CONSERVED CHARGES OF QUADRATIC CURVATURE GRAVITY THEORIES IN ARBITRARY BACKGROUNDS

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DENİZ OLGU DEVECİOĞLU

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submitted by **DENİZ OLGU DEVECİOĞLU** in partial fulfillment of the requirements for the degree of **Master of Science in Physics Department, Middle East Technical University** by,

Prof. Dr. Canan ÖZGEN
Dean, Graduate School of **Natural and Applied Sciences** ________________

Prof. Dr. Sinan BİLİKMEN
Head of Department, **Physics** ________________

Assoc. Prof. Dr. Bahtiyar Özgür SARIOĞLU
Supervisor, **Physics Department, METU** ________________

Examining Committee Members:

Prof. Dr. Abdullah VERÇİN
Physics Department , Ankara University ________________

Assoc. Prof. Dr. Bahtiyar Özgür SARIOĞLU
Physics Department , METU ________________

Prof. Dr. Atalay KARASU
Physics Department , METU ________________

Assoc. Prof. Dr. Bayram TEKİN
Physics Department , METU ________________

Assoc. Prof. Dr. Hakan CEBECİ
Physics Department , Anadolu University ________________

Date: ________________

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Name, Last Name: DENİZ OLGU DEVECİOĞLU

Signature :

ABSTRACT

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Devecioğlu, Deniz Olgu
M.S., Department of Physics
Supervisor : Assoc. Prof. Dr. Bahtiyar Özgür Sarıoğlu

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We generalize the definition of conserved gravitational Killing charges of quadratic curvature gravity theories to arbitrary backgrounds that admit at least one global (timelike) Killing vector. This charge definition is background gauge invariant and reduces correctly to the already known limit given by [1] when the background is a space of constant curvature. As an application we use this definition to compute the charges of various black holes in New Massive Gravity; namely the BTZ blackhole, the blackhole given in [2] and the Lifshitz blackhole. Finally we compare the charges of these blackholes with the ones given in [3], which uses a different approach.

Keywords: Conserved charges, Killing vectors, General background

ÖZ

EĞRİLİĞİ İKİNCİ DERECEDEN GRAVİTASYON KURAMLARINDA GENEL UZAY-ZAMAN ARKA-PLANI İÇİN KORUNAN YÜKLER

Devecioğlu, Deniz Olgu
Yüksek Lisans, Fizik Bölümü
Tez Yöneticisi : Doç. Dr. Bahtiyar Özgür Sarıoğlu

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Eğriliğin ikinci dereceden kuvvetlerini içeren gravitasyon kuramlarında korunan Killing yükleri tanımı, arka-planı en azından bir global Killing vektörüne sahip olmak koşuluyla keyfi olan uzay-zamanlar için genişletildi. Bu yük tanımının, arka-plandaki ayar dönüşümleri altında değişmez olduğu ve arka-planı sabit eğrilikte olan uzay-zamanlar için önceden bilinen tanıma [1] indiği gösterildi. Uygulama olarak New Massive Gravity kuramındaki çeşitli karadelik çözümlerinin (BTZ, [2]'te verilen karadelik ve Lifshitz) yükleri hesaplandı. Son olarak hesaplanan bu yükler, başka yöntemler [3] kullanılarak hesaplananlarla karşılaştırıldı.

Anahtar Kelimeler: Korunan yükler, Killing vektörleri, Genel arka-plan

To my parents

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CHAPTER 1

INTRODUCTION

The concept of energy is a prominent one in physics. No matter what kind of theory one is working on, energy concept with the law of conservation yields considerable amount of information about the system studied. Gravity is not an exception, though it is a subtle one. To exemplify the energy concept let us consider the Maxwell electromagnetism. The Maxwell field equations in flat Minkowski spacetime read

$$\partial^a F_{ab} = 4\pi J_b, \tag{1.1}$$

$$\partial_{[a} F_{bc]} = 0, \tag{1.2}$$

where F_{ab} is the well known electromagnetic field tensor and J_b is the current vector. The stress-energy tensor is given by

$$T_{ab} = \frac{1}{4\pi}\left\{F_{ac}F_b{}^c - \frac{1}{4}\eta_{ab}F_{de}F^{de}\right\}. \tag{1.3}$$

When $J_b = 0$, one has $\partial_b T^{ab} = 0$ with the help of the field equations. For $J_b \neq 0$ the stress-energy tensor in (1.3) is not conserved on its own; the stress-energy tensor for the fields plus matter is conserved. With the help of (1.3), one can express the field energy density in terms of the fields involved.

In gravity one has the field equations

$$\Phi_{ab}(g, R, R^2, \nabla R, \cdots) = \kappa\tau_{ab}, \tag{1.4}$$

where Φ_{ab} is the "generalized" Einstein tensor of a local, invariant action which is assumed to be quadratic in curvature for our purposes. Here τ_{ab} denotes the matter stress-energy tensor and κ is a coupling constant. The energy content of this system can be given as the energy content of the matter τ_{ab} plus the energy that the gravitational field possesses. However, in gravity there is no such stress-energy tensor for the field as we have (1.3) in electromagnetism. One can also find a possible candidate by considering the Newtonian limit. Newtonian gravitational energy density is $\kappa|\nabla\phi|^2$, where $\nabla\phi$ corresponds to the derivative of a component of the metric in Newtonian limit. Therefore, one can associate a stress-energy tensor with the square of the first derivative of metric. However *"there is no tensor other then g_{ab} itself that can be constructed locally from only coordinate basis components of g_{ab}"* [4].

To overcome this problem one can introduce a decomposition on the full metric such as $g_{ab} = \bar{g}_{ab} + h_{ab}$. Here $\bar{g}_{ab}$ is the background metric which is associated with the index raising, lowering or covariant differentiation. For the definition that will be presented in this thesis, we assume that there exists at least one globally defined timelike Killing vector of the background $\bar{g}_{ab}$. The deviation h_{ab} is the dynamical part propagating in the background space. This scheme looks promising, but there are nontrivial conditions on h_{ab}, such as the behavior of its components as the radial coordinate goes to "infinity". This condition is not easy to satisfy since the background is not flat, so there may not be a global coordinate system, just the local patches. Therefore, it is not easy to define a "radial" coordinate. However, we will not deal with these complications. Assuming h_{ab} behaves nicely, we will decompose the "generalized" Einstein tensor Φ_{ab} as

$$\Phi_{ab} = \bar{\Phi}_{ab} + (\Phi_{ab})_L + \mathscr{O}(h^2) + \cdots . \tag{1.5}$$

Then using the charge definition given by [5, 1], one can define a conserved current

$$(\Phi^{ab})_L\bar{\xi}_b, \tag{1.6}$$

where $(\Phi^{ab})_L$ is the linearized field equations and $\bar{\xi}_b$ is the Killing vector of the background metric. However, we will investigate the charges in arbitrary backgrounds as opposed to [5, 1], where constant curvature backgrounds are considered. Writing (1.6) as a 2-form and taking the surface integral leads to a conserved Killing charge. In the rest of the thesis we will discuss this whole procedure in details.

The outline of this thesis as follows. In the first chapter we will state Stokes' theorem in a coordinate basis and give a detailed discussion of the charge definition procedure. In the second chapter we will discuss the linearization of the field equations. In the third chapter we will calculate the Killing charge of the "cosmological" Einstein tensor explicitly and give the charges coming from the quadratic curvature terms. In the fourth chapter we will discuss the gauge invariance of the charges we have found. In the fifth chapter we will compute the charges of various blackholes of New Massive Gravity using our charge definition . The last chapter will be about our conclusions. Appendices are reserved for the explicit calculation of charges and the identities we have used during the calculations. Throughout this thesis our conventions are: Signature $(-,+,+,\cdots+)$, $[\nabla_a,\nabla_b]V_c = R_{abc}{}^{d}V_d$, $R_{ab} \equiv R_{acb}{}^{c}$.

