STRUCTURE THEORY OF $\mathbb{Z}_p$-CENTRAL SIMPLE GRADED ALGEBRAS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
THE MIDDLE EAST TECHNICAL UNIVERSITY

BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF
DOCTOR OF PHILOSOPHY
IN
THE DEPARTMENT OF MATHEMATICS

SEPTEMBER 1998

Approval of the Graduate School of Natural and Applied Sciences

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosopy.

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosopy.

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ABSTRACT

STRUCTURE THEORY OF $\mathbb{Z}_p$-CENTRAL SIMPLE GRADED ALGEBRAS

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September 1998, 52 pages.

The structure theory of $\mathbb{Z}_2$-central simple graded algebras ($\mathbb{Z}_2$-CSGA) was investigated by C.T.Wall in 1964. In 1995, C.Koç introduced ω-Clifford algebras associated to d-forms on a finite dimensional vector spaces over fields containing a primitive d-th root of unity ω.Then he generalized the structure theorem, of Clifford algebras associated to quadratic forms.

In this thesis we generalize the structure theory of $\mathbb{Z}_2$-CSGA and we obtain the classification theory for $\mathbb{Z}_p$-CSGAs where p is a prime number.Then using this we get the results given by C.Koç.

Keywords:Central Simple Graded Algebra,Primitive dth Root of Unity, ω-Clifford Algebra.

ÖZ

$\mathbb{Z}_p$-MERKEZİL BASİT DERECELİ CEBİRLERİN YAPI KURAMI

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Doktora Tezi, Matematik Ana Bilim Dalı

Tez Yöneticisi: Prof. Dr. Cemal KOÇ

Eylül 1998, 52 sayfa.

$\mathbb{Z}_2$-merkezil basit dereceli cebirlerin yapı kuramı 1964 yılında C.T.Wall tarafından kurulmuştur. 1995 yılında, C.Koç, birimin d'ninci kökünü kapsayan cisimler üzerinde,sonlu boyutlu vektör uzayları altında d-formları ile ilişkili ω-Clifford cebirlerinin kuadratik formlara ilişkin yapı kuramını genellemiştir. Bu tezde, $\mathbb{Z}_2$-merkezil basit cebirlerinin yapı kuramını genelleştirdik ve $\mathbb{Z}_p$-merkezil basit cebirlerini p sayısı asalken sınıflandırdık. Daha sonra bu sonuçları kullanarak C.Koç tarafından elde edilen sonuçlara vardık.

Anahtar Sözcükler: Merkezil Basit Dereceli Cebirler, Birimin d-ninci kökü,d-formları.

ACKNOWLEDGEMENTS

I would like to express my appreciation to my supervisor Prof.Dr.Cemal Koç for his wise supervision and continuous encouragement during the preparation of this thesis.

This thesis is supported by the scholarship of TÜBİTAK. I would like to thank them for this support.

I also want to thank Burak Daşdemir for his help and Mrs.Güldane Gümüş for her great efforts and care in typing this thesis.

Finally, I would like to thank my husband for his patience and continuous moral support.

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CHAPTER 1

INTRODUCTION

Clifford algebras of vector spaces have numerous applications in various branches of science (including quantum chemistry). An orthogonal decomposition of the vector space give rise to a graded tensor product decomposition of its Clifford algebra and thus its $\mathbb{Z}_2$ -graded algebra structure is proved to be an indispensible and fruitful tool in the investigation of Clifford algebras.In 1964, C.T.C.Wall generalized the structure theory of Clifford algebras by introducing central simple $\mathbb{Z}_2$-graded algebras and investigating their structure.He proved that they can be described as the graded tensor product of certain building block central simple $\mathbb{Z}_2$-graded algebras. The concept of Clifford algebras of a vector space is also generalized from various point of views (see [14],[7], [13],[11], [12]). In order to obtain finite dimensional generalizations homomorphic images of the above mentioned generalized Clifford algebras have to be considered (see [2],[3],[16]). In the case where the base field contains a primitive d-th root of unity, ω, one of these homorphic images can be described as the algebra generated by a set $\{e_1, e_2, \cdots, e_n\}$ subject to the relations

$$e_1^d = a_1, \cdots, e_n^d = a_n \quad \text{and} \quad e_j e_i = \omega e_i e_j \quad \text{for} \quad j > i$$

where $a_1, a_2, \cdots, a_n$ are nonzero elements of F. The structure theorem for these algebras has been given in [8]. The aim of the present thesis is to generalize this structure theory to central simple $\mathbb{Z}_p$-graded algebras where p is a prime number. In Chapter 2, we recall some well-known facts for the convenience of the reader.In Chapter 3, we give fundamental results for the investigation of the structure of central simple graded algebras and we give some examples that will be used as building block algebras in the next chapter. Chapter 4 contains main results establishing the structure theory of central simple $\mathbb{Z}_p$-graded algebras.It is proved that for a prime number p, every $\mathbb{Z}_p$-graded algebra is a graded tensor product of building block algebras given in Chapter 3. The final chapter is devoted to an application of the structure theory to the investigation of generalized Clifford algebras described above and the structure theory given in [8] is obtained as a consequence of results in Chapter 4.

Throughout this thesis F will stand for a field containing ω, a primitive root of unity. By a ring, we mean an assosiative ring with identity and homomorphisms preserve identitiy. An F-algebra will always mean a finite dimensional F-algebra.We assume that subalgebras contain the identity of the algebra.

CHAPTER 2

PRELIMINARIES

In this chapter, we shall recall some well known facts for the convenience of the reader.

2.1 Rings

We begin with the following general form of the Chinese Remainder theorem.

Theorem 2.1.1 *Let $A_1, A_2, \cdots, A_n$ be ideals in a ring R and $A_i + A_j = R$ for all $i \neq j$. If $b_1, \cdots, b_n \in R$, then there exists $b \in R$ such that $b \equiv b_i \pmod{A_i}$ for $i = 1, \cdots, n$.*

Furthermore b is uniquely determined up to congruence modulo the ideal $A_1 \cap \cdots \cap A_n$.

Proof. See [5, page 131].

Corollary 2.1.2 *(Chinese Remainder Theorem)*

If $A_1 \cdots, A_n$ are ideals in a ring R, then there is an isomorphism of rings

$$\theta : R/(A_1 \cap \cdots \cap A_n) \rightarrow R/A_1 \times R/A_2 \times \cdots \times R/A_n.$$

Proof. See [5, page 132].

2.2 Graded Algebras, Graded Tensor Products

Definition 2.2.1 *Let G be an abelian group. A G-graded algebra A over a field F is an algebra over F which is given in the form*

$$A = \bigoplus_{g \in G} A_g$$

where each A_g is an F-space such that $F = F \cdot 1_A \subset A_0$ and $A_g \cdot A_h \subseteq A_{g+h}$.

For a graded algebra A, the elements of

$$h(A) = \bigcup_{g \in G} A_g$$

will be called the homogenous elements of A and for $a \in h(A)$, we write $\partial(a) = g$ if $a \in A_g$. Any subset of $h(A)$ is called a homogeneous subset of A.

A subspace $V \subset A$ is called graded if it can be written in the form

$$V = \bigoplus_{g \in G} V \cap A_g.$$

We write $h(V) = V \cap h(A)$. A graded ideal of A is an ideal which is graded as a subspace.

A graded algebra homomorphism is a homomorphism of algebras which also preserves grading.

Note that every graded subspace (in particular every graded ideal) is spanned by its homogeneous elements so that if it has a nonzero element then it contains a nonzero homogeneous element.

Definition 2.2.2 *A G-graded algebra $A \neq 0$ is called a simple graded algebra (SGA) over F if A has no proper (i.e. $\neq 0, \neq A$) graded two sided ideals.*

The graded tensor product of $\mathbb{Z}_d$-graded algebras can be given by means of the following universal property (see [1] and [15]).

Theorem 2.2.3 *Let A and B be $\mathbb{Z}_d$-graded algebras. There exists a graded algebra T and graded homomorphisms*

$$i_A : A \to T,\ i_B : B \to T$$

satisfying

$$i_A(a) i_B(b) = \omega^{\partial a \partial b} i_B(b) i_A(a)$$

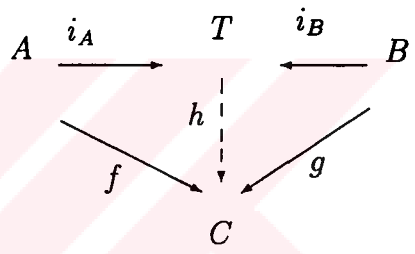
such that for any graded algebra C *and any graded homomorphisms*

$$f : A \to C, \;\; g : B \to C$$

satisfying

$$f(a)g(b) = \omega^{\partial a \partial b} g(b) f(a)$$

the following diagram can be completed by a unique graded homomorphism h *to a commutative diagram:*



in other words we have $f = hi_A$ *and* $g = hi_B$. *Further* T *is unique up to isomorphism, more precisely* T *is isomorphic to the algebra* $A\hat{\otimes}_\omega B$ *obtained by considering the tensor product* $A \otimes_F B$ *(as vector space) with the operation of multiplication defined by means of*

$$(a \otimes b)(a' \otimes b') = \omega^{\partial b \partial a'} aa' \otimes bb',$$

where $a, a' \in h(A)$ *and* $b, b' \in h(B)$ *and* ω *is the primitive d-th root of unity in* F. *Any graded algebra satisfying this universal property is called a graded tensor product of* A *and* B *and the algebra* $A\hat{\otimes}_\omega B$ *is referred to as the graded tensor product of* A *and* B.

The ordinary tensor product is obtained by considering the algebras A *and* B *to be graded algebras with trivial grading, i.e.* $A = A_0$ *and* $B = B_0$. *Thus the* *ordinary tensor product* $T = A \otimes B$ *is an algebra so that there are homomorphisms*

$$i_A : A \to T,\ i_B : B \to T$$

satisfiying

$$i_A(a)i_B(b) = i_B(b)i_A(a) \quad \textit{for} \quad a \in A, b \in B$$

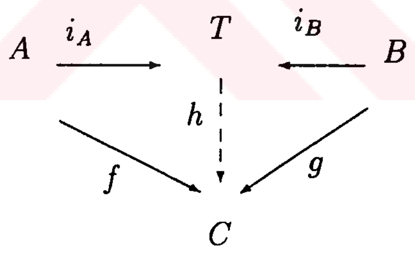
such that for any algebra C *and any homomorphisms*

$$f : A \to C,\ \ g : B \to C$$

satisfying

$$f(a)g(b) = g(b)f(a)$$

the following diagram can be completed by a unique homomorphism h *to a commutative diagram:*



2.3 Central Simple Algebras

Definition 2.3.1 *1)For any subset* S *of an* F*-algebra* A*,the centralizer of* S *is defined by*

$$C_A(S) = \{a \in A : \ as = sa \quad \textit{for all} \quad s \in S\},$$

and its center is

$$Z(A) = C_A(A).$$

2) A is called F-central (or central over F) if $Z(A) = F$.

3) A is called simple if A has no proper ($\neq 0, \neq A$) two sided ideals.

4) A is called a central simple algebra (CSA) over F if A satisfies both 2) and 3).

Theorem 2.3.2 *1) If A, B are F-algebras, and $A' \subset A$, $B' \subset B$ are subalgebras, then $C_{A \otimes B}(A' \otimes B') = C_A(A') \otimes C_A(B')$. In particular, if A, B are F-central, so is $A \otimes B$.*

2) If A is a CSA over F, and B a simple algebra, then $A \otimes B$ is simple.

3) If A, B are both CSA over F, so is $A \otimes B$.

Proof. See [4, pages 90-91].

Proposition 2.3.3 *(Double Centralizer Theorem)*

Let A be a CSA over F, and B be a simple subalgebra of A. Let $C = C_A(B)$. Then

1) C is simple

2) $B = C_A(C)$

3) $dimA = dimB \cdot dimC$.

Proof. See [9, page 73].

Corollary 2.3.4 *Suppose $B \subset A$, and both are CSAs over F. If $C = C_A(B)$, then C is also a CSA, $B = C_A(C)$ and $B \otimes C \cong A$.*

Proof. See [9, pages 73-74].

