DECOMPOSITION OF ELASTIC CONSTANT TENSOR INTO ORTHOGONAL PARTS

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submitted by **ÇİĞDEM DİNÇKAL** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Engineering Sciences Department, Middle East Technical University** by,

Prof. Dr. Canan Özgen
Dean, Graduate School of **Natural and Applied Sciences** __________

Prof. Dr. Turgut Tokdemir
Head of Department, **Engineering Sciences** __________

Prof. Dr. Y. Cevdet Akgöz
Supervisor, **Engineering Sciences, METU** __________

Examining Committee Members:

Prof. Dr. Zülfü Aşık
Engineering Sciences Dept., METU __________

Prof. Dr. Y.Cevdet Akgöz
Engineering Sciences Dept., METU __________

Prof. Dr. Hüsnü Özkan
Physics Dept., METU __________

Prof. Dr. Yusuf Aydın
Başkent U., Retired, Part Time __________

Assoc. Prof. Dr. Hakan I. Tarman
Engineering Sciences Dept., METU __________

Date: 23.07.2010

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Name, Last name	**:Çiğdem Dinçkal**
Signature	**:**

ABSTRACT

DECOMPOSITION OF ELASTIC CONSTANT TENSOR INTO ORTHOGONAL PARTS

Dinçkal, Çiğdem

PhD., Department of Engineering Sciences

Supervisor: Prof. Dr. Y. Cevdet Akgöz

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All procedures in the literature for decomposing symmetric second rank (stress) tensor and symmetric fourth rank (elastic constant) tensor are elaborated and compared which have many engineering and scientific applications for anisotropic materials. The decomposition methods for symmetric second rank tensors are orthonormal tensor basis method, complex variable representation and spectral method. For symmetric fourth rank (elastic constant) tensor, there are four mainly decomposition methods namely as, orthonormal tensor basis, irreducible, harmonic decomposition and spectral. Those are applied to anisotropic materials possessing various symmetry classes which are isotropic, cubic, transversely isotropic, tetragonal, trigonal and orthorhombic. For isotropic materials, an expression for the elastic constant tensor different than the traditionally known form is given. Some misprints found in the literature are corrected.

For comparison purposes, numerical examples of each decomposition process are presented for the materials possessing different symmetry classes. Some applications of these decomposition methods are given. Besides, norm and norm ratio concepts are introduced to measure and compare the anisotropy degree for various materials with the same or different symmetries. For these materials, norm and norm ratios are calculated. It is suggested that the norm of a tensor may be used as a criterion for comparing the overall effect of the properties of anisotropic materials and the norm ratios may be used as a criterion to represent the anisotropy degree of the properties of materials.

Finally, comparison of all methods are done in order to determine similarities and differences between them. As a result of this comparison process, it is proposed that the spectral method is a non-linear decomposition method which yields non-linear orthogonal decomposed parts. For symmetric second rank and fourth rank tensors, this case is a significant innovation in decomposition procedures in the literature.

Keywords: Elastic Constant Tensor, Orthonormal Tensor Basis Method, Complex Variable Representation, Irreducible Decomposition Method, Harmonic Decomposition Method, Spectral Method, Norm and Norm Ratio.

ÖZ

ELASTİK SABİT TENSÖRÜN ORTOGONAL KISIMLARINA AYRIŞTIRILMASI

Dinçkal, Çiğdem

Doktora, Mühendislik Bilimleri Bölümü

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Anizotrop malzemelerde, birçok mühendislik ve bilimsel uygulamalarda önemli yeri olan simetrik ikinci mertebe (stres) tensörü ve simetrik dördüncü mertebe (elastik sabit) tensörü ayrıştırılması, literatürdeki bütün metotlar için ayrıntılı çalışılmış ve karşılaştırılmıştır. Simetrik ikinci mertebe tensörleri için ayrıştırma metotları, ortonormal tensör baz metodu, karmaşık değişken gösterimi ve spektral metodudur. Simetrik dördüncü mertebe (elastik sabit) tensörü için, ortonormal tensör baz, indirgenemez, harmonik ayrıştırma ve spektral adı altında, dört tane başlıca ayrıştırma metodu vardır. Bunlar çeşitli simetri sınıflarına; izotrop, kübik, transvers izotrop, tetragonal, trigonal ve orthorhombik, sahip olan malzemelere uygulanmıştır. İzotrop malzemelerde, elastik sabit tensörü için geleneksel bilinen formdan farklı olan bir ifade verilmiştir. Literatürde bulunan bazı yanlışlar düzeltilmiştir.

Karşılaştırma amacıyla, farklı simetri sınıflarına sahip olan malzemeler için, herbir ayrıştırma işlemine ait sayısal örnekler sunulmuştur. Bu ayrıştırma metotlarının, bazı uygulamaları verilmiştir. Bunun yanında, norm ve norm oranı kavramları, aynı veya farklı simetriye sahip değişik malzemeler için, anizotrop derecesini ölçmek ve kıyaslamak amacıyla tanıtılmıştır. Bu malzemeler için, norm ve norm oranları hesaplanmıştır. Anizotrop malzeme özelliklerinin toplam etkilerini karşılaştırmada tensörün normu, anizotrop olma derecelerini karşılaştırmada ise norm oranlarının kriter olarak kullanılabileceği önerilmiştir.

Sonuç olarak, aralarındaki benzerlikleri ve farklılıkları belirlemek için bütün metotlar karşılaştırılmıştır. Bu karşılaştırma işleminin sonucunda, doğrusal olmayan, ortogonal ayrıştırılmış parçalar veren, spektral metodunun, doğrusal olmayan bir ayrıştırma metodu olduğu önerilmiştir. Simetrik ikinci mertebe ve dördüncü mertebe tensörleri için, literatürde, ayrıştırma yöntemleri için de, bu durum önemli bir yeniliktir.

Anahtar Kelimeler: Elastik Sabit Tensörü, Ortonormal Tensör Baz Metodu, Karmaşık Değişken Gösterimi, İndirgenemez Ayrıştırma Metodu, Harmonik Ayrıştırma Metodu, Spektral Metot, Norm ve Norm Oranı.

To my mother, for her deepest love and encouragement...

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LIST OF SYMBOLS

Symbol	Description
σ_{ij}	stress tensor components
σ^{ij}	contravariant stress tensor components
ε_{ij}	strain tensor components
C_{ijkl}	stiffness constants (for $i, j, k, l = 1, 2, 3$)
C_{ij}	components of elastic constant tensor in Voigt notation
$\hat{C}_{ij}$	Kelvin notation of the components of elastic constant tensor in six dimensional Cartesian space
S_{ijkl}	compliance constants (for $i, j, k, l = 1, 2, 3$)
S_{ij}	components of compliance tensor in six dimensional notation
$\hat{S}_{ij}$	Kelvin notation of the components of compliance tensor in six dimensional Cartesian space
$\mathbf{A}_{ijkl}$	components of asymmetric part of elastic constant tensor

$\mathbf{S}_{ijkl}$ components of totally symmetric part of elastic constant tensor

δ_{ij} kronecker delta (for $i, j = 1, 2, 3$)

v_a unit vectors (for $a = 1, 2, 3$)

v_{ai} components of the unit vector (for $i = 1, 2, 3$)

z^k complex coordinates (for $k = 1, 2.$)

i imaginary number ($\sqrt{-1}$)

$\mathbf{r}$ the position vector

A_α covariant base vector

A^α contravariant base vector

$\mathbf{X}^{cont}$ contravariant components of a vector

$\mathbf{X}_{cov}$ covariant components of a vector

$\mathbf{m}_j$ transformation matrix for rank j tensors

$\mathbf{m}_j^*$ inverse of the transformation matrix for rank j tensors

$\overline{\mathbf{m}}_j$ conjugate transformation matrix for rank j tensors

λ_k eigenvalues of elastic constant tensor (for $k = 1, 2, 3, 4, 5, 6$)

$\mathbf{I}$ identity matrix

$\overrightarrow{n}^{(k)}$ the corresponding eigenvectors of elastic constant tensor (for $k = 1, 2, 3, 4, 5, 6$)

CHAPTER 1

INTRODUCTION

A material is isotropic if its mechanical and elastic properties are the same in all directions. When this is not true, the material is anisotropic. Anisotropic materials become the material of choice in a variety of engineering applications in the last century. Many materials are anisotropic and inhomogeneous due to the varying composition of their constituents. For instance, polycrystalline materials generally show an elastic anisotropy due to texture and the anisotropy of single crystallites. The polycrystalline and composite materials which show high anisotropy are used in many applications in industry. Everyday passed, the number of anisotropic materials is increasing by the addition of man-made anisotropic single crystals and technologically developed anisotropic materials. In order to understand the physical properties of the anisotropic materials, use of tensors by decomposing them is important. Tensors are the most significant mathematical entities to describe direction dependent physical properties of solids and the tensor components characterizing physical properties which must be specified without reference to any coordinate system.

For symmetric second rank tensors, there are three mainly decomposition methods namely as, orthonormal tensor basis, complex variable representation and spectral in the literature. There are few works for orthonormal tensor basis and complex variable representation methods, these can be summarized as; Srinivasan and Nigam[1] and Srinivasan[2] used orthonormal tensor basis method and then Ghaith[3] and Ghaith and Akgöz[4] developed it based on a theory found by Gazis et al.[5] and Tu[6]. For the complex variable representation method, firstly Green and Zerna[7] expressed symmetric second rank tensors in terms of complex variable components by appropriate tensor transformation. Unlike Green and Zerna[7], Verchery[8] used a different transformation named transformation of Verchery and then Surrel[9] worked on this method, finally Vannucci[10,11] starting from the complex vector representation which is obtained from symmetric plane stress relations and also called the method 'polar method'. Contrary to the two previous methods, spectral method has been studied more extensively. To name some Gurtin[12] and Luehr and Rubin[13] showed that it was an orthogonal decomposition method, Hartmann[14] mainly worked on eigenvalue computation for spectral method, Vallée et al.[15] and Itskov[16] applied the spectral method to the symmetric second rank tensors and gave the closed form of eigenvalues and eigenvectors.

Besides, there are four mainly decomposition methods for elastic constant tensor which are orthonormal tensor basis, irreducible, harmonic decomposition and spectral in the literature. The works for orthonormal tensor basis can be summarized as; it was first proposed by Gazis et al.[5], developed by Tu[6] who gave name as integrity basis treated the strain energy function as a polynomial in the strain components and lead to determination integrity basis for invariant functions of the strain components for each one of the 32 crystallographic point groups. Using the integrity basis, orthonormal tensor basis which spans the space of elastic constants was derived. Orthonormal tensor basis is also obtained by another way which is form invariant. Srinivasan and Nigam[1] identified invariant elastic constants for each crystal class and form invariant method was used by Ghaith[3] and Ghaith and Akgöz[4]. We apply this method to elastic constant tensor and it is proved that, integrity basis method (by Tu[6]) and form invariant method (by Srinivasan[1,2]) give the same results for elastic constant tensor under the title of orthonormal tensor basis method in this thesis.

Irreducible method was first used by Sirotin[17,18], according to Sirotin, elastic constant tensor was decomposed with respect to general linear group and then orthogonal group O(3). Contrary to Sirotin, Spencer[19] and Jarić[20] employed only elementary algebra and makes no group theory, he gave a method in which a general tensor of any rank n can be expressed in terms of traceless symmetric tensors of rank n or less. Zou et al.[21] realized orthogonal decomposition for any n rank tensors. These are general decomposition methods for any rank tensors. Jerphagnon et al.[22] derived certain results for the irreducible tensors in their natural form and then Surrel[9] worked on the method by decomposing elastic constant tensor in terms of complex variable components under the title of irreducible representations. Mochizuki[23] decomposed elastic constant tensor by using spherical harmonics. Same equations are used in Surrel and Mochizuki. Andrews and Ghoul[24] followed the technique of Jerphagnon et al. and gave the reduction of a fourth rank cartesian tensor into irreducible parts under the three-dimensional rotation group. Finally Radwan[25,26] developed the method and decomposed the elastic constant tensor into orthogonal and physically meaningful parts which is a different property among the other irreducible decomposition methods. There are other works in which elastic constant tensor is decomposed into five irreducible parts, those are performed by Onat (quoted by Cowin, Forte, Vianello, Ting, He, Zheng and Annin, Ostrosablin[27-34]). In these methods only results of decomposition for elastic constant tensor are given. For harmonic decomposition method, firstly, Backus[35] proposed a representation of elastic constant tensor in terms of harmonic tensors. These are based on an isomorphism between the space of homogeneous harmonic polynomials of degree q and the space of totally symmetric tensors of order q. Furthermore according to Sirotin[17,18], elastic constant tensor was decomposed with respect to general linear group and then orthogonal group O(3). Baerheim[36] followed Backus[35] and developed the method. Derivations for decomposition procedures of Spencer, Backus and Baerheim are taken by the other decomposition methods[27-34] in literature. In spectral method, to name some by Kelvin, Cowin et al.[37], the concept of eigentensors was first introduced and then Sutcliffe[38] and Ryclewski[39] rediscovered and described it.

A general outline of the thesis is given as, in chapter 2, anisotropic Hooke's law is summarized and Kelvin inspired formulation of anisotropic Hooke's law is presented. Stress tensor, as an example to symmetric second rank tensor, is decomposed by orthonormal tensor basis method, decomposition by complex variable representation and spectral method and numerical example by each of these methods for stress tensor is presented in chapter 3 and chapter 4. In addition, in chapter 5, elastic constant tensor, in triclinic symmetry as well as materials with other symmetry types, is decomposed by orthonormal tensor basis, irreducible, harmonic decomposition and spectral methods. Numerical examples for each decomposition method are given for anisotropic elastic symmetry types such as isotropic, cubic, transversely isotropic, tetragonal, trigonal and orthorhombic in chapter 6. The crystal classes and the properties of anisotropic elastic symmetry types are summarized in Table 1.1 and Table 1.2, respectively.

Chapter 7 covers the norm, norm ratio and anisotropy degree for each decomposition method and elastic symmetry types. All decomposition methods; first for symmetric second rank tensors and elastic constant tensor are compared in chapter 8. Finally, in the last chapter, the developed procedures, results of comparisons and calculated results are discussed and conclusions pertinent to this study are listed.

Table 1.1: The elastic symmetries and the crystal classes [40]

Crystal System	**International Symbol of Classes**
Triclinic	$1, \overline{1}$
Monoclinic	$2, m, 2/m$
Orthorhombic	$222, mm2, mmm$
Tetragonal	$4, \overline{4}, 4/m, 422, 4mm, \overline{4}2m, 4/mmm$
Cubic	$23, m3, 432, \overline{4}3m, m3m$
Trigonal	$3, \overline{3}, 32, 3m, \overline{3}m$
Hexagonal	$6, \overline{6}, 6/m, 622, 6mm, \overline{6}2m, 6/mmm$

Table 1.2: The distinct symmetries of linear anisotropic elasticity[40]

Type of Material Symmetry	**Number of Planes of Mirror Symmetry**	**Number of Planes of Isotropy**	**Number of Independent Elastic Constans**
Triclinic	0	0	18 (21)
Monoclinic	1 plane of reflective symmetry	0	12 (13)
Orthorhombic or Orthotropic	3 mutually perpendicular planes of mirror symmetry	0	9
Tetragonal	5(four of whose normals lie in the fifth plane and make angles of 45° with respect to one another.)	0	6 (7)
Cubic	9(all intersecting at 90°, 60°or 45°)	0	3
Trigonal	3(whose normals all lie in one plane and make angles of 60° with respect to one another)(3-fold symmetry)	0	6 (7)
Hexagonal	7(6-fold symmetry)	0	5
Transversely Isotropy	∞	1	5
Isotropy	∞	∞	2

1.1 Objective of the Thesis

There are five main objectives of this work:

1) To study and develop the existing orthogonal methods for decomposition of symmetric second rank tensors such as stress and strain tensors.

2) To study and elaborate on each decomposition method for symmetric fourth rank tensor which is elastic constant tensor found in the literature and compare these methods.

3) To apply these decomposition procedure to all types of symmetries which are isotropic, cubic, transversely isotropic, tetragonal, trigonal, orthorhombic and give numerical examples from anisotropic materials in order to observe which one is more isotropic or anisotropic which are applications for each decomposition procedures.

4) To introduce norm and norm ratio concepts for elastic constant tensor and compute the anisotropy degree for different engineering materials.

5) Finally, to understand the idea behind these decomposition methods as well as the physical properties of anisotropic materials.

CHAPTER 2

ANISOTROPIC HOOKE'S LAW

The anisotropic form of Hooke's law in linear elasticity is often written in indicial notation as

$$\sigma_{ij=}C_{ijkl}\varepsilon_{kl} \tag{2.1}$$

where σ_{ij} are components of stress tensor, ε_{kl} are components of infinitesimal strain tensor and C_{ijkl} are the components of elasticity tensor[41]. In other words, C_{ijkl} are the components of a fourth-rank tensor called the elastic constant tensor (stiffness tensor) and $i, j, k, l = 1, 2, 3$.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} \tag{2.2}$$

The indices are abbreviated according to the replacement rule given in the following table[42]:

Table 2.1: Abbreviation of indices for four index and double index notations

four index notation	11	22	33	23, 32	13, 31	12, 12
double index notation	1	2	3	4	5	6

Alternatively, Hooke's law is also written in indicial notation as

$$\varepsilon_{ij=}S_{ijkl}\sigma_{kl} \tag{2.3}$$

where S_{ijkl} are the compliance constants of the anisotropic material. There are three important symmetry restrictions on the elastic constant tensor. These are

$$C_{ijkl=}C_{jikl} \qquad C_{ijkl=}C_{ijlk} \qquad C_{ijkl=}C_{klij} \tag{2.4}$$

which follow from the symmetry of the stress tensor, the symmetry of the strain tensor and the elastic strain energy. These restrictions reduce the number of independent elastic constants C_{ijkl} from 81 to 21 [42].

2.1 A Tensorial Presentation of the Kelvin Formulation

It is written as a linear transformation in six dimensions, Hooke's law has the representation $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$ and in Voigt notation equation (2.2) is represented as follows

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} \tag{2.1.1}
$$

The relationships of the components of C_{ijkl} to the components of the symmetric matrix $\mathbf{C}$ are given Table 2.2. By introducing new notation, equation (2.1.1) can be rewritten in the form $\widehat{\sigma} = \hat{C}\widehat{\varepsilon}$, where the shearing components of these new six dimensional stress and strain vectors which are denoted by $\widehat{\sigma}$ and $\widehat{\varepsilon}$, respectively. They are multiplied by $\sqrt{2}$, $\hat{C}$ a new six-by-six matrix is obtained [40,43]. The matrix form of $\widehat{\sigma} = \hat{C}\widehat{\varepsilon}$ is given by

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\sigma_{23} \\ \sqrt{2}\sigma_{13} \\ \sqrt{2}\sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & \sqrt{2}C_{14} & \sqrt{2}C_{15} & \sqrt{2}C_{16} \\ C_{12} & C_{22} & C_{23} & \sqrt{2}C_{24} & \sqrt{2}C_{25} & \sqrt{2}C_{26} \\ C_{13} & C_{23} & C_{33} & \sqrt{2}C_{34} & \sqrt{2}C_{35} & \sqrt{2}C_{36} \\ \sqrt{2}C_{14} & \sqrt{2}C_{15} & \sqrt{2}C_{16} & 2C_{44} & 2C_{45} & 2C_{46} \\ \sqrt{2}C_{24} & \sqrt{2}C_{25} & \sqrt{2}C_{26} & 2C_{45} & 2C_{55} & 2C_{56} \\ \sqrt{2}C_{34} & \sqrt{2}C_{35} & \sqrt{2}C_{36} & 2C_{46} & 2C_{56} & 2C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \sqrt{2}\varepsilon_{23} \\ \sqrt{2}\varepsilon_{13} \\ \sqrt{2}\varepsilon_{12} \end{bmatrix} \tag{2.1.2}
$$

The matrix $\hat{C}$ is called the *matrix of elastic constants* and its inverse $\hat{S}$, $\widehat{\varepsilon} = \hat{S}\widehat{\sigma}$, $\hat{S} = \hat{C}^{-1}$ is called the *compliance* matrix. A table relating these various notations for the specific elastic constants is given in Table 2.2. The symmetric matrices $\hat{C}$ and $\hat{S}$ can be shown to represent the components of a second-rank tensor in a six dimensional space. In this thesis, these matrices are used in the following chapters. Since the components of the matrix $\mathbf{C}$ appearing in equation (2.1.1) do not form a tensor [40,43].

Table 2.2: The elasticity and compliance in different notations

1	2	3	1	2	3
C_{1111}	C_{11}	$\hat{C}_{11}$	S_{1111}	S_{11}	$\hat{S}_{11}$
C_{2222}	C_{22}	$\hat{C}_{22}$	S_{2222}	S_{22}	$\hat{S}_{22}$
C_{3333}	C_{33}	$\hat{C}_{33}$	S_{3333}	S_{33}	$\hat{S}_{33}$
C_{1122}	C_{12}	$\hat{C}_{12}$	S_{1122}	S_{12}	$\hat{S}_{12}$
C_{1133}	C_{13}	$\hat{C}_{13}$	S_{1133}	S_{13}	$\hat{S}_{13}$
C_{2233}	C_{23}	$\hat{C}_{23}$	S_{2233}	S_{23}	$\hat{S}_{23}$
C_{2323}	C_{44}	$\frac{1}{2}\hat{C}_{44}$	S_{2323}	$\frac{1}{4}S_{44}$	$\frac{1}{2}\hat{S}_{44}$
C_{1313}	C_{55}	$\frac{1}{2}\hat{C}_{55}$	S_{1313}	$\frac{1}{4}S_{55}$	$\frac{1}{2}\hat{S}_{55}$
C_{1212}	C_{66}	$\frac{1}{2}\hat{C}_{66}$	S_{1212}	$\frac{1}{4}S_{66}$	$\frac{1}{2}\hat{S}_{66}$
C_{1323}	C_{54}	$\frac{1}{2}\hat{C}_{54}$	S_{1323}	$\frac{1}{4}S_{54}$	$\frac{1}{2}\hat{S}_{54}$
C_{1312}	C_{56}	$\frac{1}{2}\hat{C}_{56}$	S_{1312}	$\frac{1}{4}S_{56}$	$\frac{1}{2}\hat{S}_{56}$
C_{1223}	C_{64}	$\frac{1}{2}\hat{C}_{64}$	S_{1223}	$\frac{1}{4}S_{64}$	$\frac{1}{2}\hat{S}_{64}$
C_{2311}	C_{41}	$\frac{1}{\sqrt{2}}\hat{C}_{41}$	S_{2311}	$\frac{1}{2}S_{41}$	$\frac{1}{\sqrt{2}}\hat{S}_{41}$
C_{1311}	C_{51}	$\frac{1}{\sqrt{2}}\hat{C}_{51}$	S_{1311}	$\frac{1}{2}S_{51}$	$\frac{1}{\sqrt{2}}\hat{S}_{51}$
C_{1211}	C_{61}	$\frac{1}{\sqrt{2}}\hat{C}_{61}$	S_{1211}	$\frac{1}{2}S_{61}$	$\frac{1}{\sqrt{2}}\hat{S}_{61}$
C_{2322}	C_{42}	$\frac{1}{\sqrt{2}}\hat{C}_{42}$	S_{2322}	$\frac{1}{2}S_{42}$	$\frac{1}{\sqrt{2}}\hat{S}_{42}$
C_{1322}	C_{52}	$\frac{1}{\sqrt{2}}\hat{C}_{52}$	S_{1322}	$\frac{1}{2}S_{52}$	$\frac{1}{\sqrt{2}}\hat{S}_{52}$
C_{1222}	C_{62}	$\frac{1}{\sqrt{2}}\hat{C}_{62}$	S_{1222}	$\frac{1}{2}S_{62}$	$\frac{1}{\sqrt{2}}\hat{S}_{62}$
C_{2333}	C_{43}	$\frac{1}{\sqrt{2}}\hat{C}_{43}$	S_{2333}	$\frac{1}{2}S_{43}$	$\frac{1}{\sqrt{2}}\hat{S}_{43}$
C_{1333}	C_{53}	$\frac{1}{\sqrt{2}}\hat{C}_{53}$	S_{1333}	$\frac{1}{2}S_{53}$	$\frac{1}{\sqrt{2}}\hat{S}_{53}$
C_{1233}	C_{63}	$\frac{1}{\sqrt{2}}\hat{C}_{63}$	S_{1233}	$\frac{1}{2}S_{63}$	$\frac{1}{\sqrt{2}}\hat{S}_{63}$

In Table 2.2, column 1 illustrates the Voigt notation of these quantities as fourth rank tensor components in a three dimensional Cartesian space. Column 2 represents the same Voigt matrix in double index notation. Column 3 illustrates the Kelvin-inspired notation for these quantities as second rank tensor components in a six dimensional cartesian space.

CHAPTER 3

DECOMPOSITION OF SYMMETRIC SECOND RANK TENSORS INTO ORTHOGONAL PARTS

In this chapter, we have presented the procedures of decomposing symmetric second rank tensors, using orthonormal tensor basis, complex variable representation and spectral methods. Stress tensor as an example to symmetric second rank tensor is decomposed by each of these methods. In the mechanics of continuous media i.e. in elasticity studies; the stress and strain tensors are decomposed into spherical (hydrostatic) and deviatoric parts each of which have important meanings. Besides, stress tensor is decomposed into six orthonormal parts by orthonormal tensor basis and complex variable representation methods while spectral method gives three non-linear orthonormal decomposed parts.

3.1 By Orthonormal Tensor Basis Method

In literature, orthonormal tensor basis method has been developed in different names such as integrity basis and form-invariant. In form-invariant method, a physical property of tensor is resolved along the triad $v_1, v_{2,} v_3$ denoting the unit vectors along the crystallographic axes. It should be noted that $v_1, v_{2,} v_3$ form a linearly independent basis in three dimensions.[42] The process of resolution yields the invariants. Forming invariant is an indispensable step to construct orthonormal tensor basis needed for decomposition process. Form invariant method is used in this paper since it is valid for any rank tensor decomposition. Form invariant method is named as orthonormal tensor basis method in this work. The reason is that decomposed parts are constructed by use of orthonormal tensor basis which are made up of form invariants. According to orthonormal tensor basis method, the procedure is as follows:

The form invariant expressions for symmetric second rank tensor as stress tensor is

$$\sigma_{ij} = v_{ai} v_{bj} A_{ab}, \tag{3.1.1}$$

where summation is implied by repeated indices and this convention is followed throughout. This expression is referred to a Cartesian system $Oxyz$; v_{ai} are the components of the unit vectors v_a $(a = 1, 2, 3)$ along the crystallographic axes. A_{ab} is invariant in the sense that when the Cartesian system is rotated to a new orientation $Ox'y'z'$ then equation (3.1.1) takes the form

$$\sigma'_{ij} = v'_{ai} v'_{bj} A_{ab}, \tag{3.1.2}$$

Corresponding reciprocal triads must also satisfy the relations

$$v^{ai} v_{aj} = \delta_{ij}. \tag{3.1.3}$$

As an illustration, for the uniaxial crystal system, equation (3.1.1) takes the form[1]

$$\sigma_{ij} = \sigma_1 \delta_{ij} + \sigma_2 v_{3i} v_{3j}, \tag{3.1.4}$$

where v_3 is the unique axis and σ_1 corresponds to isotropy.

