	31 (55%)	40 (71%)
Procedural	26 (72%)	30 (83%)	31 (86%)
Conditional	14 (58%)	11 (45%)	16 (66%)
Total	68	72	87

Maximum score for declarative questions:56 points
Maximum score for procedural questions:36 points
Maximum score for conditional questions:24 points
Maximum score for total survey :116 points
Percentages in parenthesis reflect the percentage score of each type with respect to maximum score

4.1.1 Yeliz's SMK

When Yeliz's scores of the survey of function knowledge was analyzed it was seen that she was more successful on procedural questions compared to declarative and conditional ones. She confirmed this result in the follow-up interview by saying "I prefer and like questions based on calculations because they are easy for me". Here, by saying the questions based on calculations she pointed and meant the procedural questions in the survey. Out of 9 procedural questions in the survey, she got full mark for six of them. Furthermore, when the scores of declarative and conditional questions compared, it was seen that she got similar points. This is reasonable since conditional knowledge requires existence of declarative knowledge.

In order to understand her SMK further the survey of function knowledge questions were analyzed with respect to similar objectives through declarative, procedural and conditional questions.

4.1.1.1 Knowledge about the Definitions and the Applications of Definitions

The knowledge about the definitions and their applications were searched through all relevant instruments.

4.1.1.1.1 The Survey of Function Knowledge

The first declarative question in the survey asks for a definition of function. As it can be seen from the Figure 4.1, she answered the question properly and got a score of 4.

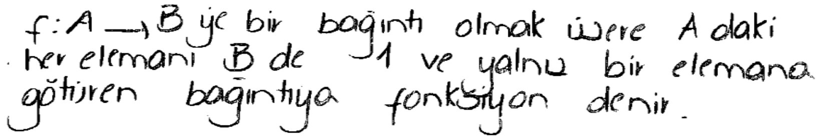


Figure 4.1: Yeliz's answer for the question 1

Related to the definition of function, question 4 asks for whether the given relations are functions and the question was declarative in nature. Although she gave a correct definition for the function, she faced difficulty in determining which functions are relations in question 4. When the items that she got low grades analyzed it was seen that she experienced problems when the relation was given in the table format (4f) and in words (real life examples) (4a & 4b). From these three items she got either 0 or 1 point. The rest of the questions were given in mathematical format (4c, 4d & 4e) and she got either 3 or 4. This situation was consistent with her answer to second question (See Figure 4.2) which was related to listing different representations of functions. It was evident from the answer that she was not aware of the existence of different representations which affected her answer to question 4.

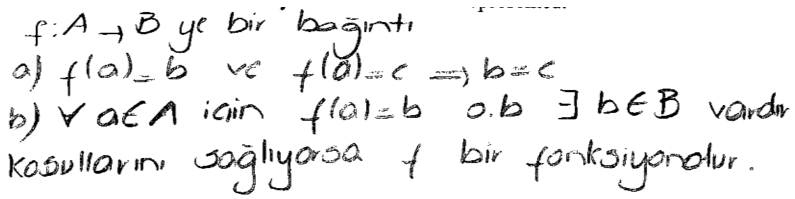


Figure 4.2: Yeliz's answer to question 2

When she was asked to define composition of functions similar to inverse functions instead of giving a formal definition in words she used the Venn diagrams given in Figure 4.3.

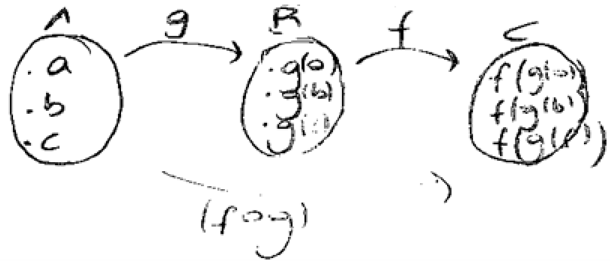


Figure 4.3: Yeliz's answer to the definition of composite function

In contrary to functions and inverse functions, question related to existence of composite functions is conditional in nature and is given in mathematical notation. Although her graphical explanation of definition of composition of functions was not clear enough, when she was asked to identify the existence composition of functions in question 17, she successfully did it and got 3 in the first part and 4 in the second one (See Figure 4.4).

Consider the set of functions whose domain and set of images are the real numbers. K assigns to each pair of such functions to their composition.

a. Is K a function? Explain.

b. Is K^{-1} a function? Explain.

a) Yes, K is a function.
f and g are functions.
K(f,g)=fog is a function because its domain is larger than its range. We can find a value for each element which we take from domain.
b) No, K^{-1} is not a function
K^{-1}(fog)=(f,g) There can be two range for some elements

Figure 4.4: Yeliz's answer to question 17

In a similar vein, when she was asked to define inverse function she gave a graphical answer as seen in Figure 4.5. In this question she chose not to gave a definition in words and she confirmed in the follow up interview that she could not put in to words the figure she imagined.

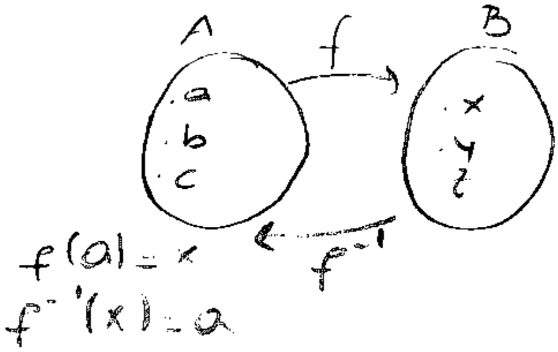


Figure 4.5: Yeliz's answer to definition of inverse function

When she was asked to decide the existence of inverse functions in question 12, she even did not attempt to answer the question. Similar to questions of functions, these questions were also given in words and related to real life examples and declarative in nature.

For the existence of functions, inverse functions, and composite functions, knowledge of domain and range is compulsory for this reason in the survey, question 5 asks for the meaning of domain and range and their importance. In Figure 4.6 her answer was given.

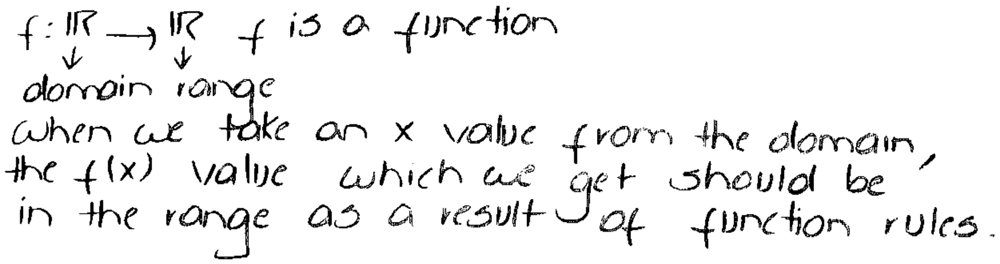


Figure 4.6: Yeliz's answer to question 5

This question was a declarative one with this answer she got 3 points. Although it seems that there are no gaps in her understanding a further explanation is required in order to see the completeness of her understanding. In line with this question, preceding questions 6 and 7 ask for the domain and range of a given function in mathematical notation and they are procedural questions. As stated before, she was very successful on procedural questions, and surprisingly, these two questions were among the three procedural questions where she got score of either 0 or 1. She got score of 1 for both of them (See Figure 4.7 and 4.8).

'. Find the domain of the function $f(x)=\frac{\sqrt[4]{x-2}}{x-3}+\frac{\sqrt[3]{x^2+1}}{\sqrt[5]{x^2-16}}$

$\mathbb{R}-\left[\{0,1,3,4\}\cup\mathbb{Z}^-\right]$

Figure 4.7: Yeliz’s answer to question 6

If $f(x)=x^2-9$ find $f([-4,3])$.

$f([-4,3])=\{-9,-8,-5,0,7\}$

Figure 4.8: Yeliz’s answer to question 7

When her answers to both of the questions were analyzed it was seen that she got the idea, as in the definition, but she approached them pointwise (in terms of integers) and missed the whole picture.

Apart from existence of domain and range, being 1-1 and onto is also required for functions to have an inverse. In the survey these properties were investigated through one procedural (question 10) and one conditional question (question 11) as seen in Figure 4.9 and 4.10 respectively. She got 4 points from the procedural question whereas got 0 from the conditional question which also explains her failure in the existence of inverse function questions, since 1-1 and onto are prerequisites for existence.

If $f:R-\{3\}\rightarrow R-\{-2\}$ and $f(x)=\frac{ax-2}{4x-b}$ is a one to one and onto function, find a and b.

$4x-b=0 \Rightarrow x=\frac{b}{4} \Rightarrow 3=\frac{b}{4} \Rightarrow \boxed{b=12}$

$f^{-1}(x)=\frac{bx-2}{4x-a}$

$4x-a=0 \Rightarrow x=\frac{a}{4} \Rightarrow -2=\frac{a}{4} \Rightarrow \boxed{a=-8}$

Figure 4.9: Question 10 and Yeliz’s answer

Let f and g be two functions whose domains and ranges are subsets of the set of real numbers. Prove or find a counter-example to the following to statements.

a. If f and g are both 1-1 then it follows that f+g is 1-1.

b. If f and g are both onto then it follows that f+g is onto

Figure 4.10: Question 11 and Yeliz's answer

As a result, it can be concluded that even though Yeliz had some gaps in her knowledge of composite and inverse functions. These gaps were checking the conditions for the existence of functions, composite and inverse functions.

4.1.1.1.2 Responses to Non-Routine Questions

A similar picture was evidenced through the analysis of non-routine questions interview. For the composition of functions as it can be seen from the Table 4.3 first two questions are given in mathematical notation and she attempted to solve the questions procedurally without any hesitation. However, solution of the questions requires more than that. For the first question, she was also required to draw the graph of the composition function, since she approached the question procedurally and ignored the domain and range of the given functions she draw the first one wrong and did not even attempted to draw the second one. Like in the survey, she faced difficulty while using different representations, graph for this question. The second question just asks for composition of two functions. She solved the question similar to previous one by ignoring the domain and range and because of that she could not identify the problem in the question without a hint. The third and fourth questions are related with composition of two functions which are given in graphs. She just expressed their domain and range but unable to create a solution.

Table 4.3: Composition of functions questions in the non-routine interview, Yeliz's answers and scores

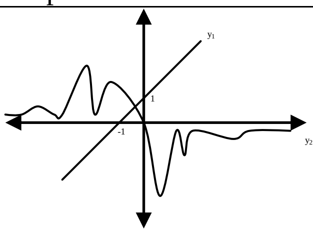
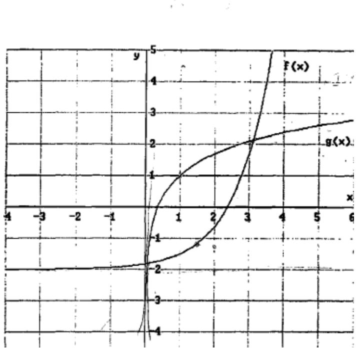
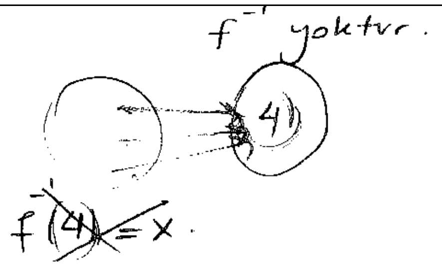
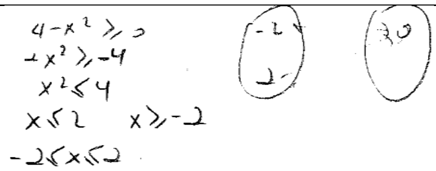
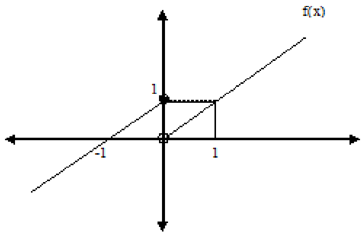
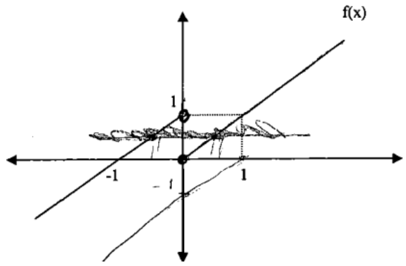
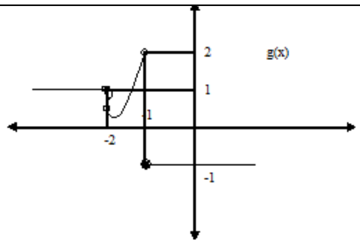
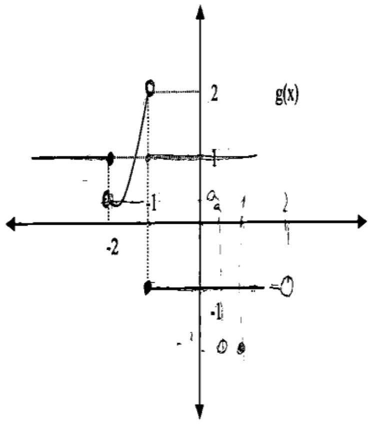
		Questions	Answers	Scores
Composition	1 a	$f(x)=x^2-2$ and $g(x)=-\sqrt{x+1}$ answer each of the following. (a) Determine (fog)(x) in simplified form and sketch a graph of this new function	$f(x)=x^2-2$ $g(x)=-\sqrt{x+1}$ $f(g(x))=(-\sqrt{x+1})^2-2$ $=x+1-2$ $=x-1.$	1
	1 b	(b) Determine (gof)(x) in simplified form and sketch a graph of this new function.	$g(f(x))=-\sqrt{x^2-2+1}$ $=-\sqrt{x^2-1}$ $=-\sqrt{(x-1)(x+1)}$ $x=0 \Rightarrow y=1$ $y=0 \Rightarrow x=1 \vee x=-1$	1
	2	$f(x)=\sqrt{4-x^2}$ and $g(x)=\sqrt{x^2-9}$ Determine (gof)(x) in simplified form.	$g(f(x))=\sqrt{4-x^2-9}=\sqrt{-x^2-5}<0$	1
	3 a	 (a) Use the given graphs to sketch y_2oy_1.	D.K ↓ T.K $(-\infty,\infty)$ o sketch y_2oy_1.	0
	3 b	(b) Use the given graphs to sketch y_1oy_2.	No more solution is attempted before the hint	0
	4	Use the given graphs to sketch $f\circ g$. 	Just talked about domain and range no solution is attempted	0

Table 4.4: Inverse functions questions in the non-routine interview, Yeliz's answers and scores

		Questions	Answers	Scores
Inverse	5 a	Find, the inverse of the following functions, if exists. a. f(x)=4, x∈R		4
	5 b	b. $f(x)=\sqrt{4-x^2}$		4
	6 a	Use the given graphs to sketch the inverse of given functions. 		1
	6 b			1

The fifth question was related with the existence of inverse function given in the mathematical notation (See Table 4.4). Yeliz attempted to solve both parts of the question procedurally. This approach led to incorrect solution at first but she realized the error by herself and found the correct answer so realizing the conditions for existence of an inverse function. The sixth question was related with finding inverse functions of a linear and non-linear functions when their graphs were given. Yeliz approached the question graphically and used the property of the inverse functions that they are symmetric with respect to $y = x$ line by assuming that the inverse of the functions exist. In part a, this approach led to incorrect solution but after the hint of checking the conditions of existence was provided she easily realized that the inverse does not exist. In part b, she felt that there is a violation of error but had some difficulty to explain the reason and needed a hint to complete the solution.

From the evidences we got from the survey of function knowledge and the non-routine questions interview it can be concluded that regardless of the topic and the type of the question, Yeliz experienced difficulties when questions were not given in mathematical notation. Her main difficulty is checking the conditions for the existence of function, inverse functions and composite functions.

4.1.1.1.3 The Analysis of the Definitions Used Through the Instruments

In order to see the reason for her main difficulty, the definitions Yeliz used through the instruments the survey of function knowledge, the journals about the definitions, the lesson plans, and the teaching practices were analyzed.

Definitions she used for composition of functions were given in Table 4.5. When they were analyzed it was seen that she used formal definitions both in the lesson plan and teaching practice. However, she used informal definitions in the survey, journal and teaching practices. Her informal definition in the teaching practice is an analogy for explaining the composition of functions. Moreover, even though not exists in the lesson plans she used two examples (See Table 4.6) during the teaching practice that would foster the understanding of the conditions for existence of composition of functions.

Table 4.5: Yeliz's definition of composite functions used through the instruments

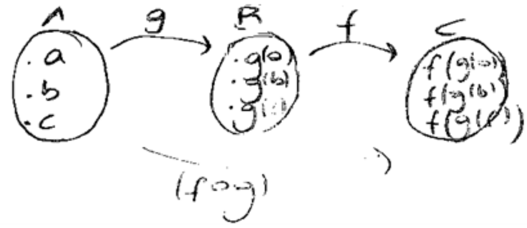
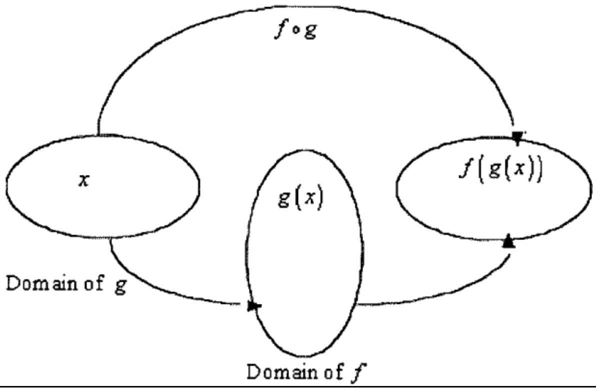
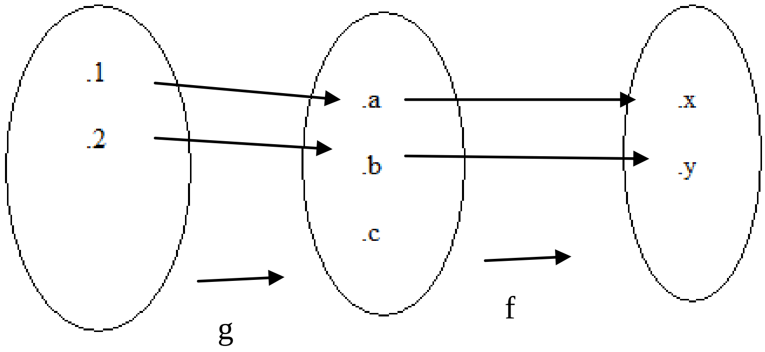
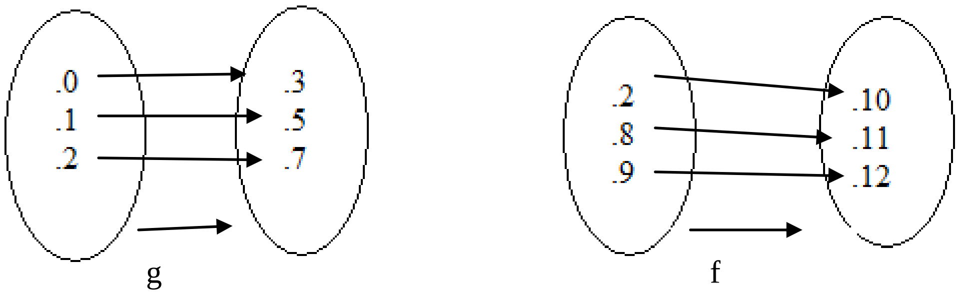
Instruments	Definition
Survey of Function Knowledge	
Journal about the Composite Function Definitions (Her choice among the given list)	Given any two functions f and g, we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f, then we can calculate the value of $f(g(x))$. The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f, and called composition of f and g.
Journal about the Composite Function Definitions (Her definition if she would teach)	$f: B \to C$ $g: A \to B$ f and g are two functions. fog is a new function who takes an element from A and carries it to C. If this element is x, (fog)(x) will be equal to $f(g(x))$.
Lesson Plan	$f: A \to B$ ve $g: B \to C$ iki fonksiyon olsun. $gof: A \to C$ olmak üzere, $gof(x) = g(f(x))$ şeklinde tanımlanan fonksiyona g ve f nin **bileşke fonksiyonu** denir. g ve f nin bileşkesi gof biçiminde gösterilir ve "g bileşke f" şeklinde okunur. 
Teaching Practice	She used the same definition with the lesson plan furthermore she used an analogy to explain the definition. Analogy is as follows "mouse eats the cheese, cat eats the mouse, so indirectly cat also eats the cheese". Apart from telling the analogy by using Venn diagrams she showed it on the board.

Table 4.6 : Yeliz's examples of composition functions in the teaching practices

Example
 Find fog if exists, give your reasons.
 Find fog and gof if exists, give your reasons.

Except for the survey of function knowledge, (See Table 4.7) she used formal definitions for definition of inverse functions. However, in the teaching practices she also gave informal definitions and even analogies to support understanding of the conditions for the existence of the inverse functions. Moreover, she carefully chose procedural questions (See Table 4.8) that would foster the understanding of the conditions for existence of inverse functions.

During the informal talk about the teaching practices, Yeliz admitted that she felt ashamed after the non-routine questions interview and she realized her weakness about the composite and inverse functions, so while preparing the lesson plans and during teaching practices she put more emphasis on the conditions for existence of composite and inverse functions. This statement approves the findings from the survey and the non-routine questions interview. She mentioned the effect of the non-routine questions in the evaluation interview by saying I think we should see these kinds of questions for every topic in the school curriculum.

Table 4.7: Yeliz's definition of inverse functions used through the instruments

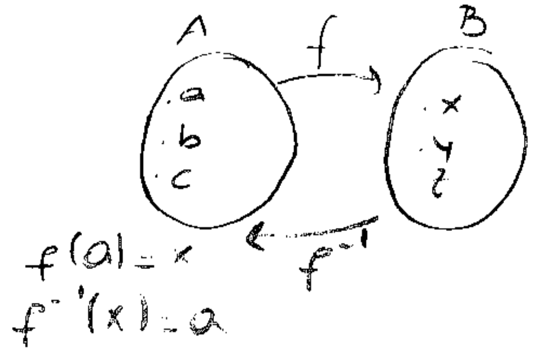
Instruments	Definition
Survey of Function Knowledge	
Journal about the Inverse Function Definitions (Her choice among the given list)	If f: A→B is one-to-one and onto function then there exists the inverse of f denoted by f^{-1} such that f^{-1}: B→A, $f(x) = y$, and $f^{-1}(y) = x$.
Journal about the Inverse Function Definitions (Her definition if she would teach)	My best definition; If f: A→B is one-to-one and onto function then (g be the inverse of f) f(x)=y and g(y)=x denoted by f: A→B, x→y and g: B→A, y→x.
Lesson Plan	**Tanım:** $f: A \to B, f = \{(x,y): x \in A \wedge y \in B\}$ birebir ve örten bir fonksiyon olmak üzere; $f^{-1}: B \to A, f^{-1} = \{(y,x): y \in B \wedge x \in A\}$ fonksiyonuna f **nin ters fonksiyonu** denir. $(x,y) \in f \Leftrightarrow (y,x) \in f^{-1}$ olduğundan, $y = f(x) \Leftrightarrow x = f^{-1}(y)$ olur.
Teaching Practice	She used the same definition with the lesson plan and moreover used set notation to represent the functions. Furthermore, she used an analogy to explain the definition. Analogy is as follows "Suppose everyday you are coming to school with your daddy's car and turn back with the school bus". Here, we can say that school bus does the opposite of the daddy's car so this case can be an example for an inverse function. Apart from telling the analogy by using Venn diagrams she showed it on the board.

Table 4.8 : Yeliz's examples of inverse functions in the teaching practices

Example	Explanations
Write f and $f^{-1}$for f:A to B where A={0,1,2}, B={1,2,3} and $f(x) = x + 1$	Provided in the lesson plan and solved during teaching practice
Write f and f^{-1} for f:A to B where A={0,1,2,3}, B={0,1,4,9,16} and $f(x) = x^2$	Not provided in the lesson plan, generated during teaching practice

Regardless of the her choice in the other instruments, she used different combinations of her knowledge (formal and informal) while teaching the concepts composition and inverse of functions. Her informal choice includes explanations for definitions, Venn diagrams and analogies.

4.1.1.2 Applications of the Rules about Composite and Inverse Functions

The rest of the questions not mentioned above are one conditional (question 18), one declarative (question 8), and six procedural (questions 7, 13, 14, 15, 16, and 19) questions. When the questions and their objectives were analyzed it was seen that all of these questions were related with the application of the concepts discussed above. It was also seen from the Table 4.2 that only in question 15 she got 0 points, for the rest she got 4 points for the declarative and procedural questions, and in conditional question she got 4 and 3 points from each part. However, when Yeliz' answers to question 8 (See Figure 4.11) and question 15 (See Figure 4.12) were compared, it was seen that there exists an inconsistency. Although she was aware of the addition operation on functions in question 8, it seems like she confused addition operation with composition in question 15.

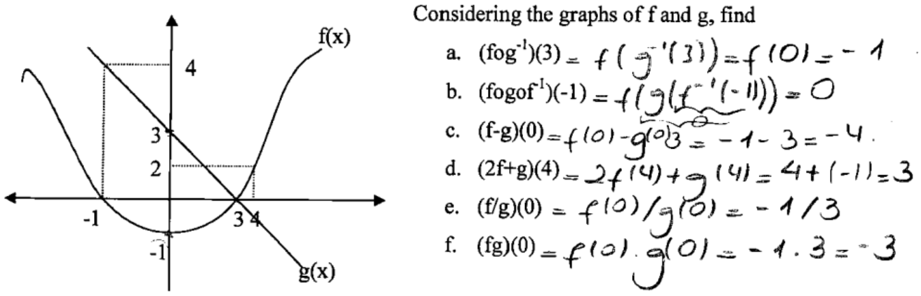


Figure 4.11: Question 8 and Yeliz's answer

This situation was investigated in the follow-up interview by first asking her to solve the question 15, she solved the question correctly and she stated that "I might have been saw in a wrong way during the survey".

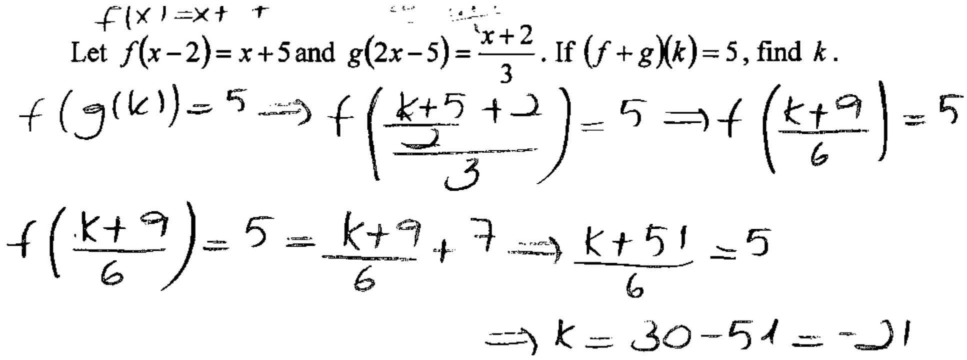


Figure 4.12: Question 15 and Yeliz's answer

As a result, it can be concluded that even though Yeliz had some gaps in her knowledge of composite and inverse functions. These gaps were checking the conditions for the existence of functions, composite, and inverse functions. Although she did not faced any difficulty when applying the rules through the questions given in mathematical notation, she experienced difficulty when the questions were not given in mathematical notation.

4.1.1.3 Connectedness of Yeliz's Knowledge of Functions, Composite and Inverse Functions

Previous results led to the fact that Yeliz did not show any evidence for the connectedness of her knowledge of composite and inverse functions. For this reason her concept maps were analyzed. Participants were asked to prepare two concept maps about functions. In the first one, they were free to choose the terms that will be used in the concept map, whereas in the second, the terms were provided but also they are free to use the terms that they prefer. After that, they wrote an essay about the concept maps they prepared and lastly focus group interview was conducted in order to share the participants' views about their concept maps, each others concept maps and concept mapping. Concepts maps were analyzed in terms of organization and accuracy whereas concept map essays were analyzed in terms of communication, organization and mechanics (Bolte, 1999).

For the first concept map (See Figure 4.13), Yeliz's organization score was 3 (fair) out of 6 based on the following reasons: she omitted some important terms like domain and range; mostly she was unable to construct meaningful clusters which would make the organization of the map more clear; although she used some cross-links, she missed many of them, like links between composition and inverse; some of the linking words lacked the mathematical terminology like she used "is a shown of" instead of "representation" when talking about representation of composite functions. Because the links between the definition of composite function and functions were wrong the accuracy score was rated 3 (fluent) instead of 4 (excellent) with no errors.

In the second concept map (See Figure 4.14), her organization score was to 2 out of 6 since she did not use any clusters, omitted some terms like composition of functions, missed cross-links between the terms like the relation between identity function and inverse function, one-to-one function, onto function, lacked use of appropriate mathematical linking words like "$f(x) = y$ is a formula of function" instead of representation of function. Her accuracy score was rated as 3 out of 4 since she wrote "x-axis shows domain, y-axis shows range" which is wrong. One can only say that elements of domain and range lie on the corresponding axis.

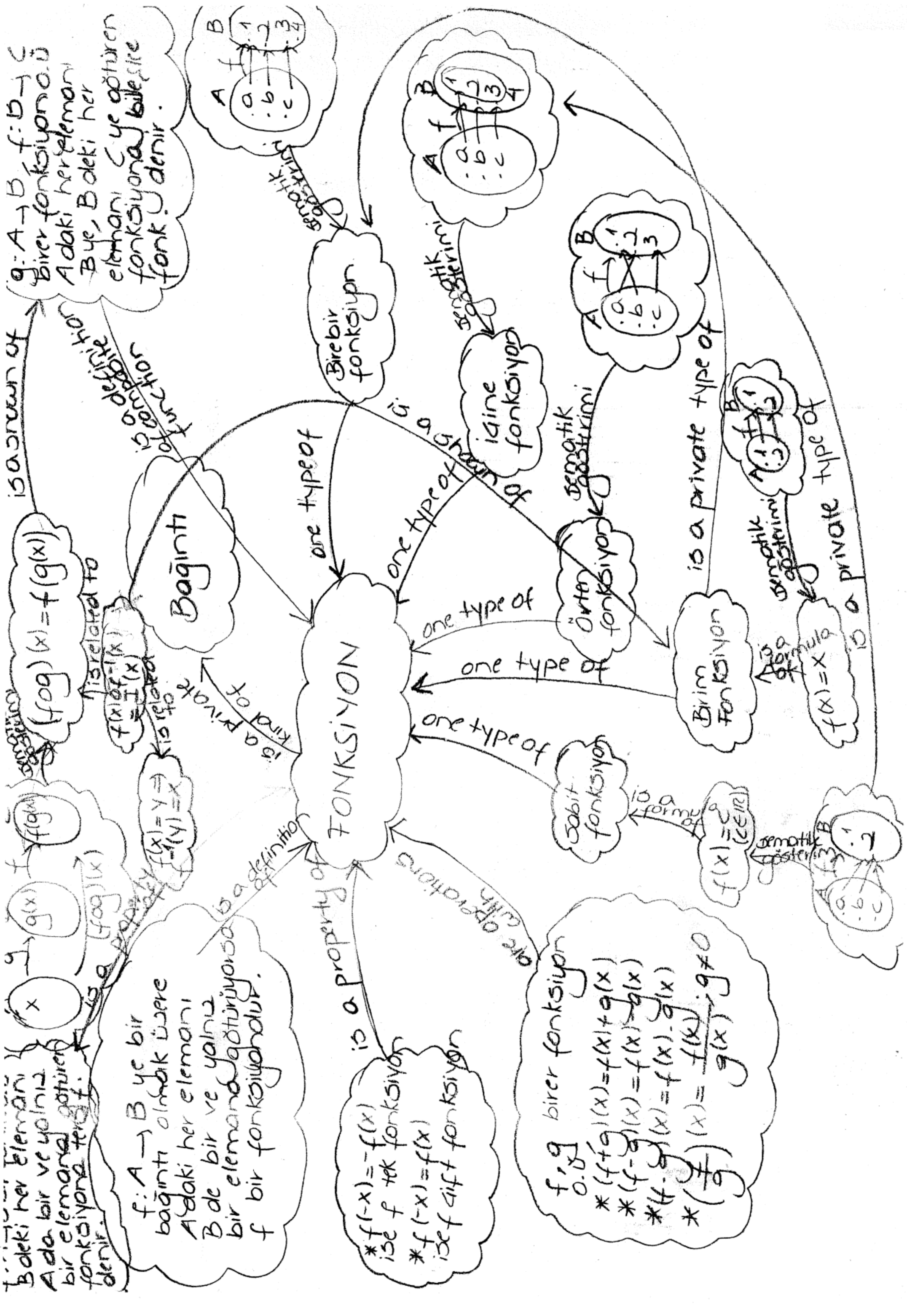


Figure 4.13: Yeliz’s first concept map

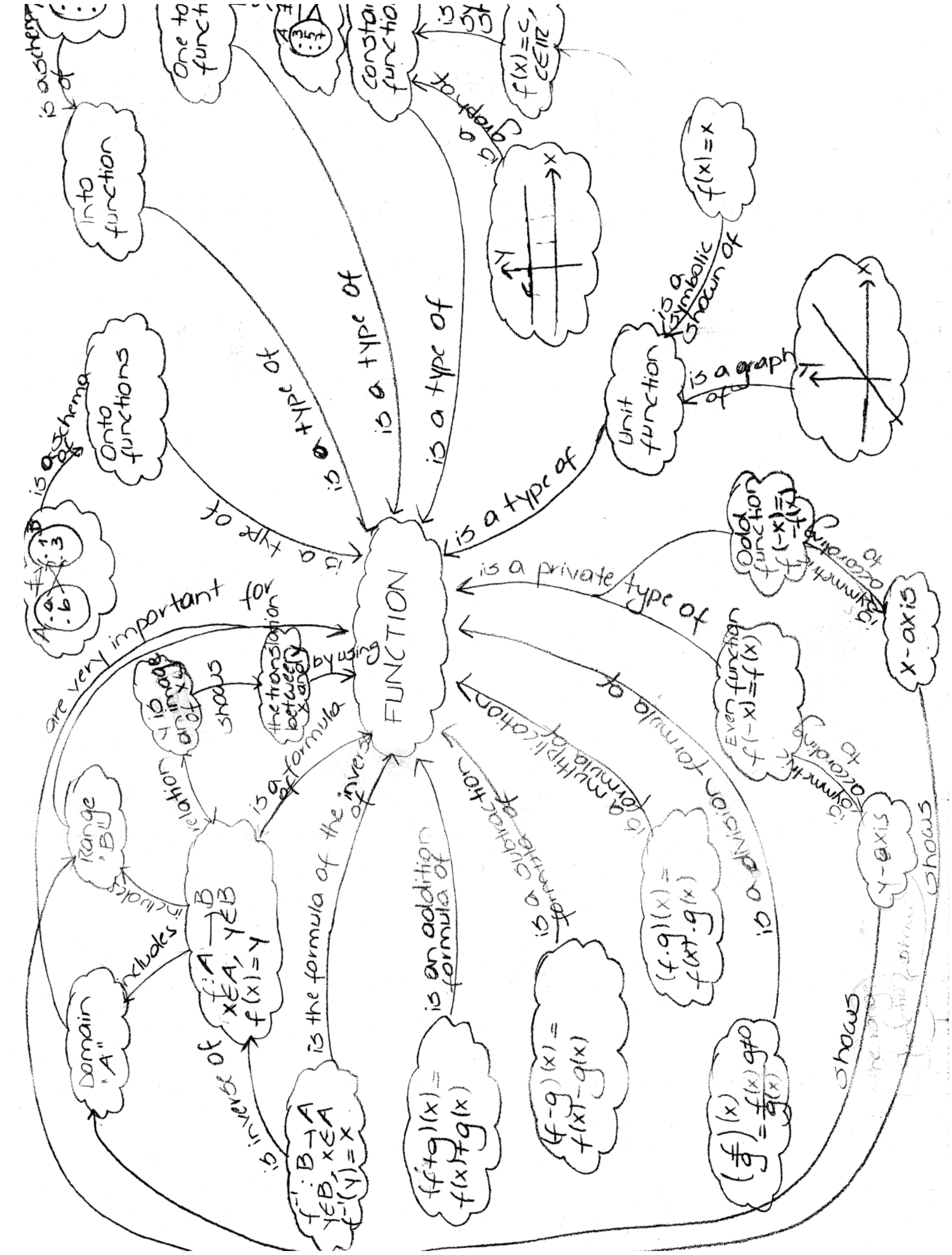


Figure 4.14: Yeliz's second concept map

In her concept map essay, she did not mention the reasons behind her mathematical choices and reasons for her links, cross-links, and linking words. She mentioned about the general process of how she constructed the map as follows: "…I started with making a brainstorm about the topic, I took some notes about descriptions, properties, formulas, and types of functions. After that, I decided my starting point. Then, I made my map spread out…". After this she mentioned about the differences and similarities between two concept maps but again she only compared the general structure not the mathematical content as follows: "…At first it seemed easier than the first one but after starting, it (second map) forced me a little bit. Although it did not limit me, I needed to check the terms and tried to use them… So, I wrote the same things in different forms…". Moreover, she talked about the some differences about two concept maps like in the first map she organized all operations in one cluster however in the second one she used all operations separately; the term "graphs of functions" used only in the second map since the list of terms reminded her to use it.

In the concept map interview, she again mentioned the above points moreover she criticized herself because of not constructing clusters which would make the map more understandable and she said that " I will never give these concept maps to any of my students". Overall evaluation of the three-staged concept map activity revealed that Yeliz was generally unable to construct meaningful subtopics (clusters) and connect the related subunits with meaningful linking words. As a result, she was either not aware of the cross-links between subtopics or unable to create cross-links since she could not see the picture clearly. Also, her linking words were rather weak or lack mathematical terminology which also explains her non-existent cross-links.

Especially, when her connectedness of knowledge about the composite and inverse functions were also investigated, it was seen that the general problem in functions also reflected in these two subunits. In the first concept map, the terms domain and range (or image) not exist, there seems to be a relationship between inverse and composition but it was not clear. The one to one and onto functions were presented but they were not shown in relation with the inverse functions. In the second concept map, the composition of functions was not presented at all, the inverse function was presented and its relation to domain and range were represented however, again the relationship between the 1-1 and onto and inverse functions were not presented. This lack of connectedness is similar to the that of survey of function knowledge and non-routine questions interview. This is because concept mapping activity was administered before the non-routine questions interview. Up to know it was observed that in the instruments administered before the non-routine questions interview, Yeliz's SMK about the composite and inverse functions had problems about the definitions and conditions of existence namely connectedness of knowledge. After that she realized and admitted her weakness as stated before. Also, it was seen in the lesson plan and vignettes that she put an emphasis on the existence f composite and inverse functions.

The sequencing the questions and the subtopics was taken as an evidence for the connectedness of her SMK during the teaching practices. When her sequencing of the subtopics of composite and inverse functions was examined it was seen that there was not any fault. However, her sequencing of the questions in the teaching practices were mostly not effective. For example, after introducing the concept composite functions her second example was a non-routine question which was not suitable since this kind of question requires students to completely understand the concept which was impossible at the second example of a newly introduced topic. Although not this much drastic, sequencing of the questions were also caused problems for the rest of the teaching practices.

4.1.1.4 Evidences of SMK from the Perspective of the Instruments Having Integration of Knowledge Components

Evidences of SMK were also searched through the instruments where there is an integration of all knowledge components exists. These instruments were vignettes, lesson plans, and teaching practices all of which assessed through the combined framework. Since Yeliz said that she was influenced from the non-routine questions interview, while analyzing the instruments it was kept in mind. This is, because, these three instruments were collected after the administration of the non-routine questions interview (See Table 3.2).

4.1.1.4.1 Lesson Plans

In the lesson plans, participants were asked to teach composite and inverse functions but they were not specifically given an order which one to teach first. She started with composition of functions and her reason was as follows "since finding an inverse of a function in under the composition of functions I prefer to teach composition first". As stated before, she used formal definitions and Venn diagrams both for the composite and inverse function definitions in the lesson plans. She mostly used declarative and procedural questions in the lesson plans. The only conditional type question was used in the first lesson plan. The Table 4.9 summarizes representative sample of the example types used in that lesson; that is, if in a lesson only procedural questions were used only a procedural example was provided and if there were more than one type of example were used one example for each type was provided.

Table 4.9: Question Excerpts from Yeliz's Lesson Plans

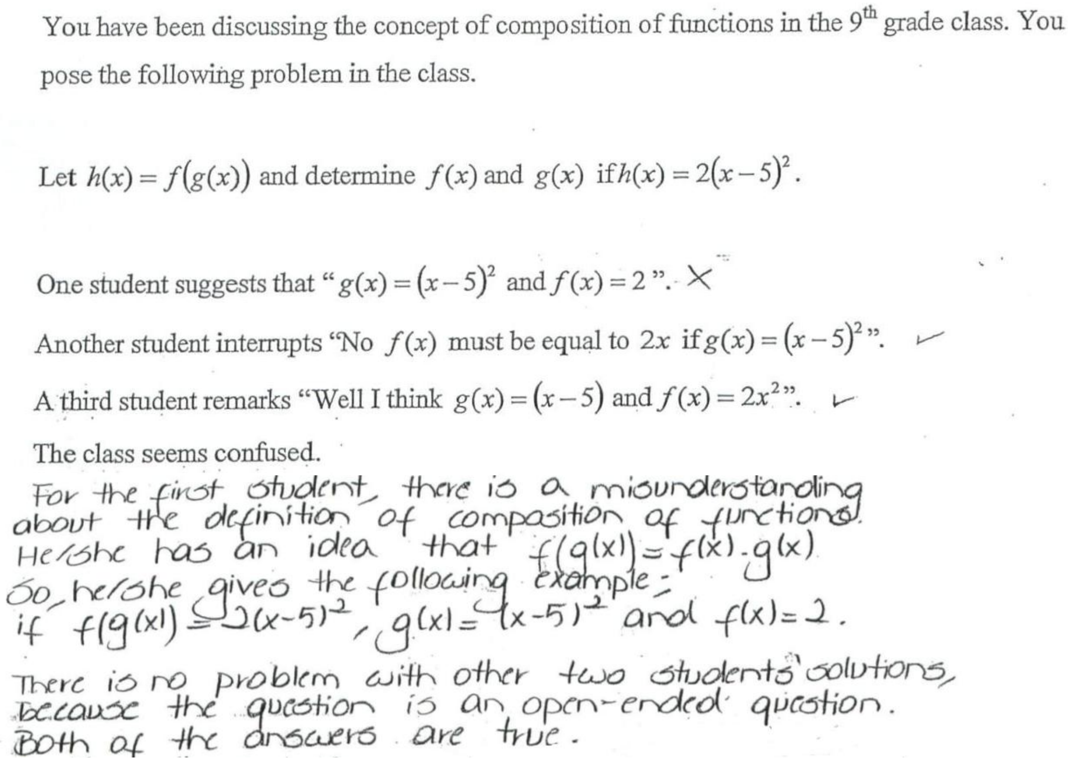
Lesson Plan	Questions	Knowledge Type
1	$f=\{(-1,1),(0,0),(2,4)\}$ $g=\{(1,3),(0,1),(4,9)\}$ fonksiyonları verilsin. Bu fonksiyonları ok diyagramı yöntemi ile gösterelim. **Örnek 1:** $f:R\rightarrow R$, $f(x)=x^2$ ve $g:R\rightarrow R, g(x)=x+1$ ise fog, gof ve fof fonksiyonlarını bulunuz. **Örnek 4:** $f(x)=x^2$ ve $g(x)=\sqrt{x-1}$ fonksiyonları için fog fonksiyonunu kuralını ve tanım kümesini bulunuz.	Declarative Procedural Conditional
2	**Örnek:** $f:R\rightarrow R, f(x)=x^2+3$ fonksiyonu ve $I(x)=x$ birim fonksiyonları verildiğine göre, $foI=Iof=f$ olduğunu gösteriniz. $f(x)=2x^2-x+1$ ve $(gof)(x)=8x^2-4x+2$ ise $g(x)$ nedir?	Declarative Procedural
3	**Örnek:** $A=\{k,m,n\}$ ve $B=\{3,5,7,9\}$ kümeleri veriliyor. $f:A\rightarrow B$ ye tanımlı $f=\{(k,3),(m,5)(n,5)\}$ fonksiyonu birebir ve örten olmadığından tersi yoktur.	Declarative
4	**Örnek:** $f:R\rightarrow R, f(x)=3x+5\Rightarrow f^{-1}(3x+1)$ fonksiyonunun kuralını bulunuz.	Procedural
5	**Soru 2:** $f(x-1)=3x-3\Rightarrow f(3x)$ in $f(x)$ türünden değerini bulunuz.	Procedural

By considering the evidences provided above and Yeliz's SMK performances on each lesson plan it was seen that she rarely used different representations and did not represent any evidence for her connectedness of knowledge, so she got Level 0-1 from all lesson plans.

4.1.1.4.2 Vignettes Related to Composite Functions

In line with the previous discussion Yeliz's vignettes were analyzed for evidences of SMK and those evidences were categorized according to the combined framework.

Firstly, the vignettes only related with the composition of functions were analyzed. The first vignette was intended to see to what extend participants resolve the conflict about the misunderstanding of the notation $h(x) = f(g(x))$ and mixing it with the ordinary multiplication $f(x).\,g(x)$. Yeliz grasped the conflict given in the vignette correctly which requires from her to know the definition and notation of composition of functions (See Figure 4.14). This was taken as an evidence for SMK.



You have been discussing the concept of composition of functions in the 9th grade class. You pose the following problem in the class.

Let $h(x) = f(g(x))$ and determine $f(x)$ and $g(x)$ if $h(x) = 2(x-5)^2$.

One student suggests that "$g(x) = (x-5)^2$ and $f(x) = 2$". ×

Another student interrupts "No $f(x)$ must be equal to $2x$ if $g(x) = (x-5)^2$".

A third student remarks "Well I think $g(x) = (x-5)$ and $f(x) = 2x^2$".

The class seems confused.

For the first student, there is a misunderstanding about the definition of composition of functions. He/she has an idea that $f(g(x)) = f(x).g(x)$ So, he/she gives the following example: if $f(g(x)) = 2(x-5)^2$, $g(x) = (x-5)^2$ and $f(x) = 2$.

There is no problem with other two students' solutions, because the question is an open-ended question. Both of the answers are true.

Figure 4.15: Excerpts from the Yeliz's vignette # 1

Yeliz used a procedural question in order to solve the conflict and show the difference between the $f(g(x))$ and $f(x).\,g(x)$ as seen in Figure 4.16.

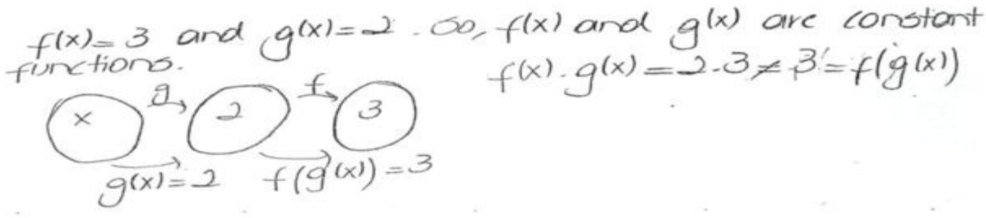


Figure 4.16: Excerpts from the Yeliz's vignette # 1

Although the example in the vignette did not include constant function, she preferred to use constant functions in her answer to make the distinction more clear. Furthermore, she showed her knowledge that there can be more than one combination to get the same composite function while making the following explanation in Figure 4.17.

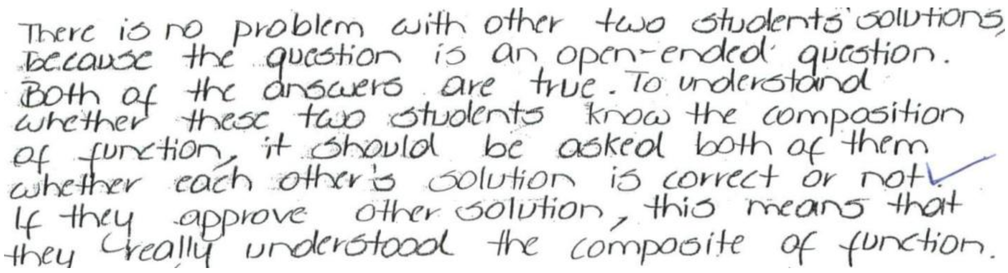
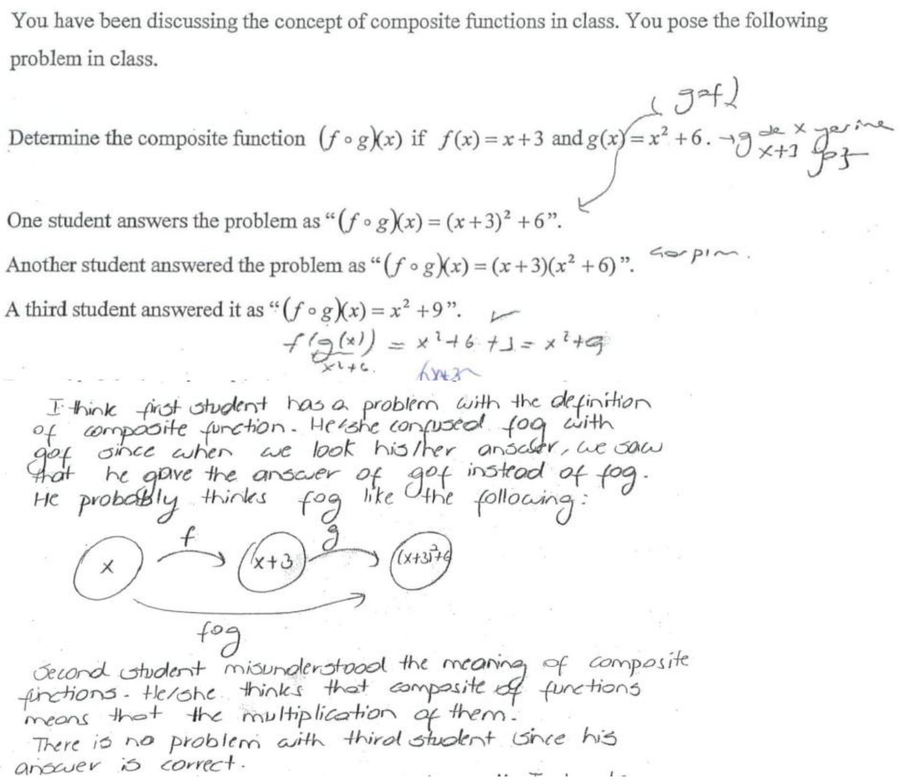

There is no problem with other two students' solutions, because the question is an open-ended question. Both of the answers are true. To understand whether these two students know the composition of function, it should be asked both of them whether each other's solution is correct or not. If they approve other solution, this means that they really understood the composite of function.

Figure 4.17: Excerpts from the Yeliz's vignette # 1

By considering the evidences of SMK found in vignette 1, it was rated as Level 1 according to the combined framework.

The second vignette was intended to see to what extend participants resolve the conflict about the mixing order of operations when taking compositions of functions and mixing it with the ordinary multiplication $f(x).\,g(x)$. As seen in Figure 4.18 she correctly identified the students' misunderstandings which requires a knowledge of definition and notation of composition of functions, which provided the researcher with the evidence of SMK.



You have been discussing the concept of composite functions in class. You pose the following problem in class.

Determine the composite function $(f \circ g)(x)$ if $f(x) = x + 3$ and $g(x) = x^2 + 6$.

One student answers the problem as "$(f \circ g)(x) = (x+3)^2 + 6$".

Another student answered the problem as "$(f \circ g)(x) = (x+3)(x^2+6)$".

A third student answered it as "$(f \circ g)(x) = x^2 + 9$".

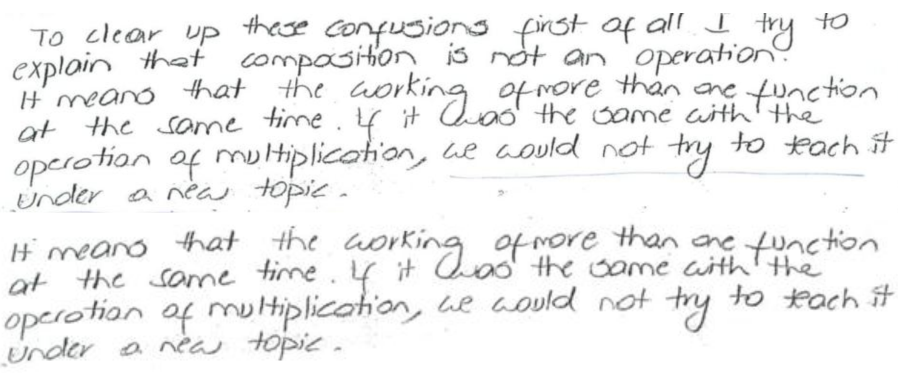
$f(g(x)) = x^2 + 6 + 3 = x^2 + 9$

I think first student has a problem with the definition of composite function. He/she confused fog with gof since when we look his/her answer, we saw that he gave the answer of gof instead of fog. He probably thinks fog like the following:

Second student misunderstood the meaning of composite functions. He/she thinks that composite of functions means that the multiplication of them.
There is no problem with third student since his answer is correct.

Figure 4.18: Excerpts from the Yeliz's vignette # 2

Furthermore, while making explanations (See Figure 4.19) she used an informal explanation about the composition of functions , which also implies a knowledge of composite functions. However, her explanation also showed a lack of knowledge about the composition of functions. She stated that composition is not an operation, however, it is a special operation among the sets of functions.



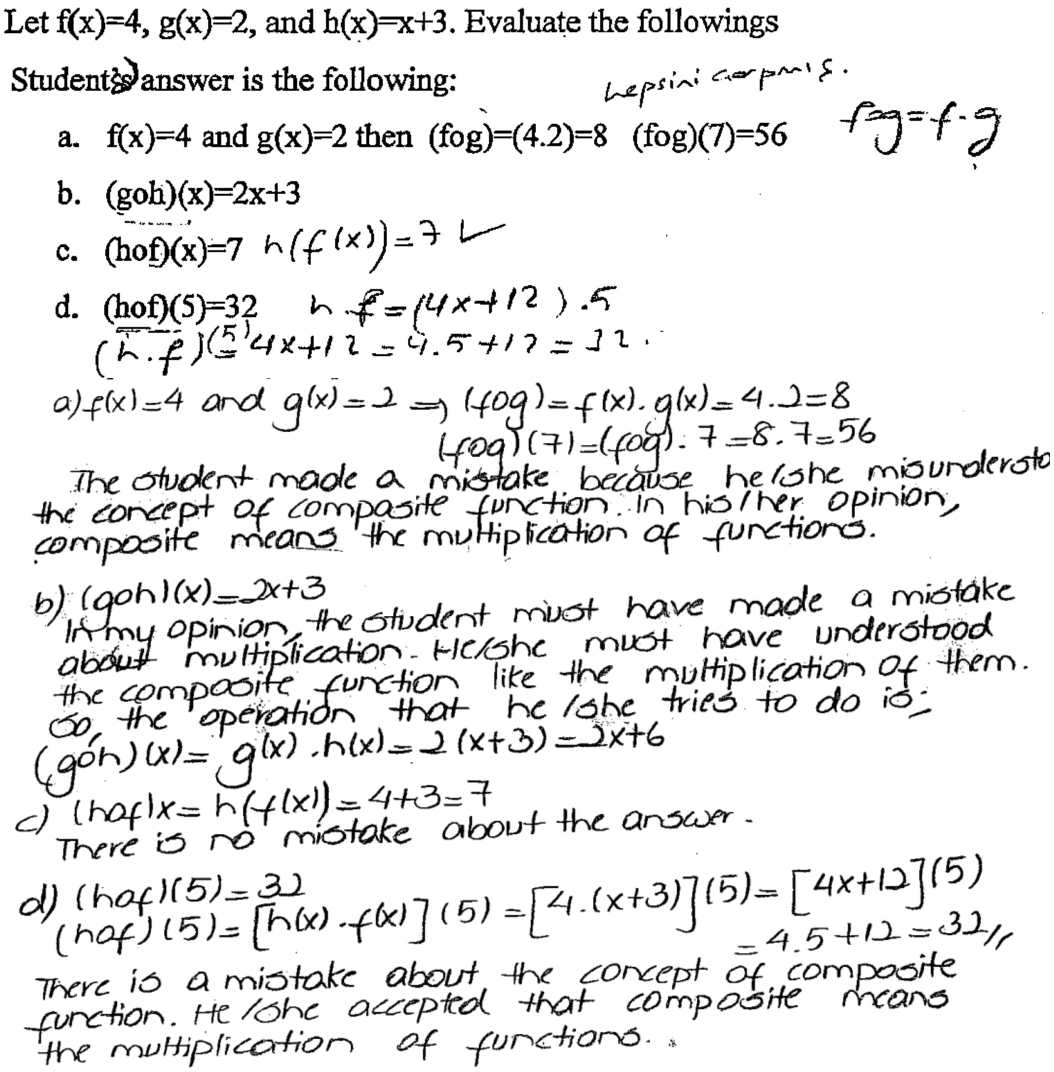
To clear up these confusions first of all I try to explain that composition is not an operation.
It means that the working of more than one function at the same time. If it was the same with the operation of multiplication, we would not try to teach it under a new topic.

It means that the working of more than one function at the same time. If it was the same with the operation of multiplication, we would not try to teach it under a new topic.

Figure 4.19: Excerpts from the Yeliz's vignette # 2

The evidences of SMK found in the vignette 2 resulted in Level 0-1 in the combined framework. Since she has some problems with the correct use of terminology.

The third vignette was intended to see to what extend participants resolve the conflict about the mixing composition with the ordinary multiplication when one of the functions is a constant function. Like in the previous two vignettes, she correctly identified the errors (See Figure 4.20) which gave the researcher evidence of knowledge about the definition of composition of functions and use of notation. Eventually, in the first vignette in order to solve the conflict she used a similar example.



Let f(x)=4, g(x)=2, and h(x)=x+3. Evaluate the followings

Student's answer is the following:

a. f(x)=4 and g(x)=2 then (fog)=(4.2)=8 (fog)(7)=56 fog=f.g

b. (goh)(x)=2x+3

c. (hof)(x)=7 h(f(x))=7

d. (hof)(5)=32 h.f=(4x+12).5

(h.f)(5)=4x+12=4.5+12=32.

a) f(x)=4 and g(x)=2 ⇒ (fog)=f(x).g(x)=4.2=8
(fog)(7)=(fog).7=8.7=56

The student made a mistake because he/she misunderstood the concept of composite function. In his/her opinion, composite means the multiplication of functions.

b) (goh)(x)=2x+3
In my opinion, the student must have made a mistake about multiplication. He/she must have understood the composite function like the multiplication of them. So, the operation that he/she tries to do is;
(goh)(x)=g(x).h(x)=2(x+3)=2x+6

c) (hof)x=h(f(x))=4+3=7
There is no mistake about the answer.

d) (hof)(5)=32
(hof)(5)=[h(x).f(x)](5)=[4.(x+3)](5)=[4x+12](5)
=4.5+12=32

There is a mistake about the concept of composite function. He/she accepted that composite means the multiplication of functions.

Figure 4.20: Excerpts from the Yeliz's vignette # 3

For clearing up confusion, she used two questions (See Figure 4.21) where the first one was in conditional nature because it asks for the reason behind the choice and the second one was a procedural question. Her SMK level for the third vignette was rated as Level 1-2 since by writing a conditional question she showed evidences from the Level 2.

To clear up this confusion, I try to ask some questions to the students to realise the difference between composite and multiplication.

Q1: What is the difference between $(fog)(x)$ and $f(x).g(x)$. If there is no difference why we use the sign (o) instead of ($\cdot$)?

Q2: If $f(x)=4$ and $g(x)=x+3$, you can multiply them easily. However, how can you multiply f and g when $f=\{(1,1),(2,3),(3,4)\}$ and $g=\{(1,1),(4,2),(3,3)\}$

Figure 4.21: Excerpts from the Yeliz's vignette # 3

The fourth vignette was intended to see to what extend participants resolve the conflict about misunderstanding of the notation $h(x) = (fog)(x)$ while working backwards in composition of function problems. Yeliz identified the problem in the question partly as seen in Figure 4.22.

If $h(x) = (fog)(x)$ where $h(x) = x^2 + 1$ and $g(x) = x$, then find $f(x)$. Show your work and explain your answer.

One of the students voluntarily comes to the board and the solved the question as follows:

$x^2 + 1 = f(g(x))$ (*) $\quad x^2+1 = f(g(x))$

$x^2 + 1 = (fx)(x)$ $\quad x^2+1 = f(x)$

$f(x) = \dfrac{x^2+1}{x}$

The student knew the shown of composite function, (*) but there is a problem with the other step of solution. He/she accepted $f(g(x)) = f(x).g(x) = f(x).x$ So, he/she found $f(x) = \dfrac{x^2+1}{x}$.

To clear up this confusion, I again prefer questions to make students realise the difference.

Figure 4.22: Excerpts from the Yeliz's vignette # 4

She identified the error just as mixing combination with multiplication. However, in the question there is more than that here in $(fog)(x)$ the meaning of (x) is also not clearly understood by the student. Her lack of understanding of the source of mistake is an evidence for a lack of SMK. Besides, she barely gave no evidence of SMK while clearing up the confusion, which resulted in getting Level 0-1 from the combined framework.