Theorem 2.3.5 *Let A be a CSA over F, and B a simple algebra. If f, g are algebra homomorphisms from B to A, then there exists an invertible element $s \in A$, such that $f(b) = s^{-1}g(b)s$ for every $b \in B$ (i.e. f and g differ by an inner automorphism of A).*

Proof. See [9, pages 73-75].

Corollary 2.3.6 *(Skolem-Noether Theorem)*

If A is a CSA over F, every algebra endomorphism of A is an inner automorphism.

2.4 Clifford Algebras of d-Forms:

In this section we shall give some results of (generalized) Clifford algebras following [8].

Definition 2.4.1 *Let V be a vector space over F and let $\{e_1, e_2, \cdots, e_n\}$ be a basis for V. If $a_1, a_2, \cdots, a_n$ are nonzero elements of F, then the ω-Clifford algebra of V is the algebra generated $e_1, e_2, \cdots, e_n$ subject to the relations*

$$e_i^d = a_i \; ; \; i = 1, \cdots, n,$$

$$e_j e_i = \omega e_i e_j \quad \textit{for} \quad j > i,$$

and it is denoted by $(a_1, \cdots, a_n)_\omega^d$ or simply by $(a_1, \cdots, a_n)$.

An ω-Clifford algebra becomes a $\mathbb{Z}_d$-graded algebra over the field F.

In the case $n = 1$ we have

$$(a)^d \cong F[x]/\langle x^d - a\rangle$$

This algebra is denoted by $F\langle \sqrt[d]{a}\rangle$. In particular if $x^d - a$ is irreducible, we obtain an extension field of F and if this is the case we use the notation $F(\sqrt[d]{a})$.

In the case $n = 2$, we obtain the *norm-residue algebra* $(a_1, a_2)_\omega^d$. *This algebra is a CSA over F (c.f. M). If F contains a root of $(x^d - a_1)(x^d - a_2)$ then the algebra (a_1, a_2) is isomorphic to the algebra of $d \times d$ matrices over F. Supposing that $x^d - a_2$ has a root α in F, it can be verified directly that the* *matrices*

$$E_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & a \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \quad E_2 = \alpha \begin{bmatrix} 1 & & & \\ & \omega & & 0 \\ & & \ddots & \\ & 0 & & \omega^{d-1} \end{bmatrix}$$

generate the algebra of $d \times d$ *matrices and satisfy the equalities*

$$E_1^d = a_1 I,\ E_2^d = a_2 I \quad \text{and} \quad E_2 E_1 = \omega E_1 E_2$$

Theorem 2.4.2 *If* $a_1, a_2, \cdots, a_n$ *are non-zero elements of a field containing a primitive d-th root of unity,* ω*, then*

1) We have canonical graded isomorphisms:

$$\begin{aligned} (a_1, a_2, \cdots, a_n)_\omega^d &\cong (a_1, \cdots, a_s)_\omega^d \hat{\otimes} (a_{s+1}, \cdots, a_n)_\omega^d \quad \text{for any} \quad 1 < s < n \\ &\cong (a_1)^d \hat{\otimes} \cdots \hat{\otimes} (a_n)^d \end{aligned}$$

where $\hat{\otimes}$ *stands for the graded tensor product defined by* ω.

2) $dim(a_1, \cdots, a_n)_\omega^d = d^n$.

3) The vector space of homogeneous elements of $\mathbb{Z}_d$*-graded algebra* $(a_1, \cdots, a_n)_\omega^d$ *of a fixed degree is of dimension* d^{n-1}.

Proof. See [8, Theorem 7].

Theorem 2.4.3 *Let* F *be a field containing a primitive d-th root of unity,* ω*, and let* $C = (a_1, \cdots, a_n)_\omega^d$,

$$a = \begin{cases} (-1)^{m(d-1)} & a_1 a_2^{d-1} \cdots a_{2m}^{d-1} a_{2m+1} \quad \text{if} \quad n = 2m+1 \\ (-1)^{(m+1)(d-1)} & a_1 a_2^{d-1} \cdots a_{2m+1} a_{2m+2^{d-1}} \quad \text{if} \quad n = 2m+2 \end{cases}$$

and $L = F[x]/\langle x^d - a \rangle$. *Then*

1) *If n is odd, the center of C is L, and it is a CSA over L when $x^d - a$ is irreducible over F. The homogeneous part C_0 of degree zero is a central simple F-algebra.*

2) *If n is even, C is a CSA over F. The center of the homogenous part C_0 of degree zero is L and C_0 is a CSA over L when $x^d - a$ is irreducible over F.*

CHAPTER 3

$\mathbb{Z}_d$-CENTRAL SIMPLE GRADED ALGEBRAS

In this chapter we introduce fundamental concepts on central simple graded algebras and exhibit some examples that will be proved to be building blocks in our structure theory. To begin with, we recall some basic definitions.

3.1 Basic Concepts

Definition 3.1.1 *a) Let S be a homogenous subset of A, we define*
$S' = \{a \in h(A) : as = \omega^{\partial a \partial s} sa$ *for all* $s \in S$ *or* $sa = \omega^{\partial s \partial a} as$ *for all* $s \in S\}$.
The centralizer of S in A is defined to be the graded subspace spanned by S' i.e. $\hat{C}_A(S) = < S' >$.

b) The graded center of A is defined by
$\hat{Z}(A) = \hat{C}_A(h(A))$ *and A is said to be central if* $\hat{C}_A(h(A)) = \hat{Z}(A) = F$. *If in addition, A is a simple graded algebra, we say that it is a central simple graded algebra (CSGA).*

Note that the graded center of A is different from the ordinary center of A as ungraded algebra. Nevertheless, $Z(A)$ is also a graded subalgebra and $Z(A)_0 = \hat{Z}(A)_0$.

From now on, the grading group will be taken to be the additive group $\mathbb{Z}_d$ and our main concern will be $\mathbb{Z}_d$-central simple graded algebras.

Theorem 3.1.2 *If A is a CSGA and B is SGA then $A \hat{\otimes}_\omega B$ is a SGA.*

Proof. Let I be a nonzero graded ideal in $E = A \hat{\otimes}_\omega B$. Our aim is to show that $I = E$ by verifying $1 \otimes 1 \in I$. First, by definition, each nonzero element of the graded ideal I is a sum of homogeneous elements in I and at least one of them is nonzero. Thus I contains a nonzero homogenous element z which is obviously in the form

$$z = \sum_{i=1}^{r} a_i \otimes b_i \quad \text{where} \quad a_i \in h(A) \quad \text{and} \quad b_i \in h(B).$$

Among all non-zero homogeneous elements of I, let us pick z as above such that r is as small as possible. Clearly, all a_i, b_i are non-zero and since z has a fixed degree in E, the sum $\partial a_i + \partial b_i \pmod d$ is independent of i. First we observe that a_i's (similarly b_i's) are linearly independent over F. Assume that it is not true. Then after renumbering if necessary we have a relation

$$a_1 = \sum_{i=2}^{s} e_i a_i, \quad e_i \in F,$$

where $a_2, \cdots, a_s$ are of the same degree. Taking the first s summand, we get

$$\begin{aligned}
\sum_{i=1}^{s}(a_i \otimes b_i) &= (\sum_{i=2}^{s} e_i a_i \otimes b_1) + (a_2 \otimes b_2) + \cdots + (a_s \otimes b_s) \\
&= e_2 a_2 \otimes b_1 + e_3 a_3 \otimes b_1 + \cdots + e_s a_s \otimes b_1 + a_2 \otimes b_2 + \cdots + a_s \otimes b_s \\
&= a_2 \otimes e_2 b_1 + a_3 \otimes e_3 b_1 + \cdots + a_s \otimes e_s b_1 + a_2 \otimes b_2 + \cdots + a_s \otimes b_s \\
&= a_2 \otimes (e_2 b_1 + b_2) + a_3 \otimes (e_3 b_1 + b_3) + \cdots + a_s \otimes (e_s b_1 + b_s) \\
&= \sum_{i=2}^{s} a_i \otimes (e_i b_1 + b_i)
\end{aligned}$$

and this yields an expression for z with smaller r and condradicts the choice of r, because $b_1, b_2, \cdots, b_s$ above are of the same degree as $a_1, \cdots, a_s$ so each $e_i b_1 + b_i$ remains homogenous. Therefore the a_i are linearly independent. Similarly so are b_i's.

Secondly we try to make a_1 equal to 1. The graded algebra A is simple so the graded ideal Aa_1A must be equal to A and hence we have

$$\sum_{j=1}^{m} c_j a_1 d_j = 1$$

where c_j and d_j's are homogeneous elements so that $\partial c_j + \partial a_1 + \partial d_j \equiv 0 \pmod{d}$ for all j. Now using these c_j and d_j's we get

$$\sum_{j=1}^{m} \omega^{(d-1)\partial b_1 \partial d_j} c_j z d_j = 1 \otimes b_1 + \sum_{j=1}^{m} \sum_{i=2}^{r} \omega^{((d-1)\partial b_1 + \partial b_i)\partial d_j} c_j a_i d_j \otimes b_i$$

Thus, we obtain a homegenous element in I of the form

$$z_1 = 1 \otimes b_1 + \sum_{i=2}^{r} a_i' \otimes b_i$$

where

$$a_i' = \sum_{j=1}^{m} \omega^{((d-1)\partial b_1 + \partial b_i)\partial d_j} c_j a_i d_j$$

Since $\partial c_j + \partial d_j \equiv -\partial a_1 \pmod{d}$ is independent of j, we see that $\partial(c_j a_i d_j) \equiv \partial a_i - \partial a_1$ for all j and hence each a_i' is homogeneous of degree $\partial a_i - \partial a_1$. Further $z_1 \neq 0$ since the b_i are linearly independent over F. Also in the passage from z to z_1 the b_i remain unchanged. Doing the same thing on b_i, we may find a nonzero homogeneous element in I of the form

$$z' = 1 \otimes 1 + \sum_{i=2}^{r} a_i' \otimes b_i'$$

where a_i', b_i' are homogenous so that $\partial a_i' + \partial b_i' \equiv 0$ (mod d).

Finally if $a \in h(A)$

$$(a \otimes 1)z' - z'(a \otimes 1) \in I \cap h(E)$$

and

$$(a \otimes 1)z' - z'(a \otimes 1) = \sum_{i=2}^{r}(aa'_i \otimes b'_i - \omega^{\partial b'_i \partial a} a'_i a \otimes b'_i)$$

$$= \sum_{i=2}^{r}(aa'_i - \omega^{\partial b'_i \partial a} a'_i a) \otimes b'_i$$

Since r is the smallest number of non-zero homogenous components, we have

$$aa'_i = \omega^{\partial b'_i \partial a} a'_i a = \omega^{-\partial a'_i \partial a} a'_i a$$

implying

$$\omega^{\partial a'_i \partial a} aa'_i = a'_i a$$

i.e. a'_i is in the graded center of A, namely F. Thus, each a'_i must be a scalar. But linear independence of $\{1, a'_2, \cdots, a'_r\}$ over F and the choice of r show that r must be 1, thas is, $z' = 1 \otimes 1 \in I$. □

Theorem 3.1.3 *If A and B are both $\mathbb{Z}_d$-CSGAs over F, then so is $A \hat{\otimes}_\omega B$.*

Proof. Regarding Theorem 3.1.2 we must only show that $A \hat{\otimes}_\omega B$ is graded central over F. Clearly $F(1 \otimes 1) \subset \hat{Z}(A \hat{\otimes} B)$. To see the inverse inclusion let $z \in h(A \hat{\otimes} B)$ We have two cases:

Case 1: $zz' = \omega^{\partial z \partial z'} z'z$ for all $z' \in h(A \hat{\otimes} B)$. Then we write $z = \sum_{i=1}^{r} a_i \otimes b_i$ where $a_i \in A$ and $b_i \in B$ are homogeneous with $\partial a_i + \partial b_i = \partial z$ and we have $z(a \otimes 1) = \omega^{\partial z \partial a}(a \otimes 1)z$ for all $a \in h(A)$. The b_i are linearly independent. Computing left and right hand sides we obtain $\sum_{i=1}^{r}(a_i \otimes b_i)(a \otimes 1) = \omega^{\partial b_i \partial a} \sum_{i=1}^{r} a_i a \otimes b_i$ and $\omega^{\partial z \partial a}(a \otimes 1) \sum_{i=1}^{r}(a_i \otimes b_i) = \omega^{\partial a_i \partial a + \partial b_i \partial a} \sum_{i=1}^{r} aa_i \otimes b_i$. Independence of the b_i yields for each i that $a_i a = \omega^{\partial a_i \partial a} aa_i$ for all $a \in h(A)$ implying that $a_i \in \hat{Z}(A) = F$.