The first step for constructing the orthonormalized basis elements is to write the δ_{ai} in the place of v_{ai} in equation (3.1.1). So the following form is obtained which is

$$\sigma_{ij} = \delta_{ai} \delta_{bj} A_{ab}, \tag{3.1.5}$$

Instead of the form-invariant expression given in equation (3.1.1) for any given class and it is possible to replace the v_{ai} by the δ_{ai} to obtain the elements of the basis. According to the expression in equation (3.1.4), the elements of the basis are δ_{ij} and $\delta_{3i}\delta_{3j}$. Similar to equation (3.1.4), for monoclinic system, with v_2 normal to the v_3v_1-plane, the form-invariant expression is[1]

$$\sigma_{ij} = A_{11}v_{1i}v_{1j} + A_{22}v_{2i}v_{2j} + A_{33}v_{3i}v_{3j} + A_{31}(v_{3i}v_{1j} + v_{1i}v_{3j}) \tag{3.1.6}$$

By making the replacement to the above expression, we obtain the elements; $\delta_{1i}\delta_{1j}, \quad \delta_{2i}\delta_{2j}, \quad \delta_{3i}\delta_{3j}, \quad (\delta_{3i}\delta_{1j} + \delta_{1i}\delta_{3j}).$

By using equation (3.1.3) and orthonormalization by well known Gram-Schmidt scheme, these basis elements are

$$\begin{aligned}
&T_{ij}^{I} = \frac{1}{\sqrt{3}}\delta_{ij}, \quad T_{ij}^{II} = \frac{1}{\sqrt{2}}(2\delta_{1i}\delta_{1j} + \delta_{3i}\delta_{3j} - \delta_{ij}),\\
&T_{ij}^{III} = \frac{1}{\sqrt{6}}(-3\delta_{3i}\delta_{3j} + \delta_{ij}), \quad T_{ij}^{IV} = \frac{1}{\sqrt{2}}(\delta_{3i}\delta_{1j} + \delta_{1i}\delta_{3j}),\\
&T_{ij}^{V} = \frac{1}{\sqrt{2}}(\delta_{1i}\delta_{2j} + \delta_{2i}\delta_{1j}), \quad T_{ij}^{VI} = \frac{1}{\sqrt{2}}(\delta_{3i}\delta_{2j} + \delta_{2i}\delta_{3j}).
\end{aligned} \tag{3.1.7}$$

In constructing this basis, following identity is used

$$\delta_{1i}\delta_{1j} + \delta_{2i}\delta_{2j} + \delta_{3i}\delta_{3j} = \delta_{ij} \tag{3.1.8}$$

This is a particular case of a more general identity which is[1]

$$v_{1i}v_{1j} + v_{2i}v_{2j} - \cos\theta(v_{1i}v_{2j} + v_{2i}v_{1j}) + \sin^2\theta v_{3i}v_{3j} = \sin^2\theta\delta_{ij} \tag{3.1.9}$$

with v_{ai} is replaced by δ_{ai} and $\theta = 90^o$. Hence we obtain the orthonormal basis elements presented in equation (3.1.7).

A complete orthonormal basis for the symmetric second rank tensor (i.e., stress tensor) will be the set $\{I, II, ..., VI\}$. The decomposition of σ_{ij} is given in terms of these basis elements as

$$\sigma_{ij} = \sum_k(\boldsymbol{\sigma}, T_{ij}^k)T_{ij}^k, \quad (k = I, II, ..., VI) \tag{3.1.10}$$

where $(\boldsymbol{\sigma}, T_{ij}^k)$ represents the inner product of σ_{ij} and the k^{th} elements, T_{ij}^k of the basis. It is well known that inner product is different from multiplication of two matrices. For second rank tensors, it is defined by $(\boldsymbol{\sigma}, T_{ij}^k) = \sigma_{ij} T_{ij}^k$. The orthonormal of the decomposed parts can be proved by taking inner products of orthonormalized basis elements (T_{ij}^I, T_{ij}^{II}) i.e., $(T_{ij}^I, T_{ij}^{II}) = 0$ and $(T_{ij}^I, T_{ij}^I) = 1$ and the results of inner products for other elements are the same. So this method is an orthonormal method.

By using equation (3.1.10) and matrix forms of the orthonormalized basis elements, decomposed parts can be obtained and by adding all decomposed parts, we obtain the stress tensor which is

$$\sigma_{ij} = \frac{\sigma_{pp}}{\sqrt{3}} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{bmatrix} + \frac{1}{\sqrt{2}}(2\sigma_{11} + \sigma_{33} - \sigma_{pp}) \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$+ \frac{1}{\sqrt{6}}(-3\sigma_{33} + \sigma_{pp}) \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix} + \frac{2\sigma_{13}}{\sqrt{2}} \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$
$$+ \frac{2\sigma_{12}}{\sqrt{2}} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{2\sigma_{23}}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}. \qquad (3.1.11)$$

In equation (3.1.11), T_{ij}^k $(k = I, II, ..., VI)$ are the matrix parts of the orthonormalized basis elements. They are orthonormal to each other. Moreover above form of stress tensor given in Ghaith et al.[4] contains misprints in matrix form of second and third decomposed parts. In equation (3.1.11), these parts are corrected.

Furthermore stress tensor is virtually decomposed into two parts

$$\sigma_{ij} = \frac{1}{3}\sigma_{pp}\delta_{ij} + \left(\sigma_{ij} - \frac{1}{3}\sigma_{pp}\delta_{ij}\right), \tag{3.1.12}$$

where $\sigma_{pp} = \sigma_{11} + \sigma_{22} + \sigma_{33}$,
$\boldsymbol{\sigma}^D = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{pp}$(deviatoric part) and $\boldsymbol{\sigma}^S = \frac{1}{3}\delta_{ij}\sigma_{pp}$(spherical part).

Deviatoric space D consists of pure shear tensors constructing by requiring orthonormal basis which are orthogonal to each other[44,45]. $\boldsymbol{\sigma}^D$ can be decomposed into five parts by orthonormal tensor basis method.

It is well known in literature that when $i = j$, $\sigma_{ij}^D = 0$, which means that the deviatoric part $\boldsymbol{\sigma}^D$ is traceless. When $\boldsymbol{\sigma}^D = 0$, equation (3.1.12) reduces to the form: $\sigma_{ij} = \frac{1}{3}\delta_{ij}\sigma_{pp}$.

Here $\frac{1}{3}\sigma_{pp} = -p$, where p is the hydrostatic pressure and $\sigma_{ij} = -p\delta_{ij}$ is called pure hydrostatic state of stress.

3.2 By Complex Variable Representation

In this method, complex coordinate system in terms of polar components is used to represent symmetric second rank tensor components in Cartesian coordinates. Complex coordinates (z^1,z^2) can be represented by the following formulae:

$$z^1 = x_1 + ix_2, \qquad z^2 = \overline{z}^1 = x_1 - ix_2, \tag{3.2.1}$$

where $i = \sqrt{-1}$. In general, covariant and contravariant components are used to transform the tensor components from complex coordinate to Cartesian coordinate system. In this thesis, this is why we have used covariant and contravariant components.

To be able to derive transformation for symmetric second rank tensors, we have transformed first order tensors (vectors) by performing the following steps:

At first, covariant and contravariant base vectors are denoted in this system of coordinates by A_α and A^α respectively. The position vector $\mathbf{r}$ is written as[7]

$$\mathbf{r} = X^\alpha A_\alpha = X_\alpha A^\alpha \tag{3.2.2}$$

By taking derivative of $\mathbf{r}$, we get $d\mathbf{r} = \dfrac{\partial \mathbf{r}}{\partial x^k} dx^k = g_k dx^k$ which is also equal to ds. By tensor transformations, the following relations are obtained

$$X^1 = \frac{\partial z^1}{\partial x_1} x_1 + \frac{\partial z^1}{\partial x_2} x_2 = x_1 + i x_2 = z^1, \qquad X^2 = \frac{\partial z^2}{\partial x_1} x_1 + \frac{\partial z^2}{\partial x_2} x_2 = x_1 - i x_2 = z^2. \tag{3.2.3}$$

For completeness to three dimensional space, $X^3 = x_3$.

A complex variable change is interpreted as a change of frame[7,11]. This transformation (also called transformation of Verchery) gives the contravariant components

$\mathbf{X}^{cont} = (X^1, X^2)$ of a vector $\mathbf{x} = (x_1,\ x_2)$ as

$$X^1 = \frac{1}{\sqrt{2}} \overline{k} z^1 = \frac{x_1 + x_2 - i(x_1 - x_2)}{2}, \qquad X^2 = \frac{1}{\sqrt{2}} k z^2 = \frac{x_1 + x_2 + i(x_1 - x_2)}{2} = \overline{X}^1, \tag{3.2.4}$$

where $k = e^{i\frac{\pi}{4}}$.

Transformation rule for first rank tensors (vectors) can be written as

$$\mathbf{X}^{cont} = \mathbf{m}_1 \mathbf{x}, \tag{3.2.5}$$

where $\mathbf{m}_1$ is the transformation matrix.

$$\mathbf{m}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} \overline{k} & k \\ k & \overline{k} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}. \tag{3.2.6}$$

The matrix form of the transformation is

$$\begin{bmatrix} X^1 \\ X^2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{3.2.7}$$

Element of an arc is in the form $ds = g_k dx^k$ $(k = 1, 2)$. The relations between components in Cartesian system and complex coordinates are summarized as follows:

$$dX^i = \frac{\partial X^i}{\partial x^j}\, dx^j, \qquad\qquad dX_j = \frac{\partial x^i}{\partial X^j}\, dx_i.$$

By using above relations, g_k are found as $\vec{g_k} = \dfrac{\partial X^m}{\partial x^k}\hat{\imath}_m$, $(k, m = 1, 2)$,(Where $\vec{g_k}$ are the functions of complex coordinates) from this formula, the following relations are obtained:

$$\vec{g_1} = \frac{\partial X^1}{\partial x^1}\hat{\imath}_1 + \frac{\partial X^2}{\partial x^1}\hat{\imath}_2 = \frac{(1-i)}{2}\hat{\imath}_1 + \frac{(1+i)}{2}\hat{\imath}_2,$$
$$\vec{g_2} = \frac{\partial X^1}{\partial x^2}\hat{\imath}_1 + \frac{\partial X^2}{\partial x^2}\hat{\imath}_2 = \frac{(1+i)}{2}\hat{\imath}_1 + \frac{(1-i)}{2}\hat{\imath}_2.$$

From the above relations metric tensor components are found as

$$g_{11} = \vec{g_1} \cdot \vec{g_1} = 0,\; g_{22} = \vec{g_2} \cdot \vec{g_2} = 0,\; g_{12} = g_{21} = \vec{g_1} \cdot \vec{g_2} = 1,$$

which yields the metric tensor:

$$\mathbf{g}_{cov} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.2.8}$$

So by using metric tensor components, the square of the line element of an infinitesimal arc is expressed as

$$ds^2 = d\mathbf{r}\cdot d\mathbf{r} = dz^1 dz^2 = 2dX^1 dX^2. \tag{3.2.9}$$

Metric tensor is needed to obtain covariant components from contravariant components. So covariant components $\mathbf{X}_{cov} = (X_1, X_2)$ are

$$\mathbf{X}_{cov} = \mathbf{g}_{cov}\mathbf{X}^{cont}, \tag{3.2.10}$$

which gives

$$X_1 = X^2 = \frac{1}{\sqrt{2}} kz^2, \qquad\qquad X_2 = X^1 = \frac{1}{\sqrt{2}} \overline{k} z^1.$$

The important feature used in this thesis is that the properties of the matrix $\mathbf{m}_1$ which operates the transformation for first rank tensors, related with the matrices $\mathbf{m}_j$ that operate the transformation for higher rank tensors. These properties can be summarized as;

$$\mathbf{m}_j = \mathbf{m}_j^T, \;\; \mathbf{m}_j^* = \overline{\mathbf{m}}_j, \quad \mathbf{m}_j^*\mathbf{m}_j = \mathbf{I}, \quad \mathbf{m}_j^{-1} = \overline{\mathbf{m}}_j, \quad \mathbf{m}_j \neq \mathbf{m}_j^*.$$

So $\mathbf{m}_2$ is constructed by the aid of $\mathbf{m}_1$. When we apply it to a second rank tensor, transformation of Verchery yields:

$$\mathbf{m}_2 = \left[\begin{array}{c|c} m_1^{11}\mathbf{m}_1 & m_1^{12}\mathbf{m}_1 \\ \hline m_1^{21}\mathbf{m}_1 & m_1^{22}\mathbf{m}_1 \end{array}\right] = \frac{1}{2}\begin{bmatrix} \overline{k}\overline{k} & \overline{k}k & k\overline{k} & kk \\ \overline{k}k & \overline{k}\overline{k} & kk & k\overline{k} \\ k\overline{k} & kk & \overline{k}\overline{k} & \overline{k}k \\ kk & k\overline{k} & \overline{k}k & \overline{k}\overline{k} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -i & 1 & 1 & i \\ 1 & -i & i & 1 \\ 1 & i & -i & 1 \\ i & 1 & 1 & -i \end{bmatrix}$$

,

For a symmetric second rank tensor such as stress tensor, $\mathbf{m}_2$ takes the following form:

$$\mathbf{m}_2 = \frac{1}{2}\begin{bmatrix} -i & 2 & i \\ 1 & 0 & 1 \\ i & 2 & -i \end{bmatrix}.$$

For stress tensor $\boldsymbol{\sigma}$, transformation of Verchery[8] is constructed as

$$\boldsymbol{\sigma}^{cont} = \mathbf{m}_2\boldsymbol{\sigma}, \tag{3.2.11}$$

In matrix form, the transformation is

$$\begin{bmatrix} \sigma^{11} \\ \sigma^{12} \\ \sigma^{22} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -i & 2 & i \\ 1 & 0 & 1 \\ i & 2 & -i \end{bmatrix}\begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \end{bmatrix}. \tag{3.2.12}$$

Transformation of Verchery[8] is characterized by transformation matrices ($\mathbf{m}_i$). Despite of the transformation of Green and Zerna[7], that of Verchery[8] is unitary. Furthermore, frame is changed from (x_1, x_2) to new frame (x_1', x_2'), by rotating through the rotation angle θ, if $r = e^{-i\theta}$, then in the new reference frame the complex variable is $z^{1'} = rz^1$. By applying again the transformation of Verchery[8] given in equation (3.2.4), the new contravariant complex components will be

$$X^{1'} = \frac{1}{\sqrt{2}}\overline{k}z^{1'} = \frac{1}{\sqrt{2}}\overline{k}rz^1 = rX^1, \qquad X^{2'} = \frac{1}{\sqrt{2}}kz^{2'} = \frac{1}{\sqrt{2}}k\overline{r}z^2 = \overline{r}X^2. \tag{3.2.13}$$

By equation (3.2.13), we obtain transformation for first order tensor;

$\mathbf{X}^{cont'} = \mathbf{R}_1\mathbf{X}^{cont}$ with rotation matrix $\mathbf{R}_1 = \begin{bmatrix} r & 0 \\ 0 & \overline{r} \end{bmatrix}$.

For symmetric second rank tensors, the rotation matrix $\mathbf{R}_2$ can be constructed by the same rule used for $\mathbf{m}_2$ in order to obtain

$$\boldsymbol{\sigma}^{cont'} = \mathbf{R}_2\boldsymbol{\sigma}^{cont}. \tag{3.2.14}$$

In matrix form, it is equal to

$$\begin{bmatrix} \sigma^{11'} \\ \sigma^{12'} \\ \sigma^{22'} \end{bmatrix} = \begin{bmatrix} r^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \overline{r}^2 \end{bmatrix} \begin{bmatrix} \sigma^{11} \\ \sigma^{12} \\ \sigma^{22} \end{bmatrix}. \tag{3.2.15}$$

From equation (3.2.15), the following relations are obtained:

$$\sigma^{12'} = \sigma^{12}, \qquad \sigma^{11'}\sigma^{22'} = \sigma^{11}\sigma^{22}.$$

Where those are invariants which can be expressed with the Cartesian components of σ^{ij}; for a symmetric tensor, through equation (3.2.12), we obtain

$$\sigma^{12} = \frac{1}{2}(\sigma_{11} + \sigma_{22}), \qquad \sigma^{11}\sigma^{22} = \frac{1}{4}(\sigma_{11} - \sigma_{22})^2 + \sigma_{12}^2. \tag{3.2.16}$$

The relationship between the complex coordinates and Cartesian coordinates can be obtained by appropriate tensor transformations. For instance, stress tensor as a symmetric second rank tensor, σ^{rs} can be expressed in terms of σ_{mn} by the following tensor transformations (transformation of Green and Zerna[7]) :

$$\sigma^{rs} = \frac{\partial z^r}{\partial x_m}\frac{\partial z^s}{\partial x_n}\sigma_{mn} \tag{3.2.17}$$

By applying equations (3.2.1) and (3.2.17), σ^{rs} are found as

$$\sigma^{11} = \sigma_{11} - \sigma_{22} + 2i\sigma_{12}, \tag{3.2.18}$$

$$\sigma^{22} = \sigma_{11} - \sigma_{22} - 2i\sigma_{12}, \tag{3.2.19}$$

$$\sigma^{12} = \sigma_{11} + \sigma_{22}, \tag{3.2.20}$$

$$\sigma^{13} = \sigma_{13} + i\sigma_{23}, \tag{3.2.21}$$

$$\sigma^{23} = \sigma_{13} - i\sigma_{23}, \tag{3.2.22}$$

$$\sigma^{33} = \sigma_{33}. \tag{3.2.23}$$

Furthermore three Cartesian components of stress tensor can be expressed by three parameters which are T (scalar), R (moduli) and ϕ (polar angle) and Vannucci[11] called them as polar components. Cartesian components of σ_{ij} can be expressed in terms of these components[10,11,46]

$$\sigma_{11} = T + R\cos 2\phi, \qquad \sigma_{12} = R\sin 2\phi, \qquad \sigma_{22} = T - R\cos 2\phi. \tag{3.2.24}$$

The inverse of the expressions given in equation (3.2.24) are respectively in the following form:

$$T = \frac{\sigma_{11} + \sigma_{22}}{2} \qquad \text{and} \qquad \mathrm{Re}^{2i\phi} = \frac{\sigma_{11} - \sigma_{22}}{2} + i\sigma_{12}. \tag{3.2.25}$$

By multiplying $\mathrm{Re}^{2i\phi}$ given in equation (3.2.25) with $-i$, we obtain

$$\sigma^{11} = \frac{i}{2}(\sigma_{22} - \sigma_{11}) + \sigma_{12}, \tag{3.2.26}$$

which is also obtained from transformation in equation (3.2.12). So σ^{11} is expressed in terms of polar components as

$$\sigma^{11} = -i\,\mathrm{Re}^{2i\phi}. \tag{3.2.27}$$

Moreover, in Cartesian coordinate system, the basis consists of mutually orthogonal unit vectors that is covariant and contravariant basis are identical.

As a result, only subscripts are used in Cartesian or rectangular coordinates. Whereas in the system of complex coordinates z^r, there are contravariant and covariant components which are not the same.

By multiplying right hand sides of the expression in equations (3.2.26) and (3.2.27) with i, we obtain $\mathrm{Re}^{2i\phi}$ found in equation (3.2.25) which is also represented by ${}^2\sigma_2$. In similar manner, ${}^2\sigma_1$ is obtained by multiplying right hand side of equation (3.2.21) with $-i$, and scalar part corresponding to ${}^2\sigma_0$ are found by subtracting equation (3.2.23) from T found in equation (3.2.25). ${}^2\sigma_2$, ${}^2\sigma_1$, ${}^2\sigma_0$ (which are Surrel's notations[9]) are the representations of deviatoric stress tensor components of weight 2 (superscript) and subscripts are $0, 1$ and 2 which are integers. Explicit forms of these representations are

$$ {}^2\sigma_2 = \frac{1}{2}(\sigma_{11}-\sigma_{22})+i\sigma_{12}, \qquad {}^2\sigma_1 = \sigma_{23}-i\sigma_{13}, \qquad {}^2\sigma_0 = \frac{1}{2}(\sigma_{11}+\sigma_{22})-\sigma_{33}. \tag{3.2.28} $$

A representation for hydrostatic part of stress tensor, well known in literature (which is invariant), of weight 0 is

$$ {}^0\sigma = \sigma_{11} + \sigma_{22} + \sigma_{33} = -3p, \tag{3.2.29} $$

where p is the hydrostatic pressure.

By inverting equations (3.2.28) and (3.2.29), stress components can be found as

$$
\begin{aligned}
\sigma_{11} &= \frac{1}{3}({}^0\sigma) + \mathrm{Re}({}^2\sigma_2) + \frac{1}{3}({}^2\sigma_0), \\
\sigma_{22} &= \frac{1}{3}({}^0\sigma) - \mathrm{Re}({}^2\sigma_2) + \frac{1}{3}({}^2\sigma_0), \\
\sigma_{33} &= \frac{1}{3}({}^0\sigma) - \frac{2}{3}({}^2\sigma_0), \\
\sigma_{23} &= \mathrm{Re}({}^2\sigma_1), \\
\sigma_{13} &= -\,\mathrm{Im}({}^2\sigma_1), \\
\sigma_{12} &= \mathrm{Im}({}^2\sigma_2).
\end{aligned}
$$

Above relations for stress tensor can be expressed in matrix form which is the same as the form in equation (3.1.11) given in orthonormal tensor basis method. These results are the same as those in Surrel[9]. Like orthonormal tensor basis method, this method gives orthonormal parts.

3.3 By Spectral Method

For comparison purposes, we give a brief description of the well known spectral decomposition method in the literature. In spectral decomposition of a symmetric second rank tensor in three dimensional space, the stress tensor is represented in a decomposed form using its eigenvalues and eigenvectors. In Gurtin[12], Luehr and Rubin[13], Hartmann[14], Vallée et al.[15], Itskov[16], the main notation used for spectral decomposition of the stress tensor is

$$\boldsymbol{\sigma} = \sum_{k=1}^{3} \lambda_k \vec{n}^{(k)} \otimes \vec{n}^{(k)}, \tag{3.3.1}$$

where $\boldsymbol{\sigma}$ represents the stress tensor in three dimensional space, λ_k are its eigenvalues and $\vec{n}^{(k)}$ are its corresponding eigenvectors. The symbol $\otimes$ is the dyadic (tensor) product. The multiplication of eigenvectors of stress tensor gives eigenprojections denoted by P_i, where $i = 1, 2, 3$.

We need to compute eigenvalues and eigenprojections (idempotent tensors) in order to perform spectral decomposition.

The eigenvalues of the matrix $\boldsymbol{\sigma}$ should satisfy the following equation:

$$\boldsymbol{\sigma} \vec{n}^{(k)} = \lambda_k \vec{n}^{(k)}. \tag{3.3.2}$$

By rewriting equation (3.3.2), it takes the well known form

$$(\boldsymbol{\sigma} - \lambda_k \mathbf{I}) \vec{n}^{(k)} = 0, \tag{3.3.3}$$

where $\mathbf{I}$ is three by three identity matrix. It is obvious that for $\vec{n}^{(k)} \neq 0$, the determinant of the tensor $\boldsymbol{\sigma} - \lambda_k \mathbf{I}$ has to vanish, leading to the characteristic polynomial of third order

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0, \qquad (3.3.4)$$

where

$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33},$

$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - (\sigma_{12})^2 - (\sigma_{13})^2 - (\sigma_{23})^2,$

$I_3 = \sigma_{11}\sigma_{22}\sigma_{33} - \sigma_{11}(\sigma_{23})^2 - \sigma_{33}(\sigma_{12})^2 - \sigma_{22}(\sigma_{13})^2 + 2\sigma_{12}\sigma_{13}\sigma_{23}.$

I_1 is linear type invariant, I_2 is quadratic type invariant and I_3 is cubic type invariant.

The roots of equation (3.3.4) can be found in terms of invariants as[16]

$$\lambda_k = \frac{1}{3}\left\{I_1 + 2\sqrt{I_1^2 - 3I_2}\cos\frac{1}{3}[\theta + 2\pi(k-1)]\right\}, \quad (k = 1, 2, 3.) \qquad (3.3.5)$$

where $\theta = \arccos\left[\dfrac{2I_1^3 - 9I_1I_2 + 27I_3}{2\sqrt{(I_1^2 - 3I_2)^3}}\right]$.

By using the expression given in equation (3.3.5), the eigenvalues of $\boldsymbol{\sigma}$ are obtained as

$$\lambda_1 = \frac{1}{3}\left\{I_1 + 2\sqrt{I_1^2 - 3I_2}\cos(\frac{\theta}{3})\right\},$$

$$\lambda_2 = \frac{1}{3}\left\{I_1 + 2\sqrt{I_1^2 - 3I_2}\cos\frac{1}{3}(\theta + 2\pi)\right\},$$

$$\lambda_3 = \frac{1}{3}\left\{I_1 + 2\sqrt{I_1^2 - 3I_2}\cos\frac{1}{3}(\theta + 4\pi)\right\}.$$

The eigenprojections for corresponding three distinct eigenvalues are found by the following formula called as Sylvester formula[16]

$$P_i = \prod_{j=1, j\neq i}^{3} \frac{\boldsymbol{\sigma} - \lambda_j \mathbf{I}}{\lambda_i - \lambda_j}, \quad i = 1, 2, 3.$$

Cases for repeated eigenvalues:

i) If there is only one distinct eigenvalue; λ_1, the spectral representation given in (3.3.1) is trivial, because

$$P_1 = \mathbf{I}, \quad C = \lambda_1 \mathbf{I}. \qquad (3.3.6)$$

ii) If there are two of the eigenvalues which are identical such as $\lambda_1 \neq \lambda_2 = \lambda_3$, we obtain two eigenprojections;

$$P_1 = \prod_{j=1, j\neq 1}^{2} \frac{\boldsymbol{\sigma} - \lambda_j \mathbf{I}}{\lambda_i - \lambda_j} = \frac{\boldsymbol{\sigma} - \lambda_2 \mathbf{I}}{\lambda_1 - \lambda_2}, \qquad P_2 = \prod_{j=1, j\neq 2}^{2} \frac{\boldsymbol{\sigma} - \lambda_j \mathbf{I}}{\lambda_i - \lambda_j} = \frac{\boldsymbol{\sigma} - \lambda_1 \mathbf{I}}{\lambda_2 - \lambda_1}.$$

So spectral decomposition of $\boldsymbol{\sigma}$ is obtained as (for repeated eigenvalues)

$$\boldsymbol{\sigma} = \sum_{i=1}^{2} \lambda_i P_i = \lambda_1 P_1 + \lambda_2 P_2 \tag{3.3.7}$$

iii) If all eigenvalues are distinct then eigenvectors of stress tensor corresponding to different eigenvalues are linearly independent. Besides eigenvectors of stress tensor corresponding to these distinct eigenvalues are mutually orthogonal. This means that that eigenspaces of a symmetric matrix $\boldsymbol{\sigma}$ are orthogonal.

Hence spectral representation of $\boldsymbol{\sigma}$ takes the form

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \lambda_i P_i = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3. \tag{3.3.8}$$

If we rewrite equation (3.3.8), we obtain

$$\boldsymbol{\sigma} = \lambda_1(I_1, I_2, I_3) P_1 + \lambda_2(I_1, I_2, I_3) P_2 + \lambda_3(I_1, I_2, I_3) P_3. \tag{3.3.9}$$

From equation (3.3.5), it is seen that the eigenvalues are functions which are depend on invariants I_1, I_2, I_3. These invariants are expressed in terms of the different combinations of stress components. Since the combinations of stress components contributed to I_2, I_3 are not linear, we see that spectral decomposition is **non-linear** so it gives three non-linear orthonormal decomposed parts.

3.4 Numerical Example of Symmetric Second Rank Tensors

To support the analytic results of three methods, we give a simple numerical example.

Let $\boldsymbol{\sigma}$ be a stress tensor which has the matrix form in appropriate units

$$\boldsymbol{\sigma} = \begin{bmatrix} 5 & -1 & -1 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}. \qquad (3.4.1)$$

By applying orthonormal tensor basis method and complex variable representation, six decomposed parts are calculated and all parts are added then stress tensor in terms of these decomposed parts takes the following form:

$$\boldsymbol{\sigma} = \frac{11}{\sqrt{3}} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{bmatrix} + \frac{2}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{2}{\sqrt{6}} \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

$$-\frac{2}{\sqrt{2}} \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix} - \frac{2}{\sqrt{2}} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{2}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}. \qquad (3.4.2)$$

It is easily verified that these six decomposed parts form an orthonormal set. Equation (3.4.2) is also equivalent to

$$\boldsymbol{\sigma} = \begin{bmatrix} \frac{11}{3} & 0 & 0 \\ 0 & \frac{11}{3} & 0 \\ 0 & 0 & \frac{11}{3} \end{bmatrix} + \begin{bmatrix} \frac{4}{3} & -1 & -1 \\ -1 & -\frac{2}{3} & 1 \\ -1 & 1 & -\frac{2}{3} \end{bmatrix}, \qquad (3.4.3)$$

where first decomposed part of equation (3.4.3) is spherical part and second one is the deviatoric part of stress tensor. The physical meanings of these two parts and each of six decomposed are presented in the last chapter. On the other hand, by performing the steps given in spectral method that numerical decomposition of it is very well known for symmetric second rank tensors, spectral representation of $\boldsymbol{\sigma}$ takes the form

$$\boldsymbol{\sigma} = 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + 3\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} + 6\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}, \qquad (3.4.4)$$

where $2, 3$ and 6 are eigenvalues and 3×3 matrices are corresponding eigenprojections (idempotent tensors) of $\boldsymbol{\sigma}$ in equation (3.4.4). Since the eigenprojections of $\boldsymbol{\sigma}$ are orthonormal to each other, this makes spectral method an orthonormal method. So we obtain three orthonormal decomposed parts given in equation (3.4.4). The physical significance of these three parts are given in section 7.1.