The fifth vignette was intended to see to what extend participants resolve the conflict about the usage of analogy for definition of composite functions. As it was seen in Figure 4.23, she decided that given analogy for the definition of composite function is true. Giving such a decision requires a necessary knowledge of the definition of the composite function, so this decision was taken as an evidence.

A teacher gave the definition of the composite function and explained it on the board to his/her students. However, some of his/her students stated that they did not understand it completely. Then teacher gave the following example to the students.

In order to clean and dry our clothes in a laundry we use two machines, washing machine and dryer, respectively.

Dry&Wash (clothes)

Dry[Wash(clothes)]=Dry[cleaned and wet clothes]=dried and cleaned clothes

Combination of these machines works can be considered as a composition of functions

I think that it is a really clear and good example for the definition of composite functions.
As I see, this example does not cause any misunderstanding.

Figure 4.23: Excerpts from the Yeliz's vignette # 5

Developing on this knowledge, she also gave an alternative true analogy (See Figure 4.22). This vignette was rated as Level 1 according to the combined framework since it reflected understanding of the required topic for that vignette.

If I were to explain the composite function by using a real life example, I would choose cat→mouse→chees
In fact, I used it in my class and students understood what I was trying to explain.
The example is exactly like the following:

f: Cheese → Mouse and g: Mouse → Cat

gof: Cheese → Cat

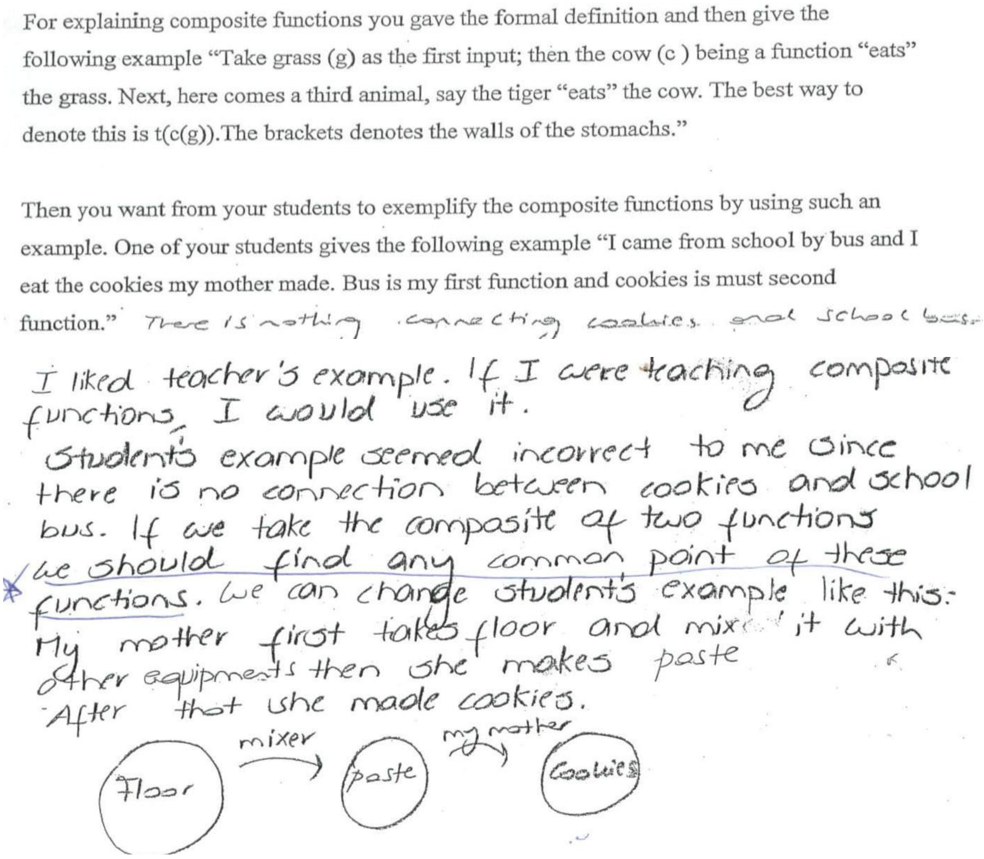
Cat[Mouse (Cheese)] = Cat[Mouse who ate cheese]

= Cat who ate cheese

We took gof because cat follows mouse and mouse follows cheese. Therefore, first mouse eats cheese, then cat eats mouse.

Figure 4.24: Excerpts from the Yeliz's vignette # 5

The last vignette related with the composite functions is the thirteenth vignette, which is similar to the fifth vignette since it was intended to see to what extend participants resolve the conflict about the usage of analogy for definition of composite functions. As it can be seen from the Figure 4.25 this time an example was provided by the teacher and student's analogy needs correction. She identified the error correctly by saying "we should find any common points of these function" which serves as an evidence for the existence of the knowledge of the definition of composite functions. Her SMK was categorized as Level 0-1 since her explanation in quote needs some clarification.


For explaining composite functions you gave the formal definition and then give the following example "Take grass (g) as the first input; then the cow (c) being a function "eats" the grass. Next, here comes a third animal, say the tiger "eats" the cow. The best way to denote this is t(c(g)).The brackets denotes the walls of the stomachs."

Then you want from your students to exemplify the composite functions by using such an example. One of your students gives the following example "I came from school by bus and I eat the cookies my mother made. Bus is my first function and cookies is must second function." There is nothing connecting cookies and school bus.

I liked teacher's example. If I were teaching composite functions, I would use it.
Student's example seemed incorrect to me since there is no connection between cookies and school bus. If we take the composite of two functions we should find any common point of these functions. We can change student's example like this: My mother first takes floor and mixed it with other equipments then she makes paste After that she made cookies.

Figure 4.25: Excerpts from the Yeliz's vignette # 13

The tenth vignette was aimed to explain the participants' understanding of combined use of inverse and composition of functions in questions. The case in the vignette includes a student's answer to a question and a dialogue between the student and the teacher about the answer (See Figure 4.26).

If $f(2x+1)=2x-1$ find $f(3x)$ in terms of $f(x)$ and explain the steps of your solution.

$-f(x)$

Then the students solved the question correctly as follows:

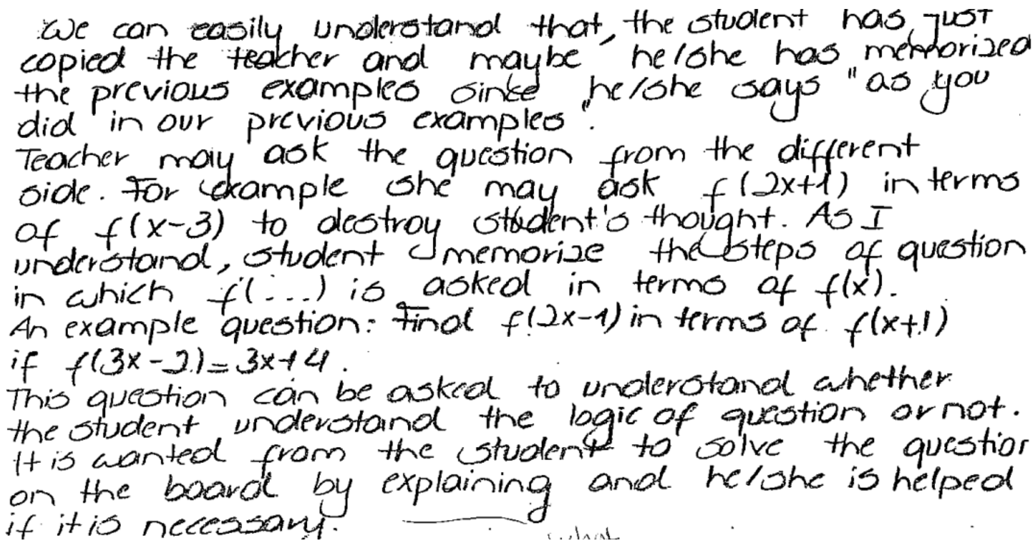
$$\left.\begin{array}{l} y=2x+1 \\ x=2y+1 \\ x-1=2y \\ y=\dfrac{x-1}{2} \end{array}\right\} \underbrace{f(x)=2.\frac{x-1}{2}-1=x-2}_{*} \; then \; f(3x)=3x-2$$

$$f(x)+2=x \Rightarrow f(3x)=3(f(x)+2)+2 \Rightarrow f(3x)=3f(x)+4$$

After the solution made, teacher wants from student to explain what she did in the step indicated by *. She said that "I have to get f(x) so that I could calculate f(3x). For getting f(x) I made the necessary calculations as you did in our previous examples".
Furthermore, teacher wants from student to explain what she did in the $f(x)+2=x$ step. She said that "we have to single out x from the equation as you did in our previous examples". However, she couldn't explain what she did.

Figure 4.26: Excerpts from the Yeliz's vignette # 10

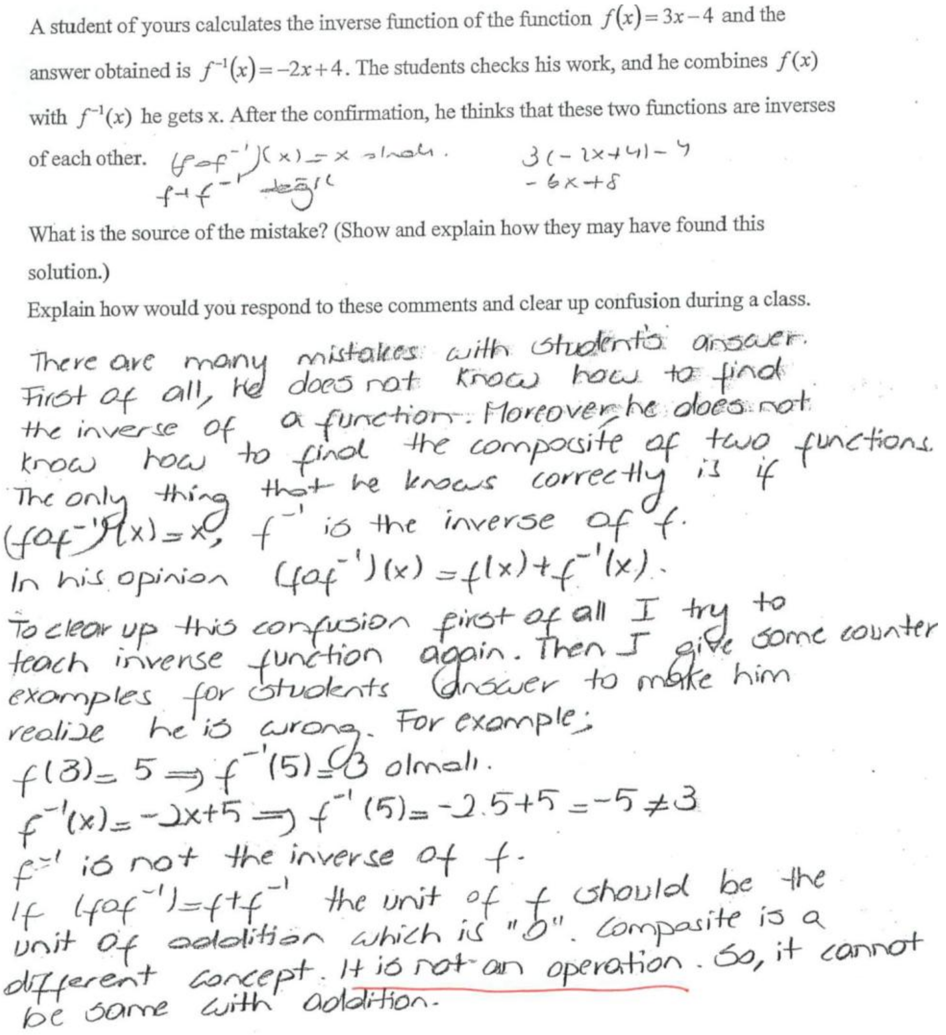
In her example, she provided an evidence for her knowledge of the applications of composite and inverse functions by writing " …teacher may ask the question from different side…" and generating a similar example in different difficulty " …an example question find $f(2x-1)$ in terms of $f(x+1)$ if $f(3x-2)=3x+4$ …", which was rated as the Level 0-1 since there were no concrete evidences even in the further explanations (See Figure 4.27).



We can easily understand that, the student has just copied the teacher and maybe he/she has memorized the previous examples since he/she says "as you did in our previous examples".
Teacher may ask the question from the different side. For example she may ask $f(2x+1)$ in terms of $f(x-3)$ to destroy student's thought. As I understand, student memorize the steps of question in which $f(\ldots)$ is asked in terms of $f(x)$.
An example question: Find $f(2x-1)$ in terms of $f(x+1)$ if $f(3x-2)=3x+4$.
This question can be asked to understand whether the student understand the logic of question or not. It is wanted from the student to solve the question on the board by explaining and he/she is helped if it is necessary.

Figure 4.27: Excerpts from the Yeliz's vignette # 10

The eleventh vignette was intended to see to what extend participants resolve the conflict about the students' misunderstandings about the use of the fact $fof^{-1} = I$ while solving questions in relation to composite and inverse functions. Her identification of the problem of the student's solution and her explanations (See Figure 4.28) gave us an evidence that she knows what a composite and an inverse function is and use the appropriate notation for both of them. Her explanations were not clear enough to understand how connected her knowledge is, so her SMK was rated as Level 0-1 from the combined framework.


A student of yours calculates the inverse function of the function $f(x)=3x-4$ and the answer obtained is $f^{-1}(x)=-2x+4$. The students checks his work, and he combines $f(x)$ with $f^{-1}(x)$ he gets x. After the confirmation, he thinks that these two functions are inverses of each other.

$(f \circ f^{-1})(x) = x$ olmalı. $\quad 3(-2x+4)-4$

$f+f^{-1}$ değil $\quad -6x+8$

What is the source of the mistake? (Show and explain how they may have found this solution.)

Explain how would you respond to these comments and clear up confusion during a class.

There are many mistakes with student's answer. First of all, he does not know how to find the inverse of a function. Moreover, he does not know how to find the composite of two functions. The only thing that he knows correctly is if $(f \circ f^{-1})(x)=x$, f^{-1} is the inverse of f.

In his opinion $(f \circ f^{-1})(x) = f(x)+f^{-1}(x)$.

To clear up this confusion first of all I try to teach inverse function again. Then I give some counter examples for students answer to make him realize he is wrong. For example;

$f(3)=5 \Rightarrow f^{-1}(5)=3$ olmalı.

$f^{-1}(x)=-2x+5 \Rightarrow f^{-1}(5)=-2.5+5=-5 \neq 3$

f^{-1} is not the inverse of f.

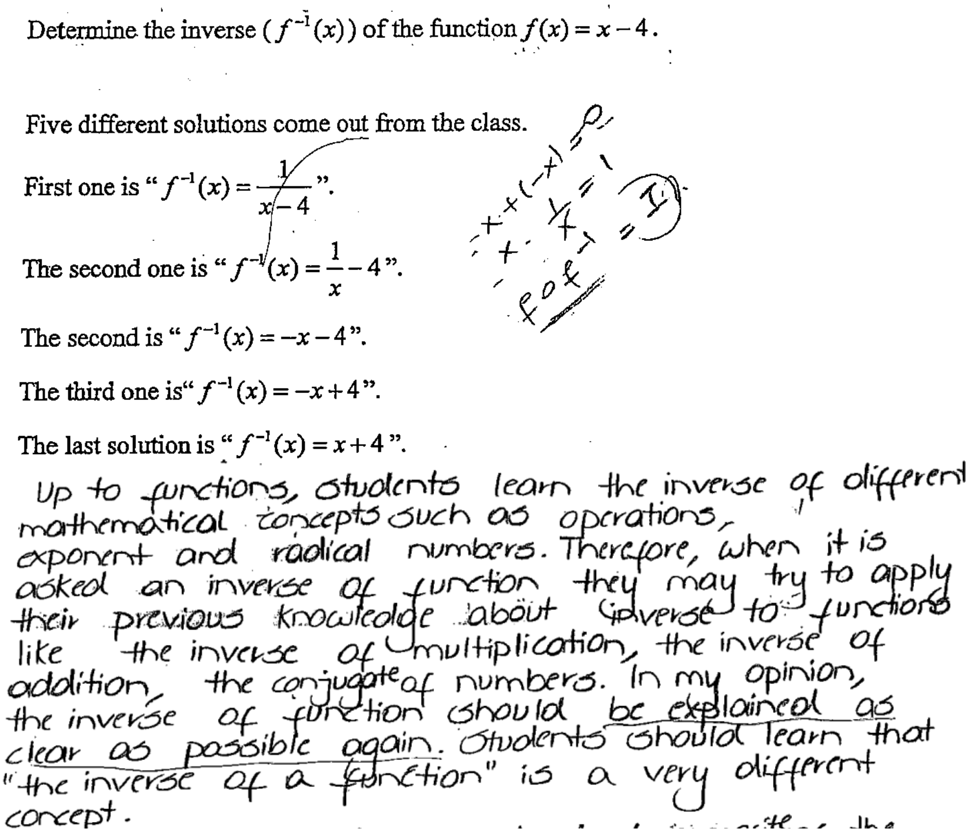
If $(f \circ f^{-1})=f+f^{-1}$ the unit of f should be the unit of addition which is "0". Composite is a different concept. It is not an operation. So, it cannot be same with addition.

Figure 4.28: Excerpts from the Yeliz's vignette # 11

When Yeliz knowledge of composite functions were evaluated through the vignettes (1, 2, 3, 4, 5, 13, 10, and 11 where last two also include knowledge of inverse functions) it was seen that she knows the definition of composition of functions and its notation good enough to resolve conflicts in different cases. However, she showed her limited understanding of composition by stating composition is not an operation. Moreover, she showed no evidence for the connectedness of her knowledge of composition of functions. She used procedural questions during her explanations and for once used a conditional question in vignette 2. Her use of different representations were also limited just to use of Venn diagrams only in the vignettes 1, 2 and 13. As stated above for each vignette, her SMK was mostly rated as Level 0-1, similar to her levels in the lesson plans. Also, her SMK was rated as once for Level 1-2, and twice for Level 1.

4.1.1.4.3 Vignettes Related to Inverse Functions

In a similar vein, the vignettes related with the inverse functions were analyzed. The sixth vignette was intended to see to what extend participants resolve the conflict about the usage of the power -1 in the function notation. This is because as it was seen in the Figure 4.24 students mixed inverse functions with multiplicative and additive inverse of real numbers. Yeliz grasped the students' misunderstandings easily and described where the error lies (See Figure 4.29), which gave us an evidence for Yeliz's knowledge of the term "inverse" in mathematics. Her further explanation (See Figure 4.30) for clearing up the confusion showed that she also knew the definition of inverse function in relation to other topics showing her connectedness of knowledge so her SMK was rated as Level 2.



Determine the inverse ($f^{-1}(x)$) of the function $f(x) = x - 4$.

Five different solutions come out from the class.

First one is "$f^{-1}(x) = \frac{1}{x-4}$".

The second one is "$f^{-1}(x) = \frac{1}{x} - 4$".

The second is "$f^{-1}(x) = -x - 4$".

The third one is "$f^{-1}(x) = -x + 4$".

The last solution is "$f^{-1}(x) = x + 4$".

Up to functions, students learn the inverse of different mathematical concepts such as operations, exponent and radical numbers. Therefore, when it is asked an inverse of function they may try to apply their previous knowledge about inverse to functions like the inverse of multiplication, the inverse of addition, the conjugate of numbers. In my opinion, the inverse of function should be explained as clear as possible again. Students should learn that "the inverse of a function" is a very different concept.

Figure 4.29: Excerpts from the Yeliz's vignette # 6

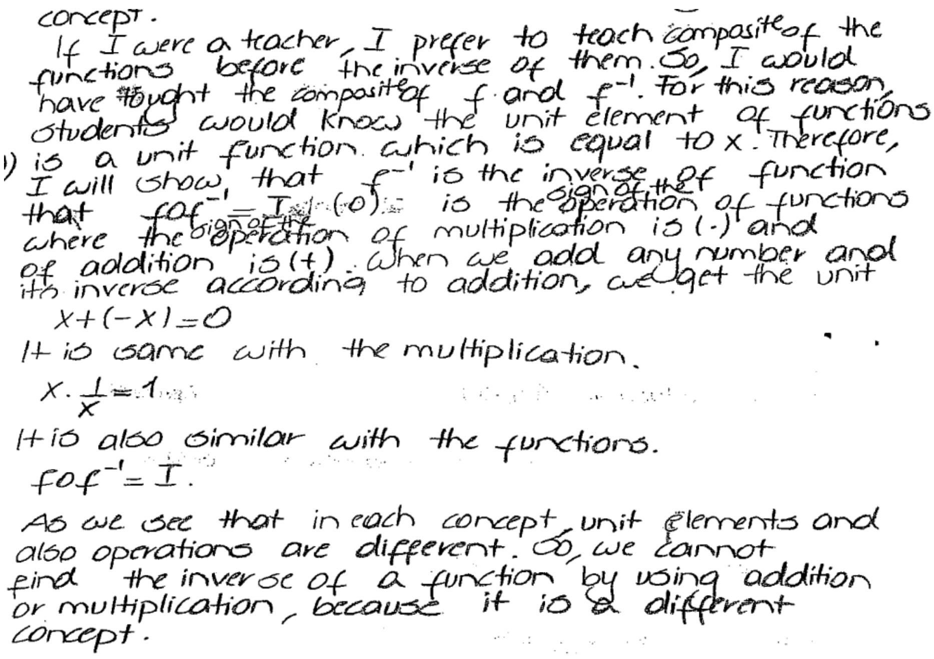
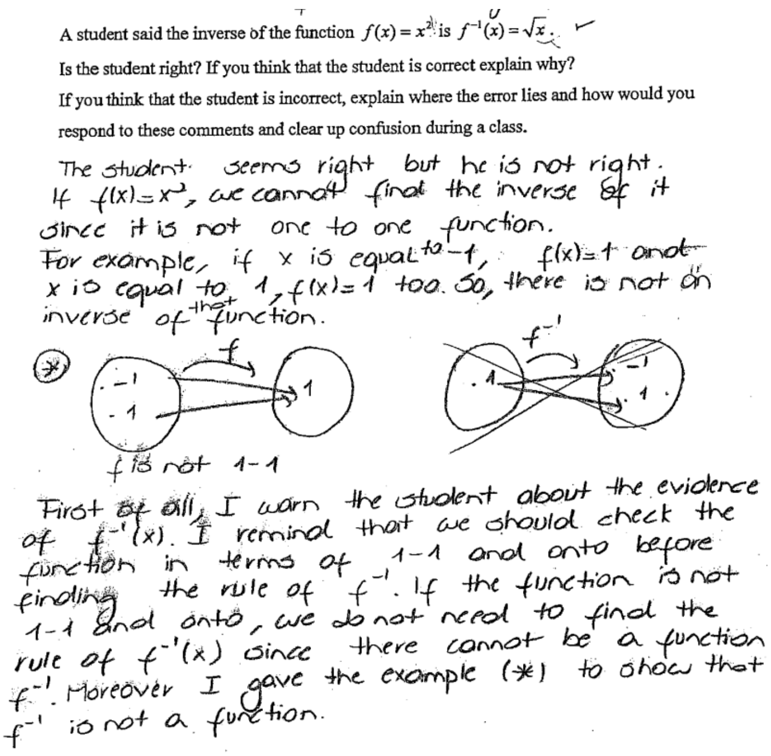

concept.
If I were a teacher, I prefer to teach composite of the functions before the inverse of them. So, I would have tought the composite of f and f^{-1}. For this reason, students would know the unit element of functions is a unit function which is equal to x. Therefore, I will show that f^{-1} is the inverse of function that $f o f^{-1} = I$. (o) is the sign of the operation of functions where the sign of the operation of multiplication is (·) and of addition is (+). When we add any number and its inverse according to addition, we get the unit
$x + (-x) = 0$
It is same with the multiplication.
$x \cdot \frac{1}{x} = 1$
It is also similar with the functions.
$f o f^{-1} = I$.
As we see that in each concept, unit elements and also operations are different. So, we cannot find the inverse of a function by using addition or multiplication, because it is a different concept.

Figure 4.30: Excerpts from the Yeliz's vignette # 6

The seventh vignette was intended to see to what extend participants resolve the conflict about the students' misunderstandings about the existence of inverse functions. Her identification of the problem of the student's solution gave us an evidence that she knows that for existence of an inverse function it has to be a one to one function (See Figure 4.31). While clearing up the confusion in the next paragraph, she once more showed her knowledge of the definition of the inverse functions by providing an example in Venn diagram which is a declarative question. Her SMK was rated as Level 1.



A student said the inverse of the function $f(x)=x^2$ is $f^{-1}(x)=\sqrt{x}$.

Is the student right? If you think that the student is correct explain why?

If you think that the student is incorrect, explain where the error lies and how would you respond to these comments and clear up confusion during a class.

The student seems right but he is not right. If $f(x)=x^2$, we cannot find the inverse of it since it is not one to one function. For example, if x is equal to -1, $f(x)=1$ and x is equal to 1, $f(x)=1$ too. So, there is not an inverse of that function.

First of all, I warn the student about the evidence of $f^{-1}(x)$. I remind that we should check the function in terms of 1-1 and onto before finding the rule of f^{-1}. If the function is not 1-1 and onto, we do not need to find the rule of $f^{-1}(x)$ since there cannot be a function f^{-1}. Moreover I gave the example (*) to show that f^{-1} is not a function.

Figure 4.31: Excerpts from the Yeliz's vignette # 7

In the eighth vignette the aim was to see to what extend participants understand the idea of inverse function as undoing. As seen in Figure 4.32 she diagnosed that the student is not correct and gave an appropriate explanation for it. Combining this evidence with her further explanation it was seen that she knows the definition of inverse functions, and differentiates between functions and non-functions. Her SMK was rated as Level 1, since further connections of knowledge was not recognized.

A student tells you that the binary operations of multiplication and division are inverse functions because they undo each other.

Is the student right? If you think that the student is correct explain why?

If you think that the student is incorrect, explain where the error lies and how would you respond to these comments and clear up confusion during a class.

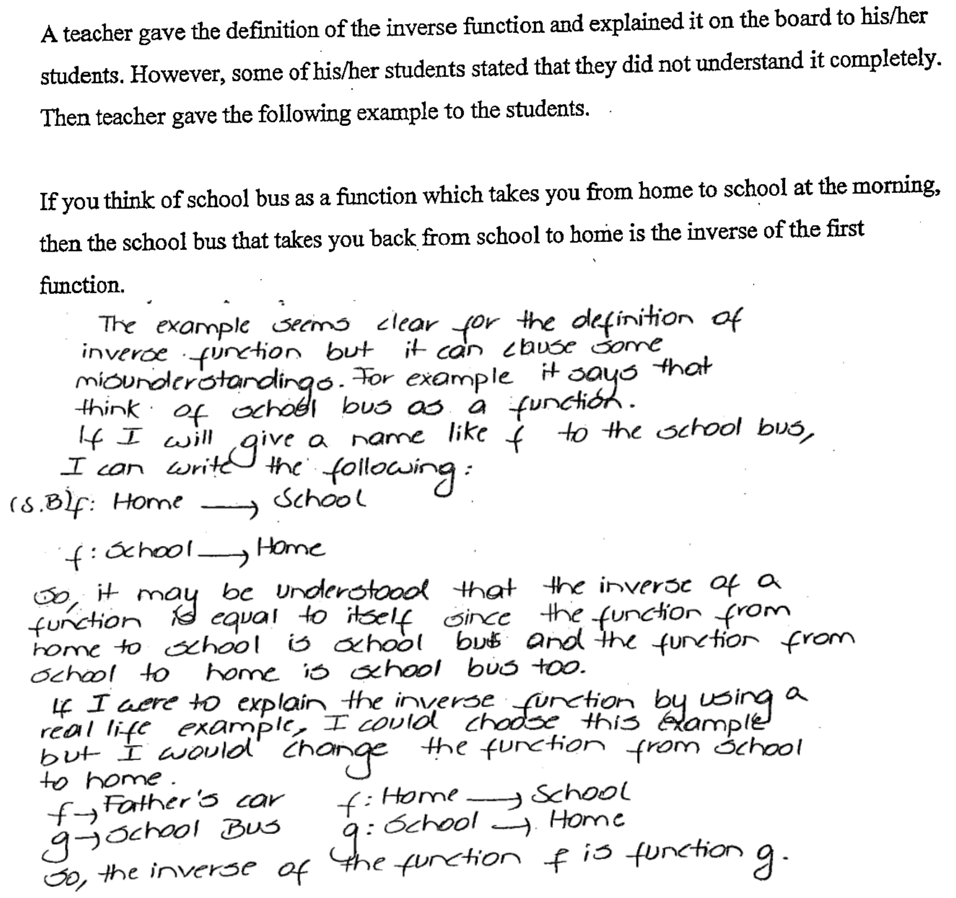
The student is incorrect because multiplication and division are just operations not functions. We say that the inverse of multiplication is division but we do not say it by thinking them as a function.

To clear up student's confusion, I use unit function. I will say that multiplication and division are inverse of each other but they are not functions. Their unit is "1" but it is not equal to unit function which is specified $f(x)=x$.

Moreover, if multiplication and division are functions when I take "0" from the set of multiplication I have to find any value for "0" from the set division. It is not possible.

Figure 4.32: Excerpts from the Yeliz's vignette # 8

The ninth vignette was intended to see to what extend participants resolve the conflict about the usage of analogy for definition of inverse functions. As it can be seen from the Figure 4.33 teacher's example was provided and her ideas about the analogy was asked. She identified the error correctly by saying " … the function from home to school and the function from school to home is school bus too…".


A teacher gave the definition of the inverse function and explained it on the board to his/her students. However, some of his/her students stated that they did not understand it completely. Then teacher gave the following example to the students.

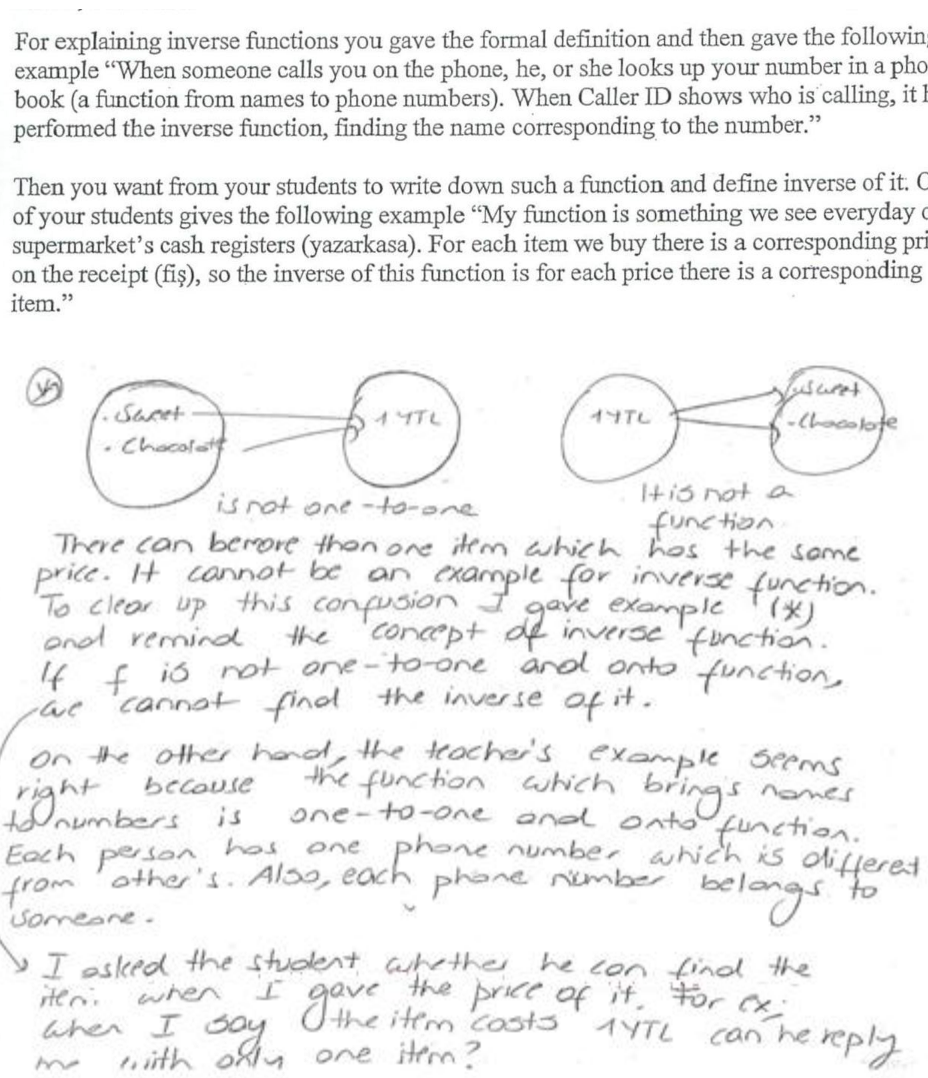
If you think of school bus as a function which takes you from home to school at the morning, then the school bus that takes you back from school to home is the inverse of the first function.

The example seems clear for the definition of inverse function but it can cause some misunderstandings. For example it says that think of school bus as a function.
If I will give a name like f to the school bus, I can write the following:
(S.B) f: Home → School
f: School → Home
So, it may be understood that the inverse of a function is equal to itself since the function from home to school is school bus and the function from school to home is school bus too.
If I were to explain the inverse function by using a real life example, I could choose this example but I would change the function from school to home.
f → Father's car   f: Home → School
g → School Bus   g: School → Home
So, the inverse of the function f is function g.

Figure 4.33: Excerpts from the Yeliz's vignette # 9

Her explanation and diagnose of the teacher's analogy gave us an evidence that she knew the definition of functions, inverse functions and their notations well enough and gave a similar analogy to correct the teacher's. Her SMK level for this vignette was identified as 1.

The twelfth vignette was also related with the use of analogy for the definition of inverse functions but this time there exists two analogies about inverse functions one from the teacher and one from a student (See Figure 4.32). She analyzed both of them and diagnosed the error in the student's analogy correctly, then, she gave an explanation for the existence of inverse function and used an example with Venn diagrams for clarifying the situation. All of these provided an evidence that she knows the definition of inverse function and conditions for existence of inverse functions at Level 1, since there exists no further evidence for connectedness of her knowledge.



For explaining inverse functions you gave the formal definition and then gave the following example "When someone calls you on the phone, he, or she looks up your number in a phone book (a function from names to phone numbers). When Caller ID shows who is calling, it has performed the inverse function, finding the name corresponding to the number."

Then you want from your students to write down such a function and define inverse of it. One of your students gives the following example "My function is something we see everyday on supermarket's cash registers (yazarkasa). For each item we buy there is a corresponding price on the receipt (fiş), so the inverse of this function is for each price there is a corresponding item."

There can bemore than one item which has the same price. It cannot be an example for inverse function. To clear up this confusion I gave example (*) and remind the concept of inverse function. If f is not one-to-one and onto function, we cannot find the inverse of it.

On the other hand, the teacher's example seems right because the function which brings names to numbers is one-to-one and onto function. Each person has one phone number which is different from other's. Also, each phone number belongs to someone.

I asked the student whether he can find the item when I gave the price of it. For ex: when I say "the item costs 1YTL" can he reply me with only one item?

Figure 4.34: Excerpts from the Yeliz's vignette # 12

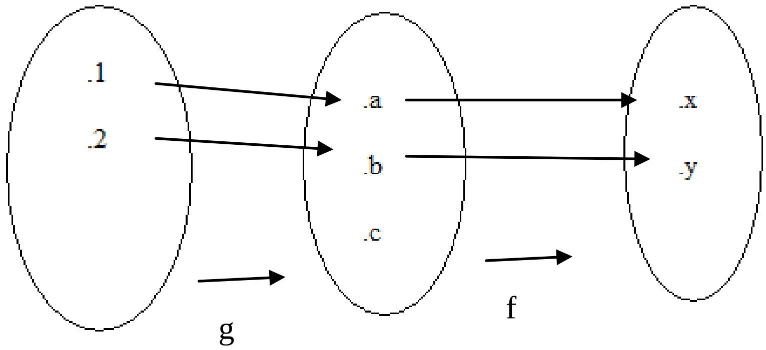
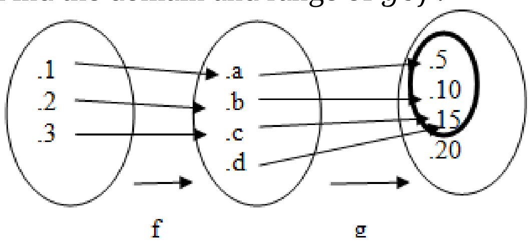
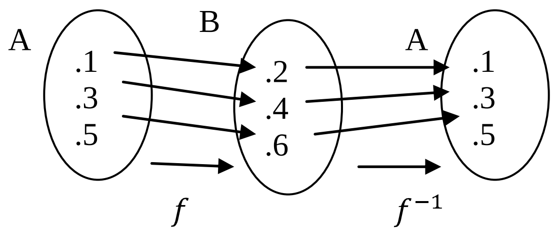
When Yeliz knowledge of inverse functions were evaluated through the vignettes (6, 7, 8, 9, 12, 10, and 11 where last two also include knowledge of composite functions) it was seen that she knows the definition of inverse function, conditions for existence of inverse functions, its notation, the term "inverse" in mathematics good enough to resolve conflicts in different cases. She rarely used different representations and as in the composition of functions she only used Venn diagrams. As stated above for each vignette, her SMK was mostly rated as Level 0-1, similar to her levels in the lesson plans. Also, her SMK was rated once for Level 1-2, and twice for Level 1.

4.1.1.4.4 Teaching Practices

Keeping all these in mind, evidences of SMK were also searched in the teaching practices thorough the examples solved and explanations Yeliz provided to the class or a student.

When the examples solved through the teaching practices were analyzed it was seen that she mostly used the same examples within the lesson plans with the same order. Her additional examples were generally analogies and Venn diagrams for explaining the concepts composite and inverse functions and procedural questions aiming at reviewing the same concept (See Table 4.10).

Table 4.10: Yeliz's Examples from Teaching Practices

Teaching Practice	Example(s)
December 1	• Find fog if exists, give your reasons.  • Mouse eats the cheese, cat eats the mouse, so indirectly cat also eats the cheese". Apart from telling the analogy by using Venn diagrams she showed it on the board.
December 4	• Find the domain and range of gof. 
December 8	• "Suppose everyday you are coming to school with your daddy's car and turn back with the school bus". Here, we can say that school bus does the opposite of the daddy's car so this case can be an example for an inverse function. Apart from telling the analogy by using Venn diagrams she showed it on the board. • Write f and f^{-1} for f:A to B where A={0,1,2,3}, B={0,1,4,9,16} and $f(x) = x^2$ and she showed it in Venn diagrams
December 12	•  $f^{-1}of$ ={1,3,5}=I_A
December 13	• $f(x-1) = 2x + 3$ and $g(2x+1) = 4x + 5$. Find $(gof^{-1})^{-1}(x)$.
December 15	No additional questions

Her explanation to the class or a student was changed according to the representation of the question. If the question was given in mathematical format her explanation was generally an oral review of the procedures just completed. Moreover, even the wording of the explanation was generally the same with the previous one. And also, her explanations in the vignettes and teaching practices were similar to each other. If the question raised at the beginning of a newly introduced concept or a rule for that concept; for the composite functions, she generally referred to an analogy given and also used Venn diagrams; for the inverse functions, she only used Venn diagrams for explanations. This case was similar to her explanations given in the vignettes for composite and inverse functions. Her SMK levels in the teaching practices were rated as Level 0-1, same as her levels in the lesson plans.

4.1.1.5 Summary of the Yeliz's SMK

Evidences of SMK were searched through two groups of instruments. First group (survey of function knowledge, non-routine questions, definitions journals, and concept mapping activity) was only related with assessing the SMK in certain aspects. In the second group (vignettes, lesson plans, and teaching practices), SMK was searched through instruments where there was an integration of knowledge, so they were designed for assessing all components of pedagogical content knowledge.

First group of instruments revealed that Yeliz had some gaps in her understanding of composite and inverse functions specifically in checking conditions of existence and this gap affected her performance on the related items of the instruments. Another difficulty she had was that when the questions were not given in mathematical notation, i.e. in different representations, she couldn't solve it.

In all phases of the assessment she showed very limited evidences for connectedness of her knowledge, this non-connectedness was confirmed during the analysis of concept maps.

Similar to this picture in the first group, through the lesson plans, vignettes, and teaching practices, her emphasis was on procedural questions and how to teach procedures and mathematical notation, not on meaning construction. Even her explanations to conflicts both in the vignettes and the teaching practices were procedural.

Even though she showed lack of knowledge about the conditions of existence in the first group of instruments, from the impressions they got from those instruments through the lesson plans, vignettes, and teaching practice she put emphasis on the conditions for the existence of composite and inverse functions via explanations and examples.

Moreover, her sequencing of the subtopics and questions were analyzed. The results revealed that she sequenced the sub topics in a logical order, however, the sequencing of the questions were not good since they were not following a logical order from easy to difficult.

Apart from these the instruments in the second group were analyzed with respect to the same framework. The analysis revealed that her SMK levels in the vignettes mostly rated as Level 0-1, similarly in all lesson plans and teaching practices her level was rated as 0-1.

4.1.2 Gizem's SMK

In order to understand her SMK further the survey of function knowledge questions were analyzed with respect to similar objectives through declarative, procedural and conditional questions. When Gizem's overall scores through the survey were analyzed it was seen that she was more successful on the procedural questions compared to declarative and conditional ones. In the follow up interview she said that "during the courses I focused on solving as many questions as possible from each subject but I see that we missed the main point, while thinking about the definition questions I felt like I forgot everything". And her scores on declarative and conditional questions were similarly low.

4.1.2.1 Knowledge about the Definitions and the Applications of Definitions

4.1.2.1.1 The Survey of Function Knowledge

The first declarative question in the survey asks for a definition of function. As it can be seen from the Figure 4.35, her answer lacks some properties like what is the difference between the relation and function. So, she got a score of 3.

They are special relations. They have equations. If we put a number to a function, we can get the output.

Figure 4.35: Gizem's answer for the question 1

Related to the definition of function, question 4 asks for whether the given relations are functions and the question was declarative in nature. Like her definition, her reasons for the existence of the functions in this questions are rather weak and inconsistent. Since, she couldn't state the difference between the relation and the function clearly her reasons were also not clear in the question 4 (See Figure 4. 36).

Please state whether or not each of the following is a function and why.

a. The People's Republic of China is the country with the largest population in the world, over 1.1 billion in 1990. Despite the efforts to limit families to one child, the population of China was still growing at a rate of 1.5% per year in 1990. Is there a function? Why or why not?

Not function. Function are decreasing, increasing or constant. If the question would be a function, because of the limits, the rate should be decrease. But it is growing.

b. A rental company charges 32YTL per day (100km free per day) and an additional 0,10 YTL per km. Is there a function? Why or why not?

It is a function. We can compute for example the charge of the 3rd day and 90th kms. ✓

c. A(p,q,r)=3,5p+6q+3r=1500. Is there a function? Why or why not?

Not. The equation is not related to A(p,q,r). I could not define a specific domain for it.

Figure 4.36: Question 4 a, b, & c and Gizem's answers

The question 4 has 6 subitems and she did not get full points on any of them. Each subitem was given in different representation of functions. When the items that she got low grades analyzed it was not seen that regardless of the representation of the function, she experienced problems. Her answer to second question (See Figure 4.37) which was related to listing different representations of functions, was inconsistent with this finding since it can be deduced from the answer that she was not aware of the different representations. This situation was investigated in the follow-up interview, she said that she did not know that these things are called representations. This means that she was aware of the different representations of functions but since she had a vague definition for the definition of the function she could not discriminate between functions and non-functions.

The functions are represented usually like f, g, h, such as $f(x) = \ldots$

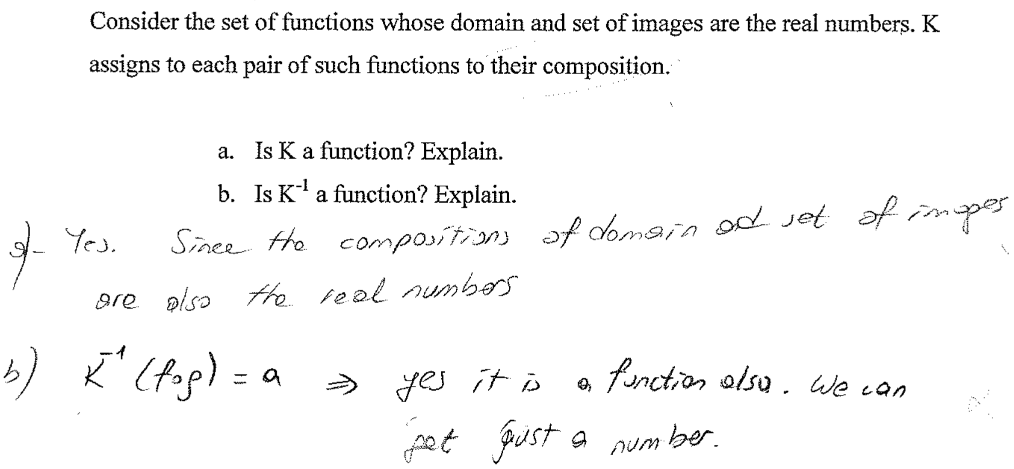
Figure 4.37: Gizem's answer to question 2

When she was asked to define composition of functions in question 3(a), at first her answer seems correct but when it was analyzed it was seen that even though she was aware of the conditions for existence, she experienced problems with the order of the composition of functions and so with its notation (See Figure 4.38). Therefore, her score for this question was rated as 2. During the follow up interview, she stated that in the survey she was shocked about how she could not put into words the things that she knew, so she understood that she must work hard before the teaching practice. This statement belongs not only to a specific question but explains her errors in the definition.

Let $f: A \to B$, $g: B \to C$ then their composition is fog will be $fog: A \to C$.

Figure 4.38: Gizem's answer to the definition of composite function

In the survey question 17 was related to existence of composite functions and conditional in nature. Although her definition of composition of functions includes errors, when she was asked to identify the existence composition of functions in, she successfully did it and got 3 in the first part, because of her weak explanation. In this question the for the function K the domain is the set of functions and range is the set of compositions of those functions. However, as it can be seen from her explanation to part a she seemed to expect functions to be defined on numbers only and so she changed the given information to fit into her own understanding. However, this partial understanding of the function K lead to totally incorrect solution in part b with score of 0 (See Figure 4.39).


Consider the set of functions whose domain and set of images are the real numbers. K assigns to each pair of such functions to their composition.

a. Is K a function? Explain.
b. Is K^{-1} a function? Explain.

a- Yes. Since the compositions of domain and set of images are also the real numbers

b) $K^{-1}(f \circ g) = a$ ⇒ yes it is a function also. We can get just a number.

Figure 4.39: Question 17 and Gizem's answer

In a similar vein, when she was asked to define inverse function she gave a definition which did not reflect any of the conditions for existence of inverse (See Figure 4.40). She got a score of 1 for this question.

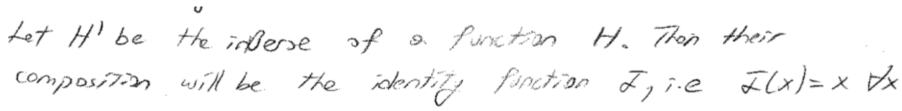
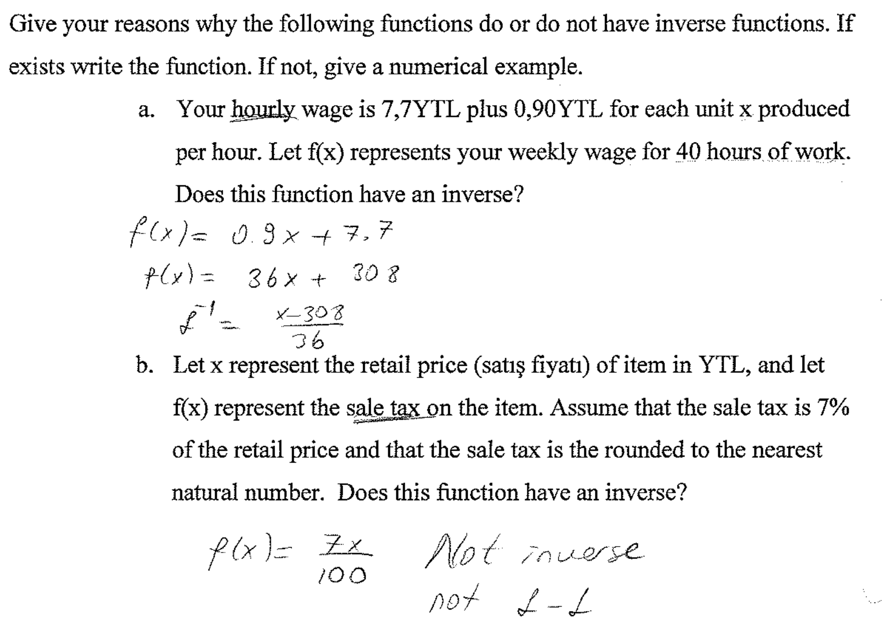

Let H^{1} be the inverse of a function H. Then their composition will be the identity function I, i.e $I(x) = x \ \forall x$

Figure 4.40: Gizem's answer to definition of inverse function

When she was asked to decide the existence of inverse functions in question 12, she answered the question correctly with weak explanations so the question was scored as 3 for both parts (See Figure 4.41). Although in the definition she did not mentioned about being 1-1 as a condition for the existence of inverse functions she stated it during her explanation in part b.



Give your reasons why the following functions do or do not have inverse functions. If exists write the function. If not, give a numerical example.

a. Your hourly wage is 7,7YTL plus 0,90YTL for each unit x produced per hour. Let f(x) represents your weekly wage for 40 hours of work. Does this function have an inverse?

$f(x) = 0.9x + 7,7$

$f(x) = 36x + 308$

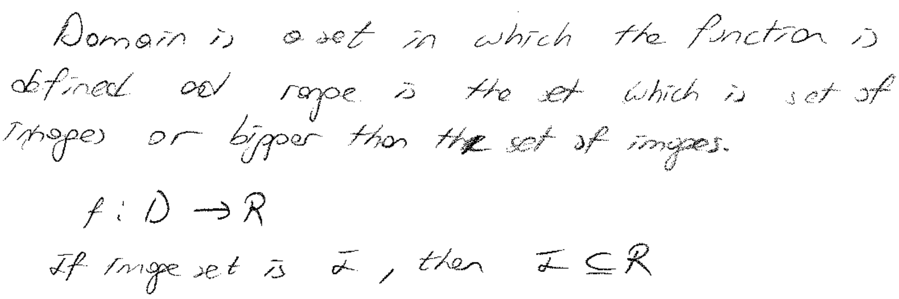
$f^{-1} = \frac{x-308}{36}$

b. Let x represent the retail price (satış fiyatı) of item in YTL, and let f(x) represent the sale tax on the item. Assume that the sale tax is 7% of the retail price and that the sale tax is the rounded to the nearest natural number. Does this function have an inverse?

$f(x) = \frac{7x}{100}$ Not inverse not 1-1

Figure 4.41: Question 12 and Gizem's answer

For the existence of functions, inverse functions, and composite functions, knowledge of domain and range is compulsory for this reason in the survey, question 5 asks for the meaning of domain and range and their importance. In Figure 4.42 her answer was given.



Domain is a set in which the function is defined and range is the set which is set of images or bigger than the set of images.

$f: D \rightarrow R$

If image set is I, then $I \subseteq R$

Figure 4.42: Gizem's answer to question 5

This question was a declarative one with this answer she got 4 points. In line with this question, preceding questions 6 and 7 ask for the domain and range of a given function in mathematical notation and they are procedural questions. As her definitions were complete, her answers to questions about the domain and range were near to correct with minor errors so she scored 3 for both (See Figure 4.43 and 4.44).

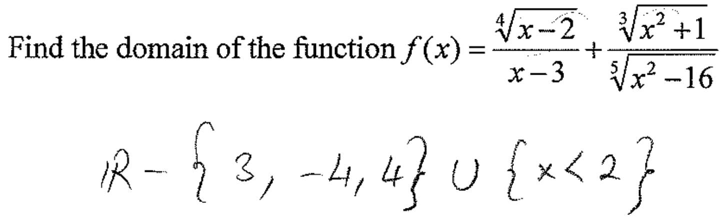


Figure 4.43: Gizem’s answer to question 6

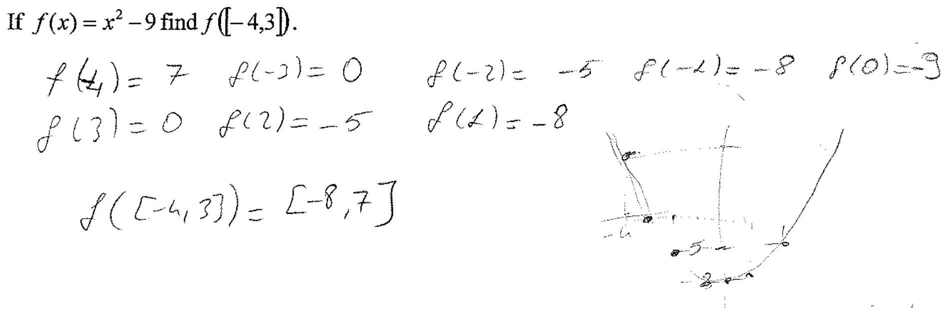


Figure 4.44: Gizem’s answer to question 7

Apart from existence of domain and range, being 1-1 and onto is also required for functions to have an inverse. In the survey these properties were investigated through one procedural (question 10) and one conditional question (question 11) as seen in Figure 4.45 and 4.46 respectively. She got 2 points from the procedural question whereas got 1 from the conditional question.

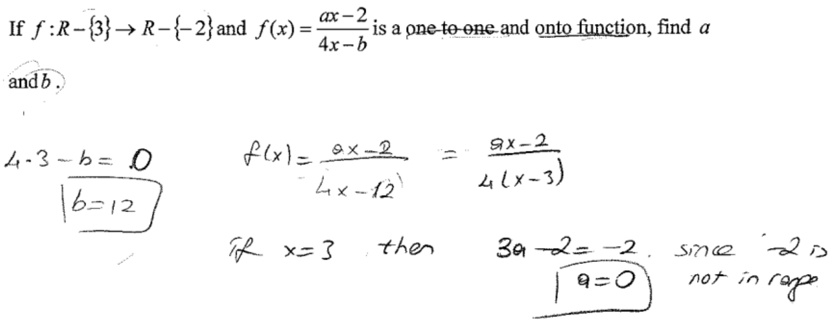
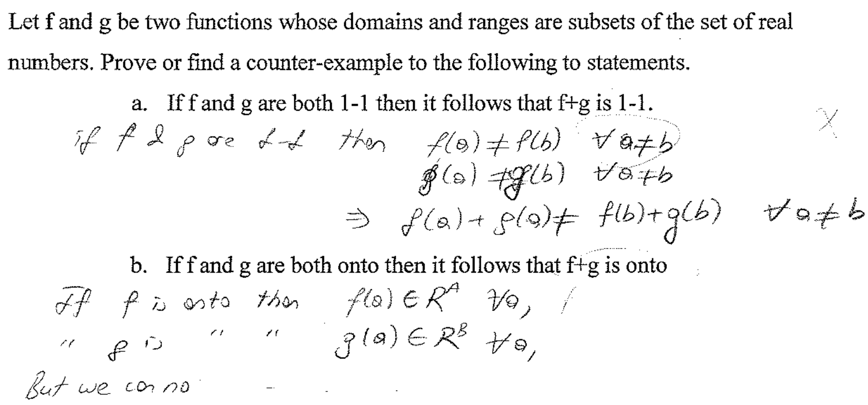


Figure 4.45: Question 10 and Gizem’s answer


Let f and g be two functions whose domains and ranges are subsets of the set of real numbers. Prove or find a counter-example to the following to statements.

a. If f and g are both 1-1 then it follows that f+g is 1-1.

If f & g are 1-1 then $f(a) \neq f(b)\ \forall a \neq b$
$g(a) \neq g(b)\ \forall a \neq b$
$\Rightarrow f(a)+g(a) \neq f(b)+g(b)\ \forall a \neq b$

b. If f and g are both onto then it follows that f+g is onto

If f is onto then $f(a) \in R^A\ \forall a$,
" g is " " $g(a) \in R^B\ \forall a$,
But we canno

Figure 4.46: Question 11 and Gizem’s answer

As a result, it can be concluded that Gizem had difficulty in expressing the definitions of the concepts of functions, composition of functions and inverse functions. The effects of this difficulty was mostly seen on the questions which require a knowledge of definitions of functions, composite and inverse functions. On the other hand, as stated before she performed very well on the questions which were procedural in nature. During the evaluation interview, she said that after the survey of function knowledge I was panicked I felt like I forgot to write definitions and use mathematical notation, and before coming to PBH I worked about the functions unit. Therefore, by this explanation she showed her awareness of her difficulty in expressing the definitions of the concepts functions, composition of functions and inverse functions.

4.1.2.1.2 Responses to Non-Routine Questions

In the non-routine problems interview, there were five questions related with the composition of functions and she got full score for 3 of them. Even though her approach to all the questions were procedural at first, she could not complete the question 1 (a) and 3 (a) (See Table 4.11). In part a of the first question, the problem was she omitted the fact that she must check the conditions for existence of composition of functions while drawing the final composite function. In part b of the same question, it seems that she checked the condition and find the correct answer, so, during the interview she was asked why did not she check the same thing in part a. She said that since the result is a linear function I did not think that there could be a problem but in the second one there is a square root at the resulting function and I should give some values in order to draw it, so I realized that there is a problem, so I checked some more points, my intent was not to check the conditions I was just trying to draw the graph. By this explanation, she also confessed that she was not checking the conditions for existence of composite functions while taking the composition. In the third and fourth questions, functions were given as graphs their composition was asked. In third question, she again attempted to solve the question procedurally, and write the equation for the linear function but since she could not write the second function she could not move any further in part a. However, when the order of the composition changed she easily move to the second step and realized the upwards shift. In the fourth question, since she could not write any of the functions in mathematical notation she did not attempt any solutions.

When the Gizem's answers to inverse function questions (See Table 4.12) in the non-routine questions interview analyzed it was seen that she easily solved the question 5 which was given in mathematical notation. This case is similar to that of the survey of function knowledge. Because, in there even though her definition did not include necessary elements for the existence of inverse functions she correctly identified the existence of inverse functions. In the sixth question, again the inverse of functions were asked however this time questions were given as graphs. Gizem attempted the questions procedurally and tried to write given functions as piecewisely defines functions. She did it for part a but experienced difficulty in writing the inverse function, like in composition of functions since the functions were linear she did not feel a reason to check the conditions for existence. However, in part b since some part of the question was quadratic she said that this function does not have an inverse because the middle part is like a quadratic so same image will go to two different numbers.

From the evidences we got from survey of function knowledge and non-routine questions interview, it can be concluded that she experienced problems in expressing definitions for functions, composite functions and inverse functions. Her answers to questions about the composite and inverse functions revealed that she knows the conditions for existence but she applied these rules if she feels a problem (like root in the composite functions or quadratic in inverse function) after the procedural steps. She always tried to convert graphical questions to mathematical notation, and otherwise she could not solve it. During the evaluation interview she admitted that she saw herself attempting to every question procedurally regardless of the question type. This statement approved the findings from the survey and the non-routine questions interview. Moreover, she mentioned the effect of the non-routine questions by stating I think we should see these kinds of questions for every topic in the school curriculum because these questions affected me in the positive way and made me think about the concept and how should I teach it.