So z is of the form $z = 1 \otimes b_1$ where $b_1 \in h(B)$. Thus we have,

$$(1 \otimes b_1)(1 \otimes b) = \omega^{\partial b_1 \partial b}(1 \otimes b)(1 \otimes b_1)$$

which gives

$$1 \otimes b_1 b = \omega^{\partial b_1 \partial b}(1 \otimes bb_1) \quad \text{and} \quad b_1 b = \omega^{\partial b_1 \partial b} bb_1 \quad \text{for all} \quad b \in h(B).$$

Therefore $b_1 \in \hat{Z}(B) = F$ and $z \in F(1 \otimes 1)$.

Case 2: $z'z = \omega^{\partial z' \partial z} zz'$ for all $z' \in h(A \hat{\otimes} B)$. Write $z = \sum_{i=1}^{s} a_i \otimes b_i$ where $a_i \in A$ and $b_i \in B$ are homogeneous with $\partial z = \partial a_i + \partial b_i$ and the a_i are linearly independent. Computing both sides of $(1 \otimes b)z = \omega^{\partial b \partial z} z(1 \otimes b)$ for all $b \in h(B)$ we get

$(1 \otimes b)z = \sum_{i=1}^{s}(1 \otimes b)(a_i \otimes b_i) = \omega^{\partial b \partial a_i} \sum_{i=1}^{s} a_i \otimes bb_i$ and

$$\omega^{\partial b \partial z} z(1 \otimes b) = \omega^{\partial b \partial z} \sum_{i=1}^{s}(a_i \otimes b_i)(1 \otimes b) = \omega^{\partial b \partial a_i + \partial b \partial b_i} \sum_{i=1}^{s}(a_i \otimes b_i b).$$

Comparing these by using independence of the a_i we obtain

$bb_i = \omega^{\partial b \partial b_i} b_i b$ implying that $b_i \in \hat{Z}(B) = F$. Thus we have $z = (a_1 \otimes 1)$ where $a_1 \in h(A)$.Then for all $a \in h(A)$, $(a \otimes 1)(a_1 \otimes 1) = \omega^{\partial a \partial a_1}(a_1 \otimes 1)(a \otimes 1)$ which gives $aa_1 \otimes 1 = \omega^{\partial a \partial a_1} a_1 a \otimes 1$ and $aa_1 = \omega^{\partial a \partial a_1} a_1 a$. Thus $a_1 \in \hat{Z}(B) = F$ and $z \in F(1 \otimes 1)$ and hence $F(1 \otimes 1) = \hat{Z}(A \hat{\otimes} B)$. □

3.2 Fundamental Examples

In this section we specify some examples of central simple $\mathbb{Z}_d$-graded algebras. In the subsequent chapters we shall develop the theory to show that every central simple graded algebra is obtained as a kind of composition of these specific central simple graded algebras.

Example 3.2.1 Let A be any ungraded algebra. We may consider it as a graded algebra, denoted (A), with components $(A)_0 = A$ and $(A)_i = 0$ for all $i = 1, \cdots, d-1$. This graded algebra (A) is said to be *trivially graded* or *concentrated at degree zero.*

Note that if A is central simple as ungraded algebra then (A) is CSGA.

Example 3.2.2 Let $K = F[x]/\langle x^d - a\rangle$ be the factor algebra of polynomials over F for some $a \neq 0$ in F. Its elements are of the form $a_0 + a_1\overline{x} + \cdots + a_{d-1}\overline{x}^{d-1}$ where $\overline{x} = x + \langle x^d - a\rangle$. We define the homogenous components of K naturally by setting

$$K_i = F\overline{x}^i, \quad i = 0, 1, \cdots, d-1$$

and thus make K into a $\mathbb{Z}_d$-graded algebra.

Let k be a nonzero homogeneous element in the graded center of K, say $k = b\overline{x}^i, b \neq 0,\ 0 \leq i \leq d-1$. Then the requirement $k \cdot h = \omega^{\partial k \partial h} h \cdot k$ for all $h \in h(K)$ or $h \cdot k = \omega^{\partial h \partial k} k \cdot h$ for all $h \in h(K)$ yields for $h = \overline{x}$ that $\overline{x}^{i+1} = \omega^{i \cdot 1}\overline{x}^{i+1}$ giving $i = 0$. This means that $k = b \in F$ and that K is a central algebra over F.

It is also graded simple. In fact, let I be a nonzero graded ideal of K and let $k = b\overline{x}^i$, $0 \leq i \leq d-1$ be a nonzero homogenous element of I. Since $a \neq 0$ we have

$$(a^{-1}b^{-1}\overline{x}^{d-i})b\overline{x}^i = a^{-1}\overline{x}^d = 1 \in I.$$

Note that if $a = 0$ then $K = F[x]/\langle x^d\rangle$ will not be graded simple since the ideal generated by $\overline{x}$ is a graded ideal.

Example 3.2.3 Let a and b be nonzero elements of F. The F-algebra generated by two elements e_1 and e_2 subject to the relations

$$e_1^d = a,\ e_2^d = b \quad \text{and} \quad e_2 e_1 = \omega e_1 e_2$$

is called a *generalized quaternion algebra* or *norm residue algebra* and it is denoted by $\langle \frac{a,b}{F} \rangle_\omega$.

This algebra turns out to be a central simple algebra as an ungraded algebra, see [10]. Further it can be described as an ω-Clifford algebra of the two dimensional vector space $V = Fe_1 \oplus Fe_2$ and as such it is a $\mathbb{Z}_d$-graded F-algebra spanned by the homogeneous elements $e_1^i e_2^j$, $i, j = 0, \cdots, d-1$ with $\partial e_1 = \partial e_2 = 1$. If the polynomial $x^d - b$ has a root α in F there is a unique algebra isomorphism from $\langle \frac{a,b}{F} \rangle$ onto $\mathbb{M}_d(F)$ given by

$$e_1 \mapsto E_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & a \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad e_2 \mapsto E_2 = \alpha \begin{bmatrix} 1 & & & \\ & \omega & & 0 \\ & & \ddots & \\ & 0 & & \omega^{d-1} \end{bmatrix}$$

See [8]. This isomorphism induces a grading on $\mathbb{M}_d(F)$. In particular the isomorphism $\langle \frac{a,b}{F} \rangle \cong \mathbb{M}_d(F)$ induces the grading of the matrix algebra $\mathbb{M}_d(F)$ obtained by taking elements of degree r to be matrices which are linear combinations of $E_1^i E_2^j$ with $i + j \equiv r \pmod d$. A straightforward computation shows that, in this grading, matrices of degree r has entries A_{ij} of the form

$$A_{ij} = \begin{cases} a_{i-j}\omega^{j(r-i+j)} & \text{if } j \leq i \\ a a_{d-j+i}\omega^{j(r-i+j)} & \text{if } j > i. \end{cases}$$

Thus one way of grading the matrix algebra $\mathbb{M}_d(F)$ is to pick up a nonzero arbitrary element a in F and to define homogeneous elements according to the formula we established just above. Some other gradings will be given in the subsequent examples.

Example 3.2.4 Given any $\mathbb{Z}_d$-graded algebra A, to put a grading on the matrix algebra $\mathbb{M}_n(A)$, assign a matrix degree i if all its entries have degree i. We shall denote this graded matrix algebra by $\tilde{\mathbb{M}}_n(A)$.

It's trivial that $\tilde{\mathbb{M}}_n(F)$ is just $(\mathbb{M}_n(F))$ and that

$$\tilde{\mathbb{M}}_n(A) \cong \tilde{\mathbb{M}}_n(F)\hat{\otimes}A \cong (\mathbb{M}_n(F)) \otimes A$$

If A is CSGA, then by Theorem 3.1.3, $\tilde{\mathbb{M}}_n(A)$ is also a CSGA, since $\mathbb{M}_n(F)$ is a CSA over F.

Example 3.2.5 Let $V = V_0 \oplus V_1 \oplus \cdots \oplus V_{d-1}$ be a finite dimensional graded vector space, that is an F space with a specified direct sum decomposition. Using this grading we can make the endomorphism ring $E = End\ (V)$ into a graded algebra by defining homogeneous components as

$$E_i = \{f \in \ End\ (V) \mid f(V_j) \subset V_k \quad \text{such that} \quad k - j \equiv i (\text{mod } d)\}$$

for $i = 0, \cdots, d-1$.

Obviously if $f \in E_i$ and $g \in E_j$, then for any $l \leq d-1$ we have

$$g(V_l) \subset V_k \quad \text{with} \quad k \leq d-1,\ k \equiv l + j \ (\text{mod } d)$$

and

$$f(V_k) \subset V_r \quad \text{with} \ \ r \leq d-1,\ r \equiv k + i \equiv l + j + i,$$

hence

$$fg(V_l) \subset V_r \quad \text{with} \quad r - l \equiv i + j \mod d$$

that is $fg \in E_{i+j}$.

Since E has no nontrivial ideals it is necessarily graded simple.

It is graded central. In fact, let $f \in E$ be a homogeneous element in $h(E)$. Then

$$fh = \omega^{\partial f \partial h} hf \quad \text{for all} \quad h \in h(E) \quad \text{or} \quad hf = \omega^{\partial h \partial f} fh \quad \text{for all} \quad h \in h(E).$$

Let $\{v_i\}$ be a basis consisting of homogeneous elements of V. The h_{ij} given by $h_{ij}(v_k) = \delta_{jk} v_i$ form a homogeneous basis for E. Writing $f = \sum_{i,j} c_{ij} h_{ij}$, $d_{ij} = \partial(h_{ij})$ and using properties of the h_{ij} for each pair k, l we get

$fh_{kl} = \sum_{i,j} c_{ij} h_{ij} h_{kl} = \sum_{i} c_{ik} h_{il}$ and $h_{kl} f = \sum_{i,j} c_{ij} h_{kl} h_{ij} = \sum_{j} c_{lj} h_{kj}$.

Now, the above condition yields that

$$c_{ik} = c_{lj} = 0 \quad \text{unless} \quad i = k \quad \text{and} \quad j = l$$

and that

$$c_{kk} = \omega^{\partial f d_{kl}} c_{ll} \quad \text{or} \quad c_{kk} = \omega^{-\partial f d_{kl}} c_{ll}.$$

On the other hand this shows that $f = c_{11} h_{11} + \cdots + c_{dd} h_{dd}$ and hence it is of degree zero as $\partial h_{ii} = 0$ for each i. Thus the last equalities take the form

$$c_{kk} = c_{ll}$$

and shows that f is a scalar multiple of the identity map.

Specifying an ordered basis

$$\mathcal{B} = \{v_{01}, \cdots, v_{0r_0}; v_{11}, \cdots, v_{1r_1}; \cdots; v_{(d-1)1}, \cdots, v_{(d-1)r_{d-1}}\}, v_{ij} \in V_i$$

for V we see that, relative to this basis $\mathcal{B}$, any $f \in E_i$ has a matrix in the block form

$$\left[\begin{array}{c|c} \bigcirc & \begin{matrix} A_{i+1} & \\ & \ddots \; A_d \end{matrix} \\ \hline \begin{matrix} A_1 & \\ & \ddots \; A_i \end{matrix} & \bigcirc \end{array}\right].$$

This matrix representation induces a grading on $\mathbb{M}_d(F)$. In the particular case where $r_0 = r_1 = \cdots = r_{d-1} = 1$ gives the grading of $\mathbb{M}_d(F)$ whose homogeneous elements of degree r are matrices whose entries A_{ij} are all zero except when $i - j = r$. The matrix algebra $\mathbb{M}_d(F)$ with this grading is denoted by $\hat{\mathbb{M}}_d(F)$ and it can be generalized in the following manner.