CHAPTER 4

DECOMPOSITION OF ELASTIC CONSTANT TENSOR INTO ORTHOGONAL PARTS

In analyzing the elastic and mechanical properties of anisotropic linear materials, elastic constant tensor is required to make up a linear constitutive relation between stress and strain tensors, each of which represents some directly detectable and measurable effect in the material (Recall Hooke's law, given in equation (2.1)). Elastic constant tensor is introduced in specification of physical properties for many anisotropic materials. Decomposition of the elastic constant tensor into orthogonal parts, offer not only valuable insight into the tensor structure but also simplify immensely the calculations of sums, products, inverses and inner products. It is decomposed by the methods which are orthonormal tensor basis, irreducible decomposition, harmonic decomposition and spectral. These methods are carried out for materials possessing symmetry classes such as isotropic, cubic, transversely isotropic, tetragonal (classes: $4mm$, $\overline{4}2m$, 422, $4/mmm$), trigonal (classes:32, $3m$, $\overline{3}m$), orthorhombic and triclinic which are defined in Appendix.

For isotropic materials, an expression for the elastic constant tensor which is different from the traditional form, is presented.

4.1 By Orthonormal Tensor Basis Method

A physical property of tensor is resolved along the triads $v_{1,}v_2, v_3$ denoting the unit vectors along the material coordinate axis.[1] The symmetry properties of the material, due to the geometric or cystallographic symmetry, can be defined by the group of orthonormal transformations which transform any of these triads ν_a into its equivalent positions. For each of the symmetry classes selected, as reference system a rectangular Cartesian coordinate system $Oxyz$ is chosen, so related to the material directions ν_1, ν_2, ν_3 in the material under consideration that the symmetry of the material may be described by one or more of the transformations. Their relative orientations in the seven crystal systems are well known[42]. Transformations in which the coefficients satisfy the orthogonality relations are called linear orthogonal transformations. In this formulation, the number of elastic constants and their values do not depend on the choice of the coordinate system.

The form invariant expression for the components of elastic constant tensor, the elastic stiffness coefficients is,

$$C_{ijkl} = \nu_{ai}\nu_{bj}\nu_{ck}\nu_{dl}A_{abcd} \tag{4.1.1}$$

where summation is implied by repeated indices, ν_{ai} are the components of the unit vectors $\nu_a(a = 1, 2, 3)$ along the material direction axes. A_{abcd} is invariant in the sense that when the Cartesian system is rotated to a new orientation $O\acute{x}\acute{y}\acute{z}$, then equation (4.1.1) takes the following form;

$$\acute{C}_{ijkl} = \acute{\nu}_{ai}\acute{\nu}_{bj}\acute{\nu}_{ck}\acute{\nu}_{dl}A_{abcd} \tag{4.1.2}$$

where ν_1, ν_2, ν_3 form a linearly independent basis in three dimensions but they are not necessarily always orthogonal (it is a general case). The orthogonality condition used in this thesis, is a particular case which has been stated for symmetric second rank tensors in equations (3.1.8) and (3.1.9). This is similar for elastic constant tensor so the corresponding reciprocal triads must satisfy the following relation

$$\nu_{ai}\nu_{aj} = \delta_{ij} \tag{4.1.3}$$

The expression given in equation (4.1.3) can be rewritten as

$$\nu\nu^T = \mathbf{I} \tag{4.1.4}$$

where $\mathbf{I}$ is identity matrix which is $\begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Since $\delta_{ij} = 1$ $(i = j)$ or $\delta_{ij} = 0$ $(i \neq j)$. These are the orthogonality relations which are also defined in equation (4.1.3).

Form invariants for all types of material symmetries are described. It is the necessary step in constructing orthonormal tensor basis of elasticity tensors. By appropriate use of δ_{ij}, elements of the orthonormal tensor basis can be constructed for each symmetry types[47].

Furthermore symmetry in crystal means simply invariance of the properties with respect to the transforms of some subgroup of the orthogonal group, whereas the properties of an isotropic medium are invariant with respect to all the transforms of the orthogonal group. In other words, it explains the form of C_{ijkl} tensor for any isotropic medium and it is invariant with respect to the all transforms of the orthogonal group. However there is a unique tensor that is not affected by all orthogonal transforms, it is a unique tensor, apart from a scalar factor, so C_{ijkl} can be expressed as combinations of the components δ_{ij}

of that tensor with certain coefficients. There are only three different such combinations which contain four subscript i, j, k, l namely $\delta_{ij}\delta_{kl}$, $\delta_{ik}\delta_{jl}$, $\delta_{il}\delta_{jk}$.[47] Because of the symmetry of C_{ijkl}, i and j are permuted. So the elements takes the new form; $\delta_{ij}\delta_{kl}$ and $\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$. For other symmetry types, these elements are used in a suitable form, when constructing orthonomalized basis. Form-invariant expression of isotropic symmetry is formed by the following two basis elements:

$$\delta_{ij}\delta_{kl}, \qquad \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \tag{4.1.5}$$

Following Ghaith[3], the decomposition of C_{ijkl} for tricilinic system with no elastic symmetries is given in terms of its orthonormalized basis elements as

$$C_{ijkl} = \sum_K (C, A^K_{ijkl})A^K_{ijkl}, (K = I...XXI), \tag{4.1.6}$$

where (C, A^K_{ijkl}) represents the inner product of C_{ijkl} and K^{th} elements, A^K_{ijkl}, the orthonormalized basis elements and given for each elastic symmetry types, besides, the inner products for tricilinic symmetry are obtained as

$$(C, A^I_{ijkl}) = \frac{1}{3}[(C_{11} + C_{22} + C_{33}) + 2(C_{12} + C_{13} + C_{23})],$$

$$(C, A^{II}_{ijkl}) = \frac{1}{3\sqrt{5}}[2(C_{11}+C_{22}+C_{33})+6(C_{44}+C_{55}+C_{66})-2(C_{12}+C_{13}+C_{23})],$$

$$(C, A^{III}_{ijkl}) = \frac{1}{\sqrt{6}}[3C_{33} - (C_{11} + C_{22} + C_{33})],$$

$$(C, A^{IV}_{ijkl}) = \frac{1}{\sqrt{3}}[3C_{13} + 3C_{23} - 2(C_{12} + C_{13} + C_{23})],$$

$$(C, A^{V}_{ijkl}) = \frac{1}{\sqrt{30}}[-2(C_{11}+C_{22}+C_{33})+4(C_{44}+C_{55}+C_{66})+2(C_{12}+C_{13}+C_{23})],$$

$$(C, A^{VI}_{ijkl}) = \sqrt{\frac{2}{3}}[C_{44} + C_{55} - 2C_{66}],\ (C, A^{VII}_{ijkl}) = \sqrt{\frac{1}{2}}[C_{11} - C_{22}],$$

$$(C, A^{VIII}_{ijkl}) = [C_{13} - C_{23}],\ (C, A^{IX}_{ijkl}) = \sqrt{2}[C_{44} - C_{55}],\ (C, A^{X}_{ijkl}) = 2\sqrt{2}C_{46},$$

$$(C, A_{ijkl}^{XI}) = 2C_{35},\ (C, A_{ijkl}^{XII}) = 2C_{15},\ (C, A_{ijkl}^{XIII}) = 2C_{25},$$

$$(C, A_{ijkl}^{XIV}) = 2\sqrt{2}C_{45}, (C, A_{ijkl}^{XV}) = 2C_{16},\ (C, A_{ijkl}^{XVI}) = 2C_{26},$$

$$(C, A_{ijkl}^{XVII}) = 2C_{36}, (C, A_{ijkl}^{XVIII}) = 2\sqrt{2}C_{56},\ (C, A_{ijkl}^{XIX}) = 2C_{24},$$

$$(C, A_{ijkl}^{XX}) = 2C_{34}, (C, A_{ijkl}^{XXI}) = 2C_{14}.$$

Here, elastic constants are given in Voigt notation.

4.1.1 For Isotropic Materials

The form invariant expressions are defined for the selected material symmetry types as follows[3]

For isotropic material:

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{4.1.1.1}$$

where λ and μ are invariant elastic constants and they are also called Lame constants and $\alpha_{ijkl} = \delta_{ij}\delta_{kl}$, $\beta_{ijkl} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$. The traditional form of stress-strain relation for isotropic solids can be defined as

$$\sigma_{ij} = \lambda\varepsilon_{rr}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{4.1.1.2}$$

It is also well known that stress tensor is decomposed into spherical and deviatoric parts and it is given as

$$\sigma_{ij} = \frac{1}{3}\sigma_{rr}\delta_{ij} + (\sigma_{ij} - \frac{1}{3}\sigma_{rr}\delta_{ij}). \tag{4.1.1.3}$$

For isotropic materials, the decomposition of C_{ijkl} for the isotropic system is given in terms of the orthonormalized basis elements as

$$C_{ijkl} = \sum_K (C, A_{ijkl}^K)A_{ijkl}^K = (C, A_{ijkl}^I)A_{ijkl}^I + (C, A_{ijkl}^{II})A_{ijkl}^{II}, (K = I, II) \tag{4.1.1.4}$$

where (C, A^{K}_{ijkl}) denotes the inner product of C_{ijkl} and $A^{I}_{ijkl} = \frac{1}{3}\alpha_{ijkl} = \frac{1}{3}\delta_{ij}\delta_{kl}$, $A^{II}_{ijkl} = \frac{1}{6\sqrt{5}}(3\beta_{ijkl} - 2\alpha_{ijkl})$. which are orthonormalized basis elements for isotropic system. The inner products are

$$(C, A^{I}_{ijkl}) = \frac{1}{3}[(C_{11} + C_{22} + C_{33}) + 2(C_{12} + C_{13} + C_{23})],$$

$$(C, A^{II}_{ijkl}) = \frac{1}{6\sqrt{5}}[4(C_{11}+C_{22}+C_{33})+12(C_{44}+C_{55}+C_{66})-4(C_{12}+C_{13}+C_{23})].$$

4.1.2 For Cubic Materials

The form invariant expression for cubic materials[1]

$$C_{ijkl} = \lambda\alpha_{ijkl} + \mu\beta_{ijkl} + \rho\nu_{ai}\nu_{aj}\nu_{ak}\nu_{al} \tag{4.1.2.1}$$

where $\alpha_{ijkl} = \delta_{ij}\delta_{kl}$, $\beta_{ijkl} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$ and $\tau_{ijkl} = \nu_{ai}\nu_{aj}\nu_{ak}\nu_{al}$,

τ_{ijkl} is also rewritten in terms of δ_{ij} as

$\tau_{ijkl} = (\delta_{i1}\delta_{j2}+\delta_{i2}\delta_{j1})(\delta_{k1}\delta_{l2}+\delta_{k2}\delta_{l1})+(\delta_{i2}\delta_{j3}+\delta_{i3}\delta_{j2})(\delta_{k2}\delta_{l3}+\delta_{k3}\delta_{l2})+(\delta_{i1}\delta_{j3}+\delta_{i3}\delta_{j1})(\delta_{k1}\delta_{l3} + \delta_{k3}\delta_{l1})$.

λ, μ and ρ are invariant elastic constants for cubic materials.

The decomposition of C_{ijkl} for cubic system is given in terms of the orthonormalized basis elements as

$$C_{ijkl} = (C, A^{I}_{ijkl})A^{I}_{ijkl} + (C, A^{II}_{ijkl})A^{II}_{ijkl} + (C, A^{III}_{ijkl})A^{III}_{ijkl}, \tag{4.1.2.2}$$

where $A^{III}_{ijkl} = \frac{1}{2\sqrt{30}}(5\tau_{ijkl} - 3\beta_{ijkl} + 2\alpha_{ijkl})$, which is the last orthonormalized basis elements for cubic system.

Since first two orthonormalized basis elements of cubic system are identical with isotropic symmetry, inner products are also same, the other inner product for cubic system is

$(C, A^{III}) = \frac{1}{2\sqrt{30}}[-4(C_{11}+C_{22}+C_{33})+8(C_{44}+C_{55}+C_{66})+4(C_{12}+C_{13}+C_{23})]$.

4.1.3 For Transversely Isotropic Materials

The form invariant expression for transversely isotropic materials[1]

$$C_{ijkl} = \lambda_1\alpha_{ijkl} + \lambda_2\beta_{ijkl} + \lambda_3\gamma_{ijkl} + \lambda_4\delta_{ijkl} + \lambda_5\varepsilon_{ijkl} \tag{4.1.3.1}$$

where $\gamma_{ijkl} = \delta_{3i}\delta_{3j}\delta_{3k}\delta_{3l}, \quad \delta_{ijkl} = \delta_{ij}\delta_{k3}\delta_{l3} + \delta_{i3}\delta_{j3}\delta_{kl}$

and $\varepsilon_{ijkl} = (\delta_{i2}\delta_{j3} + \delta_{i3}\delta_{j2})(\delta_{k2}\delta_{l3} + \delta_{k3}\delta_{l2}) + (\delta_{i1}\delta_{j3} + \delta_{i3}\delta_{j1})(\delta_{k1}\delta_{l3} + \delta_{k3}\delta_{l1})$.

λ_1, λ_2, λ_3, λ_4 and λ_5 are invariant elastic constants for transversely isotropic system.

The decomposition of C_{ijkl} for transversely isotropic system is given in terms of the orthonormalized basis elements as

$$C_{ijkl} = (C, A^{I}_{ijkl})A^{I}_{ijkl} + (C, A^{II}_{ijkl})A^{II}_{ijkl} + (C, A^{III}_{ijkl})A^{III}_{ijkl} + (C, A^{IV}_{ijkl})A^{IV}_{ijkl} + (C, A^{V}_{ijkl})A^{V}_{ijkl} \tag{4.1.3.2}$$

where $A^{III}_{ijkl} = \dfrac{1}{6\sqrt{5}}(15\gamma_{ijkl} - \beta_{ijkl} - \alpha_{ijkl})$,
$A^{IV}_{ijkl} = \dfrac{1}{12}(9\delta_{ijkl} - 15\gamma_{ijkl} + \beta_{ijkl} - 5\alpha_{ijkl})$,
$A^{V}_{ijkl} = \dfrac{1}{4}(2\varepsilon_{ijkl} - \delta_{ijkl} + 3\gamma_{ijkl} - \beta_{ijkl} + \alpha_{ijkl})$ which are orthonormalized basis elements for transversely isotropic system.

Since first two orthonormalized basis elements of transversely isotropic system are the same as isotropic symmetry, inner products are also identical, the other inner products for transversely isotropic system are

$(C, A^{III}) = \dfrac{1}{6\sqrt{5}}[-3C_{11} - 3C_{22} + 12C_{33} - 2C_{12} - 2C_{13} - 2C_{23} - 4C_{44} - 4C_{55} - 4C_{66}]$,

$(C, A^{IV}) = \dfrac{1}{12}[-3C_{11} - 3C_{22} - 10C_{12} + 8C_{13} + 8C_{23} + 4C_{44} + 4C_{55} + 4C_{66}]$,

$(C, A^{V}) = \dfrac{1}{4}[-C_{11} - C_{22} + 2C_{12} + 4C_{44} + 4C_{55} - 4C_{66}]$.

4.1.4 For Tetragonal Materials

The form invariant expression for tetragonal materials[1]

$$C_{ijkl} = \lambda_1\alpha_{ijkl} + \lambda_2\beta_{ijkl} + \lambda_3\gamma_{ijkl} + \lambda_4\delta_{ijkl} + \lambda_5\varepsilon_{ijkl} + \lambda_6\tau_{ijkl} \qquad (4.1.4.1)$$

λ_1, λ_2, λ_3, λ_4, λ_5 and λ_6 are invariant elastic constants for tetragonal system.

The decomposition of C_{ijkl} for tetragonal system is given in terms of the orthonormalized basis elements as

$$C_{ijkl} = (C, A^{I}_{ijkl})A^{I}_{ijkl} + (C, A^{II}_{ijkl})A^{II}_{ijkl} + (C, A^{III}_{ijkl})A^{III}_{ijkl} + (C, A^{IV}_{ijkl})A^{IV}_{ijkl} + (C, A^{V}_{ijkl})A^{V}_{ijkl} + (C, A^{VI}_{ijkl})A^{VI}_{ijkl}, \qquad (4.1.4.2)$$

where $A^{VI}_{ijkl} = \dfrac{1}{2\sqrt{2}}(2\tau_{ijkl} - \varepsilon_{ijkl} - \delta_{ijkl} + 3\gamma_{ijkl} - \beta_{ijkl} + \alpha_{ijkl})$ which is the last orthonormalized basis elements for tetragonal system.

Since first five orthonormalized basis elements of tetragonal system are the same as transversely isotropic symmetry, inner products are also common, the last inner product for tetragonal system is

$$(C, A^{VI}) = \frac{1}{2\sqrt{2}}[-C_{11} + 2C_{12} - C_{22} + 4C_{66}].$$

4.1.5 For Trigonal Materials

For trigonal materials, the form invariant expression is[1,3]

$$C_{ijkl} = \lambda_1\alpha_{ijkl} + \lambda_2\beta_{ijkl} + \lambda_3\gamma_{ijkl} + \lambda_4\delta_{ijkl} + \lambda_5\varepsilon_{ijkl} + \lambda_6\psi_{ijkl} \qquad (4.1.5.1)$$

where $\psi_{ijkl} = (\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1})(\delta_{k1}\delta_{l3} + \delta_{k3}\delta_{l1}) + (\delta_{i1}\delta_{j3} + \delta_{i3}\delta_{j1})(\delta_{k1}\delta_{l2} + \delta_{k2}\delta_{l1}) + (\delta_{i2}\delta_{j3} + \delta_{i3}\delta_{j2})(\delta_{k1}\delta_{l1} - \delta_{k2}\delta_{l2}) + (\delta_{i1}\delta_{j1} - \delta_{i2}\delta_{j2})(\delta_{k2}\delta_{l3} - \delta_{k3}\delta_{l2})$,

λ_1, λ_2, λ_3, λ_4, λ_5 and λ_6 are invariant elastic constants for trigonal system.

The decomposition of C_{ijkl} for trigonal system is given in terms of the orthonormalized basis elements as

$$C_{ijkl} = \sum_K (C, A^K_{ijkl}) A^K_{ijkl}, \ (K = I, .., VI) \tag{4.1.5.2}$$

where $A^{VI}_{ijkl} = \frac{1}{4}\psi_{ijkl}$ which is the last orthonormalized basis elements for trigonal system.

Since first five orthonormalized basis elements of trigonal system are the same as transversely isotropic symmetry, inner products are also common, the last inner product for trigonal system are
$(C, A^{VI}) = 2C_{56} + C_{14} - C_{24}$.

4.1.6 For Orthorhombic Materials

For orthorhombic materials, the form invariant expression is[1]

$$C_{ijkl} = \lambda_1\alpha_{ijkl}+\lambda_2\beta_{ijkl}+\lambda_3\gamma_{ijkl}+\lambda_4\delta_{ijkl}+\lambda_5\varepsilon_{ijkl}+\lambda_6\tau_{ijkl}+\lambda_7\varrho_{ijkl}+\lambda_8\sigma_{ijkl}+\lambda_9\theta_{ijkl} \tag{4.1.6.1}$$

where $\varrho_{ijkl} = \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}, \quad \sigma_{ijkl} = \delta_{i3}\delta_{j3}\delta_{k1}\delta_{l1} + \delta_{i1}\delta_{j1}\delta_{k3}\delta_{l3}$,
$\theta_{ijkl} = (\delta_{i2}\delta_{j3} + \delta_{i3}\delta_{j2})(\delta_{k2}\delta_{l3} + \delta_{k3}\delta_{l2})$.
$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8$ and λ_9 are invariant elastic constants for orthorhombic system.

The decomposition of C_{ijkl} for orthorhombic system is given in terms of its orthonormalized basis elements as

$$C_{ijkl} = \sum_K (C, A^K_{ijkl}) A^K_{ijkl}, (K = I...IX) \tag{4.1.6.2}$$

Where $A^{VII}_{ijkl} = \frac{1}{\sqrt{2}}(2\varrho_{ijkl} + \frac{1}{2}\pi_{ijkl} + \gamma_{ijkl} - \frac{1}{2}\beta_{ijkl})$, $A^{VIII}_{ijkl} = \frac{1}{2}(2\sigma_{ijkl} - \delta_{ijkl} + 2\gamma_{ijkl})$, $A^{IX}_{ijkl} = \frac{1}{2\sqrt{2}}(2\theta_{ijkl} - \varepsilon_{ijkl})$, which are orthonormalized basis elements for orthorhombic system. (C, A^K_{ijkl}) represents the inner product of C_{ijkl} for orthorhombic symmetry and the K^{th} elements, A_{ijkl} of the basis, since first six orthonormalized basis elements of orthorhombic system are the same as tetragonal symmetry, inner products are also common, the other two inner products for orthorhombic materials are

$$(C, A^{VII}) = \frac{1}{\sqrt{2}}[C_{11} - C_{22}],$$

$$(C, A^{VIII}) = C_{13} - C_{23},$$

$$(C, A^{IX}) = \sqrt{2}(C_{44} - C_{55}).$$

4.2 By Irreducible Decomposition Method

Any rank-n cartesian tensor can be written as the direct sum of irreducible tensors in the cartesian representation. The term "irreducible" indicates sets that cannot be resolved into subsets with seperate linear transformations.

The irreducible tensors of the first five ranks have special names[48]; **Scalar** (zero-rank tensor of valance 0), **vector** (first-rank tensor of valence 1.), **deviator** (second-rank tensor of valence 2.), **septor** (third-rank tensor of valence 3.), **nonor** (a fourth-rank tensor of valence 4). The irreducible decomposition method can be investigated under the title of groups and reflection symmetries. The group of rotations associated with elastic symmetry provides an irreducible representation. There are various related ways of considering elasticity tensors in terms of rotational group properties of tensors for example based on subgroups of O(3) or SO(3). These ideas are closely related to definitions of elastic symmetry in terms of a single symmetry element: reflection about a plane.

By looking at Table 4.1, number of components for n-rank tensors can be obtained. So for second-rank tensor, there are three irreducible parts which are 1 scalar, 1 vector and 1 deviator. For third-rank tensor, there are seven irreducible parts which are 1 scalar, 3 vectors, 2 deviators, 1 septor and for fourth-rank tensor, there are 3 scalars, 6 vectors, 6 deviators, 3 septors and 1 nonor.

The reduction of a (rank-n) cartesian tensor $T_{(n)}$ generaly results in a sum of irreducible tensors, with some weights (j) represented more than once. (where $0 \leq j \leq n$), it is can be accomplished by the following formula [22,24,25]:

$$T_{(n)} = \sum_{j=0}^{n} \sum_{q=1}^{N_n^{(j)}} T_{(n)}^{(j;q)},$$

Where q is called the seniority index of the irreducible tensor $T_{(n)}^{(j;q)}$(irreducible cartesian tensor which is symmetric and traceless) and $N_n^{(j)}$ is the multiplicity of weight j in this reduction, it denotes the number of independent weight-j irreducible tensor parts.

$$N_n^{(j)} = \sum_k (-1)^k \binom{n}{k} \binom{2n-3k-j-2}{n-2} \tag{4.2.1}$$

Where $0 \leq k \leq [(n-j)/3]$. Each irreducible tensor has $(2j+1)$ independent components. So that the total number of components in the reduction is

$$\sum_{j=0}^{n} (2j+1) N_n^{(j)} = 3^n$$

Table 4.1: Multiplicities $N_n^{(j)}$.

j \ n	0	1	2	3	4
0	1	0	1	1	3
1		1	1	3	6
2			1	2	6
3				1	3
4					1

These numbers in the table are found by applying the formula given in equation (4.2.1).

The natural projection of x^j onto the irreducible subspace H_j^j of traceless symmetric tensors of order j is denoted by $E^{(j)} = E^{(j)}_{k_1k_2\ldots k_j;l_1l_2\ldots l_j.}$. The principal element in the reduction procedure is the mappings $Q^{(0;q)}_{i_1i_2\ldots i_n;k_1k_2\ldots k_j}$ of the minimal rank tensor subspace $H^j_{j,q}$ onto $H^n_{j,q}$, we will choose the mappings $Q^{(0;q)}_{i_1i_2\ldots i_n;k_1k_2\ldots k_j}$ such that they are orthonormal and g_{pq} will be reduced to identity matrix, δ_{ij}, where g_{pq} is a symmetric matrix which was used and defined in Andrews and Ghoul [24] through the relation;

$$g_{pq}E^{(j)}_{k_1k_2\ldots.k_j;l_1l_2\ldots.l_j}{=}Q^{(0;p)}_{i_1i_2\ldots i_n;k_1k_2\ldots k_j}Q^{(0;q)}_{i_1i_2\ldots i_n;k_1k_2\ldots k_j}. \tag{4.2.2}$$

In this work, this relation is reduced to

$$\delta_{pq}E^{(j)}_{k_1k_2\ldots.k_j;l_1l_2\ldots.l_j}{=}Q^{(0;p)}_{i_1i_2\ldots i_n;k_1k_2\ldots k_j}Q^{(0;q)}_{i_1i_2\ldots i_n;k_1k_2\ldots k_j}. \tag{4.2.3}$$

The mappings $\widetilde{Q}^{(0;p)}_{k_1k_2\ldots k_j;i_1i_2\ldots i_n}$ dual to $Q^{(0;p)}_{i_1i_2\ldots i_n;k_1k_2\ldots k_j}$ are defined by the relation

$$\widetilde{Q}^{(0;p)}_{k_1k_2\ldots k_j;i_1i_2\ldots i_n} = Q^{(0;p)}_{i_1i_2\ldots i_n;k_1k_2\ldots k_j}. \tag{4.2.4}$$

The dual mappings extract the natural forms $t^{(j;p)}_{\lambda_1\lambda_2\ldots\lambda_j}$ from the tensor $T_{i_1i_2\ldots i_n}$ as

$$t^{(j;p)}_{\lambda_1\lambda_2\ldots\lambda_j} = \widetilde{Q}^{(0;p)}_{\lambda_1\lambda_2\ldots\lambda_j;i_1i_2\ldots i_n}T_{i_1i_2\ldots i_n}. \tag{4.2.5}$$

These tensors can be embedded in the tensor space of order n through the mapping

$$T^{(j;q)}_{i_1i_2\ldots i_n} = Q^{(0;q)}_{i_1i_2\ldots i_n;k_1k_2\ldots k_j}t^{(j;p)}_{k_1k_2\ldots k_j}, \tag{4.2.6}$$

or

$$T^{(j;q)}_{i_1i_2\ldots i_n} = Q^{(0;q)}_{i_1i_2\ldots i_n;k_1k_2\ldots k_j}\widetilde{Q}^{(0;p)}_{k_1k_2\ldots k_j;l_1l_2\ldots l_n}T_{l_1l_2\ldots l_n}. \tag{4.2.7}$$

Before giving the results of the reduction of elastic constant tensor, it was illustrated that the orthogonality condition is satisfied during the decomposition procedure. [25]

Following Radwan[25], elastic constant tensor is decomposed as;

$$C_{ijkl} = C_{ijkl}^{(0;1)} + C_{ijkl}^{(0;2)} + C_{ijkl}^{(2;1)} + C_{ijkl}^{(2;2)} + C_{ijkl}^{(4;1)} \tag{4.2.8}$$

where

$$C_{ijkl}^{(0;1)} = \frac{1}{9}\delta_{ij}\delta_{kl}C_{ppqq}, \tag{4.2.9}$$

$$C_{ijkl}^{(0;2)} = \frac{1}{90}(3\delta_{ik}\delta_{jl} + 3\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl})(3C_{pqpq} - C_{ppqq}), \tag{4.2.10}$$

$$C_{ijkl}^{(2;1)} = \frac{1}{5}(\delta_{ik}C_{jplp} + \delta_{ik}C_{jplp} + \delta_{il}C_{jpkp} + \delta_{il}C_{ipkp}) - \frac{2}{15}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})C_{pqpq}, \tag{4.2.11}$$

$$\begin{aligned} C_{ijkl}^{(2;2)} = {} & \frac{1}{7}\delta_{ij}(5C_{klpp} - 4C_{kplp}) + \frac{1}{7}\delta_{kl}(5C_{ijpp} - 4C_{ipjp}) - \frac{2}{35}\delta_{ik}(5C_{jlpp} - 4C_{jplp}) \\ & - \frac{2}{35}\delta_{jl}(5C_{ikpp} - 4C_{ipkp}) - \frac{2}{35}\delta_{il}(5C_{jkpp} - 4C_{jpkp}) - \frac{2}{35}\delta_{jk}(5C_{ilpp} - 4C_{iplp}) \\ & + \frac{2}{105}(2\delta_{jk}\delta_{il} + 2\delta_{ik}\delta_{jl} - 5\delta_{ij}\delta_{kl})(5C_{ppqq} - 4C_{pqpq}), \end{aligned} \tag{4.2.12}$$

$$\begin{aligned} C_{ijkl}^{(4;1)} = {} & (C_{ijkl} + C_{iklj} + C_{iljk})/3 - [(C_{ijmm} + 2C_{imjm})\delta_{kl} + (C_{klmm} + 2C_{kmlm})\delta_{ij} + \\ & (C_{ikmm} + 2C_{imkm})\delta_{jl} + (C_{jlmm} + 2C_{jmlm})\delta_{ik} + (C_{ilmm} + 2C_{imlm})\delta_{jk} + (C_{jkmm} + \\ & C_{jmkm})\delta_{il}]/21 + (C_{ppmm} + 2C_{pmpm})(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/105. \end{aligned} \tag{4.2.13}$$

This is the decomposition for elastic constant tensor in triclinic materials which are anisotropic materials with no elastic symmetries.