Table 4.11: Composition of functions questions in the non-routine interview, Gizem's answers and scores

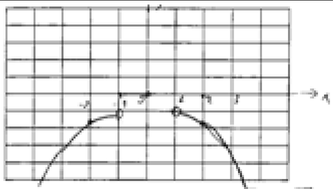
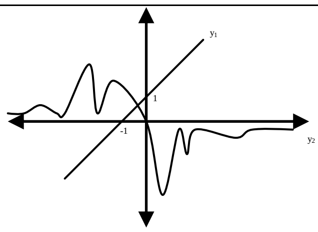
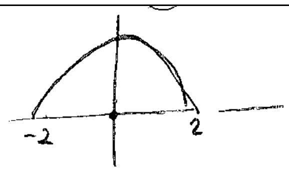
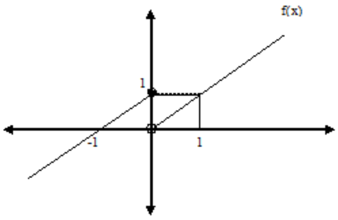
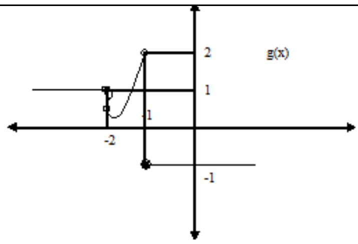
		Questions	Answers	Scores
Composition	1 a	$f(x) = x^2 - 2$ and $g(x) = -\sqrt{x+1}$ answer each of the following. (a) Determine (fog)(x) in simplified form and sketch a graph of this new function	$f \circ g(x) = f(g(x)) = (-\sqrt{x+1})^2 - 2$ $= x+1-2 = x-1$ $(f \circ g)(x) = x-1$ She draw the line for all real numbers	1
	1 b	(b) Determine (gof)(x) in simplified form and sketch a graph of this new function.	 $g \circ f(x) = g(x^2-2) = -\sqrt{x^2-2+1} = -\sqrt{x^2-1}$ $h(x) = -\sqrt{x^2-1}$ $y = -\sqrt{x^2-1}$	4
	2	$f(x) = \sqrt{4-x^2}$ and $g(x) = \sqrt{x^2-9}$ Determine (gof)(x) in simplified form.	$g \circ f$ $f: A \to B$ $g: B \to C$ $4-x^2 \geq 0$ $x^2 \leq 4$ $[-2,2] = A$ $[0,2] = B$ $g(f(x)) = \sqrt{4-x^2-9} = \sqrt{-x^2-5}$ < 0 She stated that the inverse does not exist since the inside is always negative. Also, the range of f B is not in the domain of g	4
	3 a	 (a) Use the given graphs to sketch $y_2 o y_1$.	$y_1 = x+1$ $y_2(x+1)$	1
	3	(b) Use the given graphs to sketch $y_1 o y_2$.	$y_1 = x+1$ $y_1 \circ y_2 = y_2 + 1$ After writing this she stated that the graph will move one unit upwards	4
	4		No answer is attempted even after the hint	0

Table 4.12: Inverse functions questions in the non-routine interview, Gizem's answers and scores

		Questions	Answers	Scores
Inverse	5 a	Find, the inverse of the following functions, if exists. a. f(x)=4, x∈R	Just said in the interview that the inverse does not exist	4
	5 b	b. $f(x)=\sqrt{4-x^2}$	 She draw the graph and stated that since it is not 1-1 we cannot find its inverse	4
	6 a	Use the given graphs to sketch the inverse of given functions. 	$f(x)=\begin{cases} x+1 & x<0 \\ x & x\geq 0 \end{cases}$ $\begin{cases} x-1 & x<1 \\ \ast & x\geq \end{cases}$	1
	6 b		$g(x)=\begin{cases} 1 & x\leq -2 \\ f(x) & -2<x<-1 \\ -1 & \end{cases}$ After writing the function, she directly said that this function does not have an inverse because the middle part is like a quadratic so same image will go to two different numbers	4

4.1.2.1.3 The Analysis of the Definitions Used through the Instruments

Because of her weak definitions in the survey, her definitions were compared through the instruments the survey of function knowledge, the journals about the definitions, the lesson plans, and teaching practices were analyzed.

Definitions she used for composition of functions were given in Table 4.13. When they were analyzed it was seen that she used formal definitions in all the instruments. However, it was seen that when she wrote the definition like in the survey and her definition in the journal she mixed the order of the composition and instead of $(gof)(x)$ she wrote $(fog)(x)$.

Moreover, even though not exists in the lesson plans she used an analogy to start the composition of functions (See Table 4.14) during the teaching. Also in order to foster the understanding of the conditions for existence of composition of functions she made additions for the existing questions in the lesson plan (See Table 4.14).

Table 4.13: Gizem's definition of composite functions used through the instruments

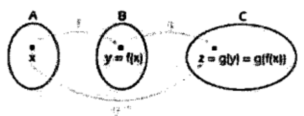
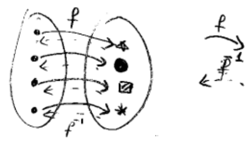
Instruments	Definition
Survey of Function Knowledge	Let $f: A \to B$, $g: B \to C$ then their composition as fog will be $fog: A \to C$.
Journal about the Composite Function Definitions (Her choice among the given list)	Let f: A→B and g: B→C be two functions. Then gof: A→C defined as $(gof)(x) = g(f(x))$ is called composition of f and g.
Journal about the Composite Function Definitions (Her definition if she would teach)	Let $f: A \to B$, $g: B \to C$ be functions. Then the function $h: A \to C$ s.t $h = fog$ is the composition of f and g and it is defined by $h(x) = (fog)(x) = f(g(x))$, $\forall x \in A$
Lesson Plan	**TANIM:** f:A→B, g: B→C olsun. gof: A→C, ***gof(x)=g(f(x))*** şeklinde tanımlanan gof fonksiyonuna ***g ile f nin bileşke fonsiyonu*** denir. 
Teaching Practice	Before giving the definition she used the analogy of washing machine and drier to work simultaneously She used the same definition with the lesson plan

Table 4.14 : Gizem’s examples of composite functions in the teaching practices

Example	Explanations
Örnek 1: A = {-1,0,2} , B = {0,1,4} ve C ={-1,0,1,2,3} kümeleri veriliyor. f:A→B , f(x) = x^2 ve g : B→C , g(x) = x-1 ise gof kümesini bulalım.	Provided in the lesson plan and she also asked $(gof)(3)$ asked for existence
Örnek 4: A = {-2, -1, 0, 1}, B = {1, 2, 5} ve C = {2, 3, 6, 8} kümeleri ve f: A → B, f(x) = x^2 + 1 ve g : B → C, g(x) = x + 1 fonksiyonları veriliyor. A kümesinin (gof) bileşke fonksiyonu altındaki görüntüsünü bulunuz.	Provided in the lesson plan and she also asked $(gof)(4)$ asked for existence

Except for the survey of function knowledge, (See Table 4.15) she used formal definitions for definition of inverse functions. Apart from being informal, her definition in the survey of function knowledge lacks the conditions for existence. However, in the teaching practices she started the concept with an analogy to support understanding of the concept of inverse function. Moreover, she used several functions given in Venn diagrams in order to explain and foster the understanding of the conditions for existence of inverse functions. These examples were not given in the lesson plans.

Table 4.15: Gizem’s definition of inverse functions used through the instruments

Instruments	Definition
Survey of Function Knowledge	Let H' be the inverse of a function H. Then their composition will be the identity function I, i.e $I(x)=x$ $\forall x$
Journal about the Inverse Function Definitions (Her choice among the given list)	If f: A→B is one-to-one and onto function then there exists the inverse of f denoted by f^{-1} such that f^{-1}: B→A, $f(x) = y$, and $f^{-1}(y) = x$.
Journal about the Inverse Function Definitions (Her definition if she would teach)	Let $f: A\to B$ be one-to-one and onto function. The inverse of f (denoted by f^{-1}) is a function such that $f^{-1}: B\to A$ and $\forall y\in B, \exists x\in A$ $f^{-1}(y)=x$. 
Lesson Plan	**TANIM:** f: A→B, ve f = {(x, y) | x ∈ A ve y ∈ B} fonksiyonu birebir ve örten ise, ***f^{-1} : B → A ve f^{-1} = {(y, x) | y ∈ B ve x ∈A} fonksiyonu f nin ters fonksiyonudur.*** ***(x,y) ∈f ⇔ (y,x) ∈f^{-1} olduğu için*** ***y=f(x) ⇔ x=f^{-1}(y) olur.***
Teaching Practice	She started the concept of inverse functions by using analogy of zoom-in and zoom-out from computers and then she used the same definition with the lesson plan

4.1.2.2 Applications of the Rules about Composite and Inverse Functions

The rest of the questions not mentioned up to now are one conditional (question 18), one declarative (question 8), and six procedural (questions 7, 13, 14, 15, 16, and 19) questions in the survey. When the questions and their objectives were analyzed it was seen that all of these questions were related with the application of the concepts discussed above. It was also seen from the Table 4.2 that she got scores 4 or 3 from these questions and she lose points only because of non-clarity of the answer. Therefore, it can be concluded that she did not experienced any problems while applying the rules about the concepts even if she had problems about the definitions.

4.1.2.3 Connectedness of Gizem's Knowledge of Composite and Inverse Functions

Previous results led to the fact that Gizem showed limited evidence for the connectedness of her knowledge of composite and inverse functions. For this reason her concept maps were analyzed. Participants were asked to prepare to concept maps about functions. In the first one, it was free to choose the terms that will be used in the concept map, whereas in the second, the terms were provided but also they are free to use the terms that they prefer. After that they wrote an essay about the concept maps they prepared and lastly focus group interview was conducted in order to share the participants' views about their concept maps, each others concept maps and concept mapping. Concepts maps were analyzed in terms of organization and accuracy whereas concept map essays were analyzed in terms of communication, organization and mechanics (Bolte, 1999).

First concept map of Gizem (See Figure 4.47) was rated as 3 (fair) out of 6 because she constructed some meaningful clusters but unable to connect all subunits by appropriate cross-links like composition of a functions are kinds of operations of functions. Some terms were missing in the concept map like identity function, independent and dependent variables. The accuracy score was rated as fluent since she wrote all one-to-one functions are onto, into and symmetric.

In the second concept map (See Figure 4.48), she lost her meaningful clusters, instead, she had only subunits approaching to functions. Although there were some meaningful links and linking words, the concept map still lacked the cross-links necessary and also had meaningless linking words since it did not have meaningful clusters. Therefore, second map was rated as 2 (weak) out of 6 for its organization. The accuracy score of the second concept map was rated as 2 out of 4 since there were some errors like vertical line test was taken as a test for one-to-oneness of a function, and functions were separated into two kinds dependent and independent.

Throughout the concept map essay, Gizem also talked about her process of constructing concept maps but she admitted that she had some difficulties to remember the terms and definitions related to the function concept in the first concept map. She mentioned this as follows: "…I tried to write what I remembered. I had some difficulties since I couldn't remember the terms or definitions exactly.

Therefore, I had difficulty to construct the links between the terms and to write complete sentences between them". Similarly, she had a different view for the second concept map as follows: "I was feeling myself more comfortable since all the important terms were in front of me. I could make some relations with another concept when I saw the terms for example when I saw the term image I remembered much more things. I believe that the terms were very good clues for me although they are given mixed". In the interview she also mentioned the same things. Then, she compared the construction of two concept maps in the interview and in the essay that her second map was more detailed than the first one in terms of number of terms used, consistency of links, and meaningfulness of the linking words. Apart from these, during the interview she criticized her concept maps as being useless. There were mistakes in her concept maps and when her mistakes about the usage of vertical line test were addressed, interestingly Gizem first denied the error and then had a difficulty to understand the case. Overall evaluation of the three-staged concept map activity revealed that Gizem was first unable to find all related terms and generally unable to construct meaningful subtopics (clusters) and connect the related subunits with meaningful linking words.

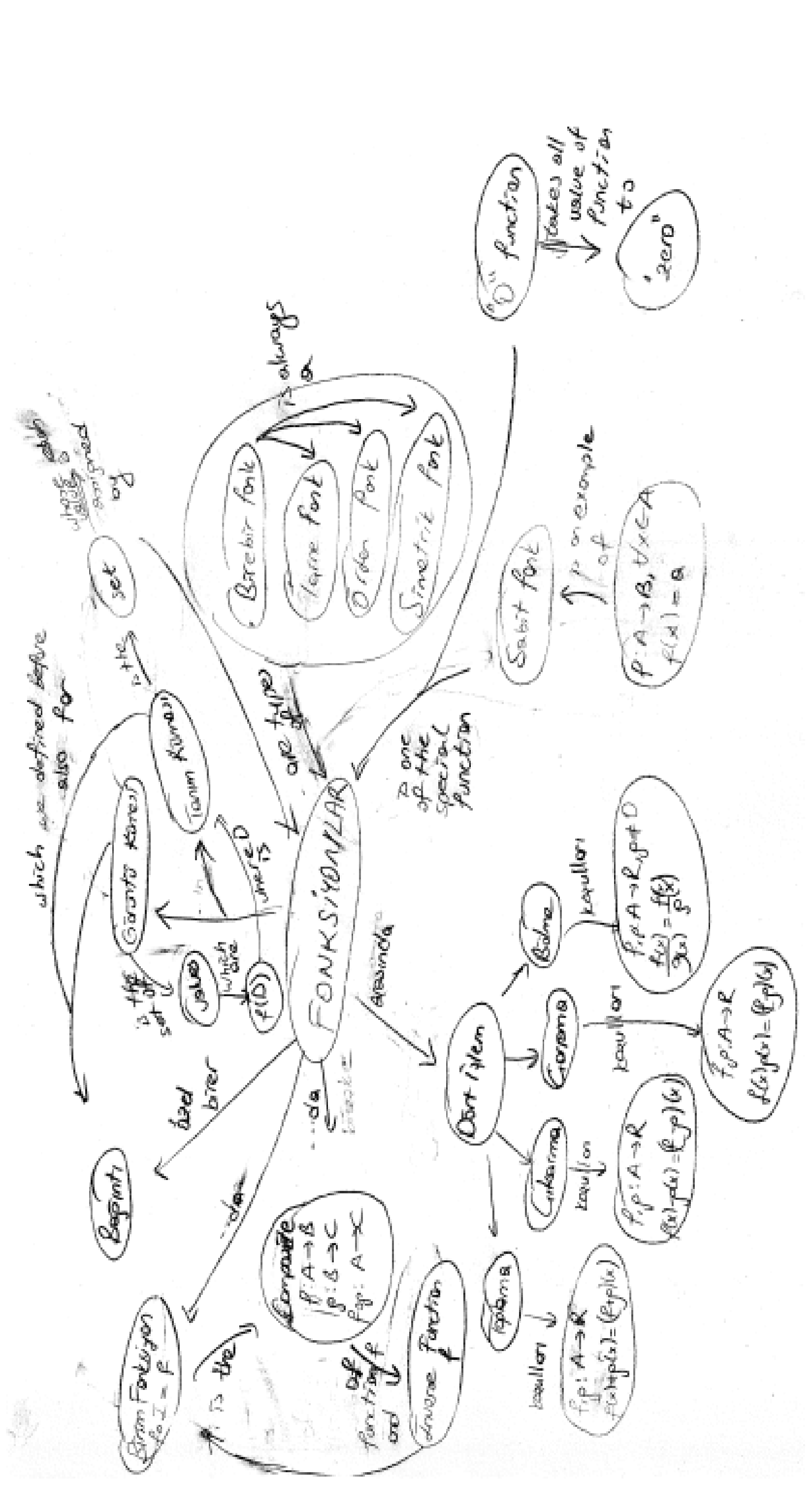


Figure 4.47: Gizem's first concept map

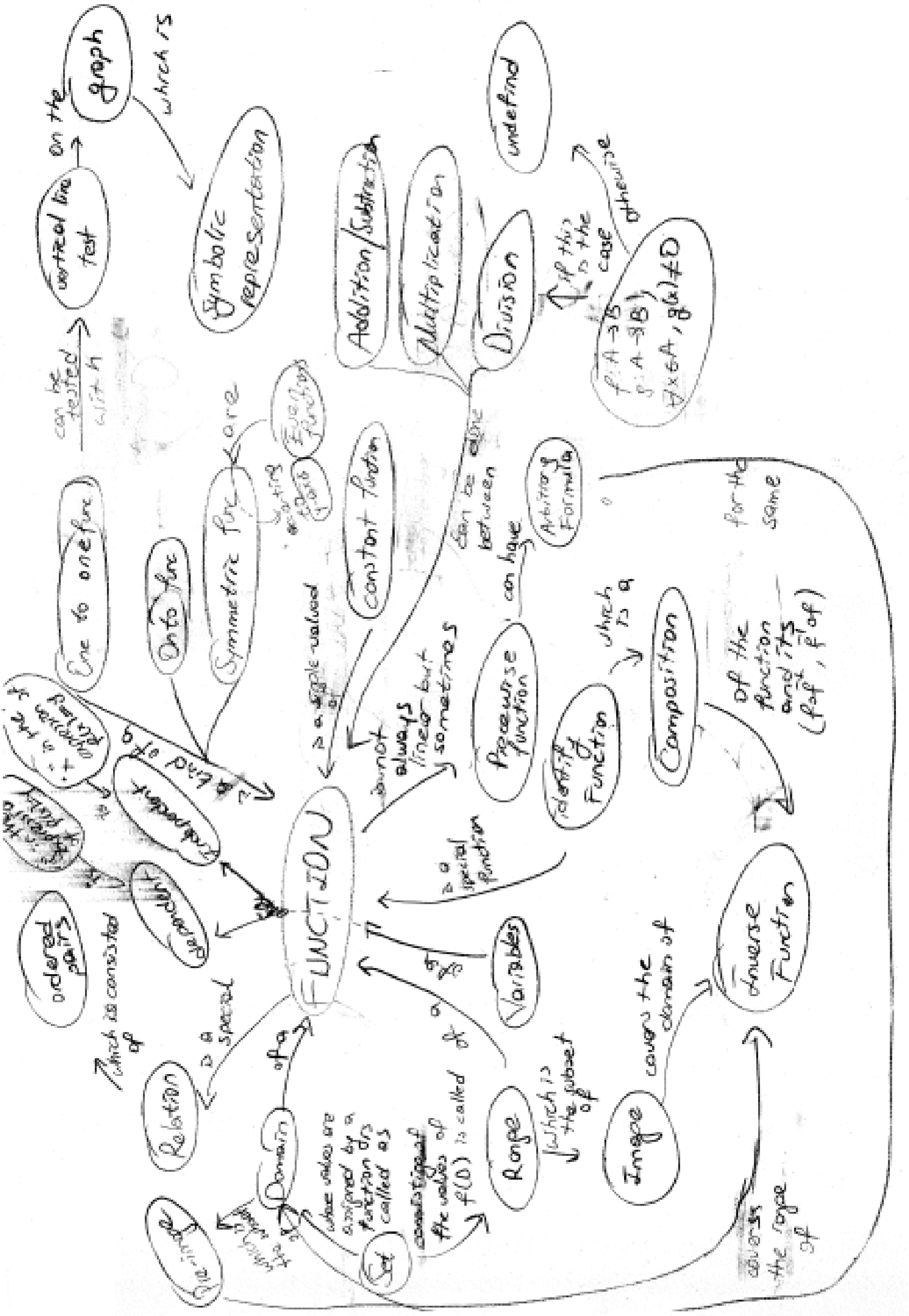


Figure 4.48: Gizem’s second concept map

For examining the connectedness of SMK in the teaching practices sequencing of the subtopics and examples were analyzed. Her sequencing of the subtopics and examples were accurate similar to her lesson plans. Moreover, she added some questions, analogies to improve the understanding which was also an evidence for connectedness of knowledge. Besides, she made connections to previous mathematical topics. For example, while teaching commutative, associative properties of composite functions and identity element of composition of functions , she referred to properties of addition and multiplication operations. Moreover, while teaching inverse functions she referred to inverse of relations.

4.1.2.4 Evidences of SMK through the Instruments Having Integration of Knowledge Components

Evidences of SMK were also searched through the instruments where there is an integration of all knowledge components exists. These instruments were vignettes, lesson plans, and teaching practices all of which assessed through the combined framework. Since Yeliz said that she was influenced from the non-routine questions interview, while analyzing the instruments it was kept in mind. This is, because, these three instruments were collected after the administration of the non-routine questions interview (See Table 3.2).

4.1.2.4.1 Lesson Plans

In the lesson plans, participants were asked to teach composite and inverse functions but they were not specifically given an order which one to teach first. She started with composition of functions and her reason was as follows "since finding an inverse of a function in under the composition of functions I prefer to teach composition first". As stated before, she used formal definitions and Venn diagrams both for the composite and inverse function definitions in the lesson plans. She mostly used declarative and procedural questions in the lesson plans. The only conditional type question was used in the first lesson plan. The Table 4.16 summarizes representative sample of the example types used in that lesson; that is, if in a lesson only procedural questions were used only a procedural example was provided and if there were more than one type of example were used one example for each type was provided. By considering the evidences provided above and Gizem's SMK performances on each lesson plan it was seen that she used different representations (Venn diagrams, graphs of functions, functions given in listing method) and represented some evidence for her connectedness of knowledge by representing the questions in a mathematical hierarchy. Therefore, she got Level 1 from all lesson plans, except for the last one in which she got Level 0-1.

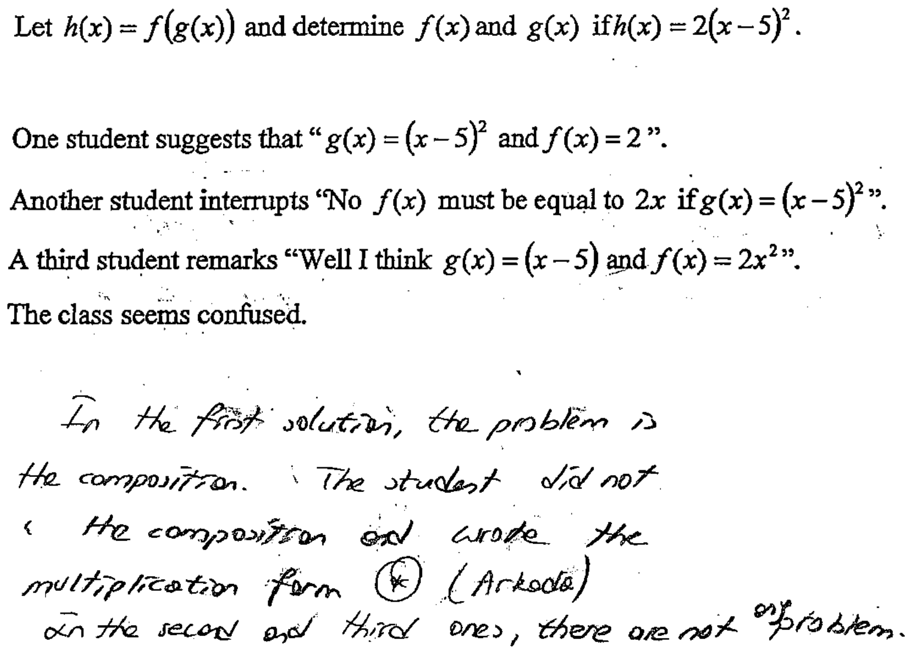
Table 4.16: Question Excerpts from Gizem's Lesson Plans

Lesson Plan	Questions	Knowledge Type
1	**Örnek:** f= {(-1,1), (0,0), (1,1), (2, 4)} ve g = { (1, 3), (0,1), (4, 9)} fonksiyonları verilsin, bu iki fonksiyonun tanım ve değer kümeleri nedir?	Declarative
	Örnek 1: A = {-1,0,2} , B = {0,1,4} ve C ={-1,0,1,2,3} kümeleri veriliyor. $f:A\rightarrow B$, $f(x) = x^2$ ve $g: B\rightarrow C$, g(x) = x-1 ise gof kümesini bulalım.	Procedural
	Örnek 4: $f(x)=x^2$ ve $g(x)=\sqrt{x-1}$ fonksiyonları için fog fonksiyonunu kuralını ve tanım kümesini bulunuz.	Conditional
2	**Örnek:** Reel sayılar kümesinde, f(x) = 2x + 1, g(x) = 3x-2 ve h (x) = 4x - 1 fonksiyonları veriliyor. Her $x \in R$, (fo(goh)) (x) = ((fog)oh) (x) olduğunu gösteriniz.	Procedural
3	**4.** (fof) (x) = 9x + 4 ise f(x) fonksiyonunu bulunuz.	Procedural
	Örnek: A = {-2, 0, 1, 2} kümesinden B = {-1, 1, 2, 3} kümesine tanımlanan f (x) = x + 1 fonksiyonunu şu şekilde yazalım: f: $A\rightarrow B$, f(x) = x + 1 ise f = {(-2, -1), (0, 1), (1, 2), (2,3)} olur. Bu fonksiyonun elemanlarının birinci ve ikinci bileşenlerinin yerlerini değiştirdiğimizde f nin tersi olan f^{-1} fonksiyonunu(f fonksiyonunun tersi) elde ederiz. Bu ters fonksiyon B kümesinden A kümesine tanımlanır. $f^{-1}: B\rightarrow A$, f^{-1} = {(-1,-2), (1,0), (2, 1), (3, 2)}	Declarative
4	**Örnek 1:** A = { a,b,c } kümesinden B = { 1,2,3 } ne tanımlı f ={(a,2) , (b,3), (c,1)} fonksiyonunun şemasını çizip ters fonksiyonunu bulalım.	Declarative
	Örnek 5: f: R-{1} $\rightarrow$ R-{2} ve g: R-{-1} $\rightarrow$ R-{-2} , olmak üzere, $f(x)=\frac{2x-3}{x-1}$ ve $g(x)=\frac{5-2x}{x+1}$ ise $f^{-1}(x)$, $g^{-1}(x)$ ve $(fog)^{-1}(x)$ fonksiyonlarını bulunuz.	Procedural
5	**Örnek 6:** R den R ye (fog) (x) = 5x - 2 ve f (x) = 4x - 3 fonksiyonları veriliyor, g ve g^{-1} fonksiyonlarını bulunuz.	Procedural
6	2. $f(x) =\frac{2x+u}{x+1}$ ve $(fof)(x)=\frac{u}{x-1}$ olduğuna göre, u'nun değerini bulunuz.	Procedural

4.1.2.4.2 Vignettes Related with the Composite Functions

In line with the previous discussion Gizem's vignettes were analyzed for evidences of SMK and those evidences were categorized according to the combined framework.

Firstly, the vignettes only related with the composition of functions were analyzed. The first vignette was intended to see to what extend participants resolve the conflict about the misunderstanding of the notation $h(x) = f(g(x))$ and mixing it with the ordinary multiplication like $f(x).g(x)$. Gizem grasped the conflict given in the vignette correctly which requires from her to know the definition and notation of composition of functions (See Figure 4.49). This was taken as an evidence for SMK.

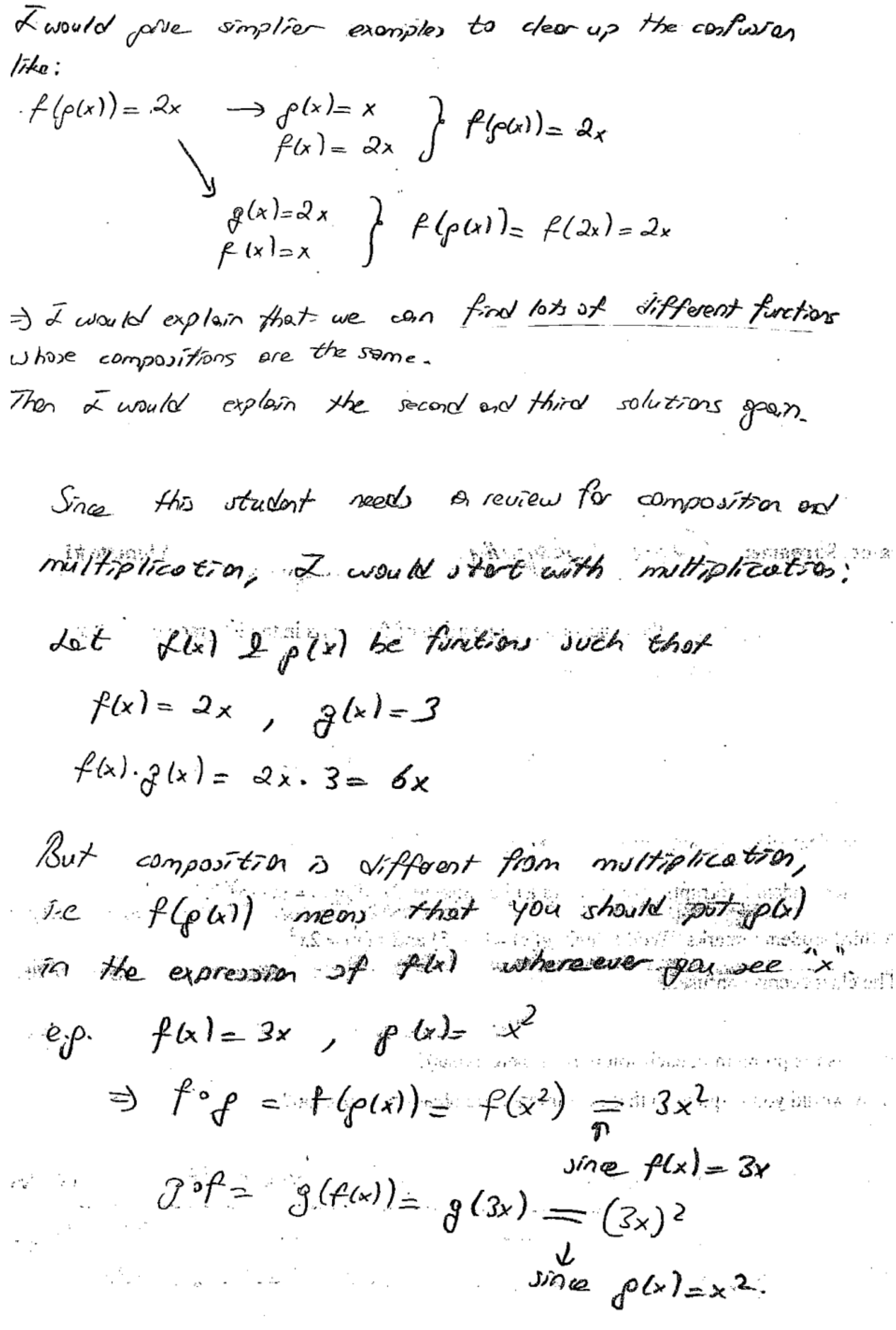

Let $h(x) = f(g(x))$ and determine $f(x)$ and $g(x)$ if $h(x) = 2(x-5)^2$.

One student suggests that "$g(x) = (x-5)^2$ and $f(x) = 2$".
Another student interrupts "No $f(x)$ must be equal to $2x$ if $g(x) = (x-5)^2$".
A third student remarks "Well I think $g(x) = (x-5)$ and $f(x) = 2x^2$".
The class seems confused.

In the first solution, the problem is the composition. The student did not (the composition and wrote the multiplication form (*) (Arkada)
In the second and third ones, there are not any problem.

Figure 4.49: Excerpt from Gizem's Vignette #1

For clearing up the confusion in the class she used simpler procedural questions (See Figure 4.50) so that students can imagine the case easily. As a result, her SMK for the first vignette was rated as Level 1.



I would give simpler examples to clear up the confusion like:

$f(g(x)) = 2x \rightarrow g(x) = x$, $f(x) = 2x$ } $f(g(x)) = 2x$

$g(x) = 2x$, $f(x) = x$ } $f(g(x)) = f(2x) = 2x$

⇒ I would explain that we can find lots of different functions whose compositions are the same.

Then I would explain the second and third solutions again.

Since this student needs a review for composition and multiplication, I would start with multiplication:

Let $f(x)$ & $g(x)$ be functions such that

$f(x) = 2x$, $g(x) = 3$

$f(x) \cdot g(x) = 2x \cdot 3 = 6x$

But composition is different from multiplication, i.e. $f(g(x))$ means that you should put $g(x)$ in the expression of $f(x)$ wherever you see "x".

e.g. $f(x) = 3x$, $g(x) = x^2$

$\Rightarrow f \circ g = f(g(x)) = f(x^2) = 3x^2$ since $f(x) = 3x$

$g \circ f = g(f(x)) = g(3x) = (3x)^2$ since $g(x) = x^2$.

Figure 4.50: Excerpt from Gizem’s Vignette #1

In her explanation she used an informal explanation of the definition of the composition and constant functions in order to make the distinction between the composition and multiplication.

The second vignette was intended to see to what extend participants resolve the conflict about the mixing order of operations when taking compositions of functions and mixing it with the ordinary multiplication $f(x).g(x)$. As seen in Figure 4.51 she correctly identified the students' misunderstandings which requires a knowledge of definition and notation of composition of functions, which provided the researcher with the evidence of SMK.

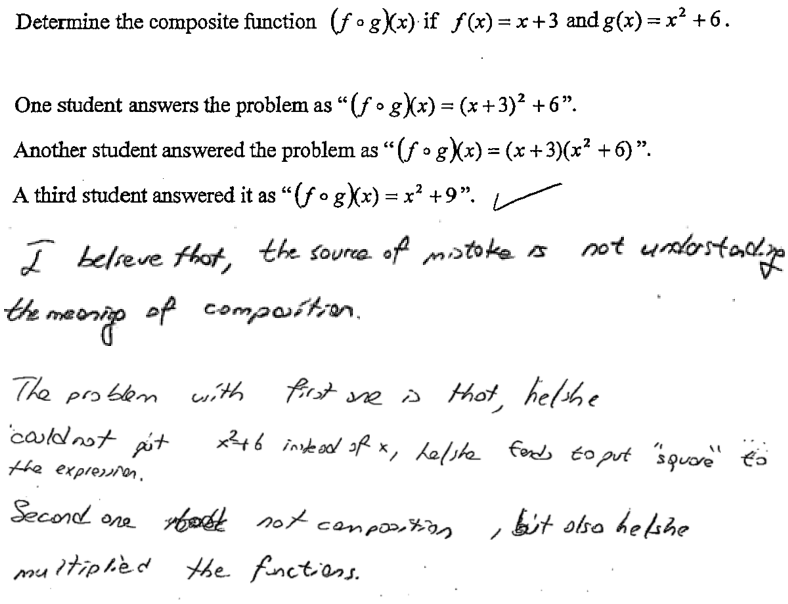


Figure 4.51: Excerpt from Gizem's Vignette #2

In her explanation (See Figure 4.52) she used a machine analogy in order to show how the order of composition of functions is important and changes the result. From these evidences her SMK was rated as Level 1.

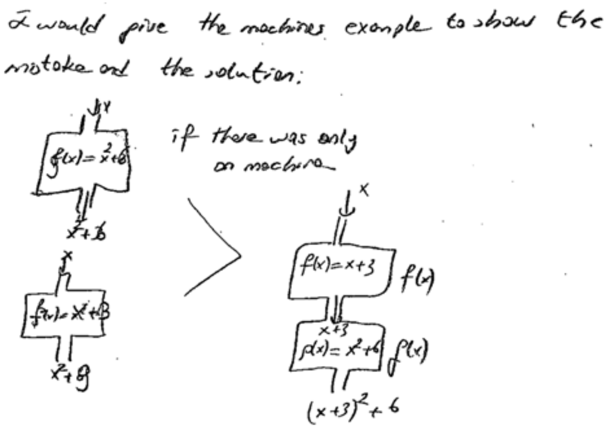


Figure 4.52: Excerpts from Gizem's vignette #2

The third vignette was intended to see to what extend participants resolve the conflict about the mixing composition with the ordinary multiplication when one of the functions is a constant function. Like in the previous two vignettes, she correctly identified the mistakes student made (See Figure 4.53) which gave the researcher evidence of knowledge about the definition of composition of functions and use of notation.

A student asked the following question.

Let f(x)=4, g(x)=2, and h(x)=x+3. Evaluate the followings

a. (fog)(7) = f(g(7)) = f(2) = 4
b. (goh)(x) = g(h(x)) = g(x+3) = 2
c. (hof)(x) = h(f(x)) = h(4) = 7
d. (hof)(3) = h(f(3)) = h(4) = 7

Student's answer is the following:

a. f(x)=4 and g(x)=2 then (fog)=(4.2)=8 (fog)(7)=56
b. (goh)(x)=2x+3
c. (hof)(x)=7
d. (hof)(5)=32

The main problem is that the student makes multiplication if he/she any number and the composition.
Sometimes, he/she could take composition, but only if he was studying with variable x.
For example, in (c) he wrote right solution ;
h(f(x)) = h(4) = 4+3 = 7.

Figure 4.53: Excerpts from Gizem's vignette #3

Her explanation includes procedural examples which provides examples and counter-examples of the composition with constant functions (See Figure 4.54). Also, an informal explanation about the constant functions were provided. In light of these evidences, her SMK was rated as Level 1.

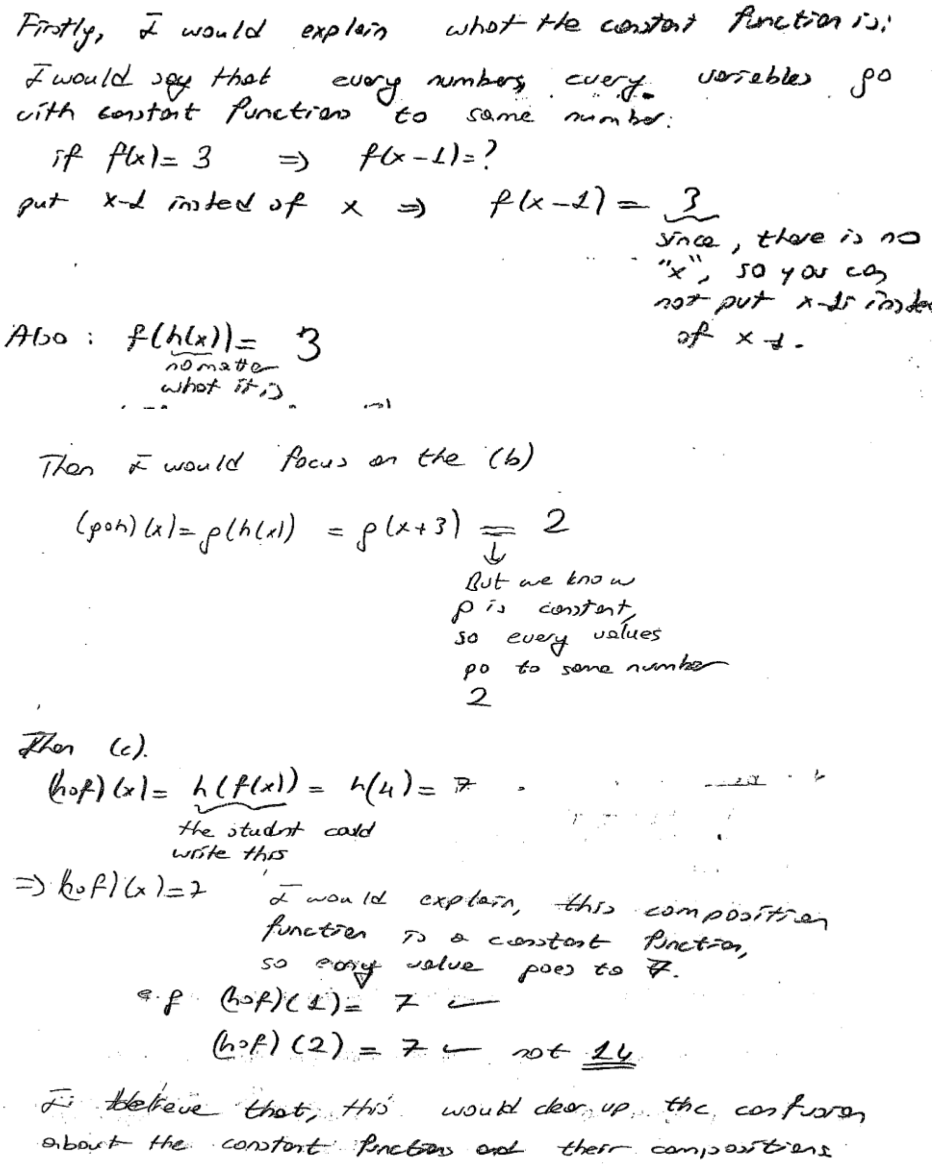
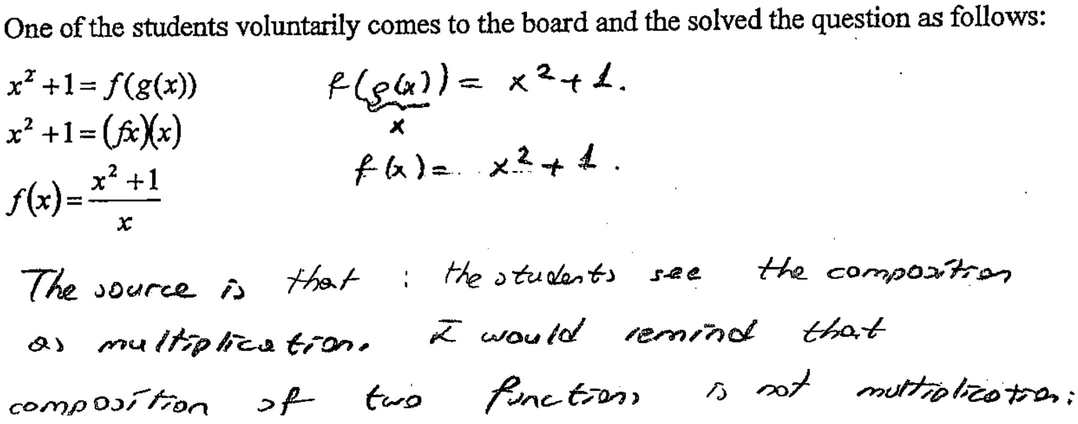


Figure 4.54: Excerpts from Gizem's vignette #3

The fourth vignette was intended to see to what extend participants resolve the conflict about misunderstanding of the notation $h(x) = (fog)(x)$ while working backwards in composition of function problems. Gizem identified the source of the mistake just as taking composition as multiplication (See Figure 4.55). However, in the answer there is more than that student had difficulty understanding the meaning of (x) in the representation $(fog)(x)$.


One of the students voluntarily comes to the board and the solved the question as follows:

$x^2+1=f(g(x))$

$x^2+1=(fx)(x)$

$f(x)=\frac{x^2+1}{x}$

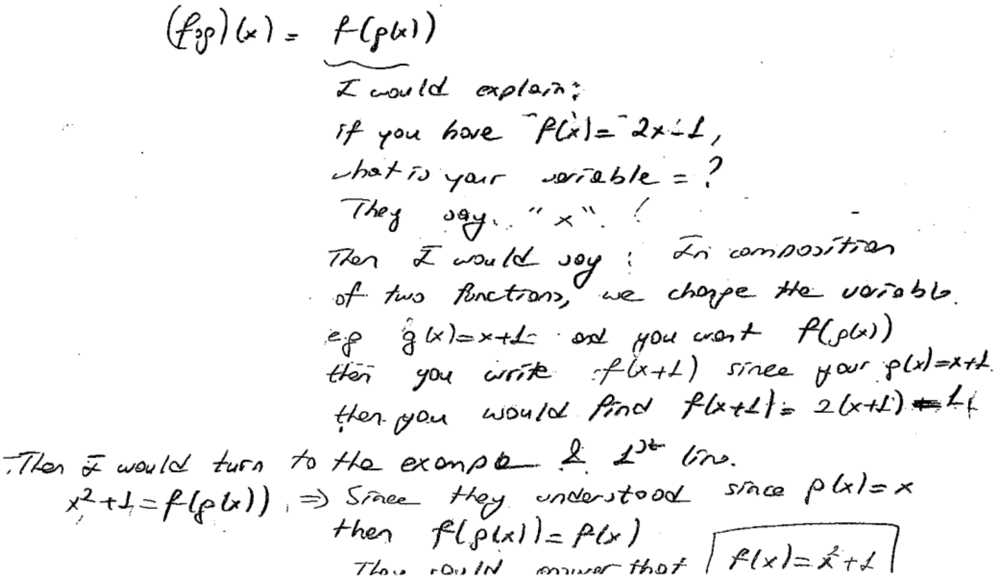
$f(g(x)) = x^2+1.$ ($g(x)$ → x)

$f(x) = x^2+1.$

The source is that: the students see the composition as multiplication. I would remind that composition of two functions is not multiplication:

Figure 4.55: Excerpts from the Gizem's vignette # 4

In her explanation (See Figure 4.56), similar to previous vignettes she used procedural examples. However, her lack of understanding of the source of mistake is an evidence for a lack of SMK. Therefore, her SMK was rated as Level 0-1 from the combined framework.


$(f \circ p)(x) = f(p(x))$

I would explain;

If you have $f(x) = 2x+1$,

what is your variable = ?

They say "x".

Then I would say: In composition of two functions, we change the variable.

e.g. $g(x)=x+1$ and you want $f(p(x))$

then you write $f(x+1)$ since your $p(x)=x+1$

then you would find $f(x+1) = 2(x+1) \ldots$

Then I would turn to the example & 1st line.

$x^2+1=f(p(x))$ ⇒ Since they understood since $p(x)=x$

then $f(p(x)) = f(x)$

They would answer that $f(x)=x^2+1$

Figure 4.56: Excerpts from the Gizem's vignette # 4

The fifth vignette was intended to see to what extend participants resolve the conflict about the usage of analogy for definition of composite functions. She decided that given analogy for the definition of composite function is true, however, she also thought that this analogy might cause some problems (See Figure 4.57).

Giving such a decision requires a necessary knowledge of the definition of the composite function, so this decision was taken as an evidence.

A teacher gave the definition of the composite function and explained it on the board to his/her students. However, some of his/her students stated that they did not understand it completely. Then teacher gave the following example to the students.
In order to clean and dry our clothes in a laundry we use two machines, washing machine and dryer, respectively.
Dry&Wash (clothes)
Dry[Wash(clothes)]=Dry[cleaned and wet clothes]=dried and cleaned clothes
Combination of these machines works can be considered as a composition of functions

• I think it is a good example, but it may cause some confusion! Since then they would be confused since we make in the composition functions: if $f(x)=x^2$
then $(f \circ g)(x) = f(g(x)) = g^2(x)$?
Since we put the $g(x)$ in the equation of $f(x)$, then some of students may say we should put the wash machine in the dry machine, they could not imagine that in fact we use the outcomes.

Figure 4.57: Excerpts from the Gizem's vignette # 5

Developing on this knowledge, she also gave an alternative true analogy (See Figure 4.58) by mentioning that she worked with her friends for this analogy. This vignette was rated as Level 1.

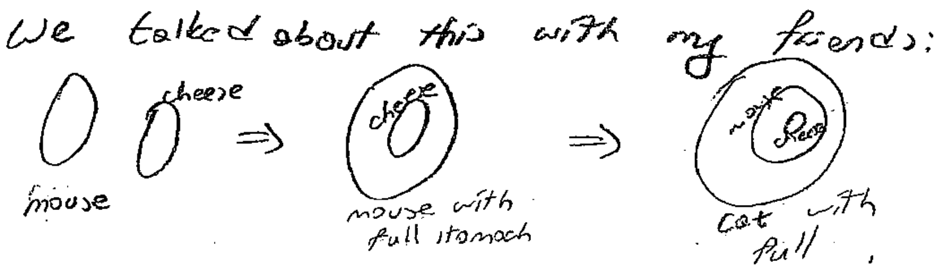


Figure 4.58: Excerpts from the Gizem's vignette # 5

The last vignette related with the composite functions is the thirteenth vignette, which is similar to the fifth vignette since it was intended to see to what extend participants resolve the conflict about the usage of analogy for definition of composite functions. As it can be seen from the Figure 4.59 this time an example was provided by the teacher and student's analogy needs correction. She identified the error correctly by saying "to be a composite function you need two functions an input for the first one and its output should be the input for the second one". This statement was an evidence for the existence of the knowledge of the definition of composite functions and awareness of the conditions of existence of composite function.

For explaining composite functions you gave the formal definition and then give the following example "Take grass (g) as the first input; then the cow (c) being a function "eats" the grass. Next, here comes a third animal, say the tiger "eats" the cow. The best way to denote this is t(c(g)).The brackets denotes the walls of the stomachs."

Then you want from your students to exemplify the composite functions by using such an example. One of your students gives the following example "I came from school by bus and I eat the cookies my mother made. Bus is my first function and cookies is must second function."

Maybe this example seems at the first glance as a good one. However in my opinion this example is not a sufficient one which is trying to explain the meaning of composite functions. To be a composite function, you need two functions, an input for the first one and its output should be the input for the second one. The grass may be the input for the cow. The function is the eating the grass by the cow. The output which is the grass in the stomach is not the input for the tiger eating the cow. At the end, the last output is not clear. If I were teaching composite functions, I would not use this example. Maybe I would use the fallowing example "the first input is a photo (p) on the computer screen. Your first function is the rotation (r) function. Firstly you rotate the photo r (p). Your second function is zooming (z) function.

When you press on zoom in bottom, you will enlarge the rotated photo which is your output from the first function and the input for the second function. Your last output will be enlarged and rotated photo z(r(p)).

Figure 4.59: Excerpts from the Gizem's vignette # 13

Furthermore, by stating the same reasons she also analyzed the student's analogy (See Figure 4.60). Also, she generated a new analogy satisfying the conditions for the composite functions showing he connectedness of knowledge. Therefore, her SMK was categorized as Level 1.

In my opinion, the example is not correct. As I mentioned above, the output for the first function should be the input for the second one. The student said that the bus is the first function and he is the input. Coming from school by bus may be an example to explain the definition of a function. However to explain the composite function, considering the cookies as the second function is not correct. Since the student eats the cookies, which may be seemed as input, the cookies can not be the second function. In my opinion, the base idea of this example is wrong so correction would be very difficult and it may cause other confusions. To clear up confusion during a class, I would try to explain that, as I mentioned before, second function should use the output of first function as its input. For the student's example, first output is the student coming by bus, which should be the input for the second function. Additionally, I would give another example which is clearer than the first one.

Figure 4.60: Excerpts from the Gizem's vignette # 13

The tenth vignette was aimed to explain the participants' understanding of combined use of inverse and composition of functions in questions. The case in the vignette includes a student's answer to a question and a dialogue between the student and the teacher about the answer (See Figure 4.61).

If $f(2x+1)=2x-1$ find $f(3x)$ in terms of $f(x)$ and explain the steps of your solution.

Then the students solved the question correctly as follows:

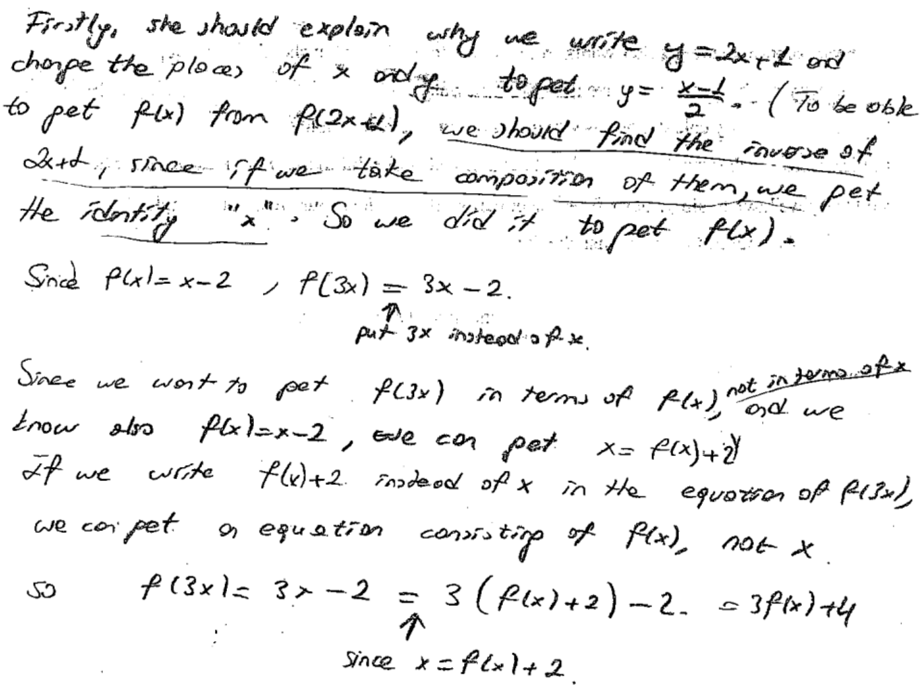
$$\left.\begin{aligned} y&=2x+1\\ x&=2y+1\\ x-1&=2y\\ y&=\frac{x-1}{2}\end{aligned}\right\}\ \underbrace{f(x)=2.\frac{x-1}{2}-1=x-2}_{*}\ \textit{then}\ f(3x)=3x-2$$

$$f(x)+2=x \Rightarrow f(3x)=3(f(x)+2)+2 \Rightarrow f(3x)=3f(x)+4$$

After the solution made, teacher wants from student to explain what she did in the step indicated by *. She said that “I have to get f(x) so that I could calculate f(3x). For getting f(x) I made the necessary calculations as you did in our previous examples”.
Furthermore, teacher wants from student to explain what she did in the $f(x)+2=x$ step. She said that “we have to single out x from the equation as you did in our previous examples”. However, she couldn’t explain what she did.

Figure 4.61: Excerpts from the Gizem’s vignette # 10

First Gizem solved the question step by step by providing explanation and then she gave her explanation for the confusion (See Figure 4.62). Her statements were taken as an evidence for understanding of the application of the composition of functions and her SMK was rated as Level 1.


Firstly, she should explain why we write $y=2x+1$ and change the places of x and y to get $y=\frac{x-1}{2}$. (To be able to get f(x) from f(2x+1), we should find the inverse of 2x+1, since if we take composition of them, we get the identity "x". So we did it to get f(x).

Since $f(x)=x-2$, $f(3x)=3x-2$.
put 3x instead of x.

Since we want to get f(3x) in terms of f(x), not in terms of x, and we know also $f(x)=x-2$, we can get $x=f(x)+2$
If we write f(x)+2 instead of x in the equation of f(3x), we can get an equation consisting of f(x), not x.

so $f(3x)=3x-2=3(f(x)+2)-2=3f(x)+4$
since $x=f(x)+2$.

Figure 4.62: Excerpts from the Gizem's vignette # 10

The eleventh vignette was intended to see to what extend participants resolve the conflict about the students' misunderstandings about the use of the fact fof^{-1}=I while solving questions in relation to composite and inverse functions. This time she did not explicitly state the problem in the student's answer however her explanation (See Figure 4.63) gave us an evidence that she identified the problem and knows what a composite and an inverse function is and use the appropriate notation for both of them. Also, she showed her connectedness of knowledge by stating that "functions are special relations". She used procedural question and Venn diagram during her explanation. Her SMK was rated as Level 1 from the combined framework, since no evidence of Level 2 were identified in the vignette.

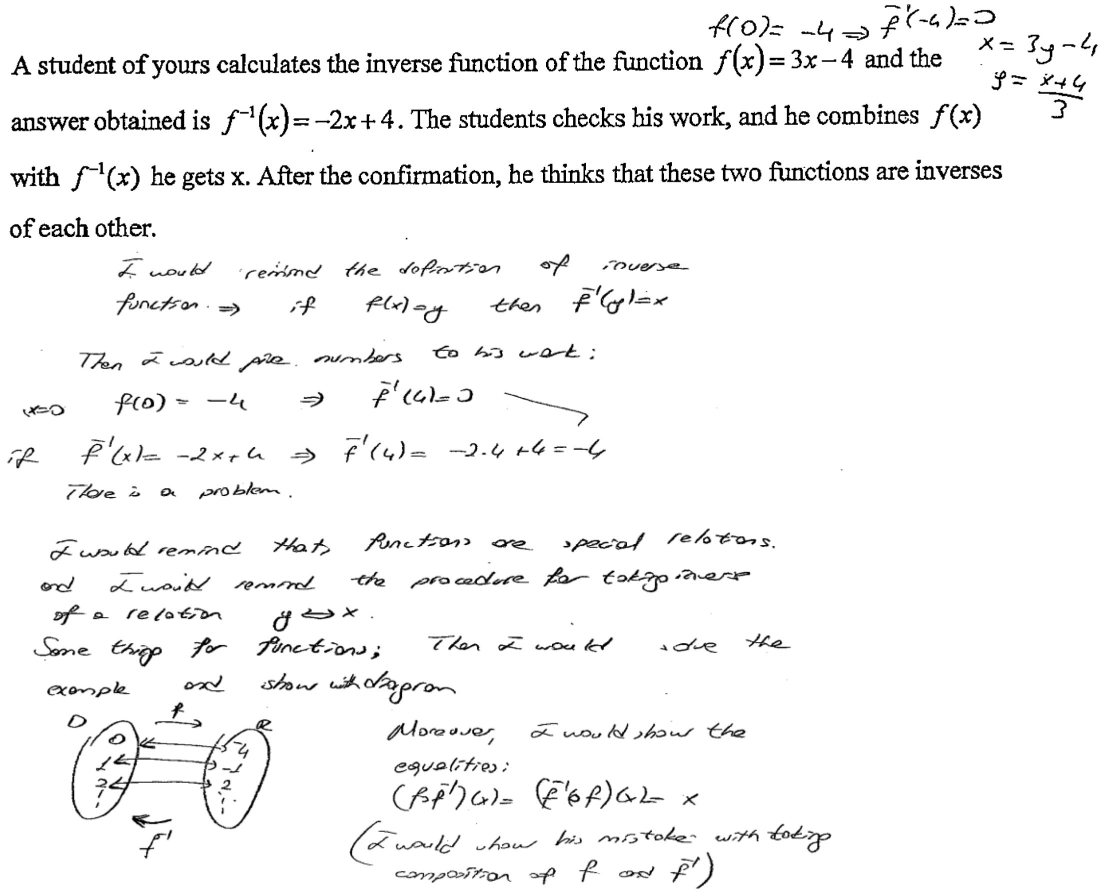
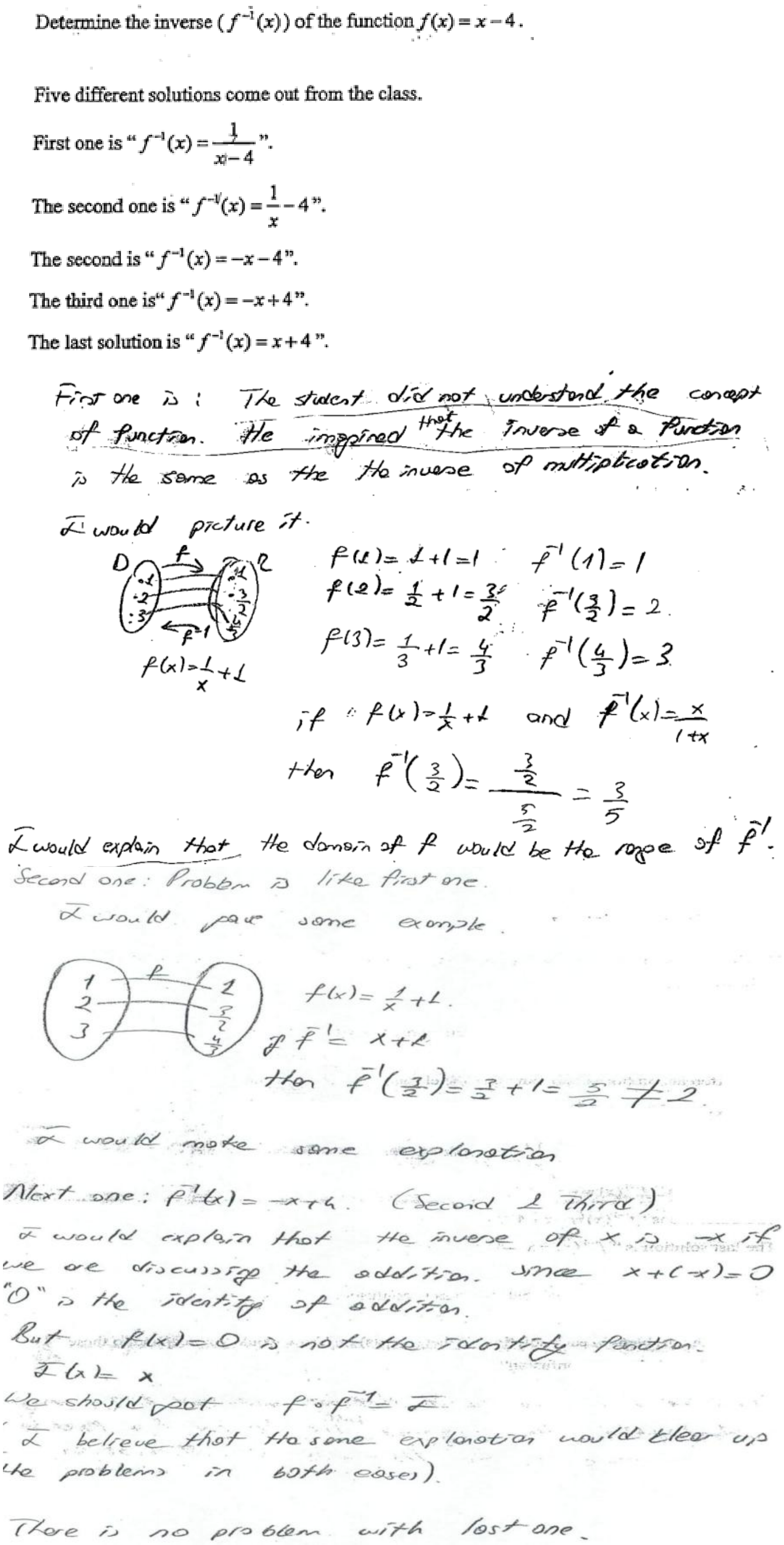


Figure 4.63: Excerpts from the Gizem's vignette # 11

When Gizem's knowledge of composite functions were evaluated through the vignettes (1, 2, 3, 4, 5, 13, 10, and 11 where last two also include knowledge of inverse functions) it was seen that she knows the definition of composition of functions and its notation good enough to resolve conflicts in different cases. She tend to use procedural examples, informal definitions, and Venn diagrams through her explanations. Moreover, she showed no evidence for the connectedness of her knowledge of composition of functions. Her use of different representations were limited just to use of Venn diagrams only in the vignettes 1, 2 and 13. Her SMK was rated as Level 1, except for the vignette 4 which was rated as Level 0-1 similar to her levels in the lesson plans.

4.1.2.4.3 Vignettes Related to Inverse Functions

In a similar way, the vignettes related with the inverse functions were analyzed. The sixth vignette was intended to see to what extend participants resolve the conflict about the usage of the power -1 in the function notation. Gizem diagnosed the error (See Figure 4.64), which gave us an evidence for her knowledge of the term “inverse” in mathematics. Her further explanation for clearing up the confusion includes some explanations and a procedural question. However, the link between the error and the question was not clear enough. As a result, her SMK was rated as Level 0-1.



Determine the inverse ($f^{-1}(x)$) of the function $f(x) = x - 4$.

Five different solutions come out from the class.

First one is "$f^{-1}(x) = \frac{1}{x-4}$".

The second one is "$f^{-1}(x) = \frac{1}{x} - 4$".

The second is "$f^{-1}(x) = -x - 4$".

The third one is "$f^{-1}(x) = -x + 4$".

The last solution is "$f^{-1}(x) = x + 4$".