CHAPTER 2

STOKES' THEOREM

In this chapter we will derive the essential tools for defining conserved charges. First we will state Stokes' theorem, then consider an application from electrodynamics. Finally we give the procedure of defining conserved charges, which is the heart of this thesis. Most of our discussion follows [6].

2.1 Stokes' theorem in a coordinate basis

Let us start with Stokes' theorem (which we state without proof)

$$\int_{\mathscr{M}} d\omega = \int_{\partial\mathscr{M}} \omega, \tag{2.1}$$

where $\mathscr{M}$ is a n-dimensional manifold with $(n-1)$-dimensional boundary $\partial\mathscr{M}$, and ω is an $(n-1)$-form on $\mathscr{M}$. We will need the coordinate basis version of this theorem.

Let us first write the $(n-1)$-form ω as the Hodge dual of a 1-form V

$$\omega = *V, \tag{2.2}$$

$$\omega_{a_1\cdots a_{n-1}} = \epsilon^{b}{}_{a_1\cdots a_{n-1}} V_b, \tag{2.3}$$

where $\epsilon_{a_1\cdots a_{n-1}} = \widetilde{\epsilon}_{a_1\cdots a_{n-1}}\sqrt{-g}$ and $\widetilde{\epsilon}_{a_1\cdots a_{n-1}}$ is antisymmetric in all indices. Taking the exterior derivative we get an n-form $d\omega$

$$d\omega = d(*V), \tag{2.4}$$

$$\begin{aligned}(d\omega)_{ca_1\cdots a_{n-1}} &= n\nabla_{[c}(\epsilon^{|b|}{}_{a_1\cdots a_{n-1}]}V_b), \\ &= n\epsilon^{|b|}{}_{[a_1\cdots a_{n-1}}\nabla_{c]}V_b,\end{aligned} \tag{2.5}$$

where we have used $\nabla_c \epsilon^b{}_{a_1 \cdots a_{n-1}} = 0$. Since any n-form can be written as the Hodge dual of some function f

$$d\omega = f\epsilon = *f, \tag{2.6}$$

it is clear that $d\omega$ can be written as some function f times the invariant volume element, so that we have a familiar integral in coordinate basis

$$\int_{\mathscr{M}} d\omega = \int_{\mathscr{M}} f\epsilon = \int_{\mathscr{M}} f\sqrt{-g}\ d^n x. \tag{2.7}$$

In order to find f, take the dual of (2.6)

$$*d\omega = **f = (-1)^s f, \tag{2.8}$$

$$f = (-1)^s * d\omega. \tag{2.9}$$

Here s is the signature of the metric on the local coordinate chart. Using (2.9)

$$\begin{aligned} *d\omega &= \frac{1}{n!}\epsilon^{ca_1 \cdots a_{n-1}} n\epsilon_{b[a_1 \cdots a_{n-1}} \nabla_{c]} V^b, \\ &= \frac{1}{(n-1)!}(n-1)!(-1)^s \delta^c_b \nabla_c V^b, \\ &= (-1)^s \nabla_b V^b, \end{aligned} \tag{2.10}$$

where $\epsilon^{a_1 \cdots a_p b_1 \cdots b_{n-p}} \epsilon_{a_1 \cdots a_p c_1 \cdots c_{n-p}} = (-1)^s p!(n-p)! \delta^{[b_1 \cdots}{}_{c_1} \ldots \delta^{b_{n-p}]}{}_{c_{n-p}}$ has been used.

So $f = \nabla_b V^b$ and $d\omega = \nabla_b V^b \sqrt{-g}\ d^n x$. Having found the left hand side of (2.1), we can follow similar steps to obtain the right hand side.

As before, since ω is an $(n-1)$-form it can be written as the Hodge dual of some function g

$$\omega = *g = g\widehat{\epsilon}, \tag{2.11}$$

where $\widehat{\epsilon}_{a_1 \cdots a_{n-1}} = n^c \epsilon_{ca_1 \cdots a_{n-1}}$ and n^c is the normal vector to the boundary. Hence $\widehat{\epsilon} = \sqrt{\gamma}\, d^{n-1} y$ is the induced volume element on the hypersurface and γ is the induced metric on the hypersurface. We have defined ω as

$$\omega = *V, \tag{2.12}$$

with (2.11) g reads

$$\begin{aligned} g &= (-1)^s * * V \\ &= (-1)^s * \left[\epsilon^b{}_{a_1 \cdots a_{n-1}} V_b\right] \\ &= (-1)^s \frac{1}{(n-1)!} \epsilon^{a_1 \cdots a_{n-1}} \epsilon^b{}_{a_1 \cdots a_{n-1}} V_b \\ &= (-1)^s \frac{1}{(n-1)} n_c \epsilon^{c a_1 \cdots a_{n-1}} \epsilon_{b a_1 \cdots a_{n-1}} V^b \\ &= (-1)^s n_c \delta^c_b (-1)^s V^b = n_b V^b. \end{aligned} \tag{2.13}$$

Therefore ω can be written as

$$\omega = n_b V^b \hat{\epsilon} = n_b V^b \sqrt{-\gamma}\ d^{n-1} y. \tag{2.14}$$

Hence Stokes' Theorem takes the following form in a coordinate basis

$$\int_{\mathscr{M}} d^n x \sqrt{-g}\ \nabla_b V^b = \int_{\partial \mathscr{M}} d^{n-1} y \sqrt{-\gamma}\ n_b V^b. \tag{2.15}$$

From this relation we can define conserved Killing charges. As an illustrative example, let us remind ourselves of Maxwell electrodynamics.

Assume that we have a conserved current J^a

$$\nabla_a J^a = 0. \tag{2.16}$$

From our previous discussion, we know that the divergence of a vector can be cast in the form

$$d(*J) = 0. \tag{2.17}$$

Now the charge on a hypersurface reads

$$Q_\Sigma = \int_\Sigma *J = \int_\Sigma d^{n-1} x \sqrt{\gamma}\ n_a J^a, \tag{2.18}$$

where we used (2.15) on the last equality.

Now assume that we have a manifold $\mathscr{M}$ with its boundary divided into two parts as Σ_1, Σ_2. Let there be part of the boundary connecting these two at infinity where all fields vanish. Then Stokes' theorem gives us

$$\begin{aligned} \int_{\mathscr{M}} d(*J) = \int_{\partial \mathscr{M}} *J &= \int_{\Sigma_1} *J - \int_{\Sigma_2} *J \\ &= Q_1 - Q_2 = 0. \end{aligned} \tag{2.19}$$

Therefore Stokes' theorem and the existence of conserved current guarantees the existence of a conserved charge. As an example, the 2-form (electromagnetic field tensor) F_{ab} can be used to define charge. Electromagnetic field equations can be cast into the form

$$d(*F) = *J \Longrightarrow \nabla_a F^{ab} = J^b. \tag{2.20}$$

Now taking the integral of this over the hypersurface Σ, one finds

$$Q = \int_\Sigma *J = \int_\Sigma d(*F) = \int_{\partial\Sigma} *F, \tag{2.21}$$

where in the last line Stokes' theorem has been used, since $*F$ is an $(n-2)$-form. In a coordinate basis

$$\int_\Sigma d^{n-1}y \, \sqrt{|\gamma|}\, n_a \, \nabla_b F^{ab} = \int_{\partial\Sigma} d^{n-2}x \, \sqrt{|\gamma^{\partial\Sigma}|}\, n_a \, \sigma_b \, F^{ab}, \tag{2.22}$$

where γ is the induced metric and n_a is the unit normal for Σ, and likewise for $|\gamma|^{\partial\Sigma}$ and σ_b for the boundary $\partial\Sigma$. Although the boundary of a boundary is zero for a simply connected manifold, vanishing fields at infinity divides the hypersurface into two parts which are not simply connected, therefore the integral in (2.22) is different from zero.