Let A be any $\mathbb{Z}_d$-graded algebra. The matrix algebra $\mathbb{M}_d(A)$ is graded by defining homogeneous elements of degree i to be matrices in the form

$$\begin{bmatrix} a_1^{(i)} & \cdots & \cdots & \cdots & a_{n-1}^{(i-2)} & a_n^{(i-1)} \\ a_1^{(i-1)} & a_2^{(i)} & \cdots & \cdots & \cdots & a_n^{(i-2)} \\ a_1^{(i-2)} & a_2^{(i-1)} & a_3^{(i)} & \cdots & \cdots & \cdots \\ \ddots & \ddots & \ddots & & & \\ \cdots & \cdots & \cdots & a_{n-2}^{(i-2)} & a_{n-1}^{(i-1)} & a_n^{i} \end{bmatrix},$$

$a_i^k \in A_k$ for each $i = 0, \cdots, d-1$. It is verified at once that $\mathbb{M}_n(A)$ with this grading becomes a graded algebra. This graded algebra is denoted by $\hat{\mathbb{M}}_n(A)$.Entries of its homogenous elements of degree i can also be described by

$$A_{kl} \in A_j \Longleftrightarrow j \equiv l - k + i \pmod d$$

If A is trivially graded, homogenous elements of $\hat{\mathbb{M}}_n(A)$ of degree i take the form

$$\left[\begin{array}{c|c} \bigcirc & \begin{matrix} a_{n-i+1} & & \bigcirc \\ & \ddots & \\ \bigcirc & & a_n \end{matrix} \\ \hline \begin{matrix} a_1 & & \bigcirc \\ & \ddots & \\ \bigcirc & & a_{n-i} \end{matrix} & \bigcirc \end{array}\right]$$

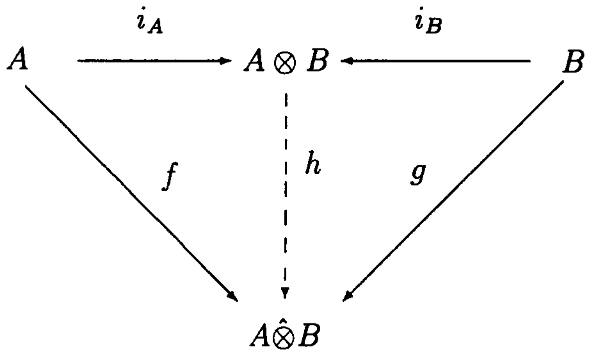
This applies to the particular case $A = F$ and furnishes the grading of $\hat{\mathbb{M}}_n(F)$ for an arbitrary n.

3.3 Ordinary and Graded Tensor Products

Computations in ordinary tensor product $\otimes$ are of course easier than those in graded tensor product. So that it is reasonable to look for some cases for which $A\hat{\otimes}B \cong A \otimes B$. It is the aim of this section to give some results in this direction.

Theorem 3.3.1 *Let* $A = \bigoplus_{i=0}^{d-i} A_i$ *and* $B = \bigoplus_{j=0}^{d-1} B_j$ *be* $\mathbb{Z}_d$ *graded algebras. Suppose that there exists a homogenous element* z *in the graded center of* A *of degree 1 such that* $z^d = 1$ *and* $a_i z = \omega^i z a_i$ *for all* $a_i \in A_i$*. Then there exists a graded algebra isomorphism* $A\hat{\otimes}B \cong A \otimes B$*.*

Proof. We use the universal mapping property of ordinary tensor product. To this end we shall construct (ungraded) algebra homomorphisms so that $f(a)g(b) = g(b)f(a)$ for all $a \in A$ and $b \in B$ and the universal mapping property will assume the existence of $h : A \otimes B \to A\hat{\otimes}B$ so that each part of the following diagram is commutative:



where $i_A(a) = a \otimes 1$ and $i_B(b) = 1 \otimes b$. We define

$$f(a) = a \hat{\otimes} 1 \quad \text{for all} \quad a \in A$$

and

$$g(b) = g(b_0 + b_1 + \cdots + b_{d-1}) = 1 \otimes b_0 + z \otimes b_1 + \cdots + z^{d-1} \otimes b_{d-1} \quad \text{for} \quad b_i \in B_i.$$

We see at once that they are algebra homomorphisms for each homogeneous element $a_i \in A_i$ and $b_j \in B_j$ we see that

$$\begin{aligned} f(a_i)g(b) &= (a_i \otimes 1)(1 \otimes b_0 + z \otimes b_1 + \cdots + z^{d-1} \otimes b_{d-1}) \\ &= (a_i \otimes 1)(1 \otimes b_0) + (a_i \otimes 1)(z \otimes 1) + \cdots + (a_i \otimes 1)(z^{d-1} \otimes b_{d-1}) \\ &= (a_i \otimes b_0) + (a_i z \otimes b_1) + (a_i z^2 \otimes b_2) + \cdots + (a_i z^{(d-1)} \otimes b_{d-1}) \\ &= (a_i \otimes b_0) + (\omega^i z a_i \otimes b_1) + (\omega^{2i} z^2 a_i \otimes b_2) + \cdots + (\omega^{(d-1)i} z^{d-1} a_i \otimes b_{d\cdot} \end{aligned}$$

and at the same time

$$\begin{aligned} g(b)f(a_i) &= (1 \otimes b_0 + z \otimes b_1 + \cdots + z^{d-1} \otimes b^{d-1})(a_i \otimes 1) \\ &= (1 \otimes b_0)(a_i \otimes 1) + (z \otimes b_1)(a_i \otimes 1) + \cdots + (z^{d-1} \otimes b_{d-1})(a_i \otimes 1) \\ &= (a_i \otimes b_0) + (\omega^i z a_1 \otimes b_1) + (\omega^{2i} z^{d-1} a_i \otimes b_{d-1}) + \cdots + (\omega^{(d-1)i} z^{d-1} a_i \end{aligned}$$

Therefore $f(a_i)g(b) = g(b)f(a_i)$. Hence $f(a)g(b) = g(b)f(a)$ for all $a \in A$ and $b \in B$.By the universal property of ordinary tensor product there exists an algebra homomorphism

$$h : A \otimes B \longrightarrow A \hat{\otimes} B \quad \text{so that} \quad hi_A = f \quad \text{and} \quad hi_B = g$$

Further we have

$$h(a_i \otimes b_j) = h((a_i \otimes 1)(1 \otimes b_j)) = hi_A(a_i)hi_B(b_j) = f(a_i)g(b_j) = (a_i \otimes 1)(z^j \otimes b_j),$$

that is

$$\partial h(a_i \otimes b_j) = \partial a_i + \partial z^j + \partial b_j = i + 0 + j = i + j = \partial(a_i \otimes b_j).$$

Thus h is a graded homomorphism. It is obviously onto since
$a_i \otimes b_j = h(a_i z^{d-i} \otimes b_j)$. Finally $\dim_F(A \hat{\otimes} B) = \dim_F(A \otimes B)$ shows that h is a graded isomorphism. □

Corollary 3.3.2 *Let B, C be $\mathbb{Z}_d$-graded algebras where C is trivally graded. Then there exists a graded algebra isomorphism*

$$\hat{\mathbb{M}}_r(C) \hat{\otimes} B \cong \hat{\mathbb{M}}_r(C) \otimes B.$$

Proof. We shall show that $A = \hat{\mathbb{M}}_r(C)$ satisfies the hypothesis of

Theorem 3.3.1 with $z = diag(1, \omega, \omega^2, \cdots, \omega^{r-1})$.

In terms of usual matrix units e_{ij} we write

$$z = \sum_{l=0}^{r-1} \omega^l e_{ll}.$$

Since C is trivially graded, a homogeneous element $a \in \hat{\mathbb{M}}_r(C)$ of degree k is of the form

$$a = \sum_{j-i\equiv k} c_{ij}e_{ij}, \quad c_{ij} \in C.$$

Properties of the matrix units e_{ij} yield that

$$e_{ij}z = \sum_{l=0}^{r-1} \omega^l e_{ij}e_{ll} = \omega^j e_{ij}$$

$$\omega^k z e_{ij} = \sum_{l=0}^{r-1} \omega^{k+l} e_{ll}e_{ij} = \omega^{i-k} e_{ij}.$$

If $j - i \equiv k \pmod{d}$ we obtain $e_{ij}z = \omega^k z e_{ij}$ and hence

$$az = \sum_{j-i\equiv k} \sum_{l=0}^{r-1} e_{ij}z = \sum_{j-i\equiv k} c_{ij}\omega^k z e_{ij} = \omega^k z \sum_{j-i\equiv k} c_{ij}e_{ij} = \omega^k za$$

that is the hypothesis of the previous theorem follows. □

Corollary 3.3.3 *For any $\mathbb{Z}_d$ graded algebra B, there are graded algebra isomorphisms*

$$\hat{\mathbb{M}}_r(F)\hat{\otimes}B \cong \hat{\mathbb{M}}_r(F) \otimes B \cong \hat{\mathbb{M}}_r(B)$$

Proof. The first isomorphism follows from Corollary 3.3.2 by letting $C = F$. The second isomorphism follows from the immeadiate observation that the usual identification $\hat{\mathbb{M}}_r \otimes B \cong \hat{\mathbb{M}}_r(B)$ is homogeneous of degree zero relative to the gradings on $\hat{\mathbb{M}}_r(F) \otimes B$ and on $\hat{\mathbb{M}}_r(B)$. □

Corollary 3.3.4 $\hat{\mathbb{M}}_r(F)\hat{\otimes} \hat{\mathbb{M}}_s(C) \cong \hat{\mathbb{M}}_r(F) \otimes \hat{\mathbb{M}}_s(C) \cong \hat{\mathbb{M}}_{rs}(C)$ *for any $\mathbb{Z}_d$-graded algebra C.*

Proof. Take $B = \hat{\mathbb{M}}_s(C)$ in Corollary 3.3.3 and simply note that $\hat{\mathbb{M}}_r(\hat{\mathbb{M}}_s(C)) \cong \hat{\mathbb{M}}_{rs}(C)$. □

CHAPTER 4

STRUCTURE OF Z_p-CENTRAL SIMPLE GRADED ALGEBRAS

In this chapter, we shall give the complete structure teory of $\mathbb{Z}_p$-central simple graded algebras where p is a prime number. We describe them in terms of the building block algebras introduced in the previous chapter.

4.1 Some Results on Z_d-Central Simple Graded Algebras

In this section, we establish certain general results on $\mathbb{Z}_d$-central simple graded algebras that will be used in the next section to develop the structure theory of $\mathbb{Z}_p$-central simple graded algebras.

Lemma 4.1.1 *Let d be any positive integer and let A be a $\mathbb{Z}_d$-CSGA.If u_k is any homogenous element in A_k, then for each $t = 0, \cdots, d-1$ we have*

$$A_t = \sum_{r+k+\ell \equiv t \pmod{d}} A_r u_k A_\ell.$$

Proof. Consider the graded subspace $L = \sum_{t=0}^{d-1} \sum_{r+k+\ell \equiv t \pmod{d}} A_r u_k A_\ell$ which is a non-zero ideal of A generated by a homogenous element $u_k \in A_k$. Clearly L is a graded ideal and hence $A = L$, namely

$$A = \sum_{t=0}^{d-1} A_t = \sum_{t=0}^{d-1} \sum_{r+k+\ell \equiv t \pmod{d}} A_r u_k A_\ell$$

implying that

$$A_t = \sum_{r+k+\ell \equiv t \pmod{d}} A_r u_k A_\ell \quad \text{for all} \quad t = 0, \cdots, d-1$$

since the sum is a direct sum. □

Proposition 4.1.2 *Let A be a simple $\mathbb{Z}_d$-graded algebra with $A_k \neq 0$ for some $k \geq 1$. Then $A_0 = \sum_{k=1}^{d-1} A_k A_{d-k}$.*

Proof. Since $A_k \neq 0$ for some $k = 1, \cdots, d-1$ the set

$$\sum_{k=1}^{d-1} A_k A_{d-k} + A_1 + \cdots + A_{d-1}$$

is a nonzero graded ideal.Therefore it must be equal to A itself.This forces $A_0 = \sum_{k=1}^{d-1} A_k A_{d-k}$. □

Theorem 4.1.3 *Let A be a $\mathbb{Z}_d$-CSGA. Assume that A is not simple as an ordinary algebra. Then A contains a homogenous element $u \in Z(A) \cap A_k$ for some $k \geq 1$ such that $u^d \in F^\bullet$. When this k is relatively prime to d, A has a central element z of degree 1 then $A_i = A_0 z^i$ for $i = 1, \cdots, d-1$ and A_0 is a simple algebra.*

Proof. Let J be any non-zero proper ideal of A with a nonzero element $\tilde{u}$. This $\tilde{u}$ is a sum of non-zero homogenous elements,say $\tilde{u} = u_{\ell_1} + u_{\ell_2} + \cdots + u_{\ell_r}$. We pick up the one with least number of non-zero homogenous components. Take any homogeneous component u_{k_j} of $\tilde{u}$ and use Lemma 4.1.1 to get

$$A_0 = \sum_{r+k_j+\ell \equiv 0 \pmod{d}} A_r u_{k_j} A_\ell.$$

and hence

$$1 = \sum_{r+k_j+\ell \equiv 0 \pmod{d}} a_r u_{k_j} a_l \quad \text{with} \quad a_r \in A_r, a_l \in A_l.$$

Thus J contains an element u' of the form $u' = 1 + u'_{k_2} + \cdots + u'_{k_r}$ where

$$u'_{k_m} = \sum_{r+k_j+\ell \equiv 0 \pmod{d}} a_r u_{l_m} b_l \quad \text{with the } a_r \text{ and } b_l \text{ used above} \quad .$$

If $r = 1$ then $u' = 1 \in J$ and hence the ideal J cannot be proper.So $r \geq 2$ and trivially each u_{k_m} is homogenous and for each $a_i \in A_i$ we have

$$u'a_i - a_iu' = \sum_{j=2}^{r}(u'_{k_j}a_i - a_iu'_{k_j}) \in J.$$

By the choice of r, $u'_{k_j}a_i - a_iu'_{k_j} = 0$ for all $j = 2, \cdots, r$ gives the result that $u'_{k_j}a_i = a_iu'_{k_j}$ for all $a_i \in A_i$, $i = 0, \cdots, d-1$. Hence $u'_{k_j} \in Z(A)$.