First one is : The student did not understand the concept of function. He imagined that the Inverse of a Function is the same as the the inverse of multiplication.

I would picture it.

$f(1) = 1 + 1 = 1$ $f^{-1}(1) = 1$

$f(2) = \frac{1}{2} + 1 = \frac{3}{2}$ $f^{-1}(\frac{3}{2}) = 2$

$f(3) = \frac{1}{3} + 1 = \frac{4}{3}$ $f^{-1}(\frac{4}{3}) = 3$

$f(x) = \frac{1}{x} + 1$

if $f(x) = \frac{1}{x} + 1$ and $f^{-1}(x) = \frac{x}{1+x}$

then $f^{-1}(\frac{3}{2}) = \frac{\frac{3}{2}}{\frac{5}{2}} = \frac{3}{5}$

I would explain that the domain of f would be the range of f^{-1}.

Second one : Problem is like first one.

I would give some example.

$f(x) = \frac{1}{x} + 1$.

if $f^{-1} = x + 1$

then $f^{-1}(\frac{3}{2}) = \frac{3}{2} + 1 = \frac{5}{2} \neq 2$.

I would make some explanation

Next one : $f^{-1}(x) = -x + 4$. (Second & Third)

I would explain that the inverse of x is $-x$ if we are discussing the addition. since $x + (-x) = 0$ "0" is the identity of addition.

But $f(x) = 0$ is not the identity function.

$I(x) = x$

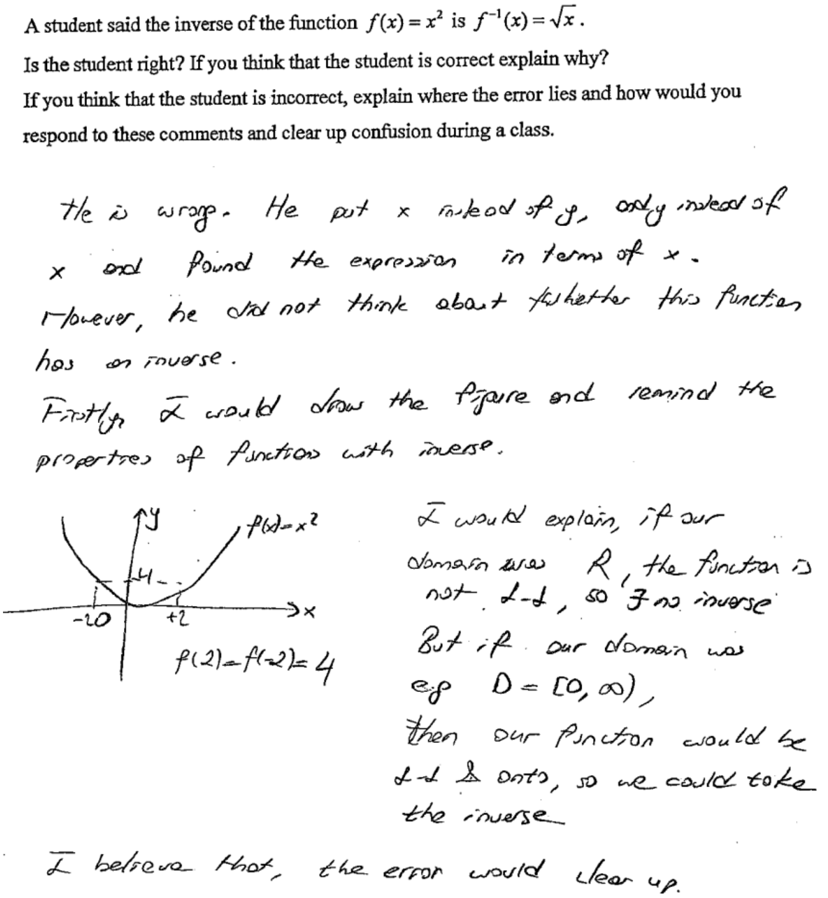
We should got $f \circ f^{-1} = I$

I believe that the same explanation would clear up the problems in both cases).

There is no problem with last one.

Figure 4.64: Excerpts from the Gizem's vignette # 6

The seventh vignette was intended to see to what extend participants resolve the conflict about the students' misunderstandings about the existence of inverse functions. Her identification of the problem of the student's solution gave us an evidence that she knows that for existence of an inverse function it has to be a one to one function (See Figure 4.65). While clearing up the confusion in the next paragraph, she once more showed her knowledge of the definition of the inverse functions by providing a graphical procedural question and relating it with the domain of functions. Hence, her SMK was rated as Level 1-2.

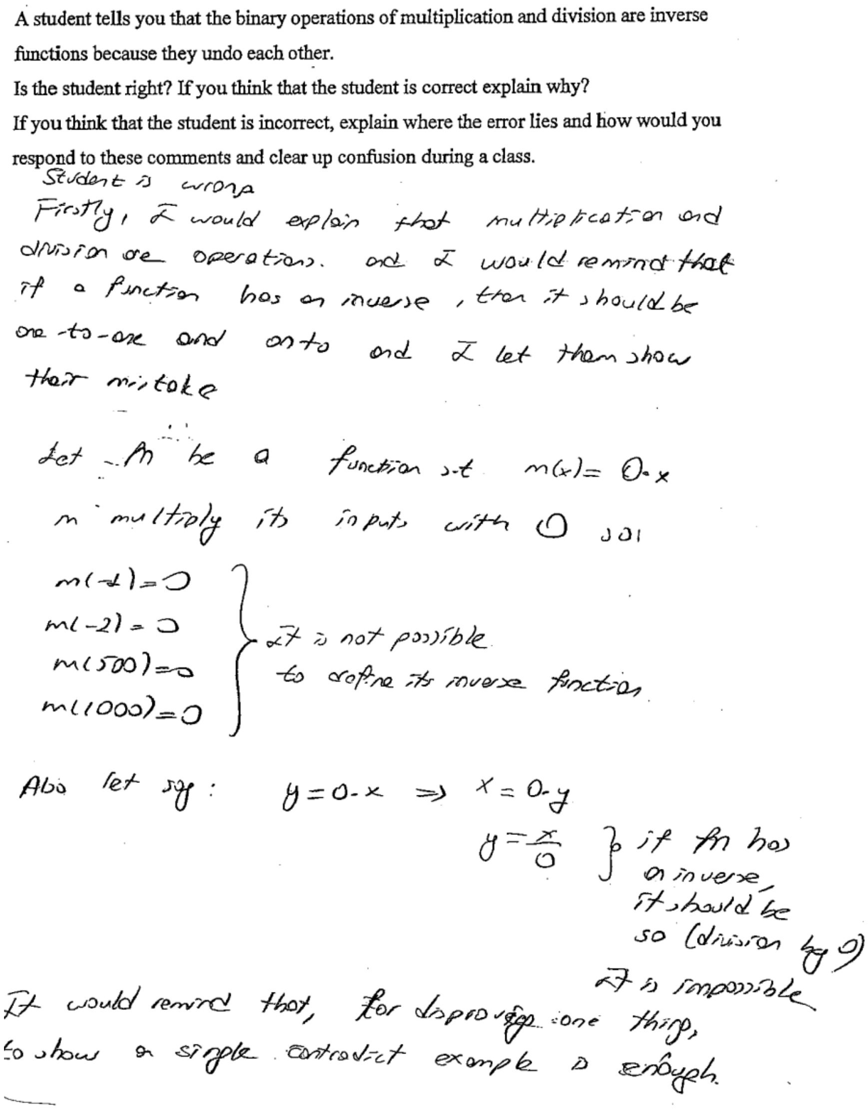

A student said the inverse of the function $f(x) = x^2$ is $f^{-1}(x) = \sqrt{x}$.
Is the student right? If you think that the student is correct explain why?
If you think that the student is incorrect, explain where the error lies and how would you respond to these comments and clear up confusion during a class.

He is wrong. He put x instead of y, only instead of x and found the expression in terms of x. However, he did not think about whether this function has an inverse.
Firstly, I would draw the figure and remind the properties of function with inverse.

I would explain, if our domain was R, the function is not 1-1, so ∃ no inverse. But if our domain was e.g. D = [0, ∞), then our function would be 1-1 & onto, so we could take the inverse.
I believe that, the error would clear up.

Figure 4.65: Excerpts from the Gizem's vignette # 7

In the eighth vignette the aim was to see to what extend participants understand the idea of inverse function as undoing. As seen in Figure 4.66 she diagnosed that the student's error however the procedural example she used in the explanation reflected that she did not get the idea of undoing. So, she gave an appropriate explanation for it. Combining these evidences, her SMK was rated as Level 0-1.



A student tells you that the binary operations of multiplication and division are inverse functions because they undo each other.

Is the student right? If you think that the student is correct explain why?

If you think that the student is incorrect, explain where the error lies and how would you respond to these comments and clear up confusion during a class.

Student is wrong

Firstly, I would explain that multiplication and division are operations. and I would remind that if a function has an inverse, then it should be one-to-one and onto and I let them show their mistake

Let m be a function s.t. $m(x) = 0 \cdot x$

m multiply its inputs with 0 so

$m(-1) = 0$
$m(-2) = 0$
$m(500) = 0$
$m(1000) = 0$

it is not possible to define its inverse function.

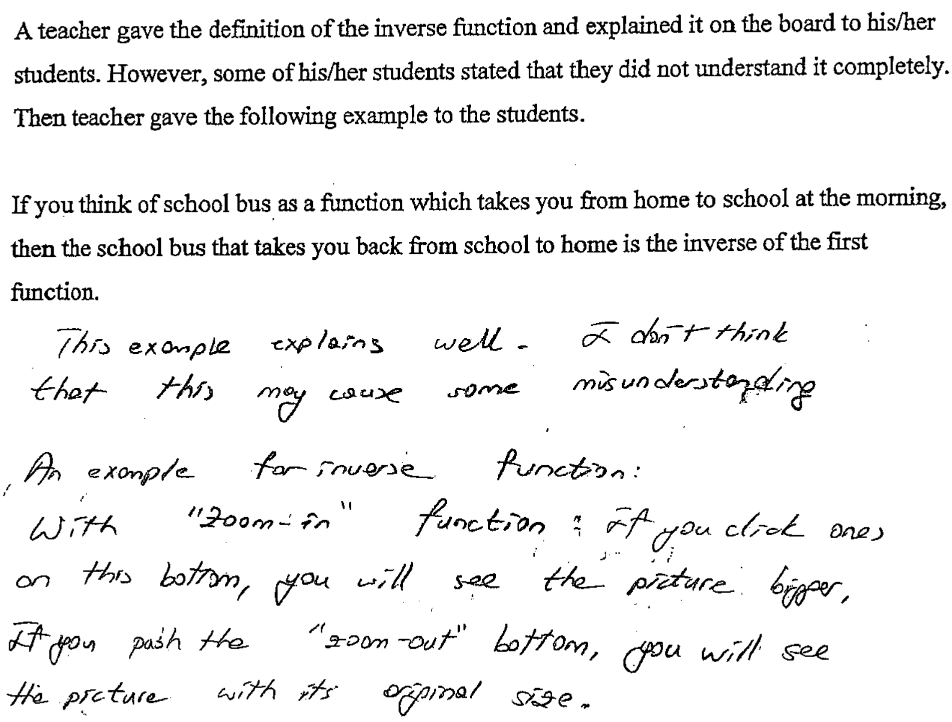
Also let say: $y = 0 \cdot x \Rightarrow x = 0 \cdot y$

$y = \frac{x}{0}$ } if fn has an inverse, it should be so (division by 0) it is impossible

It would remind that, for disproving one thing, to show a single contradict example is enough.

Figure 4.66: Excerpts from the Gizem's vignette # 8

The ninth vignette was intended to see to what extend participants resolve the conflict about the usage of analogy for definition of inverse functions. As it can be seen from the Figure 4.67 teacher's example was provided and her ideas about the analogy was asked. She did not notify the error by saying " … this example explains well. I don't think it will cause any problems…".


A teacher gave the definition of the inverse function and explained it on the board to his/her students. However, some of his/her students stated that they did not understand it completely. Then teacher gave the following example to the students.

If you think of school bus as a function which takes you from home to school at the morning, then the school bus that takes you back from school to home is the inverse of the first function.

This example explains well. I don't think that this may cause some misunderstanding

An example for inverse function:
With "zoom-in" function: if you click ones on this bottom, you will see the picture bigger, If you push the "zoom-out" bottom, you will see the picture with its original size.

Figure 4.67: Excerpts from the Gizem's vignette # 9

However, her own analogy example gave us an evidence that she knew the definition of inverse functions well enough. Combining all evidences, her SMK level for this vignette was identified as 0-1.

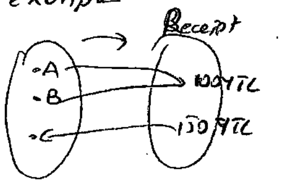
The twelfth vignette was also related with the use of analogy for the definition of inverse functions but this time there exists two analogies about inverse functions one from the teacher and one from a student (See Figure 4.68). She analyzed both of them and diagnosed the error in the student's analogy correctly, then, used an example with Venn diagrams for clarifying the situation. On the other hand, her explanation "..it may not have inverse…" did not reflect any concrete knowledge. From all these evidences her SMK was rated as Level 0-1.

For explaining inverse functions you gave the formal definition and then gave the following example "When someone calls you on the phone, he, or she looks up your number in a phone book (a function from names to phone numbers). When Caller ID shows who is calling, it has performed the inverse function, finding the name corresponding to the number."

Then you want from your students to write down such a function and define inverse of it. One of your students gives the following example "My function is something we see everyday on supermarket's cash registers (yazarkasa). For each item we buy there is a corresponding price on the receipt (fiş), so the inverse of this function is for each price there is a corresponding item."

The teacher's example is good. I would use it. However I don't like student's example, since sometimes two things may have same price.

For example



But ∃ no inverse of this function

I would explain that, it may not have inverse so it is not a good example.

Figure 4.68: Excerpts from the Gizem's vignette # 12

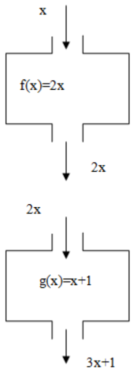
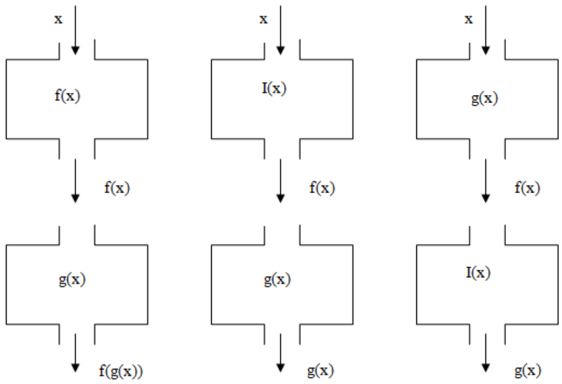
When Gizem's knowledge of inverse functions were evaluated through the vignettes (6, 7, 8, 9, 12, 10, and 11 where last two also include knowledge of composite functions) it was seen that she knows the definition of inverse function, conditions for existence of inverse functions, its notation, the term "inverse" in mathematics good enough to resolve conflicts in different cases. She rarely used different representations and as in the composition of functions she used Venn diagrams and graphs of functions. She used procedural and declarative questions. Sometimes, she was not able reflect her knowledge while choosing the procedural questions which hinders her understanding. As stated above for each vignette, her SMK was mostly rated as Level 1, similar to her levels in the lesson plans. Also, her SMK was rated once for Level 1-2, and three times for Level 0-1.

4.1.2.4.4 Teaching Practices

Keeping all these in mind, evidences of SMK were also searched in the teaching practices thorough the examples solved and explanations Gizem provided to the class or a student.

When the examples solved through the teaching practices were analyzed it was seen that she used most of the examples within the lesson plans and sometimes changed the order according to the needs of the class. Her additional examples were generally analogies, Venn diagrams, and figures for explaining the concepts composite and inverse functions and procedural questions aiming at reviewing the same concept (See Table 4.17).

Table 4.17: Gizem's Examples from Teaching Practices

Teaching Practice	Example(s)
December 5	• Functions are like machines it has inputs and outputs and sometimes more than one machines work together. Do you have examples for this? …For example, washing machine cleans the dirty clothes and drying machine dries the wet clothes and if these two work together simultaneously we got dies and clean clothes. In mathematics, two machines working together by using the outputs of the first one is called the composition of functions.  • So let's think can we put something other than laundry into the washing machine? Or 3 in gof if f is from A={-1,0,2} to B in the first question?
December 8	 • Identity function is like a pipe whatever you put at one end will go out same at the other end • For explaining that in mathematics even if you have one example not satisfying a rule/theorem etc. it is called not correct. In Turkish we have an idiom "İstisnalar kaideyi bozmaz" which contradicts with the mathematics.
December 12	• Do you know what a linear function is? Do you know what a line is? $y = x, y = 3x + 1, y = 2x - 2$ are all linear functions whereas $f(x) = x^2$ is not
December 13	• For explaining inverse functions, "Let's take a picture of Barış a student in the class and click once for zoom-in bottom and then once for zoom-out button". She also draw sketches to show students .
December 15	• In the equation $(fogoh)^{-1}(x) = h^{-1}\, og^{-1}\, of^{-1}$ the equality sign can be thought as a mirror reflecting the inverses

Her explanation to the class or a student was changed according to the representation of the question. If the question was given in mathematical format her explanation was generally resolving the question on the board via oral explanations. Even if the student did not ask for the explanation for the concept she generally referred also to concept by repeating the analogies used. And also, her explanations in the vignettes and teaching practices were similar to each other.

It was noticed that her definitions in the survey and journal had incorrect use of notation about composite functions, this problem was not seen in any phase of the teaching practices. Her SMK levels in the teaching practices were rated as Level 1, same as her levels in the lesson plans and vignettes.

4.1.2.5 Summary of Gizem's SMK

Evidences of SMK were searched through two groups of instruments. First group (survey of function knowledge, non-routine questions, definitions journals, and concept mapping activity) was only related with assessing the SMK in certain aspects. In the second group (vignettes, lesson plans, and teaching practices), SMK was searched through instruments where there was an integration of knowledge exists, so they were designed for assessing all components of pedagogical content knowledge.

First group of instruments revealed that Gizem had difficulty in expressing the definitions of the concepts functions, composition of functions and inverse functions. The effects of this difficulty was mostly seen on the questions which require a knowledge of definitions of functions, composite and inverse functions. Her answers to questions about the composite and inverse functions revealed that she knows the conditions for existence but she applied these rules if she feels a problem (like root in the composite functions or quadratic in inverse function) after the procedural steps. On the other hand, as stated before she performed very well on the questions which were procedural in nature. She always tried to convert graphical questions to mathematical notation, and otherwise she could not solve it. During the evaluation interview she admitted that she saw herself attempting to every question procedurally regardless of the question type. This statement approved the findings from the survey and the non-routine questions interview.

Gizem showed limited evidence for the connectedness of her knowledge of composite and inverse functions in all phases of the assessment, this non-connectedness was approved during the analysis of concept maps. Gizem was unable to find all related terms and generally unable to construct meaningful subtopics (clusters) and connect the related subunits with meaningful linking words through the concept mapping activity.

Even though she showed lack of knowledge about the definitions in the first group of instruments, by the effect of those during the lesson plans, vignettes, and teaching practice she put emphasis on the definitions and so as on conditions for the existence of composite and inverse functions as stated by her.

Similar to this picture, through the lesson plans, vignettes, and teaching practices, her emphasis was not only on procedural questions but also on meaning construction. Therefore, her explanations to conflicts in the teaching practices were not only procedural explanations but also included emphasis on conceptual understanding via student involvement. Through the vignettes, she tend to use procedural and declarative examples, informal definitions, and Venn diagrams through her explanations.

Moreover, her sequencing of the subtopics and questions in the teaching practices were analyzed. The results revealed that she sequenced the sub topics and the questions in a logical order.

Apart from these the instruments in the second group were analyzed with respect to the same combined framework. The analysis revealed that her SMK levels in the vignettes mostly rated as Level 1, her SMK levels in the lesson plans mostly rated as Level 1, and her SMK levels in the teaching practices rated as Level 1.

4.1.3 Deniz’s SMK

When Deniz’s scores of the survey of function knowledge was analyzed it was seen that she was more successful on procedural questions compared to declarative and conditional ones. Furthermore, when the scores of declarative and conditional questions compared, it was seen that she got similar points where her scores on the conditional questions were lower than the others. This is reasonable since conditional knowledge requires existence of declarative knowledge. When her scores on each question was analyzed it was seen that she generally got 3 or 4 for the most of the questions. The questions she got scores of 1 or 2 were question related with the definitions and the properties 1-1 and onto.

4.1.3.1 Knowledge about the Definitions and the Applications of the Definitions

In order to understand her SMK further the survey of function knowledge questions were analyzed with respect to similar objectives through declarative, procedural and conditional questions.

4.1.3.1.1 The Survey of Function Knowledge

The first declarative question in the survey asks for a definition of function. As it can be seen from the Figure 4.69, her answer was like an explanation which lacks the important properties of functions so she got a score of 2.

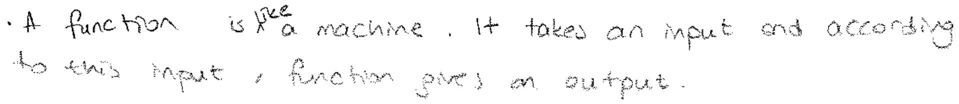
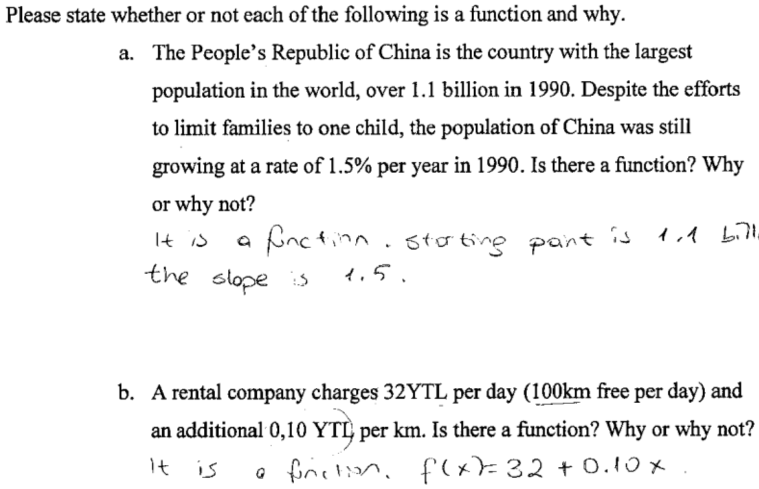

· A function is like a machine. It takes an input and according to this input, function gives an output.

Figure 4.69: Deniz’s answer for the question 1

Related to the definition of function, question 4 asks for whether the given relations are functions and the question was declarative in nature. Although she gave an explanation for the function, she did not faced difficulty in determining which functions are relations in question 4. She got either 3 or 4 points. She lost points since her explanation for the existence of the function was not clear enough (See Figure 4.70).



Please state whether or not each of the following is a function and why.

a. The People's Republic of China is the country with the largest population in the world, over 1.1 billion in 1990. Despite the efforts to limit families to one child, the population of China was still growing at a rate of 1.5% per year in 1990. Is there a function? Why or why not?

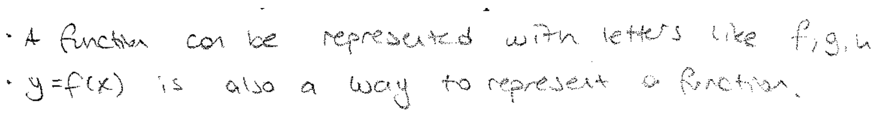
It is a function. starting point is 1.1 billion the slope is 1.5.

b. A rental company charges 32YTL per day (100km free per day) and an additional 0,10 YTL per km. Is there a function? Why or why not?

It is a function. f(x)= 32 + 0.10x.

Figure 4.70: Deniz's answer to question 2

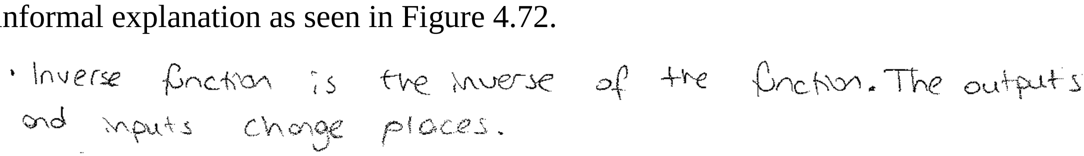
During the analyses of the question 4 it was noticed that Deniz's identified function regardless of the representation the function was given. However, when her answer to question 2 was analyzed it was seen that her answer include only one representation (See Figure 4.71). This situation was investigated during the follow-up interview and she stated that I misunderstood the question which explained that she was aware of different representations and able to use them in the questions.



· A function can be represented with letters like f, g, h
· y=f(x) is also a way to represent a function.

Figure 4.71: Deniz's answer to question 4 and b

In a similar vein, when she was asked to define inverse function she gave an informal explanation as seen in Figure 4.72.



· Inverse function is the inverse of the function. The outputs and inputs change places.

Figure 4.72: Deniz's answer to definition of inverse function

When she was asked to decide the existence of inverse functions in question 12. Similar to the case of existence of functions, even though her definition did not reflect the conditions for the existence of inverse functions, she correctly identified inverse functions as seen in Figure 4.73. She got 4 points for part b and 3 points for part a since she did not give any explanation.

Give your reasons why the following functions do or do not have inverse functions. If exists write the function. If not, give a numerical example.

a. Your hourly wage is 7,7YTL plus 0,90YTL for each unit x produced per hour. Let f(x) represents your weekly wage for 40 hours of work. Does this function have an inverse? $g(x) = 0.9x + 7.7$

$f(x) = 36x + 308$

$f^{-1}(x) = \frac{x - 308}{36}$

b. Let x represent the retail price (satış fiyatı) of item in YTL, and let f(x) represent the sale tax on the item. Assume that the sale tax is 7% of the retail price and that the sale tax is the rounded to the nearest natural number. Does this function have an inverse?

$f(x) \cong \frac{7x}{100}$ No!

for $x = 30$ the result is 2.1 but it is rounded so 2.
for $x = \frac{200}{7}$ it is again 2.
The inverse function $f^{-1}(2)$ goes to two different places which is not true.

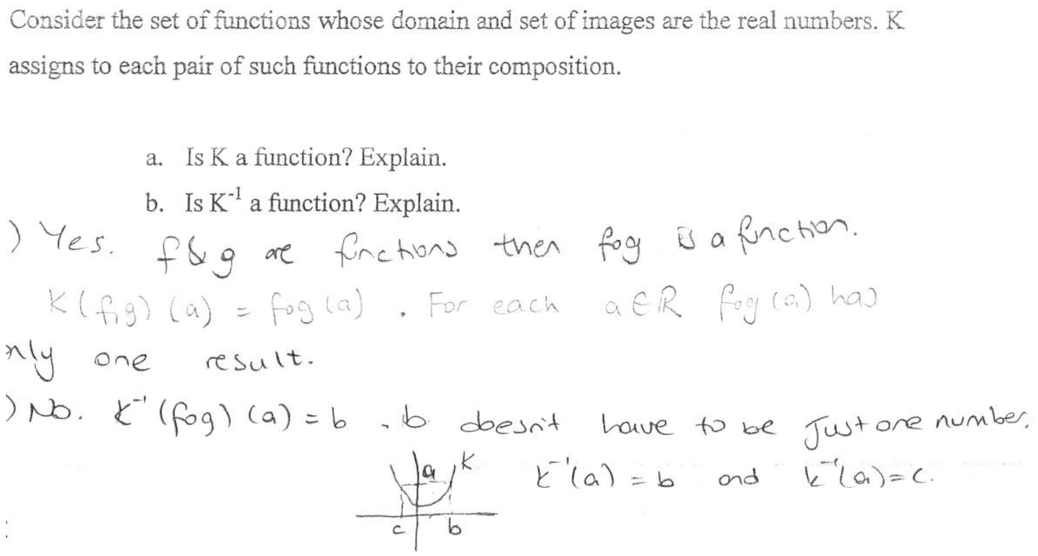
Figure 4.73 : Question 12 and Deniz's answer

When she was asked to define composition of functions similar to functions and inverse functions instead of giving a formal definition in words she used an informal explanation given in Figure 4.74.

· Composite function is combination of two functions. A functions takes input, the other functions takes the output of the previous func. as an input. And the final output is the result.

Figure 4.74: Deniz's answer to the definition of composite function

Her answers to question related to existence of composite functions are conditional in nature and is given in mathematical notation. Although her definition of composition of functions was an informal explanation without conditions for the existence, when she was asked to identify the existence composition of functions in the question 17, she successfully did it and got 4 in the first part, and 3 in the second part because of her weak explanation. In this question the domain for the function K is the set of functions and range is the set of compositions of those functions. However, as it can be seen from her explanation to part a she seemed to expect functions to be defined on numbers only and so she changed the given information to fit into her own understanding. (See Figure 4.75).


Consider the set of functions whose domain and set of images are the real numbers. K assigns to each pair of such functions to their composition.

a. Is K a function? Explain.

b. Is K^{-1} a function? Explain.

Figure 4.75: Deniz’s answer to question 17

For the existence of functions, inverse functions, and composite functions, knowledge of domain and range is compulsory for this reason in the survey, question 5 asks for the meaning of domain and range and their importance. In Figure 4.76 her answer was given.

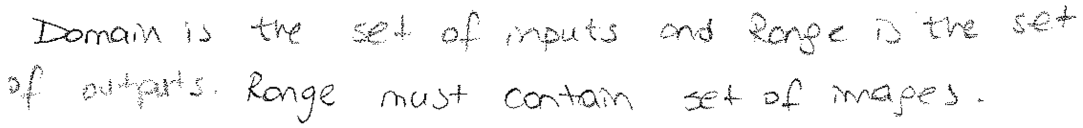


Figure 4.76: Deniz’s answer to question 5

Similar to her previous definitions her definition for the domain and range were like an explanation so she got a score of 2 from this question. In line with this question, preceding questions 6 and 7 ask for the domain and range of a given function in mathematical notation and they are procedural questions and she got score of 3 for both of them because of her weak explanation (See Figure 4.77 and 4.78).

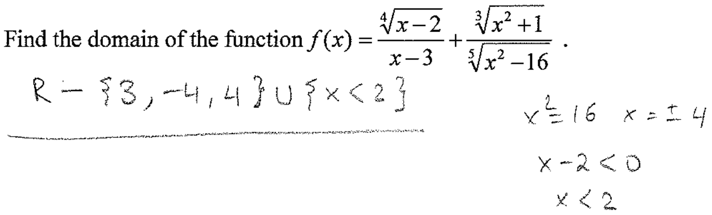


Figure 4.77: Deniz's answer to question 6

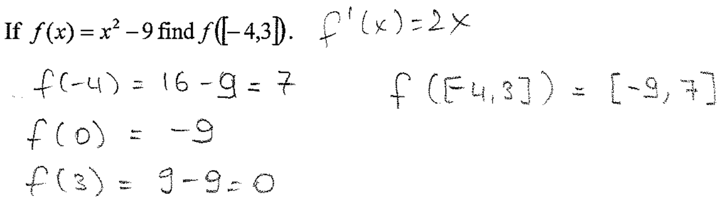


Figure 4.78: Deniz's answer to question 7

Apart from existence of domain and range, being 1-1 and onto is also required for functions to have an inverse. In the survey these properties were investigated through one procedural (question 10) and one conditional question (question 11) as seen in Figure 4.79 and 4.80 respectively. She got 2 points from question 10 and 1 point for each part of question 11. Apart from the definition questions in the survey, these are the questions that she got lowest scores. Her failure in these questions explains why both in the definition of inverse function and question about inverse functions she did not mention the terms 1-1 and onto.

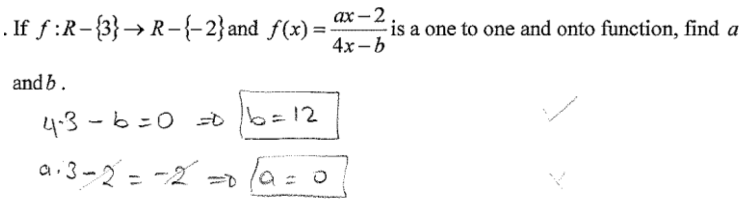
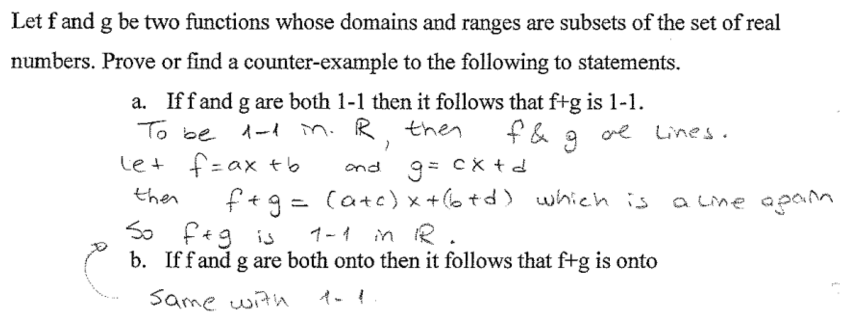


Figure 4.79: Question 10 and Deniz's answer


Let f and g be two functions whose domains and ranges are subsets of the set of real numbers. Prove or find a counter-example to the following to statements.

a. If f and g are both 1-1 then it follows that f+g is 1-1.

To be 1-1 in R, then f & g are Lines.
Let f=ax+b and g=cx+d
then f+g=(a+c)x+(b+d) which is a Line again
So f+g is 1-1 in R.

b. If f and g are both onto then it follows that f+g is onto

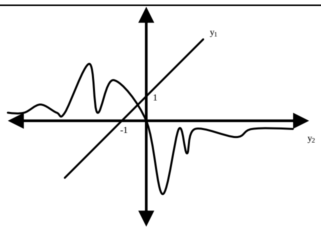
Same with 1-1.

Figure 4.80: Question 11 and Deniz's answer

4.1.3.1.2 Responses to Non-Routine Questions

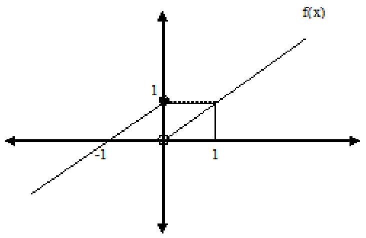
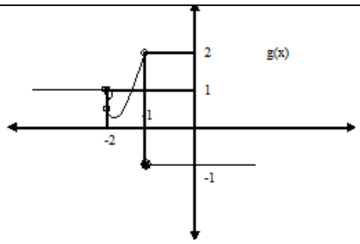
In the non-routine problems interview, there were five questions related with the composition of functions and she did not get full score for any of them (See Table 4.18). For the questions 1 & 2 which were given in mathematical notation, Deniz approached them procedurally. In part a of the first question, the problem was she omitted the fact that she must check the conditions for existence of composition of functions while drawing the final composite function. In part b of the same question, it seems that she checked the condition but she could not finalize the question, so, during the interview she was asked why did not she check the same thing in part a. She said that since the result is a linear function I did not think that there could be a problem. By this explanation, she also confessed that she was not checking the conditions for existence of composite functions. Her SMK level for these questions were rated as 1 and 2 respectively. In the second question, she experienced the same problems with the first one. Since the inside of the root is always negative in this case she realized that the composition is not possible but she could not make sufficient explanation for the existence of composition of functions. Therefore, her SMK level was rated as 3. After this question, she stated that I should work before the teaching practice for the topic we will be responsible. In the third and fourth questions, functions were given as graphs their composition was asked. In third question, she attempted to solve the question pointwise but she could not move any further in part a. However, when the order of the composition changed she easily move to the second step by writing the linear function in equation form and finding the composition with the other function. But, she couldn't move any further. In the fourth question, since she could not write any of the functions in mathematical notation she did not attempt any solutions.

Table 4.18: Composition of functions questions in the non-routine interview, Deniz's answers and scores

		Questions	Answers	Scores
Composition	1 a	$f(x)=x^2-2$ and $g(x)=-\sqrt{x+1}$ answer each of the following. (a) Determine $(fog)(x)$ in simplified form and sketch a graph of this new function	$f(g(x)) = f(-\sqrt{x+1}) = (-\sqrt{x+1})^2 - 2$ $= x+1-2 = x-1$ $x=0$ $f(g(x)) = -1$ $(-1,0)$ $y=0$ $f(g(x)) = 0 = x-1$ $x=1$ $(1,0)$	1
	1 b	(b) Determine $(gof)(x)$ in simplified form and sketch a graph of this new function.	$g(f(x)) = g(x^2-2) = -\sqrt{x^2-2+1}$ $y = -\sqrt{x^2-1}$ $x^2-1 \geq 0$ $x^2 \geq 1$ $x \geq 1$ $x \leq -1$ $x=1$ $y=0$ $x=-1$ $y=0$	2
	2	$f(x)=\sqrt{4-x^2}$ and $g(x)=\sqrt{x^2-9}$ Determine $(gof)(x)$ in simplified form.	$g(f(x)) = g(\sqrt{4-x^2}) = \sqrt{(\sqrt{4-x^2})^2-9} = \sqrt{4-x^2-9} = \sqrt{-x^2-5}$ After writing this, she said that it is not possible to take the composition it is undefined but her reasoning was not clear enough	3
	3 a	 (a) Use the given graphs to sketch $y_2 o y_1$.	She approached the question procedurally and could not move any further without a hint	1
	3 b	(b) Use the given graphs to sketch $y_1 o y_2$.	$y_1(y_2(x)) \Rightarrow y_2(x)+1$ Although she wrote the composition she could not see the upward shift	2
	4		No answer is attempted even after the hint	0

When the Deniz's answers to inverse function questions (See Table 4.19) in the non-routine questions interview analyzed it was seen that whether the question was given in mathematical notation or in graphs she attempted to solve the question procedurally. This case was similar to that of the survey of function knowledge. Even though, her definition did not include necessary conditions (1-1 and onto) for the existence of inverse functions, she correctly identified the inverse functions in the survey without an explanation. In a similar vein, in the first part of the fifth question she recognized that the constant function does not have an inverse however, she did not mention the conditions of existence. In the part b, since she was not aware of checking the conditions she just procedurally tried to find the inverse and got a score of 2. In the sixth question, again the inverse of functions were asked however this time questions were given as graphs. Deniz attempted the questions procedurally and tried to write given functions as piecewisely defined functions. She did it for part a but experienced difficulty in writing the inverse function, like in composition of functions since the functions were linear she did not feel a reason to check the conditions for existence. However, in part b since some part of the question was quadratic she said that this function does not have an inverse because the middle part is like a quadratic so same image will go to two different numbers.

Table 4.19: Inverse functions questions in the non-routine interview, Deniz's answers and scores

		Questions	Answers	Scores
Inverse	5 a	Find, the inverse of the following functions, if exists. c. $f(x) = 4$, x∈R	$y = 4$ $x = 4$ $f^{-1}(x) = 4$	4
	5 b	d. $f(x) = \sqrt{4 - x^2}$	$y = \sqrt{4 - x^2}$ $x^2 = \sqrt{4 - y^2}$ $y^2 = 4 - x^2$ $f^{-1}(x) = y = \sqrt{4 - x^2}$	2
	6 a	Use the given graphs to sketch the inverse of given functions. 	$f(x) = \begin{cases} x, & x \geq 0 \\ x+1, & x < 0 \end{cases}$ $f^{-1}(x) = x - 1$ $f^{-1}(x) = \begin{cases} x & x \geq 0 \\ x - 1 & x < 1 \end{cases}$	1
	6 b		$g(x) = \begin{cases} 1, & (-\infty, -2] \\ g(x), & (-2, 2) \\ -1, & (2, \infty) \end{cases}$	4

From the evidences we got from survey of function knowledge and non-routine questions interview, it can be concluded that she used informal explanations instead of formal definitions. These definitions lacked the necessary conditions for the existence. However, when applying the rules for composite and inverse functions and deciding the existence of them. She did not experience any problems regardless of the problem type in the survey. In contrast, in the non-routine she experienced problems when the questions were given in graphical form. Moreover, her main problem both in the survey and non-routine questions interview was that the concepts 1-1 and onto and those were as being conditions of existence for inverse functions.

4.1.3.1.3 The Analysis of the Definitions Used through the Instruments

The definitions Deniz used through the instruments the survey of function knowledge, the journals about the definitions, the lesson plans, and the teaching practices were analyzed.

Definitions she used for composition of functions were given in Table 4.20. When they were analyzed it was seen that she used formal definitions except for the survey of function knowledge and teaching practices. The definition she used in the survey was like an explanation. She used a similar terminology while explaining composite functions through a machine analogy. She emphasized the conditions for the existence of composite functions through the Venn diagram which was her first introductory example to composite functions.

Table 4.20: Deniz's definition of composite functions used through the instruments

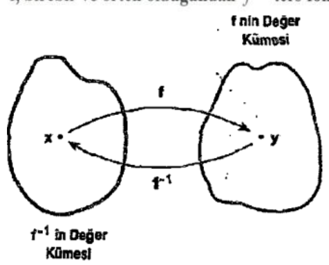
Instruments	Definition
Survey of Function Knowledge	·Composite function is combination of two functions. A functions takes input, the other functions takes the output of the previous func. as an input. And the final output is the result.
Journal about the Composite Function Definitions (Her choice among the given list)	Let f: A→B and g: B→C be two functions. Then gof: A→C defined as $(gof)(x) = g(f(x))$ is called composition of f and g.
Journal about the Composite Function Definitions (Her definition if she would teach)	Let f:A→B & g:B→C be two functions. Then g∘f:A→C defined as (g∘f)(x)=g(f(x)) is called composition of f&g."
Lesson Plan	**Bileşke fonksiyon:** f:A → B ve g:B → C iki fonksiyon olsun. g∘f: A → C olmak üzere, (g∘f)(x)=g(f(x)) şeklinde tanımlanan fonksiyona g ve f nin **bileşke fonksiyonu** denir. g ile f nin bileşkesi g∘f biçiminde gösterilir ve "g bileşke f" diye okunur.
Teaching Practice	She used the same definition with the lesson plan. However, before starting the composition of functions she talked about machines work together since inputs for one of them are outputs for the other. Also, by using Venn diagrams she gave an example and supported the conditions for the existence of composition of functions.

For the inverse functions, the picture was similar to that of composite functions. She used formal definitions except for the survey and teaching practices. The informal definition in the survey was like an explanation about how to take an inverse.

Except for the survey of function knowledge, (See Table 4.21) she used formal definitions for definition of inverse functions. However, in the teaching practices she also gave informal definitions and even analogies to support understanding of the conditions for the existence of the inverse functions.

During the follow-up interview and non-routine questions interview, Deniz said that I think we should work about the composite and inverse functions before the teaching practices, so while preparing the lesson plans and during teaching practices we can indicate on the conditions for existence of composite and inverse functions. This statement approves the findings from the survey and the non-routine questions interview.

Table 4.21: Deniz's definition of inverse functions used through the instruments

Instruments	Definition
Survey of Function Knowledge	'Inverse function is the inverse of the function. The outputs and inputs change places.
Journal about the Inverse Function Definitions (Her choice among the given list)	If f: A→B is one-to-one and onto function then there exists the inverse of f denoted by f^{-1} such that f^{-1}: B→A, $f(x) = y$, and $f^{-1}(y) = x$.
Journal about the Inverse Function Definitions (Her definition if she would teach)	"If f: A→B is 1-1 and onto function then ∃ the inverse of f denoted by f^{-1} st. f^{-1}: B→A, f(x)=y & f^{-1}(y)=x".
Lesson Plan	**Tanım:** f:A→B ve f={(x, y)| x∈A ∧ y∈B} fonksiyonu birebir ve örten ise, f^{-1}:B→A ve f={(x, y)| y∈B ∧ x∈A }fonksiyonu **f in ters fonksiyonudur.** **f, birebir ve örten olduğundan** f^{-1} **ters fonksiyonu vardır.,** 
Teaching Practice	She used the same definition with the lesson plan and moreover used set notation to represent the functions. Furthermore, she used an analogy to explain the definition. Analogy is as follows "Suppose everyday you are coming to school with your daddy's car and turn back with the school bus". Here, we can say that school bus does the opposite of the daddy's car so this case can be an example for an inverse function. Apart from telling the analogy, by using Venn diagrams she showed it on the board.

Regardless of the her choice in the other instruments, she used different combinations of her knowledge (formal and informal) while teaching the concepts composition and inverse of functions. Her informal choice includes explanations for definitions, Venn diagrams and analogies.

4.1.3.2 Application of the Rules about Functions, Composite and Inverse Functions

The rest of the questions not mentioned up to know in the survey are one conditional (question 18), one declarative (question 8), and six procedural (questions 7, 12, 13, 14, 15, and 19) questions. When the questions and their objectives were analyzed it was seen that all of these questions were related with the application of the concepts discussed above. It was also seen from the Table 4.2 that she got either 3 or 4 from these questions. She lost points due to her weak explanations. Therefore, it can be concluded that even though she did not give any formal definition with all the requirements, she was able to solve the questions related with functions, composition of functions, and inverse functions except for the topics 1-1 and onto.

4.1.3.3 Connectedness of Deniz's Knowledge of Functions, Composite and Inverse Functions

Previous results led to the fact that Deniz showed limited evidence for the connectedness of her knowledge of composite and inverse functions. For this reason her concept maps were analyzed. Participants were asked to prepare to concept maps about functions. In the first one, it was free to choose the terms that will be used in the concept map, whereas in the second, the terms were provided but also they are free to use the terms that they prefer. After that they wrote an essay about the concept maps they prepared and lastly focus group interview was conducted in order to share the participants' views about their concept maps, each others concept maps and concept mapping. Concepts maps were analyzed in terms of organization and accuracy whereas concept map essays were analyzed in terms of communication, organization and mechanics (Bolte, 1999).

First concept map (Figure 4.81) of Deniz was rated as 3 (fair) out of 6 because she constructed some meaningful clusters but unable to connect all subunits by appropriate cross-links like composition of a functions are kinds of operations of functions. Some terms were missing in the concept map like identity function, independent and dependent variables. The accuracy score was rated as fluent since she wrote all one-to-one functions are onto, into and symmetric.

In the second concept map (Figure 4.82), she lost her meaningful clusters, instead, she had only subunits approaching to functions. Although there were some meaningful links and linking words, the concept map still lacked the cross-links necessary and also had meaningless linking words since it did not have meaningful clusters. Therefore, second map was rated as 2 (weak) out of 6 for its organization. The accuracy score of the second concept map was rated as 2 out of 4 since there were some errors like vertical line test was taken as a test for one-to-oneness of a function, and functions were separated into two kinds dependent and independent.

Throughout the concept map essay, Gizem also talked about her process of constructing concept maps but she admitted that she had some difficulties to remember the terms and definitions related to the function c

The organization of the first concept map of Deniz was rated as 3 (fair) out of 6 because she constructed some meaningful clusters but unable to connect all subunits by appropriate cross-links for example composite and inverse functions were not shown to be related. Moreover, her linking words were rather weak in terms of mathematical terminology such as, "…checked by…, …have types like…" . The accuracy score for the first concept map was 3 (fluent) out of 4.

In the second concept map, her organization score was 4 (fair) out of 6 one more level, she used very meaningful clusters and links among them and used linking words among them appropriately. However, she was not good at creating cross-links among clusters. The accuracy score of the second concept map was rated as 3 (fluent) instead of 4 (excellent) since vertical line test was taken as a test for one-to-oneness of a function.

In the concept map essay, Deniz described her process of constructing concept map as follows: "…I thought all the things about functions and listed them on a piece of paper. I did not directly start writing them in a concept map. After my list was finished, I tried to group the things that I wrote. For example, I wrote the properties of functions separately but I combined them under one title. After these steps were finished I started to write them as a web-like design…". Then she compared two concept maps in terms of related mathematical terms used in the concept maps and her construction process of the second map when the words are provided. She also mentioned that she felt limited when words were provided, and she criticized herself for the mistakes she did in the first concept map.

During the interview, she mentioned the same things as in her concept map essay, moreover she talked about the mathematical mistakes she did in the concept maps more as follows: "…For example, I used the vertical line test in the wrong place as a condition for one-to-oneness, also I separated functions in two as undefined and defined I don't know what I was thinking in that moment". Her final remarks for her concept maps were as follows : "Although there are more mistakes in the first one, I liked it more because I feel that it belongs to me". Overall evaluation of the three-staged concept map activity revealed that Deniz was generally able to construct meaningful subtopics (clusters) however she could not connect the related subunits with meaningful linking words and she missed some important terms.

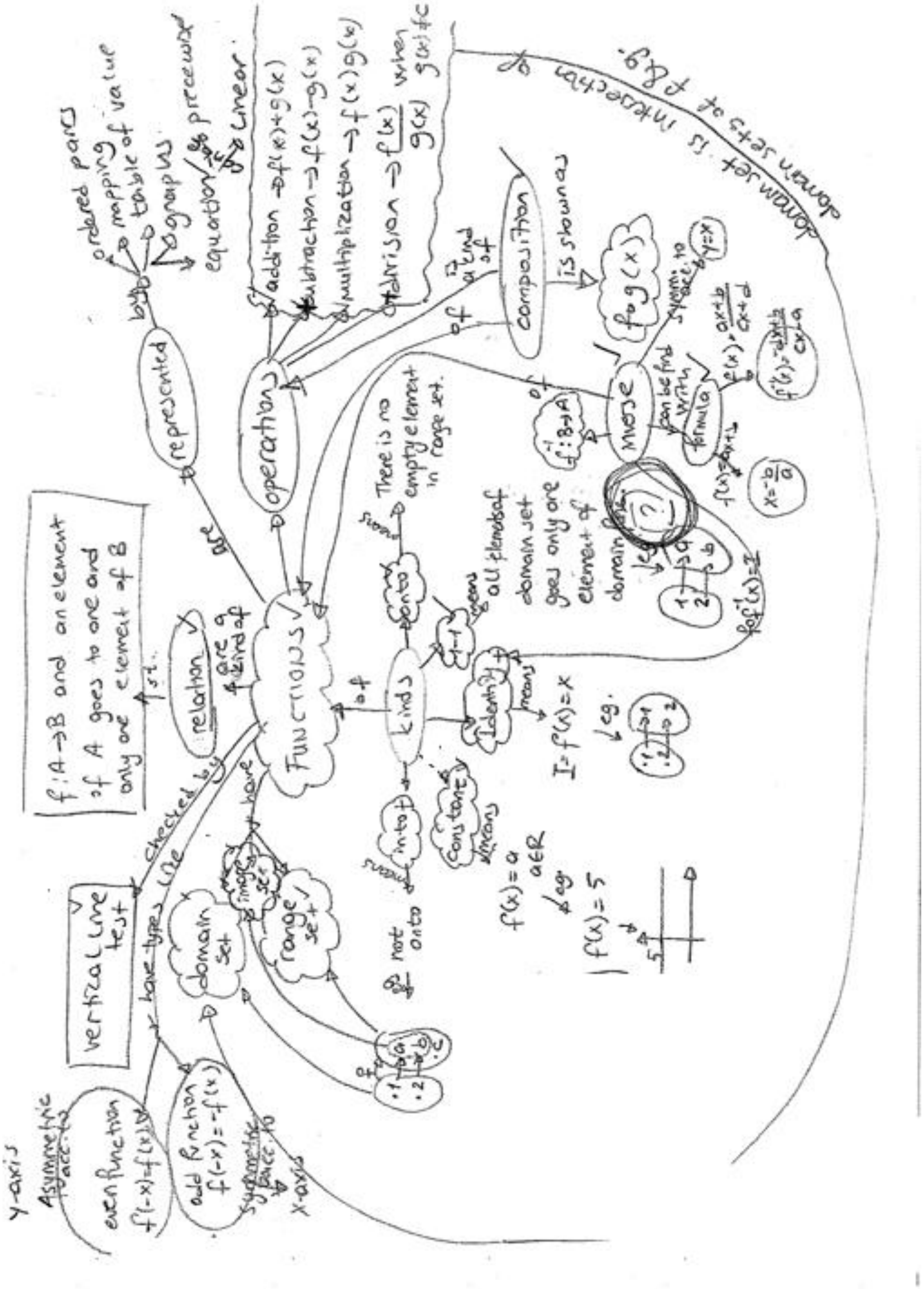


Figure 4.81: Deniz’s first concept map

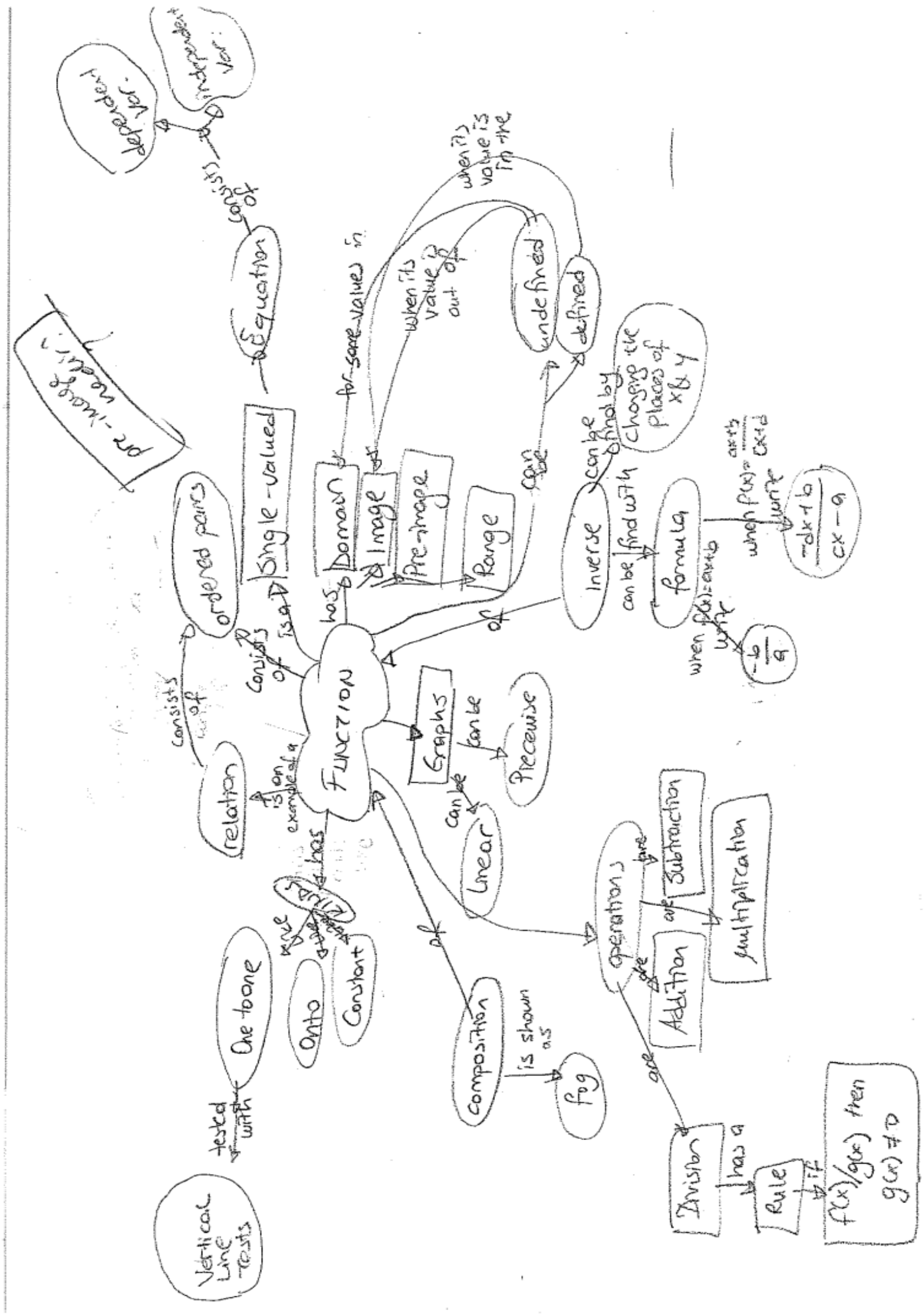


Figure 4.82: Deniz’s second concept map

For examining the connectedness of SMK in the teaching practices sequencing of the subtopics and examples were analyzed. Her sequencing of the subtopics and examples were accurate similar to her lesson plans. However, her sequencing of the examples were not connected enough all the time. Even, she used incorrect examples. Like, she defined f:A to B and g:B to C where A={-2,-1,0,1}, B={1,2,5}, C={2,3,6,8}and $f(x) = x^2 + 1$, $g(x) = x + 1$. She asked students to find $(fog)(x)$, $(gof)(x)$ and $(fof)(x)$ where according to definition of composite functions it was impossible to find $(gof)(x)$ and $(fof)(x)$. Besides, she made connections to previous mathematical topic once. For example, while teaching identity element of composition of functions, she referred to identity element of addition and multiplication operations.

4.1.3.4 Evidences of SMK from the Perspective of the Instruments Having Integration of Knowledge Components

Evidences of SMK were also searched through the instruments where there is an integration of all knowledge components exists. These instruments were vignettes, lesson plans, and teaching practices all of which assessed through the combined framework. Since Deniz said that she was influenced from the non-routine questions interview, while analyzing the instruments it was kept in mind. This is, because, these three instruments were collected after the administration of the non-routine questions interview (See Table 3.2).

4.1.3.4.1 Lesson Plans

In the lesson plans, participants were asked to teach composite and inverse functions but they were not specifically given an order which one to teach first. She started with composition of functions and her reason was as follows “I preferred to teach composition at first since in this way we can define inverse functions easily”. As stated before, she used formal definitions and Venn diagrams both for the composite and inverse function definitions in the lesson plans. She mostly used declarative and procedural questions in the lesson plans. The only conditional type question was used in the first lesson plan. The Table 4.22 summarizes representative sample of the example types used in that lesson; that is, if in a lesson only procedural questions were used only a procedural example was provided and if there were more than one type of example were used one example for each type was provided. The only conditional type question was a question inspired from the non-routine questions interview.

Table 4.22: Question Excerpts from Deniz's Lesson Plans

Lesson Plan	Questions	Knowledge Type
1	f={(-1, 1), (0, 0), (1, 1), (2, 4)} ve g={(1, 3), (0, 1), (4, 9)} fonksiyonları verilsin. f ve g fonksiyonlarını ok diyagramı yöntemiyle gösterimi aşağıdadır. **Örnek2:** f:R→R, f(x)=4x+5 g R→R, g(x)=3x-2 fonksiyonları tanımlanıyor. $(g\circ f)(x)$ ve $(f\circ g)(x)$ i bulalım.	Declarative Procedural
2	**Örnek1:** $f(x) = x^2$ ve $g(x) = \sqrt{x-1}$ fonksiyonları için $(f\circ g)$ bileşke fonksiyonunun kuralını ve tanım kümesini bulunuz. **Örnek2:** $f(x)=\frac{x+1}{x-1}$ ve g(x)=2x+1 kuralları ile verilen f ve g fonksiyonları için ve $(f\circ g)(A)=\{2, 3, 5\}$ olduğuna göre, A kümesini bulunuz.	Conditional Procedural
3	**Örnek:** Reel sayılar kümesinde $f(x)=2x^2-1$ fonksiyonu ve I(x)=x birim fonksiyonları veriliyor. $(f\circ I)(x)=(I\circ f)(x)=f(x)$ olduğunu gösteriniz.	Procedural
4	**Örnek2:** A={a, b, c} ve B={1, 2, 3} kümeleri veriliyor. f={(a, 2), (b, 3), (c, 1)} ise f^{-1} fonksiyonunu bulunuz. **Örnek1:** A={-1, 0, 1, 2} ve B={-3, -1, 1, 3} kümeleri üzerinde f:A→B, y=f(x)=2x-1 fonksiyonu birebir ve örtendir. f ve f^{-1} fonksiyonlarının elemanlarını liste yöntemi ile yazalım:	Declarative Procedural
5	**Örnek4:** f(x)=2x-1 fonksiyonunun tersini bulalım.	Procedural
6	aşağıdaki fonksiyonların tersini bulalım. 1. f:R-{1} → R-{2}, $f(x)=\frac{2x-3}{x-1}$	Procedural
7	**Örnek 2:** tanımlı olduğu aralıkta f ve g fonksiyonları için, f(x)=2x-6 $g(x)=\frac{3x-1}{x+1}$ $(g^{-1}\circ f)(a)=3$ olduğuna göre, a kaçtır?	Procedural

4.1.3.4.2 Vignettes Related to Composite Functions

In line with the previous discussion Deniz's vignettes were analyzed for evidences of SMK and those evidences were categorized according to the combined framework.

Firstly, the vignettes only related with the composition of functions were analyzed. The first vignette was intended to see to what extend participants resolve the conflict about the misunderstanding of the notation $h(x) = f(g(x))$ and mixing it with the ordinary multiplication $f(x).g(x)$. Deniz grasped the conflict given in the vignette correctly which requires from her to know the definition and notation of composition of functions (Figure 4.83). This was taken as an evidence for SMK.

Let $h(x) = f(g(x))$ and determine $f(x)$ and $g(x)$ if $h(x) = 2(x-5)^2$.

One student suggests that "$g(x) = (x-5)^2$ and $f(x) = 2$".
Another student interrupts "No $f(x)$ must be equal to $2x$ if $g(x) = (x-5)^2$".
A third student remarks "Well I think $g(x) = (x-5)$ and $f(x) = 2x^2$".
The class seems confused.