2.2 Abbott- Deser-Tekin (ADT) charge

In this section we follow the procedure given by [1]. We will consider the gravitational field equation

$$\Phi_{ab}(g, R, R^2) = \kappa \tau_{ab}, \tag{2.23}$$

which we assume follows from an arbitrary quadratic curvature action that guarantees the covariant conservation [7]

$$\nabla_a \Phi^{ab} = 0, \tag{2.24}$$

where τ_{ab} is the conserved energy-momentum tensor coupled to the "generalized Einstein tensor" with a coupling κ. Following the ADT procedure, we will write the metric g_{ab} as

$$g_{ab} = \bar{g}_{ab} + h_{ab}, \quad g^{ab} = \bar{g}^{ab} - h^{ab}, \tag{2.25}$$

where $\bar{g}_{ab}$ is the background metric, which is assumed to satisfy (2.23) for $\tau_{ab} = 0$. The deviation h_{ab} must be such that it vanishes rapidly at infinity, so that one can use Stokes' theorem. With the minus sign in the contravariant metric in (2.25) $g^{ab}g_{bc} = \delta^{a}{}_{c}$ condition is satisfied to first order in h_{ab}. By assumption, all operations such as index raising, lowering or the covariant differentiation will be with respect to $\bar{g}_{ab}$. With that decomposition, the field equations (2.23) can be written as

$$\Phi_{ab} = \bar{\Phi}_{ab} + (\Phi_{ab})_L + \mathscr{O}(h^2) + \cdots, \tag{2.26}$$

where $\bar{\Phi}_{ab} = 0$ is assumed to be satisfied by background metric $\bar{g}_{ab}$ and $(\Phi_{ab})_L$ is first order in h_{ab}. If one defines all other nonlinear terms and τ_{ab} as the gravitational energy momentum tensor T_{ab}, then the linearized version of (2.23) reads

$$(\Phi_{ab})_L = (T_{ab})_L + \mathscr{O}(h^2). \tag{2.27}$$

In order to define a conserved current as in electromagnetism, one needs a 2-form. However $(T_{ab})_L$ is a symmetric tensor. To overcome this problem, let us introduce a Killing vector $\bar{\xi}_c$ associated with the isometry of the background metric $\bar{g}_{ab}$. The Killing vector chosen, must be globally well defined. Now one can construct the current with the Killing vector $\bar{\xi}_b$

$$\sqrt{-\bar{g}}(T^{ab})_L\bar{\xi}_a. \tag{2.28}$$

In order to define a conserved charge, this current must be covariantly conserved

$$\bar{\nabla}_b\left(\bar{\xi}_a(T^{ab})_L\right) = \partial_b\left(\sqrt{-\bar{g}}(T^{ab})_L\bar{\xi}_a\right) = 0,$$

which can also be written as

$$\bar{\nabla}_b\left(\bar{\xi}_a(T^{ab})_L\right) = \bar{\nabla}_a\left((T^{ab})_L\right)\bar{\xi}_b + (T^{ab})_L(\bar{\nabla}_a\bar{\xi}_b). \tag{2.29}$$

Here the second term vanishes due to the antisymmetry of the Killing equation. However, the vanishing of the first term is not obvious. At this point, we depart from the procedure given by [8, 1], here it was assumed that the linearized field equations are covariantly conserved

$$\bar{\nabla}_a(T^{ab})_L = 0. \tag{2.30}$$

This condition guarantee the conservation of the current in (2.28). Though it is shown that this condition holds for constant curvature backgrounds, it is too strong for general ones. For us the first condition holds (antisymmetry of the Killing equation), however the covariant conservation assumption can be weakened. So we only assume

$$\bar{\nabla}_b\left(\bar{\xi}_a(T^{ab})_L\right) = \partial_b\left(\sqrt{-\bar{g}}(T^{ab})_L\bar{\xi}_a\right) = 0 \tag{2.31}$$

As written above, we set the contraction of the Killing vector with the covariant derivative of $(T^{ab})_L$, rather than setting only $\bar{\nabla}_b(T^{ab})_L = 0$. Evidently this assumption depends on the properties of the Killing vector considered.

With this assumption, we construct the following conserved vector density current

$$\bar{\nabla}_a \left(T^{ab} \bar{\xi}_b \right) = \partial_b \left(\sqrt{-\bar{g}} (T^{ab})_L \bar{\xi}_a \right) = 0. \tag{2.32}$$

Having a conserved current in hand, the conserved charge as a surface integral can be expressed, if $T^{ab}\bar{\xi}_b$ can be cast as divergence of a 2-form $\bar{\nabla}_b \mathcal{F}^{ab}$. The surface integral has the form

$$\begin{aligned} Q^a(\bar{\xi}) &= \int_\Sigma d^{n-1}x \sqrt{-\bar{g}} (T^{ab})_L \bar{\xi}_b \\ &= \int_\Sigma d^{n-1}x \sqrt{-\bar{g}} \bar{\nabla}_b \mathcal{F}^{ab} = \int_{\partial\Sigma} d^{n-2}x \sqrt{-\bar{g}^{\partial\Sigma}} \mathcal{F}^{ab} n_b \\ &= \int_{\partial\Sigma} d\bar{S}_b \mathcal{F}^{ab}. \end{aligned} \tag{2.33}$$

Here, Σ is the $(n-1)$-dimensional hypersurface of an n dimensional manifold $\mathscr{M}$. Likewise $\partial\Sigma$ is an $(n-2)$-dimensional boundary with the surface element $dS = \sqrt{-\bar{g}} d^{N-2}x$ on the boundary. Although the manifold $\mathscr{M}$ needs to satisfy some nontrivial conditions to define integration properly, (e.g. compactness, orientability) we won't go into these explicitly [7]. Another important issue is the diffeomorphism invariance of the charge which we will examine in detail, in Chapter 5.

In this chapter, we gave the tools and the procedure to construct the conserved Killing charges in the generic, higher order curvature gravity models. To prepare the necessary objects that go into the charge calculation, we will now continue with the details of the linearization in the next chapter.

CHAPTER 3

LINEARIZATION OF THE FIELD EQUATIONS

3.1 The Linearization Procedure

For our purposes we will deal with an action at quadratic order in curvature given as

$$I = \int d^D x \sqrt{-g} \left\{ \frac{1}{\kappa}(R + 2\Lambda_0) + \alpha R^2 + \beta R_{ab} R^{ab} \right\}. \tag{3.1}$$

One can also add the Gauss-Bonnet term $\gamma(R^2_{abcd} - 4R_{ab}R^{ab} + R^2)$ to this, however in $D = 3$ it is zero and in $D = 4$ it is a surface integral and does not contribute to the field equations [9]. We will not be dealing with the Gauss-Bonnet term, since we will be mostly interested in the charges of three dimensional blackholes in New Massive Gravity [10] (NMG) as an application. The variation of (3.1) with respect to metric gives the field equations as [1]

$$\begin{aligned} T_{ab} = {} & \frac{1}{\kappa}\left(R_{ab} - \frac{1}{2} g_{ab} R - \Lambda_0 g_{ab}\right) + 2\alpha R \left(R_{ab} - \frac{1}{4} g_{ab} R\right) + (2\alpha + \beta)\left(g_{ab}\Box - \nabla_a \nabla_b\right) R \\ & + \beta \Box \left(R_{ab} - \frac{1}{2} g_{ab} R\right) + 2\beta \left(R_{acbd} - \frac{1}{4} g_{ab} R_{cd}\right) R^{cd}. \end{aligned} \tag{3.2}$$

We have to follow a somewhat different procedure from [1] to compute the linearized field equations, where they express the whole linearized equations as a function of the linearized 'cosmological' Einstein tensor $(\mathcal{G}^{ab})_L$ and the linearized curvature scalar R_L. However with a general background $\bar{g}_{ab}$ it is not possible to do that. So we keep the linearized terms as general as possible.