4.2.1 For Isotropic Materials

There are only two irreducible parts for isotropic materials which are the scalar parts; $C_{ijkl}^{(0;1)}$ and $C_{ijkl}^{(0;2)}$ mentioned in equations (4.2.9) and (4.2.10). By rewriting these parts in matrix form, we get

$$C_{mn}^{(0;1)} = K \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C_{mn}^{(0;2)} = \frac{G}{3} \begin{bmatrix} 4 & -2 & -2 & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & 0 & 0 \\ -2 & -2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix},$$

If we add those two parts, isotropic elastic constant tensor is obtained by irreducible decomposition method, isotropic elastic constant tensor can be rewritten instead of traditional form given in equation (4.1.1.1) as follows

$$C_{ijkl} = K\delta_{ij}\delta_{kl} + 2G(\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl}) \tag{4.2.1.1}$$

Where $K = (C_{11} + 2C_{12})/3$ and $G = \mu = (C_{11} - C_{12})/2$, where K is the bulk modulus and G is the shear modulus. In equation (4.2.1.1) the decomposed parts are orthogonal to each other whereas the decomposed parts in traditional form given in equation (4.1.1.1) are not orthogonal. From equation (2.1) and (4.2.1.1), the stress-strain relation, for isotropic materials, can be obtained as

$$\sigma_{ij} = K\varepsilon_{rr}\delta_{ij} + 2G(\varepsilon_{ij} - \frac{1}{3}\varepsilon_{rr}\delta_{ij}) \qquad (4.2.1.2)$$

This is also different than the traditional form given in equation (4.1.1.2) and it is a new form in stress-strain relations in the literature.

Equation (4.2.1.2) was also obtained in [49] by a different method which was based on the expansion of the strain energy density function in powers of ε_{ij}.

4.2.2 For Cubic Materials

For cubic symmetry, there are three irreducible parts which are two scalar parts; $C^{(0;1)}_{ijkl}$, $C^{(0;2)}_{ijkl}$ (given in equations (4.2.9) and (4.2.10)), and a nonor part $C^{(4;1)}_{ijkl}$ (4.2.13). By arranging and rewriting these parts, we obtain

$$C^{(0;1)}_{ijkl} = \frac{1}{9}\delta_{ij}\delta_{kl}(C_{11} + 2C_{12}), \qquad (4.2.2.1)$$

$$C^{(0;2)}_{ijkl} = \frac{1}{15}(3\delta_{ik}\delta_{jl} + 3\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl})(C_{11} - C_{12} + 3C_{44}), \qquad (4.2.2.2)$$

$$C^{(4;1)}_{ijkl} = \frac{1}{5}(5\delta_{ij}\delta_{kl}\delta_{jk} - \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl})(C_{11} - C_{12} - 2C_{44}), \qquad (4.2.2.3)$$

If we add equations (4.2.2.1), (4.2.2.2) and (4.2.2.3), elastic constant tensor for cubic material is obtained by irreducible decomposition method as

$$C_{ijkl} = C^{(0;1)}_{ijkl} + C^{(0;2)}_{ijkl} + C^{(4;1)}_{ijkl} \qquad (4.2.2.4)$$

4.2.3 For Transversely Isotropic Materials

There are five irreducible parts for transversely isotropic materials which are two scalar parts; $C^{(0;1)}_{ijkl}$, $C^{(0;2)}_{ijkl}$, two deviators; $C^{(2;1)}_{ijkl}$, $C^{(2;2)}_{ijkl}$, and a nonor part $C^{(4;1)}_{ijkl}$, mentioned in equations (4.2.9), (4.2.10); (4.2.11), (4.2.12) and (4.2.13) respectively. These parts are as follows:

$$C^{(0;1)}_{ijkl} = \frac{a}{9}\delta_{ij}\delta_{kl,} \qquad (4.2.3.1)$$

$$C_{ijkl}^{(0;2)} = \frac{b}{90}\left(3\delta_{ik}\delta_{jl} + 3\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}\right), \tag{4.2.3.2}$$

$$C_{ijkl}^{(2;1)} = \frac{c}{30}\left(2\delta_{ik}\delta_{jl} + 2\delta_{il}\delta_{jk} - 3\delta_{3i}\delta_{3k}\delta_{jl} - 3\delta_{3j}\delta_{3k}\delta_{il} + \delta_{3i}\delta_{3l}\delta_{jk} + \delta_{3j}\delta_{3l}\delta_{ik}\right), \tag{4.2.3.3}$$

$$C_{ijkl}^{(2;2)} = \frac{d}{105}(-4\delta_{ik}\delta_{jl} - 4\delta_{il}\delta_{jk} + 10\delta_{ij}\delta_{kl} - 15\delta_{3i}\delta_{3j}\delta_{kl} - 15\delta_{3k}\delta_{3l}\delta_{ij} + 6\delta_{3i}\delta_{3k}\delta_{jl} + 6\delta_{3j}\delta_{3l}\delta_{ik} + 6\delta_{3j}\delta_{3k}\delta_{il} + 6\delta_{3i}\delta_{3l}\delta_{jk}), \tag{4.2.3.4}$$

$$C_{ijkl}^{(4;1)} = \frac{e}{35}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \delta_{ij}\delta_{kl} - 5\delta_{3i}\delta_{3j}\delta_{kl} - 5\delta_{3k}\delta_{3l}\delta_{ij} - 5\delta_{3i}\delta_{3k}\delta_{jl} - 5\delta_{3j}\delta_{3k}\delta_{il} - 5\delta_{3i}\delta_{3l}\delta_{jk} - 5\delta_{3j}\delta_{3l}\delta_{ik} + 35\delta_{3i}\delta_{3j}\delta_{3k}\delta_{3l}). \tag{4.2.3.5}$$

Where $a = 2C_{11} + 2C_{12} + 4C_{13} + C_{33}$, $b = 7C_{11} - 5C_{12} + 2C_{33} + 12C_{44} - 4C_{13}$, $c = 3C_{11} - 2C_{33} - 2C_{44} - C_{12}$, $d = 7C_{12} - C_{33} - C_{11} - 5C_{13} + 4C_{44}$ and $e = C_{11} + C_{33} - 2C_{13} - 4C_{44}$.

If we add those five parts, elastic constant tensor for transversely isotropic material is obtained by irreducible decomposition method as

$$C_{ijkl} = C_{ijkl}^{(0;1)} + C_{ijkl}^{(0;2)} + C_{ijkl}^{(2;1)} + C_{ijkl}^{(2;2)} + C_{ijkl}^{(4;1)} \tag{4.2.3.6}$$

4.2.4 For Tetragonal Materials

For tetragonal materials, there are five irreducible parts which are two scalar parts; $C_{ijkl}^{(0;1)}$, $C_{ijkl}^{(0;2)}$, two deviators; $C_{ijkl}^{(2;1)}$, $C_{ijkl}^{(2;2)}$, and a nonor part $C_{ijkl}^{(4;1)}$ (given in equations (4.2.9), (4.2.10), (4.2.11), (4.2.12) and (4.2.13) respectively). These parts are arranged and rewritten to obtain the followings:

$$C_{ijkl}^{(0;1)} = \frac{a}{9}\delta_{ij}\delta_{kl}, \tag{4.2.4.1}$$

$$C_{ijkl}^{(0;2)} = \frac{b}{45}\left(3\delta_{ik}\delta_{jl} + 3\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}\right), \tag{4.2.4.2}$$

$$C_{ijkl}^{(2;1)} = \frac{c}{15}\left(2\delta_{ik}\delta_{jl} + 2\delta_{il}\delta_{jk} - 3\delta_{3i}\delta_{3k}\delta_{jl} - 3\delta_{3j}\delta_{3k}\delta_{il} + \delta_{3i}\delta_{3l}\delta_{jk} + \delta_{3j}\delta_{3l}\delta_{ik}\right), \tag{4.2.4.3}$$

$$C_{ijkl}^{(2;2)} = \frac{d}{105}(-4\delta_{ik}\delta_{jl} - 4\delta_{il}\delta_{jk} + 10\delta_{ij}\delta_{kl} - 15\delta_{3i}\delta_{3j}\delta_{kl} - 15\delta_{3k}\delta_{3l}\delta_{ij} + 6\delta_{3i}\delta_{3k}\delta_{jl} + 6\delta_{3j}\delta_{3l}\delta_{ik} + 6\delta_{3j}\delta_{3k}\delta_{il} + 6\delta_{3i}\delta_{3l}\delta_{jk}), \tag{4.2.4.4}$$

$$C_{ijkl}^{(4;1)} = \frac{1}{35}(e(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \delta_{ij}\delta_{kl}) - (e+f)(\delta_{3i}\delta_{3j}\delta_{kl} + \delta_{3k}\delta_{3l}\delta_{ij}) - (e+f)(\delta_{3i}\delta_{3k}\delta_{jl} + \delta_{3j}\delta_{3k}\delta_{il} + \delta_{3i}\delta_{3l}\delta_{jk} + \delta_{3j}\delta_{3l}\delta_{ik}) + 35(C_{33} + C_{12} + 2C_{66} - 2C_{13} - 4C_{44})\delta_{3i}\delta_{3j}\delta_{3k}\delta_{3l} + 35(C_{11} - C_{12} - 2C_{66})(\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l} + \delta_{2i}\delta_{2j}\delta_{2k}\delta_{2l})). \tag{4.2.4.5}$$

Where $a = 2C_{11} + 2C_{12} + 4C_{13} + C_{33}$, $b = 2C_{11} - C_{12} + C_{33} + 3C_{66} + 6C_{44} - 2C_{13}$, $c = C_{11} - C_{33} - C_{44} + C_{66}$, $d = C_{11} - C_{33} + 5C_{12} - 5C_{13} + 4C_{44} - 4C_{66}$ and

$e = 9C_{12} + 18C_{66} - 8C_{11} - 2C_{13} - 4C_{44} + C_{33}$, $f = 4C_{33} + 3C_{11} - 8C_{13} + C_{12} - 16C_{44} + 2C_{66}$.

The sum of those five parts gives elastic constant tensor for tetragonal material, obtained by irreducible decomposition method as

$$C_{ijkl} = C_{ijkl}^{(0;1)} + C_{ijkl}^{(0;2)} + C_{ijkl}^{(2;1)} + C_{ijkl}^{(2;2)} + C_{ijkl}^{(4;1)} \tag{4.2.4.6}$$

4.2.5 For Trigonal Materials

For trigonal materials, there are five irreducible parts which are two scalar parts; $C_{ijkl}^{(0;1)}$ (4.2.9), $C_{ijkl}^{(0;2)}$ (4.2.10), two deviators; $C_{ijkl}^{(2;1)}$ (4.2.11), $C_{ijkl}^{(2;2)}$ (4.2.12), and a nonor part $C_{ijkl}^{(4;1)}$(4.2.13). These parts are arranged and then we obtain

$$C_{ijkl}^{(0;1)} = \frac{a}{9}\delta_{ij}\delta_{kl}, \tag{4.2.5.1}$$

$$C_{ijkl}^{(0;2)} = \frac{b}{90}\left(3\delta_{ik}\delta_{jl} + 3\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}\right), \tag{4.2.5.2}$$

$$C_{ijkl}^{(2;1)} = \frac{c}{30}\left(2\delta_{ik}\delta_{jl} + 2\delta_{il}\delta_{jk} - 3\delta_{3i}\delta_{3k}\delta_{jl} - 3\delta_{3j}\delta_{3k}\delta_{il} + \delta_{3i}\delta_{3l}\delta_{jk} + \delta_{3j}\delta_{3l}\delta_{ik}\right), \tag{4.2.5.3}$$

$$C_{ijkl}^{(2;2)} = \frac{d}{105}(-4\delta_{ik}\delta_{jl} - 4\delta_{il}\delta_{jk} + 10\delta_{ij}\delta_{kl} - 15\delta_{3i}\delta_{3j}\delta_{kl} - 15\delta_{3k}\delta_{3l}\delta_{ij} + 6\delta_{3i}\delta_{3k}\delta_{jl} + 6\delta_{3j}\delta_{3l}\delta_{ik} + 6\delta_{3j}\delta_{3k}\delta_{il} + 6\delta_{3i}\delta_{3l}\delta_{jk}), \tag{4.2.5.4}$$

$$C_{ijkl}^{(4;1)} = \frac{e}{35}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \delta_{ij}\delta_{kl} - 5\delta_{3i}\delta_{3j}\delta_{kl} - 5\delta_{3k}\delta_{3l}\delta_{ij} - 5\delta_{3i}\delta_{3k}\delta_{jl} - 5\delta_{3j}\delta_{3k}\delta_{il} - 5\delta_{3i}\delta_{3l}\delta_{jk} - 5\delta_{3j}\delta_{3l}\delta_{ik} + 35\delta_{3i}\delta_{3j}\delta_{3k}\delta_{3l}) + C_{14}(\delta_{1i}\delta_{1j}\delta_{2k}\delta_{3l} + \delta_{1i}\delta_{1j}\delta_{3k}\delta_{2l} + \delta_{3i}\delta_{2j}\delta_{1k}\delta_{1l} + \delta_{2i}\delta_{3j}\delta_{1k}\delta_{1l} + \delta_{3i}\delta_{1j}\delta_{2k}\delta_{1l} + \delta_{3i}\delta_{1j}\delta_{1k}\delta_{2l} + \delta_{1i}\delta_{3j}\delta_{2k}\delta_{1l} + \delta_{1i}\delta_{3j}\delta_{1k}\delta_{2l} + \delta_{1i}\delta_{2j}\delta_{1k}\delta_{3l} + \delta_{1i}\delta_{2j}\delta_{3k}\delta_{1l} + \delta_{2i}\delta_{1j}\delta_{1k}\delta_{3l} + \delta_{2i}\delta_{1j}\delta_{3k}\delta_{1l} - \delta_{2i}\delta_{2j}\delta_{2k}\delta_{3l} - \delta_{2i}\delta_{2j}\delta_{3k}\delta_{2l} - \delta_{3i}\delta_{2j}\delta_{2k}\delta_{2l} - \delta_{2i}\delta_{3j}\delta_{2k}\delta_{2l}), \tag{4.2.5.5}$$

where $a = 2C_{11} + 2C_{12} + 4C_{13} + C_{33}$, $b = 7C_{11} - 5C_{12} + 2C_{33} + 12C_{44} - 4C_{13}$, $c = 3C_{11} - 2C_{33} - 2C_{44} - C_{12}$, $d = 7C_{12} - C_{33} - C_{11} - 5C_{13} + 4C_{44}$ and $e = C_{11} + C_{33} - 2C_{13} - 4C_{44}$.

If we add those five parts, elastic constant tensor for trigonal material, represented by irreducible decomposition method as

$$C_{ijkl} = C_{ijkl}^{(0;1)} + C_{ijkl}^{(0;2)} + C_{ijkl}^{(2;1)} + C_{ijkl}^{(2;2)} + C_{ijkl}^{(4;1)} \tag{4.2.5.6}$$

4.2.6 For Orthorhombic Materials

There are five irreducible parts for orthorhombic materials which are two scalar parts; $C_{ijkl}^{(0;1)}$ (4.2.9), $C_{ijkl}^{(0;2)}$ (4.2.10), two deviators; $C_{ijkl}^{(2;1)}$ (4.2.11), $C_{ijkl}^{(2;2)}$ (4.2.12), and a nonor part $C_{ijkl}^{(4;1)}$(4.2.13). These parts are rewritten as

$$C_{ijkl}^{(0;1)} = \frac{a}{9}\delta_{ij}\delta_{kl}, \tag{4.2.6.1}$$

$$C_{ijkl}^{(0;2)} = \frac{b}{45}\left(3\delta_{ik}\delta_{jl} + 3\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}\right), \tag{4.2.6.2}$$

$$C_{ijkl}^{(2;1)} = \frac{1}{15}\left(2j(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + (2c+d)(\delta_{1i}\delta_{1k}\delta_{jl} + \delta_{1j}\delta_{1k}\delta_{il} + \delta_{1i}\delta_{1l}\delta_{jk} + \delta_{1j}\delta_{1l}\delta_{ik}) + \right.$$
$$(d-j)(\delta_{2i}\delta_{2k}\delta_{jl} + \delta_{2j}\delta_{2k}\delta_{il} + \delta_{2i}\delta_{2l}\delta_{jk} + \delta_{2j}\delta_{2l}\delta_{ik}),$$

(4.2.6.3)

$$C_{ijkl}^{(2;2)} = \frac{1}{105}(-4k(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + 10k\delta_{ij}\delta_{kl} - (4e+2f)(\delta_{1i}\delta_{1k}\delta_{jl} + \delta_{1j}\delta_{1k}\delta_{il} +$$
$$\delta_{1i}\delta_{1l}\delta_{jk} + \delta_{1j}\delta_{1l}\delta_{ik}) + (4k+2e)(\delta_{2i}\delta_{2k}\delta_{jl} + \delta_{2j}\delta_{2k}\delta_{il} + \delta_{2i}\delta_{2l}\delta_{jk} + \delta_{2j}\delta_{2l}\delta_{ik}) +$$
$$(5f-5k)(\delta_{2k}\delta_{2l}\delta_{ij} + \delta_{2i}\delta_{2j}\delta_{kl}) + (5e-5k)(\delta_{1k}\delta_{1l}\delta_{ij} + \delta_{1i}\delta_{1j}\delta_{kl}),$$

(4.2.6.4)

$$C_{ijkl}^{(4;1)} = \frac{1}{35}(-(n+\imath)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + (3n+2\imath)\delta_{ij}\delta_{kl} + (\imath-g)(\delta_{1i}\delta_{1k}\delta_{jl} + \delta_{1j}\delta_{1k}\delta_{il} +$$
$$\delta_{1i}\delta_{1l}\delta_{jk} + \delta_{1j}\delta_{1l}\delta_{ik}) + (\imath-h)(\delta_{2i}\delta_{2k}\delta_{jl} + \delta_{2j}\delta_{2k}\delta_{il} + \delta_{2i}\delta_{2l}\delta_{jk} + \delta_{2j}\delta_{2l}\delta_{ik}) +$$
$$(2m-3n+3g)(\delta_{2k}\delta_{2l}\delta_{ij} + \delta_{2i}\delta_{2j}\delta_{kl}) + (2l-3n+3h)(\delta_{1k}\delta_{1l}\delta_{ij} + \delta_{1i}\delta_{1j}\delta_{kl}) +$$
$$35(C_{12} + C_{33} + 2C_{66} - C_{23} - C_{13} - 2C_{44} - 2C_{55})(\delta_{1i}\delta_{1j}\delta_{2k}\delta_{2l} + \delta_{2i}\delta_{2j}\delta_{1k}\delta_{1l}) +$$
$$35(C_{11} + C_{33} - 2C_{13} - 4C_{55})\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l} + 35(C_{33} + C_{22} - 2C_{23} - 4C_{44})\delta_{2i}\delta_{2j}\delta_{2k}\delta_{2l}.$$

(4.2.6.5)

where $a = C_{11} + C_{22} + C_{33} + 2C_{12} + 2C_{13} + 2C_{23}$,

$b = C_{11} + C_{22} + C_{33} - C_{12} - C_{13} - C_{23} + 3C_{44} + 3C_{55} + 3C_{66}$,

$c = 2C_{11} - C_{22} - C_{33} - 2C_{44} + C_{55} + C_{66}$,

$d = 2C_{22} - C_{11} - C_{33} + C_{44} - 2C_{55} + C_{66}$,

$e = 2C_{11} - C_{22} - C_{33} + 5C_{12} + 5C_{13} - 10C_{23} + 8C_{44} - 4C_{55} - 4C_{66}$,

$f = 2C_{22} - C_{11} - C_{33} + 5C_{12} - 10C_{13} + 5C_{23} - 4C_{44} + 8C_{55} - 4C_{66}$,

$g = C_{11} - 4C_{22} - 4C_{33} - C_{12} - C_{13} + 9C_{23} + 18C_{44} - 2C_{55} - 2C_{66}$,

$h = C_{22} - 4C_{11} - 4C_{33} - C_{12} + 9C_{13} - C_{23} - 2C_{44} + 18C_{55} - 2C_{66},$

$\imath = C_{33} - 4C_{11} - 4C_{22} + 9C_{12} - C_{13} - C_{23} - 2C_{44} - 2C_{55} + 18C_{66},$

$j = -(c + d), \quad k = -(e + f), \quad l = -(h + \imath), \quad m = -(j + \imath),$

$n = -(j + h).$

If we add those five parts, elastic constant tensor for orthorhombic material, obtained by irreducible decomposition method as

$$C_{ijkl} = C_{ijkl}^{(0;1)} + C_{ijkl}^{(0;2)} + C_{ijkl}^{(2;1)} + C_{ijkl}^{(2;2)} + C_{ijkl}^{(4;1)} \tag{4.2.6.6}$$

4.3 By Harmonic Decomposition Method

In harmonic decomposition, the action of $SO(3)$ on a vector space is said to be irreducible when there are no proper invariant subspaces. It is deduced that there is a decomposition of the space of elastic constant tensors (**E**la) into a direct sum of orthogonal subspaces on which the action of $SO(3)$ is irreducible. An important theorem of group representation theory can be summarized as: every space on which the group of rotations acts irreducibly is isomorphic through an $SO(3)$-invariant map with an appropriate space of harmonic polynomials. In view of isomorphism, there is a decomposition of **E**la into a direct sum of spaces of harmonic tensors.[30,31]

Besides, there is an $SO(3)$-invariant isomorphism between **E**la and the direct sum

R⊕R⊕Dev⊕Dev⊕**H**rm. We give a brief description for this method as follows:

Any elastic constant tensor with fourth rank in three dimensions, can be written in the form

$$\mathbf{C} = \mathbf{S} + \mathbf{A} \tag{4.3.1}$$

Elastic constant tensor is decomposed into totally symmetric and asymmetric parts[35,36]:

symmetric part is expressed as

$$\mathbf{S}_{ijkl} = \frac{1}{3}(C_{ijkl} + C_{iklj} + C_{iljk}), \tag{4.3.2}$$

asymmetric part is expressed as

$$\mathbf{A}_{ijkl} = C_{ijkl} - \mathbf{S}_{ijkl} = \frac{2}{3}C_{ijkl} - \frac{1}{3}C_{iklj} - \frac{1}{3}C_{iljk}. \tag{4.3.3}$$

The total symmetric part is rewritten in terms of H and H_{ij}:

$$\mathbf{S}_{ijkl} = H_{ijkl} + [\delta_{ij}H_{kl} + \delta_{kl}H_{ij} + \delta_{ik}H_{jl} + \delta_{jl}H_{ik} + \delta_{il}H_{jk} + \delta_{jk}H_{il}] + H(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{4.3.4}$$

The total asymmetric part is rewritten in terms of h and h_{ij} is

$$\mathbf{A}_{ijkl} = \delta_{ij}(h_{kl} + \frac{1}{2}h\delta_{kl}) + \delta_{kl}(h_{ij} + \frac{1}{2}h\delta_{ij}) - \frac{1}{2}\delta_{ik}(h_{jl} + \frac{1}{2}h\delta_{jl}) - \frac{1}{2}\delta_{jl}(h_{ik} + \frac{1}{2}h\delta_{ik}) - \frac{1}{2}\delta_{il}(h_{jk} + \frac{1}{2}h\delta_{jk}) - \frac{1}{2}\delta_{jk}(h_{il} + \frac{1}{2}h\delta_{il})$$

$$= \delta_{ij}h_{kl} + \delta_{kl}h_{ij} - \frac{1}{2}\delta_{ik}h_{jl} - \frac{1}{2}\delta_{jl}h_{ik} - \frac{1}{2}\delta_{il}h_{jk} - \frac{1}{2}\delta_{jk}h_{il} + h(\delta_{ij}\delta_{kl} - \frac{1}{2}\delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{il}\delta_{jk}). \tag{4.3.5}$$

By adding equations (4.3.4) and (4.3.5) gives the decomposition of elastic constant tensor for anisotropic materials possessing triclinic symmetry, which is

$$C_{ijkl} = H_{ijkl} + [\delta_{ij}H_{kl} + \delta_{kl}H_{ij} + \delta_{ik}H_{jl} + \delta_{jl}H_{ik} + \delta_{il}H_{jk} + \delta_{jk}H_{il}] + H[\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}] + \delta_{ij}h_{kl} + \delta_{kl}h_{ij} - \frac{1}{2}\delta_{ik}h_{jl} - \frac{1}{2}\delta_{jl}h_{ik} - \frac{1}{2}\delta_{il}h_{jk} - \frac{1}{2}\delta_{jk}h_{il} + h(\delta_{ij}\delta_{kl} - \frac{1}{2}\delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{il}\delta_{jk}), \tag{4.3.6}$$

where $H = \frac{1}{45}(C_{ppqq} + 2C_{pqpq}), \quad h = \frac{1}{9}(C_{ppqq} - C_{pqpq})$,

$H_{ij} = \frac{1}{21}\left[C_{ijkk} - \frac{1}{3}C_{jjkk}\delta_{ij} + 2\left(C_{ikjk} - \frac{1}{3}C_{jkjk}\delta_{ij}\right)\right]$,

$h_{ij} = \frac{2}{3}(C_{ijpp} - C_{ipjp}) - \frac{2}{9}\delta_{ij}(C_{rrpp} - C_{rprp})$.

In this method, we use the notation of Baerheim[36].

The total scalar (isotropic) part of harmonic decomposition (obtained from equation (4.3.6)) is (denoted as S)

$$S = \frac{\delta_{ij}\delta_{kl}}{15}(2C_{ppqq} - C_{pqpq}) + \frac{(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})}{30}(3C_{pqpq} - C_{ppqq}). \tag{4.3.7}$$

Furthermore the total deviatoric part or second rank traceless tensor is composed of summation of the linear combination of second order tensors(H_{ij} and h_{ij}) given in equation (4.3.6), which is (denoted as D)

$$D = \frac{5}{7}\delta_{kl}C_{ijpp} + \frac{5}{7}\delta_{ij}C_{klpp} - \frac{2}{7}\delta_{ik}C_{jlpp} - \frac{2}{7}\delta_{il}C_{jkpp} - \frac{2}{7}\delta_{jl}C_{ikpp} - \frac{2}{7}\delta_{jk}C_{ilpp} - \frac{4}{7}\delta_{kl}C_{ipjp} - \frac{4}{7}\delta_{ij}C_{kplp} + \frac{3}{7}\delta_{jk}C_{iplp} + \frac{3}{7}\delta_{il}C_{jpkp} + \frac{3}{7}\delta_{ik}C_{jplp} + \frac{3}{7}\delta_{jl}C_{ipkp} - \frac{10}{21}\delta_{ij}\delta_{kl}C_{pprr} + \frac{4}{21}\delta_{ik}\delta_{jl}C_{pprr} + \frac{4}{21}\delta_{il}\delta_{jk}C_{pprr} + \frac{8}{21}\delta_{ij}\delta_{kl}C_{rprp} - \frac{2}{7}\delta_{jl}\delta_{ik}C_{rprp} - \frac{2}{7}\delta_{il}\delta_{jk}C_{rprp}. \tag{4.3.8}$$

Equation (4.3.8) is simplified in order to obtain the following form:

$$D = \frac{1}{7}\delta_{kl}(5C_{ijpp} - 4C_{ipjp}) + \frac{1}{7}\delta_{ij}(5C_{klpp} - 4C_{kplp}) + \frac{1}{7}\delta_{ik}(3C_{jplp} - 2C_{jlpp}) + \frac{1}{7}\delta_{il}(3C_{jpkp} - 2C_{jkpp}) + \frac{1}{7}\delta_{jl}(3C_{ipkp} - 2C_{ikpp}) + \frac{1}{7}\delta_{jk}(3C_{iplp} - 2C_{ilpp}) + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\left(\frac{4}{21}C_{pprr} - \frac{2}{7}C_{rprp}\right) + \delta_{ij}\delta_{kl}(\frac{8}{21}C_{rprp} - \frac{10}{21}C_{pprr}). \tag{4.3.9}$$

From equation (4.3.6), harmonic (nonor) part is obtained as

$$H_{ijkl} = (C_{ijkl}+C_{iklj}+C_{iljk})/3-[(C_{ijmm}+2C_{imjm})\delta_{kl}+(C_{klmm}+2C_{kmlm})\delta_{ij}+ (C_{ikmm} + 2C_{imkm})\delta_{jl} + (C_{jlmm} + 2C_{jmlm})\delta_{ik} + (C_{ilmm} + 2C_{imlm})\delta_{jk}+ (C_{jkmm} + C_{jmkm})\delta_{il}]/21 + (C_{ppmm} + 2C_{pmpm})(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/105. \tag{4.3.10}$$

Moreover, the results for elastic constant tensor decomposition are given by Onat[27,28] and quoted by Cowin[29], Forte and Vianello[30,31], Ting and

He[32], Zheng[33], Annin and Ostrosablin[34] in which elastic constant tensor is decomposed into two scalar, two deviatoric and nonor parts. These decompositions are the same as harmonic decomposition method since scalar, deviatoric and nonor parts are common and they are identical with those obtained from harmonic decomposition method, only difference here is notations used for scalar, traceless symmetric second rank tensors and nonor parts.