Further, for each component u'_{k_j} and for each $0 \neq a_i \in A_i$ we have $a_iu'_{k_j} \neq 0$ and $u'_{k_j}a_i \neq 0$ because if we had $a_iu'_{k_j} = 0$ for some i and k_j we would get

$$a_iu' = a_i + a_iu'_{k_2} + \cdots + \underbrace{a_iu'_{k_j}}_{0} + \cdots + a_iu'_{k_r} \in J$$

and this would give the result that $a_i = 0$ by the minimality of r.Thus we conclude in particular that u'_{k_j} cannot be nilpotent and so that $(u'_{k_j})^d \neq 0$.

Now, $(u'_{k_j})^d$ is a nonzero element of $(Z(A))_0 = \hat{Z}(A)_0 = F$ and the existence of $u \in Z(A) \cap A_k$ for some $k \geq 1$ with $u^d \in F^\bullet$ follows.

Suppose now that $k = \partial u$ is relatively prime to d.Then $kk' + dd' = 1$ for some integers k' and d' and hence

$$\partial u^{k'} = k'\partial u = k'k \equiv 1(\text{mod } d).$$

Therefore $z = u^{k'}$ is in $Z(A) \cap A_1$ and it satisfies $z^d = (u^d)^{k'} \in F^\bullet$ and further

$$A_i = A_iz^d = (A_iz^{d-i})z^i \subseteq A_0z^i \subseteq A_i$$

implying $A_i = A_0z^i$.

Finally A_0 is simple, because for any nonzero ideal I of A_0 picking up $0 \neq a_0 \in I$ we get by Lemma 4.1.1 that

$$A_0 = \sum_{k+\ell\equiv 0 \ (\text{ mod } d)} A_ka_0A_\ell$$

and hence $A_0 \subseteq I$ giving $A_0 = I$. $\square$

Corollary 4.1.4 *If p is a prime number and A has a central homogenous element u of degree 1 so that $u^p \in F^\bullet$ then A_0 is a simple algebra and we have $A_i = A_0 u^i$ for each $i = 0, \cdots, p-1$.*

Proof. It is an immediate consequence of the theorem since any $k = 1, \cdots, p-1$ is relatively prime to p. $\square$

Theorem 4.1.5 *Let A be a $\mathbb{Z}_d$-CSGA. Then $Z(A) = F$ if and only if A is a CSA over F (as ungraded algebra).*

Proof.

($\Rightarrow$): If $Z(A) = F$, then $Z(A)$ has no homogenous element of nonzero degree.On the other hand if A is not simple, by Theorem 4.1.3 it contains a central homogenous element u of nonzero degree and gives a contradiction.

($\Leftarrow$) : This implication is obvious. $\square$

Theorem 4.1.6 *Let A be a $\mathbb{Z}_p$-CSGA with a nontrivial grading and $Z(A) = F \oplus Z'$ where $Z' = Z_1 \oplus \cdots \oplus Z_{p-1}$ and $Z_i \subset A_i$, for $i = 1, \cdots, p-1$. Then $Z' \neq 0$ if and only if A_0 is a CSA over F, and if this is the case there exists $z \in Z_1$ such that $z^p \in F^\bullet$ and $A = A_0 \oplus A_0 z \oplus \cdots \oplus A_0 z^{p-1}$.*

Proof. ($\Rightarrow$) : Suppose $Z' \neq 0$. Any nonzero $z_k \in Z_k$ is invertible since $Az_kA = A$ gives that $az_k = 1$ for some $a \in A$. Since p is prime, any $k = 1, \cdots, p-1$ is relatively prime to p and as in the proof of Theorem 4.1.3 some power of z_k is contained in Z_1 and hence there exists $z_1 \in Z_1$ such that $z_1^p \in F^\bullet$and that $A_i = A_0 z_1^i$ for each $i = 0, \cdots, p-1$.
Now $Z(A_0) \subset Z(A)$ since $A_i = A_0 \cdot z_1^i$ and $z_1 \in Z(A)$ and thus $Z(A_0) \subset Z(A) \cap A_0 = F$ showing that A_0 is a central F-algebra.

If $I \neq 0$ is an ideal of A_0, $I + Iz_1 + Iz_1^2 + \cdots + Iz_1^{p-1}$ is a graded ideal of A, hence it must be $A = A_0 + A_1 + \cdots + A_{p-1}$ and consequently we must have $A_k = Iz_k$ for each k and in particular $A_0 = I$. So A_0 has no proper ideals and it is simple. Thus we have seen that A_0 is a central simple algebra.

($\Leftarrow$): Suppose that A_0 is central simple and $Z' = 0$. Then by Theorem 4.1.5. A itself is a CSA over F.

Let $C = C_A(A_0)$.It is a graded subalgebra of A because for any $a = a_0 + a_1 + \cdots + a_{p-1} \in C_A(A_0)$ the equality

$$(a_0 + a_1 + \cdots + a_{p-1})b_0 = b_0(a_0 + a_1 + \cdots + a_{p-1}) \quad \text{for all} \quad b_0 \in A_0$$

implies that

$$a_i b_0 = b_0 a_i \quad \text{for all} \quad b_0 \in A_0 \quad \text{and} \quad i = 0, \cdots, p-1$$

in other words homogenous components of a are all in $C_A(A_0)$. Using double centralizer theorem, we know that C is also central simple and $A \cong A_0 \otimes C$ as ungraded algebras. This forces $C_i \neq 0$ for some $i \geq 1$.Also using the facts $F = Z(A_0) = C_A(A_0) \cap A_0 = C_0$ and by Proposition 4.1.2 we have $F = C_0 = \sum_{k=1}^{p-1} c_k c_{p-k}$ shows that for some $k = 1, \cdots, p-1$ there exist $u \in C_k$ and $v \in C_{p-k}$ so that $0 \neq uv \in F$. This implies that $u \in C_k$ is invertible in C, since k and p are relatively prime for some integer k' we have $v = u^{k'} \in C_1$ and it is also invertible.Therefore $v^p \in F^\bullet$ and thus $C_i = C_i v^p = C_i v^{p-i} v^i \subset C_0 v^i = F v^i \subset C_i$. Consequently $C_i = F v^i$ and

$$\begin{aligned} C &= C_0 \oplus C_1 \oplus \cdots \oplus C_{p-1} \\ &= F \oplus Fv \oplus Fv^2 \oplus \cdots \oplus Fv^{p-1}. \end{aligned}$$

This implies that C is commutative, contradictory to the fact that C is a CSA over F.

Further if this is the case, there exists $z \in Z_1$ such that $0 \neq z^p \in F^{\bullet}$ and $A = A_0 \oplus A_0 z \oplus \cdots \oplus A_0 z^{p-1}$ because, $Z' \neq 0$ shows that Z' has a homogenous element z' of degree $k \geq 1$.If z' is nilpotent $Z(A) = F \oplus Z'$ cannot be a field and hence A is not simple and Corollary to Theorem 4.1.3 completes the proof.If z' is not nilpotent then $0 \neq z = (z')^l \in Z_1$ for some l and $0 \neq z^p \in Z(A) \cap A_0 = F$ and further $A_i = A_i z^p = A_i z^{p-i} z^i \subset A_0 z^i \subset A_i$ giving $A_i = A_0 z^i$ for all i. □

Definition 4.1.7 *Let A be a $\mathbb{Z}_p$-CSGA over F and $Z(A) = F \oplus Z'$ where $Z' = Z_1 \oplus \cdots \oplus Z_{p-1}$ and each $Z_i \subset A_i$, $i = 1, \cdots, p-1$.*

(**1**)*If $Z' = 0$ we shall say that A is of the even type.*

(**2**)*If $Z' \neq 0$ we shall say that A is of the odd type.*

By Theorem 4.1.5; A is of even type iff A is a CSA over F as an ungraded algebra and by Theorem 4.1.6; A is of odd type iff A_0 is a CSA over F as an ungraded algebra.

4.2 Structure Theory of Z_p-CSGA

The purpose of this section is to classify all CSGA's, by showing that each CSGA is a graded tensor product of the building block algebras $F\langle\sqrt[p]{a}\rangle$, $\tilde{\mathbb{M}}_n(F)$, $\langle\frac{a,b}{F}\rangle_\omega$ and (A).

Theorem 4.2.1 *Let A be a $\mathbb{Z}_p$-CSGA of odd type. Then the following statements hold:*

(1) $Z(A) = C_A(A_0) = F \oplus Fz \oplus \cdots \oplus Fz^{p-1}$ *for some* $z \in Z_1$ *and* $z^p = a \in F^{\bullet}$.

Also $Z(A) \cong F\langle\sqrt[p]{a}\rangle$ *as graded algebras.*

(2) *There are graded algebra isomorphisms* $A \cong (A_0)\hat{\otimes} F\langle \sqrt[p]{a}\rangle \cong A_0 \otimes F\langle \sqrt[p]{a}\rangle$.

(3) *If* $x^p - a$ *is irreducible then* $Z(A) \cong F(\sqrt[p]{a})$.

If $x^p - a$ *is not irreducible, then* $Z(A) \cong F \times \cdots \times F$ *and* $A \cong A_0 \times \cdots \times A_0$.

Proof.

1) Since A is a CSGA of odd type, $Z' \neq 0$ and by Theorem 4.1.3. there exists an element $z \in Z_1$ with $z^p = a \in F^{\bullet}$ and $A_i = A_0 \cdot z^i$, $i = 0, \cdots, p-1$. Thus any element of A centralizing A_0, commutes also with z, since z is central and so with $A_i = A_0 z^i$ and thus central in A that is $Z(A) = C_A(A_0)$. The rest of (1) follows from

$$Z_k = Z_k a = Z_k z^{p-k} z^k \subset Z_0 z^k = F z^k \subset Z_k$$

and the fact that the map

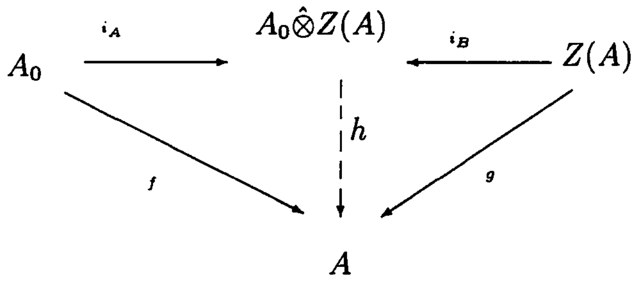
$$\phi : Z(A) \rightarrow F\langle \sqrt[p]{a}\rangle = F[x]/\langle x^p - a\rangle$$

given by

$$\phi(c_0 + c_1 z + \cdots + c_{p-1} z^{p-1}) = c_0 + c_1 \bar{x} + \cdots + c_{p-1} \bar{x}^{p-1}$$

is trivially a graded isomorphism.