Then, lets examine our friends results. In the first one if $f(x)=2$ and $g(x)=(x-5)^2$ we have to write $(x-5)^2$ in the place of x in $f(x)$. When we write it in the right side of $f(x)$ there is no x. It is a constant function so the result has to be 2 whatever we write instead of x in $f(x)$. Can it be? No because the result has to be $h(x)=2(x-5)^2$. What is wrong here? Our friend understood it as the multiplication but it is not the case. Let's look to the second one. The result of $g(x)$ is $(x-5)^2$. When we put it instead of x in $f(x)$, we get $f(g(x)) = 2\cdot(x-5)^2$ It is the result that we want to reach. Is it the only

solution that we can find? Let's look to the third answer that our friend had given. He defined $g(x)$ as $(x-5)$ and $f(x)$ as $2x^2$. Then again we will put $g(x)$ instead of x in $f(x)$. Let's try to find the result, $f(g(x))=2(x$ Again we reached to the same result. Is that all or can we find another value for f and g functions?

Figure 4.83: Excerpt from Deniz's Vignette #1

For clearing up the confusion in the class she preferred explaining the steps to take composition once more, and showed her awareness about having multiple answers for the same question (See Figure 4.84). As a result her SMK for the first vignette was rated as Level 1.

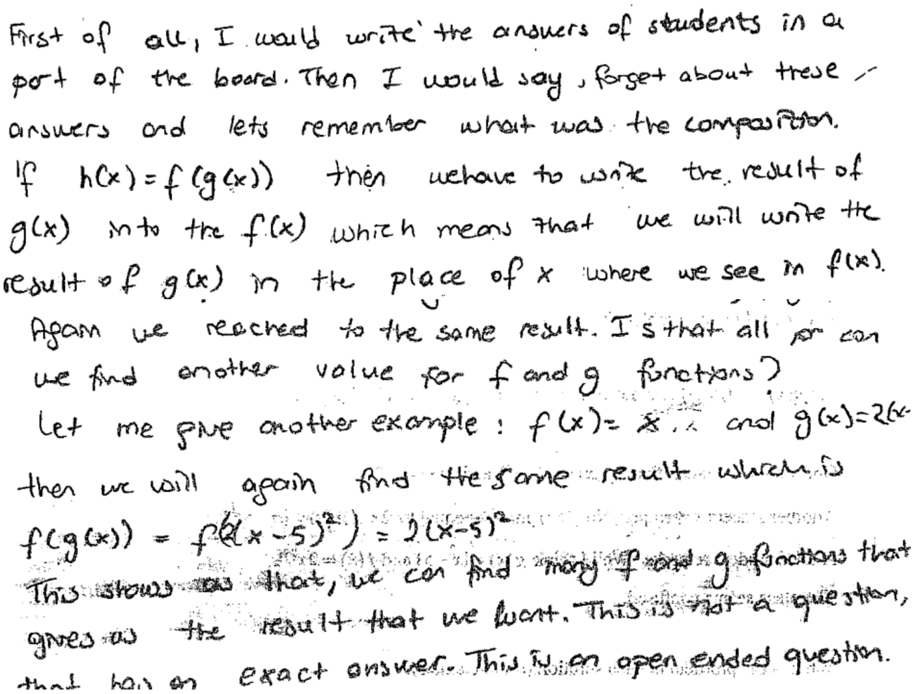
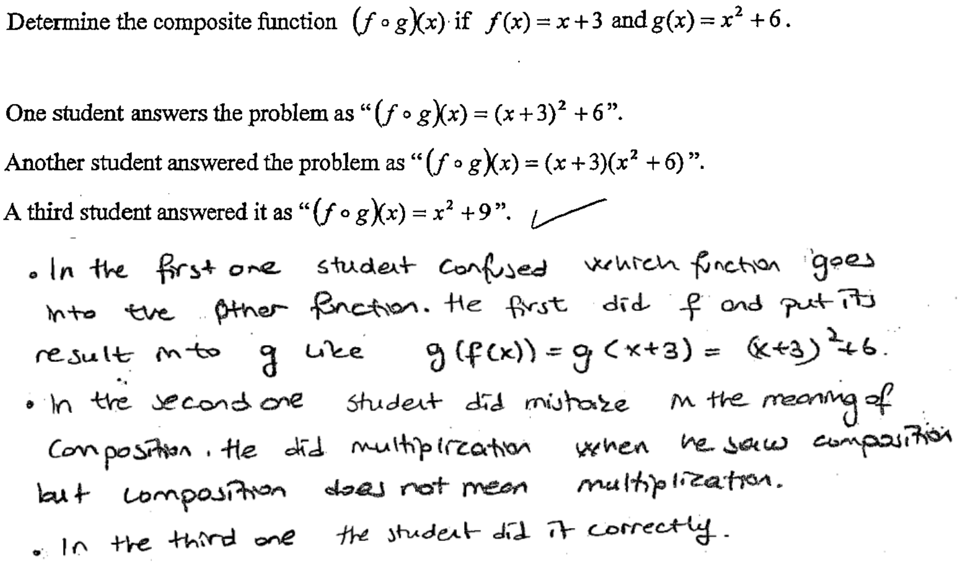

First of all, I would write the answers of students in a part of the board. Then I would say, forget about these answers and lets remember what was the composition.
If $h(x) = f(g(x))$ then we have to write the result of $g(x)$ into the $f(x)$ which means that we will write the result of $g(x)$ in the place of x where we see in $f(x)$.
Again we reached to the same result. Is that all or can we find another value for f and g functions?
Let me give another example: $f(x) = x$ .. and $g(x) = 2(x$
then we will again find the same result which is
$f(g(x)) = f((x-5)^2) = 2(x-5)^2$
This shows us that, we can find many f and g functions that gives us the result that we want. This is not a question, that has an exact answer. This is an open ended question.

Figure 4.84: Excerpt from Deniz's Vignette #1

In her explanation she used an informal explanation of the definition of the composition and constant functions in order to make the distinction between the composition and multiplication.

The second vignette was intended to see to what extend participants resolve the conflict about the mixing order of operations when taking compositions of functions and mixing it with the ordinary multiplication $f(x).g(x)$. As seen in Figure 4.85 she correctly identified the students' misunderstandings which requires a knowledge of definition and notation of composition of functions, which provided the researcher with the evidence of SMK.


Determine the composite function $(f \circ g)(x)$ if $f(x) = x + 3$ and $g(x) = x^2 + 6$.

One student answers the problem as "$(f \circ g)(x) = (x+3)^2 + 6$".

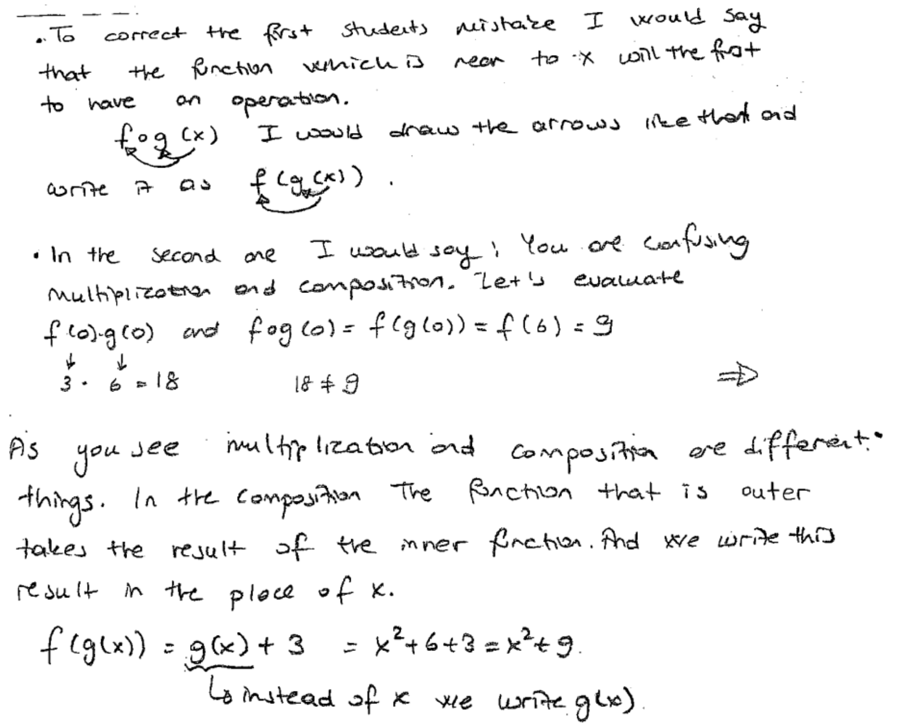
Another student answered the problem as "$(f \circ g)(x) = (x+3)(x^2+6)$".

A third student answered it as "$(f \circ g)(x) = x^2 + 9$". ✓

- In the first one student confused which function goes into the other function. He first did f and put its result into g like $g(f(x)) = g(x+3) = (x+3)^2 + 6$.
- In the second one student did mistake in the meaning of composition. He did multiplication when he saw composition but composition does not mean multiplication.
- In the third one the student did it correctly.

Figure 4.85: Excerpt from Deniz's Vignette #2

In her explanation (See Figure 4.86) she used an informal explanation and procedural questions in order to show how the order of composition of functions is important and changes the result. From these evidences her SMK was rated as Level 1.


- To correct the first students mistake I would say that the function which is near to x will the first to have an operation.

$f \circ g(x)$ I would draw the arrows like that and write it as $f(g(x))$.

- In the second one I would say: You are confusing multiplication and composition. Let's evaluate $f(0) \cdot g(0)$ and $f \circ g(0) = f(g(0)) = f(6) = 9$

$3 \cdot 6 = 18$ $18 \neq 9$ ⇒

As you see multiplication and composition are different things. In the composition the function that is outer takes the result of the inner function. And we write this result in the place of x.

$f(g(x)) = g(x) + 3 = x^2 + 6 + 3 = x^2 + 9$.

↳ instead of x we write g(x).

Figure 4.86: Excerpts from Deniz's vignette #2

The third vignette was intended to see to what extend participants resolve the conflict about the mixing composition with the ordinary multiplication when one of the functions is a constant function. Like in the previous two vignettes, she correctly identified the mistakes student made (See Figure 4.87) which gave the researcher evidence of knowledge about the definition of composition of functions and use of notation.

A student asked the following question.

Let f(x)=4, g(x)=2, and h(x)=x+3. Evaluate the followings

a. (fog)(7)
b. (goh)(x)
c. (hof)(x)
d. (hof)(3)

Student's answer is the following:

a. f(x)=4 and g(x)=2 then (fog)=(4.2)=8 (fog)(7)=56
b. (goh)(x)=2x+3
c. (hof)(x)=7
d. (hof)(5)=32 (h.f)(5)=(4x+12)(5) = 20+12 = 32

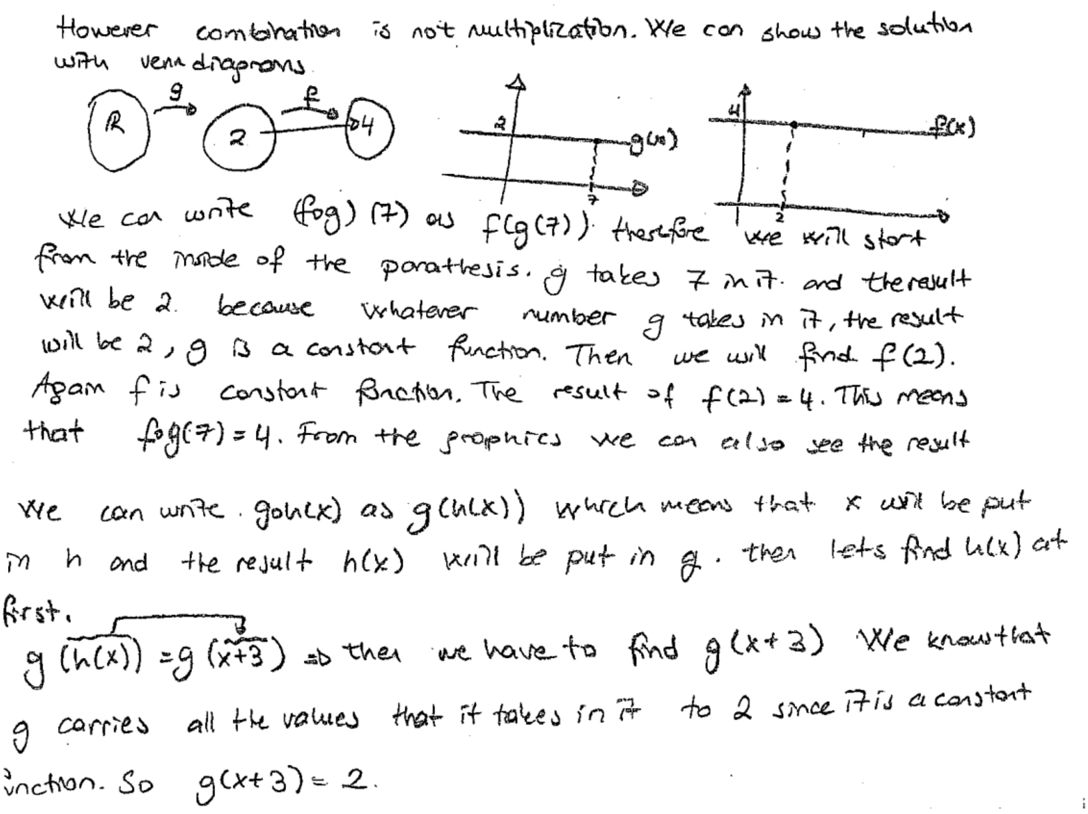
a) He thought combination as multiplication.
(f∘g) = f.g = 4·2 = 8
(f∘g) (7) = 8·7 = 56

b) g∘h.(x) = 2x+3 to find this result stu. multiplied g with the x of h. He has a misconception in multiplication and combination.

) hof (x) = 7 This time he puts the value of f(x) in to h and finds the correct result. h(f(x)) = h(4) = 4+3=7

Figure 4.87: Excerpts from Deniz's vignette #3

Her explanations included explanations for the composition with constant functions via Venn diagrams and graphs (See Figure 4.88). Also, informal explanations about how to find composition of two functions. In light of these evidences, her SMK was rated as Level 1.


However combination is not multiplication. We can show the solution with venn diagrams.

We can write (fog)(7) as f(g(7)). therefore we will start from the inside of the parathesis. g takes 7 in it. and the result will be 2. because whatever number g takes in it, the result will be 2, g is a constant function. Then we will find f(2). Again f is constant function. The result of f(2) = 4. This means that fog(7) = 4. From the graphics we can also see the result

We can write goh(x) as g(h(x)) which means that x will be put in h and the result h(x) will be put in g. then lets find h(x) at first.
g(h(x)) = g(x+3) ⇒ then we have to find g(x+3) We know that g carries all the values that it takes in it to 2 since it is a constant function. So g(x+3) = 2.

Figure 4.88: Excerpts from Deniz’s vignette #3

The fourth vignette was intended to see to what extend participants resolve the conflict about misunderstanding of the notation $h(x) = (fog)(x)$ while working backwards in composition of function problems. Deniz identified the source of the mistake just as taking composition as multiplication (See Figure 4.89). However, in the answer there is more than that student had difficulty understanding the meaning of (x) in the representation$(fog)(x)$.

One of the students voluntarily comes to the board and the solved the question as follows:

$$x^2+1=f(g(x))$$
$$x^2+1=(fx)(x)$$
$$f(x)=\frac{x^2+1}{x}$$

He had a mistake because he did combination as multiplization. He multiplied f(x) with x and equaled to $f(x)\cdot x = x^2+1$ which is wrong.
Combination means after finding the result of the inner function, putting this result into the other function.
Whıch is: $x^2+1 = f(g(x))$
$x^2+1 = f(x)$ ($\downarrow$)

This easily gives us the result.

Figure 4.89: Excerpts from the Deniz's vignette # 4

In her explanation, she just gave an informal explanation of how to take composition of two functions similar to other vignettes. However, her lack of understanding of the source of mistake is an evidence for a lack of SMK. Therefore, her SMK was rated as Level 0-1 from the combined framework.

The fifth vignette was intended to see to what extend participants resolve the conflict about the usage of analogy for definition of composite functions. She decided that given analogy for the definition of composite function is true, however, she also thought that this analogy might cause some problems due to the notation used (See Figure 4.90). Giving such a decision requires a necessary knowledge of the definition of the composite function, so this decision was taken as an evidence for SMK.

A teacher gave the definition of the composite function and explained it on the board to his/her students. However, some of his/her students stated that they did not understand it completely. Then teacher gave the following example to the students.

In order to clean and dry our clothes in a laundry we use two machines, washing machine and dryer, respectively.

Dry&Wash (clothes)

Dry[Wash(clothes)]=Dry[cleaned and wet clothes]=dried and cleaned clothes

Combination of these machines works can be considered as a composition of functions

• Since they are daily used machines, students can easily understand this examples. Therefore in my opinion it is a perfect example.

• This example will not cause a problem in my opinion. However on the notation we have to be careful. If we write Dry & Wash(clothes) there may be misunderstandings, if we write Dry ∘ Wash(clothes) it can be better.

• Dikiş makinesi ∘ Kesim makinesi (kumaş)
= Dikiş makinesi (kesilmiş kumaş)
= Ceket.

Figure 4.88: Excerpts from the Deniz's vignette # 5

Developing on this knowledge, she also gave an alternative true analogy (See Figure 4.90). SMK of vignette #5 was rated as Level 1.

The last vignette related with the composite functions is the thirteenth vignette, which is similar to the fifth vignette since it was intended to see to what extend participants resolve the conflict about the usage of analogy for definition of composite functions. As it can be seen from the Figure 4.91, this time an example was provided by the teacher and student's analogy needs correction. Even though she identified the error in student's answer and gave an informal explanation "the result of the first action is not goes into the other one" she did not attempted to provide an an alternative example. This statement was an evidence for the existence of the knowledge of the definition of composite functions and awareness of the conditions of existence of composite function. Therefore, her SMK was categorized as Level 1.

For explaining composite functions you gave the formal definition and then give the following example "Take grass (g) as the first input; then the cow (c) being a function "eats" the grass. Next, here comes a third animal, say the tiger "eats" the cow. The best way to denote this is t(c(g)).The brackets denotes the walls of the stomachs."

Then you want from your students to exemplify the composite functions by using such an example. One of your students gives the following example "I came from school by bus and I eat the cookies my mother made. Bus is my first function and cookies is must second function."

Teacher's example is good. But the letters can confuse student's mind, because we generally denote function as g but here it is a variable.
The student's example is not correct.
According to his thought, he is the variable and bus and the cookies are the functions.
However "coming" to the house and "eating" the cookies are functions.
This example cannot be corrected because it is like listing the things that he did, the result of the first action is not goes in to the other one.

Figure 4.91: Excerpts from the Deniz's vignette # 13

The tenth vignette was aimed to explain the participants' understanding of combined use of inverse and composition of functions in questions. The case in the vignette includes a student's answer to a question and a dialogue between the student and the teacher about the answer (See Figure 4.92).

If $f(2x+1)=2x-1$ find $f(3x)$ in terms of $f(x)$ and explain the steps of your solution.

Then the students solved the question correctly as follows:

$$\left.\begin{aligned} y&=2x+1\\ x&=2y+1\\ x-1&=2y\\ y&=\frac{x-1}{2}\end{aligned}\right\} \underbrace{f(x)=2.\frac{x-1}{2}-1=x-2}_{*} \; then \; f(3x)=3x-2$$

$$f(x)+2=x \Rightarrow f(3x)=3(f(x)+2)+2 \Rightarrow f(3x)=3f(x)+4$$

After the solution made, teacher wants from student to explain what she did in the step indicated by *. She said that "I have to get f(x) so that I could calculate f(3x). For getting f(x) I made the necessary calculations as you did in our previous examples".

Furthermore, teacher wants from student to explain what she did in the $f(x)+2=x$ step. She said that "we have to single out x from the equation as you did in our previous examples". However, she couldn't explain what she did.

Figure 4.92: Excerpts from the Deniz's vignette # 10

Deniz solved the question step by step by providing verbal explanations (See Figure 4.93). Her statements were taken as an evidence for understanding of the application of the composition of functions and her SMK was rated as Level 1.

First of all as the student mentioned before, we have to find f(x). To find it we write x instead y, and y instead of x and leave y alone. By this way we find, $y=\frac{x-1}{2}$ and write this value to the places of x. At the end we find f(x) as x-2. Now it is easy to find f(3x) as 3x-2. To find f(3x) in terms of f(x), in the right hand side of f(3x) there mustn't be x, there must be f(x). To cancel x we have to find what we will write instead of x, in terms of f(x). To find it in the equation f(x)=x-2 we can leave x alone and find the value of x in terms of f(x): x=f(x)+2. This value will help us to find f(3x) in terms of f(x).

$f(3x)=3x-2$ and $x=f(x)+2$

$f(3x)=3(f(v)+2)-2=3f(x)+4$

At the end we find f(3x) in terms of f(x).

Figure 4.93: Excerpts from the Deniz's vignette # 10

The eleventh vignette was intended to see to what extend participants resolve the conflict about the students' misunderstandings about the use of the fact $fof^{-1}=$

I while solving questions in relation to composite and inverse functions. This time she stated the student's problem however her explanation (See Figure 4.94) did not explain why student was not correct. Since she was able to identify the problem and knows what a composite and an inverse function is and use the appropriate notation for both of them and also used Venn diagram during her explanation. Her SMK was rated as Level 1 from the combined framework.

A student of yours calculates the inverse function of the function $f(x) = 3x - 4$ and the answer obtained is $f^{-1}(x) = -2x + 4$. The students checks his work, and he combines $f(x)$ with $f^{-1}(x)$ he gets x. After the confirmation, he thinks that these two functions are inverses of each other.

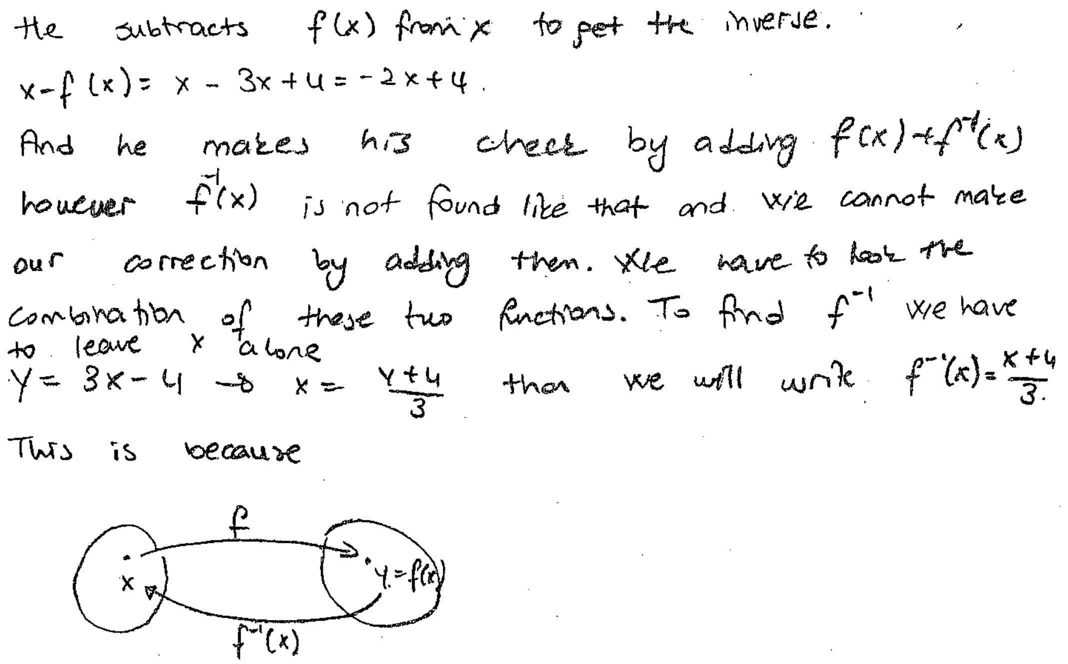


Figure 4.94: Excerpts from the Deniz's vignette # 11

When Deniz's knowledge of composite functions were evaluated through the vignettes (1, 2, 3, 4, 5, 13, 10, and 11 where last two also include knowledge of inverse functions) it was seen that she knows the definition of composition of functions in different vignettes. She mostly used informal explanations which were like explaining procedures, and used Venn diagrams twice (vignettes # 3 and 11) and once graphs in vignette # 3 her explanations. Moreover, she showed no evidence for the connectedness of her knowledge of composition of functions. Her SMK was rated as Level 1, except for the vignette 4 which was rated as Level 0-1 which were similar to her levels in the lesson plans.

4.1.3.4.3 Vignettes Related to Inverse Functions

Similarly, the vignettes related with the inverse functions were analyzed. The sixth vignette was intended to see to what extend participants resolve the conflict about the usage of the power -1 in the function notation. Deniz diagnosed the error (See Figure 4.95), which gave us an evidence for her knowledge of the term "inverse" in mathematics. Her further explanation (See Figure 4.96) for clearing up the confusion includes some rule based explanations and a procedural question. As a result, her SMK was rated as Level 1.

Determine the inverse ($f^{-1}(x)$) of the function $f(x) = x - 4$.

Five different solutions come out from the class.

First one is "$f^{-1}(x) = \frac{1}{x-4}$".

The second one is "$f^{-1}(x) = \frac{1}{x} - 4$".

The second is "$f^{-1}(x) = -x - 4$".

The third one is "$f^{-1}(x) = -x + 4$".

The last solution is "$f^{-1}(x) = x + 4$".

In the first solution student find multiplicative inverse.
In the second " " " multiplicative inverse of x.
In the 3rd " " " additive inverse of x.
In the 4th " " " addive inverse of (x−4)
In the last solution student did in the correct way.

Figure 4.95: Excerpts from the Deniz's vignette # 6

The inverse of functions can be find by changing the places of x and y, let change the place of x and y in our problem. In this question $y = x - 4$ then lets write it as $x = y - 4$ → from this equation when we make y alone we get $x + 4 = y$. Our inverse function becomes $f^{-1}(x) = x + 4$. Or the other thing that we can do is that; since the problem is easy here we can easily see what we have to put instead of x. to make right hand side of the equation x. It is easily seen,

$f(x) = x - 4$

$(x+4)$

$f^{-1}(x) = x + 4$

Our function is $f(x) = x - 4$.

If its inverse is $f^{-1}(x) = \frac{1}{x-4}$ as it is said in the first one

$f(2) = -2$ → $f^{-1}(-2) = 2$ the inverse must be 2

If we do it with the eqn. $f^{-1}(-2) = \frac{1}{-2-4} = -\frac{1}{6} \neq 2$

∴ the inverse of the function cannot be multiplicative inverse,

if it is $f^{-1}(x) = \frac{1}{x} - 4$ then $f^{-1}(-2) = -\frac{1}{2} - 4 = -\frac{9}{2} \neq 2$

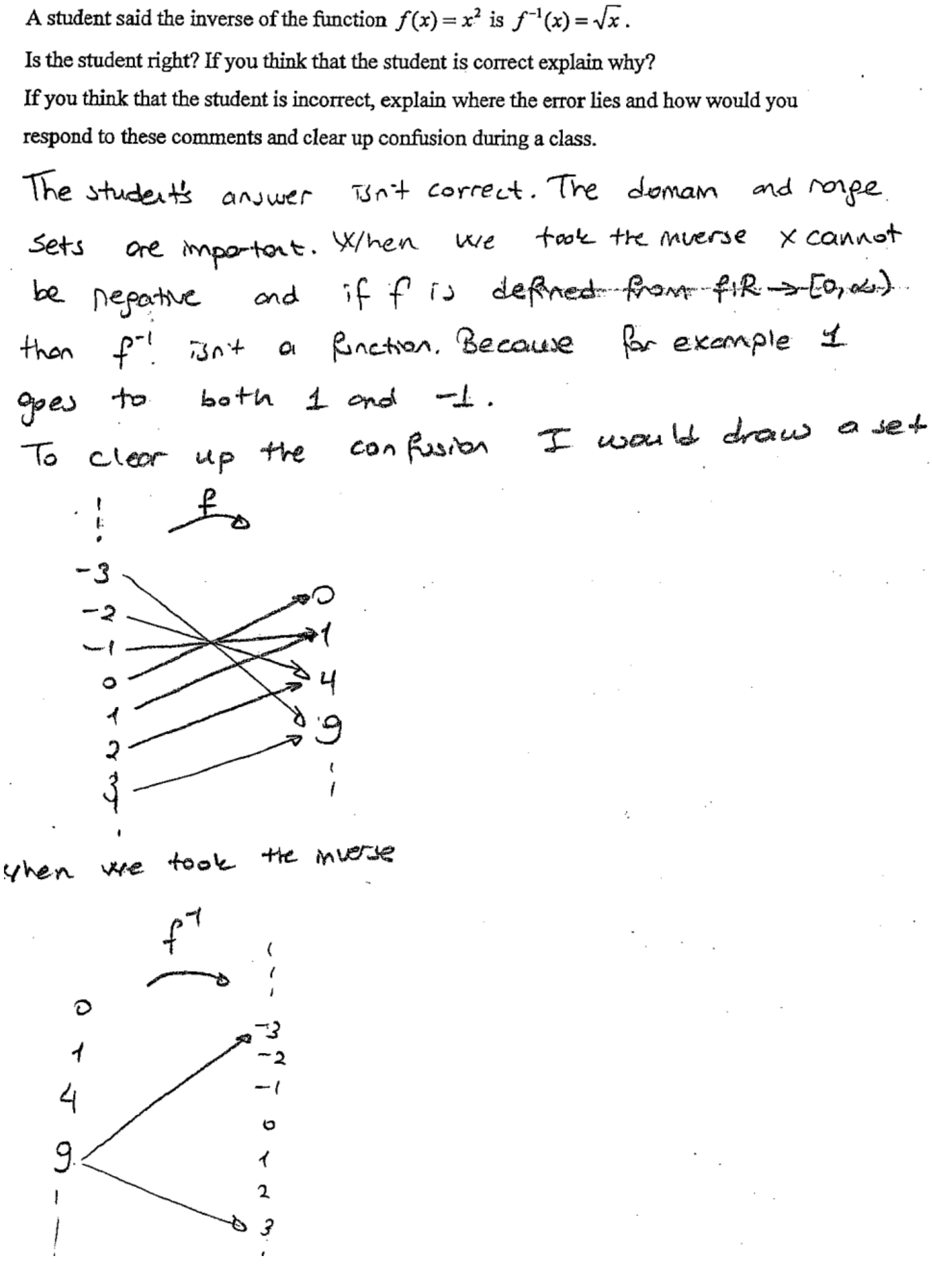
if it is $f^{-1}(x) = -x - 4$ then $f^{-1}(-2) = 2 - 4 = -2 \neq 2$

if it is $f^{-1}(x) = -x + 4$ then $f^{-1}(-2) = 2 + 4 = 6 \neq 2$

therefore these equations for inverse are not true

Figure 4.96: Excerpts from the Deniz's vignette # 6

The seventh vignette was intended to see to what extend participants resolve the conflict about the students' misunderstandings about the existence of inverse functions. Her identification of the problem of the student's solution gave us an evidence that she knows that for existence of an inverse function it has to be a one to one function (See Figure 4.97). While clearing up the confusion, she explained the case via a numerical example. Hence, her SMK was rated as Level 0-1.


A student said the inverse of the function $f(x)=x^2$ is $f^{-1}(x)=\sqrt{x}$.

Is the student right? If you think that the student is correct explain why?

If you think that the student is incorrect, explain where the error lies and how would you respond to these comments and clear up confusion during a class.

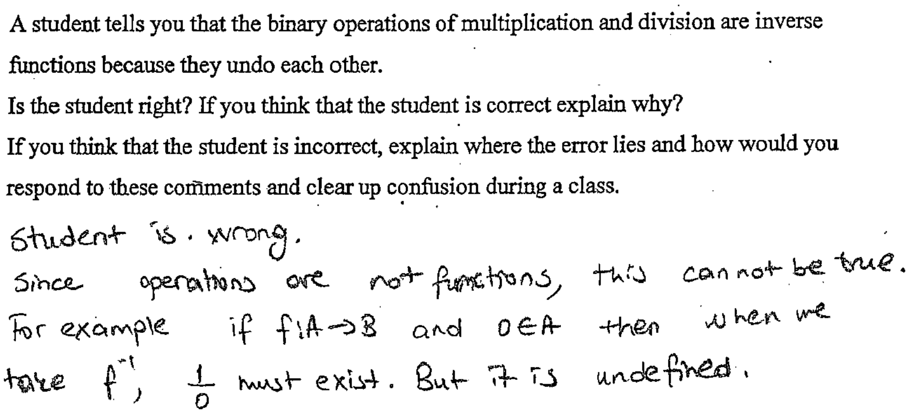
The student's answer isn't correct. The domain and range sets are important. When we took the inverse x cannot be negative and if f is ~~defined from f:R → [0,∞)~~ than f^{-1} isn't a function. Because for example 1 goes to both 1 and −1.

To clear up the confusion I would draw a set

when we took the inverse

Figure 4.97: Excerpts from the Deniz's vignette # 7

In the eighth vignette the aim was to see to what extend participants understand the idea of inverse function as undoing. As seen in Figure 4.98 she diagnosed that the student had difficulties in recognizing the difference between the operations and functions. However, he explanation was rather weak. Combining these evidences, her SMK was rated as Level 0-1.



A student tells you that the binary operations of multiplication and division are inverse functions because they undo each other.
Is the student right? If you think that the student is correct explain why?
If you think that the student is incorrect, explain where the error lies and how would you respond to these comments and clear up confusion during a class.

Student is wrong.
Since operations are not functions, this can not be true.
For example if $f: A \to B$ and $0 \in A$ then when we take f^{-1}, $\frac{1}{0}$ must exist. But it is undefined.

Figure 4.98: Excerpts from the Deniz's vignette # 8

The ninth vignette was intended to see to what extend participants resolve the conflict about the usage of analogy for definition of inverse functions. As it can be seen from the Figure 4.99 teacher's example was provided and her ideas about the analogy was asked. She did not notify the error that both functions have the same name by saying " … it is a good example I think all of them can understand it easily…".

A teacher gave the definition of the inverse function and explained it on the board to his/her students. However, some of his/her students stated that they did not understand it completely. Then teacher gave the following example to the students.

If you think of school bus as a function which takes you from home to school at the morning, then the school bus that takes you back from school to home is the inverse of the first function.

- It is a good example in my opinion. All of then can understand it easily.
- This will not cause any problem I think.
- When we surf in internet we pass to other page when we click on next page button (f function). To go back we press on back page button (f^{-1} function)

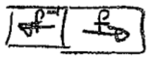


Figure 4.99: Excerpts from the Deniz's vignette # 9

However, her own analogy example gave us an evidence that she knew the definition of inverse functions well enough. Combining all evidences, her SMK level for this vignette was identified as 0-1.

The twelfth vignette was also related with the use of analogy for the definition of inverse functions but this time there exists two analogies about inverse functions one from the teacher and one from a student (See Figure 4.100). She analyzed both of them and diagnosed the error in the student's analogy correctly, then, gave an explanation for the case. Then, she corrected the student's example and showed her knowledge of the conditions for the existence of inverse functions without saying 1-1 and onto. From all these evidences her SMK was rated as Level 1.

For explaining inverse functions you gave the formal definition and then gave the following example "When someone calls you on the phone, he, or she looks up your number in a phone book (a function from names to phone numbers). When Caller ID shows who is calling, it has performed the inverse function, finding the name corresponding to the number."

Then you want from your students to write down such a function and define inverse of it. One of your students gives the following example "My function is something we see everyday on supermarket's cash registers (yazarkasa). For each item we buy there is a corresponding price on the receipt (fiş), so the inverse of this function is for each price there is a corresponding item."

Teacher's example is good in my opinion. It can be used in the class easily.

I didn't like the student's example.

item $\xrightarrow{f}$ price on receipt, with f^{-1} going back from price on receipt to item

It seems the function can be correct however there can be two items which has the same price than f^{-1} cannot be a function.

The example can be corrected as, each item has a code on the receipt their codes can be seen than when we took the inverse each code will go to its own item.

But if there are items with same code this example will not work again.

Figure 4.100: Excerpts from the Deniz's vignette # 12

When Deniz's knowledge of inverse functions were evaluated through the vignettes (6, 7, 8, 9, 12, 10, and 11 where last two also include knowledge of composite functions) it was seen that she knows the explanation for the concept of inverse function, its notation, the term "inverse" in mathematics good enough to try to solve conflicts in different cases. She used procedural and declarative questions. Sometimes, she was not able reflect her knowledge while choosing the procedural questions which hinders her understanding. As stated above for each vignette, her SMK was mostly rated as Level 1, similar to her levels in the lesson plans. Also, her SMK was rated three times for Level 0-1.

4.1.3.4.4 Teaching Practices

Keeping all these in mind, evidences of SMK were also searched in the teaching practices thorough the examples solved and explanations Deniz provided to the class or a student.

When the examples solved through the teaching practices were analyzed it was seen that she mostly used the same examples within the lesson plans with the same order. Her additional examples were generally analogies for explaining the concepts composite and inverse (See Table 4.23). Her use of the analogies were limited to just the beginning of the concept, she did not refer them at the rest of the teaching practices.

Table 4.23: Deniz's Examples from Teaching Practices

Teaching Practice	Example(s)
December 1	• "Functions are like machines and sometimes a few machines can work together. This situation can be considered as combination of functions."
December 7	• "Suppose everyday you are coming and turning back from school with the school bus. This case is called as inverse functions in mathematics". Apart from telling the analogy by using Venn diagrams she showed it on the board.
December 15	• If a function f is going one step front inverse of f is going one step back.

The definitions she used in the teaching practices were the same with the lesson plans. Except, she had a mistake in the inverse function definition she did not make the same mistake in the teaching practice. Even though she was aware of the fact that she struggled with the conditions of existence for the composite and inverse functions, she did not put much emphasis on these during the teaching practices.

Her explanation to the class or a student was changed according to the representation of the question. If the question was given in mathematical format her explanation was generally an oral review of the procedures just completed. Moreover, even the wording of the explanation was generally the same with the previous one. And also, her explanations in the vignettes and teaching practices were similar to each other. She rarely got questions about the concepts when she got her answer was restating the definition by giving oral explanations like in the explanations of the questions. This case was similar to her explanations given in the vignettes for composite and inverse functions. Her SMK levels in the teaching practices were rated mostly as Level 0-1 and once for Level 0.

4.1.3.5 Summary of Deniz's SMK

Evidences of SMK were searched through two groups of instruments. First group (survey of function knowledge, non-routine questions, definitions journals, and concept mapping activity) was only related with assessing the SMK in certain aspects. In the second group (vignettes, lesson plans, and teaching practices), SMK was searched through instruments where there was an integration of knowledge exists, so they were designed for assessing all components of pedagogical content knowledge.

First group of instruments revealed that even though Deniz was not able to give any formal definition with all the requirements, when applying the rules for composite and inverse functions and deciding the existence of them, she did not experience any problems except for the questions given in graphical form. Her main problem both in the survey and non-routine questions interview was that the concepts 1-1 and onto and those were as being conditions of existence for inverse functions.

Except for the survey of function knowledge, she used formal definitions for definition of inverse functions. However, in the teaching practices she also gave informal definitions and even analogies to support understanding of the conditions for the existence of the inverse functions.

Deniz showed very limited evidence for the connectedness of her knowledge of composite and inverse functions in all phases of the assessment. However, she was generally able to construct meaningful subtopics (clusters) but could not connect the related subunits with meaningful linking words and she missed some important terms.

Her lack of knowledge about the conditions of existence in the first group of instruments, was not evidenced directly through the lesson plans, vignettes, and teaching practice. However, she did not put much emphasis on these topics and her explanations to students were not satisfactory.

Moreover, her sequencing of the subtopics and questions in the teaching practices were analyzed. The results revealed that she sequenced the sub topics and the questions in a logical order.

Apart from these the instruments in the second group were analyzed with respect to the same combined framework. The analysis revealed that her SMK levels in the vignettes and lesson plans mostly rated as Level 1, whereas her SMK levels in the teaching practices rated as Level 1.

4.1.4 Comparisons of Participants' SMK

When the scores analyzed it was seen that all participants performances on procedural questions far more than the other types of knowledge questions. However, their performances on declarative and conditional questions were close to each other which was meaningful since conditional knowledge requires good understanding of declarative knowledge namely definitions. While Gizem and Deniz were more successful on the declarative questions compared to conditional ones, for Yeliz the order was vice versa. In order to elaborate on participants' views about the questions, during the follow up interview on survey of function knowledge they were generally talked about the procedural questions as easy questions. This supported the high points they got in the survey from procedural questions.

Even though there exists a few differences among the participants, they all stated that numerical questions are easier than the verbal ones and they felt more comfortable when dealing with such questions. Moreover, during the follow-up interview all participants talked about some of the questions as important regardless of the question was solved by solved the participant correctly or not. When these questions were analyzed it was seen that these questions were declarative questions related with real life conversions and all conditional questions.

Participants of the study faced with the conditional type questions also during the non-routine questions interview. There were 6 questions and total of 10 items in the interview. First and second question in the interview were related with the definition of composition of functions. Third and fourth questions again related with the definition of composition of functions but this time functions were given as graphs. Fifth question was related with definition of an inverse function and six question was again related with the definition of inverse functions but this time functions were given as graphs.

Preservice teachers scores before a hint provided by the researcher were given in Table 4.22. When these scores were analyzed it was seen that participants were more successful when the functions were given in equation format (questions 1,2, and 5) rather than graph (3, 4, and 6).

Table 4.24: The participants' scores for the items in the non-routine questions interview

		Yeliz	Gizem	Deniz
Composition	1 a	1	1	1
	1 b	1	4	2
	2	1	4	4
	3 a	0	1	1
	3 b	0	4	2
	4	0	0	0
Inverse	5 a	4	4	4
	5 b	4	4	2
	6 a	1	1	1
	6 b	3	4	4
	Total	15 37%	25 63%	21 53%

Maximum points:10*4= 40 points

Also, when Tables 4.1, 4.2, and 4.22 are compared in terms of the conditional questions it can be concluded that while Yeliz and Deniz had lower scores in the non-routine interview, whereas Gizem had higher scores during the non-routine questions interview, she mentioned this in the interview as "I liked working with graphs". Conditional questions in survey of function knowledge were different from the ones in the non-routine questions interview in that questions in the survey of function knowledge did not require a change in the representation of function. Besides, all participants gave a deficient answer for the question about the different representations of functions (question 2) in the survey of function. This can be an evidence for why Yeliz and Deniz decreased their scores.

From the evidences we got from the survey of function knowledge and the non-routine questions interview it can be concluded that all participants difficulties were mostly related with concepts. Yeliz's main difficulty was checking the conditions for the existence of function, inverse functions and composite functions. Gizem experienced problems in expressing definitions for functions, composite functions and inverse functions. Deniz's main problem both in the survey and non-routine questions interview was that the concepts 1-1 and onto and those were as being conditions of existence for inverse functions.

Yeliz experienced difficulties when questions were not given in mathematical notation. Gizem always tried to convert graphical questions to mathematical notation, and otherwise she could not solve it. Deniz did not experience any problems regardless of the problem type in the survey. In contrast, in the non-routine she experienced problems when the questions were given in graphical form.

Moreover, they all mentioned the effect of the survey and the non-routine questions by stating I think we should see these kinds of questions for every topic in the school curriculum because these questions affected me in the positive way and made me think about the concept and how should I teach it.

Preservice teachers' lesson plans and teaching practices were analyzed in order to see the how the knowledge types declarative, conditional, and procedural were reflected on the examples chosen, the order of questions, the homework questions etc. It was seen that there were two conditional questions in the all lesson plans of each participant and they were inspired from the non-routine questions interview and survey of function knowledge. Although they all used the same questions how they used it in the lesson plan and applied it during their teaching practice differentiated too much. Deniz and Yeliz put the questions in their lesson plans without any connection to the previous or the next question whereas Gizem used similar questions in order to lead students to solve an unfamiliar problem. Therefore, when their teaching practices of the related lesson plans were analyzed it was seen that the lesson was more fluent an understandable in Gizem's class and using a conditional question was reached its' aim whereas in Yeliz and Deniz classes conditional questions were not fully understood and students complained about it. In the lesson plans there were only a few declarative questions like "Do you remember what a function is?".

Table 4.25: Participants' Scores of Concept Map 1 and Concept Map 2

	Concept Map 1		Concept Map 2	
	Organization	Accuracy	Organization	Accuracy
Yeliz	Fair *(3)*	Fluent *(3)*	Weak *(2)*	Fluent *(3)*
Deniz	Fair *(3)*	Fluent *(3)*	Good *(4)*	Fluent *(3)*
Gizem	Fair *(3)*	Fluent *(3)*	Weak *(2)*	Good *(2)*

When the scores of the first concept map were analyzed it was seen that all participants got the same score which was 6 out of 10 (See Table 4.23). For the second concept map, Yeliz and Gizem were decreased their scores, whereas Deniz increased her score. Examples of scored concept maps can be seen in Appendix M.

Table 4.26: Participants' Scores of Concept Map Essay

	Concept Map Essay		
	Communication	Organization	Mechanics
Yeliz	Weak *(2)*	Adequate *(2)*	Acceptable *(1)*
Deniz	Weak *(2)*	Adequate *(2)*	Acceptable *(1)*
Gizem	Weak *(2)*	Adequate *(2)*	Acceptable *(1)*

When the concept map essay scores were analyzed it was seen that participants communicated their thought processes in a weak way since none-of the participants did not mention how they constructed the mathematical connections in their concept maps. All of their concept map essays' organization was rated as adequate since all essay lacks the knowledge of how they made the transitions. There were only a few violations in grammar; all essays were rated as acceptable in terms of mechanics.

In line with this, when the other instruments were analyzed in terms of connectedness of SMK, the picture for Yeliz and Deniz was similar. They showed very limited evidence for connectedness of their SMK where Gizem was one step beyond them.

The second group of instruments in the study was designed for assessing all components of pedagogical content knowledge. While deciding the SMK levels of the participants through the second group of instruments, sequencing of the subtopics and the questions used, types of questions used, correctness of the knowledge presented, and their type of their explanations.

In terms of sequencing of the subtopics and questions presented, Deniz and Gizem showed a similar picture in which they sequenced the subtopics and questions in a logical order, whereas Yeliz sequenced only subtopic in a logical order not the questions. Type of the questions used showed similarity among all participants. They mainly used procedural questions, from time to time used declarative questions and rarely used conditional type questions.

Throughout the instruments only Gizem and Deniz had a notation mistake in the lesson plan and they did not make the same mistake in the teaching practices.

When the type of the explanations provided during the vignettes and the teaching practices were analyzed it was seen that throughout the vignettes their explanations were all procedural review of the concept via numerical examples and sometimes questions in different representations. Only the frequency of the usage of the different representations changed according to participants, where Gizem was the most frequent user. In the teaching practices, however, the explanations the participants used was different for all of them. Yeliz used mostly used procedural explanations. If the explanation was about a solution of a question then she just used procedural oral overview of the solution and if the explanation was about a concept she preferred the analogies used and Venn diagrams. Gizem also mostly used procedural explanations, and similar to Yeliz her answer was changed in accordance with what she was going to explain. If she was going to explain a solution of a question she preferred resolving the questions via explanations and even by referring the concept and if she was going to explain a concept she always referred back to the basic analogy she gave about the concept and also used Venn diagrams. Lastly, Deniz only used oral procedural explanations for the solutions on the board.

The participants' SMK were assessed through the combined framework. According to that Yeliz's SMK level was mostly rated as 0-1, Gizem's and Deniz's SMK levels were mostly rated as 1. Only once Yeliz's and Gizem's vignettes were rated as Level 1-2. None of the participants SMK level was rated as Level 2 through these instruments since they did not show evidences of connectedness of their knowledge, interpretation and use of different representations of questions in a logical order in different knowledge types.

4.2 General Pedagogical Knowledge

This section summarizes preservice teachers knowledge of the general pedagogy reflected through different instruments as a component of their pedagogical content knowledge. It was assessed through three instruments (vignettes, lesson planning activity, and teaching practices).

For describing the GPK of the participants teaching practices and lesson plans were analyzed simultaneously. First, general characteristics of each participants' teaching were described. Then, some mechanics of the lesson plan (prerequisite skills, objectives, methods, materials) and how participants completed those were discussed. Next, each component of the lesson plan (introduction, development, closure, and evaluation) were discussed in relation to teaching practices. During the development part, participants' vignette evaluations were also included. Lastly, the scores of the lesson plans and teaching practices were compared.

4.2.1 Yeliz's GPK

Yeliz's use of OHP needs improvement. This is because the transparencies prepared was too small, and while using those instead of showing the necessary part she show the whole transparency which caused students deal with the previous or next question and/or topic. She always used students' names, dealt with the students individually and checked the students' work by wandering in the class. She always gave enough time for students to take in notes to their notebooks. She warned the students who were trying to disturb the lesson. After the first two lesson, she checked whether students were obeying the school uniform rule, and also she warned students who were late for the class to take the permission paper.

Most of the objectives she used was problematic because they were written as a teacher to do list. For example, one of her objectives was as follows "to make a clear beginning to the composition of functions". As it can be seen from this objective, it was not a real objective for a lesson, further examples of her objectives were given in Table 4.27. It can be concluded from this table that she was not aware of the verbs that should be used while writing the objectives. In a similar vein, her methods/techniques section also lacked the necessary terminology and they were all like what teacher will do during the lesson (See Table 4.28).

Table 4.27: Objectives Yeliz used through the lesson plans

Lesson Plan	Objectives
1	•To make a clear beginning to "Composition of Functions" •To solve the basic questions related the topic
2	•To remind the definitions, range and domain of composite functions •To teach the properties of composite functions •To solve mixed questions about the topic
3	•To remind one to one and onto function •To remind the previous lessons with the help of small competition •To teach inverse of functions clearly •To solve questions to reinforce the knowledge about the topic
4	•To remind the definition of inverse function •To teach how to find the inverse of function •To solve questions to make the topic clear
5	•To make a review of composite and inverse functions •To teach writing one function in terms of another function •To evaluate students' knowledge about the topic

Table 4.28: Methods/Techniques Yeliz used through the lesson plans

Lesson Plan	Methods/Techniques
1	•To use the students' previous knowledge (cartesian product) for the beginning of the topic •To use diagrams to make the students understand concretely •To use real life examples to provide concrete understanding
2	•To use the real life example which was used in the previous lesson as a review •To use questions related the topic to make the properties concrete enough
3	•To use a competition to take students' attention •To ask questions to the students about their previous knowledge on one to one and onto function •To use real life examples
4	•To check the students whether they learned the definition of inverse function or not •To choose different kinds of questions
5	•To apply a small competition to make an enjoyable review and to take students' attention •To let students solve questions on the board to make them to learn better

For each lesson, she listed prerequisite skills required like "knowing the definition of the function", materials needed like "whiteboard, board markers, and OHP". Moreover, she distinguished the introduction, development, and closure in the procedure part of the lesson plan as indicated in the lesson plan format.

When her introductions were analyzed it was seen that except for the first lesson her introductions aimed at reviewing the previous lesson. In the first lesson plan, by giving a numerical example she introduced the new topic procedurally. When these introductions were compared with the teaching practices, it was seen that her introductions in the teaching practices showed some differences. She also gave the agenda of the day and sometimes included students in the review part.

In the development part of the lesson plans, she did not give any clue about how she is going to precede the written material during teaching. Although the ordering of the subtopic were good, the orders of the questions were always bad both in the lesson plans and teaching practice. This caused transition problems to students from question to question, and resulted in students not to cover the main idea of the concept. In one of the lessons students said that “why are we moving so fast can we do a similar example for the last one, I couldn’t understand it in this way” rest of the class agreed with this students’ ideas.

Most of the teaching practices were teacher centered even if she asked a question mostly she did not wait for students to answer and she answered herself. Focus of the lesson was not meaning construction but to solve questions. She was always procedural even when she was introducing the concepts. Instead of reaching definitions, she directly provides definitions to students. After that students generally said that “I did not understand” after rewording the definition she generally said “If you understand the examples don’t worry about the definition”. In one of the examples she used the symbol “∀” and when students asked what it is, she explained once after that students asked it again and in a similar vein she said that don’t confuse yourself with this notation it is not important for us now.

Generally while explaining the students’ misunderstandings she used both oral and written explanations, and gave a similar numerical example for the problem. From time to time she only used oral explanations which caused problems in students’ understandings. Twice she organized a knowledge contest including questions in different difficulty levels in specific categories to review the previous topics. Although these two were the only two student-centered activities in all of her teaching, the aim was not to increase students’ understanding; it was just solving procedural questions which could also be used for evaluative purposes.

In a similar vein, when her vignettes were analyzed in terms of GPK it was noticed that mostly she used the statements like “I gave example…, I try to explain…, I gave counter examples…” and then continue with the procedural explanations. In two of the vignettes, she wrote about students’ realizing the difference, and students’ checking each other’s work (See Table 4.29).

Table 4.29: Evidences of GPK in Yeliz's Vignettes

Vignette	Evidence
1	Both of the answers are true. To understand whether these two students know the composition of function, it should be asked both of them whether each other's solution is correct or not. If they approve other solution, this means that they really understood the composite of function.
3	To clear up this confusion, I try to ask some questions to the students to realise the difference between composite and multiplication. Q1: What is the difference between (fog)(x) and f(x).g(x). If there is no difference why we use the sign (o) instead of (·)? Q2: If f(x)=4 and g(x)=x+3, you can multiply them easily. However how can you multiply f and g when f={(1,1),(2,3),(3,4)} and g={(1,1),(4,2),(3,3)}

These two vignettes were the ones she got Level 0-1. For the rest, she got Level 0. From these evidences it can be concluded that she was aware of the others methods to resolve conflicts in a class. However, both in the vignettes and teaching practices she preferred to use teacher-centered procedural explanations.

In a similar manner, most of the closures in the lesson plans just included the following statement of yes-no question, "At the end of the lesson students will asked whether they have any questions about the lesson. If exists, the questions will be answered." which is not an effective question. Only in the first and the last lesson plan she gave a question as seen in Figure 4.101 and in Figure 4.102 both to review the lesson and evaluate students' understanding. Apart from this example, she did not use any evaluation at the end of the lesson plans, but gave worksheets in accordance with the lessons. In practice, there were no closures generally lesson ends when the bell rings, she gave homework in three lessons out of six. Even though she gave two evaluations in the lesson plans, she used only the second one (See Figure 4.102) in the teaching practice.

Öğrencilere bileşke fonksiyon, tanım ve değer kümesini kapsayacak şekilde derste öğrendiklerini görmek amacıyla bir soru sorulur. Cevabını kağıda yazmaları istenir ve belli bir süre sonunda kağıtlar toplanarak değerlendirmeye alınır.

Soru: $f(x)=4^x$ ve g(x)=x-2 olmak üzere aşağıda verilen fonksiyonların tanım ve değer kümelerini bularak, kurallarını yazınız.

a. $(fof)(x)$ **b.** $(fog)(x)$ **c.** $(gof)(x)$

Figure 4.101: Excerpt from Yeliz's Lesson Plan 1

Dersin sonunda öğrencilerin konuyla ilgili bir soru yazmaları ve çözmeleri istenir. Daha sonra bu kağıtlar toplanır, değerlendirilir ve en güzel soru seçilerek bir sonraki ders sınıfta çözülür.

Figure 4.102: Excerpt from Yeliz's Lesson Plan 5

In all of the teaching practices her GPK level was rated as Level 0-1, even in the lessons where students played a game since the aim of the game was no to create meaning but to provide more applications on procedural questions. Even though, she did not provide enough detail in the lesson plans about the procedure of the lessons. Her GPK was rated twice as Level 0-1, and three times as Level 0 similar to teaching practices. The case was more dramatic for the vignettes where in two of them her GPK was rated as Level 0-1 and for the rest Level 0. It can be concluded that on the written material her GPK level was mostly lower than the teaching practices but the difference was not dramatic since the difference between the Level 0 and 1 was not dramatic.

4.2.2 Gizem's GPK

Gizem's use of OHP needs improvement. This is because the transparencies prepared were too small, and while using those instead of showing the necessary part she showed the whole transparency which caused a mass on the board. She always used students' names, dealt with the students individually and checked the students' work by wandering in the class. She created a positive friendly classroom environment and tried to involve all students in the lesson. She always gave enough time for students to take in notes to their notebooks. She warned the students who were trying to disturb the lesson. She always checked whether students were obeying the school rules like uniform, being late for the class.

Some of the objectives in the lesson plans were seem like written for the teacher like first objective of the first lesson plan (See Table 4.30). In a similar vein, her methods/techniques section also lacked the necessary terminology and they were all like steps that teacher will do during the lesson (See Table 4.31).

Table 4.30: Objectives Gizem used through the lesson plans

Lesson Plan	Objectives
1	• Decide whether they can take the composition of the given two functions • Solve the simple examples on taking composition of functions.
2	• Take the composition of the given two functions • Solve the examples on taking composition of functions. • Use the properties of the composition operation.
3	• Solve the examples on taking composition of functions. • Use the properties of the composition operation. • Have an idea about the meaning of inverse function.
4	• Find the inverse of a function. • Solve the examples on finding inverse of a function. • Decide whether they can find the inverse of given function or not.
5	• Decide whether they can take the inverse the given function. • Solve the examples related to the topic (mixed problems)
6	• Solve all types of examples related to the topics: Composition of Functions and Inverse Function

Table 4.31: Methods/Techniques Gizem used through the lesson plans

Lesson Plan	Methods/Techniques
1	• Showing the domain and range of functions by diagram on board and the composition of them • After writing the definition of composition of functions, solving examples on board (some of them are solved by teacher, some of them are solved by students.)
2	• After writing the properties of composition operation, solving examples on board related to this topic. (some of them are solved by teacher, some of them are solved by students.)
3	• After solving some questions from the worksheet 2, I will explain the meaning of inverse function with relating to the real life.
4	• Review for one-to-one and onto functions on board • Explaining the conditions and definition of inverse function (with using OHP, the students follow the directions from their handouts.) • Solving examples related to the topic.
5	• Making a review for the properties of inverse functions • Solving questions on the sheet which is handed out at the beginning of the lesson.
6	• Solving questions on the sheet which is handed out at the beginning of the lesson. • Solving the homework questions

For each lesson, she listed prerequisite skills required like "knowing the definition of the function, being able to take composition of two functions…", materials needed like "Board markers, white board, OHP, handouts, and worksheets". Moreover, she distinguished the introduction, development and closure in the procedure part of the lesson plan as indicated in the lesson plan format.

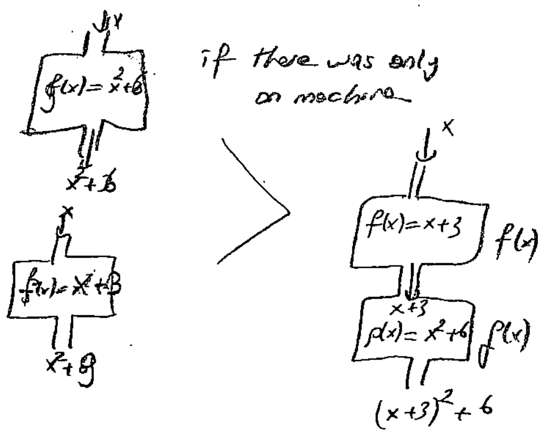
When her introductions were analyzed it was seen that except for the first lesson her introductions aimed at reviewing the previous lesson and solving questions which cannot be solved by the students. When these introductions were compared with the teaching practices it was seen that her introductions in the teaching practices showed some differences. In the first lesson plan, by giving a numerical example and reminding domain and range of a function on Venn diagram she introduced the new topic procedurally. However, her introduction in the first teaching practice included analogies about machines and machines working together, and while introducing this analogy she used students ideas and involved them in the lesson. In the rest of the teaching practices, she always included students in the review, during the review she used both oral and written explanations, and also gave the agenda of the day.

In the development part of the lesson plans, generally in parentheses she gave the activities that the teacher should do while implementing the lesson plan. For example, in the lesson plan aiming at teaching composition of functions she wrote "Gave two functions in the list form and ask for the domain and range of those two functions". The sequencing of the subtopics and questions were good, but the later one can be improved. Because of that during the teaching practices students followed the lessons easily and did not experience any problems of transition. She helped this by referring the examples and properties done in the previous lessons frequently.

Most of the lessons were teacher centered and flow of information was from teacher to student. However, she tried to create an environment that could increase the understandings of the students. She tried to involve students in the lesson by using their names in the explanations or making them to help her while creating a new question by suggesting numbers. Regardless of what she was introducing to the class, she never gave the definition directly instead she gave some examples, explanations, analogies or previous knowledge that would lead them to the definitions. Generally while explaining the students' misunderstandings she used both oral and written explanations, and flashback to the analogies given at the beginning of the concept and also gave a similar numerical example for the problem.

In a similar vein when her vignettes were analyzed in terms of GPK it was noticed that mostly she used the statements like "I would remind that …, I would explain…, I would show that…". Even though she used statements which shows teacher at the center and use procedural questions, she also tried to involve students (vignette #4, and #10) and gave examples at different representations (vignette #13, #2, #7, #9, and #5) to increase understanding (See Table 4.32).