Let us start from the Einstein equation

$$\mathcal{G}_{ab} = R_{ab} - \frac{1}{2} g_{ab} R - \Lambda_0 g_{ab}. \tag{3.3}$$

Since R_{ab}, R are functions of g_{ab}, they can be linearized in the following fashion

$$R_{ab} = \bar{R}_{ab} + (R_{ab})_L + \mathscr{O}(h^2),$$
$$R \equiv (R_{ab}g^{ab}) = \bar{R} + R_L + \mathscr{O}(h^2),$$

where barred terms are functions of $\bar{g}_{ab}$ only, and $(R_{ab})_L$, R_L are first order in h_{ab}. Then the whole Einstein equation reads

$$\mathscr{G}_{ab} = \bar{R}_{ab} + (R_{ab})_L - \frac{1}{2}\bar{g}_{ab}\bar{R} - \frac{1}{2}\bar{R}h_{ab} - \frac{1}{2}\bar{g}_{ab}R_L - \Lambda_0\bar{g}_{ab} - \Lambda_0 h_{ab} + \mathscr{O}(h^2),$$
$$\mathscr{G}_{ab} = \bar{\mathscr{G}}_{ab} + (\mathscr{G}_{ab})_L + \mathscr{O}(h^2), \tag{3.4}$$

where $(\mathscr{G}_{ab})_L = (R_{ab})_L - \frac{1}{2}\bar{R}h_{ab} - \frac{1}{2}\bar{g}_{ab}R_L - \Lambda_0 h_{ab}$. For a second example, let us take $g_{ab}\Box R$

$$g_{ab}\Box R = (\bar{g}_{ab} + h_{ab})\left(\bar{\Box}\bar{R} + (\Box R)_L + \mathscr{O}(h^2)\right)$$
$$= \bar{g}_{ab}\bar{\Box}\bar{R} + h_{ab}\bar{\Box}\bar{R} + \bar{g}_{ab}(\Box R)_L + \mathscr{O}(h^2) + \cdots,$$
$$(g_{ab}\Box R)_L = \bar{g}_{ab}(\Box R)_L + h_{ab}\bar{\Box}\bar{R}. \tag{3.5}$$

Proceeding in this fashion, the full linearized equations read

$$T_{ab} = \bar{T}_{ab} + \frac{1}{\kappa}\left[(R_{ab})_L - \frac{1}{2}h_{ab}\bar{R} - \frac{1}{2}\bar{g}_{ab}R_L - \Lambda_0 h_{ab}\right]$$
$$+ 2\alpha\left[\bar{R}(R_{ab})_L + \bar{R}_{ab}R_L - \frac{1}{2}\bar{g}_{ab}\bar{R}R_L - \frac{1}{4}h_{ab}\bar{R}^2\right]$$
$$+ (2\alpha + \frac{\beta}{2})\left(\bar{g}_{ab}(\Box R)_L + h_{ab}\bar{\Box}\bar{R} - (\nabla_a\nabla_b R)_L\right) \tag{3.6}$$
$$- \frac{\beta}{2}\left[(\nabla_a\nabla_b R)_L - 2(\Box R_{ab})_L - 4(R_{acbd}R^{cd})_L\right]$$
$$- \frac{\beta}{2}\left[(R_{cd}R^{cd})_L\bar{g}_{ab} + h_{ab}\bar{R}_{cd}\bar{R}^{cd}\right] + \mathscr{O}(h^2) + \cdots$$
$$= \bar{T}_{ab} + (T_{ab})_L + \mathscr{O}(h^2) + \cdots, \tag{3.7}$$

where $\bar{T}_{ab}$ is satisfied with background metric, so without matter $\bar{T}_{ab} = 0$ and $(T_{ab})_L$ will be first order terms in h_{ab}. For future use, we will define

$$\frac{1}{\kappa}\bar{\mathscr{G}}_{ab} \equiv \bar{R}_{ab} - \frac{1}{2}\bar{g}_{ab}\bar{R} - \Lambda_0\bar{g}_{ab}, \tag{3.8}$$
$$\alpha\bar{A}_{ab} \equiv \alpha\left[2\bar{R}\bar{R}_{ab} - \frac{1}{2}\bar{g}_{ab}\bar{R}^2 + 2\bar{g}_{ab}\bar{\Box}\bar{R} - 2\bar{\nabla}_a\bar{\nabla}_b\bar{R}\right], \tag{3.9}$$
$$\beta\bar{B}_{ab} \equiv \beta\left[2\bar{R}_{acbd}\bar{R}^{cd} - \bar{\nabla}_a\bar{\nabla}_b\bar{R} - \frac{1}{2}\bar{g}_{ab}\bar{R}^2_{cd} + \bar{\Box}\bar{R}_{ab} + \frac{1}{2}\bar{g}_{ab}\bar{\Box}\bar{R}\right]. \tag{3.10}$$

Likewise the linearized parts read

$$\begin{aligned}
\frac{1}{\kappa}(\mathscr{G}_{ab})_L &\equiv \frac{1}{\kappa}\left[(R_{ab})_L - \frac{1}{2}h_{ab}\bar{R} - \frac{1}{2}\bar{g}_{ab}R_L - \Lambda_0 h_{ab}\right] \\
\alpha(A_{ab})_L &\equiv 2\alpha\Big[\bar{R}(R_{ab})_L + \bar{R}_{ab}R_L - \frac{1}{2}\bar{g}_{ab}\bar{R}R_L \\
&\qquad - \frac{1}{4}h_{ab}\bar{R}^2 + \bar{g}_{ab}(\Box R)_L + h_{ab}\bar{\Box}\bar{R} - (\nabla_a\nabla_b R)_L\Big] \qquad (3.11) \\
\beta(B_{ab})_L &\equiv \frac{\beta}{2}\left[\bar{g}_{ab}(\Box R)_L + h_{ab}\bar{\Box}\bar{R} - 2(\nabla_a\nabla_b R)_L\right] \\
&\qquad + \frac{\beta}{2}\left[2(\Box R_{ab})_L 4(R_{acbd}R^{cd})_L\right] \\
&\qquad - \frac{\beta}{2}\left[(R_{cd}R^{cd})_L\bar{g}_{ab} + h_{ab}\bar{R}_{cd}\bar{R}^{cd}\right], \qquad (3.12)
\end{aligned}$$

so that

$$(T_{ab})_L = \frac{1}{\kappa}(\mathscr{G}_{ab})_L + \alpha(A_{ab})_L + \beta(B_{ab})_L. \qquad (3.13)$$

In the next section we will show how to write these linearized equations in terms of the deviation h_{ab} of the metric.