(2) To see the first isomorphism, we use the universal mapping property of graded tensor product and by taking f and g in the following commutative diagram to be inclusion maps.



For $a_0 \in A_0$ and $b \in Z(A)$, the condition

$$f(a_0)g(b) = a_0 b = \omega^0 b a_0 = \omega^{\partial f(a_0)\partial g(b)} g(b) f(a_0)$$

is trivially satisfied and hence there exists a unique graded homomorphism h completing the diagram to a commutative diagram.Further for any $a_k \in A_k$ we have

$$a_k = a_0 z^k = f(a_0)g(z^k) = h(i_A(a_0))h(i_B(z^k)) = h((a_0 \otimes 1)(1 \otimes z^k)) = h(a_0 \otimes z^k)$$

that is h is surjective.Comparing dimensions we conclude that h is a (graded) isomorphism.The second isomorphism is obvious since A_0 is trivially graded.

(3) If $x^p - a$ is irreducible then $Z(A) \cong F(\sqrt[p]{a})$is a field extension over F.

If $x^p - a$ is not irreducible then $a = c^p$ for some $c \in F$ and hence it splits over F that is

$x^p - a = (x - c)(x - \omega c) \cdots (x - \omega^{p-1} c)$ see [18]. Then

$$\begin{aligned} F[x]/\langle x^p - a \rangle &\cong F[x]/\langle x - a_0 \rangle \times \cdots \times F[x]/\langle x - a_{p-1} \rangle \\ &\cong F \times F \times \cdots \times F \end{aligned}$$

by Chinese Remainder Theorem.

Using (1), we have $Z(A) \cong F\langle \sqrt[p]{a} \rangle \cong F \times \cdots \times F$ and by (2),

$$A \cong A_0 \otimes Z(A) \cong A_0 \otimes (F \times F \times \cdots \times F) \cong A_0 \times \cdots \times A_0. \qquad \square$$

One immediate consequence of this theorem is that, in the odd case, each component A_k in A, $k = 0, \cdots, p-1$, must have the same dimension $dim\ A_k = \frac{\text{dim A}}{\text{p}}$.

Lemma 4.2.2 *Suppose that $x^p - a$ has no root in D where D is a central division algebra over F and B is a splitting field of this polynomial over F. Then $E = B \otimes_F D^{op}$ is a central division algebra over B.*

Proof. First we note that $x^p - a$ has no root in F and hence it is irreducible see [18]. Let us write $T = E^{op} = B \otimes D$.By Theorem 3.1.6, T is a central simple algebra over B as such we can write $T \cong \mathbb{M}_r(S)$ where S is a central division algebra over B. To complete the proof it suffices to show that $r = 1$.

Let $s = dim_B S$ and $d = dim_F D = dim_B T$ and let M be the irreducible left T-module then,

$$d = dim_B T = r^2 \cdot dim_B S = r^2 \cdot s.$$

Also we have

$$dim_B M = dim_S M \cdot dim_B S = r \cdot s = \frac{r^2 \cdot s}{r} = \frac{d}{r}$$

which gives

$$dim_F M = dim_B M \cdot dim_F B = \frac{d}{r} \cdot p = \frac{pd}{r}.$$

On the other hand, we can say that $dim_F M$ must be a multiple of $d = dim_F D$ because M, as a left T-module, can be viewed as a left D-vector space (more precisely $(1 \otimes D)$ vector space) so that

$$dim_F M = dim_F D \cdot dim_D M.$$

Thus r must be either 1 or p. If $r = p$ then $d = dim_F M$ implies that $M \cong D$ as a left D-module.But, $z \otimes 1 \in B \otimes D$ commutes with $1 \otimes D$ so the left multiplication map f by $1 \otimes z$ is a D-linear map on M:

$$f(dm) = f((1 \otimes d)m) = (1 \otimes z)(1 \otimes d)m = (1 \otimes d)(1 \otimes z)m = (1 \otimes d)f(m) = df(m).$$

This shows that $f \in End_D(M) \cong End_D(D) = D^{op}$. Since $z^p = a$ we have $f^p = a$ and D^{op} contains a root of $x^p - a$ which contradicts the assumption of our lemma. Therefore $r = 1$ and hence $T \cong S$ is a central division algebra over B. □

Definition 4.2.3 *Let $A = A_0 \oplus A_1 \oplus \cdots \oplus A_{p-1}$ be any $\mathbb{Z}_p$-graded algebra. For each $x_i \in A_i$ for $i = 0, \cdots, p-1$ we define*

$$\nu(x_0 + x_1 + \cdots + x_{p-1}) = x_0 + \omega x_1 + \cdots + \omega^{p-1} x_{p-1}.$$

It can be verified that this is a graded algebra automorphism of A and it has order p.

Theorem 4.2.4 *Let A be a $\mathbb{Z}_p$-CSGA of even type with a nontrivial grading. Let D be a central division algebra over F for which A, as an ungraded central simple algebra, is isomorphic to $\mathbb{M}_n(D)$. Then the following hold:*

1) *$Z(A_0) = C_A(A_0)$ and there exists $z \in Z(A_0)$ such that $Z(A_0) = F \oplus Fz \oplus \cdots \oplus Fz^{p-1}$ and $z^p = a \in F^\bullet$.*

2) *Suppose that $x^p - a$ is not irreducible i.e. $a = b^p$ for some $b \in F^\bullet$. Then $Z(A_0) \cong F \times F \times \cdots \times F$. Then there exists a graded F-vector space $V = V_0 \oplus V_1 \oplus \cdots \oplus V_{p-1}$ such that $A \cong End\ V \hat{\otimes} (D)$ as graded algebras. (Here, $End\ V$ is graded in the manner of Example 3.2.5).*
Further,

$$A_0 \cong \mathbb{M}_{r_0}(D) \times \mathbb{M}_{r_1}(D) \times \cdots \times \mathbb{M}_{r(p-1)}(D) \quad where \quad r_i = dim\ V_i,\ i = 0, \cdots, p-1.$$

3) *Suppose that $x^p - a$ is irreducible i.e. $a \notin F^{\bullet p}$ and that the field $Z(A_0) \cong F(\sqrt[p]{a})$ can be imbedded into D. Then there exists a grading on D such that $A \cong \tilde{\mathbb{M}}_n(D)$. In this case, $A_0 \cong \mathbb{M}_n(D_0)$ is a CSA over $Z(A_0)$.*

4) *Suppose that $x^p - a$ is irreducible and that the field $Z(A_0) \cong F(\sqrt[p]{a})$ can not be imbedded into D. Then for some integer n, $n = pm$ and $A \cong (\mathbb{M}_m(D))\hat{\otimes}\langle\frac{a,1}{F}\rangle_\omega$ as graded algebras. In this case $A_0 \cong \mathbb{M}_m(D) \otimes F(\sqrt[p]{a})$ is a CSA over $Z(A_0)$.*

Proof. Let us prove (1) and (2) simultaneously. We proved that; A is a $\mathbb{Z}_p$-CSGA of even type iff is a CSA over F as an ungraded algebra. By Skolem-Noether Theorem, every algebra endomorphism of A is an inner automorphism, so the automorphism introduced above is inner,i.e. there exists an invertible element $z \in A$ such that $\nu(x) = z^{-1}xz$ for all $x \in A$.
Now we claim that $A_0 = C_A(z)$:
First from $\nu(z) = z^{-1}zz = z$ it follows that $z \in A_0$ by the definition of ν.
Since $\nu(a) = z^{-1}az$ for all $a \in A$, writing $a = \sum_{i=0}^{p-1} a_i$ with $a_i \in A$ we get

$$z\sum_{i=0}^{p-1}\omega^i a_i = (\sum_{i=0}^{p-1} a_i)z$$

which yields that

$$a_i z = \omega^i z a_i \quad \text{for all} \quad a_i \in A \quad \text{and} \quad i = 0, \cdots, p-1$$

and in particular $A_0 \subseteq C_A(z)$.
Conversely, if $a \in C_A(z)$ is of the form $a = a_0 + a_1 + \cdots + a_{p-1}$ then $az = za$ and $a = z^{-1}az = \nu(a)$ gives that

$$a_0 + a_1 + \cdots + a_{p-1} = a_0 + \omega a_1 + \cdots + \omega^{p-1}a_{p-1}.$$

Since ω is a primitive p-th root of unity this equality holds if and only $a_i = 0$ for all $i = 1, \cdots, p-1$. Hence $a \in A_0$ and $A_0 = C_A(z)$.

Using the fact that ν is of order p we have

$$x = \nu^p(x) = \nu^{p-1}(\nu(x)) = \nu^{p-1}(z^{-1}xz) = \nu^{p-2}(z^{-2}xz^2) = \cdots = z^{-p}xz^p = x,$$

so $z^p x = xz^p$ for all $x \in A$ which means that $z^p \in Z(A) = F$. Since z is invertible, $z^p \neq 0$ and hence $z^p = a \in F^\bullet$. We claim that

$Z(A_0) = C_A(A_0)$. For this, $Z(A_0) = C_{A_0}(A_0) = C_A(A_0) \cap A_0$ and

$C_A(A_0) \subset C_A(z) = A_0$ gives that $C_A(A_0) \cap A_0 = C_A(A_0)$. Hence

$Z(A_0) = C_A(A_0)$ as claimed.

Let B denote the subalgebra $F \oplus Fz \oplus \cdots Fz^{p-1}$ of A_0.

First, assume that $x^p - a$ is an irreducible polynomial, then $B \cong F(\sqrt[p]{a})$ is a simple algebra, (as a field) contained in the central simple algebra A. Using the double centralizer theorem, we have

$$B = C_A(C_A(B)) \quad \text{and} \quad C_A(B) = C_A(z) = A_0.$$

Thus $B = C_A(A_0) = Z(A_0)$ which gives that $Z(A_0) = F \oplus Fz \oplus \cdots \oplus Fz^{p-1}$.

Also, A_0 is simple by the same theorem.

Secondly, assume that $x^p - a$ is **not** irreducible, while trying to prove 2) at the same time. We have $b^p = a \in F^{\bullet^p}$ and $z^p = a$. Then if necessary, changing z by $(\frac{z}{b})$, we may assume that $z^p = 1$.

Fix any F-isomorphism $\phi : A \cong \mathbb{M}_n(D)$. Since $\omega \in F$ then $x^p - 1$ can be written as a product of distinct linear polynomials.