According to these studies, decomposition of elastic constant tensor for anisotropic materials possessing triclinic symmetry is expressed as follows:

$$C_{ijkl} = \frac{1}{15}(2C_{ppqq} - C_{pqpq})\delta_{ij}\delta_{kl} + \frac{1}{30}(3C_{pqpq} - C_{ppqq})(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \delta_{ij}A_{kl} + \delta_{kl}A_{ij} + \delta_{ik}B_{jl} + \delta_{jl}B_{ik} + \delta_{il}B_{jk} + \delta_{jk}B_{il} + Z_{ijkl}, \tag{4.3.11}$$

From equation (4.3.11), total scalar part is

$$\frac{1}{15}(2C_{ppqq} - C_{pqpq})\delta_{ij}\delta_{kl} + \frac{1}{30}(3C_{pqpq} - C_{ppqq})(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{4.3.12}$$

The total deviatoric part is

$$\delta_{ij}A_{kl} + \delta_{kl}A_{ij} + \delta_{ik}B_{jl} + \delta_{jl}B_{ik} + \delta_{il}B_{jk} + \delta_{jk}B_{il}. \tag{4.3.13}$$

The components of deviatoric part are

$A_{ij} = (15C_{ijkk} - 12C_{ikjk} - 5\delta_{ij}C_{ppkk} + 4\delta_{ij}C_{pkpk})/21,$

$B_{ij} = (-6C_{ijkk} + 9C_{ikjk} + 2\delta_{ij}C_{ppkk} - 3\delta_{ij}C_{pkpk})/21.$

Finally nonor part is

$$Z_{ijkl} = (C_{ijkl}+C_{iklj}+C_{iljk})/3-[(C_{ijmm}+2C_{imjm})\delta_{kl}+(C_{klmm}+2C_{kmlm})\delta_{ij}+ (C_{ikmm} + 2C_{imkm})\delta_{jl} + (C_{jlmm} + 2C_{jmlm})\delta_{ik} + (C_{ilmm} + 2C_{imlm})\delta_{jk}+ (C_{jkmm} + C_{jmkm})\delta_{il}]/21 + (C_{ppmm} + 2C_{pmpm})(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/105 \tag{4.3.14}$$

Furthermore the decomposition of elastic constant tensor given in Forte and Vianello[30,31] contain misprints in components of scalar part and total deviatoric part. In equations (4.3.12) and (4.3.13), these parts are corrected.

4.3.1 For Isotropic Materials

Like irreducible decomposition method, there are two irreducible parts constituted total isotropic (scalar) parts which are found by using equations (4.3.11) and (4.3.12) and these decomposed parts are

$$\frac{1}{15}(2C_{ppqq} - C_{pqpq})\delta_{ij}\delta_{kl} \tag{4.3.1.1}$$

and

$$\frac{1}{30}(3C_{pqpq} - C_{ppqq})(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{4.3.1.2}$$

The sum of equations (4.3.1.1) and (4.3.1.2) gives total scalar part which is obtained in equations (4.3.7) and (4.3.12).

4.3.2 For Cubic Materials

There exists three decomposed parts which are two scalars and a harmonic part. The decomposed parts of total scalar part are identical with those given in equations (4.3.1.1) and (4.3.1.2). Nonor part is equal to $C^{(4;1)}_{ijkl}$ given in equation (4.2.2.3).

If we add those parts, elastic constant tensor for cubic materials is obtained by harmonic decomposition method.

4.3.3 For Transversely Isotropic Materials

There are five irreducible parts for transversely isotropic materials which are two scalars and two deviators and a nonor (harmonic) part. The decomposed parts of total scalar part are the same as those given in equations (4.3.1.1) and (4.3.1.2) and decomposed parts of total deviatoric part which are $C^{(2;1)}_{ijkl}$ and

$C_{ijkl}^{(2;2)}$ found in equations (4.2.3.3) and (4.2.3.4) respectively, are obtained by arranging equation (4.3.9) so we get total deviatoric part in harmonic decomposition method.

Nonor part is equal to $C_{ijkl}^{(4;1)}$ given in equation (4.2.3.5), we obtain harmonic part for transversely isotropic materials in harmonic decomposition method.

4.3.4 For Tetragonal Materials

By applying harmonic decomposition method, elastic constant tensor is decomposed into the parts namely as two scalar, two deviatoric and nonor. Decomposed parts are the same as those given in equations (4.3.1.1) and (4.3.1.2). Parts of total deviatoric part which are $C_{ijkl}^{(2;1)}$ and $C_{ijkl}^{(2;2)}$ found in equations (4.2.4.3) and (4.2.4.4) respectively, are obtained by arranging equation (4.3.9) so we get total deviatoric part in harmonic decomposition method.

Nonor part is equal to $C_{ijkl}^{(4;1)}$ given in equation (4.2.4.5), we obtain harmonic part for tetragonal materials in harmonic decomposition method.

4.3.5 For Trigonal Materials

For trigonal materials, five decomposed parts, which are two scalars, two deviatoric and nonor, are obtained by harmonic decomposition method. Decomposed parts of total scalar part are identical with the expressions given in equations (4.3.1.1) and (4.3.1.2). Terms of total deviatoric parts can be obtained by rearranging equation (5.3.9) so we get these parts which are the same as those given in equations (4.2.5.3) and (4.2.5.4). Nonor part is equal to $C_{ijkl}^{(4;1)}$ given in equation (4.2.5.5), we obtain harmonic part for trigonal materials in harmonic decomposition method.

So elastic constant tensor for trigonal materials can be represented by summing equations (4.3.1.1), (4.3.1.2), (4.2.5.3), (4.2.5.4) and (5.2.5.5) in harmonic decomposition method.

4.3.6 For Orthorhombic Materials

There are five irreducible parts constituted total scalar, total deviatoric and nonor parts. Total scalar part is composed of the parts given in (4.3.1.1) and (4.3.1.2). By rearranging equation (4.3.9) so we get two parts forming total deviatoric part which are the same as those given in equations (4.2.6.3) and (4.2.6.4). Nonor part is equal to $C_{ijkl}^{(4;1)}$ given in equation (4.2.6.5), we obtain harmonic part for orthorhombic materials in harmonic decomposition method.

So elastic constant tensor for orthorhombic materials can be represented by summing equations (4.3.1.1), (4.3.1.2), (4.2.6.3), (4.2.6.4) and (4.2.6.5) in harmonic decomposition method.

4.4 By Spectral Method

The spectral decomposition of C (elastic constant tensor) is given by [37-39]

$$C = \lambda_k(\overrightarrow{n}^{(k)} \otimes \overrightarrow{n}^{(k)}) \quad \text{summation over } k, \quad (k = 1, ..., 6.) \tag{4.4.1}$$

where λ_k are the eigenvalues of elastic constant tensor and $\overrightarrow{n}^{(k)}$ are the corresponding eigenvectors of elastic constant tensor, $\overrightarrow{n}^{(k)} \otimes \overrightarrow{n}^{(k)} = N$ are the eigentensors and they are orthogonal to each other.

This decomposition method is a non-linear orthogonal method since eigenvalues constituting decomposed parts, are expressed in terms of elastic constants.

In spectral method; the orthogonality condition is checked by using orthonormalized basis elements.

In order to construct orthonormalized basis, the following properties must be satisfied:

$$\mathbf{I} = \overrightarrow{n}^{(1)} \otimes \overrightarrow{n}^{(1)} + ... + \overrightarrow{n}^{(6)} \otimes \overrightarrow{n}^{(6)}$$

$$\Rightarrow$$

$$\mathbf{I} = N_1 + N_2 + N_3 + N_4 + N_5 + N_6$$

(where $\mathbf{I}$ is the identity matrix.)

$$N_1,\ N_2, ..., N_6, \quad N_K \cdot N_K = N_K$$

$$N_K \cdot N_L = 0 \quad (\text{if } K \neq L)$$

So by using spectral method, decomposed parts of elastic constant tensor is orthogonal to each other.

4.4.1 For Isotropic Materials

For isotropic symmetry, there are two eigenvalues which are $C_{11}+2C_{12}(\lambda_1)$ and $C_{11}-C_{12}(\lambda_2)$. Where these eigenvalues are found by the formula; $|C - \lambda_k \mathbf{I}| = 0$, the normalized eigenvectors are found by the formula; $(C-\lambda_k\mathbf{I})\overrightarrow{n}^{(k)} = 0$, where $\overrightarrow{n}^{(k)}$ are normalized eigenvectors.

The elastic constant tensor for isotropic materials can be written as

$$C = \lambda_1(\overrightarrow{n}^{(1)} \otimes \overrightarrow{n}^{(1)}) + \lambda_2(\sum_{k=2}^{6} \overrightarrow{n}^{(k)} \otimes \overrightarrow{n}^{(k)}) \tag{4.4.1.1}$$

Recall that $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6$, and tensors that have the same eigenvalues can be combined and the elastic constant tensor can be written as two parts, we obtain

$$C = \frac{(C_{11}+2C_{12})}{3}\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{(C_{11}-C_{12})}{2}\begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 & 0 & 0 \\ -\frac{2}{3} & \frac{4}{3} & -\frac{2}{3} & 0 & 0 & 0 \\ -\frac{2}{3} & -\frac{2}{3} & \frac{4}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \tag{4.4.1.2}$$

And by recalling that $K = (C_{11} + 2C_{12})/3$ and $G = \mu = (C_{11} - C_{12})/2$. The above equation can be written as

$$C = K \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + G \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 & 0 & 0 \\ -\frac{2}{3} & \frac{4}{3} & -\frac{2}{3} & 0 & 0 & 0 \\ -\frac{2}{3} & -\frac{2}{3} & \frac{4}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}. \tag{4.4.1.3}$$

4.4.2 For Cubic Materials

There exists three eigenvalues for cubic materials which are

$\lambda_1 = C_{11} + 2C_{12},$

$\lambda_{2,3} = C_{11} - C_{12},$

$\lambda_{4,5,6} = 2C_{44}.$

And normalized eigenvectors corresponding to these distinct eigenvalues are as follows:

For $\lambda_1 = C_{11} + 2C_{12}$,

$$\overrightarrow{n}^{(1)} = \frac{1}{\sqrt{3}} [1, 1, 1, 0, 0, 0]^T ,$$

For $\lambda_{2,3} = C_{11} - C_{12}$,

$$\overrightarrow{n}^{(2)} = \frac{1}{\sqrt{6}} [-1, -1, 2, 0, 0, 0]^T ,$$

$$\overrightarrow{n}^{(3)} = \frac{1}{\sqrt{2}} [1, -1, 0, 0, 0, 0]^T ,$$

For $\lambda_{4,5,6} = 2C_{44}$,

$$\overrightarrow{n}^{(4)} = [0, 0, 0, 1, 0, 0]^T ,$$

$$\overrightarrow{n}^{(5)} = [0, 0, 0, 0, 1, 0]^T ,$$

$$\overrightarrow{n}^{(6)} = [0, 0, 0, 0, 0, 1]^T .$$

By using these eigenvalues and eigenvectors, elastic constant tensor for a cubic material can be decomposed by spectral method as

$$C = \sum_{k=1}^{6} \lambda_k (\overrightarrow{n}^{(k)} \otimes \overrightarrow{n}^{(k)}) \qquad (4.4.2.1)$$

4.4.3 For Transversely Isotropic Materials

For transversely isotropic symmmetry like the other symmetry types, eigenvalues are found by the formula:

$|C - \lambda_k \mathbf{I}| = 0$, corresponding normalized eigenvectors are found by the formula: $(C - \lambda_k \mathbf{I})\overrightarrow{n}^{(k)} = 0$. From these formulas, eigenvalues are

$$\lambda_1 = (C_{11} + C_{12} + C_{33} - \sqrt{(C_{11} + C_{12} - C_{33})^3 + 8C_{13}^2})/2,$$

$$\lambda_2 = (C_{11} + C_{12} + C_{33} + \sqrt{(C_{11} + C_{12} - C_{33})^3 + 8C_{13}^2})/2,$$

$$\lambda_3 = C_{11} - C_{12}, \quad \lambda_4 = \lambda_5 = 2C_{44} \qquad \lambda_6 = C_{11} - C_{12}.$$

The corresponding normalized eigenvectors are

$$\overrightarrow{n}^{(1)} = \frac{1}{\left(\sqrt{2C_{13}^2 + b_1^2}\right)} \left[C_{13}, C_{13}, b_1, 0, 0, 0\right]^T,$$

$$\overrightarrow{n}^{(2)} = \frac{1}{\left(\sqrt{2C_{13}^2 + b_2^2}\right)} \left[C_{13}, C_{13}, b_2, 0, 0, 0\right]^T,$$

$$\overrightarrow{n}^{(3)} = \frac{1}{\sqrt{2}} \left[1, -1, 0, 0, 0, 0\right]^T,$$

$$\overrightarrow{n}^{(4)} = \left[0, 0, 0, 0, 1, 0\right]^T,$$

$$\overrightarrow{n}^{(5)} = \left[0, 0, 0, 1, 0, 0\right]^T,$$

$$\overrightarrow{n}^{(6)} = \left[0, 0, 0, 0, 0, 1\right]^T,$$

where $b_1 = \left(-C_{11} - C_{12} + C_{33} - \sqrt{(C_{11} + C_{12} - C_{33})^3 + 8C_{13}^2})/2\right)/2$ and $b_2 = \left(-C_{11} - C_{12} + C_{33} + \sqrt{(C_{11} + C_{12} - C_{33})^3 + 8C_{13}^2})/2\right)/2$.

By using these eigenvalues and eigenvectors, elastic constant tensor for a transversely isotropic material can be represented by spectral method as

$$C = \sum_{k=1}^{6} \lambda_k (\overrightarrow{n}^{(k)} \otimes \overrightarrow{n}^{(k)}) \tag{4.4.3.1}$$

4.4.4 For Tetragonal Materials

The eigenvalues for tetragonal materials are

$$\lambda_1 = (C_{11} + C_{12} + C_{33} - d)/2,$$

$$\lambda_2 = (C_{11} + C_{12} + C_{33} + d)/2,$$

$$\lambda_3 = C_{11} - C_{12}, \quad \lambda_4 = \lambda_5 = 2C_{55}, \quad \lambda_6 = 2C_{66},$$

where $d = \sqrt{(C_{11} + C_{12} - C_{33})^3 + 8C_{13}^2}$.

Normalized eigenvectors correponding to these eigenvalues are as follows:

For $\lambda_1 = (C_{11} + C_{12} + C_{33} - d)/2$,

$$\overrightarrow{n}^{(1)} = \frac{1}{\left(\sqrt{2C_{13}^2 + b_1^2}\right)} [C_{13}, C_{13}, b_1, 0, 0, 0]^T ,$$

For $\lambda_2 = (C_{11} + C_{12} + C_{33} + d)/2$,

$$\overrightarrow{n}^{(2)} = \frac{1}{\left(\sqrt{2C_{13}^2 + b_2^2}\right)} [C_{13}, C_{13}, b_2, 0, 0, 0]^T ,$$

For $\lambda_3 = C_{11} - C_{12}$,

$$\overrightarrow{n}^{(3)} = \frac{1}{\sqrt{2}} [1, -1, 0, 0, 0, 0]^T ,$$

For $\lambda_{4,5} = 2C_{55}$,

$$\overrightarrow{n}^{(4)} = [0, 0, 0, 1, 0, 0]^T ,$$

$$\overrightarrow{n}^{(5)} = [0, 0, 0, 0, 1, 0]^T ,$$

For $\lambda_6 = 2C_{66}$,

$$\overrightarrow{n}^{(6)} = [0, 0, 0, 0, 0, 1]^T .$$

By using these eigenvalues and eigenvectors, elastic constant tensor for a material possessing tetragonal symmetry can be represented by spectral method as

$$C = \sum_{k=1}^{6} \lambda_k (\overrightarrow{n}^{(k)} \otimes \overrightarrow{n}^{(k)}) \tag{4.4.4.1}$$

4.4.5 For Trigonal Materials

For materials possessing trigonal symmetry, the eigenvalues are

$$\lambda_1 = \frac{1}{2}\left[C_{11} + C_{12} + C_{33} + \sqrt{8(C_{13})^2 + (C_{11} + C_{12} - C_{33})^2}\right],$$

$$\lambda_2 = \frac{1}{2}\left[C_{11} + C_{12} + C_{33} - \sqrt{8(C_{13})^2 + (C_{11} + C_{12} - C_{33})^2}\right],$$

$$\lambda_{3,6} = \frac{1}{2}\left[C_{11} - C_{12} + C_{44} + \sqrt{8(C_{14})^2 + (C_{11} - C_{12} - C_{44})^2}\right],$$

$$\lambda_{4,5} = \frac{1}{2}\left[C_{11} - C_{12} + C_{44} - \sqrt{8(C_{14})^2 + (C_{11} - C_{12} - C_{44})^2}\right].$$

The corresponding normalized eigenvectors are

$$\overrightarrow{n}^{(1)} = \left[-\frac{(-\frac{C_{11}}{2} + \frac{C_{12}}{2} + C_{44}) - \sqrt{e}}{2C_{14}}, \frac{(-\frac{C_{11}}{2} + \frac{C_{12}}{2} + C_{44}) - \sqrt{e}}{2C_{14}}, 0, 1, 0, 0\right]^T,$$

$$\overrightarrow{n}^{(2)} = \left[-\frac{(-\frac{C_{11}}{2} + \frac{C_{12}}{2} + C_{44}) + \sqrt{e}}{2C_{14}}, \frac{(-\frac{C_{11}}{2} + \frac{C_{12}}{2} + C_{44}) + \sqrt{e}}{2C_{14}}, 0, 1, 0, 0\right]^T,$$

$$\overrightarrow{n}^{(3)} = \left[0, 0, 0, 0, 1, -\frac{(\frac{C_{11}}{4} + \frac{C_{12}}{4} + C_{44} - \frac{\sqrt{f}}{4})}{C_{14}}\right]^T,$$

$$\overrightarrow{n}^{(4)} = \left[0, 0, 0, 0, 1, -\frac{(\frac{C_{11}}{4} + \frac{C_{12}}{4} + C_{44} + \frac{\sqrt{f}}{4})}{C_{14}}\right]^T,$$

$$\overrightarrow{n}^{(5)} = \left[-\frac{(-\frac{C_{11}}{2} - \frac{C_{12}}{2} + \frac{C_{33}}{2}) - \sqrt{g}}{2C_{13}}, -\frac{(-\frac{C_{11}}{2} - \frac{C_{12}}{2} + \frac{C_{33}}{2}) - \sqrt{g}}{2C_{13}}, 1, 0, 0, 0\right]^T,$$

$$\overrightarrow{n}^{(6)} = \left[-\frac{(-\frac{C_{11}}{2} - \frac{C_{12}}{2} + \frac{C_{33}}{2}) + \sqrt{g}}{2C_{13}}, -\frac{(-\frac{C_{11}}{2} - \frac{C_{12}}{2} + \frac{C_{33}}{2}) + \sqrt{g}}{2C_{13}}, 1, 0, 0, 0 \right]^T .$$

where $e = C_{11}^2 - 2C_{12}C_{11} - 4C_{11}C_{44} + C_{12}^2 + 4C_{12}C_{44} + 4C_{44}^2 + 8C_{14}^2$,

$f = C_{11}^2 - 2C_{12}C_{11} - 8C_{11}C_{44} + C_{12}^2 + 8C_{12}C_{44} + 16C_{44}^2 + 16C_{14}^2$,

$g = C_{12}^2 + 2C_{12}C_{11} - 2C_{12}C_{33} + C_{11}^2 - 2C_{33}C_{11} + C_{33}^2 + 8C_{13}^2$.

By using these eigenvalues and eigenvectors, elastic constant tensor for a trigonal material can be decomposed by spectral method as

$$C = \sum_{k=1}^{6} \lambda_k (\overrightarrow{n}^{(k)} \otimes \overrightarrow{n}^{(k)}) \qquad (4.4.5.1)$$

4.4.6 For Orthorhombic Materials

Symbolic results for eigensystems including eigenvalues and corresponding normalized eigenvectors, are too complicated and difficult to calculate so explicit solutions for them are not presented. However they can easily be calculated numerically. For this reason, a numerical example for an orthorhombic material is given in section (5.6.4).

The eigenvalues for orthorhombic materials are

λ_1, λ_2, λ_3 are the roots of the characteristic equation

$$\begin{vmatrix} C_{11} - \lambda & C_{12} & C_{13} \\ C_{12} & C_{22} - \lambda & C_{23} \\ C_{13} & C_{23} & C_{33} - \lambda \end{vmatrix} = 0,$$

$\lambda_4 = 2C_{44}$,

$\lambda_5 = 2C_{55}$,

$\lambda_6 = 2C_{66}$.

Spectral representation for elastic constant tensor of orthorhombic material is presented as

$$C = \sum_{k=1}^{6} \lambda_k (\overrightarrow{n}^{(k)} \otimes \overrightarrow{n}^{(k)}) \tag{4.4.6.1}$$

CHAPTER 5

NUMERICAL EXAMPLES OF THE DECOMPOSITION PROCESS

In order to understand each decomposition process in detail, numerical examples are given for various materials which possess isotropic, cubic, transversely isotropic, trigonal (classes:32, $3m$, $\overline{3}m$), tetragonal (classes:$4mm$, $\overline{4}2m$, 422, $4/mmm$) and orthorhombic symmetry. Decomposition processes, orthonormal tensor basis, irreducible and harmonic decomposition are performed by simple computer programs written in MATLAB in order to ease the calculations.

5.1 For Isotropic Material

Especially textured and non-crystalline materials show isotropic symmetry. There are two independent elastic constants for isotropic symmetry which are C_{11}, C_{12}. Reactor pressure vessel (RPV) steel is presented as isotropic material and the elastic coefficients in GPa(10^{10} dyn·cm^{-2}) for RPV steel are given[50]

$$C_{ij} = \begin{bmatrix} 277.001 & 118.715 & 118.715 & 0 & 0 & 0 \\ 118.715 & 277.001 & 118.715 & 0 & 0 & 0 \\ 118.715 & 118.715 & 277.001 & 0 & 0 & 0 \\ 0 & 0 & 0 & 79.143 & 0 & 0 \\ 0 & 0 & 0 & 0 & 79.143 & 0 \\ 0 & 0 & 0 & 0 & 0 & 79.143 \end{bmatrix} \tag{5.1.1}$$

Each method is applied to the material for decomposition of its elastic constants.

5.1.1 By Orthonormal Tensor Basis

By orthonormal tensor basis method, we apply the formula given in equation (4.1.1.4). For this reason inner products must be calculated as presented below

$$(C, A^{I}) = 514.431 \tag{5.1.1.1}$$

$$(C, A^{II}) = 353.94 \tag{5.1.1.2}$$

The symmetric fourth rank tensor for RPV steel can be represented in the form

$$C_{ijkl} = 514.431 A^{I}_{ijkl} + 353.94 A^{II}_{ijkl} \tag{5.1.1.3}$$

If we substitute the values of the elastic constants into the right-hand side of equation (5.1.1.3), we end up with the identical matrix of equation (5.1.1), which shows the validity of the decomposed terms. First part of isotropic system is

$$I_1 = 514.431 A^{I}_{ijkl} \tag{5.1.1.4}$$

If we put the appropriate values into equation (5.1.1.4), first isotropic part is found as

$$I_1 = \begin{bmatrix} 171.477 & 171.477 & 171.477 & 0 & 0 & 0 \\ 171.477 & 171.477 & 171.477 & 0 & 0 & 0 \\ 171.477 & 171.477 & 171.477 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.1.1.5}$$

The second part of the isotropic system is

$$I_2 = 353.94 A^{II}_{ijkl} \tag{5.1.1.6}$$

If we put the appropriate values into equation (5.1.1.6), second isotropic part is found as

$$I_2 = \begin{bmatrix} 105.524 & -52.762 & -52.762 & 0 & 0 & 0 \\ -52.762 & 105.524 & -52.762 & 0 & 0 & 0 \\ -52.762 & -52.762 & 105.524 & 0 & 0 & 0 \\ 0 & 0 & 0 & 79.143 & 0 & 0 \\ 0 & 0 & 0 & 0 & 79.143 & 0 \\ 0 & 0 & 0 & 0 & 0 & 79.143 \end{bmatrix} \tag{5.1.1.7}$$

If we add the matrices given in (5.1.1.5) and (5.1.1.7) we obtain the original matrix for RPV steel given in (5.1.1).

5.1.2 By Irreducible Decomposition Method

For isotropic materials, there are only two irreducible parts which are scalars $C_{mn}^{(0;1)}, C_{mn}^{(0;2)}$ given in section 4.2.1. For RPV steel these parts are calculated as

$$C_{mn}^{(0;1)} = \begin{bmatrix} 171.5 & 171.5 & 171.5 & 0 & 0 & 0 \\ 171.5 & 171.5 & 171.5 & 0 & 0 & 0 \\ 171.5 & 171.5 & 171.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.1.2.1}$$

$$C_{mn}^{(0;2)} = \begin{bmatrix} 105.524 & -52.762 & -52.762 & 0 & 0 & 0 \\ -52.762 & 105.524 & -52.762 & 0 & 0 & 0 \\ -52.762 & -52.762 & 105.524 & 0 & 0 & 0 \\ 0 & 0 & 0 & 79.143 & 0 & 0 \\ 0 & 0 & 0 & 0 & 79.143 & 0 \\ 0 & 0 & 0 & 0 & 0 & 79.143 \end{bmatrix}. \tag{5.1.2.2}$$

If we add the matrices given in equations (5.1.2.1) and (5.1.2.2) we obtain the original matrix for RPV steel given in (5.1.1).

5.1.3 By Harmonic Decomposition Method

Like irreducible decomposition method, there are only two decomposed parts which are scalars (denoted as S_1 and S_2) given in section (4.3.1). For RPV steel these parts are calculated as

$$S_1 = \begin{bmatrix} 118.715 & 118.715 & 118.715 & 0 & 0 & 0 \\ 118.715 & 118.715 & 118.715 & 0 & 0 & 0 \\ 118.715 & 118.715 & 118.715 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.1.3.1}$$

$$S_2 = \begin{bmatrix} 158.286 & 0 & 0 & 0 & 0 & 0 \\ 0 & 158.286 & 0 & 0 & 0 & 0 \\ 0 & 0 & 158.286 & 0 & 0 & 0 \\ 0 & 0 & 0 & 79.143 & 0 & 0 \\ 0 & 0 & 0 & 0 & 79.143 & 0 \\ 0 & 0 & 0 & 0 & 0 & 79.143 \end{bmatrix}. \tag{5.1.3.2}$$

If we add the matrices given in equations (5.1.3.1) and (5.1.3.2) we obtain the original matrix for RPV steel given in (5.1.1).