Table 4.32: Evidences of GPK in Yeliz's Vignettes

Vignette	Evidence
2	I would give the machines example to show the mistake and the solution: 
4	$(f \circ p)(x) = f(p(x))$ I would explain; if you have $f(x)=2x-1$, what is your variable = ? They say "x". Then I would say: In composition of two functions, we change the variable. e.g. $g(x)=x+1$ and you want $f(p(x))$ then you write $f(x+1)$ since your $p(x)=x+1$. then you would find $f(x+1) = 2(x+1)$

From these evidences it can be concluded that her resolutions to conflicts in the class in the vignettes, and teaching practices were similar to each other in that she preferred to use procedural questions and /or different representations and analogies to increase understanding. For those vignettes discussed above her GPK was rated as Level 0-1, for the rest Level 0.

In a similar manner, every lesson plan had a closure and mostly she preferred to made revision in addition to that twice she asked whether students have any misunderstandings and once gave information about the next lesson. She never used precise evaluation throughout the lesson plans but gave worksheets as a homework at the end of every lesson. In practice, there were no closures generally lesson ends when the bell rings, she gave homework in all of the teaching practices. The only thing that can be considered as an evaluation was the game that was used to review the composite functions.

In the teaching practices her GPK level was mostly rated as Level 1 once for Level 0-1 and once for Level 1-2. Since she provided details in the lesson plans about the procedure of the lessons in parentheses (See Figure 4.103), her GPK evaluations of the lesson plans were close to teaching practices. Her GPK was rated as Level 1, except for the last lesson one which was rated as Level 0. On contrary, out of 13 vignettes she got Level 0-1 for 7 of them, and Level 0 for the rest. As a result, it can be concluded that her GPK level gets the highest scores on the teaching practices even though her scores on the other instruments did not differentiated too much.

FONKSİYONLARIN BİLEŞKESİ

(İlk olarak liste biçiminde iki fonksiyon verilir ve öğrencilere bu iki fonksiyonunun tanım ve görüntü kümeleri sorulur.)

Örnek: f= {(-1,1), (0,0), (1,1), (2, 4)} ve

g = { (1, 3), (0,1), (4, 9)} **fonksiyonları verilsin, bu iki fonksiyonun tanım ve değer kümeleri nedir?**

(Bu bölüm fotokopi şeklinde öğrencilere dağıtılır ve OHP yardımıyla asetatten özerinden geçilir.)

Figure 4.103: Excerpts' from Gizem's lesson plan

4.2.3 Deniz's GPK

Deniz always used students' names, from time to time dealt with the students individually and checked the students' work by wandering in the class. She always gave enough time for students to take in notes to their notebooks. She sometimes warned the students who were trying to disturb the lesson. She never checked whether students were obeying the school uniform rule, or warned the late comers.

The objectives (See Table 4.33) and methods (See Table 4.34) she used during the lesson plans were appropriate but did not show variety they were all similar to each other.

Table 4.33: Objectives Deniz used through the lesson plans

Lesson Plan	Objectives
1	At the end of the lesson students: • Know composition of functions • Solve questions about composition of functions
2	At the end of the lesson students: • Know composition of functions • Solve questions about composition of functions • Find the domain set of composition function
3	At the end of the lesson students: • Know the properties of composition function • Know transition property • Know union property • Know identity function
4	At the end of the lesson students: • Know definition of inverse function • Finding values of inverse functions
5	At the end of the lesson students: • Know how to find inverse of a function
6	At the end of the lesson students: • Know how to find inverse of a function • Know how to find inverse of a function like $f(x)=\frac{ax+b}{cx+d}$
7	At the end of the lesson students: • Solve problems about inverse of functions, composition of functions • Know the property $(g \circ f)^{-1}=f^{-1} \circ g^{-1}$ • Solve graphic problems • Know how to find functions in term of other functions

Table 4.34: Methods/Techniques Deniz used through the lesson plans

Lesson Plan	Methods/Techniques
1	**Questioning, giving the definition.**
2	Questioning, giving the definition.
3	Questioning, giving definition
4	Giving the definition, solving questions and asking questions.
5	Giving the definition, solving questions and asking questions.
6	Giving the definition, solving questions and asking questions.
7	Giving the definition, solving questions and asking questions.

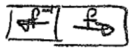
For each lesson, she listed prerequisite skills required like "Definition of the function, Types of functions", materials needed like "White board, board markers". However, she never distinguished the introduction, development and closure in the procedure part of the lesson plan. Also, she never included an evaluation at the end of the lesson plans, just in the first lesson plan she wrote about distributing the prepared worksheet. In practice, she always have introduction at the beginning of the lessons, but the structure of the lesson plan was always a quick oral review of the past lesson. She never had a chance to make a closure at the end of the lesson whenever the bell rings the lesson ended. Just for the first two weeks, she gave homework. She never had a special evaluation about the lessons during the teaching practices.

In the development part of the lesson plans, she did not give any clue about how she is going to proceed the written material during teaching. The sequencing of the subtopics and the questions were mostly good in the lesson plans and she followed the same sequence during the teaching practice.

Most of the lessons were teacher centered even if she asked a question mostly she did not wait for students to answer and she answered herself which contradicts with what she wrote in the method section of the lesson plans. Because, questioning does not mean solving questions on the board and asking any kind of question. Focus of the lesson was not meaning construction but to solve questions. She was always procedural even when she was introducing the concepts. Instead of reaching definitions, she directly provide definitions to students and made them write the definition while reading the definition. Because she did not put much emphasis on the definitions students generally did not asked any questions about the definitions regardless of whether they understood it or not. When she asked a question to the class she never let them answer, or solve individually. One of the students reacted this by saying please let us solve from time to time. After she solved, if the class or a student had a misunderstandings, she just used oral explanations. She always said directed well the students through the lesson plans by saying listen before writing, I'll give time for writing. Moreover, she constantly warned students about their mistakes in the basic operations.

In a similar vein when her vignettes were analyzed in terms of GPK it was noticed that mostly during her explanations to students confusions she used the statements like "Let me explain composition…, to correct students mistake I would say that…" and then continue with the procedural explanations. Only in two of the vignettes, she gave a non-procedural example with using an analogy (See Table 4.35).

Table 4.35: Evidences of GPK in Deniz's Vignettes

Vignette	Evidence
5	• Dikiş makinesi o Kesim makinesi (kumaş) = Dikiş makinesi (kesilmiş kumaş) = Ceket
9	• When we surf in internet we pass to other page when we click on next page button (f function). To go back we press on back page button (f^{-1} function) 

From these evidences it can be concluded that she was aware of the others ways to resolve conflicts in a class. However, both in the vignettes and teaching practices she preferred to use teacher-centered procedural explanations.

In all of the teaching practices her GPK level was rated as Level 0 since the aim of the lessons was no to create meaning but to provide more applications on procedural questions. Even though, she did not provide enough detail in the lesson plans about the procedure of the lessons. Her GPK was rated mostly as Level 1, and once as Level 0 and once as Level 0-1. This level difference is because of the fact that in the analysis of the lessons plans it was seen that she was trying to create meaning but in practice she was only trying to solve questions and teach procedures. In a similar vein, her GPK levels in the vignettes were mostly Level 0. In vignette # 5 and #9 where she used analogies for explanations and in vignettes #1 and # 2 where she used questioning her GPK was rated as Level 0-1. The similarity between the teaching practices and the vignettes was evident since in both of them she was trying to teach procedures not the concepts.

4.2.4 Comparisons of Participants' GPK

Preservice teachers enrolled in this study had just taken the courses Classroom Management and Mathematics Teaching Methods I and II courses where they learned special teaching method and techniques and classroom management. In Table 4.36 participants' grades from the mentioned courses were summarized.

Table 4.36: Participant' grades of Classroom Management and Mathematics Teaching Methods I and II courses

	Classroom Management	**Mathematics Teaching Methods I**	**Mathematics Teaching Methods II**
Deniz	B+ (3.3)	A (4.0)	A (4.0)
Gizem	A- (3.7)	A (4.0)	A (4.0)
Yeliz	B (3.0)	A- (3.7)	A- (3.7)

In the parenthesis what are those letters stand for out of 4 was given.

When the above table was analyzed regarding the method courses it was seen that participants' grades were nearly the same. However, in the Classroom Management course there is a difference but their grades are not too low. Apart from the course grades vignettes, lesson plans, and teaching practices were analyzed in order to see whether preservice teachers integrated general pedagogical knowledge they already had.

Analysis of the vignettes for the purpose of general pedagogical content knowledge according to combined framework revealed that participants' responses were mostly taken the category Level 0 since they dominated the teaching and showed no evidence of importance of meaning construction and students involvement in learning process. Whenever they emphasized those their level was rated as 0-1. The vignettes which got different scores were tabulated in Table 4.37 and examples of scored vignettes were given in Appendix N.

Table 4.37: The general pedagogical knowledge levels of the participants in vignettes

	Yeliz	**Gizem**	**Deniz**
Vignette # 1	0-1	0	0-1
Vignette # 2	0	0-1	1
Vignette # 3	0-1	0	0
Vignette # 13	NA	0	NA

NA: Not applicable

When participants' general characteristics of teachings were compared it was seen that participants differentiated. Yeliz's use of body language, voice, and board was effective. However, her use of OHP needs improvement. She always used students' names, dealt with the students individually and checked the students' work by wandering in the class. She always gave enough time for students to take in notes to their notebooks. She warned the students who were trying to disturb the lesson. She mostly checked whether students were obeying the school uniform rule, and also she warned students who were late for the class to take the permission paper. Gizem's use of body language, voice, and board was mostly good. However, her use of voice and OHP needs improvement. She always used students' names, dealt with the students individually and checked the students' work by wandering in the class. She created a positive friendly classroom environment and tried to involve all students in the lesson. She always gave enough time for students to take in notes to their notebooks. She warned the students who were trying to disturb the lesson. She always checked whether students were obeying the school rules like uniform, being late for the class. Deniz's use of voice was effective but use of body language and the board can be improved. She always used students' names, from time to time dealt with the students individually and checked the students' work by wandering in the class. She always gave enough time for students to take in notes to their notebooks. She sometimes warned the students who were trying to disturb the lesson. She never checked whether students were obeying the school uniform rule, or warned the late comers.

In terms of mechanics of the lesson plans and how they were applied during the teaching practice, all participants were able to write down the necessary materials and prerequisite skills appropriately. However, the objectives and methods written by Yeliz and Gizem were not appropriate. The objectives and methods were like a teacher to do list. Deniz on the other hand wrote more appropriate objectives and methods. However, during the teaching practice it was observed that by the method "questioning" she meant question solving which was not the meaning of the method of questioning.

In all lesson plans, Yeliz and Gizem put a separate section of introduction, closure, and evaluation where Deniz did not. However, during the teaching practices they allotted time for introduction in all lessons only how they conducted it was differentiated among the participants. Yeliz's introductions included an oral review of the previous lesson and an agenda of the day. During the review sometimes she encourages student involvement. Gizem's introductions were a review of the previous lesson by the students with guidance and questions of the teacher. She also gave an agenda of the day. Sometimes, she solved an example from the previous lesson. On the other hand, Deniz's introductions only included an oral representation of what was done in the previous lesson and an agenda of the day.

Even though Yeliz and Gizem put separate section for closure, both of them had a chance to apply it only a few times, where Deniz never did it. For the evaluation part, they tried to gave a homework at the end of the lessons but couldn't do it for all of them. Only Gizem and Yeliz were tried to do a different evaluation via a problem solving game.

In the development part of the lesson plans, Yeliz and Deniz did not give any clue about how they were going to precede the written material during teaching, whereas Gizem gave explanations in parentheses. The structure of the lesson mostly determined by the how well prepared the sequence of the subtopics and the questions. In this respect Gizem and Deniz showed similarity. The sequencing of the subtopics and questions were good, but the later one can be improved. Because of that during the teaching practices students followed the lessons easily and did not experience any problems of transition. For Yeliz, even though the ordering of the subtopic were good, the order of the questions was always bad both in the lesson plans and teaching practice. This caused transition problems to students from question to question, and resulted in students not to cover the main idea of the concept. For Yeliz and Deniz, most of the teaching practices were teacher centered even if they asked a question mostly they did not wait for students to answer and answered themselves. Focus of the lesson was not meaning construction but to solve questions. Instead of reaching definitions, they directly provided definitions to students. For Gizem, the case was a little bit different. Similarly, most of the teaching practices were teacher centered but regardless of what she was introducing to the class, she never gave the definition directly instead she gave some examples, explanations, analogies or previous knowledge that would lead them to the definitions.

Generally while explaining the students' misunderstandings Yeliz's and Deniz's approach showed similarity. Yeliz used both oral and written explanations, where Deniz used only oral explanations and they both gave a similar numerical example for the problem. Oral explanations caused problems in students' understandings. On the other hand, Gizem used both oral and written explanations, and flashback to the analogies given at the beginning of the concept and also gave a similar numerical example for the problem.

4.3 Value of Teaching Functions, Inverse and Composite Functions

This section summarized the results obtained from journal about value of teaching, vignettes, lesson plans and teaching practices where first one assessed the awareness of the value of functions as a part of pedagogical content knowledge, composite functions and inverse functions and the others assessed whether this awareness was reflected while completing the instruments and teaching practices. For the analysis of journal about the value of teaching functions, inverse and composite and inverse functions firstly results were tabulated (See Table 4.38). In the table the points participants allocated for each different value statement in the journal were presented.Then, these results were compared with the interview transcripts. Then, evidences of awareness of value were searched through vignettes, lesson plans and teaching practices.

Table 4.38: The distribution of points given to statements in the journals

Value of	Statements	Category	Yeliz	Gizem	Deniz
Functions	A	Pedagogical	15	15	20
	B	Pedagogical	30	10	35
	C	Intrinsic	20	30	20
	D	Pedagogical	5	5	10
	E	Pedagogical	10	15	10
	F	Excitement&Beauty	20	25	5
Composite Functions	G	Intrinsic	50	50	50
	H	Pedagogical	10	10	20
	I	Excitement&Beauty	25	20	20
	J	Pedagogical	15	20	10
Inverse Functions	K	Pedagogical	30	10	5
	L	Pedagogical	15	15	5
	M	Intrinsic	40	20	45
	N	Intrinsic	10	30	35
	O	Excitement&Beauty	5	25	10

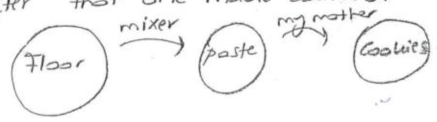
4.3.1 Yeliz's Possesion of the Value of Teaching Functions, Composite and Inverse Functions

When the Yeliz's journals analyzed it was seen that there were no consistency among all journals about which type of value she prefers as her favorite. However, it can be concluded that she mainly preferred intrinsic journals over the others both as her first and second choice. Only in the functions journal her favorite choice was pedagogical, which includes statements related with the importance of the certain subject to mathematics. In her explanations in the functions journal and in the interview she stated that "I mainly preferred statements related to the real life connections (intrinsic) of the topic however since functions and its symbolism is very central for the rest of the mathematics courses. I preferred pedagogical statement for the first place in the functions journal". When her second choice for the functions journal analyzed it was seen that they were intrinsic and excitement and beauty value statements. So, these choices supported her first choice and explanations.

Like her favorite choices, her least favorite choices were not consistent through all journals. In inverse functions journal her least favorite choice was excitement and beauty statement, and in the others pedagogical statements. Even though during the interview, she stated that value statements describing connections only for the importance to other mathematical concepts (pedagogical) would not be enough for the students since they would need more concrete things (intrinsic and excitement & beauty). This statement approved her least favorite choice in the functions and composite functions journal. The reason for choice of excitement and beauty value statement as the least one in the inverse function journal may be due to the fact that beauty was emphasized in this journal through a mathematical example.

Furthermore, the analysis of the vignettes revealed that she used three intrinsic and one pedagogical value statements which were related with composition of functions and inverse functions respectively (See Table 4.39). She emphasized the importance of learning inverse functions correctly because of the danger of confusing it with the other inverses in mathematics. Her intrinsic value statements were always in reaction to an analogy given in the vignette.

Table 4.39: Evidences of Value in Yeliz's Vignettes

Vignette	Evidence	Value Type
6	addition, the conjugate of numbers. In my opinion, the inverse of function should be explained as clear as possible again. Students should learn that "the inverse of a function" is a very different concept. If I were a teacher, I prefer to teach composite of the functions before the inverse of them. So, I would have taught the composite of f and f^{-1}. For this reason, students would know the unit element of functions which is equal to x. Therefore,) is a unit function which is equal to x. Therefore, I will show that f^{-1} is the inverse of function	Pedagogical
5	If I were to explain the composite function by using a real life example, I would choose cat → mouse → chees In fact, I used it in my class and students understood what I was trying to explain. The example is exactly like the following: f: Cheese → Mouse and g: Mouse → cat gof: Cheese → Cat cat [Mouse (Cheese)] = cat [Mouse who ate cheese] = cat who ate cheese we took gof because cat follows mouse and mouse follows cheese. Therefore, first mouse eats cheese, then cat eats mouse.	Intrinsic
9	If I were to explain the inverse function by using a real life example, I could choose this example but I would change the function from school to home. f → Father's car f: Home → School g → School Bus g: School → Home So, the inverse of the function f is function g.	Intrinsic
13	My mother first takes floor and mix it with other equipments then she makes paste After that she made cookies. 	Intrinsic

Yeliz's main preference of intrinsic value statements were also observed in her teaching practices. She taught six times and in four of them she used value statements where three of them were intrinsic and one of them was pedagogical. First intrinsic value statement was used in order to explain composition of functions in two consecutive lessons. The example was as follows "take cheese (c) as the first input; then the mouse (m) being a function "eats" the cheese. Next, here comes a third step, say the cat "eats" the cheese. The best way to denote this is c(m(c)).The brackets denotes the walls of the stomachs". She also gave this example in her lesson plan and the vignette and this was the only value statements seen on the lesson plans. Her second intrinsic value statement was about inverse functions which explain it as undoing behavior of the car and the school bus who took a student from home to school, and school to home respectively. This analogy was also seen in the vignettes. Last value statement was pedagogical and it was about the importance of the identity function for the future mathematics topics. A student asked why do we need identity function and she answered as "Identity function will be very useful while you are learning inverse functions". Although she allocated moderate points for the excitement and beauty she never used excitement and beauty statements in her teaching practices.

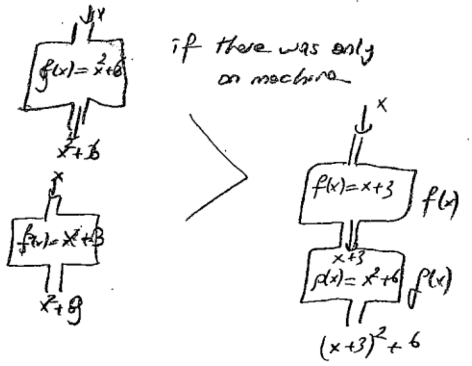
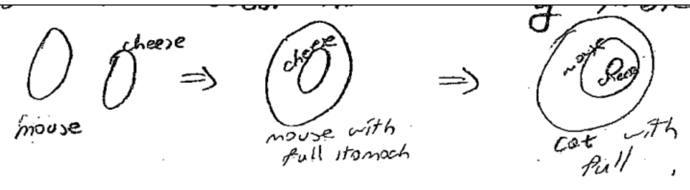
4.3.2 Gizem's Possesion of the Value of Teaching Functions, Composite and Inverse Functions

Gizem preferred intrinsic value statements for the first place and her second choices were mainly excitement and beauty statements. The points she allocated to excitement and beauty statements were very close to those of intrinsic value. Only in value of functions journal a pedagogical statement and excitement and beauty statement were allocated equal points. Her least favorite statements were always procedural ones.

For her favorite choices as intrinsic statements she gave consistent explanations through three journals emphasizing the importance of showing students connections of mathematics to the real life. For example, she wrote in the functions journal "I find it the best since this teacher talks about not only mathematical side of functions but also their places in our lives". She consistently supported her favorite choice in her explanations of least favorite choices like in the inverse functions journal “The teacher gives good examples to explain the importance however the examples were related to mathematical topics only. Discussing about the meaning of inverse functions according to real life would be better”. During the interview she supported her view by stating “I purposefully gave more points to the statements related to the real world examples”. Moreover, after the discussion with the group she admitted the importance of functions as a turning point in mathematics education and she said that I may allocate more points to pedagogical statement in the functions journal. She also stated that relating the value of teaching a topic only to another mathematical topic is not good enough so she gave fewer points to the pedagogical statements in each journal.

Gizem used four intrinsic value statements and none of the other types in the vignettes (See Table 4.40).

Table 4.40: Evidences of Value in Gizem's Vignettes

Vignette	Evidence	Value Type
2	I would give the machines example to show the mistake and the solution: 	Intrinsic
5		Intrinsic
9	An example for inverse function: With "zoom-in" function; if you click ones on this bottom, you will see the picture bigger, if you push the "zoom-out" bottom, you will see the picture with its original size.	Intrinsic
13	"the first input is a photo (p) on the computer screen. Your first function is the rotation (r) function. Firstly you rotate the photo r (p). Your second function is zooming (z) function. When you press on zoom in bottom, you will enlarge the rotated photo which is your output from the first function and the input for the second function. Your last output will be enlarged and rotated photo z(r(p)).	Intrinsic

First one was related with explaining composite functions by real life examples to correct and clarify students' mistake. It was as follows: "the first input is a photo (p) on the computer screen. Your first function is the rotation (r) function. Firstly you rotate the photo r (p). Your second function is zooming (z) function. When you press on zoom in bottom, you will enlarge the rotated photo which is your output from the first function and the input for the second function. Your last output will be enlarged and rotated photo z(r(p))." Second one was also related with using real life examples but this time with inverse function she gave the same example "zoom-in out" she used during teaching practices. Third one was using more than one machine to get composite functions. Fourth one was again related with explaining composite function by "mouse east cheese, cat eats mouse, so cat indirectly eats cheese.". Except for the second vignette, she was inspired from the analogies given in the vignettes.

The analyses of teaching practices supported the results of the Gizem's value journals. She taught six times and in five of them she used different value statements. She used intrinsic value statements six times. First one was related with the explaining composition of two functions via washing machine and drier. Second one explained the identity function as a hose since whatever you put in one side of a hose it will come as the same. In order to explain inverse functions, she used the example of using zoom in-out in the computers or cameras. Fourth one was related with the any two machines one of which takes the outputs of the other again to explain composition of functions. Fifth one was related with the taking the inverse of composite functions in parentheses, like $(fogoh)^{-1}=h^{-1}og^{-1}o\ f^{-1}$. In this case she wanted from students to imagine the equality as a mirror showing the inverse of every function. Last one was for explaining the meaning of inverse she said that "think of a washing machine is that logical to put something other than clothes inside the machine?...Things that could be put in to the washing machine are called the domain of that machine in mathematics function". Moreover, she used three pedagogical value statements. For example, she stated in her first teaching that "Friends be careful, functions are very important for the rest of your mathematical life, if you did not learn it properly you could not move, after 9^{th} grade you are going to see kinds of functions every year, even in the 12^{th} grade.".

Even though she mentioned the importance of excitement and beauty statements she never used it during her teaching practices. Although, she was frequently used the value statements in the teaching practices, she never mentioned them in her lesson plans.

4.3.3 Deniz's Possesion of the Value of Teaching Functions, Composite and Inverse Functions

When the Deniz's journals analyzed according to the points allocated it was seen that there were no consistency among all journals. However, it can be concluded that she mainly preferred intrinsic journals over the others both as her first and second choice. Only in functions journal her favorite choice was procedural and when her reason for choosing procedural statement as favorite one analyzed, it was seen that she put an additional meaning to the statement by saying "…representing the real world problem with the mathematical symbols…". Therefore, it can be said that she considered this statement as having intrinsic components. This inference was approved during the interview because when asked why she gave the most points to the procedural statement she said that "…because in their work life may be students will be an architect and they will need to transfer some data into mathematical symbols….". Similarly, her least favorite choices were not consistent through all journals. In functions journal her least favorite choice was excitement and beauty statement, and in the others procedural statements. During the interview, she explained her reason of choosing excitement and beauty value statement as the least favorite one as functions are not only iterations her explanation in the journal was the same. After the discussion, she said that I could increase the points of this statement because the statement only gave one example type for the application of functions.

The analyses of the vignettes revealed that she just used two intrinsic value statements through the vignettes and those were inspired from the analogies given in the associated vignette (See Table 4.41).

Table 4.41: Evidences of Value in Deniz's Vignettes

Vignette	Evidence	Value Type
5	• Dikiş makinesi o Kesim makinesi (kumaş) = Dikiş makinesi (kesilmiş kumaş) = Ceket	Intrinsic
9	• When we surf in internet we pass to other page when we click on next page button (f function). To go back we press on back page button (f^{-1} function) f^{-1} | f_o	Intrinsic

The analyses of teaching practices supported the results of the Deniz' value journals. She taught five times and in only three of them she used intrinsic value statements. First one was related with the any two machines one of which takes the outputs of the other to explain composition of functions. Second one was about inverse functions which explain it as undoing behavior of the car and the school bus who took a student from home to school, and school to home respectively. Last one was related with comparison of f and f^{-1} as going one step front and back respectively. Although the examples were good, since she did not adopt the examples to the lesson properly and explained them in isolation from the rest of the lesson, students did not seemed to internalize the idea of using such examples and connect them with the associated topics. Furthermore, she did not use any examples of pedagogical and excitement and beauty value statements during teaching practices. Apart from this, no evidence of value was identified in the lesson plans.

4.3.4 Comparisons of Participants' Possesion of the Value of Teaching Functions, Composite and Inverse Functions

When the points allocated to the statements in each category analyzed across all preservice teachers it was seen that all statements belonging to intrinsic value was given the most points for 7 times, and procedural value category 2 times. On the other hand, procedural value statements were given least points for 7 times, and excitement and beauty for two times. For the most-value statements, when each preservice teachers' journals analyzed, it was seen that Gizem chose intrinsic value statements for all journals, whereas Deniz and Yeliz chose procedural statements for the functions journal and intrinsic statements for the others. Although they chose procedural statements, for their second choice they preferred intrinsic statements and the points they allocated for them were far more than the others types and close to the procedural statements. Therefore, it can be concluded that preservice teachers preferred intrinsic statements over pedagogical and excitement and beauty ones, which means they believe in representing connections of mathematics to real world to the students.

Having pedagogical statements both in the most and least part seem like a contradiction although the statements were procedural their scope were different, and the reason for this was searched through the reasons the preservice teachers gave in the journals and the interview. When the reasons for allocating the lowest points to the pedagogical statements analyzed it was seen that all preservice teachers appreciate the importance of these subtopics for the further mathematics courses, and other branches like physics. However, they believe that this cannot be the most important reason for teaching functions, composite functions or inverse functions. For instance, Yeliz stated that “I did not like this perspective because it only sees the functions are necessary for school mathematics”. Similarly Gizem stated that “I agree if students understood functions in general then they can appreciate specific functions ,… but functions are not enough for them and also this is not enough to explain the importance of functions”. In a similar vein, Deniz stated that “…its(functions) importance cannot be because of another mathematics topic”. When the reasons for allocating the lowest points to the excitement and beauty value statements analyzed it was seen that Yeliz in inverse functions journal and Deniz in functions journal allocated lowest points to the excitement and beauty value statements preservice teachers found those statements too deficient and specific compared to the other value statements. For example, Yeliz said that “it is very deficient since it is only about the visual side of inverse functions” and Deniz said that “since functions are not just iteration, explanation seemed not enough to me”.

When the participants’ value journals analyzed it was inferred that they all preferred intrinsic journals over the others and their use of intrinsic journals in their teaching practices approved their preference. Their second choice was seen as pedagogical value statements during teaching practices although they allocated more points to excitement and beauty statements in the journals.

When the lesson plans were analyzed in terms of the value statements included it was seen that none of the participants showed their awareness of the value statements through the lesson plans. On the other hand, participants used value statements through some of their vignettes where all of them were inspired from the intrinsic value statements in the associated vignettes whereas Deniz and Gizem also used some additional statements.

Through the teaching practices, it was noticed that all participants used value statements but Gizem was the most frequent user whereas Deniz was the least. None of the participants used excitement and beauty value statements even though they allocated major points through the value journals. Only Yeliz and Gizem mentioned the pedagogical value whereas all of them mentioned the intrinsic value statements in their teaching practices in a differentiating frequency.

4.4 Knowledge of Context

This section summarizes the results obtained from knowledge of context interview, vignettes, lesson planning activity and teaching practices where first one assessed the knowledge and awareness of preservice teachers about knowledge of context as a component of pedagogical content knowledge and the others assessed how these awareness reflected while completing the instruments and teaching practices.

From the analysis of the knowledge of context interview transcripts, three categories were emerged: school-related, student-related, and class-related context. School-related context was defined as issues concerning school's policies, administration, departments, opportunities, and atmosphere. Student-related context was defined as issues concerning student's mathematics level, SES, family, and attention. Class-related context was defined as issues concerning class time, size, and climate. In the first step of the analysis, whether each preservice teacher mentioned each category and its subdimensions or not were tabulated (See Table 4.42).

Table 4.42: Distribution of participants' responses to knowledge of context categories in the interview

		Yeliz	Gizem	Deniz
School –Related	Opportunities provided by the school		x	x
	Mathematics department	x		
	Students admission policy		x	
	School administration		x	
	School climate	x		x
	External exam	x	x	x
Student-Related	Mathematics level of the students	x	x	x
	Attention of the students	x		
	SES	x	x	x
	Parents	x	x	x
Class – Related	Classroom size	x	x	x
	Classroom climate	x	x	x
	Class time	x	x	x

4.4.1 Yeliz's Knowledge of Context

Yeliz showed no evidence of knowledge of context through the lesson plans and her answers to vignettes. Her evidences of knowledge of context from the interview and the teaching practices were given simultaneously.

When Yeliz's responses were checked through Table 4.42, it was seen that she did not mention three sub-dimensions, opportunities provided by the school, school student admission policy, and school administration, in the whole interview. However, after reading the transcripts when she was asked about whether the opportunities provided by the school affects teaching or not, she said that they affect teaching since in some point they become tools for the teacher. Moreover, she said that she did not really know about the effect of school administration. For the school's student admission policy, she did not comment but she referred several times the importance of students' mathematics level which is a result of the admission policy. After reading the transcripts she was asked about whether admission policy affects teaching and she said that since it resulted in the level of students in the classes, it also affects teaching. In order to get a more detail view about her awareness of knowledge of context, her view was described via examples she gave in the interview.

For the school related context, Yeliz first mentioned the effect of mathematics department to teaching like this "…for example mathematics department in one school ordered the topics in one way in the other school in other way. In of those for example that department did not include details as a department principle. If you are member of that department you should follow these kinds of principles". Furthermore, she mentioned that teachers should explain to students that they need rules in order to maintain school climate. In relation to this statement, after her second teaching practice she warned the students about the school uniform rules. Lastly, she talked about the effect of external exams on her teaching as follows: "… for example, if I am going to start a new topic which is not included in the OSS, even I as a teacher would think before entering to class how I am going to teach this topic and ensure students to listen that". As explained before, during the interview she did not mention the issues related to school administration and opportunities provided by the school. However, during the check with the researcher she said that school administration must have some effect but I cannot give a specific example, because of that I did not give any example during the interview. For the opportunities provided by the school, she mentioned after the check that opportunities do affect the teaching for example, if there does not exist any technological equipment in the school, teacher cannot use it. During the teaching practice, she showed the same awareness and use OHP whenever it is needed.

Yeliz commented on every sub-dimension of student-related context during the interview. In order to describe her view her comments was exemplified. She commented about the mathematics level of the students several times. She mentioned it from different perspectives. First, she talked about in how many repetitions student understand the topic and time elapsed during understanding of the student. Then, she said that students level also affects the examples chosen by the teacher, and added that students level also affects their perception of the realities of the class like relatedness of the exam questions to the class-work. Although she showed this awareness during the interview, her sequencing of the questions were not appropriate for the class but she did not changed it . She commented on attention of the students as "how carefully students listened the teacher also affects teaching". The results of this statement was seen in every teaching practice of Yeliz. She tried to take attention of the class either by using an analogy related to the concept or just by saying "look at the board it is important" or organizing games to make students solve questions. SES of the students was another category and she mentioned two things first she said that since families pay money for education some students think that they pay teachers wages and mentioned these kinds of things in the school and as a teacher this will irritate me. On the other side, as a result of SES students take private tutoring which might result in discomfort during teaching since student do not listen the lesson. The last student related category was the parents. She said that parents must be conscious about their child's characteristics and students reflect their parents' attitude in the school.

Like the student-related context, Yeliz commented on every sub-dimension of the class-related context during the interview. In terms of classroom climate, she talked about the students and classes adaptation to every teacher and behavior change according to teacher. Similarly, she meant the same thing during the teaching practice while students were talkative during the class by stating "are you behaving like this since your class teacher is not teaching write now?". She thought that class time would be a problem for teaching and proposed a solution like "…if the class is at the end of the day teacher might motivate students psychologically to the class like if you listen the lesson time will pass more quickly…". Moreover, she talked about the effect of physical facilities of the class, like having lockers, and she said that the number of students in the class also affects teaching.

As a result, her evidences of knowledge of context in the teaching practices were not abundant when compared with her results of the knowledge of context interview. She used school rules and opportunities, showed effort to take students' attention, and stated her awareness of class climate during teaching practices. Her knowledge of context level was rated as Level 0 for the first two teaching practices and Level 0-1 for the rest.

4.4.2 Gizem's Knowledge of Context

Gizem showed no evidence of knowledge of context through her lesson plans and her answers to vignettes. Her evidences of knowledge of context from the interview and the teaching practices were given simultaneously.

When Gizem's responses were checked through Table 4.42, it was seen that she did not mention three sub-dimensions, attention of the students, school climate, and mathematics department. After reading the transcripts, her views about these three sub-dimensions were asked. For attention of the students sub-dimension, she said that I am sure that having students with weak attention levels will cause problem during teaching but since I only gave private tutoring up to know I couldn't imagine and exemplify what kind of problems exist during teaching. For the school climate sub-dimension, she said that I agree with all of my friends views so I just don't see any point to talk during interview. For the mathematics department sub-dimension, she said that I suppose this might affect teachers but I don't know in what way and to what extent.

For the school related context, Gizem elaborated about the opportunities provided by the school like hands on materials, models, colored papers and add that they can change the way a teacher gives the lesson. She also showed her awareness in her teaching practices by using an over head projector which is also an opportunity provided by the school. Moreover, she said that the school's student admission policy affects teaching since it results in having students with high or low perception. Apart from these, she mentioned that the atmosphere the school administration provided to teacher can affect teaching since if for example, administration puts too much stress on teacher this would definitely affect teacher and teaching. She mentioned the last sub-dimension, external exam, by approving the comments of the other preservice teachers in the group. She hold the general view of the group about the external exams that external exams affects teaching since teachers worries about whether including test techniques in their lesson plans or not and also including activity based lesson plans may cause problems when parents and students are connected to the external exams. Even though she did not mention about the school rules during the interview she always warned students about the uniform rules and asked for permission paper from the late comers.

For the student related context, Gizem commented about the mathematics level of the students several times. First, she talked about the effect of students' level of mathematics to teaching in different level classes. She said that students level also affects their perception of the realities of the class like relatedness of the exam questions to the class-work. Her awareness of students' level of mathematics was also seen on the teaching practices through her sequencing of the lessons which were ordered from easy to difficult. Although Gizem did not mention students' attention during the interview, she organized a game and used analogies efficiently during teaching practices in order to take students attention. Moreover, she used statements like "these are important questions look at the board please". She elaborated on SES of the students two times. She mentioned that if the SES of the students are very high, they may behave disrespectful to the teacher since they might think that by paying money for education they are actually paying wages of the teacher. Gizem said that this kind of thoughts might affect her performance. On the other hand, if SES of the students is low as a teacher you cannot make them buy a book for a project for example. During the teaching practice, while giving analogies to take students' attention she sometimes preferred technological examples which were related with the students' life, so as with SES. For example, she used the term "technological shortcut" while giving a formula for finding the inverse of $f(x) = \frac{ax+b}{cx+d}$ and use zoom-in and out in the computers and cameras. These analogies affected students' attention more than the others. The last sub-dimension of the student related context is the effects of parents. She mentioned the effect of parents attitude toward school and teacher. She said that if she experienced a negative attitude toward herself from parent she might be affected toward student. Apart from these, during the teaching practices she showed her awareness about the students' mathematics anxiety by saying the following statements "…don't worry you already learned this property through the examples but you are not aware of it…this is the prettiest property…you learned this already…". Since she used such statements students seemed not worried about and tried to understand the case and connect it to their previous understandings.

Gizem commented on every sub-dimension of the class-related context. In terms of classroom climate she talked about students and classes' change of behavior according to teacher. During the teaching practice, just for once she warned students about being talkative because she is teaching and she shared with the students her feelings about this by saying "now I don't want to teach anymore in this case I hope we can continue the lesson". Also, she talked about the effect of class time, she said that having lessons at the end of the day affects both students and teachers performance and because of that teachers must be awake and energetic in order to make students concentrate to the lesson. Even at the beginning of the day students are sleepy, again here it is the responsibility of the teacher to make students listen the lesson. She confirmed the other preservice teachers' view which was the number of students in the class effects teaching. For example, in governmental schools the class sizes are generally more than 40, and I private high schools class size is between 12-30, which is more comfortable for teaching.

As a result, her evidences of knowledge of context in the teaching practices were showed some differences from the interview. She used school rules and opportunities, sequenced examples appropriate to their mathematics levels, showed effort to take students' attention and lower their anxiety levels, stated her awareness of class climate, and selected examples appropriate to their SES during teaching practices. Her knowledge of context level was rated as Level 0-1 for the teaching practices.

4.4.3 Deniz's Knowledge of Context

Deniz showed no evidence of knowledge of context through her lesson plans and her answers to vignettes. Her evidences of knowledge of context from the interview and the teaching practices were given simultaneously.

When Deniz's responses were checked through Table 4.42, it was seen that she did not mention the effect of mathematics department, school administration, and attention of the students. After reading the transcripts, her views about these three sub-dimensions were asked. For attention of the students, she said that during my private tutoring sessions I really don't have any problems with attention because it was very easy to take attention I really don't know anything about class situation because of that I did not talk during the conversation but during teaching practice. For the effect of mathematics department and school administration, she said that in the schools I have observed up to know I really don't see and observed any problems related to them, I think they have effect on teachers but I cannot exemplify it.

For the school related context, Deniz said that materials provided by the school are very important for teachers because teachers cannot afford these materials themselves. During teaching practice she used overhead projector for once which showed awareness of opportunities provided by the school but after the first lesson because of the students' reactions, which also shows awareness of class climate, she did not use it for the rest of the lessons. Furthermore, she confirmed the views about the effects of the schools admission policy said by the other preservice teachers in the group. Moreover, her views about the school climate were based on the application of school rules, for her proper application of the rules creates a positive school climate. For the effect of external exams, she said that when teacher want to apply an activity, students might say that we should solve problems related to the exam it would be more useful, even parents say the similar things to teachers.

For the student related context, she mentioned every sub-dimension except the effect of attention of the students. Even though she never mentioned about the students' attention, during teaching practice she used analogies only at the beginning of a new concept but they weren't used efficiently. Also, from time to time she said "…look at the board…be careful…" for taking attention of the students. For the mathematics level of the students, Deniz mentioned the effects of it on the examples solved in the lesson and activities applied during the lesson. In this manner, during teaching practices her sequencing of the questions was good. She said that SES of the students might affect the activities you are going to apply, examples you are going to give and even homeworks. She exemplified it like this you can give an internet search as homework and students with high SES can easily do it since they have access to internet at home, but if you give it in a class with low SES, it will not work properly. Moreover, she mentioned that parents also affects teaching because if you collaborate with the parents well than it affect the relationship with the student so the teaching and the students success.

Deniz commented on every sub-dimension of the class-related context. In terms of class time she thought that mathematics lessons should be at the morning hours since students get tired through the end of the day so it will be hard to grasp a mathematics lesson in that time. For the classroom climate and size she confirmed the views of the other preservice teachers that students change behavior depending on the teacher and mentioned that the class size affects teaching activities. As mentioned before, during teaching practice since students reacted to the use of OHP she preferred not to use it for the rest of her teaching by showing an awareness of classroom climate.

As a result, her evidences of knowledge of context in the teaching practices were not abundant when compared with her results of the knowledge of context interview. She used opportunities provided by the school, sequenced examples appropriate to their mathematics levels, showed a weak effort to take students' attention, and understand the class-climate. Her knowledge of context level was rated as Level 0 for the teaching practices.

4.4.4 Comparisons of Participants' Knowledge of Context

Through the interview, participants agreed that teaching might differ according to the schools. Moreover, they mentioned all the sub-dimensions of school-related context. The sub-dimensions they mentioned most were school's climate, and external exams. For the sub-dimensions school's student admission policy, opportunities provided by the school, effects of mathematics department worked, and school administration only a few excerpts were seen in the transcript. This may be because of the fact that they were students previously and it is always easy to talk about your experiences instead of hypothetical things. After reading the transcripts, this point was identified and the researcher asked the preservice teachers whether they have any idea about the sub-dimensions they did not mention, they all said that these sub-dimensions also affects teaching, however, they were unable to give specific examples like in the other categories.

When context categories related to students were analyzed, it was seen that like in the sub-dimensions of the school-related context, preservice teachers mostly mentioned the sub-dimensions they experienced before. The sub-dimensions mostly mentioned were mathematics level, SES, and parents of the students. The sub-dimensions least mentioned were attention level, and previous knowledge of the students. After reading the transcripts, it was seen that only Yeliz mentioned the attention of the students sub-dimension. When Deniz and Gizem asked why they did not give any comment on that Gizem said that I only had a few private tutoring up to now I think attention of the students is important for the lesson but I never had any experience related to this topic so I did not talked during the interview, and Deniz have similar view with Gizem.

When context categories related to class were analyzed, it was seen that all preservice secondary mathematics teachers commented on each sub-dimension, and reason was stated through the interview by all participants as follows: since we experienced those things as a student for years.

The analysis of the teaching practices resulted in a different picture (See Table 4.43) even a new sub-dimension was emerged under the student-related context which was called mathematics anxiety. When the * in the Table 4.43 analyzed it was seen that Gizem showed her knowledge of context during the teaching practice more than the other participants, and observations also led to the fact that her use of knowledge of context was more effective.

Table 4.43: Distribution of participants' evidences to knowledge of context sub-dimensions in the teaching practices

		Yeliz	Gizem	Deniz
School –Related	Opportunities provided by the school	*	x*	x*
	Mathematics department	x		
	Students admission policy		x	
	School administration		x	
	School climate	x*	*	x
	External exam	x	x	x
Student-Related	Mathematics level of the students	x	x*	x*
	Attention of the students	x*	*	*
	SES	x	x*	x
	Parents	x	x	x
	Mathematics anxiety		*	
Class – Related	Classroom size	x	x	x
	Classroom climate	x*	x*	x*
	Class time	x	x	x

X represents mentioned in the interview. * represents observed in the teaching practice

The sub-dimensions observed in all participants were opportunities provided by the school, students' attention, and the classroom climate. For opportunities provided by the school, they realized that it was possible to use OHP in the school and adopt this technology in their teaching practices. Even though they all aware of the importance of the students' attention and tried to take attention, their approach and effectiveness can be discussed. Yeliz and Gizem used phrases, games, and analogies whereas Deniz only used analogies and phrases for taking attention. Even though both Yeliz and Gizem used the same methods, the way Gizem used the analogies and phrases through the teaching practices were more efficient and more consistent through the whole teaching practice. Since Deniz only used analogies at the beginning of a new concept it was not as effective as Yeliz or Gizem. Use of phrases for taking the attention was consistent through all participants throughout the teaching practices. For the classroom climate, Yeliz's and Gizem's evidences showed similarity again. They were both mentioned and/or complained about students' change of behavior due to preservice teacher teaching. Deniz on the other hand, became aware of the class reaction to use of OHP in the first lesson, and never used it for the rest of her teaching practices.

The sub-dimensions observed only in two participants were school climate and mathematics level of the students. For school climate, Yeliz and Gizem checked the school uniform rules and Gizem asked late comers to the class for the permission paper, whereas Deniz never dealt with such issues during her teaching practices. For the mathematics level of the students, Gizem and Deniz were sequenced the questions in the teaching practices in a logical order.

Two sub-dimensions only observed during Gizem's teaching practices were SES and the mathematics anxiety which was emerged during her observations. She showed an evidence of her awareness of SES by selecting analogies which were appropriate for students at this SES. Moreover, she always tried to calm down students by using phrases which tell them that they can do it or they already understand it, so a tried to reduce their math anxiety.

As a result of this analysis, even though all participants stated their knowledge of context through the interview, it was seen that in the teaching practices they rarely tried to adopt their knowledge of context. Because of their approach, they failed to use their knowledge effectively. In accordance with that, Gizem applied her knowledge of context during the teaching practices efficiently which was evident from her Levels of knowledge of context according to the combined framework.

4.5 Knowledge of Learners

This section summarizes the results obtained from three instruments (vignettes, lesson planning activity, and teaching practices) administered to the participants in order to assess their knowledge of learners as a component of their pedagogical content knowledge. All three instruments were analyzed by using the combined framework. This data provides to what extend preservice teachers diagnose students' errors, misconceptions and misunderstandings; realize students' needs for understanding and how they respond to them through the instruments.

4.5.1 Yeliz's Knowledge of Learners

Through the vignettes Yeliz always diagnosed the students' errors and tried to resolve the conflict the student or the class have through her explanations in the form of mostly in numerical examples and/or sometimes in procedural review of the concept (See Table 4.44). Besides, very rarely she preferred asking questions to students in order to involve students in the lesson (See Table 4.45). As a result of her consciousness of students' needs of understanding from time to time she used different representations in her explanations (See Table 4.46). In light of these evidences her knowledge of learner levels for all vignettes was rated as Level 0-1 according to the combined framework.

Table 4.44: Excerpts from Yeliz's vignettes which shows the diagnose of the students errors and resolution to the case

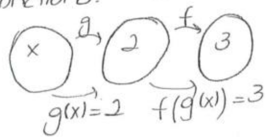
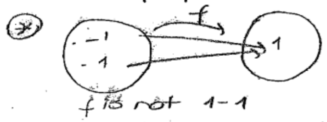
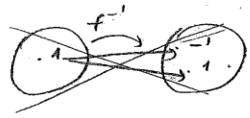
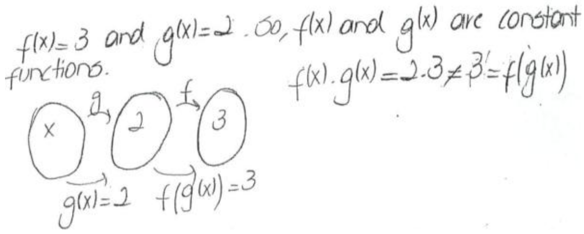
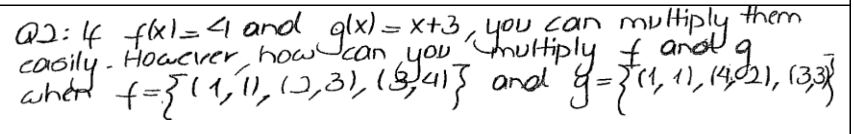
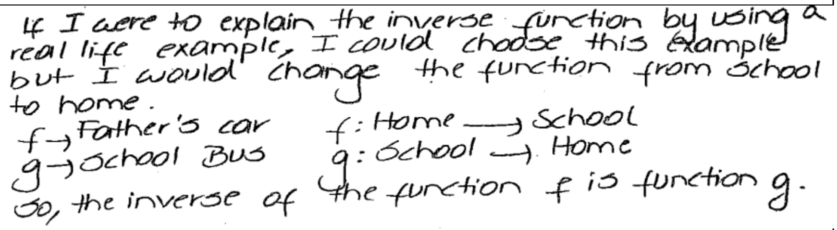
Vignette number	Excerpt
1	Let $h(x) = f(g(x))$ and determine $f(x)$ and $g(x)$ if $h(x) = 2(x-5)^2$. One student suggests that "$g(x) = (x-5)^2$ and $f(x) = 2$". ✗ Another student interrupts "No $f(x)$ must be equal to $2x$ if $g(x) = (x-5)^2$". ✓ A third student remarks "Well I think $g(x) = (x-5)$ and $f(x) = 2x^2$". ✓ For the first student, there is a misunderstanding about the definition of composition of functions. He/she has an idea that $f(g(x)) = f(x).g(x)$ So, he/she gives the following example: if $f(g(x)) = 2(x-5)^2$, $g(x) = (x-5)^2$ and $f(x) = 2$. There is no problem with other two students solutions, because the question is an open-ended question. Both of the answers are true. To understand First student needs some counter examples to understand the difference between $f(x).g(x)$ and $f(g(x))$. For example the following example can be given. $f(x) = 3$ and $g(x) = 2$. So, $f(x)$ and $g(x)$ are constant functions. $f(x).g(x) = 2.3 \neq 3 = f(g(x))$ 
7	A student said the inverse of the function $f(x) = x^2$ is $f^{-1}(x) = \sqrt{x}$. ✓ Is the student right? If you think that the student is correct explain why? The student seems right but he is not right. If $f(x) = x^2$, we cannot find the inverse of it since it is not one to one function. For example, if x is equal to −1, f(x)=1 and x is equal to 1, f(x)=1 too. So, there is not an inverse of that function. (*)   First of all, I warn the student about the evidence of $f^{-1}(x)$. I remind that we should check the function in terms of 1−1 and onto before finding the rule of f^{-1}. If the function is not 1−1 and onto, we do not need to find the rule of $f^{-1}(x)$ since there cannot be a function f^{-1}. Moreover I gave the example (*) to show that f^{-1} is not a function.

Table 4.45: Excerpts from Yeliz's vignettes about involving students to class discussion

Vignette number	Excerpt
1	Both of the answers are true. To understand whether these two students know the composition of function, it should be asked both of them whether each other's solution is correct or not. If they approve other solution, this means that they really understood the composite of function.
3	To clear up this confusion, I try to ask some questions to the students to realize the difference between composite and multiplication. Q1: What is the difference between (fog)(x) and f(x).g(x). If there is no difference why we use the sign (o) instead of (·)? Q2: If f(x)=4 and g(x)=x+3, you can multiply them easily. However, how can you multiply f and g when f={(1,1), (2,3), (3,4)} and g={(1,1), (4,2), (3,3)}
12	Then you want from your students to write down such a function and define inverse of it. One of your students gives the following example "My function is something we see everyday on supermarket's cash registers (yazarkasa). For each item we buy there is a corresponding price on the receipt (fiş), so the inverse of this function is for each price there is a corresponding item." I asked the student whether he can find the item when I gave the price of it. For ex: when I say the item costs 1YTL can he reply me with only one item?

Table 4.46: Excerpts from Yeliz's vignettes about use of different representations

Vignette number	Excerpt
1	
3	
9	

Through the lesson plans it was not possible to detect evidences for knowledge of learners since lesson plans were written like a book chapter. Therefore, no level was assigned to lesson plans according to the combined framework.

Similar to vignettes, her knowledge of learners was consistent through the teaching practices. She was aware of the fact that students needs different representations for understanding and always used Venn diagrams, sometimes used analogies and rarely used functions in graphs and ordered pairs through the teaching practices.

She always answered students' misunderstandings. If it was about an example solved, her answer was procedural explanation of the steps to follow for the solution. Sometimes, her explanation was just an oral review of the solution on the board which was hard for students to follow. If students' question was about a concept learned she always return to basic analogy she gave and used Venn diagrams. During her explanations she sometimes ask questions to the class, but generally she answered herself. Her aim in asking question was not to create a class discussion which would reinforce students' understandings but to keep students' attention on the board.

Sometimes students asked questions that were not directly related with not understanding the solution or the concept excerpts showing the such dialogues between the student and teacher were as follows:

Yeliz's Teaching Practice December 1

Student: Teacher does composition has distributive property?

Teacher: Don't think about it right know

…

Given that $f(x) = x^2$

Student: Teacher is $(fof)(x)$ equals x^2+x^2

Teacher: No answer

Yeliz's Teaching Practice December 4

Teacher: The definition of the domain of composite functions is too complicated

Students: Yes

Teacher: Let's show it with Venn diagrams

…

Teacher: Don't worry about the definition of the domain of composite functions if you understand the related example

...

Students: What is the symbol ∀ means

Teacher: Don't confuse yourself with this notation

…

Student: Why do we need identity function?

Teacher: It is important for the inverse functions

…

While showing $foI = Iof = f$

Student: Teacher doing this has no purpose why are we doing this

Teacher: We are going to see the importance in the inverse functions.

Yeliz's Teaching Practice December 8

Inverse of a function was found via Venn diagrams, and then they wrote it as ordered pairs. One of the students realized that the result of the second element always one less than the first

Student: It is always 1 less than the first one

Teacher: Don't worry about this

Yeliz's Teaching Practice December 12

Teacher: Ok friends this two lines (f and f^{-1}) have to be symmetric with respect to y=x

Student: Why it is so?

Teacher: Because they have to.

After this answer, students asked the same question then she showed the points are symmetric on two functions f and f^{-1} by picking points on them.

When the dialogues were analyzed it was seen that from time to time instead of providing answer she used answers like "don't worry about it" which never explains the students' conflict and also gave students a message that the thing the student asked was not important. For once she ignored students' question which was very important since students mixed the composition operation with the addition.

As a result, Yeliz's knowledge of learner was rated as Level 0-1 through the teaching practices which is the same as her levels in the vignettes.

4.5.2 Gizem's Knowledge of Learners

Through the vignettes Gizem always diagnosed the students' errors and tried to resolve the conflict the student or the class have through her explanations in the form of mostly in numerical examples and/or sometimes in procedural review of the concept (See Table 4.47). Moreover, as a result of her consciousness of students' needs of understanding from time to time she used different representations (diagrams, graphs, Venn diagrams, real life examples) in her explanations (See Table 4.48). In light of these evidences her knowledge of learner levels for all vignettes was rated as Level 0-1 according to the combined framework.

Table 4.47: Excerpts from Gizem's vignettes which shows the diagnose of the students errors and resolution to the case

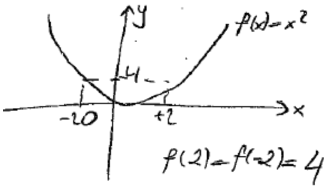
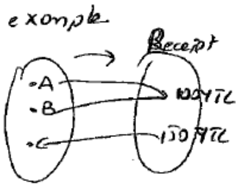
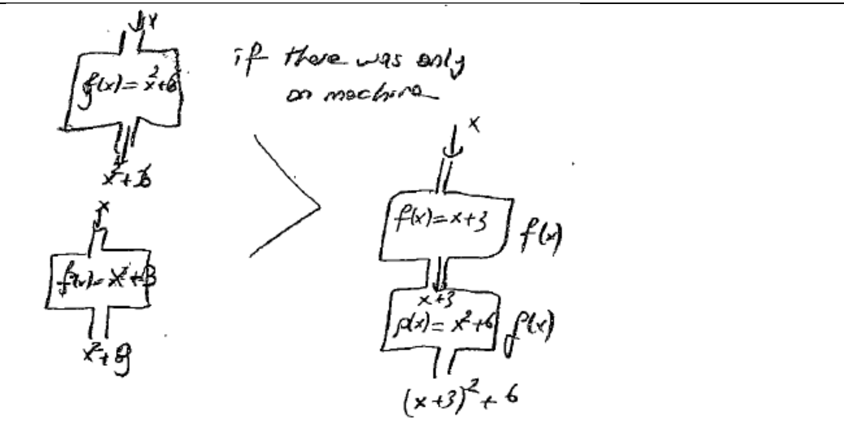
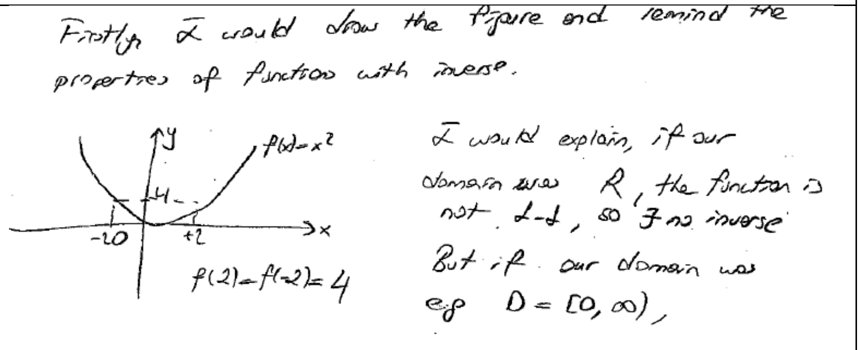
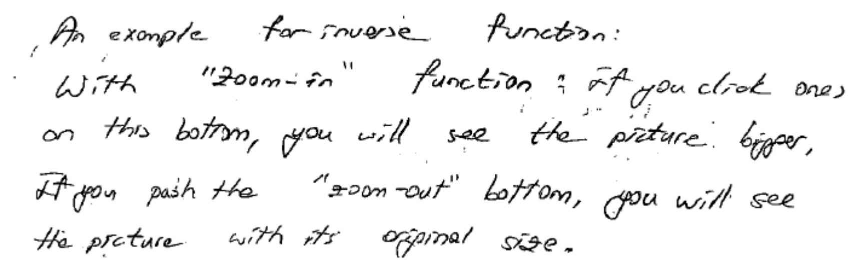
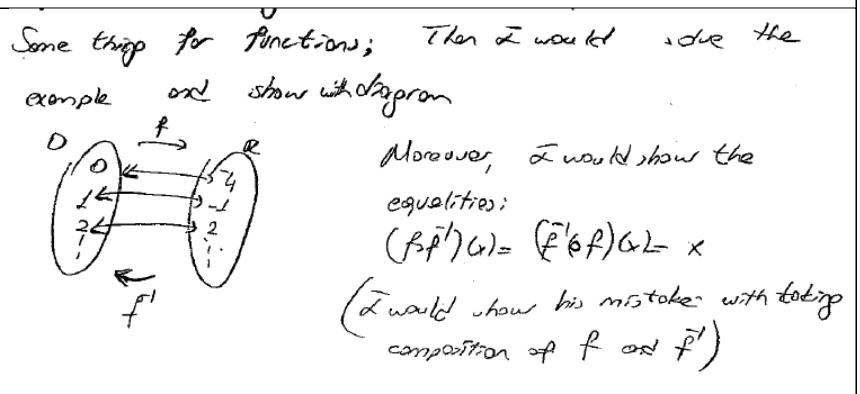
Vignette number	Excerpt
7	A student said the inverse of the function $f(x) = x^2$ is $f^{-1}(x) = \sqrt{x}$. He is wrong. He put x instead of y, only instead of x and found the expression in terms of x. However, he did not think about whether this function has an inverse. Firstly I would draw the figure and remind the properties of function with inverse.  I would explain, if our domain was R, the function is not 1-1, so ∄ no inverse. But if our domain was e.g. D = [0, ∞), then our function would be 1-1 & onto, so we could take the inverse I believe that, the error would clear up.
12	Then you want from your students to write down such a function and define inverse of it. One of your students gives the following example "My function is something we see everyday on supermarket's cash registers (yazarkasa). For each item we buy there is a corresponding price on the receipt (fiş), so the inverse of this function is for each price there is a corresponding item." However I don't like student's example, since sometimes two things may have same price. For example  } But ∃ no inverse of this function I would explain that, it may not have inverse so it is not a good example.

Table 4.48: Excerpts from Gizem's vignettes about use of different representations

Vignette number	Excerpt
2	
7	
9	
11	

Through the lesson plans it was not possible to detect evidences for knowledge of learners since lesson plans were written like a book chapter. Therefore, no level was assigned to lesson plans according to the combined framework.

Similar to vignettes, her knowledge of learners was consistent through the teaching practices. She was aware of the fact that students needs different representations for understanding and always used Venn diagrams and analogies through the teaching practices and rarely used functions in graphs and ordered pairs.

She always answered students' misunderstandings. If it was about an example solved, her answer was generally resolving the question on board by the help of the students. If students' question was about a concept learned she always return to basic analogy she gave and used Venn diagrams. During her explanations she sometimes ask questions to the class and she always waited for students answers and constructed her explanations on students' answers. By this way she gains the attention of the students for understanding the missing point.

She realized that students in the class had a difficulty in meaning construction. Accordingly throughout the teaching practices she used statements, question, and examples which could promote meaning construction:

Gizem's Teaching Practice December 5

Teacher: Could you give examples of functions/machines from our daily life?

Students: Computer, iron, dishwasher

Teacher: Can we use washing machine and drier as our functions machines? What are the inputs for the first one?

Students: Dirty clothes…

Teacher: What are the outputs?

Students: Wet and clean clothes…

Teacher: Are those outputs are in relation with the drier?

Students: Yes we put them in the direr

Teacher: So they are inputs for our second machine if they work together we call this composite functions.

Then she explained via Venn diagram and asked:

Teacher: Did you get the idea behind it?

…

Teacher: Did you get the idea behind it ?

…

Teacher: Are there anyone who did not understand? Even if you have a hesitation please ask.

After this statement no one answered hen she asked one of the students that she thought did not understand. Since he could not answer, she reviewed once more.

In an example asking for $(fog)(3)$ and $(gof)(3)$ she wanted students to elaborate on the answer

Teacher: Are these two are equal?

Students: No

Teacher: Then this must mean something

…

Teacher: When we take composition of two functions is the result still a function?

Given the question $f(x) = x^2$, and $g(x) = \sqrt{x-1}$ find $(fog)(x)$ and domain of it. Then she said:

Teacher: Let's do a mistake together

After finding $(fog)(x)$ she asked:

Teacher: Can we put 2 in place of x in $(fog)(x)$?