Now, since p is a prime number for every $1 \leq i \leq p-1$ the element ω^i in F is a primitive p-th root of unity and hence

$$\sum_{k=0}^{p-1} \omega^{ik} = \begin{cases} p & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

and hence letting

$$e_i = \frac{1}{p}\sum_{l=0}^{p-1}(\omega^i z)^l$$

we see that

$$e_0 e_i = \begin{cases} e_0 & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

which yields that $e_0, e_1, \cdots, e_{p-1}$ are orthogonal idempotents of A.It also yields that

$$\sum_{i=0}^{p-1} e_i = 1 \quad \text{and} \quad \sum_{j=0}^{p-1} \omega^{-j} e_j = z.$$

On the other hand it is well known that if $E_0, E_1, \cdots, E_{p-1}$ are orthogonal idempotents in $\mathbb{M}_n(D)$, there exists an invertible matrix $P \in \mathbb{M}_n(D)$ such that $P^{-1}E_iP$ is the standard idempotent,

$$P^{-1}E_iP = diag(0, \cdots, 0, \underbrace{1, \cdots, 1}_{r_i-times}, 0, \cdots, 0)$$

for all $i = 0, \cdots, p-1$, where $r_i = rank(E_i)$, see [6]. Applying this to the $\theta(e_i)$ and and considering the composition of θ and the inner authomorphism ρ of $\mathbb{M}_n(D)$ corresponding to P, we see that there is an isomorphism

$$\varphi : A \mapsto \mathbb{M}_n(D)$$

so that for some sequence $r_0, r_1, \cdots, r_{p-1}$ of integers we have

$$\varphi(e_i) = diag(0, \cdots, 0, \underbrace{1, \cdots, 1}_{r_i-times}, 0, \cdots, 0)$$

and

$$\varphi(1) = I = diag(\underbrace{1, \cdots, 1}_{r_0-times}, \underbrace{1, \cdots, 1}_{r_1-times}, \cdots, \underbrace{1, \cdots, 1}_{r_{p-1}-times})$$

Therefore we get

$$\begin{aligned}\varphi(z) &= diag(\underbrace{1,\cdots,1}_{r_0-times},\underbrace{\omega^{-1},\cdots,\omega^{-1}}_{r_1-times},\underbrace{\omega^{-2},\cdots,\omega^{-2}}_{r_2-times},\cdots,\underbrace{\omega^{-p+1},\cdots,\omega^{-p+1}}_{r_{p-1}-times}) \\ &= \begin{bmatrix} I_{r_0} & & & \\ & \omega^{-1}I_{r_1} & & \\ & & \ddots & \\ & & & \omega^{-p+1}I_{r_{p-1}} \end{bmatrix}\end{aligned}$$

where $0 \leq r_k \leq n$ and I_k stands for the $k \times k$ identity matrix.
Since z is a homogenous element for which $az = \omega^{\partial a}za$ for all $a \in A$ we can describe the homogenous components A_i of A by

$$A_i = \{a \in A \mid az = \omega^i za\}$$

and we can use this to make $\mathbb{M}_n(D)$ into a graded algebra by setting

$$(\mathbb{M}_n(D))_i = \varphi(A_i) = \{a \in \mathbb{M}_n(D) \mid a\varphi(z) = \omega^i\varphi(z)a\} \quad \text{for} \quad i = 0,\cdots,p-1.$$

Using the block form

$$a = \begin{bmatrix} A_{11} & \cdots & A_{1p} \\ & \cdots & \\ A_{p1} & \cdots & A_{pp} \end{bmatrix}$$

where A_{kk} is an $r_k \times r_k$ matrix over D, we verify at once that elements of $(\mathbb{M}_n(D))_i$ are of the form

$$\left[\begin{array}{c|c} \bigcirc & \begin{matrix} A_{1(n-i+1)} & & \\ & \ddots & \\ & & A_{ip} \end{matrix} \\ \hline \begin{matrix} A_{(i+1)1} & & \\ & \ddots & \\ & & A_{p(n-i)} \end{matrix} & \bigcirc \end{array}\right]$$

This gives us the graded algebra $\hat{\mathbb{M}}_n(D)$ established in Example 3.2.5. Now, to constract the space V we take $V = F^{n\times 1}$ and V_i to be its subspace consisting of column matrices in block form,

$$\begin{bmatrix} C_0 \\ \vdots \\ C_i \\ \vdots \\ C_{p-i} \end{bmatrix}, \quad C_i \in F^{r_i \times 1} \quad \text{with} \quad C_k \neq 0 \quad \text{when} \quad k \neq i.$$

Obviously matrices of $(\mathbb{M}_n(F))_i$ with corresponding linear transformations, we see that $End(V)$ has grading established in Example 3.2.5 and it is graded isomorphic to $\hat{\mathbb{M}}_n(F)$. Thus we have

$$A \cong \hat{\mathbb{M}}_n(D) \cong \hat{\mathbb{M}}_n(F) \otimes (D) \cong End(V) \hat{\otimes} (D)$$

as asserted.The subalgebra $(\mathbb{M}_n(D))_0$ consists of matrices in block form

$$\begin{bmatrix} A_{00} & & & \\ & A_{11} & & \bigcirc \\ \bigcirc & & \ddots & \\ & & & A_{(p-1)(p-1)} \end{bmatrix}; \quad A_{ii} \in \mathbb{M}_{r_i}(D)$$

and hence trivally

$$A_0 \cong (\mathbb{M}_n(D))_0 \cong \mathbb{M}_{r_0}(D)) \times \cdots \times \mathbb{M}_{r_{p-1}}(D))$$

Since D is F-cenral we have $Z(A_0) = F \times \cdots \times F$. It is of dimension p and it follows from this and $B = F \oplus Fz \oplus \cdots \oplus Fz^{p-1} \subset Z(A_0)$ that

$$Z(A_0) = F \oplus \cdots \oplus Fz^{p-1}$$

and the proof of (2) is completed.

(3) Suppose $a \notin F^{\bullet p}$ and hence by 1), $B \cong Z(A_0) \cong F(\sqrt[p]{a})$. Since we assume $Z(A)_0 \subset D$ there exists $z_0 \in D$ such that $z_0^p = a \in F^{\bullet}$. Let θ be any F-isomorphism between A and $\mathbb{M}_n(D)$. We shall identify D as the subalgebra of scalar matrices in $\mathbb{M}_n(D)$.

The two subfields $F(z_0)$ and $F(\theta(z))$ of the matrix algebra must be conjugate under an inner automorphism α of $\mathbb{M}_n(D)$ by Skolem-Noether Theorem, $\alpha(\theta(z)) = z_0$.

If necessary, changing θ to $\alpha\theta$, we may assume that $\theta(z) = z_0$. Our aim is to identify A with $\mathbb{M}_n(D)$ using this θ. To do this, first we note that the involution ν stabilizes D, since $\nu(D) = z_0^{-1} D z_0 = D$.

Now using ν, we may put a grading on D by setting

$$D_i = \{d \in D \mid \nu(d) = \omega^i d\} = \{d \in D \mid dz_0 = \omega^i z_0 d\}, \quad 0 \le i \le p-1.$$

Thus we have,

$$A_0 \cong C_A(z) \cong C_{\mathbb{M}_{n(D)}}(z_0) = \mathbb{M}_n(C_D(z_0)) = \mathbb{M}_n(D_0).$$

Similarly for each i and for each $a_i \in A_i$ we have

$$\omega^i a_i = \nu(a_i) = z^{-1} a_i z$$

giving

$$a_i z = \omega^i z a_i;$$

and this yields that

$$\theta(a_i) z_0 = \omega^i z_0 \theta(a_i) \quad \text{and} \quad \theta(a_i) \in \mathbb{M}_n(D_i)$$

and $\theta : A \mapsto \mathbb{M}_n(D)$ becomes a graded isomorphism. So each $A_i = \mathbb{M}_n(D_i)$ and $A \cong \tilde{\mathbb{M}}_n(D)$, which proves (3).

(4) Let V be the irreducible right module over the simple algebra A. The ring of A-endomorphisms of V, written as left operators is precisely the division algebra D i.e. $D = \mathrm{End}_A(V)$. Therefore, V itself is a left D-vector space and we may view V as a right module over E by setting

$$m(\sum_{i=1}^{r} b_i \otimes d_i) = \sum_{i=1}^{r} d_i m b_i; \quad d_i \in D^{op}, b \in B$$

because for $d, d' \in D^{op}$ and $b, b' \in B$ we have

$$m((b \otimes d)(b' \otimes d')) = m(bb' \otimes d'd)) = (d'd)m(bb') = d'(dmb)b' = (m(b \otimes d))(b' \otimes d').$$

Since E is a central divison algebra over B by the Lemma 4.2.2, V is a right E-vector space of some dimension m. We therefore have

$$n = dim_{D^{op}} V = dim_E V \cdot dim_{D^{op}} E = mp.$$

Let us fix some F-isomorphism $\theta : A \cong \mathbb{M}_{pm}(D)$ and z_0 be the matrix

$$z_o = \begin{bmatrix} \epsilon & & \bigcirc \\ & \ddots & \\ \bigcirc & & \epsilon \end{bmatrix}$$

with diagonal blocks of the form

$$\epsilon = \begin{bmatrix} 0 & 0 & \dots & 0 & a \\ \omega^{-1} & 0 & \dots & 0 & 0 \\ 0 & \omega^{-2} & & & \\ \vdots & & \ddots & & \\ 0 & & \dots & \omega^{-(p-1)} & 0 \end{bmatrix} = E_1 E_2^{-1}$$

where E_1 and E_2 are matrices in Example 3.2.3 with $\alpha = 1$. As indicated in this example,being matrices corresponding to e_1 and e_2, these matrices E_1 and E_2 satisfy

$$E_1^p = a, E_2^p = 1 \quad \text{and} \quad E_2 E_1 = \omega E_1 E_2,$$

hence

$$E_1 E_2^{-1} = \omega E_2^{-1} E_1$$

giving

$$(E_1 E_2^{-1})^k = \omega^k E_2^{-k} E_1^k.$$

In particular,

$$(E_1 E_2^{-1})^p = E_2^{-p} E_1^p = 1 \cdot a = a.$$

Thus we have $z_0^p = a$ and $F(z_0)$ is a subfield of $\mathbb{M}_n(D)$. Since both of $F(z)$ and $F(z_0)$ can be embedded into $\mathbb{M}_n(D)$, as in the proof of (3), by composing θ with an inner authomorphism of $\mathbb{M}_n(D)$ we can say that there exists an isomorphism $\varphi : A \mapsto \mathbb{M}_n(D)$ so that $\varphi(z) = z_0$. By using this φ and the description

$$A_i = \{a \in A \mid az = \omega^i za\},$$

we naturally define a grading on $\mathbb{M}_n(D)$. Thus we obtain the homogenous components of $\mathbb{M}_n(D)$ as

$$(\mathbb{M}_n(D))_k = \{x \in \mathbb{M}_n(D) \mid xz_0 = \omega^k z_0 x\}.$$

To be more explicit about the homogenous elements x of degree k we write $x \in \mathbb{M}_n(D) = \mathbb{M}_{pm}(D_k)$ in the block form of $p \times p$ matrices x_{ij} over D and obtain the equalities

$$x_{ij}\epsilon = \omega\epsilon x_{ij} \quad , i, j = 1, \cdots, m$$

On the other hand we verify that

$$E_1^j E_2^j \epsilon = \omega^{i+j} \epsilon E_1^i E_2^j.$$

Thus using the isomorphism

$$\phi : \mathbb{M}_p(F) \mapsto \langle \frac{a, 1}{F} \rangle_\omega$$

we get

$$e_1^i e_2^j \phi(\epsilon) = \omega^{i+j} \phi(\epsilon) e_1^i e_2^j$$

namely homogenous elements of degree k in $Q = \langle \frac{a,1}{F} \rangle_\omega$ are determined by

$$Q_k = \{q \in Q \mid q\phi(\epsilon) = \omega^k q\phi(\epsilon)\}$$

and the k-th homogenous component of $(\mathbb{M}_m(D)) \otimes_F Q$ is simply $(\mathbb{M}_m(D)) \otimes_F Q_k$, namely it consists of elements a of $(\mathbb{M}_m(D)) \otimes_F Q$ satisfying

$$a(1 \otimes \phi(\epsilon)) = \omega^k (1 \otimes \phi(\epsilon))a.$$

Now this shows that the isomorphism given by the composite map ψ

$$\mathbb{M}_n(D) = \mathbb{M}_{mp}(D) \stackrel{\sigma}{\to} \mathbb{M}_m(D) \otimes \mathbb{M}_p(F) \stackrel{1\otimes\phi}{\to} \mathbb{M}_m(D) \otimes \langle \frac{a,1}{F} \rangle_\omega$$

is a graded isomorphism since

$$\psi(z_0) = \begin{bmatrix} \epsilon & & \bigcirc \\ & \ddots & \\ \bigcirc & & \epsilon \end{bmatrix} = (1 \otimes \phi)(1 \otimes \epsilon) = 1 \otimes \phi(\epsilon)$$

and homogenous components are given by the corresponding computation rule to z_0 and $1 \otimes \phi(\epsilon)$.Thus we have graded isomorphisms

$$A \cong \mathbb{M}_n(D) \cong (\mathbb{M}_m(D)) \otimes \langle \frac{a,1}{F} \rangle_\omega.$$

As for the zeroth component A_0, it corresponds to $\mathbb{M}_m(D) \otimes Q_0$ and Q_0 is spanned by $\{e_1^i e_2^j \mid i+j \equiv 0 \pmod p\}$. But, when $i+j \equiv 0 (\text{mod } p)$ we have

$$e_1^i e_2^j = b e_1^i e_2^{-i} = c(e_1 e_2^{-1})^k$$

for some $b, c \in F$. Since $e_1 e_2^{-1}$ is a root of the irreducible polynomial $x^p - a$ in Q, we see that Q_0 is the algebra $F[e_1 e_2^{-1}] \cong F(\sqrt[p]{a})$. Thus $A_0 \cong \mathbb{M}_m(D) \otimes F(\sqrt[p]{a})$ and is central simple over $F(\sqrt[p]{a}) = Z(A_0)$. □

CHAPTER 5

STRUCTURE OF CLIFFORD ALGEBRAS OF p-FORMS

In this chapter, we apply the results of $\mathbb{Z}_p$-graded algebras into the Clifford algebras of p-forms to derive their complete structure theory.