3.2 The Linearized Christoffel Symbols

Since the linearized field equations are satisfied by h_{ab} in first order, and field equations consist of $(R_{abcd})_L$, $(R_{ab})_L$ $\cdots$ we will begin calculating the linearized Christoffel symbols. These are defined as

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}\left[\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right] \qquad (3.14)$$

The decomposition of the metric yields

$$\begin{aligned}
\Gamma^a{}_{bc} &= \frac{1}{2}(\bar{g}^{ad} - h^{ad})\left[\partial_b(\bar{g}_{cd} + h_{cd}) + \partial_c(\bar{g}_{bd} + h_{bd}) - \partial_d(\bar{g}_{cb} + h_{cb})\right] \\
&= \bar{\Gamma}^a{}_{bc} + \frac{1}{2}\bar{g}^{ad}\left[\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc}\right] - \frac{1}{2}h^{ad}\left[\partial_b\bar{g}_{cd} + \partial_c\bar{g}_{bd} - \partial_d\bar{g}_{bc}\right] + \mathscr{O}(h^2) \quad (3.15)
\end{aligned}$$

Notice that the partial derivatives are not affected by this decomposition. Let us now take the partial derivatives to covariant ones in the first parenthesis and write it as

$$\begin{aligned}
\Gamma^a{}_{bc} &= \frac{1}{2}\bar{g}^{ad}\left[\bar{\nabla}_b h_{cd} + \bar{\nabla}_c h_{bd} - \bar{\nabla}_d h_{bc}\right] \\
&\quad + \frac{1}{2}\bar{g}^{ad}\left[\bar{\Gamma}^e{}_{bc}h_{ed} + \bar{\Gamma}^e{}_{bd}h_{ce} + \bar{\Gamma}^e{}_{bc}h_{ed} + \bar{\Gamma}^e{}_{cd}h_{be} - \bar{\Gamma}^e{}_{db}h_{ec} - \bar{\Gamma}^e{}_{dc}h_{be}\right] \qquad (3.16) \\
&\quad - \frac{1}{2}h^{ad}\left[\partial_b\bar{g}_{cd} + \partial_c\bar{g}_{bd} - \partial_d\bar{g}_{bc}\right] + \mathscr{O}(h^2).
\end{aligned}$$

Using metric compatibility $\bar{\nabla}_b \bar{g}_{cd} = 0 \implies \partial_b g_{cd} = \bar{\Gamma}^e{}_{bc}\bar{g}_{ed} + \bar{\Gamma}^e{}_{bd}\bar{g}_{ec}$ the third parenthesis cancels the second one, and the Linearized Christoffel symbols read

$$(\Gamma^a{}_{bc})_L = \frac{1}{2}\bar{g}^{ad}\left[\bar{\nabla}_b h_{cd} + \bar{\nabla}_c h_{bd} - \bar{\nabla}_d h_{bc}\right]. \tag{3.17}$$

Notice that although the Christoffel symbol is not a tensor its linearized version is. Using the linearized Christoffel symbols, we can compute the linearized Riemann tensor which is in our conventions defined as

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{ce}\Gamma^e{}_{bd} - \Gamma^a{}_{de}\Gamma^e{}_{bc}. \tag{3.18}$$

Following similar steps, one has

$$\begin{aligned} R^a{}_{bcd} &= \partial_c[\bar{\Gamma}^a{}_{bd} + (\Gamma^a{}_{bd})_L] - \partial_d[\bar{\Gamma}^a{}_{bc} + (\Gamma^a{}_{bc})_L] + \bar{\Gamma}^a{}_{ce}\bar{\Gamma}^e{}_{bd} - \bar{\Gamma}^a{}_{de}\bar{\Gamma}^e{}_{bc} \\ &\quad + \bar{\Gamma}^a{}_{ec}(\Gamma^e{}_{bd})_L + \bar{\Gamma}^e{}_{bd}(\Gamma^a{}_{ec})_L - \bar{\Gamma}^a{}_{ed}(\Gamma^e{}_{bc})_L - \bar{\Gamma}^e{}_{bc}(\Gamma^a{}_{ed})_L + \mathscr{O}(h^2). \end{aligned} \tag{3.19}$$

The background Christoffel symbols will yield $\bar{R}^a{}_{bcd}$. Taking the partial derivatives of linearized Christoffel symbols to the covariant ones by adding extra terms amounts to

$$\partial_c(\Gamma^a{}_{bd})_L = \bar{\nabla}_c(\Gamma^a{}_{bd})_L - \bar{\Gamma}^a{}_{ce}(\Gamma^e{}_{bd})_L + \bar{\Gamma}^e{}_{cb}(\Gamma^a{}_{ed})_L + \bar{\Gamma}^e{}_{cb}(\Gamma^a{}_{be})_L. \tag{3.20}$$

Using this in (3.19), the Riemann tensor can be decomposed into background part plus linearized part

$$R^a{}_{bcd} = \bar{R}^a{}_{bcd} + \bar{\nabla}_c(\Gamma^a{}_{bd})_L - \bar{\nabla}_d(\Gamma^a{}_{bc})_L + \mathscr{O}(h^2), \tag{3.21}$$

yielding

$$(R^a{}_{bcd})_L = \bar{\nabla}_c(\Gamma^a{}_{bd})_L - \bar{\nabla}_d(\Gamma^a{}_{bc})_L. \tag{3.22}$$

Contracting the first and the third indices, the linearized Ricci tensor can be found as

$$(R_{bd})_L = \bar{\nabla}_a(\Gamma^a{}_{bd})_L - \bar{\nabla}_d(\Gamma^a{}_{ab})_L. \tag{3.23}$$

Inserting (3.17) in (3.23), one finds the linearized Ricci tensor as a function of h_{ab}

$$\begin{aligned} (R_{ab})_L &= \frac{1}{2}\bar{\nabla}_c\left[\bar{g}^{cd}(\bar{\nabla}_b h_{ad} + \bar{\nabla}_a h_{bd} - \bar{\nabla}_d h_{ab})\right] - \frac{1}{2}\bar{\nabla}_b\left[\bar{g}^{cd}(\bar{\nabla}_a h_{cd} + \bar{\nabla}_c h_{ad} - \bar{\nabla}_d h_{ac})\right] \\ &= \frac{1}{2}\left[\bar{\nabla}_c\bar{\nabla}_b h^c{}_a + \bar{\nabla}_c\bar{\nabla}_a h^c{}_b - \bar{\Box} h_{ab} - \bar{\nabla}_a\bar{\nabla}_b h\right], \end{aligned} \tag{3.24}$$

where $\bar{g}^{cd}h_{cd} \equiv h$. Before computing, the linearized curvature scalar R_L, we must point out one crucial fact about linearization. One must be very careful about the position of indices in the linearized terms, for example $(R^{ab}{}_{cd})_L \neq \bar{g}^{eb}(R^a{}_{ecd})_L$. This can be easily seen as

$$
\begin{aligned}
R^{ab}{}_{cd} &= g^{eb}R^a{}_{ecd} \\
&= (\bar{g}^{eb} - h^{eb})[\bar{R}^a{}_{ecd} + (R^a{}_{ecd})_L + \mathscr{O}(h^2)] \\
&= \bar{R}^{ab}{}_{cd} + \bar{g}^{eb}(R^a{}_{ecd})_L - h^{eb}\bar{R}^a{}_{ecd} + \mathscr{O}(h^2) + \cdots \\
(R^{ab}{}_{cd})_L &= \bar{g}^{eb}(R^a{}_{ecd})_L - h^{eb}\bar{R}^a{}_{ecd}.
\end{aligned} \tag{3.25}
$$

Therefore for every index raised or lowered, we get an extra term. Considering this, R_L follows from

$$
\begin{aligned}
R &= g^{ab}R_{ab} \\
&= \bar{R} + \bar{g}^{ab}(R_{ab})_L - h^{ab}\bar{R}_{ab} + \mathscr{O}(h^2),
\end{aligned} \tag{3.26}
$$

so that $R_L = \bar{g}^{ab}(R_{ab})_L - h^{ab}\bar{R}_{ab}$. In terms of h_{ab}, the linearized curvature scalar can be easily computed

$$
R_L = \bar{\nabla}^a\bar{\nabla}^b h_{ab} - \bar{\Box}h - h^{ab}\bar{R}_{ab}. \tag{3.27}
$$

(3.27) and (3.24) are enough for the linearized Einstein term. However we also have contractions and derivatives of these tensors, which need to be linearized accordingly. Now we will compute them in a way that will help us to put them easily on a CADABRA[11] code.