Gizem's Teaching Practice December 8

Given the example $f(x) = x + 1$ and $g(x) = x$ calculate $(fog)(x)$ and $(gof)(x)$.

Teacher: What does having equal answers mean?

Students: They are equal

Teacher: We just learned the commutative property which is not true for composite functions. This example showed us that that are exceptions.

…

Teacher: Be careful while distributing numbers over parentheses

…

Teacher: When we are adding a number with 0 what is the result?

Students: The number itself

Teacher: Ok then what is 0 called

Students: Identity element

Then pass to the identity element of the composition operation

Gizem's Teaching Practice December 12

Teacher: Did you heard of a linear function?

Students: No

Teacher: Do you know how to draw a line?

Students: Yes

Teacher: OK then equation of a line in the form of $y = ax + b$ is called a linear function

…

Gizem's Teaching Practice December 14

Teacher: Do you remember relations and inverse relations ?

Students: Yes, functions are relations

Teacher: So if we have functions as special relations can they also have an inverse?

…

Teacher: Does every function has an inverse?

…

Teacher: Up to know we have learned the basic idea of inverse functions. Do you have any questions?

Gizem's Teaching Practice December 15

Teacher: What is the inverse of an inverse function

Students: The function itself, we learned it both in logic and sets

As it was stated before, the analysis of the dialogues revealed that Gizem put emphasis on meaning construction a lot. Moreover, many times in all lessons she warned the students about being careful with the operations through calculations. Besides, she answered every question the students asked regardless of the type or relatedness of the topic. As a result, Gizem's knowledge of learner was rated as Level 1 through the teaching practices which are higher than her levels in the vignettes since through the teaching practices she also showed her understating of the students involvement in meaning construction.

4.5.3 Deniz's Knowledge of Learners

Through the vignettes Deniz always diagnosed the students' errors and tried to resolve the conflict the student or the class have through explanations in the form of mostly in numerical examples and/or sometimes in procedural review of the concept (See Table 4.49). As a result of her consciousness of students' needs of understanding from time to time she used different representations in her explanations (See Table 4.50). In light of these evidences her knowledge of learner levels for all vignettes was rated as Level 0-1 according to the combined framework.

Through the lesson plans it was not possible to detect evidences for knowledge of learners since lesson plans were written like a book chapter. Therefore, no level was assigned to lesson plans according to the combined framework.

Similar to vignettes, her knowledge of learners was consistent through the teaching practices. Even though she showed her awareness of the fact that students needs different representations for understanding, she sometimes used Venn diagrams and rarely used functions in graphs and ordered pairs. Even though she used analogies, they were not integrated into the lesson in a good way so students were unable to see the reason why they used it.

Table 4.49: Excerpts from Deniz's vignettes which shows the diagnose of the students errors and resolution to the case

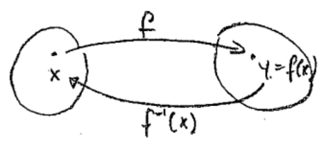
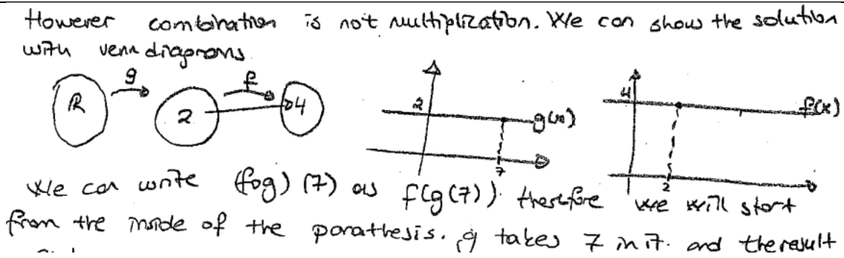
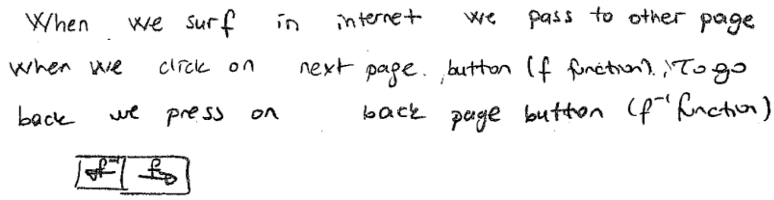
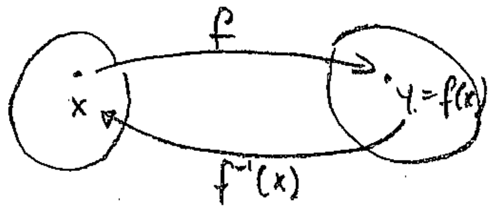
Vignette number	Excerpt
2	• In the first one student confused which function goes into the other function. He first did f and put its result into g like $g(f(x)) = g(x+3) = (x+3)^2+6$. • In the second one student did mistake in the meaning of composition. He did multiplication when he saw composition but composition does not mean multiplication. • In the third one the student did it correctly. • To correct the first students mistake I would say that the function which is near to x will the first to have an operation. $f \circ g(x)$ I would draw the arrows like that and write it as $f(g(x))$. • In the second one I would say; You are confusing multiplication and composition. Let's evaluate $f(0) \cdot g(0)$ and $f \circ g(0) = f(g(0)) = f(6) = 9$ $3 \cdot 6 = 18$ $\quad 18 \neq 9$ $\Rightarrow$
	A student of yours calculates the inverse function of the function $f(x) = 3x - 4$ and the answer obtained is $f^{-1}(x) = -2x + 4$. The students checks his work, and he combines $f(x)$ with $f^{-1}(x)$ he gets x. After the confirmation, he thinks that these two functions are inverses of each other. He subtracts f(x) from x to get the inverse. $x - f(x) = x - 3x + 4 = -2x + 4$. And he makes his check by adding $f(x) + f^{-1}(x)$ however $f^{-1}(x)$ is not found like that and we cannot make our correction by adding them. We have to look the combination of these two functions. To find f^{-1} we have to leave x alone $y = 3x - 4 \rightarrow x = \frac{y+4}{3}$ than we will write $f^{-1}(x) = \frac{x+4}{3}$. This is because 

Table 4.50: Excerpts from Deniz's vignettes about use of different representations

Vignette number	Excerpt
3	
9	
11	

She mostly answered students' misunderstandings. If it was about an example solved, her answer was oral review of the solution on the board which was hard for students to follow. During this review, she sometimes mentioned the concept the example was related. During her explanations, she sometimes asked questions to the class and generally answered herself as seen in the following dialogues.

Deniz's Teaching Practice December 1

Teacher: Is this new thing defined a function?

Student: Yes

…

Student: I did not understand according to what we complete the following Venn diagrams?

Teacher: No answer

Deniz's Teaching Practice December 4

Students: Teacher what does ∀ mean?

Teacher: Don't worry about it means for all x and y

…

For finding the domain of $(fog)(x)$ given that $f(x) = x^2$ and $g(x) = \sqrt{x-1}$

Teacher: Can a value which is not in the domain of g reach a value in the resulting set of (fog)(x)

Teacher: No

…

Given that $f(x) = \frac{x+1}{x}$ and $g(x) = x + 1$, find the domain of $(fog)(x)$ and $(gof)(x)$.

Teacher: Does the domain of g important for me? Yes. Because something not inside the domain of g cannot go outside.

Deniz's Teaching Practice December 8

Student: Teacher why are we doing this (finding the inverse of an inverse function) it is like adding with 0.

Teacher: No answer

Deniz's Teaching Practice December 15

In the question If $f(x) = 2x + 4$ and $(fog)(x) = -x + 3$, find $g(x)$ she asked:

Teacher: In order to find g what should I get rid of?

Teacher: f

Student: Why can't we get rid of g ?

Teacher: We are deciding according to what is asked in the question.

After adding f^{-1} to both sides one of the students asked:

Student: Am I going to multiply these?

Teacher: Carefully look at the board is our operation multiplication or composition?

The other thing that came out from the teacher student dialogues was that because of the emphasis on the procedural explanations sometimes students even did not realize what they found or reached at the end of the questions as seen in the following dialogues:

Deniz's Teaching Practice December 7

After she dictated the definition of the associative property, one of the students asked:

Student: Is this an example?

Teacher: No I will explain.

After her explanation another student asked

Student: Is this the solution?

Deniz's Teaching Practice December 15

In the question of finding a function in terms of f(x)

Student: Is this the result?

Teacher: Yes, let's look if the result does not contain any variable other than f(x) then yes.

After this explanation a similar question was solved another student asked:

Student: Is this the result?

Teacher: Yes

As it was stated before, the analysis of the dialogues revealed that Deniz did not put emphasis on meaning construction and even sometimes students couldn't understand whether they were learning a property or solving an example.

As a result, Deniz's knowledge of learner was rated as Level 0-1 through the teaching practices which were the same as her levels in the vignettes.

4.5.4 Comparisons of Participants' Knowledge of Learners

No comparison was made for lesson plans among the participants since through the lesson plans it was not possible to detect evidences for knowledge of learners.

For the vignettes, the participants' knowledge of learners were nearly the same. Only Yeliz showed some differences. Through the vignettes, all participants diagnosed the students' errors and tried to resolve the conflict the student or the class have through her explanations in the form of mostly in numerical examples and/or sometimes in procedural review of the concept. As a result of their consciousness of students' needs of understanding, from time to time they used different representations in their explanations. Besides, very rarely Yeliz preferred asking questions to students in order to involve students in the lesson. In light of these evidences their knowledge of learner levels for all vignettes was rated as Level 0-1 according to the combined framework.

For the teaching practices, Yeliz and Deniz showed similarity about how they presented their knowledge of learner. They were aware of the fact that students needs different representations for understanding and used Venn diagrams, analogies and functions in graphs and ordered pairs through the teaching practices. However, when the frequency and effectiveness of usage compared, it can be concluded that Yeliz used different representations more effectively. The analysis of dialogues in the teaching practices revealed that they mostly answered students' misunderstandings. Besides, dialogues revealed for Deniz that she did not put much emphasis on meaning construction and lessons were so much procedural that sometimes students had difficulty to understand for what reason they were doing the exercises. For Yeliz, dialogues revealed that if the students' question was not about the solution of a question or about a definition of a concept she did not give enough attention.

Gizem was aware of the fact that students needs different representations for understanding and always used Venn diagrams and analogies through the teaching practices and rarely used functions in graphs and ordered pairs. She always answered students' misunderstandings. If students' question was about a concept learned she always return to basic analogy she gave and used Venn diagrams. During her explanations, she sometimes asks questions to the class and always waited for students' answers and constructed her explanations on students' answers. By this way, she gains the attention of the students for understanding the missing point. Accordingly the analysis of the dialogues revealed that throughout the teaching practices she used statements, question, and examples which could promote meaning construction.

As a result, the knowledge of learner levels of Yeliz and Deniz in the teaching practices were 0-1 similar to their levels in the vignettes, the knowledge of learner level of Gizem was 1, which was higher than her level in the vignettes. Gizem's knowledge of learner level in the teaching practices was higher since she constantly prepared learning environments which promoted meaning construction.

CHAPTER 5

DISCUSSION, CONCLUSION, AND IMPLICATIONS

The purpose of this study was to investigate the preservice secondary mathematics teachers' pedagogical content knowledge of composite and inverse functions. In light of this aim, this chapter presents the discussion and the conclusion of the results, educational implications, recommendations for future research studies, and the limitations of the research study.

5.1 The Nature of Preservice Teachers' Pedagogical Content Knowledge

The literature has shown that pedagogical content knowledge plays a central role in a teacher's development from learning mathematics to teaching mathematics (Ball, 1990; Borko et al, 1992; Chamberlin, 2005; Shulman, 1987). This section presents the how the reported results about the components of pedagogical content knowledge addressed the research questions.

5.1.1 Subject Matter Knowledge

Research has shown the need for strong preparation in content prior to the teaching practice (Brown & Borko, 1992). In line with this fact, through different instruments preservice teachers' subject matter knowledge of composite and inverse functions were analyzed and it was seen that the picture was not as it would be required. Their knowledge includes some gaps and mostly not connected enough to reflect it to the students through teaching practices.

All participants expressed a definition for both composite and inverse functions either formally or informally. Their weak performances on the questions directly related with the definitions through the survey and the non-routine questions interview increased their awareness about the importance of the definitions and underlying principles. As a result of this awareness, through the vignettes, lesson plans, and teaching practices, they tried to put much more emphasis on the definitions and underlying principles. Even though they seemed affected from the instruments, the results also showed that the efficiency of use changed with respect to participants. The important point here is that as a result of the survey and non-routine questions interview participants questioned their own knowledge of concepts and tried to change themselves and this observation was confirmed by the participants during the evaluation interview.

Knowledge about definitions and their underlying principals were also related with participants' performances on three types of questions: declarative, conditional, and procedural. Since the participants' main emphasis was not on the concepts, they were procedurally focused and their scores on the procedural questions were higher than the others; that is, although preservice teachers could solve problems related to rules about composite and inverse functions, namely procedural questions, their reasoning on the declarative and conditional questions were low and close to each other. This is meaningful since in order to complete a conditional question one needs to know the related declarative knowledge. These findings support the findings of some previous research (Byrnes & Wasik, 1991; Mack, 1990; Perry 1991; Star, Glasser, Lee, Gucler, Demir, & Chang, 2005) that preservice teachers should have a rich store of definitions and facts to explain the relationships among principles and adopt adequate procedures to the solution process. Here, one can easily infer that preservice teachers do not put specific emphasis on the conceptual understanding of the concepts since they can easily solve procedural questions, which means the adequacy in procedural knowledge inhibits their reasoning on the declarative and conditional knowledge.

Moreover, when the questions the participants faced with difficulty were analyzed, it was seen that most of the questions were given in different representations. Research has shown that representation is one of the important constructs in research on the teaching and learning of mathematics (Cobb, Yackel, & Wood, 1992; Cai, 2005). Teachers need to be able to represent a mathematical idea in multiple ways and they need to be flexible in their understanding so that they can interpret and correct their students' misunderstandings (Ball, 1990) because this influences the ways that students understand and learn mathematics (Cooney, Shealy,

& Arvold, 1998; Greeno, 1987; Thompson, 1992). Analysis of the representations used by the participants revealed the fact that preservice teachers had a limited knowledge of representations based on their limited knowledge of the concept of composite and inverse functions. Specifically, while all participants showed difficulty in solving question in graphical notation, only Yeliz experienced difficulties with the other types. This might be because of the difference between the universities the Yeliz and the other participants attended for undergraduate study, where Gizem and Deniz attended the same university.

In relation with the previous discussion, participants’ use of question types in the lesson plans and teaching practices showed a similar picture with their performances on the questions of the survey. As expected all participants make use of different types of procedural questions in every lesson and explanation, and very rarely used declarative and conditional type questions. In terms of use of different representations even though they did not give place in the lesson plans too often, they used different representations through the vignettes and the teaching practices. However, their efficiency and frequency of use was changed among the participants. Gizem was the most frequent and effective user of different representations, although not much in number she also used the conditional questions effectively. Deniz rarely used different representations and conditional questions and even though she successfully adopted the conditional questions, the representations were not effectively adopted into the lessons. Even though Yeliz also rarely used different representations and conditional questions, her adaptation of representations were better than that of conditional questions.

In terms of the combined framework used in the study, one of the indicators of SMK was being able to sequence the subtopics and questions of a unit in a mathematically hierarchical order. For sequencing subtopics, none of the participants experienced problems. However, for the questions while Gizem and Deniz managed this through the lesson plans, Yeliz’s sequencing was not successful. Since they followed their lesson plans through the teaching practices, the picture did not change for them. However, Gizem changed the order of some of the questions through the teachings in order to make it better.

The importance of connectedness of knowledge was also stated by many researchers (Baxter & Lederman, 1999; Bolte, 1999). However, when the participants' connectedness of the knowledge of functions, composite and inverse functions were investigated through the instruments it was seen that none of the participants' knowledge was connected enough. Even though they performed well on the procedural questions, this performance was not an indicator of subject matter knowledge which is reach in relations. However, Gizem's knowledge of connectedness was one step beyond the others since her ability to use different representations, analogies, value statement were more appropriate than the others.

Another indicator of SMK is correct use of definitions, terminology, and questions. The evidences of this were also searched through the results revealed that participants made mistakes once or twice through the instruments up to the teaching practices. Only, Deniz carried one of her wrong example to the class, and she was not aware of her mistake up to the point she was warned. Even though Yeliz and Gizem did not show mathematically wrong statements during teaching practices, Yeliz couldn't sequence the questions in a mathematical hierarchy. On the other hand, due to giving less importance to definitions, Deniz used some examples which were impossible to solve in the given context.

Use of different types of questions and representations through the lessons and explanations provided were also an indicator of SMK. Accordingly, evidences of those were also searched through and results revealed that participants used different representations of questions and question types in varying frequency and efficiency through their explanations. It was interesting to note that participants' explanations provided through the vignettes and the teaching practices showed similarity. Therefore, SMK of the participants with respect to the instruments assessing only SMK showed that they all had nearly the same knowledge about composite and inverse functions. However, when the integration of this knowledge was explored through the vignettes, lesson plans, and teaching practices, it was seen that Gizem was able to integrate all her knowledge into the lessons, whereas Yeliz and Deniz experienced difficulties.

The participants' SMK were assessed through the combined framework through the instruments having integration of components of PCK. According to that

Yeliz's SMK level was mostly rated as 0-1, Gizem's and Deniz's SMK levels were mostly rated as 1. Only once Yeliz's and Gizem's vignettes were rated as Level 1-2. None of the participants SMK level was rated as Level 2 through these instruments since they did not show evidences of connectedness of their knowledge, interpretation and use of different representations of questions in a logical order in different knowledge types. Therefore, these consistency of the results were taken as an evidence for the reliability of the framework.

Even though all these discussions are important part of teacher education, subject matter knowledge alone does not ensure effective teaching performance (Kahan, Cooper, & Bethea, 2003).Therefore, the other components of pedagogical content knowledge were also discussed in the following sections.

5.1.2 General Pedagogical Knowledge

General pedagogical knowledge was taken as a component in most of the studies dealing with teachers' knowledge. It was defined as a general body of knowledge, skills related to: knowledge of teaching strategies, methods, approaches and techniques, knowledge and skills related to classroom management, and knowledge of assessment techniques (Abd Rahman & Scaife, 2005; Cochran, King, & DeRuiter, 1993; Ebert, 1994; Grossman, 1990; Shulman, 1987). In line with this definition participants' general pedagogical knowledge was indirectly assessed through lesson plans, vignettes, and teaching practices.

The analysis of the results revealed that even though the grades the participants took from the courses: classroom management and teaching methods courses were close to each other, general pedagogical knowledge was the component where participants' knowledge varied more than the others. This difference can be attributable to the fact that GPK was tried to be assessed through the observed evidences and behaviors observed have tendency to affect by the beliefs of the participants (Thompson, 1992). Since, beliefs and attitudes were not the main concern of the study, the description of the pedagogical content knowledge of the participants were limited. Further research studies related with pedagogical content knowledge by also including these components were suggested in the recommendation part.

When the general characteristics of their teachings were compared, it was seen that they always used students' names, dealt with the students individually and checked the students' work by wandering in the class, gave enough time to students to take in notes, and used OHP improperly. In terms of classroom management, although Yeliz and Gizem were tried to act like a class teacher and applied some of the school rules, Deniz never tried to adopt them into the class.

In terms of the parts of the lesson, in the lesson plans and teaching practices while Gizem and Yeliz allocated different sections for all in the lesson plans, Deniz never mentioned about the sections. For the teaching practices, all of the participants made an introduction to the lesson in different ways, unable to conduct a closure, and sometimes gave homework.

All participants' teachings were teacher-centered. However, Gizem paid attention to meaning construction and tried to create such learning environments. In the lessons of Yeliz and Deniz the purpose of asking question was to get the attention of the students whereas in Gizem's lessons mostly she asked questions to involve students in the lesson and in the learning process.

The results of the combined framework evidenced the same picture. While Gizem's general pedagogical scores were mostly 1, Yeliz scores were mostly 0-1 and Deniz's scores were mostly 0 which in turn means that overall levels for the GPK was not at the desired levels. Two interrelated factors seemed to be a reason for the lack of observable impact of classroom management and teaching method courses. First, these courses were generally considered to be having a content which preservice teachers would learn for the first time and find interesting, applicable, and challenging. Even though the participants experiences through these courses included cases given in papers or via videos and after that discussion within groups and class the results revealed that not all participants felt the challenge of applying those methods. So, the second factor explaining the lack of effect of those courses was the how that knowledge was presented and assessed.

Above discussion lead to the fact that, all participants were teacher centered and unable to show the gained knowledge through the courses in the teacher education program. In terms of integration of GPK into the vignettes and teaching practices Gizem was in the first, Yeliz was in the second, and Deniz was in the third order even though their grades in the teacher education programs were close to each other. Accepting the possible effect of the beliefs of preservice teachers, in order to minimize this difference ways of assessing GPK through assessments different than written ones might be used.

5.1.3 Value of Teaching Composite and Inverse Functions

A teacher's presentation of mathematics is an indication of what the teacher believes to be most essential in it, and hence, influences the ways that students understand and learn mathematics (Cooney, Shealy, & Arvold, 1998; Greeno, 1987; Thompson, 1992). Unfortunately, teachers are not aware of the fact that their presentations include values which influences students (Bishop, 2001).

When the participants' value journals analyzed it was inferred that they all preferred intrinsic journals over the others and their use of intrinsic journals in their teaching practices approved their preference. Their second choice was seen as pedagogical value statements during teaching practices although they allocated more points to excitement and beauty statements in the journals. They never used excitement and beauty statements through the teaching practices. The result of this study similar to that of Cha (1999) in that participant in that study showed awareness of three types like in this study but prefered the real life use of the concepts namely intrinsic value over the others and reflected this through the teaching practices. They all also gave evidences for the importance of the topic for the related mathematics topics, namely pedagogical value. Even though the participants were aware of the excitement and beauty value of teaching functions, through the instruments having integration of knowledge components they never used statements related to excitement and beauty. Besides, the efficiency of use was changed according to the participants, similar to GPK, Gizem was in the first, Yeliz was in the second and Deniz was in the third order. These could be because of two reasons. First since participants were provided the value statements while completing the journals they found those statements as important but since they internally don't believe in such things, during integration of knowledge they did not use it. Second, since they don't know how to integrate that kind of an example into teaching. Because, examples of intrinsic and pedagogical value statements were more abundant through their teaching experiences as students and even in books.

5.1.4 Knowledge of Context

Knowledge of context was mentioned by many researchers as an essential component of pedagogical content knowledge (Abd Rahman & Scaife, 2005; Grossman, 1990; Marks, 1990; Veal & MaKinster, 1999). The effect of context on teachers' knowledge was reported by Wilcox, Lanier, Schram, and Lappan (1992). Because of that, awareness of knowledge of context was investigated through the study. The results lead to the fact that participants showed awareness about three emerged categories from the interview: school-related, class-related, and student-related context. Their awareness was seen more through the categories they had experienced as a student: class-related and student-related contexts. They all mentioned this during the interview by stating they know that it affects teaching but they couldn't give specific examples.

The results lead to the fact that participants showed awareness about three emerged categories from the knowledge of context interview: school-related, class-related, and student-related context. Their awareness was seen more through the categories they had experienced as a student: class-related and student-related contexts. Unfortunately, results revelaled that even though all participants stated their knowledge of context through the interview, they couldn't integrate it to the teaching practices or vignettes. In accordance with that, knowledge of context levels according to the combined framework showed that only Gizem had Level 0-1 from all teaching practices whereas the others mostly got Level 0. Participants observed many schools during School Experience I, II, and Teaching Practice courses. Besides, preservice teachers attending to this program spend more time at schools and made teaching practice more compared to other universities in Turkey. And also, they were writing weekly reflections and having a seminar course regarding those lessons. Even though they shared their experiences through these, more emphasis should be given regarding the knowledge of context categories since context identifies many things in their teachings like the examples chosen (Abd Rahman & Scaife, 2005; Cochran, King, & DeRuiter, 1993; Grossman, 1990; Magnusson, Krajcik, & Borko, 1999; Morine-Dershimer & Kent, 1999).

5.1.5 Knowledge of Learners

Recent reform efforts in mathematics education stress the importance of teachers attending to and understanding their students' mathematical thinking (Chamberlin, 2005). Moreover, the same efforts defines effective teaching that requires understanding of what students know and need to learn and then challenging and supporting them to learn it well (Cai, 2005). Fennema, Carpenter, Franke, Levi, Jacobs, and Empson (1996) mentioned that one major way to improve mathematics instruction and learning is to help teachers to understand the mathematical thought processes of their students. In line with this aim, participants' knowledge of learner was discussed concerning lesson plans, vignettes, and teaching practices.

Through the vignettes, all participants diagnosed the students' errors and tried to resolve the conflict the student or the class have through her explanations in the form of mostly in numerical examples and/or sometimes in procedural review of the concept. As a result of their consciousness of students' needs of understanding, from time to time they used different representations in their explanations in varying frequencies among the participants where Gizem was the most frequent user. As a result, her explanations were more student-friendly and got understood by the students. However, during the teaching practices diagnosing students' misunderstandings was not at the first priority like in the vignettes. Participants dealt with the students' misunderstandings whenever they were asked a question. Even though the first aim in teaching is students' understanding (Grossman, 1990), participants continued teachings without assuring students' understandings.

Results of the analysis of teaching practices also revealed that Gizem was different from Yeliz and Deniz in terms of knowledge of learners. Because, while Deniz and Yeliz answered most of the students' questions, Gizem answered all of them. Besides, dialogues revealed for Deniz that she did not put much emphasis on meaning construction and lessons were so much procedural that sometimes students had difficulty to understand for what purpose they were doing the exercises. For Yeliz, dialogues revealed that if the students' question was not about the solution of a question or about a definition of a concept she did not give enough attention. Gizem was aware of the fact that students needs different representations for understanding and always used Venn diagrams and analogies through the teaching practices and rarely used functions in graphs and ordered pairs. If students' question was about a concept learned, she always returns to basic analogy she gave and used Venn diagrams. During her explanations, she sometimes asks questions to the class and always waited for students' answers and constructed her explanations on students' answers. Besides, she frequently asked whether they have any questions. Accordingly the analysis of the dialogues revealed that throughout the teaching practices she used statements, questions, and examples which could promote meaning construction.

As a result, the knowledge of learner levels of Yeliz and Deniz in the teaching practices were 0-1 similar to their levels in the vignettes, the knowledge of learner level of Gizem was 1, which was higher than her level in the vignettes. Gizem's knowledge of learner level in the teaching practices was higher since she constantly prepared learning environments which promoted meaning construction. Again, the framework resulted in consistent results. Through the method courses participants were taught what a misconception is, why it is important, but specifically not talked the content-specific misconceptions and how they would overcome those. The results revealed that preservice teachers need more than this to feel the importance of identifying and resolving misconceptions through the lessons.

In addition to these discussions, there is more thing that should be discussed as a natural cause of action of this study. Through the data collection, participants devote a considerable amount of time and energy in order to complete the instruments. Data analysis revealed that this process had some influences on their awareness about the concepts but this influence was not affected all of them in the same manner due to their knowledge and beliefs. Thus, data revealed that to be involved in such a process had challenged them and might have developed their knowledge components of pedagogical content knowledge.

When the overall picture was evaluated it was seen that all participants have some knowledge in every component but how they integrated it varied. In terms of integration it was seen that they put much effort to create variety while applying SMK. For example, in terms of GPK they never tried to use methods like cooperative learning, in terms of knowledge of learners they never tried to assess students' understanding or misunderstandings something other than verbal questions, in terms of value they never bring some materials that could show students some kind of value visually, in terms of knowledge of context they could consider the effect of external exams or students’ attention level. Even though these criticisms, Gizem was one step beyond the others and Yeliz was one step beyond Yeliz in terms of integration of knowledge. So, the question comes out from this study was that even though they had similar knowledge levels why there is a difference at the integration stage and how this difference is minimized?

5.2 Implications for the Mathematics Teacher Educators

This study provides information that can be useful in implementing rational future changes for mathematics teacher education. According to the conclusions and the literature review, the educational and pedagogical suggestions can be presented as the followings.

Recent study supported the argument in the literature that preservice teachers should have a highly-connected and well-formed subject matter knowledge (Lederman, Gess-Newsome, & Latz, 1994; Gess-Newsome & Lederman, 1999; Shulman, 1987). In line with this, teacher educators were faced with the question of how to promote deeper understanding of mathematics during teacher education programs (Crespo & Nicol, 2006). Thus, teacher education programs are expected to develop both depth and breadth in subject matter knowledge of the preservice teachers which is only possible through a course work that covers the content relevant to students. In the particular context of this study, there were two courses (Mathematics Curriculum Review I and II) designed for covering the contents of the school curriculum. The coursework includes review of the concepts mainly in high school curriculum, and since there is so much to cover in a limited time only selected topics were covered. Even though the participants took these courses they experienced difficulties while completing the instruments related to subject matter knowledge. One of the participants Gizem explained this as follows: “During the curriculum courses we focused covering as many topics and solving as many questions as we can, so during the survey I felt like I forgot the definitions”. Therefore, the point is not offering a course on curriculum, such courses should be challenging, require critical exploration and should be accompanied by required course work in curriculum and methods of teaching. The analysis of the results shown that when the participants were faced with challenging tasks like the survey and the non-routine questions they were positively influenced and they questioned their conceptions of the concepts. Therefore, involving challenging tasks related to school curriculum in such courses would be beneficial for preservice teachers. Besides, research has shown that methods courses and content courses based on actual mathematics to be taught can be directly "transportable" into classrooms (Borko & Putnam, 2000). The researchers show in a case study that a preservice teacher

> was able to take specific tasks introduced in her mathematics methods course and successfully adapt them to her own classroom and students. These tasks were transportable across situations, despite the fact that she was a student in one situation and a teacher in the other. Such was not the case with discourse patterns... [the novice teacher] was responsible for initiating and establishing expectations for inquiry-based discourse in her own classes, as opposed to simply participating in such discourse as a student. Her teacher education program provided [her] with little knowledge of how to manage this responsibility; thus, she found it much more difficult to transport discourse patterns, than instructional tasks, into her teaching practice. (p. 203)

Research has also shown that there is little relationship between increased advance content courses that are not based on elementary or secondary school mathematics and improved teaching competence (Borko & Putnam, 2000). Since curriculum review courses do not exists in the teacher education programs in other universities of Turkey, adding such courses to the programs with the emphasis stated above could be valuable for preservice teachers.

Research findings of the study also revealed that preservice teachers only experienced difficulty in answering declarative and conditional type questions and/or non-routine questions which also supported the argument in the literature that declarative knowledge has effect on the conditional knowledge (Aydın, 2007). Furthermore, it was seen that even though not having an appropriate concept definitions preservice teachers solved the procedural questions easily. Therefore, in order to have a competence in subject matter knowledge about a concept preservice teachers knowledge of concept definitions (declarative knowledge) must be improved which will affect directly their knowledge of building relationships among concepts and principles (conditional knowledge). There is a need to create learning environments and assessments which includes all knowledge types.

One of the most important weakness in the participants understanding of composite and inverse functions was their connectedness of mathematical functions concept with its subunits, and related units in mathematics. To the teachers, knowing functions means solving procedural questions. It will be difficult for teachers to implement connections in their teaching if their own mathematical experience does not even make them aware that such connections exists (Wilson, 1992). As a result, one obvious implication is that mathematics teacher education should devote attention to the issue of connectedness of mathematical knowledge as well as the connection in the mathematics and in the real world. In line with this, during the evaluation interview all of the participants stated that they prefer to have these kinds of activities before teaching practice and these kinds of things should be a part of teacher education program.

Analysis of the vignettes, lesson plans and teaching practices in light of the combined framework revealed that participants' levels in the vignettes and the lesson plans were close to those of teaching practices. Therefore, increasing the number of activities which mimics the classroom cases like vignettes would be beneficial for preservice teachers. Furthermore, after these activities writing reflections or sharing reflections as a class could promote preservice teachers' understanding of the cases. Similarly, participants stated during the evaluation interview that completing vignettes were very useful activity for them but they also would like hear the other participants responses and discuss with them.

In terms of GPK, all participants were teacher centered and unable to show the gained knowledge through the courses in the teacher education program. Accepting the possible effect of the beliefs of preservice teachers, in order to minimize this difference ways of assessing GPK through assessments in the methods and classroom management courses assessment techniques different than written ones might be used. Moreover, teacher education programs should consider increasing the preservice teachers' awareness about their own beliefs since they affect their actions. This awareness could be done via questionnaires or interviews throughout the program. Because, by involving in these processes preservice teachers have to think about their own beliefs. Furthermore, as a part of their beliefs values of teaching composite and inverse functions were investigated. The analysis of the results suggested that even though preservice teachers know the value of teaching a topic they experienced difficulty in integrating those into class. Therefore, through the method courses specific emphasis should be given on how to integrate those into teaching practice.

Knowledge of context categories since context identifies many things in their teachings like the examples chosen (Abd Rahman & Scaife, 2005; Cochran, King, & DeRuiter, 1993; Grossman, 1990; Magnusson, Krajcik, & Borko, 1999; Morine-Dershimer & Kent, 1999). The results of the study revealed that even if preservice teachers were experiencing different contexts through their programs, this does not ensure increased awareness. Therefore, through the seminar courses following the practice courses, more emphasis should be given regarding the categories of knowledge of context. This could be done via preparing hypothetical cases requiring solutions in a specific context.

Parallel with the implications for the mathematics teacher educators, in the following section recommendations for further research studies were presented.

5.3 Recommendations for the Future Research Studies

This research study was aimed at understanding the preservice secondary mathematics teachers' pedagogical content knowledge of composite and inverse functions. Findings of the study were believed to suggest valuable implications for the mathematics teacher educators. Based on the findings, in this section, several related research studies were suggested.

This study provides information to policy makers about in what ways preservice mathematics teacher education programs should be improved. Although this study does not a representative sample for the teacher education programs in Turkey, the results revealed that the knowledge structures of the preservice teachers were not at the desired levels. Thus, first of all further research studies should be carried in out in different mathematics teacher education programs in Turkey to see the whole picture.

The results of the study revealed that preservice teachers sometimes did not demonstrate the knowledge they hold especially GPK, value and knowledge of context and this might be attributable to the fact that they were affected by the beliefs they hold. Therefore, further research study may also investigate the pedagogical content knowledge by including the beliefs and attitudes the preservice teachers hold. Since every choice the teacher made during teaching depends on the mathematical conceptions he/she holds so by describing conceptions of teachers along with the other components.

The present study only focused on the pedagogical content knowledge about composite and inverse functions. The studies that will investigate the pedagogical content knowledge about the other significant content areas of mathematics would also be beneficial for understanding of the nature of preservice teachers' knowledge about mathematics.

The focus of this study was on the pedagogical content knowledge the teachers had about composite and inverse functions. Another study could include the relationship between the pedagogical content knowledge of presevice teachers and the students' achievement on a certain topic. Similarly, the pedagogical content knowledge of the inservice teacher could be investigated and then contrasted with the preservice teachers' pedagogical content knowledge.

This was a qualitative multi-case study. By developing tests to measure pedagogical content knowledge and applying this to preservice and inservice teachers in large numbers, quantitative research studies could be performed which could give a chance to generalize the findings of the research study to the broader context.

Lastly, while evaluating instruments having integration of knowledge components the combined framework was used. Some parts of it were already exists in the literature (Ebert, 1994; Lindgren, 1996; Thompson, 1991) and some parts were written by the researcher. Therefore, a future direction for the framework will be adding or eliminating some statements in the components of the combined framework. This framework was also the contribution of the present study to the pedagogical content knowledge literature since all of the components of a highly subjective construct were assessed through a working tool that is open to improvement.

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APPENDIX A

TEACHER EDUCATION PROGRAM DETAILS

MATHEMATICS TEACHING

FIRST YEAR

Autumn Semester

Code	Course Name	Hours			Credit	ECTS Credit
		Lec.	Prac.	Lab		
MTE 501	Mathematics Curriculum Review I	3			3	8
MTE 503	Computer Technology in Mathematics Education	2	2		3	6
TE 501	Introduction to Teaching Profession	3			3	8
TE 535	Mathematics Teaching Methods I	2	2		3	6
TE 555	School Experience I in Mathematics	1	4		3	6

Spring Semester

Code	Course Name	Hours			Credit	ECTS Credit
		Lec.	Prac.	Lab		
MTE 502	Mathematics Curriculum Review II	3			3	8
TE 519	Classroom Management	2			2	4
TE 523	Understanding Arguments	3			3	6
TE 524	Guidance	2			2	6
TE 545	Mathematics Teaching Methods II	2	2		3	6
TE 565	School Experience II in Mathematics	1	4		3	7

Summer Semester

Code	Course Name	Hours			Credit	ECTS Credit
		Lec.	Prac.	Lab		
TE 502	Development and Learning	3			3	6
TE 504	Educational Technology and Materials Development	3			3	6

SECOND YEAR

Autumn Semester

Code	Course Name	Hours			Credit	ECTS Credit
		Lec.	Prac.	Lab		
TE 506	Planning and Assessment in Teaching	4			4	8
TE 575	Teaching Practice in Mathematics	2	6		5	8

Spring Semester

Code	Course Name	Hours			Credit	ECTS Credit
		Lec.	Prac.	Lab		
ENG 404	English for Philosophy of Education	3			3	-
TE 521	History of Political and Educational Philosophy	3			3	6
TE 507	Subject Area Textbook Review	3			3	6

MATHEMATICS TEACHING

FIRST YEAR

Autumn Semester

MTE 501 Mathematics Curriculum Review I

This course provides students with knowledge and experience to assist them to become effective mathematics teachers. The major areas of mathematics taught in school will be reviewed in detail and related to the high school curriculum, focusing on grade 9 and grade 10. The skills covered include knowledge of the appropriate level of mathematical content and relevancy, together with a working knowledge of school mathematics text books, and the application of these skills in the classroom. National standards in mathematics will be discussed. *Credit units: 3 ECTS Credit units: 8.*

MTE 503 Computer Technology in Mathematics Education

The course will equip student-teachers with the skills to use computer technology to teach secondary mathematics. These skills will be used to create lesson plans, classroom demonstrations and teaching/learning materials that clarify topics in the mathematics curriculum. The topics covered will include algebra, geometry, trigonometry, calculus, probability, discrete math, and other areas. *Credit units: 3 ECTS Credit units: 6.*

TE 501 Introduction to Teaching Profession

Characteristics and principles of the teaching profession. The school as an organization. Management, leadership and decision-making in schools. School effectiveness and school improvement. Sociological, psychological and philosophical foundations of educational practice. Classroom and school environments. The curriculum. Learning theories. Domains of learning. The Turkish educational system, its history and current policies.

TE 535 Mathematics Teaching Methods I

The course explores, with practical examples, and with reference to current research, the teaching of mathematics at high school level. It considers all relevant teaching methods, and their application to a range of teaching/learning contexts. Students will engage in extensive reflection on the methods and applications considered. Credit units: 3 ECTS Credit units: 6.

TE 555 School Experience I in Mathematics

Students spend an extended period in a school, under the supervision of their school mentor and faculty supervisor. Students become members of the school for this period. They work with teachers, they attend meetings and extra-curricular activities, they observe lessons, and teach full lessons in the mathematics department. The course includes tutorials and seminars which assist students in the planning and evaluation of their school work and allows them to share experience. Credit units: 5 ECTS Credit units: 8.

Spring Semester

MTE 502 Mathematics Curriculum Review II

This course is a continuation of MTE 501. The major areas of mathematics taught in school will be reviewed in detail and related to the high school curriculum, focusing on grade 10 and later. Students gain further understanding of mathematics content, relevancy, and the application of these skills in the classroom. Discussion of national standards in mathematics will continue. Credit units: 3 ECTS Credit units: 8.

TE 503 Classroom Management

Classroom organization for effective learning. Development and implementation of effective systems for classroom management to maximize learning. Social and psychological factors which determine or affect students' attitudes, motivation and behavior in schools. Group interactions. Behavioral problems. Techniques for meeting the needs of individual learners. The analysis of events and critical incidents in the classroom.

TE 505 Guidance

General principles of guidance and counseling in schools. Nature and objectives of guidance services, and their role in education. Procedures to be observed. Special education: the special needs of individual school students, their assessment, and the education of students with such needs.

TE 522 Understanding Arguments

Language and argument. The basic structure of arguments. Validity, truth, soundness. The formal analysis of arguments: Propositional logic, categorical logic, predicate logic. Inductive reasoning. Probabilistic reasoning. Fallacies. Paradoxes. Areas of argumentation: legal, moral, scientific, philosophical. Throughout the course, there is emphasis on the uses of language in everyday reasoning.

TE 545 Mathematics Teaching Methods II

This course is a continuation of TE 535. It continues the developmental work of TE 535 in the teaching of mathematics. Students gain further understanding of the teaching and learning methods which may be used with different groups of students, and of the context in which learning is set. There will be further practical applications and classroom experience. *Credit units: 3 ECTS Credit units: 6,* *Prerequisite: TE 535.*

TE 565 School Experience II in Mathematics
Students spend one day a week in a school, under the daily supervision of their mentor. They teach classes, as well as working on structured activities related to teaching and the school environment. There is a one-hour seminar which consolidates the work done in school. Credit units: 3 ECTS Credit units: 7, Prerequisite: TE 555

Summer Semester

TE 502 Development and Learning
Physical, cognitive, psychological and social development of the individual. Learning theories and development. Application of learning theories to educational issues. Analysis of educational research with reference to the classroom and teaching/learning activities, the design of effective instruction.

TE 504 Educational Technology and Materials Development
The use of technology in teaching: computers, visual teaching aids, and all other interactive materials. The production of such materials by student teachers, and the evaluation of these materials when used in teaching.

SECOND YEAR

Autumn Semester

TE 506 Planning and Assessment in Teaching
Concepts, processes and principles of curriculum planning and program development. Production of annual, unit and daily plans. Teaching methods and strategies, and the selection of appropriate teaching materials. Introduction to the field of assessment and testing, theoretical background, and practice in test and item construction. Functions and uses of assessment.

TE 575 Teaching Practice in Mathematics
Students spend an extended period in a school, under the supervision of their school mentor and faculty supervisor. Students become members of the school for this period. They work with teachers, they attend meetings and extra-curricular activities, they observe lessons, and teach full lessons in the mathematics department. The course includes tutorials and seminars which assist students in the planning and evaluation of their school work and allows them to share experience. Credit units: 5 ECTS Credit units: 8.

Spring Semester

ENG 404 English for Philosophy of Education

This course aims to provide students with the necessary academic skills to read, analyse, discuss and write about primary political theory texts. An emphasis is placed on close reading and evaluation of key passages and on the logical and coherent structuring of short written arguments. Credit units: 3 ECTS Credit units: None.

TE 507 Subject Area Textbook Review

Review of Ministry of Education-approved textbooks. Book review in terms of the school curriculum, sequencing, ease of use by students, readability and other criteria. Contribution to the development of student understanding and skills.

TE 521 History of Political and Educational Philosophy

The course introduces students to philosophical thinking about the relation between human nature, society and education. It focuses on the study of key texts in the history of philosophy and educational thought including Aristophanes, Plato, Descartes, Voltaire, Mill and Russell. There is strong emphasis on the development of students critical reasoning skills. Students are encouraged to think about the implications of the views discussed for their own pedagogical practice. Credit units: 3 ECTS Credit units: 6.

APPENDIX B

SURVEY OF FUNCTION KNOWLEDGE

The content you provide in the questionnaire will not be disclosed, and we appreciate you are doing your best.

Name, Surname:

1. In your view, what is a function? (You may give as many definitions as you want). If you give more than one definition, please discuss the differences and similarities between the features of the definitions.

2. Describe the different ways a function can be represented.

3. Make a definition of
   (a)composite function
   (b) inverse of a function.

4. Please state whether or not each of the following is a function and why.
    a. The People's Republic of China is the country with the largest population in the world, over 1.1 billion in 1990. Despite the efforts to limit families to one child, the population of China was still growing at a rate of 1.5% per year in 1990. Is there a function? Why or why not?

    b. A rental company charges 32YTL per day (100km free per day) and an additional 0,10 YTL per km. Is there a function? Why or why not?

    c. A(p,q,r)=3,5p+6q+3r=1500. Is there a function? Why or why not?

    d. The formula for the area of a trapezoid. Is there a function? Why or why not?

e. $g(x) = \begin{cases} x, \text{ if } x \text{ is a rational number} \\ 0, \text{ if } x \text{ is an irrational number} \end{cases}$ Is there a function? Why or why not?

f.

x	0	1	2	3	4	5	6	7	8
y	0	1	2	1	0	1	2	1	0

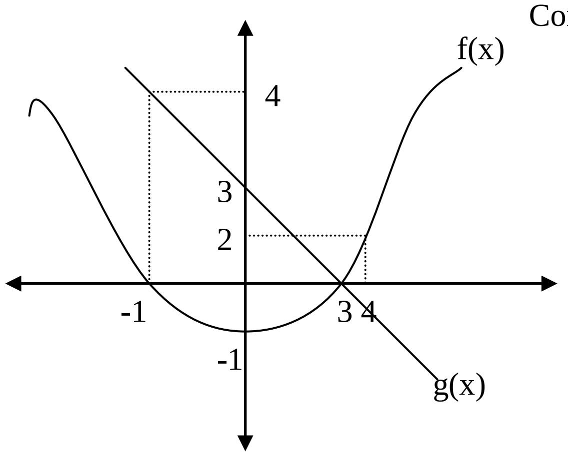
Is there a function? Why or why not?

5. What do the domain and range of the function mean? What is the importance of them?

6. Find the domain of the function $f(x) = \dfrac{\sqrt[4]{x-2}}{x-3} + \dfrac{\sqrt[3]{x^2+1}}{\sqrt[5]{x^2-16}}$.

7. If $f(x)=x^2-9$ find $f([-4,3])$.

8.



Considering the graphs of f and g, find

a. $(fog^{-1})(3)$
b. $(fogof^{-1})(-1)$
c. (f-g)(0)
d. (2f+g)(4)
e. (f/g)(0)
f. (fg)(0)

9. Given $f(x)=\dfrac{5f(x-1)+3}{5}$ and $f(0)=1$. What is f(30)?

10. If $f: R-\{3\}\to R-\{2\}$ and $f(x)=\dfrac{ax-2}{4x-b}$ is a one to one and onto function, find a and b.

11. Let f and g be two functions whose domains and ranges are subsets of the set of real numbers. Prove or find a counter-example to the following two statements.
    a. If f and g are both 1-1 then it follows that f+g is 1-1.
    b. If f and g are both onto then it follows that f+g is onto

12. Give your reasons why the following functions do or do not have inverse functions. If exists, write the function. If not, give a numerical example.
    a. Your hourly wage is 7,7YTL plus 0,90YTL for each unit x produced per hour. Let f(x) represents your weekly wage for 40 hours of work. Does this function have an inverse?
    b. Let x represent the retail price (satış fiyatı) of item in YTL, and let f(x) represent the sale tax on the item. Assume that the sale tax is 7% of the retail price and that the sale tax is the rounded to the nearest natural number. Does this function have an inverse?

13. If $f^{-1}(x) = 3x+1$ and $(fog^{-1})(x) = \dfrac{x+4}{3}$, find $g(x)$.

14. Given $f^{-1}(\dfrac{x+1}{2}) = \dfrac{x-3}{4}$, find $f(2x+3)$ in terms of $f(x-1)$.

15. Let $f(x-2) = x+5$ and $g(2x-5) = \dfrac{x+2}{3}$. If $(f+g)(k) = 5$, find k.

16. Given $f(x) = h\left(f\left(\dfrac{x+1}{3}\right)\right)$ and $(h^{-1}of)(x) = 3x-4$, find $f(4)$.

17. Consider the set of functions whose domain and set of images are the real numbers. K assigns to each pair of such functions to their composition.

a. Is K a function? Explain.
b. Is K^{-1} a function? Explain.

18. Given $(f \circ g)(x) = \sqrt[5]{x+3}$

a. Find f and g that satisfy this condition.
b. Are there more than one answer to part a. Explain.

19. If $f(x) = 0$ only when $x = 1$ and $x = 4$, then for what values of z does $f(4z) = 0$?

APPENDIX C

KNOWLEDGE OF CONTEXT FOCUS GROUP INTERVIEW PROTOCOL

Araştırma Sorusu: Matematik öğretmenliği öğrencilerinin okul deneyimi dersini geçirecekleri okulların ve öğrencilerin şartları ve genel durumu hakkındaki bilgileri ve düşünceleri nelerdir?

Görüşme Formu

Tarih / /2006 **Saat** Başlangıç ______Bitiş ______**Görüşmeci**_____________.

Merhaba, benim adım Burcu Karahasan ve Özel Bilkent Lisesi'nde matematik öğretmeni olarak görev yapmaktayım. Doktora çalışmam kapsamında yaptığım bu görüşmenin amacı matematik öğretmenliği öğrencilerinin okul deneyimi dersini geçirecekleri okulların ve öğrencilerin şartları ve genel durumları hakkındaki bilgi ve düşüncelerini ortaya çıkartmaktır. Sizinle bu konuda görüşme yapmak istiyorum. Çalışma sonuçlarının matematik öğretmen yetiştirme programlarının niteliğinin arttırılmasına katkıda bulunacağını ümit ediyorum. Bu nedenle, sizlerin görüşleri benim için çok önemlidir.

- Bana görüşme sürecinde söyleyeceklerinizin tümü gizli kalacaktır. Bu bilgileri araştırmacıların dışında herhangi bir kimsenin görmesi mümkün değildir. Ayrıca, araştırma sonuçlarını yazarken görüştüğümüz bireylerin isimlerini kesinlikle rapora yansıtmayacağım.
- Başlamadan önce, bu söylediklerimle ilgili belirtmek istediğiniz bir düşünce veya sormak istediğiniz bir soru var mı?
- Görüşmeyi izin verirseniz kaydetmek istiyorum. Bunun sizce bir sakıncası var mı?
- Bu görüşmenin yaklaşık bir saat süreceğini tahmin ediyorum. İzin verirseniz sorulara başlamak istiyorum.

1) Aranızda özel ders veren var mı?

- Hangi seviyelere veriyorsun?
- Daha önce bu seviyede öğrencin olmuş muydu?
- Bu iki ders arasında bir karşılaştırma yaparsan bize neler anlatabilirsin?

Sorulara cevap verirken okul deneyimi 1 ve 2 derslerinin kapsadığı tüm okulları ve deneyimleri göz önünde bulundurunuz. Örnek verirken okul isimi kullanabilirsiniz.

2) İki farklı okulda çalıştığınızı ve her ikisinde de lise 1. sınıflarda ders verdiğinizi düşünün. Dersinizde farklılık olabileceğini düşünüyor musunuz?

- Evet ise; Bu farklılıkların hangi sebeplerden kaynaklanacağını düşünüyorsunuz, kendi deneyimlerinizden (okul deneyimi 1 ve 2) örnek vererek anlatınız.
- Hayır ise veya açıklamalarda deyinilmemişse; Çevre'nin, okul kültürü'nün, okulun olanaklarının, veli beklentilerinin, sınavların (ÖSS, Uluslararası Bakalorya (IB) standart sınavları), bölüm içi kuralların,…etkisi olup olmayacağı konusunda ne düşünüyorsunuz.

3) Aynı okulda ve aynı seviyede iki sınıfa ders verdiğinizi düşünün. Dersinizde farklılık olabileceğini düşünüyor musunuz?

- Evet ise; Bu farklılıkların hangi sebeplerden kaynaklanacağını düşünüyorsunuz, kendi deneyimlerinizden (okul deneyimi 1 ve 2) örnek vererek anlatınız.

Hayır ise veya açıklamalarda değinilmemişse; sınıf seviyesinin, öğrenci seviyesinin, ailelerinin, ders saatinin, öğrencilerin ilgilerinin, öğrencilerin güçlü ve zayıf yönlerinin, sınıftaki öğrencilerin sosyoekonomik statüsünün, … etkisi olup olmayacağı konusunda ne düşünüyorsunuz.

APPENDIX D

CONCEPT MAP ACTIVITY

Name, Surname:

Function Concept map #1

You are required to construct a concept map for the functions unit in the 9th grade level. The following multi-step process will led you through the construction.

1. Generate (write down) the list of terms for the function unit
2. Sort the terms into clusters. There is no one correct map, do not worry about making a mistake.
3. Arrange the clusters and the terms either hierarchical or web-like. See the given examples for both types.
4. Draw linking lines between each cluster and cross-links between related clusters
5. Label the linking lines to explain the relationship being illustrated (e.g., has the property, is an example of, involves).
6. Draw directional arrows on the linking lines to indicate the direction of the relationship being expressed.

Name, Surname:

Function Concept map #2

You are required to construct a concept map for the functions unit in the 9th grade level by using the words provided in the table. The following multi-step process will lead you through the construction.

Functions (terms for concept map)

Single-valued	Domain	Graph	Pre-image
Equation	Linear	Arbitrary	Correspondence
Dependent variable	Independent variable	Mapping	Symbolic
Onto	Function	Table of values	Addition
Ordered pair	Piecewise	Inverse	Multiplication
Constant	One to one	Range	Representation
Composition	Image	Rule	Relation
Formula	Vertical line test	Undefined	Transformation

1. Read the list of terms
2. Sort the terms into clusters. There is no one correct map, do not worry about making a mistake.
3. You can add additional terms or remove (omit) any terms from the list.
4. Arrange the clusters and the terms either hierarchical or web-like
5. Draw linking lines between each cluster and cross-links between related clusters
6. Label the linking lines to explain the relationship being illustrated (e.g., has the property, is an example of, involves)
7. Draw directional arrows on the linking lines to indicate the direction of the relationship being expressed

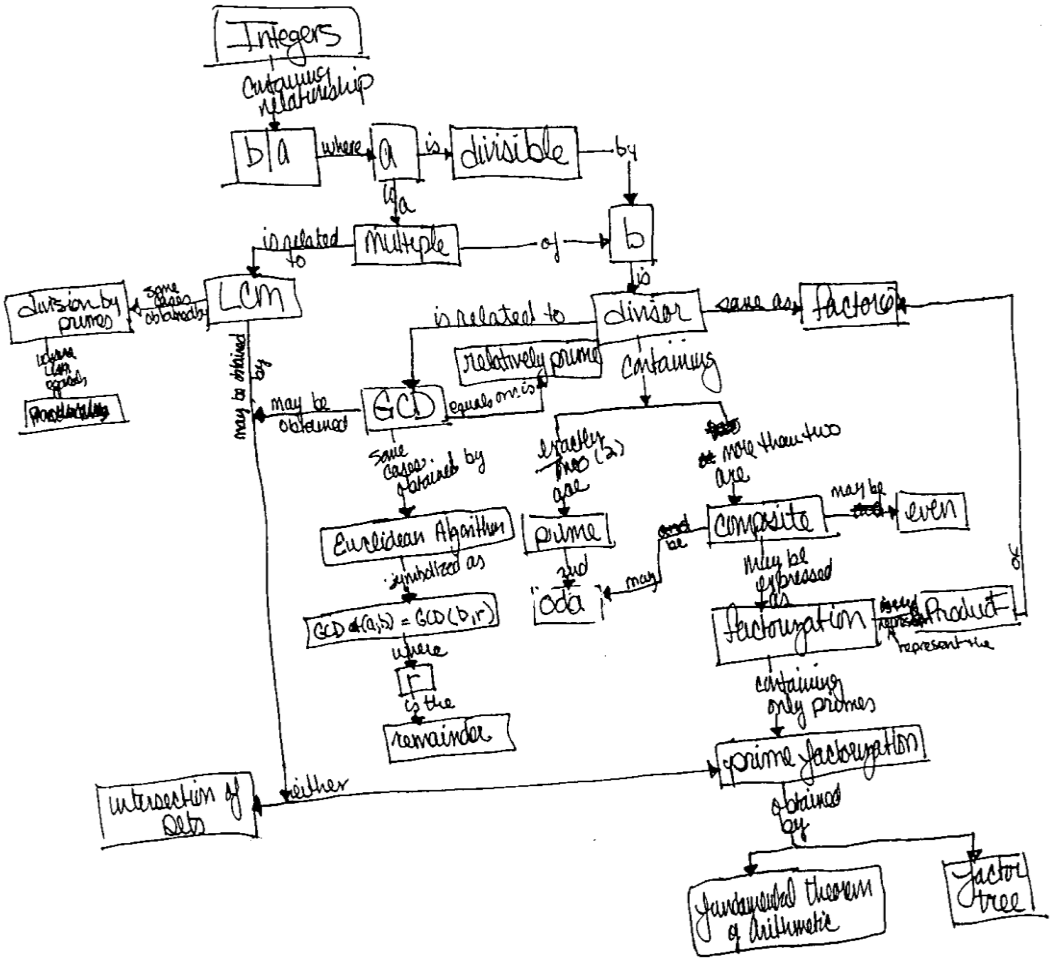
Name, Surname:

Concept map Essay

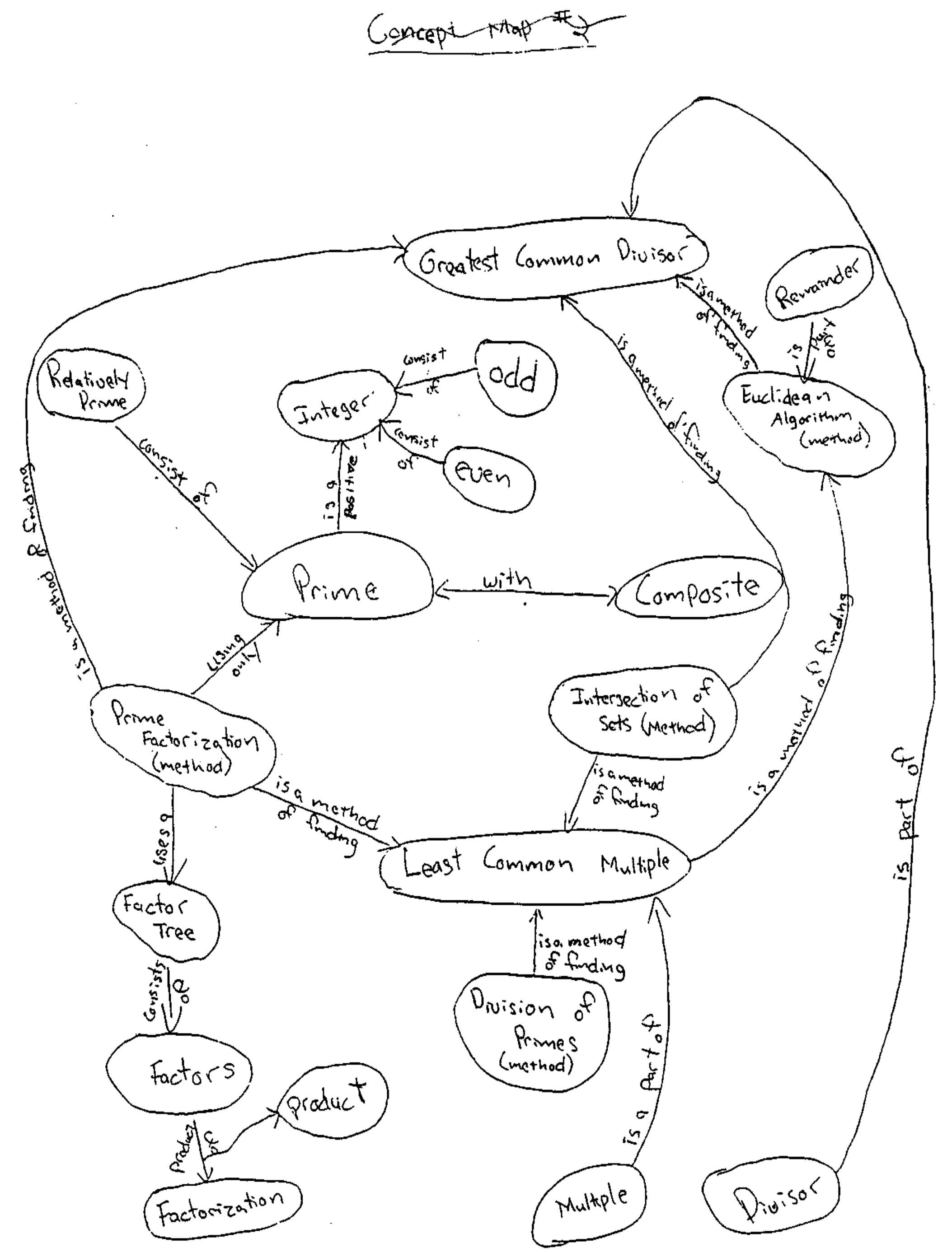
Write an interpretive essay comparing your concept maps by answering the following questions

1. Describe your thought process while constructing your first concept map.
2. Describe your thought process while constructing the second map when you were given the terms to consider incorporating?
3. Are there any differences (inconsistencies, anomalies etc.) in your maps? What are they? Where do you think these differences come from? Include additional information that might be relevant or that presents personal insights.