5.1 Graded Structure of Clifford Algebras

In this section we shall investigate some properties related with graded structure of Clifford Algebras that will be used in the next section.We adopt the notations and terminology given in Chapter 1. As we indicated there the ω-Clifford algebra $C(V) = (a_1, a_2, \cdots, a_n)^p_\omega$ is the graded algebra

$$C(V) = C_0 \oplus \cdots \oplus C_{p-1}$$

so that $C_1 = V$ and in general C_r is the subspace spanned by

$$e_1^{i_1} \cdots e_n^{i_n} \;; \quad 0 \leq i_k \leq p-1 \;\; ; i_1 + i_2 + \cdots i_n = r \;\; ; k = 1, \cdots, n.$$

see [8].This yieds in particular that these elements are not contained in F. First we describe the types of Clifford algebras by means of the parity of dimension of V.

Theorem 5.1.1 *Let $C(V) = (a_1, a_2, \cdots, a_n)^p_\omega$ be an ω-Clifford algebra. Then*

(1) *it is of odd type iff dim V is odd.*

(2) *it is of even type iff dim V is even.*

Proof. Let $n = \dim_F V$ and $\{e_1, e_2, \cdots, e_n\}$ be a basis for V. We specify an element in $C(V)$ by

$$z = e_1 e_2^{-1} \cdots e_n^{(-1)^{(n-1)}}$$

and write

$$Z(C(V)) = Z_0 \oplus Z_1 \oplus \cdots \oplus Z_{p-1}$$

where each $Z_i \subset C_i$ for all $i = 1, 2, \cdots, p-1$. We simply write

$$Z' = Z_1 \oplus Z_2 \oplus \cdots \oplus Z_{p-1}.$$

We first note that if $\dim V$ is odd, say, $n = 2m+1$, z is in the center of $C(V)$.In order to see this, it is enough to verify that

$$ze_i = e_i z \quad \text{for all} \quad i = 1, \cdots, n.$$

Infact pick up an i and consider the cases where n is even or odd:

a) If i is even, then using the rule $e_j e_i = \omega e_i e_j$ where $j > i$ we obtain

$$\begin{aligned}
ze_i &= (e_1 e_2^{-1} e_3 \cdots e_i^{-1} \underbrace{e_{i+1} \cdots e_{n-1}^{-1} e_n)}_{\text{odd number of terms}} e_i \\
&= \underbrace{(\omega\omega^{-1} \cdots \omega\omega^{-1})}_{1} \omega (e_1 e_2^{-1} \cdots e_i^{-1} e_i e_{i+1} \cdots e_{n-1}^{-1} e_n) \\
&= \omega \underbrace{(e_1 e_2^{-1} e_3 \cdots e_i}_{\text{odd number of terms}} e_i^{-1} e_{i+1} \cdots e_{n-1}^{-1} e_n) \\
&= \underbrace{\omega(\omega^{-1}\omega \cdots \omega\omega^{-1})}_{1} e_i (e_1 e_2^{-1} \cdots e_i^{-1} e_{i+1} \cdots e_{n-1}^{-1} e_n) \\
&= e_i z.
\end{aligned}$$

b) If i is odd,then

$$\begin{aligned}
ze_i &= (e_1 e_2^{-1} \cdots e_i \underbrace{e_{i+1}^{-1} \cdots e_{n-1} e_n)}_{\text{even number of terms}} e_i \\
&= \underbrace{\omega\omega^{-1} \cdots \omega\omega^{-1})}_{1} (e_1 e_2^{-1} \cdots e_{i-1}^{-1} e_i \cdots e_{n-1}^{-1} e_n) \\
&= \underbrace{(e_1 e_2^{-1} \cdots e_{i-1}^{-1}}_{\text{even number of terms}} e_i \cdots e_{n-1}^{-1} e_n) \\
&= \underbrace{(\omega\omega^{-1} \cdots \omega\omega^{-1})}_{1} e_i (e_1 e_2^{-1} \cdots e_{i-1}^{-1} e_i \cdots e_{n-1}^{-1} e_n) \\
&= e_i z.
\end{aligned}$$

Further by the remark above $z \notin F$ and it can be verified by means of computations similar to above that $z^p \in F^\bullet$.

Now, if dim $V = n$ is even as above we see that $e_i z = \omega z e_i$ for $i = 1, \cdots, n$ and $z^p \in F^\bullet$.

Thus $z \in Z(C_0(V))$ and since $z \notin F$ we get the result that $C_0(V)$ is not a central F algebra.Thus we proved by "if" parts of Theorem 5.1.1 of the assertions. Therefore $Z' = 0$, in other words, $C(V)$ is of even type.

As for the " only if " parts, suppose first that $(a_1, \cdots a_n)^p_\omega$ is of odd type.

Then $Z(C(V)) = F \oplus Z_1 \oplus \cdots \oplus Z_{p-1}$ with $Z_i \subset C_i$ and there exists $z \in Z_1$ such that $z^p \in F^\bullet$.

Let $z = \sum a_{v_1 \cdots v_n} e_1^{v_1} \cdots e_n^{v_n}$ be in $Z(C)$ with $v_1 + v_2 + \cdots + v_n \equiv 1 \pmod p$ and $ze_i = e_i z$.

We must show that n is odd. Since $ze_i = e_i z$ holds for each i, then $(e_1^{v_1} \cdots e_n^{v_n})e_i = e_i(e_1^{v_1} \cdots e_n^{v_n})$ is true if and only if

$$\omega^{(-v_n - \cdots - v_{i+1}) + v_{i-1} + \cdots + v_1} \equiv 1 \pmod p.$$

Then we have

$$-v_n - \cdots - v_{i+1} + v_{i-1} + \cdots + v_1 \equiv 0 \pmod p$$

for all $i = 1, \cdots, n$. Hence we obtain the following system of equations:

$$\begin{aligned} v_n + v_{n-1} + \cdots + v_2 + v_1 &\equiv 1 \pmod p \\ -v_n + v_{n-2} + \cdots + v_1 &\equiv 0 \pmod p \\ -v_n - v_{n-1} + v_{n-3} + \cdots + v_1 &\equiv 0 \pmod p \\ &\vdots \\ -v_n - v_{n-1} - \cdots - v_3 + v_1 &\equiv 0 \pmod p \\ -v_n - v_{n-1} - \cdots - v_2 &\equiv 0 \pmod p \end{aligned}$$

Using these equations, we have

$$v_1 \equiv 1, \quad v_2 \equiv -1, \quad v_3 \equiv 1, \quad v_4 \equiv -1, \cdots, \cdots \pmod p$$

Then,

$$v_i = \begin{cases} 1 & \text{if } i \text{ is odd} \\ -1 & \text{if } i \text{ is even.} \end{cases}$$

Hence, be able to obtain $v_1 + \cdots + v_n \equiv 1 \pmod p$, the number of v_i's, namely n, must be odd.

Secondly, let $(a_1, \cdots, a_n)_\omega^p$ be of even type.

Then there exists $z \in C_0(V)$ with $z = \sum a_{v_1 \cdots v_n} e_1^{v_1} \cdots e_n^{v_n}$ in C_0 such that $v_1 + v_2 + \cdots + v_n \equiv 0 \pmod p$.

Here we have $e_i z = \omega z e_i$ and using these equations we get

$$e_i z = e_i(e_1^{v_1} \cdots e_n^{v_n}) = \omega(e_1^{v_1} \cdots e_i^{v_i} \cdots e_n^{v_n}) e_i$$

and

$$\omega \cdot \omega^{(v_n + \cdots + v_{i+1}) - (v_{i-1} + \cdots + v_1)} = \omega^{v_n + \cdots + v_{i+1} - v_{i-1} - \cdots - v_i + 1} \equiv 1 \pmod p$$

for all $i = 1, \cdots, n$. Therefore, we have the following system of equations:

$$v_n + v_{n-1} + \cdots + v_1 \equiv 0 \pmod p$$

$$v_n + v_{n-2} - v_{n-3} - \cdots v_1 + 1 \equiv 0 \pmod p$$

$$v_n + v_{n-1} + v_{n-2} - v_{n-4} - \cdots - v_1 + 1 \equiv 0 \pmod p$$

$$\vdots$$

$$v_n + v_{n-1} + \cdots + v_3 - v_1 + 1 \equiv 0 \pmod p$$

$$v_n + v_{n-1} + \cdots + v_2 + 1 \equiv 0 \pmod p.$$

Here we get $v_1 \equiv -1$, $v_2 \equiv 1$, $v_3 \equiv -1, \cdots \pmod p$ and

$$v_i = \begin{cases} -1 & if \quad i \text{ is odd} \\ 1 & if \quad i \text{ is even.} \end{cases}$$

Hence we conclude that n must be even since $v_1 + \cdots + v_n \equiv 0 \pmod p$□.

5.2 Structure Theorems

Now, we can translate the results of Chapter 4 into the Clifford algebras of p-forms.

Theorem 5.2.1 *Suppose that dim $V = n = 2m+1$ is odd and*
$a = (-1)^{m(p-1)} a_1 a_2^{-1} \cdots a_{2m}^{-1} a_{2m+1}$.
Then

1) $C_0(V)$ is a CSA over F and

$$C(V) \cong (C_0(V)) \hat{\otimes} F\langle \sqrt[p]{a} \rangle.$$

2) If $x^p - a$ has no roots in F then $C(V)$ is a CSA over $F\langle \sqrt[p]{a} \rangle$ and $Z(C(V)) \cong F\langle \sqrt[p]{a} \rangle$.

3) If $x^p - a$ has a root in F then
$Z(C(V)) \cong F \times \cdots \times F$ *and* $C(V) \cong C_0 \times \cdots \times C_0$.

Proof. By Theorem 5.1.1 $(a_1, a_2, \cdots, a_n)_\omega^p$ is of odd type and from Theorem 4.2.1, C_0 is a CSA over F. The rest is obtained from Theorem 4.2.1.□

Theorem 5.2.2 *Suppose that dim* $V = n = 2m + 2$ *is even and*
$a = (-1)^{(m+1)(p-1)} a_1 a_2^{-1} \cdots a_{2m-1} a_{2m}^{-1}$.
Then

1) $C(V)$ is a CSA over F, say $C(V) \cong \mathbb{M}_t(D)$ as ungraded algebras, where D is a F-central division algebra.

2) If $x^p - a$ is irreducible over F then $Z(C_0) \cong F\langle \sqrt[p]{a} \rangle$ and C_0 is a CSA over $F\langle \sqrt[p]{a} \rangle$.

3) If $x^p - a$ is not irreducible then $Z(C_0) \cong F \times \cdots \times F$ and $C(V) \cong \tilde{\mathbb{M}}_t((D))$ as graded algebras.
Further, if t is a p-power then
$C_0(V) \cong \mathbb{M}_r(D) \times \mathbb{M}_r(D) \times \cdots \times \mathbb{M}_r(D)$, *(p-times) where* $r = \frac{t}{p}$.□

Proof.

1)By Theorem 5.1.1 $(a_1, a_2, \cdots, a_n)_\omega^p$ is of even type and it follows from Theorem 4.2.4 that $C(V)$ is a CSA over F.

2) If $x^p - a$ is irreducible then by Theorem 4.2.4 we have

$$Z(C_0) \cong F\langle \sqrt[p]{a} \rangle.$$

3) Using the 2^{nd} part of the same theorem we have $Z(C_0) \cong F \times \cdots \times F$. Then we have, $C(V) \cong \tilde{\mathbb{M}}_t(D)$ and

$$C_0(V) \cong \mathbb{M}_{r_1}(D) \times \mathbb{M}_{r_2}(D) \times \cdots \times \mathbb{M}_{r_p}(D),$$

taking $r_1 = r_2 = \cdots = r_p = r$ where $r_i = \dim V_i$, $i = 1, \cdots, p$ we have

$$C_0(V) \cong \mathbb{M}_r(D) \times \mathbb{M}_r(D) \times \cdots \times \mathbb{M}_r(D).□$$

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