5.1.4 By Spectral Method

Eigenvalues of RPV steel are found by the formula; $|C - \lambda_k I| = 0$, corresponding normalized eigenvectors are found by the formula; $(C - \lambda_k I)\overrightarrow{n}^{(k)} = 0$. From these formulas, eigenvalues are

$$\lambda_1 = 514.4, \quad \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 158.286.$$

The corresponding normalized eigenvectors are found as

$$\overrightarrow{n}^{(1)} = [0.5774, 0.5774, 0.5774, 0, 0, 0]^T,$$

$$\overrightarrow{n}^{(2)} = [-0.4082, -0.4082, 0.8165, 0, 0, 0]^T ,$$

$$\overrightarrow{n}^{(3)} = [0.7071, -0.7071, 0, 0, 0, 0]^T ,$$

$$\overrightarrow{n}^{(4)} = [0, 0, 0, 1, 0, 0]^T ,$$

$$\overrightarrow{n}^{(5)} = [0, 0, 0, 0, 1, 0]^T ,$$

$$\overrightarrow{n}^{(6)} = [0, 0, 0, 0, 0, 1]^T .$$

By applying the formula given in equation (4.4.1.1) and using these eigenvalues and eigenvectors, elastic constant tensor for RPV steel can be decomposed by spectral method as

$$C_{ij} = \begin{bmatrix} 171.5 & 171.5 & 171.5 & 0 & 0 & 0 \\ 171.5 & 171.5 & 171.5 & 0 & 0 & 0 \\ 171.5 & 171.5 & 171.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 105.52 & -52.76 & -52.76 & 0 & 0 & 0 \\ -52.76 & 105.52 & -52.76 & 0 & 0 & 0 \\ -52.76 & -52.76 & 105.52 & 0 & 0 & 0 \\ 0 & 0 & 0 & 79.14 & 0 & 0 \\ 0 & 0 & 0 & 0 & 79.14 & 0 \\ 0 & 0 & 0 & 0 & 0 & 79.14 \end{bmatrix} . \tag{5.1.4.1}$$

The summation in equation (5.1.4.1) gives the original matrix for RPV steel given in (5.1.1).

5.2 For Cubic Material

In cubic symmetry, four three-fold axes arranged like the body diagonals of a cube. There are three independent elastic constants for cubic symmetry which are C_{11}, C_{12}, C_{44}. Aluminium Antimonide (AlSb) is presented as cubic material and the elastic coefficients in GPa for AlSb are given[52]

$$C_{ij} = \begin{bmatrix} 87.7 & 43.4 & 43.4 & 0 & 0 & 0 \\ 43.4 & 87.7 & 43.4 & 0 & 0 & 0 \\ 43.4 & 43.4 & 87.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 40.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 40.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 40.8 \end{bmatrix} \tag{5.2.1}$$

Each method is applied to the material for decomposition of its elastic constants.

5.2.1 By Orthonormal Tensor Basis Method

By orthonormal tensor basis method, we apply the formula given in equation (4.1.2.2). For this reason inner products must be calculated as presented below

$$(C, A^{I}) = 174.5 \tag{5.2.1.1}$$

$$(C, A^{II}) = 149.1 \tag{5.2.1.2}$$

$$(C, A^{III}) = 40.86 \tag{5.2.1.3}$$

The symmetric fourth rank tensor for AlSb can be represented in the form

$$C_{ijkl} = 174.5A^{I}_{ijkl} + 149.1A^{II}_{ijkl} + 40.86A^{III}_{ijkl} \tag{5.2.1.4}$$

If we substitute the values of the elastic constants into the right-hand side of equation (5.2.1.4), we end up with the identical matrix of equation (5.2.1), which shows the validity of the decomposed terms. Isotropic part of AlSb is

$$I = 174.5A^{I}_{ijkl} + 149.1A^{II}_{ijkl} \tag{5.2.1.5}$$

If we put the appropriate values into equation (5.2.1.5), isotropic part is found as

$$I = \begin{bmatrix} 102.6 & 35.9 & 35.9 & 0 & 0 & 0 \\ 35.9 & 102.6 & 35.9 & 0 & 0 & 0 \\ 35.9 & 35.9 & 102.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 33.34 & 0 & 0 \\ 0 & 0 & 0 & 0 & 33.34 & 0 \\ 0 & 0 & 0 & 0 & 0 & 33.34 \end{bmatrix} \tag{5.2.1.6}$$

The cubic part of the material is

$$C = 40.86A^{III}_{ijkl} \tag{5.2.1.7}$$

If we put the appropriate values into equation (5.2.1.7), cubic part is found as

$$C = \begin{bmatrix} -14.9 & 7.5 & 7.5 & 0 & 0 & 0 \\ 7.5 & -14.9 & 7.5 & 0 & 0 & 0 \\ 7.5 & 7.5 & -14.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7.5 \end{bmatrix}. \tag{5.2.1.8}$$

If we add the matrices given in (5.2.1.6) and (5.2.1.8) we obtain the original matrix for Aluminium Antimonide given in (5.2.1).

5.2.2 By Irreducible Decomposition Method

For cubic materials, there are three irreducible parts which are scalars $C_{mn}^{(0;1)}, C_{mn}^{(0;2)}$ and $C_{mn}^{(4;1)}$ given in section 4.2.2. For AlSb these parts are calculated as

$$C_{mn}^{(0;1)} = \begin{bmatrix} 58.2 & 58.2 & 58.2 & 0 & 0 & 0 \\ 58.2 & 58.2 & 58.2 & 0 & 0 & 0 \\ 58.2 & 58.2 & 58.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.2.2.1}$$

$$C_{mn}^{(0;2)} = \begin{bmatrix} 44.4 & -22.2 & -22.2 & 0 & 0 & 0 \\ -22.2 & 44.4 & -22.2 & 0 & 0 & 0 \\ -22.2 & -22.2 & 44.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 33.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 33.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 33.3 \end{bmatrix}, \tag{5.2.2.2}$$

$$C_{mn}^{(4;1)} = \begin{bmatrix} -14.9 & 7.5 & 7.5 & 0 & 0 & 0 \\ 7.5 & -14.9 & 7.5 & 0 & 0 & 0 \\ 7.5 & 7.5 & -14.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7.5 \end{bmatrix}. \tag{5.2.2.3}$$

If we add the matrices given in equations (5.2.2.1), (5.2.2.2) and (5.2.2.3) we obtain the original matrix for AlSb given in (5.2.1).

5.2.3 By Harmonic Decomposition Method

For cubic materials, there are three harmonic parts which are scalars S_1, S_2 and a nonor (harmonic) part H given in section 4.3.2. For AlSb these parts are calculated as

$$S_1 = \begin{bmatrix} 35.9 & 35.9 & 35.9 & 0 & 0 & 0 \\ 35.9 & 35..9 & 35.9 & 0 & 0 & 0 \\ 35.9 & 35.9 & 35.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.2.3.1}$$

$$S_2 = \begin{bmatrix} 66.68 & 0 & 0 & 0 & 0 & 0 \\ 0 & 66.68 & 0 & 0 & 0 & 0 \\ 0 & 0 & 66.68 & 0 & 0 & 0 \\ 0 & 0 & 0 & 33.34 & 0 & 0 \\ 0 & 0 & 0 & 0 & 33.34 & 0 \\ 0 & 0 & 0 & 0 & 0 & 33.34 \end{bmatrix}, \tag{5.2.3.2}$$

$$H = \begin{bmatrix} -14.9 & 7.5 & 7.5 & 0 & 0 & 0 \\ 7.5 & -14.9 & 7.5 & 0 & 0 & 0 \\ 7.5 & 7.5 & -14.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7.5 \end{bmatrix}. \tag{5.2.3.3}$$

If we add the matrices given in equations (5.2.3.1), (5.2.3.2) and (5.2.3.3) we obtain the original matrix for AlSb given in (5.2.1).

5.2.4 By Spectral Method

Eigenvalues of AlSb are found by the formula; $|C - \lambda_k I| = 0$, corresponding normalized eigenvectors are found by the formula; $(C - \lambda_k I)\overrightarrow{n}^{(k)} = 0$. From these formulas, eigenvalues are

$$\lambda_1 = 174.5, \quad \lambda_2 = \lambda_3 = 44.3, \quad \lambda_4 = \lambda_5 = \lambda_6 = 40.8.$$

The corresponding normalized eigenvectors are found as

$$\overrightarrow{n}^{(1)} = [0.5774, 0.5774, 0.5774, 0, 0, 0]^T,$$

$$\overrightarrow{n}^{(2)} = [0.6112, -0.7745, 0.1633, 0, 0, 0]^T,$$

$$\overrightarrow{n}^{(3)} = [0.5414, 0.2586, -0.8, 0, 0, 0]^T,$$

$$\overrightarrow{n}^{(4)} = [0, 0, 0, 1, 0, 0]^T,$$

$$\overrightarrow{n}^{(5)} = [0, 0, 0, 0, 1, 0]^T,$$

$$\overrightarrow{n}^{(6)} = [0, 0, 0, 0, 0, 1]^T.$$

By applying the formula given in equation (4.4.2.1) and using these eigenvalues and eigenvectors, elastic constant tensor for AlSb can be decomposed by spectral method as

$$C_{ij} = \begin{bmatrix} 58.18 & 58.18 & 58.18 & 0 & 0 & 0 \\ 58.18 & 58.18 & 58.18 & 0 & 0 & 0 \\ 58.18 & 58.18 & 58.18 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 16.483 & -20.94 & 4.40 & 0 & 0 & 0 \\ -20.94 & 26.61 & -5.60 & 0 & 0 & 0 \\ 4.40 & -5.60 & 1.18 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 12.92 & 6.20 & -19.14 & 0 & 0 & 0 \\ 6.20 & 2.97 & -9.18 & 0 & 0 & 0 \\ -19.14 & -9.18 & 28.35 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 40.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 40.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 40.8 \end{bmatrix}. \tag{5.2.4.1}$$

The summation in equation (5.2.4.1) gives the original matrix for AlSb given in (5.2.1).

5.3 For Transversely Isotropic Material

Most textured and non-crystalline materials exhibit transversely isotropic symmetry. There are five independent elastic constants for transversely isotropic symmetry which are $C_{11}, C_{12}, C_{13}, C_{33}, C_{44}$. Polystyrene is presented as a transversely isotropic material and the elastic coefficients in GPa for Polystyrene are given[53]

$$C_{ij} = \begin{bmatrix} 5.20 & 2.75 & 2.75 & 0 & 0 & 0 \\ 2.75 & 5.20 & 2.75 & 0 & 0 & 0 \\ 2.75 & 2.75 & 5.70 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.30 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.30 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.225 \end{bmatrix} \tag{5.3.1}$$

Each method is applied to the material for decomposition of its elastic constants.

5.3.1 By Orthonormal Tensor Basis Method

By orthonormal tensor basis method, we apply the formula given in equation (4.1.3.2). For this reason inner products must be calculated as presented below

$$(C, A^{I}) = 10.87 \tag{5.3.1.1}$$

$$(C, A^{II}) = 5.7616 \tag{5.3.1.2}$$

$$(C, A^{III}) = 0.4025 \tag{5.3.1.3}$$

$$(C, A^{IV}) = 0.05 \tag{5.3.1.4}$$

$$(C, A^{V}) = 0.15 \tag{5.3.1.5}$$

The symmetric fourth rank tensor for Polystyrene can be represented in the form

$$C_{ijkl} = 10.87A^{I}_{ijkl} + 5.7616A^{II}_{ijkl} + 0.4025A^{III}_{ijkl} + 0.05A^{IV}_{ijkl} + 0.15A^{V}_{ijkl} \tag{5.3.1.6}$$

If we substitute the values of the elastic constants into the right-hand side of equation (5.3.1.6), we end up with the identical matrix of equation (5.3.1), which shows the validity of the decomposed terms.

Isotropic part of the Polystyrene:

$$I = 10.87A^{I}_{ijkl} + 5.7616A^{II}_{ijkl} \tag{5.3.1.7}$$

If we put the appropriate values into equation (5.3.1.7), isotropic part is found as

$$I = \begin{bmatrix} 5.3411 & 2.7644 & 2.7644 & 0 & 0 & 0 \\ 2.7644 & 5.3411 & 2.7644 & 0 & 0 & 0 \\ 2.7644 & 2.7644 & 5.3411 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.2883 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.2883 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.2883 \end{bmatrix} \tag{5.3.1.8}$$

Transversely isotropic part of the Polystyrene:

$$TI = 0.4025A^{III}_{ijkl} + 0.05A^{IV}_{ijkl} + 0.15A^{V}_{ijkl} \tag{5.3.1.9}$$

If we put the appropriate values into equation (5.3.1.9), transversely isotropic part is found as

$$TI = \begin{bmatrix} -0.14 & -0.0133 & -0.0133 & 0 & 0 & 0 \\ -0.0133 & -0.14 & -0.0133 & 0 & 0 & 0 \\ -0.0133 & -0.0133 & 0.36 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0117 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0117 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.063 \end{bmatrix} \tag{5.3.1.10}$$

If we add the matrices given in (5.3.1.8) and (5.3.1.10) we obtain the original matrix for Polystyrene given in (5.3.1).

5.3.2 By Irreducible Decomposition Method

For transversely isotropic materials, there are five irreducible parts which are two scalars; $C_{mn}^{(0;1)}, C_{mn}^{(0;2)}$, two deviators; $C_{mn}^{(2;1)}, C_{mn}^{(2;2)}$ and a nonor part; $C_{mn}^{(4;1)}$ given in section (4.2.3). For Polystyrene, these parts are computed as

$$C_{mn}^{(0;1)} = \begin{bmatrix} 3.6222 & 3.6222 & 3.6222 & 0 & 0 & 0 \\ 3.6222 & 3.6222 & 3.6222 & 0 & 0 & 0 \\ 3.6222 & 3.6222 & 3.6222 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.3.2.1}$$

$$C_{mn}^{(0;2)} = \begin{bmatrix} 1.7178 & -0.8589 & -0.8589 & 0 & 0 & 0 \\ -0.8589 & 1.7178 & -0.8589 & 0 & 0 & 0 \\ -0.8589 & -0.8589 & 1.7178 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.2883 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.2883 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.2883 \end{bmatrix}, \tag{5.3.2.2}$$

$$C_{mn}^{(2;1)} = \begin{bmatrix} -0.1533 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1533 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.3067 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0383 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0383 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.0767 \end{bmatrix}, \tag{5.3.2.3}$$

$$C_{mn}^{(2;2)} = \begin{bmatrix} -0.0038 & -0.0190 & 0.0095 & 0 & 0 & 0 \\ -0.0190 & -0.0038 & 0.0095 & 0 & 0 & 0 \\ 0.0095 & 0.0095 & 0.0076 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0038 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0038 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0076 \end{bmatrix}, \tag{5.3.2.4}$$

$$C_{mn}^{(4;1)} = \begin{bmatrix} 0.0171 & 0.0057 & -0.0229 & 0 & 0 & 0 \\ 0.0057 & 0.0171 & -0.0229 & 0 & 0 & 0 \\ -0.0229 & -0.0229 & 0.0457 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0229 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0229 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0057 \end{bmatrix}. \tag{5.3.2.5}$$

If we add the matrices given above we obtain the original matrix for Polystyrene given in (5.3.1).

5.3.3 By Harmonic Decomposition Method

For transversely isotropic materials, there are five harmonic parts which are two scalars; S_1, S_2, two deviators; D_1, D_2 and a nonor part; H given in section (4.3.3). For Polystyrene, these parts are computed as

$$S_1 = \begin{bmatrix} 2.7633 & 2.7633 & 2.7633 & 0 & 0 & 0 \\ 2.7633 & 2.7633 & 2.7633 & 0 & 0 & 0 \\ 2.7633 & 2.7633 & 2.7633 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.3.3.1}$$

$$S_2 = \begin{bmatrix} 2.5767 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2.5767 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2.5767 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.2883 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.2883 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.2883 \end{bmatrix}, \tag{5.3.3.2}$$

$$D_1 = \begin{bmatrix} -0.1533 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1533 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.3067 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0383 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0383 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.0767 \end{bmatrix}, \tag{5.3.3.3}$$

$$D_2 = \begin{bmatrix} -0.0038 & -0.0190 & 0.0095 & 0 & 0 & 0 \\ -0.0190 & -0.0038 & 0.0095 & 0 & 0 & 0 \\ 0.0095 & 0.0095 & 0.0076 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0038 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0038 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0076 \end{bmatrix}, \tag{5.3.3.4}$$

$$H = \begin{bmatrix} 0.0171 & 0.0057 & -0.0229 & 0 & 0 & 0 \\ 0.0057 & 0.0171 & -0.0229 & 0 & 0 & 0 \\ -0.0229 & -0.0229 & 0.0457 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0229 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0229 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0057 \end{bmatrix}, \tag{5.3.3.5}$$

If we add the matrices; S_1, S_2, D_1, D_2 and H, we obtain the original matrix for Polystyrene given in (5.3.1).

5.3.4 By Spectral Method

Eigenvalues of Polystyrene are found by the formula; $|C - \lambda_k I| = 0$, corresponding normalized eigenvectors are found by the formula:

$(C - \lambda_k I)\overrightarrow{n}^{(k)} = 0$.From these formulas, eigenvalues are

$\lambda_1 = 10.874, \quad \lambda_2 = 2.777, \quad \lambda_3 = 2.45, \quad \lambda_4 = 1.30, \quad \lambda_5 = 1.30,$

$\lambda_6 = 1.225.$

The corresponding normalized eigenvectors are found as

$\overrightarrow{n}^{(1)} = [0.5652, 0.5652, 0.6009, 0, 0, 0]^T,$

$\overrightarrow{n}^{(2)} = [0.4249, 0.4249, -0.7993, 0, 0, 0]^T,$

$\overrightarrow{n}^{(3)} = [0.7071, -0.7071, 0, 0, 0, 0]^T,$

$\overrightarrow{n}^{(4)} = [0, 0, 0, 0, 1, 0]^T,$

$\overrightarrow{n}^{(5)} = [0, 0, 0, 1, 0, 0]^T,$

$\overrightarrow{n}^{(6)} = [0, 0, 0, 0, 0, 1]^T.$

By applying the formula given in equation (4.4.3.1) and using these eigenvalues and eigenvectors, elastic constant tensor for Polystyrene can be decomposed by spectral method as

$$C_{ij} = \begin{bmatrix} 3.4736 & 3.4736 & 3.6930 & 0 & 0 & 0 \\ 3.4736 & 3.4736 & 3.6930 & 0 & 0 & 0 \\ 3.6930 & 3.6930 & 3.9262 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0.5013 & 0.5013 & -0.9430 & 0 & 0 & 0 \\ 0.5013 & 0.5013 & -0.9430 & 0 & 0 & 0 \\ -0.9430 & -0.9430 & 1.7739 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 1.225 & -1.225 & 0 & 0 & 0 & 0 \\ -1.225 & 1.225 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.30 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.30 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.225 \end{bmatrix}. \tag{5.3.4.1}$$

The above summation gives the original matrix for Polystyrene given in (5.3.1).

5.4 For Tetragonal Material

Tetragonal materials show tetragonal symmetry. There are six independent elastic constants for tetragonal materials which are $C_{11}, C_{12}, C_{13}, C_{33}, C_{44}, C_{66}$. Zircon ($ZrSiO_4$, metamict) is presented as tetragonal material and the elastic coefficients in GPa for $ZrSiO_4$ are given[54]

$$C_{ij} = \begin{bmatrix} 284 & 73 & 119 & 0 & 0 & 0 \\ 73 & 284 & 119 & 0 & 0 & 0 \\ 119 & 119 & 309 & 0 & 0 & 0 \\ 0 & 0 & 0 & 77.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 77.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 47.7 \end{bmatrix} \tag{5.4.1}$$

Each method is applied to the material for decomposition of its elastic constants.

5.4.1 By Orthonormal Tensor Basis Method

By orthonormal tensor basis method, we apply the formula given in equation (4.1.4.2). For this reason inner products must be calculated as presented below

$$(C, A^{I}) = 499.67 \tag{5.4.1.1}$$

$$(C, A^{II}) = 350 \tag{5.4.1.2}$$

$$(C, A^{III}) = -58.64 \tag{5.4.1.3}$$

$$(C, A^{IV}) = 20.4 \tag{5.4.1.4}$$

$$(C, A^{V}) = 53.12 \tag{5.4.1.5}$$

$$(C, A^{VI}) = 48.66 \tag{5.4.1.6}$$

The symmetric fourth rank tensor for Zircon can be represented in the form

$$C_{ijkl} = 499.67A^{I}_{ijkl} + 350A^{II}_{ijkl} - 58.64A^{III}_{ijkl} + 20.4A^{IV}_{ijkl} + 53.12A^{V}_{ijkl} + 48.66A^{VI}_{ijkl} \tag{5.4.1.7}$$

If we substitute the values of the elastic constants into the right-hand side of equation (5.4.1.7), we end up with the identical matrix of equation (5.4.1), which shows the validity of the decomposed terms. Isotropic part of Zircon is

$$I = 499.67A^{I}_{ijkl} + 350A^{II}_{ijkl} \tag{5.4.1.8}$$

If we put the appropriate values into equation (5.4.1.8), isotropic part is found as

$$I = \begin{bmatrix} 270.92 & 114.37 & 114.37 & 0 & 0 & 0 \\ 114.37 & 270.92 & 114.37 & 0 & 0 & 0 \\ 114.37 & 114.37 & 270.92 & 0 & 0 & 0 \\ 0 & 0 & 0 & 78.27 & 0 & 0 \\ 0 & 0 & 0 & 0 & 78.27 & 0 \\ 0 & 0 & 0 & 0 & 0 & 78.27 \end{bmatrix} \tag{5.4.1.9}$$

The cubic part of the material is

$$C = -58.64A^{III}_{ijkl} \tag{5.4.1.10}$$

If we put the appropriate values into equation (5.4.1.10), cubic part is found as

$$C = \begin{bmatrix} 21.4 & -10.7 & -10.7 & 0 & 0 & 0 \\ -10.7 & 21.4 & -10.7 & 0 & 0 & 0 \\ -10.7 & -10.7 & 21.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -10.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10.7 & 0 \\ 0 & 0 & 0 & 0 & 0 & -10.7 \end{bmatrix}. \tag{5.4.1.11}$$

Lastly, the tetragonal part of the material is

$$Tet = 20.4A^{IV}_{ijkl} + 53.12A^{V}_{ijkl} + 48.66A^{VI}_{ijkl} \tag{5.4.1.12}$$

If we put the appropriate values into equation (5.4.1.12), tetragonal part is found as

$$Tet = \begin{bmatrix} -8.33 & -30.67 & 15.33 & 0 & 0 & 0 \\ -30.67 & -8.33 & 15.33 & 0 & 0 & 0 \\ 15.33 & 15.33 & 16.67 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9.93 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9.93 & 0 \\ 0 & 0 & 0 & 0 & 0 & -19.87 \end{bmatrix}. \qquad (5.4.1.13)$$

If we add the matrices given in (5.4.1.9), (5.4.1.11) and (5.4.1.13) we obtain the original matrix for Zircon given in (5.4.1).

5.4.2 By Irreducible Decomposition Method

For tetragonal materials, there are five irreducible parts which are scalars; $C_{mn}^{(0;1)}, C_{mn}^{(0;2)}$, deviators; $C_{mn}^{(2;1)}, C_{mn}^{(2;2)}$, and a nonor $C_{mn}^{(4;1)}$ given in section 4.2.4. For Zircon, these parts are calculated as

$$C_{mn}^{(0;1)} = \begin{bmatrix} 166.56 & 166.56 & 166.56 & 0 & 0 & 0 \\ 166.56 & 166.56 & 166.56 & 0 & 0 & 0 \\ 166.56 & 166.56 & 166.56 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (5.4.2.1)$$

$$C_{mn}^{(0;2)} = \begin{bmatrix} 104.36 & -52.18 & -52.18 & 0 & 0 & 0 \\ -52.18 & 104.36 & -52.18 & 0 & 0 & 0 \\ -52.18 & -52.18 & 104.36 & 0 & 0 & 0 \\ 0 & 0 & 0 & 78.27 & 0 & 0 \\ 0 & 0 & 0 & 0 & 78.27 & 0 \\ 0 & 0 & 0 & 0 & 0 & 78.27 \end{bmatrix}, \qquad (5.4.2.2)$$

$$C_{mn}^{(2;1)} = \begin{bmatrix} -14.6 & 0 & 0 & 0 & 0 & 0 \\ 0 & -14.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 29.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.65 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.65 & 0 \\ 0 & 0 & 0 & 0 & 0 & -7.31 \end{bmatrix}, \tag{5.4.2.3}$$

$$C_{mn}^{(2;2)} = \begin{bmatrix} -2.59 & -12.9 & 6.47 & 0 & 0 & 0 \\ -12.9 & -2.59 & 6.47 & 0 & 0 & 0 \\ 6.47 & 6.47 & 5.17 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2.59 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2.59 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5.17 \end{bmatrix}, \tag{5.4.2.4}$$

$$C_{mn}^{(4;1)} = \begin{bmatrix} 16.657 & -13.11 & -3.54 & 0 & 0 & 0 \\ -13.11 & 16.657 & -3.54 & 0 & 0 & 0 \\ -3.54 & -3.54 & 7.09 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3.54 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.54 & 0 \\ 0 & 0 & 0 & 0 & 0 & -13.11 \end{bmatrix}. \tag{5.4.2.5}$$

If we add the matrices given in equations (5.4.2.1), (5.4.2.2), (5.4.2.3), (5.4.2.4) and (5.4.2.5) we obtain the original matrix for Zircon given in (5.4.1).

5.4.3 By Harmonic Decomposition Method

By applying harmonic decomposition method to tetragonal materials, there are five harmonic parts which are scalars; S_1, S_2, deviators; D_1, D_2, and a nonor H given in section 4.3.4. For Zircon, these parts are calculated as

$$S_1 = \begin{bmatrix} 114.37 & 114.37 & 114.37 & 0 & 0 & 0 \\ 114.37 & 114.37 & 114.37 & 0 & 0 & 0 \\ 114.37 & 114.37 & 114.37 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.4.3.1}$$

$$S_2 = \begin{bmatrix} 156.55 & 0 & 0 & 0 & 0 & 0 \\ 0 & 156.55 & 0 & 0 & 0 & 0 \\ 0 & 0 & 156.55 & 0 & 0 & 0 \\ 0 & 0 & 0 & 78.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 78.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 78.3 \end{bmatrix}, \tag{5.4.3.2}$$

$$D_1 = \begin{bmatrix} -14.6 & 0 & 0 & 0 & 0 & 0 \\ 0 & -14.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 29.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.65 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.65 & 0 \\ 0 & 0 & 0 & 0 & 0 & -7.31 \end{bmatrix}, \tag{5.4.3.3}$$

$$D_2 = \begin{bmatrix} -2.59 & -12.9 & 6.47 & 0 & 0 & 0 \\ -12.9 & -2.59 & 6.47 & 0 & 0 & 0 \\ 6.47 & 6.47 & 5.17 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2.59 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2.59 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5.17 \end{bmatrix}, \tag{5.4.3.4}$$

$$H = \begin{bmatrix} 16.657 & -13.11 & -3.54 & 0 & 0 & 0 \\ -13.11 & 16.657 & -3.54 & 0 & 0 & 0 \\ -3.54 & -3.54 & 7.09 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3.54 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.54 & 0 \\ 0 & 0 & 0 & 0 & 0 & -13.11 \end{bmatrix}. \tag{5.4.3.5}$$

Like irreducible decomposition method, if we add the matrices given in equations (5.4.3.1), (5.4.3.2), (5.4.3.3), (5.4.3.4) and (5.4.3.5) we obtain the original matrix for Zircon given in (5.4.1).