Let us start with the covariant derivative of the Ricci tensor

$$
\begin{aligned}
\nabla_a R_{cd} &= \partial_a R_{cd} - \Gamma^i{}_{ac}R_{id} - \Gamma^i{}_{ad}R_{ci} \\
&= \partial_a\bar{R}_{cd} + \partial_a(R_{cd})_L - \bar{\Gamma}^i{}_{ac}\bar{R}_{id} - (\Gamma^i{}_{ac})_L\bar{R}_{id} - \bar{\Gamma}^i{}_{ac}(R_{id})_L \\
&\quad - \bar{\Gamma}^i{}_{ad}\bar{R}_{ic} - (\Gamma^i{}_{ad})_L\bar{R}_{ic} - \bar{\Gamma}^i{}_{ad}(R_{ic})_L + \mathscr{O}(h^2)\cdots \\
&= \bar{\nabla}_a\bar{R}_{cd} + \bar{\nabla}_a(R_{cd})_L - (\Gamma^i{}_{ac})_L\bar{R}_{id} - (\Gamma^i{}_{ad})_L\bar{R}_{ic} + \mathscr{O}(h^2) + \cdots \\
(\nabla_a R_{cd})_L &= \bar{\nabla}_a(R_{cd})_L - (\Gamma^i{}_{ac})_L\bar{R}_{id} - (\Gamma^i{}_{ad})_L\bar{R}_{ic}.
\end{aligned} \tag{3.28}
$$

Therefore the rule for taking the linearized covariant derivative is as follows: For every lower index one gets a linearized Christoffel symbol with a minus sign, contracted with the tensor inside the covariant derivative. For upper index one has plus of that. Hence, for $\nabla_b\nabla_a R_{cd}$

$$
(\nabla_b\nabla_a R_{cd})_L = \bar{\nabla}_b(\nabla_a R_{cd})_L - (\Gamma^i{}_{ba})_L(\bar{\nabla}_i\bar{R}_{cd}) - (\Gamma^i{}_{bc})_L(\bar{\nabla}_a\bar{R}_{id}) - (\Gamma^i{}_{bd})_L(\bar{\nabla}_a\bar{R}_{ci}). \tag{3.29}
$$

The advantage of writing the linearized terms as (3.28) is that by using them recursively, one can calculate the linearized terms in the field equations in terms of h_{ab} easily (especially on a computer). For example to compute (3.29), one has to simply insert (3.28) and (3.17) in (3.29). Proceeding in the same fashion, other terms in the linearized field equations are

$$(\Box R_{cd})_L = \bar{g}^{ab}(\nabla_b\nabla_a R_{cd})_L - h^{ab}\bar{\nabla}_b\bar{\nabla}_a\bar{R}_{cd}, \tag{3.30}$$

$$(\nabla_b\nabla_a R)_L = \bar{g}^{cd}(\nabla_b\nabla_a R_{cd})_L - h^{cd}\bar{\nabla}_b\bar{\nabla}_a\bar{R}_{cd}, \tag{3.31}$$

$$(\Box R)_L = \bar{g}^{ab}(\nabla_b\nabla_a R)_L - h^{ab}(\bar{\nabla}_b\bar{\nabla}_a\bar{R}). \tag{3.32}$$

Similarly the contractions of the Riemann and the Ricci tensors, such as $(R_{acbd}R^{cd})_L$ and $(R_{cd}R^{cd})_L$ can be calculated in terms of the linearized $(R^a{}_{bcd})_L$, $(R_{cd})_L$ which were found in (3.22), (3.24). Let us compute $(R_{acbd}R^{cd})_L$:

$$\begin{aligned}
R_{acbd}R^{cd} &= g_{ae}R^e{}_{cbd}g^{ck}g^{df}R_{kf} \\
&= (\bar{g}_{ae}+h_{ae})(\bar{g}^{ck}-h^{ck})(\bar{g}^{df}-h^{df})\Big[\bar{R}^e{}_{cbd}+(R^e{}_{cbd})_L\Big]\Big[\bar{R}_{kf}+(R_{kf})_L\Big] \\
&= \bar{R}_{acbd}\bar{R}^{cd} + \bar{g}_{ae}(R^e{}_{cbd})_L\bar{R}^{cd} + \bar{R}_{acbd}\bar{g}^{ck}\bar{g}^{df}(R_{kf})_L \\
&\quad + h_{ae}\bar{R}^e{}_{cbd}\bar{R}^{cd} - h^{ck}\bar{R}_{acbd}\bar{R}_k{}^d - h^{df}\bar{R}_{acbd}\bar{R}^c{}_f + \mathcal{O}(h^2)+\cdots \\
(R_{acbd}R^{cd})_L &= \bar{g}_{ae}(R^e{}_{cbd})_L\bar{R}^{cd} + \bar{R}_{acbd}\bar{g}^{ck}\bar{g}^{df}(R_{kf})_L \\
&\quad + h_{ae}\bar{R}^e{}_{cbd}\bar{R}^{cd} - h^{ck}\bar{R}_{acbd}\bar{R}_k{}^d - h^{df}\bar{R}_{acbd}\bar{R}^c{}_f.
\end{aligned} \tag{3.33}$$

Similarly $(R_{ab}R^{ab})_L = 2(R_{ab})_L\bar{R}^{ab} - 2h^{ac}\bar{R}_{cd}\bar{R}_a{}^d$. By using all these, the linearized field equations can be written in terms of h_{ab} which is given in appendix A and B.

In this chapter we gave the details of the linearization procedure. Calculating the linearized Christoffel symbols in terms of deviation, linearized versions of the Ricci and the Riemann tensor are found. In the next chapter we will calculate the charge from 'cosmological' Einstein term.

CHAPTER 4

KILLING CHARGES

4.1 Killing charges of the Einstein equation

After computing the linearized equations in terms of h_{ab}, one can continue as described in Chapter 2, contracting the linearized equations with the background Killing vector $\bar{\xi}_b$ and arranging the terms to have a divergence of an antisymmetric tensor. Then employing the Stokes' theorem the definition of charges as surface integrals will be straightforward.

In this section, to demonstrate how the calculation follows, we will find the contribution charge of the relatively simple Einstein part to the overall charge, which is also done in [15]. The other parts ($\alpha(A_{ab})_L$ and $\beta(B_{ab})_L$) are much more cumbersome and are shown in the appendix. Let us start with the linearized Einstein tensor

$$(\mathcal{G}_{ab})_L = (R_{ab})_L - \frac{1}{2}h_{ab}\bar{R} - \frac{1}{2}\bar{g}_{ab}R_L - \Lambda_0 h_{ab}. \tag{4.1}$$

Using (3.24) and (3.27),

$$\begin{aligned}(\mathcal{G}_{ab})_L = &\frac{1}{2}\left[\bar{\nabla}_c\bar{\nabla}_b h^c{}_a + \bar{\nabla}_c\bar{\nabla}_a h^c{}_b - \bar{\Box}h_{ab} - \bar{\nabla}_a\bar{\nabla}_b h\right] \\ &- \frac{1}{2}\left[\bar{g}_{ab}\bar{\nabla}_c\bar{\nabla}_d h^{cd} - \bar{g}_{ab}\bar{\Box}h - \bar{g}_{ab}h^{cd}\bar{R}_{cd}\right] - \Lambda_0 h_{ab}.\end{aligned} \tag{4.2}$$