A.1 Hierarchical Concept Map



A.2 Web-like Concept Map



APPENDIX E

JOURNALS ABOUT THE DEFINITIONS ABOUT FUNCTIONS, INVERSE FUNCTIONS AND COMPOSITE FUNCTIONS

Functions

1. Please choose **your favorite 3 definitions** from the list and explain your reasons for choosing them. Also, indicate the order you choose the definitions.
2. Please choose your **least favorite** definition from the list and explain your reasons for choosing it.
3. With or without considering the definitions given above, how you may give the definition of a definition of a function while you are teaching to the 9th grade class? Please describe the underlying reasoning for your decision.
4. Suppose you presented the definition you gave at question 3 to the students, however, some of your students said that they did not understand it. With or without considering the definitions given above how are you going to clear up the confusion the students have?
    a. A set of number pairs that are related by a certain rule.
    b. A relationship between two variables such that changes in one variable result in change in another.
    c. A correspondence between two sets P and Q in which each element of P corresponds exactly one element of another set.
    d. A rule that assigns to each element of one set exactly to one element of another set.
    e. A machine that you put a number in, the machine changes it, and gives you an output.

f. A relation in which, for each ordered pair, the first coordinate has exactly one second coordinate.
g. A relationship between two quantities where the value of one quantity is uniquely determined by the value of the other quantity.
h. A set of ordered pairs in which each first element is paired with exactly one second element.
i. A relation with a graph that passes vertical line test.
j. A correspondence between two variables such that each value of the first variable corresponds to exactly one value of the second variable.
k. A dependence of one variable on another.
l. A function is a formula or an expression or an equation made up of variables and constants representing the relation between two variables with its graph having no sharp corners.
m. If some quantities depends on others in such a way that if the latter are changed the former undergo changes themselves then the former quantities are called the function of the latter quantities. If x denotes a variable quantity then all the quantities which depend on x in any manner whatever or are determined by it are called its functions.
n. Y is a function of variable x if for any value of x in a given interval there corresponds a unique value of y. It does not matter whether throughout this interval y depends upon x according to one law or more than one law or even whether the dependence of y on x can be expressed by mathematical operations or analytic expression (i.e., formula, equation, etc.).
o. A function is a set of ordered pairs in which each first element is paired with only one second element.
p. A function is a relation with a graph where any line drawn parallel to y axis crosses the graph of function only once.
q. A function is a special kind of dependence relationship, that is, between variables which are distinguished as dependent and independent.
r. A function is any correspondence between two sets which assigns to every element in the domain exactly one element in the range.

s. A function is an operation or a manipulation (one acts on a given number, generally by means of algebraic operation, in order to get its image).
t. A function is a relationship between two quantities where the value of one quantity is uniquely determined by the value of the other quantity.
u. A function can be considered as a machine (a black box) that has an input and output; for each value in you get a value out the other side.

Inverse Functions

1. Please choose **your favorite 3 definitions** from the list and explain your reasons for choosing them. Also, indicate the order you choose the definitions.
2. Please choose your **least favorite** definition from the list and explain your reasons for choosing it.
3. With or without considering the definitions given below, how you may give the definition of a definition of an inverse function while you are teaching to the 9^{th} grade class? Please describe the underlying reasoning for your decision.
4. Suppose you gave the definition you gave at question 3 to the students, however, some of your students said that they did not understand it. With or without considering the definitions given above how are you going to clear up the confusion the students have?

Definitions

a. Inverse function is a function which does the opposite of the things done by the original function.

b. If the graph of two functions f and g symmetric with respect to y=x then these two functions are inverses of each other and it is denoted by $f^{-1}=g$ or $g^{-1}=f$.

c. In the equation of function f, when x and y values are changed and then y written in terms of x, this new equation is called an inverse of the function f and denoted by f^{-1}.

d. Inverse functions are special class of functions that undo each other.

e. If all the y values and x values of a function f are changed place the new function is called an inverse of the function f and denoted by f^{-1}.

f. If all the input and output values of the function f are changed place, the new function is called the inverse of f.

g. If f: A→B is one-to-one and onto function then there exists the inverse of f denoted by f^{-1} such that f^{-1}: B→A, f(x)=y, and $f^{-1}(y)=x$.

h. The inverse of a function f denoted by f^{-1} is formed by reversing all of the ordered pairs in the function f.

i. Let f be a function and g be the inverse of f.

Then f(x)=y, and g(y)=x denoted by f: A→B and g: B→A

x →y   y →x

j. The two functions f and g are inverse functions, or are inverses of each other, provided that; the range of values of each function is the domain of definition of the other, and the relation f(g(x))=x and g(f(x))=x hold for all x in the domains of f and g respectively.

k. Let f be a one-to-one unction with domain A and range B. then its inverse function f^{-1}has domain B and range A and is defined by $f^{-1}(y)=x \Leftrightarrow f(x)=y$ for any y in B.

l. In the equation of one-to-one function f, by writing y for f(x) and solving this equation for x in terms of y, and interchanging x and y we get a new equation. This resulting equation is called the inverse function of and denoted by $y=f^{-1}(x)$.

m. Each output of one-to-one function f comes from just one input, so a one-to-one function can be reversed to send outputs back to the inputs from which they came. The function defined y reversing a one-to-one function f is the inverse of f.

Composition of Functions

1. Please choose **your favorite 3 definitions** from the list and explain your reasons for choosing them. Also, indicate the order you choose the definitions.
2. Please choose your **least favorite** definition from the list and explain your reasons for choosing it.
3. With or without considering the definitions given above, how you may give the definition of a definition of a composite function while you are teaching to the 9th grade class? Please describe the underlying reasoning for your decision.
4. Suppose you gave the definition you gave at question 3 to the students, however, some of your students said that they did not understand it. With or without considering the definitions given above how are you going to clear up the confusion the students have?

Definitions

a. For the function f, writing g(x) for every x in f(x), one will get f(g(x)) which is the composition of f and g.

b. In logic if a→b and b→c then a→c. Therefore, if there exists a function f which takes a to b and another function g which takes b to c, then one can talk about a third function, say h, which takes a to c. This new function is denoted by h=gof and called the composition of g and f.

c. Let f: A→B and g: B→C be two functions. Then gof: A→C defined as (gof)(x)=g(f(x)) is called composition of f and g.

d. Composition of functions is a mechanical topic which can be best explained through examples showing how to substitute the functions into each other.

e. Composition of functions are like the transformation of graphs $f(x)=3x-2$ *and* $g(x)=x^2$ *where fog is a transformation but gof is not.*

f. The composition of two functions f and g is the function h=fog defined by h(x)=f(g(x)) for all x in the domain of g such that g(x) is in the domain of f.

g. Given two functions f and g, the composite function fog, also called the composition of f and g, is defined by (fog)(x)=f(g(x)).

h. Given any two functions f and g, we start with a number x in the domain of g and find its image g(x). If this number g(x) is in the domain of f, then we can calculate the value of f(g(x)). The result is a new function h(x)= f(g(x)) obtained by substituting g into f, and called composition of f and g.

i. If some of the outputs of a function g can be used as inputs of a function f, we can then link g and f to form a new function whose inputs x are inputs of g and whose outputs are the numbers f(g(x)). We say that the function f(g(x)) is the composite of g and f.

APPENDIX F

VIGNETTES

Name, Surname:

Vignette #1

You have been discussing the concept of composition of functions in the 9th grade class. You pose the following problem in the class.

Let $h(x) = f\left(g(x)\right)$ and determine $f(x)$ and $g(x)$ if $h(x) = 2(x-5)^2$.

One student suggests that "$g(x) = (x-5)^2$ and $f(x) = 2$".

Another student interrupts "No $f(x)$ must be equal to $2x$ if $g(x) = (x-5)^2$".

A third student remarks "Well I think $g(x) = (x-5)$ and $f(x) = 2x^2$".

The class seems confused.

What is the problem in each solution (if there exists)?

Explain how would you respond to these comments and clear up confusion during a class.

Name, Surname:

Vignette #2

You have been discussing the concept of composite functions in class. You pose the following problem in class.

Determine the composite function $(f \circ g)(x)$ if $f(x) = x + 3$ and $g(x) = x^2 + 6$.

One student answers the problem as "$(f \circ g)(x) = (x+3)^2 + 6$".

Another student answered the problem as "$(f \circ g)(x) = (x+3)(x^2+6)$".

A third student answered it as "$(f \circ g)(x) = x^2 + 9$".

For each of the incorrect solutions

What is the source of the mistake? (Show and explain how they may have found this solution.)

Explain how would you respond to these comments and clear up confusion during a class.

Name, Surname:

Vignette #3

A student asked the following question.

Let f(x)=4, g(x)=2, and h(x)=x+3. Evaluate the followings

a. (fog)(7)
b. (goh)(x)
c. (hof)(x)
d. (hof)(3)

Student's answer is the following:

a. f(x)=4 and g(x)=2 then (fog)=(4.2)=8 (fog)(7)=56
b. (goh)(x)=2x+3
c. (hof)(x)=7
d. (hof)(5)=32

What is the source of the mistake? (Show and explain how they may have found this solution.)

Explain how would you respond to these comments and clear up confusion during a class.

Name, Surname:

Vignette #4

A class is asked the following question.

If $h(x) = (fog)(x)$ where $h(x) = x^2 + 1$ and $g(x) = x$, then find $f(x)$. Show your work and explain your answer.

One of the students voluntarily comes to the board and the solved the question as follows:

$$x^2 + 1 = f(g(x))$$
$$x^2 + 1 = (fx)(x)$$
$$f(x) = \frac{x^2 + 1}{x}$$

Some of the students in the class agree with this solution.

What is the source of the mistake? (Show and explain how they may have found this solution.)

Explain how would you respond to these comments and clear up confusion during a class.

Name, Surname:

Vignette #5

A teacher gave the definition of the composite function and explained it on the board to his/her students. However, some of his/her students stated that they did not understand it completely. Then teacher gave the following example to the students.

In order to clean and dry our clothes in a laundry we use two machines, washing machine and dryer, respectively.
Dry&Wash (clothes)
Dry[Wash(clothes)]=Dry[cleaned and wet clothes]=dried and cleaned clothes
Combination of these machines works can be considered as a composition of functions

What do you think of this example?
Can this example cause students to misunderstand any points in the definition? If exists, please explain these points.
If you were to explain the composite function by using a real life example, what will be your example? Explain how you will use it in class.

Name, Surname:

Vignette #6

You have been discussing the concept of inverse functions in class. You pose the following problem in class.

Determine the inverse ($f^{-1}(x)$) of the function $f(x) = x - 4$.

Five different solutions come out from the class.

First one is " $f^{-1}(x) = \dfrac{1}{x-4}$ ".

The second one is " $f^{-1}(x) = \dfrac{1}{x} - 4$ ".

The second is " $f^{-1}(x) = -x - 4$ ".

The third one is" $f^{-1}(x) = -x + 4$ ".

The last solution is " $f^{-1}(x) = x + 4$ ".

The different solutions reveal that the class is confused.

What is the problem in each solution (if there exists).

Explain how would you respond to these comments and clear up confusion during a class.

Name, Surname:

Vignette #7

A student said the inverse of the function $f(x) = x^2$ is $f^{-1}(x) = \sqrt{x}$.

Is the student right? If you think that the student is correct explain why?

If you think that the student is incorrect, explain where the error lies and how would you respond to these comments and clear up confusion during a class.

Name, Surname:

Vignette #8

A student tells you that the binary operations of multiplication and division are inverse functions because they undo each other.

Is the student right? If you think that the student is correct explain why?

If you think that the student is incorrect, explain where the error lies and how would you respond to these comments and clear up confusion during a class.

Name, Surname:

Vignette #9

A teacher gave the definition of the inverse function and explained it on the board to his/her students. However, some of his/her students stated that they did not understand it completely. Then teacher gave the following example to the students.

If you think of school bus as a function which takes you from home to school at the morning, then the school bus that takes you back from school to home is the inverse of the first function.

- What do you think of this example?
- Can this example cause students to misunderstand any points in the definition? If exists, please explain these points.
- If you were to explain the inverse function by using a real life example, what will be your example? Explain how you will use it in class.

Name, Surname:

Vignette #10

You have been discussing the concept of inverse functions in class. You pose the following problem in class.

If $f(2x+1)=2x-1$ find $f(3x)$ in terms of $f(x)$ and explain the steps of your solution.

Then the students solved the question correctly as follows:

$$\left.\begin{array}{l} y=2x+1 \\ x=2y+1 \\ x-1=2y \\ y=\dfrac{x-1}{2} \end{array}\right\} \underbrace{f(x)=2.\frac{x-1}{2}-1=x-2}_{*} \; then \; f(3x)=3x-2$$

$$f(x)+2=x \Rightarrow f(3x)=3(f(x)+2)+2 \Rightarrow f(3x)=3f(x)+4$$

After the solution made, teacher wants from student to explain what she did in the step indicated by *. She said that “I have to get f(x) so that I could calculate f(3x). For getting f(x) I made the necessary calculations as you did in our previous examples”.

Furthermore, teacher wants from student to explain what she did in the $f(x)+2=x$ step. She said that “we have to single out x from the equation as you did in our previous examples”. However, she couldn’t explain what she did.

What should teacher do to make his/her students understand the case.

Name, Surname:

Vignette #11

A student of yours calculates the inverse function of the function $f(x) = 3x - 4$ and the answer obtained is $f^{-1}(x) = -2x + 4$. The students checks his work, and he combines $f(x)$ with $f^{-1}(x)$ he gets x. After the confirmation, he thinks that these two functions are inverses of each other.

What is the source of the mistake? (Show and explain how they may have found this solution.)

Explain how would you respond to these comments and clear up confusion during a class.

Name, Surname:

Vignette #12

For explaining inverse functions you gave the formal definition and then gave the following example "When someone calls you on the phone, he, or she looks up your number in a phone book (a function from names to phone numbers). When Caller ID shows who is calling, it has performed the inverse function, finding the name corresponding to the number."

Then you want from your students to write down such a function and define inverse of it. One of your students gives the following example "My function is something we see everyday on supermarket's cash registers (yazarkasa). For each item we buy there is a corresponding price on the receipt (fiş), so the inverse of this function is for each price there is a corresponding item."

Write down your ideas about the teacher's example. If you were teaching inverse functions, would you use it?

Analyze the student's example.

Is the student's example correct? If you think that it is correct explain why?

If you think that the student's example is incorrect, explain where the error lies and whether this error can be corrected.
Explain how would you respond to this example and clear up confusion during a class.

Name, Surname:

Vignette #13

For explaining composite functions you gave the formal definition and then give the following example “Take grass (g) as the first input; then the cow (c) being a function “eats” the grass. Next, here comes a third animal, say the tiger “eats” the cow. The best way to denote this is t(c(g)).The brackets denotes the walls of the stomachs.”

Then you want from your students to exemplify the composite functions by using such an example. One of your students gives the following example “I came from school by bus and I eat the cookies my mother made. Bus is my first function and cookies is must second function.”

Write down your ideas about the teacher’s example. If you were teaching composite functions, would you use it?

Analyze the student’s example.

Is the student’s example correct? If you think that it is correct explain why?

If you think that the student’s example is incorrect, explain where the error lies and whether this error can be corrected.
Explain how would you respond to this example and clear up confusion during a class.

APPENDIX G

INTERVIEW PROTOCOL ABOUT NON-ROUTINE PROBLEMS

Araştırma Sorusu: Matematik öğretmenliği öğrencilerinin fonksiyon konusundaki sıradan olmayan sorulardaki performansları nedir?

Görüşme Formu

Tarih /11/2006 **Saat** Başlangıç ______Bitiş ______**Görüşmeci**_____________.

Merhaba, benim adım Burcu Karahasan ve Özel Bilkent Lisesi'nde matematik öğretmeni olarak görev yapmaktayım. Doktora çalışmam kapsamında yaptığım bu görüşmenin amacı matematik öğretmenliği öğrencilerinin fonksiyonların tersi ve bileşkesi konusunda sıradan olmayan sorular yardımı ile performanslarını ortaya çıkartmaktır. Sizinle bu konuda görüşme yapmak istiyorum. Çalışma sonuçlarının matematik öğretmen yetiştirme programlarının niteliğinin arttırılmasın katkıda bulunacağını ümit ediyorum. Bu nedenle, sizlerin görüşleri benim için çok önemlidir.

- Bana görüşme sürecinde söyleyeceklerinizin tümü gizlidir. Bu bilgileri araştırmacıların dışında herhangi bir kimsenin görmesi mümkün değildir. Ayrıca, araştırma sonuçlarını yazarken görüştüğümüz bireylerin isimlerini kesinlikle rapora yansıtmayacağım.
- Başlamadan önce, bu söylediklerimle ilgili belirtmek istediğiniz bir düşünce veya sormak istediğiniz bir soru var mı?
- Görüşmeyi izin verirseniz kaydetmek istiyorum. Bunun sizce bir sakıncası var mı?
- Bu görüşmenin yaklaşık bir saat süreceğini tahmin ediyorum. İzin verirseniz sorulara başlamak istiyorum.

As you work through the task, please attempt to document as many of your thoughts and ideas as possible (whether you used ideas or not). When you finish, I will ask you to reconstruct what you have done.

1. $f(x) = x^2 - 2$ and $g(x) = -\sqrt{x+1}$ answer each of the following.

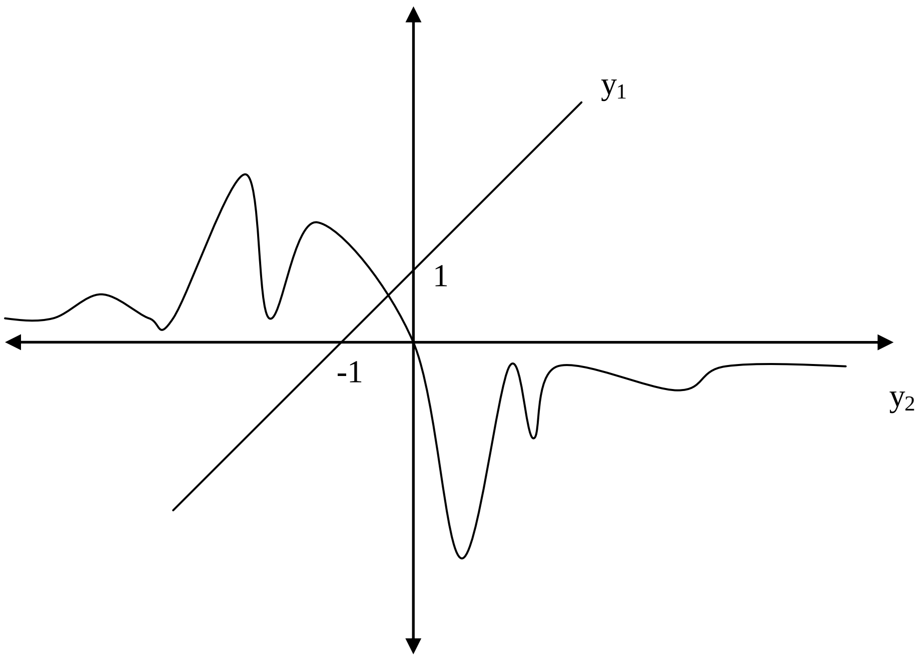
(a) Determine (fog)(x) in simplified form and sketch a graph of this new function.

(b) Determine (gof)(x) in simplified form and sketch a graph of this new function.

2. $f(x) = \sqrt{4 - x^2}$ and $g(x) = \sqrt{x^2 - 9}$

Determine (gof)(x) in simplified form.

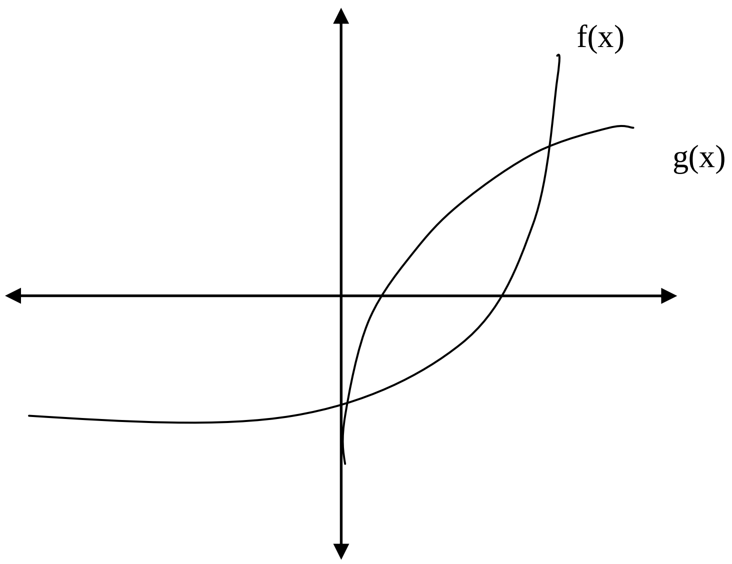
3.



(a) Use the given graphs to sketch $y_2 o y_1$.

(b) Use the given graphs to sketch $y_1 o y_2$.

4. Use the given graphs to sketch fog.



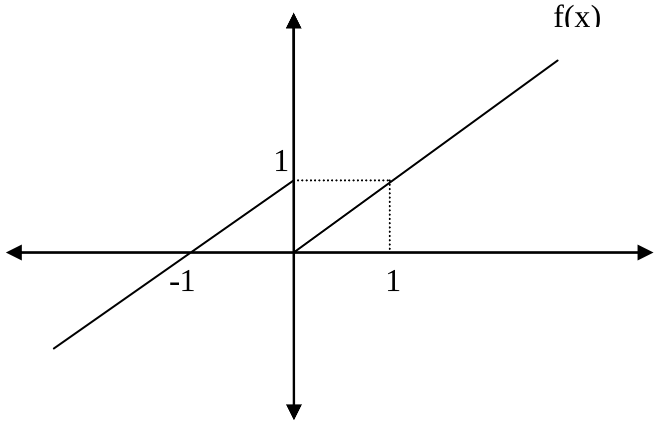
5. Find the inverse of the following functions by following the algebraic algorithm.

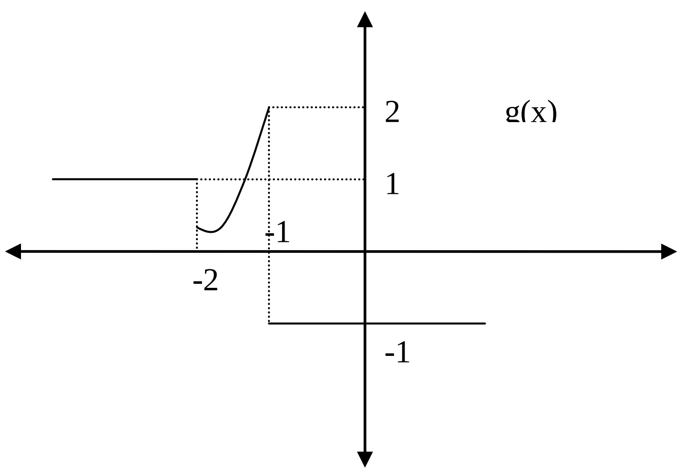
e. $f(x)=4,\ x \in R$

f. $f(x) = \sqrt{4 - x^2}$

g. $g(x) = \sqrt{x^2 - 9}$

6. Use the given graphs to sketch the inverse of given functions.





Non-routine Problems Sheet

Name, Surname:

1. $f(x) = x^2 - 2$ and $g(x) = -\sqrt{x+1}$ answer each of the following.

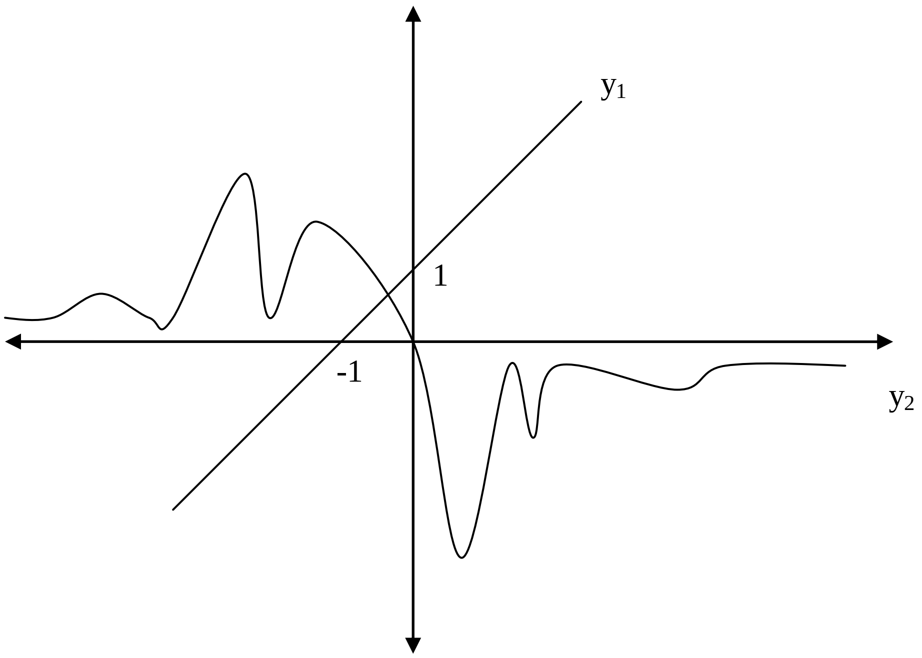
(a) Determine (fog)(x) in simplified form and sketch a graph of this new function.

(b) Determine (gof)(x) in simplified form and sketch a graph of this new function.

2. $f(x)=\sqrt{4-x^2}$ and $g(x)=\sqrt{x^2-9}$

Determine (gof)(x) in simplified form.

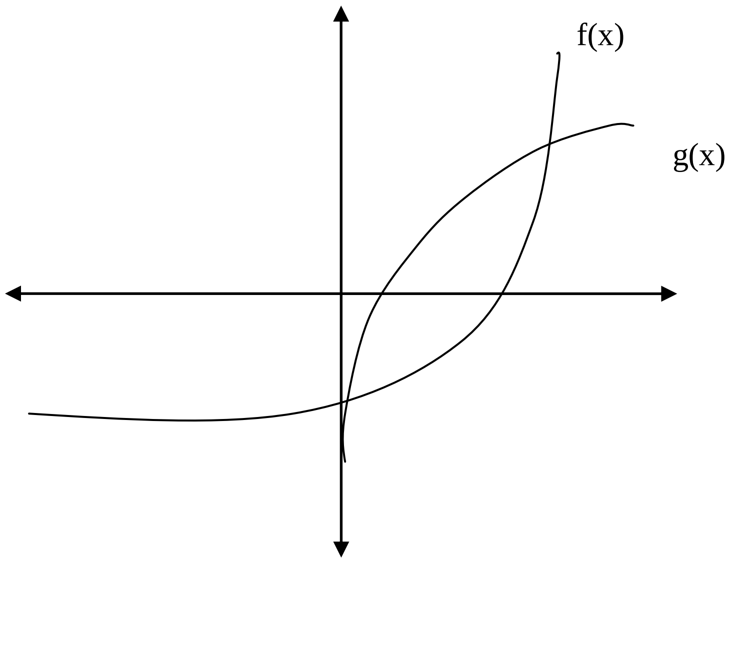
3.



(a) Use the given graphs to sketch $y_2 o y_1$.

(b) Use the given graphs to sketch $y_1 o y_2$.

4. Use the given graphs to sketch fog.



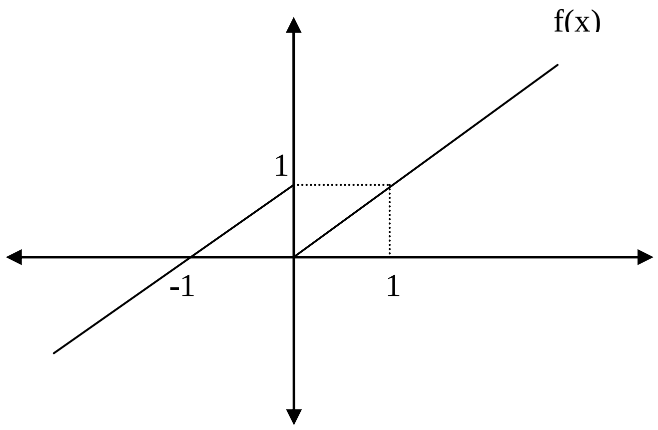
5. Find the inverse of the following functions by following the algebraic algorithm.

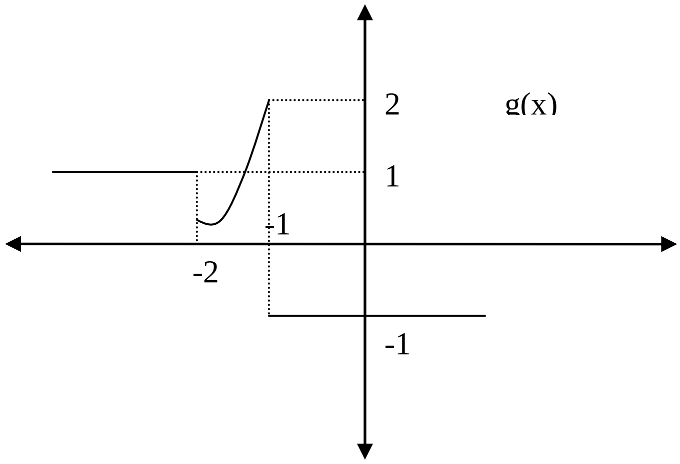
a. f(x)=4, x∈R

b. $f(x)=\sqrt{4-x^2}$

c. $g(x)=\sqrt{x^2-9}$

6. Use the given graphs to sketch the inverse of given functions.




2
g(x)
1
-1
-2
-1

APPENDIX H

LESSON PLANNING ACTIVITY

Lesson Planning Activity Instructions

Your task is to prepare lesson plans on the concept of inverse and composite functions for the 9th grade class you follow. Plan each lesson with an 80-minute class period. You may look to any source (e.g. the Internet, textbooks etc) for ideas for your lesson. You will have access to various high school textbooks. The books will be available from me or the school library. You may incorporate into your lesson plans any materials our equipment that you deem necessary to be successful. Please cite what sources you use in planning your lesson (i.e. textbooks, addresses of Internet sites, etc.)

Try to be as detailed as possible as writing your lesson plans by using the lesson plan format provided. Include in your lesson plans examples you may give to students, questions you would ask students, homework you would assign to students, any handouts or overhead transparencies you may use, as well as the prerequisite skills you are assuming that students have. Plan your lesson as if you were going to place it in a teacher portfolio as an example of your best work.

You have a week to complete your lesson plan. Please spend about two hours on preparing each 80-minute lesson plan. You have maximum five 80-minute lessons to complete topics. Your lesson plans are due back to me on 30.11.2006.

A.3 Lesson Plan Format

Lesson Plan Format
Title of the Lesson:
Name, Surname :
Grade Level :
Prerequisite skills:
Materials/Equipment:
Objectives:
Methods/Techniques:
Procedure a) Introduction b) Development c) Closure
Evaluation/Assessment/Homework

APPENDIX I

JOURNAL ABOUT VALUE OF COMPOSITE AND INVERSE FUNCTIONS

Journal about Value of Teaching Functions

Name, Surname:

Each of the following teachers has different perspective on what they considered important about the teaching of functions. Allocate a **total of 100 points** to the various teachers positions according to the extent to which you agree with each position. You may distribute the points in any size of increments. For example, you may assign all 100 points to a single position and 0 to the remaining positions or you may assign any number of points between 0 and 100 to a position. The sum total of the points must be 100 points, however.

Assign points for each of the teachers. **Write a brief explanation** on why you assigned the points as you did for the teachers above.

Teacher's Name	**Teacher's Perspective on Functions**
A	I think functions are important because they provide a context for developing basic skills such as solving equations and graphing. Let's face it; if students aren't proficient in these basics, they just aren't going to go very far in mathematics. Functions provide an ideal context for developing these skills.
B	I think functions are important because they give students an opportunity to deal with mathematical language and representations including the symbolism of mathematics. It is important for students to realize that functions are represented in various ways, including formulas, sets, mappings, and graphs. They should have experience translating among these different representations.
C	I think functions are important because they give students an opportunity to see how mathematics can describe real world phenomena. My students study functions as the relationship

	between Fahrenheit and Celsius temperature scales and other functions that represent real world phenomena.
D	I think functions are important because they are so basic to the rest of the topics in the secondary school mathematics. If students understand what a function is in general, then they can better appreciate specific functions such as linear quadratic, exponential, logarithmic, or trigonometric functions.
E	I think functions are important because they give students an opportunity to study functional relationships; that is, to see how one variable changes when another variable changes. The study of relationships is central to the study of mathematics. Functions provide an excellent opportunity for students to study various relationships.
F	I think functions are important because the topic is a good vehicle for showing some of the beauty and excitement in mathematics. For example, it is a fascinating exercise to iterate functions with a calculator and see what happens. Iterating the square root function, the sine, the cosine, and the exponential reveals four very different and interesting behaviors.

Points	**Explanation**
A	
B	
C	
D	
E	
F	

Journal about Value of Teaching Composite Functions

Name, Surname:

Each of the following teachers has different perspective on what they considered important about the teaching of composite functions. Allocate a **total of 100 points** to the various teachers positions according to the extent to which you agree with each position. You may distribute the points in any size of increments. For example, you may assign all 100 points to a single position and 0 to the remaining positions or you may assign any number of points between 0 and 100 to a position. The sum total of the points must be 100 points, however.

Assign points for each of the teachers. **Write a brief explanation** on why you assigned the points as you did for the teachers above.

Teacher's Name	**Teacher's Perspective on Composite Functions**
G	I think composite functions are important because in real life almost nothing happens in just one step. For example, take grass (g) as the first input; then the cow (c) being a function "eats" the grass. Next, here comes a third animal, say the tiger "eats" the cow. The best way to denote this is t(c(g)).The brackets denotes the walls of the stomachs. The composite functions give students an opportunity to see how mathematics related with real world phenomena.
H	I think composite functions are important because they are very important for the topics in Calculus. If students understand & use composite functions efficiently, then for example they can apply the chain rule effectively.
I	I think composite functions are important because they can make students aware of the fact that procedures we apply in mathematics can create beauties like Escher tessellations. In these tessellations rotation, translation, and iteration functions used successively.
J	I think composite functions are important because their existence help us to compute many things in physics. There are multi step processes which take a long time. By using composite functions, these multi step processes can be reduced to a single function and one can save time and energy by this way.
Points	**Explanation**
G	
H	
I	
J	

Journal about Value of Teaching Inverse Functions

Name, Surname:

Each of the following teachers has different perspective on what they considered important about the teaching of inverse functions. Allocate a **total of 100 points** to the various teachers positions according to the extent to which you agree with each position. You may distribute the points in any size of increments. For example, you may assign all 100 points to a single position and 0 to the remaining positions or you may assign any number of points between 0 and 100 to a position. The sum total of the points must be 100 points, however.

Assign points for each of the teachers. **Write a brief explanation** on why you assigned the points as you did for the teachers above.

Teacher's Name	**Teacher's Perspective on Inverse Functions**
K	I think inverse functions are important because they are very important to understand some other mathematical concepts like exponential & logarithm functions which are inverses of each other.
L	I think inverse functions are important because in Calculus there are many topics connected to inverse functions. For instance, if inverse functions are understood and used by a student effectively, then the rule and corresponding properties related to the derivative of inverse functions can be grasped by this student easily.
M	I think inverse functions are important because they give students an opportunity to see the fact that inverse relations students see in their daily life is also exists in mathematics. For example, if you think of school bus as a function which takes you from home to school at the morning, then the school bus that takes you back from school to home is the inverse of the first function.
N	I think inverse functions are important because even in cryptography, which is the most beautiful branch of mathematics for me, one can see them. While coding a message we use functions, and while decoding a message we use inverse functions.
O	I think inverse functions are important because while teaching inverse functions when a teacher showed to students graphically all functions and their inverse functions are symmetric with respect to y=x, students can easily see the beauty behind the mathematical processes.
Points	**Explanation**
K	

L	
M	
N	

APPENDIX J

EVALUATION INTERVIEW PROTOCOL

Araştırma Sorusu: Matematik öğretmenliği öğrencilerinin okul deneyimi dersi sırasında katıldıkları aktiviteler hakkındaki görüşleri nelerdir?

Görüşme Formu

Tarih /12/2006 **Saat** Başlangıç ______Bitiş ______**Görüşmeci**_____________.

Merhaba, benim adım Burcu Karahasan ve Özel Bilkent Lisesi'nde matematik öğretmeni olarak görev yapmaktayım. Doktora çalışmam kapsamında yaptığım bu görüşmenin amacı sizlerin okul denetimi dersi sıradında katıldığınız aktiviteler hakkındaki görüşleriniz almaktır.

- Bana görüşme sürecinde söyleyeceklerinizin tümü gizlidir. Bu bilgileri araştırmacıların dışında herhangi bir kimsenin görmesi mümkün değildir. Ayrıca, araştırma sonuçlarını yazarken görüştüğümüz bireylerin isimlerini kesinlikle rapora yansıtmayacağım.
- Başlamadan önce, bu söylediklerimle ilgili belirtmek istediğiniz bir düşünce veya sormak istediğiniz bir soru var mı?
- Görüşmeyi izin verirseniz kaydetmek istiyorum. Bunun sizce bir sakıncası var mı?
- Bu görüşmenin yaklaşık bir saat süreceğini tahmin ediyorum. İzin verirseniz sorulara başlamak istiyorum.

1. Okul deneyimi dersi süresince yaptığımız aktivitelerle ilgili genel edeğerlendirmenizi alabilirmiyim?
2. Katıldığınız herbir aktivite ile ilgili nasıl bir değerlendirme yaparsınız? (Aktivitelerin listesi araştırmacı tarafından katılımcıya verilir, gerekirse açıklama yapılır.)
3. Buradaki aktivitelrin içinde size en çok etkileyenler hangileriydi? Neden?

APPENDIX K

SURVEY OF FUNCTION KNOWLEDGE HOLISTIC SCORING SCHEME

A.4 Holistic Scoring Scheme for Declerative Knowledge Questions

DECLARATIVE KNOWLEDGE QUESTIONS	
Basic Skills: correct use of terminology, accurate description of concepts, recognition and observation of properties. Logical Skills: classification, recognition of essential properties of function concept	
Score	**Description**
0	_ No answer attempted. _ Copies parts of the problem without attempting a solution. _ Uses irrelevant information. _ Includes declarative knowledge which completely misrepresents the problem situation.
1	_ Shows very limited understanding and recalling of function concept definitions, symbols, notations, facts and properties. _ Misuse or fail to use function concept definitions, symbols, notations, facts and properties. _ Tries to solve the question but includes improper definitions, and unnecessary symbols, or notations. _ Limitedly defines, identifies properties, describes, and classifies the concepts. _ Rewrite the statement in the question that is not clear or writes something that does not go with the answer.
2	_ Shows some of the understanding and recalling of function concept definitions, symbols, notations, facts and properties. _ Makes significant progress towards completion of a question but the work may be ambiguous or unclear. _ Includes flawed or unclear work representing the problem situation.
3	_ Shows nearly complete understanding and recalling of function concept definitions, symbols, notations, facts and properties. _ Uses nearly correct mathematical terminology when defining a concept. _ Includes nearly complete and appropriate work when representing the problem situation.
4	_ Shows understanding and recalling of function concept definitions, symbols, notations, facts and properties. _ Uses appropriate mathematical terminology when defining a concept. _ Includes complete and appropriate work when representing the problem situation.

A.5 Holistic Scoring Scheme for Conditional Knowledge Questions

CONDITIONAL KNOWLEDGE QUESTIONS	
Basic Skills: interpreting statements, correct use of terminology, appropriate use of symbols and notations, and accurate communication in describing relationships. Logical Skills: formulating and testing hypothesis, making inferences, using counter explanations, develop mathematical arguments about functional relationships	
Score	**Description**
0	_ No answer attempted. _ Copies parts of the problem without attempting a solution. _ Uses irrelevant information. _ Includes declarative /conditional knowledge which completely misrepresent the problem situation.
1	_ Shows very limited understanding of the principles, theorems, relations, and statements. _ Fails to identify the important parts when expressing the “if-then” statements. _ Gives incomplete evidence of the explanation process. _ Places too much emphasis on unimportant relations when expressing the “if-then” statements.
2	_ Shows some understanding of the principles, theorems, relations, and statements. _ Identifies some important parts when expressing the “if-then” statements. _ The relations expressed in the “if-then” statement is difficult to interpret and the arguments given are incomplete and logically unsound.
3	_ Shows nearly complete understanding of the principles, theorems, relations, and statements. _ Identifies the most important parts when expressing the “if-then” statements. _ Shows general understanding of the relations in the “if-then” statements. _ Gives a fairly complete response with reasonably clear explanations or descriptions. _ Presents supporting logically sound arguments which may contain some minor gaps.
4	_ Shows understanding of the principles, theorems, relations, and statements. _ Identifies all the important parts when expressing the “if-then” statements. _ Shows understanding of the relations in the “if-then” statements. _ Gives a complete response with a clear, unambiguous explanation or description. _ Presents strong, supporting, logically sound and complete arguments which may include counter-explanations or different aspects.

A.6 Holistic Scoring Scheme for Procedural Knowledge Questions

PROCEDURAL KNOWLEDGE QUESTIONS	
Basic Skills: correct use of terminology, appropriate use of symbols and notations, and accurate application of the algorithm. Logical Skills: classification, recognition of essential properties of a functional concept, formulating and testing hypothesis, making inferences, using counter explanations, appropriate use of the procedures.	
Score	**Description**
0	_ No answer attempted. _ Copies parts of the problem without attempting a solution. _ Uses irrelevant information. _ Includes declarative/conditional/procedural knowledge which completely misrepresent the problem situation.
1	_ Makes major computational errors when employing the algorithms and rules. _ Reflects an inappropriate strategy for solving the problem. _ Gives incomplete evidence of a solution process. _ The solution process is missing, difficult to identify or completely unsystematic.
2	_ Makes serious computational errors when employing the algorithms and rules. _ Gives some evidence of the solution process. _ The solution process is incomplete or somewhat unsystematic. _ Makes significant progress towards completion of the problem but the algorithm is unclear.
3	_ Executes algorithms and rules completely. _ Computations are generally correct but may contain minor errors. _ Gives clear evidence of a solution process. _ The solution process is nearly complete and systematic.
4	_ Executes algorithm and rules completely and correctly. _ Reflects an appropriate and systematic strategy for solving the problem. _ Gives evidence for the solution process. _ The solution process is complete and systematic

APPENDIX L

HOLISTIC SCORING CRITERA FOR CONCEPT MAP ACTIVITY

A.7 Holistic Scoring Criteria for Concept Maps

Holistic scoring criteria for concept maps	
Points	Description
	Organization: description of clusters and connectors used
6	Excellent: shows complete, in-depth understanding of links among various terms; creates clear and insightful clusters of related terms; utilizes exemplary linking words; may add terms
5	Fluent: shows a thorough understanding of links among various terms; creates illustrative clusters of related terms; utilizes effective linking words; uses all terms
4	Good: shows a general understanding of links among various terms; creates adequate clusters of related terms; utilizes applicable linking words; may omit a few terms
3	Fair: shows a partial understanding of links among various terms; creates understandable clusters of related terms; utilizes adequate linking words; may omit some terms
2	Weak: shows a minimal understanding of links among various terms; creates deficient clusters of related terms; utilizes unsuitable linking words; omits several key terms
1	Inadequate: shows little understanding on links among various terms; creates ineffective clusters of related terms; utilizes inapplicable/no linking words; omits numerous key terms
0	Unacceptable: no attempt made or unintelligible
	Accuracy: evidence of inaccuracies/misconceptions
4	Excellent: no errors
3	Fluent: few minor errors, no conceptual errors
2	Good: some errors

1	Weak: numerous errors
0	Inadequate: numerous major conceptual errors.

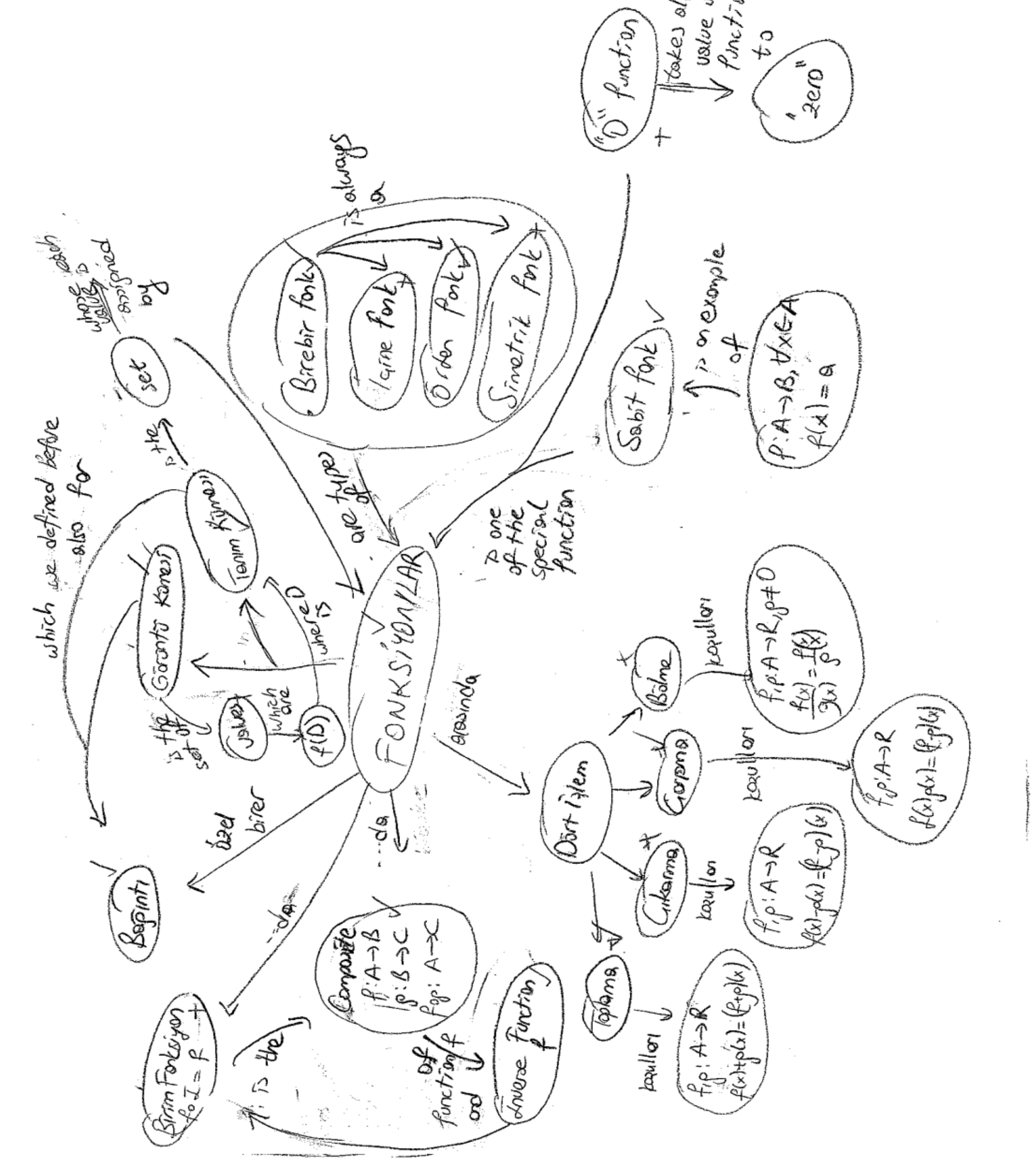
A.8 Holistic Scoring Criteria for Interpretive Essay

Holistic scoring criteria for interpretive essay	
Points	Description
	Communication: clarification of understandings and expression of mathematical ideas
6	Excellent: demonstrates interpretations and understandings in a clear, systematic, and organized manner; represents mathematical ideas accurately in an exemplary manner
5	Fluent: demonstrates interpretations and understandings in a clear and organized manner; represents mathematical ideas in an effective manner; may contain minor misconceptions
4	Good: demonstrates interpretations and understandings in an organized manner; represents mathematical ideas in a proficient manner; may contain some misconceptions
3	Fair: demonstrates interpretations and understandings in an understandable manner; represents mathematical ideas in an acceptable manner, may contain several misconceptions
2	Weak: demonstrates interpretations and understandings in a manner that is disorganized or difficult to understand; represents mathematical ideas in an inappropriate manner; may contain numerous misconceptions
1	Inadequate: demonstrates interpretations and understandings in a manner that is impossible to understand; represents mathematical ideas in an inaccurate manner; numerous misconceptions
0	Unacceptable: no attempt made or unintelligible
	Organization
3	Excellent: method of presentation clear and appropriate transitions
2	Adequate: relationships and transitions sometimes unclear
1	Poor: lacks coherence; disjointed statements
0	Unacceptable: no attempt made or unintelligible
	Mechanics
1	Acceptable: few violations in grammar, punctuation, capitalization
0	Unacceptable: errors interfere with understanding; unintelligible

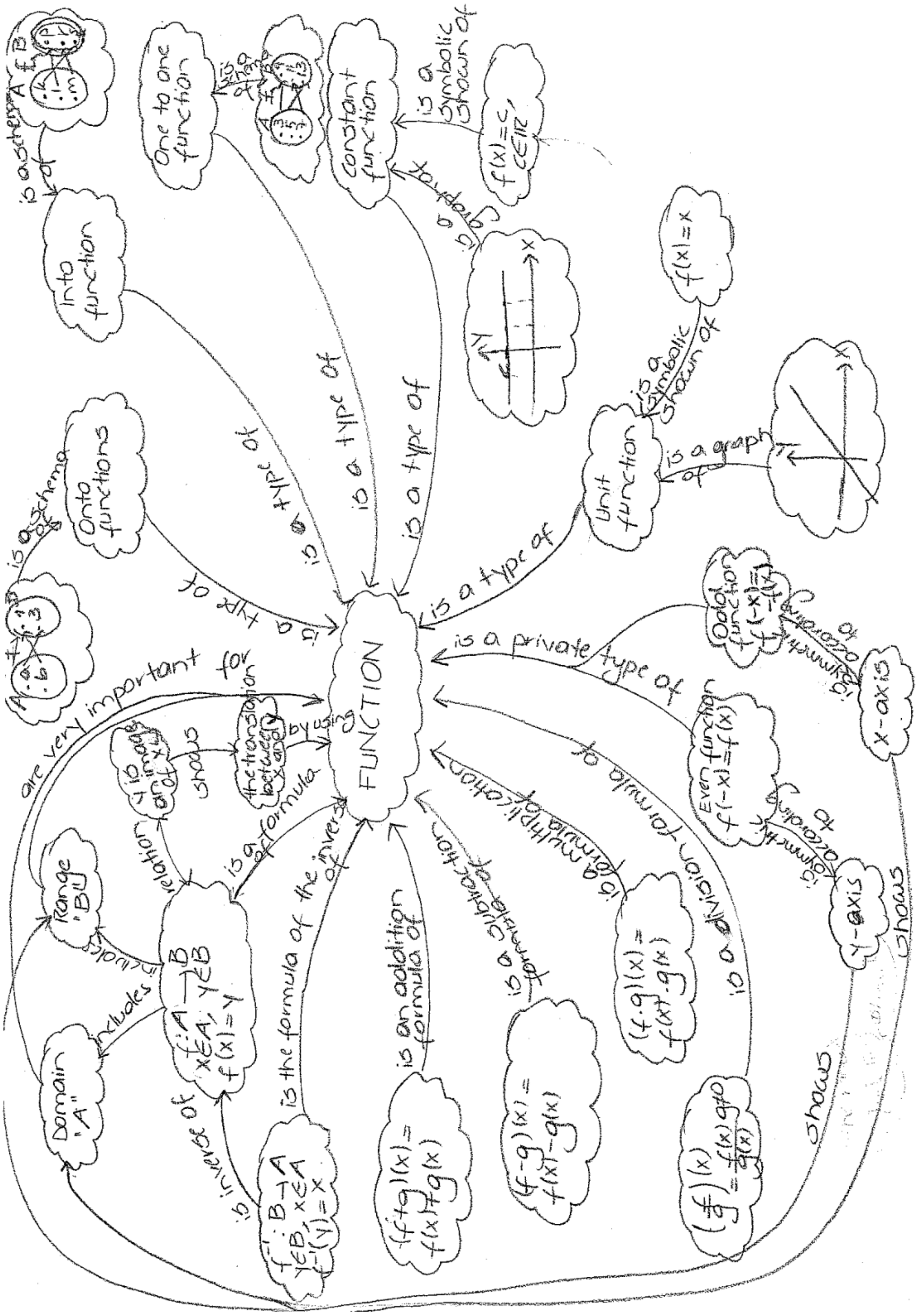
APPENDIX M

EXAMPLES OF SCORED CONCEPT MAPS AND ESSAYS

A.9 Deniz's Concept Map # 1



A.10 Yeliz's Concept Map # 2



A.11 Deniz's Concept Map Essay

Name, Surname:

Concept map Essay

Write an interpretive essay comparing your concept maps by answering the following questions

1. Describe your thought process while constructing your first concept map.
2. Describe your thought process while constructing the second map when you were given the terms to consider incorporating?
3. Are there any differences (inconsistencies, anomalies etc.) in your maps? What are they? Where do you think these differences come from? Include additional information that might be relevant or that presents personal insights.

While constructing my first concept map, I thought about all the things about functions and listed them in a piece of paper. I didn't directly started writing them in a concept map. After my list finished, I tried to group the things that I wrote. For example, I wrote the properties of functions seperately, but I combined them under one title. After these steps finished, I started to write them as a web-like concept map. It was hard to remember all the things about functions from my mind but generally when I write something this helped me to remember some other thing.

While constructing my second concept map, I was given a list about functions. At first I have read them and tried to find combinations between them. After that I started to write them as a web-like concept map. I felt my self limited and I forced my mind to find more combinations. In my opinion this not a concept map that I wrote on my own. It was like trying to make a concept map with someone elses thinkings. May be it contains all information about functions but I didn't like it.

These two concept maps are about functions, but they have some differences of course. First of all I want to say that I was more comfortable while I was preparing the first conceptmap and I am more fulfilled with it. The information that they consist are not so much different. In the second one, I wrote some more additional information like relation, ordered pair, single-valued and so on. In both maps there are things that is forgotten. For example, in both of them I wrote kinds of function in first one I wrote all the functions that come to my mind, in the second one I wrote only the functions that are on the list. Both of them has missing functions but in the first one I wrote more. In both of them there may be some mistakes. For example, in the first one I wrote the properties of relation, as properties of functions. I confused them. In my opinion these differences come from the beginning. In the first one I was free to think, so I wrote all the things that come to my mind. However in the second one my thinking was limited with the list given. Although there are more mistakes in the first one, I liked it more because I feel that it belongs to me.

APPENDIX N

VIGNETTE EXAMPLES

A. 12 Deniz Vignette # 1

Name, Surname: **Vignette #1**

You have been discussing the concept of composition of functions in the 9th grade class. You pose the following problem in the class.

Let $h(x) = f(g(x))$ and determine $f(x)$ and $g(x)$ if $h(x) = 2(x-5)^2$

One student suggests that "$g(x) = (x-5)^2$ and $f(x) = 2$"

Another student interrupts "No $f(x)$ must be equal to $2x$ if $g(x) = (x-5)^2$".

A third student remarks "Well I think $g(x) = (x-5)$ and $f(x) = 2x^2$"

The class seems confused.

What is the problem in each solution (if there exists)?

How would you respond to these comments and clear up confusion?

First of all, I would write the answers of students in a part of the board. Then I would say, forget about these answers and lets remember what was the composition. If $h(x) = f(g(x))$ then we have to write the result of $g(x)$ into the $f(x)$ which means that we will write the result of $g(x)$ in the place of x where we see in $f(x)$. Then, lets examine our friends results. In the first one if $f(x) = 2$ and $g(x) = (x-5)^2$ we have to write $(x-5)^2$ in the place of x in $f(x)$. When we write it in the right side of $f(x)$ there is no x. It is a constant function so the result has to be 2 whatever we write instead of x in $f(x)$. Can it be? No because the result has to be $h(x) = 2(x-5)^2$. What is wrong here? Our friend understood it as the multiplication but it is not the case. Let's look to the second one. The result of $g(x)$ is $(x-5)^2$. When we put it instead of x in $f(x)$, we get $f(g(x)) = 2 \cdot (x-5)^2$. It is the result that we want to reach. Is it the only solution that we can find? Let's look to the third — answer that our friend had given. He defined g(x) as (x-5) and f(x) as $2x^2$. Then again we will put g(x) instead of x in f(x). Let's try to find the result, f(g(x))= 2(x

Again we reached to the same result. Is that all or can we find another value for f and g functions?

Let me give another example : f(x)= x and g(x)=2(x

then we will again find the same result which is

$f(g(x)) = f(2(x-5)^2) = 2(x-5)^2$

This shows us that, we can find many f and g functions that gives us the result that we want. This is not a question, that has an exact answer. This is an open ended question.

A.13 Deniz Vignette # 2

Name, Surname: **Vignette #2**

You have been discussing the concept of composite functions in class. You pose the following problem in class.

Determine the composite function $(f \circ g)(x)$ if $f(x) = x + 3$ and $g(x) = x^2 + 6$.

One student answers the problem as "$(f \circ g)(x) = (x+3)^2 + 6$".

Another student answered the problem as "$(f \circ g)(x) = (x+3)(x^2+6)$".

A third student answered it as "$(f \circ g)(x) = x^2 + 9$".

For each of the incorrect solutions

What is the source of the mistake? (Show and explain how they may have found this solution.)

Explain how would you respond to these comments and clear up confusion during a class.

- In the first one student confused which function goes into the other function. He first did f and put its result into g like $g(f(x)) = g(x+3) = (x+3)^2+6$
- In the second one student did mistake in the meaning of composition. He did multiplication when he saw composition but composition does not mean multiplication.
- In the third one the student did it correctly.
- To correct the first students mistake I would say that the function which is near to x will the first to have an operation.

  $f \circ g\,(x)$ I would draw the arrows like that and write it as $f(g(x))$.

- In the second one I would say; You are confusing multiplication and composition. Let's evaluate $f(0) \cdot g(0)$ and $f \circ g\,(0) = f(g(0)) = f(6) = 9$

  $3 \cdot 6 = 18$ $\quad 18 \neq 9$ $\quad \Rightarrow$

As you see multiplization and composition are different things. In the composition The function that is outer takes the result of the inner function. And we write this result in the place of x.

$f(g(x)) = g(x) + 3 = x^2 + 6 + 3 = x^2 + 9.$

↳ instead of x we write g(x).

A. 14 Deniz Vignette # 3

Name, Surname: **Vignette #3**

A student asked the following question.

Let f(x)=4, g(x)=2, and h(x)=x+3. Evaluate the followings

a. (fog)(7)
b. (goh)(x)
c. (hof)(x)
d. (hof)(3)

Student's answer is the following:

a. f(x)=4 and g(x)=2 then (fog)=(4.2)=8 (fog)(7)=56
b. (goh)(x)=2x+3
c. (hof)(x)=7
d. (hof)(5)=32 (h.f)(5)=(4x+12)(5)
= 20+12
= 32

What is the source of the mistake? (Show and explain how they may have found this solution.)

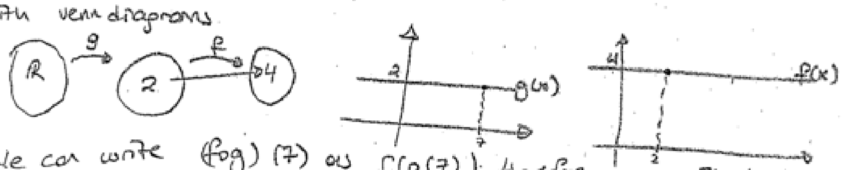
Explain how would you respond to these comments and clear up confusion during a class.

a) He thought combination as multiplication.

(f∘g) = f.g = 4·2 = 8

(f∘g) (7) = 8·7 = 56

However combination is not multiplication. We can show the solution with venn diagrams



We can write (fog) (7) as f(g(7)). therefore we will start from the inside of the parathesis. g takes 7 in it. and the result will be 2. because whatever number g takes in it, the result will be 2, g is a constant function. Then we will find f(2). Again f is constant function. The result of f(2) = 4. This means that fog(7) = 4. From the graphics we can also see the result

b) $g \circ h.(x) = 2x+3$ to find this result stu. multiplied g with the x of h. He has a misconception in multiplication and combination.
We can write $g \circ h(x)$ as $g(h(x))$ which means that x will be put in h and the result h(x) will be put in g. then lets find h(x) at first.

$g(h(x)) = g(x+3)$ ⇒ then we have to find $g(x+3)$ We know that g carries all the values that it takes in ℝ to 2 since it is a constant function. So $g(x+3) = 2$.

) $h \circ f(x) = 7$ This time he puts the value of f(x) in to h and finds the correct result. $h(f(x)) = h(4) = 4+3=7$

) $(h \circ f)(3)$

A. 15 Deniz Vignette # 13

Name, Surname: **Vignette #13**

For explaining composite functions you gave the formal definition and then give the following example "Take grass (g) as the first input; then the cow (c) being a function "eats" the grass. Next, here comes a third animal, say the tiger "eats" the cow. The best way to denote this is t(c(g)).The brackets denotes the walls of the stomachs."

Then you want from your students to exemplify the composite functions by using such an example. One of your students gives the following example "I came from school by bus and I eat the cookies my mother made. Bus is my first function and cookies is must second function."

Write down your ideas about the teacher's example. If you were teaching composite functions, would you use it?

Analyze the student's example.

Is the student's example correct? If you think that it is correct explain why?

If you think that the student's example is incorrect, explain where the error lies and whether this error can be corrected.

Explain how would you respond to this example and clear up confusion during a class.

Teacher's example is good. But the letters can confuse student's mind, because we generally denote function as g but here it is a variable.
The student's example is not correct.
According to his thought, he is the variable and bus and the cookies are the functions.
However "coming" to the house and "eating" the cookies are functions.
This example cannot be corrected because it is like listing the things that he did, the result of the first action is not goes in to the other one.

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