5.4.4 By Spectral Method

Eigenvalues of Zircon are found by the formula; $|C - \lambda_k I| = 0$, corresponding normalized eigenvectors are found by the formula; $(C - \lambda_k I)\overrightarrow{n}^{(k)} = 0$.From these formulas, eigenvalues are

$$\lambda_1 = 502.99, \quad \lambda_2 = 211, \quad \lambda_3 = 163, \quad \lambda_4 = \lambda_5 = 77.5, \quad \lambda_6 = 47.7.$$

The corresponding normalized eigenvectors are found as

$$\overrightarrow{n}^{(1)} = [0.5341, 0.5341, 0.6553, 0, 0, 0]^T,$$

$$\overrightarrow{n}^{(2)} = [0.7071, -0.7071, 0, 0, 0, 0]^T,$$

$$\overrightarrow{n}^{(3)} = [0.4634, 0.4634, -0.7554, 0, 0, 0]^T,$$

$$\overrightarrow{n}^{(4)} = [0, 0, 0, 1, 0, 0]^T,$$

$$\overrightarrow{n}^{(5)} = [0,0,0,0,1,0]^T\,,$$
$$\overrightarrow{n}^{(6)} = [0,0,0,0,0,1]^T\,.$$

By applying the formula given in equation (4.4.4.1) and using these eigenvalues and eigenvectors, elastic constant tensor for Zircon can be decomposed by spectral method as

$$C_{ij} = \begin{bmatrix} 143.49 & 143.49 & 176.05 & 0 & 0 & 0 \\ 143.49 & 143.49 & 176.05 & 0 & 0 & 0 \\ 176.05 & 176.05 & 216 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 105.5 & -105.5 & 0 & 0 & 0 & 0 \\ -105.5 & 105.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} +$$
$$\begin{bmatrix} 35 & 35 & -57.06 & 0 & 0 & 0 \\ 35 & 35 & -57.06 & 0 & 0 & 0 \\ -57.06 & -57.06 & 93 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 77.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} +$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 77.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 47.7 \end{bmatrix}. \tag{5.4.4.1}$$

The summation in equation (5.4.4.1) gives the original matrix for Zircon given in (5.4.1).

5.5 For Trigonal Material

Trigonal materials exhibit trigonal symmetry. There are six independent elastic constants for trigonal symmetry which are $C_{11}, C_{12}, C_{13}, C_{14}, C_{33}, C_{44}$. Haematite ($Fe_2O_3$) is selected as an example for trigonal materials and the elastic coefficients in GPa for Haematite are given[55]

$$C_{ij} = \begin{bmatrix} 242 & 54.9 & 15.7 & -12.7 & 0 & 0 \\ 54.9 & 242 & 15.7 & 12.7 & 0 & 0 \\ 15.7 & 15.7 & 228 & 0 & 0 & 0 \\ -12.7 & 12.7 & 0 & 85.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 85.3 & -12.7 \\ 0 & 0 & 0 & 0 & -12.7 & 93.55 \end{bmatrix} \tag{5.5.1}$$

Each method is applied to the material for decomposition of its elastic constants.

5.5.1 By Orthonormal Tensor Basis Method

By orthonormal tensor basis method, we apply the formula given in equation (4.1.5.2). For this reason inner products must be calculated as presented below

$$(C, A^{I}) = 294.87 \tag{5.5.1.1}$$

$$(C, A^{II}) = 422.8 \tag{5.5.1.2}$$

$$(C, A^{III}) = 4.08 \tag{5.5.1.3}$$

$$(C, A^{IV}) = -57.77 \tag{5.5.1.4}$$

$$(C, A^{V}) = -16.5 \tag{5.5.1.5}$$

$$(C, A^{VI}) = -50.8 \tag{5.5.1.6}$$

The symmetric fourth rank tensor for Haematite can be represented in the form

$$C_{ijkl} = 294.87A^{I}_{ijkl} + 422.8A^{II}_{ijkl} + 4.08A^{III}_{ijkl} - 57.77A^{IV}_{ijkl} - 16.5A^{V}_{ijkl} - 50.8A^{VI}_{ijkl} \tag{5.5.1.7}$$

If we substitute the values of the elastic constants into the right-hand side of equation (5.5.1.7), we end up with the identical matrix of equation (5.5.1), which shows the validity of the decomposed terms. Isotropic part of Haematite is

$$I = 294.87A^{I}_{ijkl} + 422.8A^{II}_{ijkl} \tag{5.5.1.8}$$

If we put the appropriate values into equation (5.5.1.8), isotropic part is found as

$$I = \begin{bmatrix} 224.35 & 35.26 & 35.26 & 0 & 0 & 0 \\ 35.26 & 224.35 & 35.26 & 0 & 0 & 0 \\ 35.26 & 35.26 & 224.35 & 0 & 0 & 0 \\ 0 & 0 & 0 & 94.54 & 0 & 0 \\ 0 & 0 & 0 & 0 & 94.54 & 0 \\ 0 & 0 & 0 & 0 & 0 & 94.54 \end{bmatrix} \tag{5.5.1.9}$$

The transversely isotropic part of the material is

$$TI = 4.08A^{III}_{ijkl} - 57.77A^{IV}_{ijkl} - 16.5A^{V}_{ijkl} \tag{5.5.1.10}$$

If we put the appropriate values into equation (5.5.1.10), transversely isotropic part is found as

$$TI = \begin{bmatrix} 17.65 & 19.64 & -19.56 & 0 & 0 & 0 \\ 19.64 & 17.65 & -19.56 & 0 & 0 & 0 \\ -19.56 & -19.56 & 3.65 & 0 & 0 & 0 \\ 0 & 0 & 0 & -9.24 & 0 & 0 \\ 0 & 0 & 0 & 0 & -9.24 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.99 \end{bmatrix}. \tag{5.5.1.11}$$

The trigonal part of the material is

$$TR = -50.8A^{VI}_{ijkl} \tag{5.5.1.12}$$

If we put the appropriate values into equation (5.5.1.12), trigonal part is found as

$$TR = \begin{bmatrix} 0 & 0 & 0 & -12.7 & 0 & 0 \\ 0 & 0 & 0 & 12.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -12.7 & 12.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -12.7 \\ 0 & 0 & 0 & 0 & -12.7 & 0 \end{bmatrix}. \qquad (5.5.1.13)$$

If we add the matrices given in (5.5.1.9), (5.5.1.11),and (5.5.1.13) we obtain the original matrix for Haematite given in (5.5.1).

5.5.2 By Irreducible Decomposition Method

There are five irreducible parts which are scalars; $C_{mn}^{(0;1)}, C_{mn}^{(0;2)}$, deviators; $C_{mn}^{(2;1)}, C_{mn}^{(2;2)}$ and a nonor $C_{mn}^{(4;1)}$ given in section 4.2.5. For Haematite, these parts are calculated as

$$C_{mn}^{(0;1)} = \begin{bmatrix} 98.29 & 98.29 & 98.29 & 0 & 0 & 0 \\ 98.29 & 98.29 & 98.29 & 0 & 0 & 0 \\ 98.29 & 98.29 & 98.29 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (5.5.2.1)$$

$$C_{mn}^{(0;2)} = \begin{bmatrix} 126.06 & -63.03 & -63.03 & 0 & 0 & 0 \\ -63.03 & 126.06 & -63.03 & 0 & 0 & 0 \\ -63.03 & -63.03 & 126.06 & 0 & 0 & 0 \\ 0 & 0 & 0 & 94.54 & 0 & 0 \\ 0 & 0 & 0 & 0 & 94.54 & 0 \\ 0 & 0 & 0 & 0 & 0 & 94.54 \end{bmatrix}, \qquad (5.5.2.2)$$

$$C_{mn}^{(2;1)} = \begin{bmatrix} 5.93 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5.93 & 0 & 0 & 0 & 0 \\ 0 & 0 & -11.87 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.48 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.48 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.97 \end{bmatrix}, \tag{5.5.2.3}$$

$$C_{mn}^{(2;2)} = \begin{bmatrix} 3.37 & 16.857 & -8.43 & 0 & 0 & 0 \\ 16.857 & 3.37 & -8.43 & 0 & 0 & 0 \\ -8.43 & -8.43 & -6.74 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.37 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.37 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6.74 \end{bmatrix}, \tag{5.5.2.4}$$

$$C_{mn}^{(4;1)} = \begin{bmatrix} 8.35 & 2.783 & -11.13 & -12.7 & 0 & 0 \\ 2.783 & 8.35 & -11.13 & 12.7 & 0 & 0 \\ -11.13 & -11.13 & 22.26 & 0 & 0 & 0 \\ -12.7 & 12.7 & 0 & -11.13 & 0 & 0 \\ 0 & 0 & 0 & 0 & -11.13 & -12.7 \\ 0 & 0 & 0 & 0 & -12.7 & 2.783 \end{bmatrix}. \tag{5.5.2.5}$$

If we add the matrices given in equations (5.5.2.1), (5.5.2.2), (5.5.2.3), (5.5.2.4) and (5.5.2.5), we obtain the original matrix for Haematite given in (5.5.1).

5.5.3 By Harmonic Decomposition Method

There are five irreducible parts which are scalars; S_1, S_2, deviators; D_1, D_2 and a nonor H given in section 4.3.5. For Haematite, these parts are calculated as

$$S_1 = \begin{bmatrix} 35.26 & 35.26 & 35.26 & 0 & 0 & 0 \\ 35.26 & 35.26 & 35.26 & 0 & 0 & 0 \\ 35.26 & 35.26 & 35.26 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.5.3.1}$$

$$S_2 = \begin{bmatrix} 189.09 & 0 & 0 & 0 & 0 & 0 \\ 0 & 189.09 & 0 & 0 & 0 & 0 \\ 0 & 0 & 189.09 & 0 & 0 & 0 \\ 0 & 0 & 0 & 94.54 & 0 & 0 \\ 0 & 0 & 0 & 0 & 94.54 & 0 \\ 0 & 0 & 0 & 0 & 0 & 94.54 \end{bmatrix}, \tag{5.5.3.2}$$

$$D_1 = \begin{bmatrix} 5.93 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5.93 & 0 & 0 & 0 & 0 \\ 0 & 0 & -11.87 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.48 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.48 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.97 \end{bmatrix}, \tag{5.5.3.3}$$

$$D_2 = \begin{bmatrix} 3.37 & 16.857 & -8.43 & 0 & 0 & 0 \\ 16.857 & 3.37 & -8.43 & 0 & 0 & 0 \\ -8.43 & -8.43 & -6.74 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.37 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.37 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6.74 \end{bmatrix}, \tag{5.5.3.4}$$

$$H = \begin{bmatrix} 8.35 & 2.783 & -11.13 & -12.7 & 0 & 0 \\ 2.783 & 8.35 & -11.13 & 12.7 & 0 & 0 \\ -11.13 & -11.13 & 22.26 & 0 & 0 & 0 \\ -12.7 & 12.7 & 0 & -11.13 & 0 & 0 \\ 0 & 0 & 0 & 0 & -11.13 & -12.7 \\ 0 & 0 & 0 & 0 & -12.7 & 2.783 \end{bmatrix}. \tag{5.5.3.5}$$

If we add the matrices; S_1, S_2, D_1, D_2 and H, we obtain the original matrix for Haematite given in (5.5.1).

5.5.4 By Spectral Method

Eigenvalues of Haematite are found by the formula; $|C - \lambda_k I| = 0$, corresponding normalized eigenvectors are found by the formula; $(C - \lambda_k I)\overrightarrow{n}^{(k)} = 0$. From these formulas, eigenvalues are

$\lambda_1 = 303.435, \quad \lambda_2 = 221.465, \quad \lambda_3 = 190.1758, \quad \lambda_4 = 102.778,$

$\lambda_5 = 82.2242, \quad \lambda_6 = 76.072.$

The corresponding normalized eigenvectors are found as

$\overrightarrow{n}^{(1)} = [-0.6783, -0.6783, -0.2824, 0, 0, 0]^T$,

$\overrightarrow{n}^{(2)} = [0.1997, 0.1997, -0.9593, 0, 0, 0]^T$,

$\overrightarrow{n}^{(3)} = [0.697, -0.697, 0, -0.1688, 0, 0]^T$,

$\overrightarrow{n}^{(4)} = [0, 0, 0, 0, -0.588, 0.8090]^T$,

$$\overrightarrow{n}^{(5)} = [0.1194, -0.1194, 0, 0.9857, 0, 0]^T \,,$$

$$\overrightarrow{n}^{(6)} = [0, 0, 0, 0, -0.8090, -0.588]^T \,.$$

By applying the formula given in equation (4.4.5.1) and using these eigenvalues and eigenvectors, elastic constant tensor for Haematite can be decomposed by spectral method as

$$C_{ij} = \begin{bmatrix} 139.608 & 139.608 & 58.124 & 0 & 0 & 0 \\ 139.608 & 139.608 & 58.124 & 0 & 0 & 0 \\ 58.124 & 58.124 & 24.199 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 8.832 & 8.832 & -42.427 & 0 & 0 & 0 \\ 8.832 & 8.832 & -42.427 & 0 & 0 & 0 \\ -42.427 & -42.427 & 203.805 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 92.3891 & -92.3891 & 0 & -22.3749 & 0 & 0 \\ -92.3891 & 92.3891 & 0 & 22.3749 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -22.3749 & 22.3749 & 0 & 5.4188 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 35.5349 & -48.8907 \\ 0 & 0 & 0 & 0 & -48.8907 & 67.2663 \end{bmatrix}$$

$$+ \begin{bmatrix} 1.172 & -1.172 & 0 & 9.677 & 0 & 0 \\ -1.172 & 1.172 & 0 & -9.677 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 9.677 & -9.677 & 0 & 79.889 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 49.788 & 36.187 \\ 0 & 0 & 0 & 0 & 36.187 & 26.301 \end{bmatrix} . \tag{5.5.4.1}$$

The above summation gives the original matrix for Haematite given in (5.5.1).

5.6 For Orthorhombic Material

Orthorhombic media show an orthorhombic symmetry. When the same symmetry is applied to textured or non-crystalline materials, it is generally called orthotropy. Most of the elastic bodies in engineering, biological tissues, rock structures, metal crystals (due to the symmetries of the lattice), most of the polycrystalline textured materials and fiber-reinforced composites (because of the production technique reasons), are, with acceptable accuracy, considered as orthotropic. However, for most practical purposes, orthotropic symmetry is equivalent to orthorhombic symmetry. There are nine independent elastic constants for orthorhombic symmetry which are $C_{11}, C_{22}, C_{33}, C_{12}, C_{13}, C_{23}, C_{44}, C_{55}, C_{66}$. Olivine is presented as an orthorhombic material and the elastic coefficients in GPa for Olivine are given[56]

$$C_{ij} = \begin{bmatrix} 192 & 66 & 60 & 0 & 0 & 0 \\ 66 & 160 & 56 & 0 & 0 & 0 \\ 60 & 56 & 272 & 0 & 0 & 0 \\ 0 & 0 & 0 & 60 & 0 & 0 \\ 0 & 0 & 0 & 0 & 62 & 0 \\ 0 & 0 & 0 & 0 & 0 & 49 \end{bmatrix} \tag{5.6.1}$$

Each method is applied to the material for decomposition of its elastic constants.

5.6.1 By Orthonormal Tensor Basis Method

By orthonormal tensor basis method, we apply the formula given in equation (4.1.6.2). For this reason inner products must be calculated as presented below

$$(C, A^{I}) = 329.33 \tag{5.6.1.1}$$

$$(C, A^{II}) = 284.73 \tag{5.6.1.2}$$

$$(C, A^{III}) = 86.461 \tag{5.6.1.3}$$

$$(C, A^{IV}) = -8.67 \tag{5.6.1.4}$$

$$(C, A^{V}) = 18 \tag{5.6.1.5}$$

$$(C, A^{VI}) = -8.49 \tag{5.6.1.6}$$

$$(C, A^{VII}) = 22.6 \tag{5.6.1.7}$$

$$(C, A^{VIII}) = 4 \tag{5.6.1.8}$$

$$(C, A^{IX}) = -2.83 \tag{5.6.1.9}$$

The symmetric fourth rank tensor for Olivine can be represented in the form

$$\begin{aligned} C_{ijkl} &= 329.33A^{I}_{ijkl} + 284.73A^{II}_{ijkl} + 86.461A^{III}_{ijkl} - 8.67A^{IV}_{ijkl} + 18A^{V}_{ijkl} \\ &-8.49A^{VI}_{ijkl} + 22.6A^{VII}_{ijkl} + 4A^{VIII}_{ijkl} - 2.83A^{IX}_{ijkl}. \end{aligned} \tag{5.6.1.10}$$

If we substitute the values of the elastic constants into the right-hand side of equation (5.6.1.10), we end up with the identical matrix of equation (5.6.1), which shows the validity of the decomposed terms.

Isotropic part of the Olivine:

$$I = 329.33A^{I}_{ijkl} + 284.73A^{II}_{ijkl} \tag{5.6.1.11}$$

If we put the appropriate values into equation (5.6.1.11), isotropic part is found as

$$I = \begin{bmatrix} 194.7 & 67.3 & 67.3 & 0 & 0 & 0 \\ 67.3 & 194.7 & 67.3 & 0 & 0 & 0 \\ 67.3 & 67.3 & 194.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 63.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 63.7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 63.7 \end{bmatrix} \tag{5.6.1.12}$$

Transversely isotropic part of the Olivine:

$$TI = 86.461A_{ijkl}^{III} - 8.67A_{ijkl}^{IV} + 18A_{ijkl}^{V} \tag{5.6.1.13}$$

If we put the appropriate values into equation (5.6.1.13), transversely isotropic part is found as

$$TI = \begin{bmatrix} -21.7 & 1.7 & -9.3 & 0 & 0 & 0 \\ 1.7 & -21.7 & -9.3 & 0 & 0 & 0 \\ -9.3 & -9.3 & 77.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2.7 & 0 \\ 0 & 0 & 0 & 0 & 0 & -11.7 \end{bmatrix} \tag{5.6.1.14}$$

Tetragonal part of the Olivine:

$$Tet = -8.49A_{ijkl}^{VI} \tag{5.6.1.15}$$

If we put the appropriate values into equation (5.6.1.15), tetragonal part is found as

$$Tet = \begin{bmatrix} 3 & -3 & 0 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix} \tag{5.6.1.16}$$

Orthorhombic part of the Olivine:

$$O = 22.6A_{ijkl}^{VII} + 4A_{ijkl}^{VIII} - 2.83A_{ijkl}^{IX} \tag{5.6.1.17}$$

If we put the appropriate values into equation (5.6.1.17), orthorhombic part is found as

$$O=\begin{bmatrix}16 & 0 & 2 & 0 & 0 & 0\\ 0 & -16 & -2 & 0 & 0 & 0\\ 2 & -2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} \tag{5.6.1.18}$$

If we add the matrices given in (5.6.1.12), (5.6.1.14), (5.6.1.16) and (5.6.1.18) we obtain the original matrix for Olivine given in (5.6.1).

5.6.2 By Irreducible Decomposition Method

There are five irreducible parts which are scalars; $C_{mn}^{(0;1)}, C_{mn}^{(0;2)}$, deviators; $C_{mn}^{(2;1)}, C_{mn}^{(2;2)}$ and a nonor $C_{mn}^{(4;1)}$ given in section 4.2.6. For Olivine, these parts are calculated as

$$C_{mn}^{(0;1)}=\begin{bmatrix}109.78 & 109.78 & 109.78 & 0 & 0 & 0\\ 109.78 & 109.78 & 109.78 & 0 & 0 & 0\\ 109.78 & 109.78 & 109.78 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}, \tag{5.6.2.1}$$

$$C_{mn}^{(0;2)}=\begin{bmatrix}84.89 & -42.44 & -42.44 & 0 & 0 & 0\\ -42.44 & 84.89 & -42.44 & 0 & 0 & 0\\ -42.44 & -42.44 & 84.89 & 0 & 0 & 0\\ 0 & 0 & 0 & 63.67 & 0 & 0\\ 0 & 0 & 0 & 0 & 63.67 & 0\\ 0 & 0 & 0 & 0 & 0 & 63.67\end{bmatrix}, \tag{5.6.2.2}$$

$$C_{mn}^{(2;1)} = \begin{bmatrix} -15.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -42.40 & 0 & 0 & 0 & 0 \\ 0 & 0 & 57.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -14.4 \end{bmatrix}, \tag{5.6.2.3}$$

$$C_{mn}^{(2;2)} = \begin{bmatrix} 1.10 & -0.762 & 3.52 & 0 & 0 & 0 \\ -0.762 & -1.41 & -2.76 & 0 & 0 & 0 \\ 3.52 & -2.76 & 0.305 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.10 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.41 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.305 \end{bmatrix}, \tag{5.6.2.4}$$

$$C_{mn}^{(4;1)} = \begin{bmatrix} 11.43 & -0.57 & -10.857 & 0 & 0 & 0 \\ -0.57 & 9.14 & -8.57 & 0 & 0 & 0 \\ -10.857 & -8.57 & 19.43 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8.57 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10.857 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.57 \end{bmatrix}. \tag{5.6.2.5}$$

If we add the matrices given in equations (5.6.2.1), (5.6.2.2), (5.6.2.3), (5.6.2.4) and (5.6.2.5), we obtain the original matrix for Olivine given in (5.6.1).

5.6.3 By Harmonic Decomposition Method

There are five harmonic parts which are scalars; S_1, S_2, deviators; D_1, D_2 and a nonor H given in section 4.3.6. For Olivine, these parts are calculated as

$$S_1 = \begin{bmatrix} 67.33 & 67.33 & 67.33 & 0 & 0 & 0 \\ 67.33 & 67.33 & 67.33 & 0 & 0 & 0 \\ 67.33 & 67.33 & 67.33 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.6.3.1}$$

$$S_2 = \begin{bmatrix} 127.33 & 0 & 0 & 0 & 0 & 0 \\ 0 & 127.33 & 0 & 0 & 0 & 0 \\ 0 & 0 & 127.33 & 0 & 0 & 0 \\ 0 & 0 & 0 & 63.67 & 0 & 0 \\ 0 & 0 & 0 & 0 & 63.67 & 0 \\ 0 & 0 & 0 & 0 & 0 & 63.67 \end{bmatrix}, \tag{5.6.3.2}$$

$$D_1 = \begin{bmatrix} -15.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -42.40 & 0 & 0 & 0 & 0 \\ 0 & 0 & 57.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -14.4 \end{bmatrix}, \tag{5.6.3.3}$$

$$D_2 = \begin{bmatrix} 1.10 & -0.762 & 3.52 & 0 & 0 & 0 \\ -0.762 & -1.41 & -2.76 & 0 & 0 & 0 \\ 3.52 & -2.76 & 0.305 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.10 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.41 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.305 \end{bmatrix}, \qquad (5.6.3.4)$$

$$H = \begin{bmatrix} 11.43 & -0.57 & -10.857 & 0 & 0 & 0 \\ -0.57 & 9.14 & -8.57 & 0 & 0 & 0 \\ -10.857 & -8.57 & 19.43 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8.57 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10.857 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.57 \end{bmatrix}. \qquad (5.6.3.5)$$

If we add the matrices; S_1, S_2, D_1, D_2 and H, we obtain the original matrix for Olivine given in (5.6.1).

5.6.4 By Spectral Method

Eigenvalues of Olivine are found by the formula; $|C - \lambda_k I| = 0$, corresponding normalized eigenvectors are found by the formula; $(C - \lambda_k I)\overrightarrow{n}^{(k)} = 0$. From these formulas, eigenvalues are

$\lambda_1 = 341.054, \quad \lambda_2 = 175.2736, \quad \lambda_3 = 107.6724, \quad \lambda_4 = 60, \quad \lambda_5 = 62,$
$\lambda_6 = 49.$

The corresponding normalized eigenvectors are found as

$\overrightarrow{n}^{(1)} = [0.4920, 0.4159, 0.7648, 0, 0, 0]^T$,

$\overrightarrow{n}^{(2)} = [0.6416, 0.4206, -0.6415, 0, 0, 0]^T$,

$\overrightarrow{n}^{(3)} = [0.5884, -0.8063, 0.0599, 0, 0, 0]^T$,

$\overrightarrow{n}^{(4)} = [0, 0, 0, 1, 0, 0]^T$,

$$\overrightarrow{n}^{(5)} = [0,0,0,0,1,0]^T\,,$$

$$\overrightarrow{n}^{(6)} = [0,0,0,0,0,1]^T\,.$$

By applying the formula given in equation (4.4.6.1) and using these eigenvalues and eigenvectors, elastic constant tensor for Olivine can be decomposed by spectral method as

$$C_{ij} = \begin{bmatrix} 82.557 & 69.787 & 128.332 & 0 & 0 & 0 \\ 69.787 & 58.993 & 108.483 & 0 & 0 & 0 \\ 128.332 & 108.483 & 199.489 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 72.152 & 47.299 & -72.140 & 0 & 0 & 0 \\ 47.299 & 31.007 & -47.291 & 0 & 0 & 0 \\ -72.140 & -47.291 & 72.129 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 37.278 & -51.083 & 3.795 & 0 & 0 & 0 \\ -51.083 & 69.999 & -5.200 & 0 & 0 & 0 \\ 3.795 & -5.200 & 0.386 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 60 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 62 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 49 \end{bmatrix}. \qquad (5.6.4.1)$$

The above summation gives the original matrix for Olivine given in (5.6.1).

CHAPTER 6

NORM, NORM RATIO CALCULATIONS AND ANISOTROPY DEGREE

The norm concept for elastic constant tensor is described, norm and norm ratios as well as the measure of 'nearness' to the nearest isotropic tensor are computed for several examples from various anisotropic materials possessing elastic symmetries such as cubic, transversely isotropic, tetragonal, trigonal and orthorhombic. These computations are used to compare and assess the anisotropy in various anisotropic materials by means of strength or magnitude and also determine the 'nearness' of the nearest isotropic tensor for the materials with lower symmetry types.

6.1 The Norm Concept

Norm is an invariant of the material. For instance, the magnitude of length is a norm for a vector. If it is a unit vector, its norm will be equal to 1.There are many types of norm in literature. Those norms are Euclidean, Riemannian, log-Euclidean, Taxicab, infinity, uniform, zero and so on. These norms are used in different fields of science and engineering. For instance, zero norm is related with machine learning and optimization. Log-Euclidean norm is a measure for tensors such as symmetric positive-definite matrices in medical imaging, modeling of anatomical variability i.e. human brain variability and Riemannian and log-Euclidean norms are used to find the shortest distance between an elasticity tensor of arbitrary symmetry and an elasticity tensor of lower symmetry.[80]

These two norms are effective when elastic compliance tensor is considered. Since the most appropriate and reliable norm for elastic constant tensor is Euclidean norm in literature, Euclidean norm is used for computations as a measure in this thesis. Comparison of magnitudes of the Euclidean norm give a valuable information about the origin of the physical property under examination. Euclidean norm also represents the stiffness effect in the material like fiber-reinforced composites.

Euclidean norm of a Cartesian tensor is defined as the square root of the contracted product over all the indices with itself, which is given as follows

$$N = \|C\| = \{C_{ijkl\ldots}C_{ijkl\ldots}\}^{\frac{1}{2}} \tag{6.1.1}$$

Since the basis constructed in this thesis is orthonormal and $C_{ijkl\ldots}$ is in the space spanned by that orthonormal basis $\{A^K\}$, it is straightforward to see that, now the norm

$$N = \|C\| = \{\sum_K (C, A^K_{ijkl})^2\}^{\frac{1}{2}} \tag{6.1.2}$$

The norm of nearest isotropic tensor, denoted by C^o_{iikl}, of C_{ijkl} is therefore

$$N_i = \|C^o\| = \{\sum_{K=I} (C^o, A^K_{ijkl})^2\}^{\frac{1}{2}}, (K = I, II) \tag{6.1.3}$$

In similar way, with respect to the tensor C_{ijkl}, the nearest tensors of other symmetry classes within the class spanned by the basis $\{A^K\}$ can be read off from the representation and their norms may be computed according to equation (6.1.2).

By using the norms, the nearest isotropic tensors of lower symmetries such as cubic, transversely isotropic, tetragonal, trigonal and orthorhombic can be found via the following formula[6]

$$\varepsilon^o = \frac{\|C\| - \|C^o\|}{\|C\|} \tag{6.1.4}$$

where ε^o is a scalar constant independent of the rotation of the axes. It is a measure of 'nearness' of the nearest isotropic tensor.

6.2 A Proposed Relation Between the Norm Ratio and the Anisotropy Degree

It is obvious that the anisotropy of the material, for instance, the symmetry group of the material and the anisotropy of the measured property depicted in the same materials may be quite different. Clearly, the property tensor must show, at least, the symmetry of the material. For instance, a property which is measured in a material can almost be isotropic but the material symmetry group itself may have very few symmetry elements. We know that, for isotropic materials, the elastic constant tensor has two scalar (isotropic) parts, so the norm of the elastic constant tensor for isotropic materials depends only on the norm of the scalar parts, i.e., $N = N_i$. So, the ratio $\frac{N_i}{N} = 1$ for isotropic materials. For cubic materials, the elastic constant tensor has the same two parts that consisting the isotropic symmetry and a third which is designated as the anisotropic part, hence we define two ratios: $\frac{N_i}{N}$ for the isotropic parts and $\frac{N_a}{N}$ for the anisotropic part. For lower symmetry type materials such as transversely isotropic, tetragonal, trigonal and orthorhombic, the elastic constant tensor additionally contains more anisotropic parts, so we can define $\frac{N_a}{N}$ for all the anisotropic parts.