However, one needs $(\mathcal{G}^{ab})_L$ to contract with the Killing vector, so taking two indices up

$$\begin{aligned}(\mathcal{G}^{ab})_L &= \bar{g}^{ac}\bar{g}^{bd}(\mathcal{G}_{cd})_L - h^{ac}\bar{\mathcal{G}}_c{}^b - h^{bc}\bar{\mathcal{G}}_c{}^a \\ &= \frac{1}{2}\left[\bar{\nabla}_c\bar{\nabla}^b h^{ca} + \bar{\nabla}_c\bar{\nabla}^a h^{cb} - \bar{\Box}h^{ab} - \bar{\nabla}^a\bar{\nabla}^b h\right] - \frac{1}{2}\left[\bar{g}^{ab}\bar{\nabla}_c\bar{\nabla}_d h^{cd} - \bar{g}^{ab}\bar{\Box}h - \bar{g}^{ab}h^{cd}\bar{R}_{cd}\right] \\ &+ \Lambda_0 h^{ab} - h^{ac}\bar{R}_c{}^b - h^{bc}\bar{R}_c{}^a + h^{ab}\bar{R}.\end{aligned} \tag{4.3}$$

Contracting this with $\bar{\xi}_b$

$$\bar{\xi}_b(\mathscr{G}^{ab})_L = \frac{1}{2}\left[\bar{\xi}_b\bar{\nabla}_c\bar{\nabla}^b h^{ca} + \bar{\xi}_b\bar{\nabla}_c\bar{\nabla}^a h^{cb} - \bar{\xi}_b\bar{\Box}h^{ab} - \bar{\xi}_b\bar{\nabla}^a\bar{\nabla}^b h\right]$$
$$- \frac{1}{2}\left[\bar{\xi}^a\bar{\nabla}_c\bar{\nabla}_d h^{cd} - \bar{\xi}^a\bar{\Box}h - \bar{\xi}^a h^{cd}\bar{R}_{cd}\right] \tag{4.4}$$
$$+ \bar{\xi}_b\Lambda_0 h^{ab} - \bar{\xi}_b h^{ac}\bar{R}_c{}^b - \bar{\xi}_b h^{bc}\bar{R}_c{}^a + \bar{\xi}_b h^{ab}\bar{R}.$$

From the trace parts, (i.e the fourth and the sixth terms), we can generate charge plus extra term as

$$\begin{aligned}
-\frac{1}{2}\bar{\xi}_b\bar{\nabla}^a\bar{\nabla}^b h + \frac{1}{2}\bar{\xi}^a\bar{\Box}h &= \frac{1}{2}\bar{\nabla}_b\left[\bar{\xi}^a\bar{\nabla}^b h - \bar{\xi}^b\bar{\nabla}^a h\right] - \frac{1}{2}(\bar{\nabla}_c\bar{\xi}^a)(\bar{\nabla}^c h)\\
&= \bar{\nabla}_b\left[\bar{\xi}^{[a}\bar{\nabla}^{b]}h\right] - \frac{1}{2}\bar{\nabla}_b\left[(\bar{\nabla}^b\bar{\xi}^a)h\right] + \frac{1}{2}(\bar{\Box}\bar{\xi}^a)h\\
&= \bar{\nabla}_b\left[\bar{\xi}^{[a}\bar{\nabla}^{b]}h - \frac{1}{2}(\bar{\nabla}^{[b}\bar{\xi}^{a]})h\right] - \frac{1}{2}\bar{R}^{ac}\bar{\xi}_c h\\
&= \bar{\nabla}_b\left[Q_1^{ab} + Q_2^{ab}\right] - \frac{1}{2}\bar{R}^{ac}\bar{\xi}_c h,
\end{aligned} \tag{4.5}$$

where (A.8) is used in the last line and $Q_1^{ab} \equiv \bar{\xi}^{[a}\bar{\nabla}^{b]}h$, $Q_2^{ab} \equiv \frac{1}{2}(\bar{\nabla}^{[a}\bar{\xi}^{b]})h$. From the second and the third terms in (4.4), one gets

$$\begin{aligned}
\frac{1}{2}\bar{\xi}_b\bar{\nabla}_c\bar{\nabla}^a h^{cb} - \frac{1}{2}\bar{\xi}_b\bar{\Box}h^{ab} &= \frac{1}{2}\bar{\nabla}_b\left[\bar{\xi}_c(\bar{\nabla}^a h^{bc}) - \bar{\xi}_c(\bar{\nabla}^b h^{ac})\right] + \frac{1}{2}(\bar{\nabla}_c\bar{\xi}_b)(\bar{\nabla}^c h^{ab})\\
&= \bar{\nabla}_b\left[\bar{\xi}_c(\bar{\nabla}^{[a}h^{b]c})\right] + \frac{1}{2}\bar{\nabla}_b\left[(\bar{\nabla}^b\bar{\xi}_c)h^{ac}\right] - \frac{1}{2}\bar{\Box}\bar{\xi}_b h^{ab}.
\end{aligned} \tag{4.6}$$

Adding and subtracting $\frac{1}{2}\bar{\nabla}_b\left[(\bar{\nabla}^a\bar{\xi}_c)h^{bc}\right]$ above, one finds

$$\begin{aligned}
\frac{1}{2}\bar{\xi}_b\bar{\nabla}_c\bar{\nabla}^a h^{cb} - \frac{1}{2}\bar{\xi}_b\bar{\Box}h^{ab} &= \bar{\nabla}_b\left[\bar{\xi}_c(\bar{\nabla}^{[a}h^{b]c}) + (\bar{\nabla}^{[b}\bar{\xi}_c)h^{a]c}\right] + \frac{1}{2}\bar{\nabla}_b\left[(\bar{\nabla}^a\bar{\xi}_c)h^{bc}\right] + \frac{1}{2}\bar{R}_{bc}\bar{\xi}^c h^{ab}\\
&= \bar{\nabla}_b\left[Q_3^{ab} + Q_4^{ab}\right] - \frac{1}{2}\bar{R}^a{}_{cbe}\bar{\xi}^e h^{bc} + \frac{1}{2}(\bar{\nabla}^a\bar{\xi}_c)\bar{\nabla}_b h^{bc} + \frac{1}{2}\bar{R}_{bc}\bar{\xi}^c h^{ab},
\end{aligned} \tag{4.7}$$

where (A.7) is used on the last line and $Q_3^{ab} \equiv \bar{\xi}_c(\bar{\nabla}^{[a}h^{b]c})$, $Q_4^{ab} \equiv (\bar{\nabla}^{[b}\bar{\xi}_c)h^{a]c}$. The first and the fifth terms yield

$$\begin{aligned}
\frac{1}{2}\bar{\xi}_b\bar{\nabla}_c\bar{\nabla}^b h^{ca} - \frac{1}{2}\bar{\xi}^a\bar{\nabla}_c\bar{\nabla}_d h^{cd} &= \frac{1}{2}\bar{\xi}_b[\bar{\nabla}_c, \bar{\nabla}^b]h^{ca} + \frac{1}{2}\bar{\xi}_b\bar{\nabla}^b\bar{\nabla}_c h^{ca} - \frac{1}{2}\bar{\xi}^a\bar{\nabla}_c\bar{\nabla}_d h^{cd}\\
&= \frac{1}{2}\bar{\xi}_b\bar{R}_c{}^{bc}{}_e h^{ea} + \frac{1}{2}\bar{\xi}_b\bar{R}^{cbae}h_{ce}\\
&+ \frac{1}{2}\bar{\nabla}_b\left[\bar{\xi}^b\bar{\nabla}_c h^{ca}\right] - \frac{1}{2}\bar{\nabla}_b\left[\bar{\xi}^a\bar{\nabla}_d h^{db}\right] + \frac{1}{2}(\bar{\nabla}_b\bar{\xi}^a)\bar{\nabla}_d h^{bd}\\
&= \bar{\nabla}_b[Q_5^{ab}] + \frac{1}{2}\bar{\xi}_b\bar{R}^b{}_e h^{ea} + \frac{1}{2}\bar{\xi}_e\bar{R}^{beac}h_{cb} + \frac{1}{2}(\bar{\nabla}_c\bar{\xi}^a)\bar{\nabla}_b h^{cb},
\end{aligned} \tag{4.8}$$