Although the norm ratios of different parts represent the anisotropy of that particular part, they can also be used to asses and compare the anisotropy degree of a material property as a whole.

6.3 Examples From Cubic Symmetry

Elastic constants of cubic materials are given in Table 6.1. The units are in GPa.

Table 6.1: Elastic constant data of cubic materials

Cubic Media	C_{11}	C_{12}	C_{44}
AlSb[52]	87.7	43.4	40.8
Indium Phosphide(InP)[57]	102	58	46
Gallium Arsenide(GaAs)[58]	118	53.5	59.4
Gallium Antimonide(GaSb)[59]	88.4	40.3	43.4
Indium Arsenide(InAs)[60]	84.4	46.4	39.6
Gallium Phosphide(GaP)[61]	142	63	71.6

The norm and norm ratios (the anisotropy degrees) for cubic materials are calculated in order to determine which one is close to isotropy or anisotropy. The results for norm, norm ratios and the measure of 'nearness' of the nearest isotropic tensor are presented in Table 6.2.

Table 6.2: The norm and norm ratios (the anisotropy degrees) for cubic materials

Cubic Media	N_i	N_a	N	$\frac{N_i}{N}$	$\frac{N_a}{N}$	ε^o
AlSb	229.524	40.8601	233.1328	0.9845	0.1753	0.0155
InP	272.072	52.5814	277.1065	0.9818	0.1898	0.0182
GaAs	312.646	59.4827	318.2543	0.9824	0.1869	0.0176
GaSb	232.3655	42.3937	236.2011	0.9838	0.1795	0.0162
InAs	225.9841	45.1323	230.4469	0.9806	0.1958	0.0194
GaP	375.3382	70.3276	381.87	0.9829	0.1842	0.0171

According to the calculated results in Table 6.2, the most isotropic material among the other six materials is Aluminium Antimonide (AlSb). Since mathematically, $\frac{N_i}{N}$ for AlSb is so close to 1 that implies the closeness to the isotropic behaviour of the cubic materials which agrees with the physical understanding of the materials with cubic symmetry. This case is also verified by taking into account the results of ε^o which is closer to 0 than those of other five materials which indicates that AlSb is nearest to isotropy among the other materials. The most anisotropic material is selected as Indium Arsenide (InAs). Since the value of $\frac{N_i}{N}$ for InAs is the smallest and in reverse manner, the value of $\frac{N_a}{N}$ for InAs is the largest among the cubic materials. This case shows that the property of Indium Arsenide is the most anisotropic.

6.4 Examples From Transversely Isotropic Symmetry

Elastic constants of transversely isotropic materials are given in Table 6.3. The units
are in GPa.

Table 6.3: Elastic constant data of transversely isotropic materials

Transversely isotropic Media	C_{11}	C_{12}	C_{13}	C_{33}	C_{44}
Polystyrene[53]	5.20	2.75	2.75	5.70	1.30
Hardened tool steel[62]	277	113	112	272	80.3
Zinc(Zn)[63]	165	31.1	50	61.8	39.6
Cadmium[64]	116	42	41	50.9	19.6
Normal tool steel[65]	289	116	117	284	84.5

For transversely isotropic materials, the norm and norm ratios, ε^o (the anisotropy degrees) are computed in order to determine which one is close to isotropy or anisotropy. The results for norm, norm ratios and the measure of 'nearness' of the nearest isotropic tensor are presented in Table 6.4.

Table 6.4: The norm and norm ratios (the anisotropy degrees) for transversely isotropic materials

Transversely isotropic Media	N_i	N_a	N	$\frac{N_i}{N}$	$\frac{N_a}{N}$	ε^o
Polystyrene	12.2996	0.432	12.3100	0.9994	0.0351	0.000617
Hardened tool steel	617.745	5.257	617.768	1.000	0.0085	0.000036
Zinc(Zn)	301.619	98.510	317.298	0.9506	0.3105	0.049400
Cadmium	211.340	60.330	219.7819	0.9616	0.2745	0.038400
Normal tool steel	645.282	5.367	645.3038	1.000	0.0083	0.0000346

From Table 6.4, it is seen that the ratio $\frac{N_i}{N}$ gives the same result for hardened and normal tool steel which is equal to 1 and the results for ε^o is close to each other. But ε^o of normal tool steel is smaller than ε^o of hardened tool steel which shows that normal tool steel is more isotropic than the hardened one. The same case is also proved by comparing the $\frac{N_a}{N}$ for both tool steel type. The larger ratio $\frac{N_a}{N}$ and ε^o, the more anisotropic property exists for a transversely isotropic material and in reverse manner, the smaller ratio $\frac{N_i}{N}$, a transversely isotropic material possesses the more anisotropic property. So Zinc is the most anisotropic material among the other transversely isotropic materials.

6.5 Examples From Tetragonal Symmetry

Elastic constants of tetragonal materials are presented in Table 6.5. The units are in
GPa.

Table 6.5: Elastic constant data of tetragonal materials

Tetragonal Media	C_{11}	C_{12}	C_{13}	C_{33}	C_{44}	C_{66}
Zircon, ZrSiO$_4$ (metamict)[54]	284	73	119	309	77.5	47.7
Indium-cadmium alloy, In-3.42 at %Cd[66]	44.8	41	40.5	44.1	6.86	11.25
Ammonium dihydrogen arsenate (piezoel.), NH$_4$H$_2$ASO$_4$[67]	62.2	8.6	18.4	29.6	6.69	6.22
Rolled steel[68]	284	96	112	269	82.1	68.9
Indium bismuth(InBi)[69]	51.1	37	32	34.6	19.8	15.9

The norm and norm ratios, ε^o (the anisotropy degrees) for tetragonal materials are calculated in order to determine the effect of anisotropy in other words which one is more anisotropic or isotropic. The results for norm, norm ratios and the measure of 'nearness' of the nearest isotropic tensor are summarized in Table 6.6.

Table 6.6: The norm and norm ratios (the anisotropy degrees) for tetragonal materials

Tetragonal Media	N_i	N_a	N	$\frac{N_i}{N}$	$\frac{N_a}{N}$	ε^o
Zircon, $ZrSiO_4$ (metamict)	610.083	95.11	617.45	0.9881	0.1540	0.0119
Indium-cadmium alloy, In-3.42 at %Cd	128.521	15.717	129.478	0.9926	0.1214	0.0074
Ammonium dihydrogen arsenate (piezoel.), $NH_4H_2ASO_4$	95.653	38.462	103.097	0.9278	0.3731	0.0722
Rolled steel	611.468	36.06	612.53	0.9983	0.0589	0.0017
Indium bismuth(InBi)	128.03	31.77	131.91	0.9706	0.2408	0.0294

According to Table 6.6, by comparing the ratio $\frac{N_i}{N}$ and ε^o, rolled steel exhibits the most isotropic property among the others. On the other hand, by taking into account the ratio $\frac{N_a}{N}$, Ammonium dihydrogen arsenate (piezoel.) shows the most anisotropic property among the other tetragonal materials.

6.6 Examples From Trigonal Symmetry

Elastic constants of trigonal materials are given in Table 6.7. The units are in GPa.

Table 6.7: Elastic constant data of trigonal materials

Trigonal Media	C_{11}	C_{12}	C_{13}	C_{14}	C_{33}	C_{44}
Haematite, $\mathbf{Fe_2O_3}$[55]	242	54.9	15.7	-12.7	228	85.3
Antimony[70]	99.4	30.9	26.4	21.6	44.5	39.5
Magnesite, $\mathbf{MgCO_3}$[71]	259	75.6	58.8	-19	156	54.8
As-Sb at % As 25.5[72]	106.7	48.4	28.5	18.8	48	40.8
Arsenic[73]	130.2	30.3	64.3	-3.71	58.7	22.5

For trigonal materials, the norm and norm ratios, ε^o (the anisotropy degrees) are computed in order to determine which one is close to isotropy or anisotropy. The results for norm, norm ratios and the measure of 'nearness' of the nearest isotropic tensor are shown in Table 6.8.

Table 6.8: The norm and norm ratios (the anisotropy degress) for trigonal materials

Trigonal Media	N_i	N_a	N	$\frac{N_i}{N}$	$\frac{N_a}{N}$	ε^o
Haematite, $\mathbf{Fe_2O_3}$	515.476	78.782	521.46	0.9885	0.1511	0.0115
Antimony	202.253	100.854	226.004	0.8949	0.4462	0.1051
Magnesite, $\mathbf{MgCO_3}$	457.455	116.54	472.068	0.969	0.247	0.031
As-Sb at % As 25.5	214.39	97.027	235.327	0.911	0.4123	0.089
Arsenic	250.355	85.421	264.53	0.9464	0.3229	0.0536

From Table 6.8, it is understood that Haematite is the most isotropic material among the others by comparing the ratio $\frac{N_i}{N}$ and ε^o. Besides among trigonal materials, Antimony is the most anisotropic material by investigating the effect of the ratio $\dfrac{N_a}{N}$.

6.7 Examples From Orthorhombic Symmetry

Elastic constants of orthorhombic materials are presented in Table 6.9. The units are

in GPa.

Table 6.9: Elastic constant data of orthorhombic materials

Orthorhombic Media	C_{11}	C_{12}	C_{13}	C_{22}	C_{23}	C_{33}	C_{44}	C_{55}	C_{66}
Olivine[56]	192	66	60	160	56	272	60	62	49
Pine(Softwood)[74]	1.24	0.74	0.76	17.1	0.94	1.79	1.18	0.079	0.91
Olivinite[75]	232	93	92	210	82	199	73.3	70.9	68.6
Marble[75]	119	51	52	110	47	104	29.7	30.7	32.6
Canine femora[76]	19	9.73	11.9	22.2	11.9	29.7	6.67	5.67	4.67

The norm and norm ratios, ε^o (the anisotropy degrees) for orthorhombic materials are calculated in order to determine the effect of anisotropy in other words which one is more anisotropic or isotropic. The results for norm, norm ratios and the measure of 'nearness' of the nearest isotropic tensor are summarized in Table 6.10.

Table 6.10: The norm and norm ratios (the anisotropy degress) for orthorhombic materials

Orthorhombic Media	N_i	N_a	N	$\frac{N_i}{N}$	$\frac{N_a}{N}$	ε^o
Olivine	435.35	93.267	445.228	0.9778	0.2095	0.0222
Pine(Softwood)	11.0247	15.9396	19.381	0.5688	0.8224	0.4312
Olivinite	494.479	35.1969	495.73	0.9975	0.0710	0.0025
Marble	251.9798	12.8411	252.3067	0.9987	0.0509	0.0013
Canine femora	53.0038	8.9226	53.7495	0.9861	0.1660	0.0139

In Table 6.10, by taking into account the effect of the norm ratios; $\frac{N_i}{N}$, $\frac{N_a}{N}$ and ε^o, it is obvious that marble is an orthorhombic material that possesses the most isotropic effect among the other orthorhombic materials with the largest value for ratio $\frac{N_i}{N}$ and the smallest value for ε^o. Pine, which is a softwood, exhibits the most anisotropic property among the others with the largest value for ratio $\frac{N_a}{N}$ and ε^o.

Conclusion:

The following significant notes are taken into account when we have evaluated the computed results in above tables. These notes are:

1. It can be used as a parameter representing and comparing the overall effect of a certain property of anisotropic materials of the same or different symmetry. If the norm value of a material is large, it has more effective property than the other materials of the same symmetry type.

2. When N_i is the largest among norms of the decomposed parts, if the norm ratio $\frac{N_i}{N}$ is closer to one, the material property is closer to isotropic.

3. When N_i is not the largest or not present, norms of the other parts can be used as a criterion. But in this case the situation is reverse; if the norm ratio value is larger than the others, the material property is more anisotropic.

Although we have used Euclidean norm when calculating norm, norm ratios, anisotropy degree for various anisotropic materials in this thesis, other norms found in literature such as log-Euclidean norm can be used in computations by applying above cases. In fact, the results of calculations given in tables for different materials do not significantly change depending on the norm types. As a future research, this can can be proved by comparing the results obtained from various norms for different or same elastic symmetry type materials.

CHAPTER 7

COMPARISON OF THE DECOMPOSITION METHODS

In the present chapter, we have compared the decomposition methods for both symmetric second rank tensors and elastic constant tensor to emphasize the significant effect of these methods on symmetric second rank and elastic constant tensors by regarding the results of numerical examples of both symmetric second rank tensors and elastic constant tensor.

7.1 For Symmetric Second Rank Tensors

In chapter 3, we have focussed on the three decomposition methods for stress tensor. In order to understand the similarities and differences of these methods, we make comparisons by taking into account the numerical example.

According to orthonormal tensor basis and complex variable representation methods, we have six decomposed parts which are orthonormal to each other and each has a physical meaning. Decomposed parts obtained from orthonormal tensor basis and complex variable representation methods are linear and orthonormal to each other.

Unlike the other two methods, in spectral method, the procedure of decomposing is completely different. Since we require eigenvalues and eigenvectors to form decomposed parts. Eigenvalues are the functions of invariants which are expressed by several combinations of stress tensor components (recall equation (3.3.5)), so this makes the decomposition non-linear and as a result, we get three non-linear decomposed parts. The physical meanings of these three decomposed parts are different from that of six decomposed parts obtained from the other two methods. As an applied example to spectral method, eigenvalues and eigenvectors play a key role in interpreting the geodetic phenomena like earthquakes (seismic deformations), plate motions and plate deformations among others. They classify deformation and stress patterns in earthquake regions, plate tectonics and glacial isostatic adjustment (postglacial rebounds).[77]

7.2 For Elastic Constant Tensor

For comparison purposes, we find out critical relationships between irreducible and harmonic decomposition methods and the other works found in literature. Not only these relations but also the comparison of the four decomposition methods are summarized in this section.

In section (4.3), it is seen that there are many works done harmonic decomposition method in literature. Like irreducible decomposition, two scalars, two deviators and the nonor part are obtained in harmonic decomposition method. Hence total scalar, total deviatoric parts and nonor part are identical in two methods. This is the first relationship between these methods.

In irreducible method, components of total scalar parts; $C_{mn}^{(0;1)}$ and $C_{mn}^{(0;2)}$are orthogonal to each other. Contrary to it, decomposed parts of total scalar part are not orthogonal to each other in harmonic decomposition method due to the expression for decomposition of elastic constant tensor given in equations (4.3.6) and (4.3.11). Since $\delta_{ij}\delta_{kl}$ is not orthogonal to $\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$ (denoted as

$2I_{ijkl}$). If we replace $\delta_{ij}\delta_{kl}$ and $2I_{ijkl}$ by the hydrostatic and deviatoric operators

$$I^h_{ijkl} = \frac{1}{3}\delta_{ij}\delta_{kl}, \qquad I^d_{ijkl} = I_{ijkl} - I^h_{ijkl}, \tag{7.2.1}$$

respectively, then we obtain back the irreducible decomposition method[25] in which decomposed parts of total scalar parts are orthogonal to each other. So the components of total scalar part in harmonic decomposition method takes the form of $C^{(0;1)}_{mn}$ and $C^{(0;2)}_{mn}$. This case is a significant innovation for both decomposition methods for elastic constant tensor. It is the second relationship between both decomposition methods.

Besides, in irreducible decomposition method, by using equations (4.1.1.3), (4.2.1.2), we obtain two physically meaningful parts which are

$$J_1 = 3KI_1, \tag{7.2.2}$$

and

$$(\sigma_{ij} - \frac{1}{3}\sigma_{rr}\delta_{ij}) = 2G(\varepsilon_{ij} - \frac{1}{3}\varepsilon_{rr}\delta_{ij}), \tag{7.2.3}$$

where $J_1 = \sigma_{ii}$ and $I_1 = \varepsilon_{ii}$ are the first fundamental invariants of stress and strain tensors, respectively.

Equation (7.2.2) represents volume-change without distortion under hydrostatic stress and equation (7.2.3) represents shape-change without volume-change under deviatoric stress.

So irreducible decomposition method gives two isotropic (scalar) parts which have significant physical meanings.

On the other hand, scalar parts obtained from harmonic decomposition method do not have any physical meanings like irreducible decomposition method. By using equation (7.2.1), we can make scalar parts orthogonal to each other for harmonic decomposition method and then these parts also have same physical meanings as those in irreducible decomposition method. So orthogonal decomposition of elastic constant tensor is important.

As a result, components of scalar and deviatoric parts in irreducible method are not equal to those in harmonic decomposition method, so it proves that there is not a unique decomposition for both deviatoric and scalar parts, in other words total deviatoric and scalar parts can be decomposed into infinitely many independent components. This case also indicates that total scalar, deviatoric and nonor parts of elastic constant tensor obtained from irreducible decomposition methods are the same as those of harmonic decomposition method.

In harmonic decomposition method, we are able to construct the decomposed parts, obtained from irreducible decomposition method, by making decomposed parts of total scalar part orthogonal.

One of the important contribution of this thesis is that the sum of the total scalar and deviatoric parts are the same in both irreducible and harmonic decomposition methods.

Furthermore, in order to designate the similarities between the all decomposition methods as well as differences, we compare them. The following results are found out:

1. Orthonormal tensor basis and irreducible decomposition methods are valid for any tensor of rank n on the other hand spectral and harmonic decomposition methods can be applied to only fourth rank tensors such as elastic constant and compliance tensors.

2. Orthonormal tensor basis decomposition is decomposing the tensor to a number equal in general, to the nominal number of non-vanishing distinct coefficients under that group of symmetry and spectral decomposition is decomposing the general anisotropic elastic material into six terms at most whereas irreducible and harmonic decomposition methods decompose a given tensor into a number of irreducible parts less than or equal to the nominal number of non-vanishing distinct coefficients of a given symmetry class.

3. Decomposed parts of isotropic material are identical in orthonormal tensor basis, irreducible decomposition and spectral methods, contrary to these methods, components of isotropic material are different in harmonic decomposition method.

4. For cubic symmetry, decomposed parts are the same for orthonormal tensor basis and irreducible decomposition methods but this case is not valid for the other two methods.

5. Total scalar (isotropic) parts from irreducible and harmonic decomposition methods are identical with the isotropic parts of lower symmetry types such as transversely isotropic symmetry obtained from orthonormal tensor basis method. For transversely isotropic symmetry and lower symmetry types, the decomposed parts are also different from the four methods.

Following Ryclewski[39], spectral method can be called '**non-linear invariant decomposition**' and the other three decomposition methods are named as '**linear invariant decomposition**'.

CHAPTER 8

DISCUSSION AND CONCLUSIONS

The decomposition methods of tensors have many applications in different subjects of engineering. In the mechanics of continuous media, for instance, in elasticity studies; the stress and strain tensors are decomposed into spherical (hydrostatic) and deviatoric parts each of which have important meanings.

From equation (3.1.12), it is obvious that stress tensor is decomposed into spherical (hydrostatic pressure) part which is the first term of (3.1.11) and the deviatoric part which is the sum of the other five terms of equation (3.1.11). It is decomposed into traceless tensors, each of them is related to shearing which represents a general symmetric second rank tensor (stress and strain tensors). Each of the six tensor parts has physical meanings and all decomposed parts form an orthonormal set.

The first part of equation (3.1.11) represents the spherical (hydrostatic pressure) effect which is connected to the change of volume without change of shape through the bulk modulus. The second and third part represent combined simple extension or contraction along the various symmetry axes.

The second part is a special case of biaxial stress which is plane stressed state. This part could be, for example, the stresses which are produced by torsional loading in a shaft. For Mohr's circle construction, the center coincides with the origin of axes and a rotation of 90^o (on the circle) leads to a state of stress in which the normal stresses are zero. This rotation is equivalent to a 45^o rotation in the body (real space). The magnitude of the shear stress at this orientation is equal to the radius of the circle. It shows at once that if the axes are turned through 45^o about Ox_3(the axis of shear) then, the normal stresses vanish. This is why the stress is named as pure shear stress and the tensor takes the form of fourth, fifth and sixth parts respectively.[78]. Pure shear stress fields arise in many practical cases; for example, these type of stresses occur in the torsion of linearly elastic rods.[44]

Furthermore the components of the third part are proportional to $1, 1, -2$. This case may be an example for cylindrical shear. It is axisymmetric with respect to the Ox_3 axis which means invariant under a rotation about it. Thick walled cylindrical pressure vessels are one of the most typical applications of these type of stresses. The internal pressure will cause stresses in the material such that the hoop stress component is twice as much as the axial stress components (radial stresses and longitudinal stresses) for the cylindrical pressure vessel.[79] But the hoop stress and axial stresses are in different directions. This cause an advantage for engineering materials that can be made stronger in one direction than another (the property of anisotropy).

Last three parts represent simple shearing in the symmetry planes. The sum of these three parts correspond to state of pure shear stresses which is Cauchy stress tensor. It is traceless and symmetric. Pure shear stress state has been widely described in recent studies such as Blinowski and Rychlewski[44], Hayes and Laffey[45].

It should be noted that symmetric second rank tensors such as stress tensor and strain tensors are important subjects to understand the idea behind mechanics and elasticity. This is why decomposing them into orthonormal parts plays a significant role. Decomposition methods; orthonormal tensor basis and complex variable representation help us to figure out the physical meanings of these tensors by decomposing them into six parts which introduces a new form of decomposition. This property makes these two methods different than spectral method which yields three non linear orthonormal parts.

In this thesis, it is mainly emphasized that orthonormal tensor basis and complex variable representation are linear orthonormal methods and provide deep understanding of second and third decomposed parts as well as other parts since it is not stated before in the literature. Whereas spectral method is well known in literature and it is a non-linear orthonormal decomposition method.

Decomposition methods as irreducible decomposition, orthonormal tensor basis, harmonic decomposition and spectral have many applications in different subjects of physics and engineering (atomic and molecular physics and the physics of condensed matter). The decomposition methods of elastic constant tensor are applied to different fields of science and engineering. For instance, Geophysicists have used it in geophysical applications.[56]

Furhermore for very valuable materials like diamond or quartz used in mining, it is difficult to measure its elastic constants because of its small samples. Applying orthonormal tensor basis method, we are able to specify the elastic constants of these types of materials.[6] As an application for harmonic decomposition, it is possible to decide which type of symmetry a material has when the elastic constants are measured relative to an arbitrary coordinate system. A second rank symmetric tensor associated to the elastic constant tensor can be used to verify if the coordinate axes are the symmetry axes of the material and determine a symmetry coordinate system.[36] So comprehending the decomposition methods is considerable to understand the idea behind these decomposition methods as well as the physical properties of anisotropic materials.

In conclusion, it is seen that orthonormal tensor basis, irreducible and harmonic decomposition methods give orthogonal and linear decomposed parts, while spectral method is a non-linear and orthogonal decomposition method since decomposed parts are expressed in terms of functions of elastic constants.

The contributions of this thesis to literature are to appreciate a general review of orthogonal decomposition methods such as orthonormal tensor basis for both symmetric second rank and elastic constant tensors and irreducible decomposition for only elastic constant tensor, besides, provide a deep understanding the decomposition method, complex variable representation for symmetric second rank tensors, harmonic decomposition method for elastic constant tensor and spectral decomposition method for both type of tensors, finally stating the non-linear property of spectral method. Decomposition of elastic constant tensor into orthogonal parts by various methods provides a deeper understanding about elastic and mechanical behavior of anisotropic materials. It also has more significant effects on many applications in different fields such as:

1) calculation of norm and norm ratios for assessing and comparing the anisotropic properties of materials.

2) examining the material symmetry types in detail,

3) determination of materials possessing same crystal symmetry type which are highly anisotropic or close to isotropy,

4) understanding the mechanical and elastic behaviour of natural composites such as Bone and Wood types.

Related to the subject of this thesis, the following problems can be studied for future research:

1) The decomposition methods; orthonormal tensor basis and irreducible decomposition methods can be applied to third rank tensors such as general piezoelectric tensors so a better and deeper understanding of the physical meaning of these tensors may be achieved.

2) Different norms such as log-Euclidean and Riemannian can be computed for various anisotropic materials possessing each of the elastic symmetry types in order to compare the overall effect of the norms on these materials.

3) The pure shear and pure longitudinal wave propagation in different anisotropic materials such as transversely isotropic materials can be investigated.

4) The relation between the decomposed parts and the angle of orientation of fibers and the material properties of fibers and matrix of transversely isotropic materials in fiber reinforced composites can be studied.

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APPENDIX

Material symmetry of tensors can be considered as two types; rotational and reflectional. They also can be called crystal group symmetries. These symmetries can have a powerful influence on defining the allowed structure of matter tensors. Most materials possess some form of material symmetry which are isotropic, cubic, transversely isotropic, tetragonal, trigonal, orthorhombic and triclinic. A brief information for each of them is given here.

Isotropic symmetry:

A property of a material is said to possess isotropic symmetry with respect to a point when every plane passing through the point is a plane of mirror symmetry or a plane of isotropy. A plane of isotropy is a plane in which every vector is the normal to a plane of mirror symmetry.[40] The matrix form of elastic constant tensor is given as

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & (C_{11}-C_{12})/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (C_{11}-C_{12})/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_{11}-C_{12})/2 \end{bmatrix} \quad (1)$$

Cubic symmetry:

Cubic symmetry is characterized by nine planes of mirror symmetry. The nine planes of mirror symmetry include the three orthogonal planes of mirror symmetry. So cubic symmetry has nine planes of symmetry all intersecting at 90^o,

60^o or 45^o.[40] The elastic stiffness matrix has the following form with only three elastic constants.

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \tag{2}$$

Transversely isotropic symmetry:

Transversely isotropic medium is a special case of hexagonal symmetry. The hexagonal symmetry is a six-fold symmetry with seven planes of mirror symmetry. Six of the normals to these seven planes all lie in the seventh plane and make angles of thirty degrees with one another. Furthermore transversely isotropic symmetry is characterized by a single plane of isotropy.[40] The elastic constant tensor of transversely isotropic symmetry has almost the same structure as that of hexagonal symmetry therefore the two symmetries can be considered together and it has the following matrix form:

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_{11} - C_{12})/2 \end{bmatrix} \tag{3}$$

Tetragonal symmetry(Classes: $4mm$, $\overline{4}2m$, 422, $4/mmm$**):**

Tetragonal symmetry is characterized by a four-fold axis, for instance, by invariance against a rotation of $\pi/2$ about the 3-axis. In other words, this symmetry type is specified by five planes of mirror symmetry, four of whose normals lie in the fifth plane and make angles that are multiples of $\pi/4$ with respect to one another.[40] Tetragonal symmetry has the following structure for its elastic constants given in matrix form.

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \tag{4}$$

Trigonal symmetry(Classes: 32, $3m$, $\overline{3}m$**):**

Trigonal symmetry has three planes of mirror symmetry. The normals to these three planes all lie in one plane and make angles of 60^o degrees with one another. It is also characterized by a three-fold axis, for instance, the angle is equal to $2\pi/3$.[40] It has the matrix form given as follows:

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{12} & C_{11} & C_{13} & -C_{14} & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ C_{14} & -C_{14} & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & C_{14} \\ 0 & 0 & 0 & 0 & C_{14} & (C_{11}-C_{12})/2 \end{bmatrix} \tag{5}$$

Orthorhombic symmetry:

Orthorhombic symmetry includes three mutually perpendicular two-fold axes. It is characterized by three mutually perpendicular planes of mirror symmetry.[40] The elastic stiffness matrix has the following matrix form.

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \tag{6}$$

Triclinic symmetry:

Triclinic materials possessing triclinic symmetry is identical with the case of absence of symmetry. This symmetry has no planes of mirror symmetry. Since there are no symmetry restrictions on the matrix of elastic stiffness , there are twenty-one independent elastic constants which is given as follows:

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{24} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \tag{7}$$

CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Dinçkal, Çiğdem

Nationality: Turkish (TC)

Date and Place of Birth: 11 September 1982, Ankara

Marital Status: Single

Phone: +90 505 670 94 13

email: e150501@metu.edu.tr

EDUCATION

Degree	Institution	Year of Graduation
Ph.D.	METU Dept. of Engineering Sciences	2010
M.Sc.	METU Dept. of Engineering Sciences	2008
BSc.	Çankaya University Industrial Engineering	2004

WORK EXPERIENCE

Year	Place	Enrollment
2003 July	Development Bank of Turkey	Trainee
2002 August	HAVELSAN INC.	Trainee

FOREIGN LANGUAGES

Advanced English, Basic German

PUBLICATIONS

1. 'Decomposition of elastic constant tensor into orthogonal parts' prepared for publication.

2. 'Orthonormal decomposition of symmetric second rank tensors' prepared for publication.