where dummy indices have been renamed and $Q_5^{ab} \equiv \bar{\xi}^{[b}\bar{\nabla}_d h^{a]d}$. Summing (4.7) and (4.8), the terms with the Riemann tensor and $\frac{1}{2}(\bar{\nabla}_c\bar{\xi}^a)\bar{\nabla}_b h^{cb}$ term cancels. Using (4.5), (4.7), (4.8) in (4.4) reads

$$\bar{\xi}_b(\mathscr{G}^{ab})_L = \bar{\nabla}_b\left(\sum_{i=1}^{5} Q_{\rm i}^{ab}\right) - \frac{1}{2}\bar{R}^{ac}\bar{\xi}_c h - \bar{\xi}_b h^{bc}\bar{R}_c{}^{a} + \frac{1}{2}\bar{\xi}_b h^{ab}\bar{R} + \Lambda_0\bar{\xi}_b h^{ab} - \frac{1}{2}\bar{g}^{ab}\bar{R}_{cd}h^{cd}. \quad (4.9)$$

It can be easily shown that terms except for the first one in the last equation can be written as

$$\bar{\xi}_b(\mathscr{G}^{ab})_L = \bar{\nabla}_b\left(\sum_{i=1}^{5} Q_{\rm i}^{ab}\right) - \bar{\mathscr{G}}^{ab}h_{bc}\bar{\xi}^c + \frac{1}{2}\bar{\mathscr{G}}_{cd}\bar{\xi}^a h^{cd} - \frac{1}{2}h\bar{\xi}_b\bar{\mathscr{G}}^{ab}. \quad (4.10)$$

Firstly, notice that the charge from the Einstein term is the same as the one found in AdS background [1, 8] and there is no contribution of the bare cosmological constant to charge. Secondly the very form of the terms except for the divergence part gives an important clue as how one should proceed on the more complicated cases, e.g α and β terms. Notice that when $\alpha(A^{ab})_L\bar{\xi}_b$ or $\beta(B^{ab})_L\bar{\xi}_b$ is written as a divergence of some 2-form plus extra terms, the form of the extra terms must be same as the Einstein piece above to yield terms made up of $\bar{T}^{ab}$ which is zero. More precisely it must be that

$$\bar{\xi}_b\alpha(A^{ab})_L = \alpha\bar{\nabla}_b\left(\sum_{i=1}^{n} F_{\rm i}^{ab}\right) - \alpha\bar{A}^{ab}h_{bc}\bar{\xi}^c + \alpha\frac{1}{2}\bar{A}_{cd}\bar{\xi}^a h^{cd} - \alpha\frac{1}{2}h\bar{\xi}_b\bar{A}^{ab} \quad (4.11)$$

$$\bar{\xi}_b\beta(B^{ab})_L = \beta\bar{\nabla}_b\left(\sum_{i=1}^{n} J_{\rm i}^{ab}\right) - \beta\bar{B}^{ab}h_{bc}\bar{\xi}^c + \beta\frac{1}{2}\bar{B}_{cd}\bar{\xi}^a h^{cd} - \beta\frac{1}{2}h\bar{\xi}_b\bar{B}^{ab}, \quad (4.12)$$

so that all extra terms add up to $-\bar{T}^{ab}h_{bc}\bar{\xi}^c + \frac{1}{2}\bar{T}_{cd}\bar{\xi}^a h^{cd} - \frac{1}{2}h\bar{\xi}_b\bar{T}^{ab}$, which one can set to zero. Thanks to the field equations $\bar{T}^{ab} = 0$ satisfied by the background metric. This property will help a lot to decide, what to keep in charge and the extra terms in the remaining calculations.

4.2 Killing charges of α and β terms

From the calculations given in appendix B, α contribution to the charge is

$$Q_\alpha^{ab} = 2\bar{R}Q_E^{ab} + 2(\bar{\nabla}_c\bar{R})\bar{\xi}^{[b}h^{a]c} + 4\bar{\xi}^{[a}\bar{\nabla}^{b]}R_L + 2R_L\bar{\nabla}^{[a}\bar{\xi}^{b]}, \quad (4.13)$$

where $Q_E^{ab} \equiv \bar{\xi}_c\bar{\nabla}^{[a}h^{b]c} + \bar{\xi}^{[b}\bar{\nabla}_c h^{a]c} + h^{c[b}\bar{\nabla}_c\bar{\xi}^{a]} + \bar{\xi}^{[a}\bar{\nabla}^{b]}h + \frac{1}{2}h\bar{\nabla}^{[a}\bar{\xi}^{b]}$ is the contribution to the charge coming from Einstein equation we have calculated in previous section and R_L is given in (3.27). The AdS limit of this charge is easily realized replacing $\bar{R}_{cd} = \frac{2\Lambda}{(D-2)}\bar{g}_{cd}$ and $\bar{R} = \frac{2\Lambda D}{(D-2)}$ in (4.13). The second term vanishes in AdS limit, the structure of the remaining ones are the same as [1] (equation (31)). Therefore our α charge reduces to the expected one in the AdS limit.

For the β, charge calculations in appendix C yields (which we have written more compactly)

$$
\begin{aligned}
Q^{ab}_{\beta} =& \bar{\xi}^{[a}\bar{\nabla}^{b]}R_L + \bar{\xi}^{c}\nabla^{[b}(R^{a]}{}_{c})_L + 2(R^{[b}{}_{c})_L(\bar{\nabla}^{a]}\bar{\xi}^{c}) + h_{cd}\bar{\xi}^{[b}\bar{\nabla}^{a]}\bar{R}^{cd} + 2h^{c[a}\bar{\xi}_d\bar{\nabla}_c\bar{R}^{b]d} + h\bar{\xi}_c\bar{\nabla}^{[b}\bar{R}^{a]c} \\
&+ 2(\bar{\nabla}_c\bar{\xi}_d)\bar{R}^{c[b}h^{a]d} + h\bar{R}^{c[b}(\bar{\nabla}^{a]}\bar{\xi}_c) + 2\bar{R}^{c[a}\bar{\xi}_d\bar{\nabla}_c h^{b]d} - 2\bar{R}^{c[a}\bar{\xi}^{d}\bar{\nabla}_d h^{b]}_c + \bar{\xi}^{[a}\bar{R}^{b]c}\bar{\nabla}_c h \\
&+ \bar{R}^{cd}\bar{\xi}^{[a}\bar{\nabla}^{b]}h_{cd} + 2\bar{\xi}^{d}\bar{R}_{cd}\bar{\nabla}^{[b}h^{a]c}
\end{aligned}
\tag{4.14}
$$

In this part of charge we can only write the terms with three derivatives compactly. The remaining parts must also be written in a nicer way. Again in this one, by making necessary substitutions one can see the AdS limit is the same as [1] (equation (31)). Using (4.10), (4.13) and (4.14) the total conserved Killing charge can be written as

$$
\begin{aligned}
Q^{ab}_{T} =& \frac{1}{\kappa}\left\{\bar{\xi}_c\bar{\nabla}^{[a}h^{b]c} + \bar{\xi}^{[b}\bar{\nabla}_c h^{a]c} + h^{c[b}\bar{\nabla}_c\bar{\xi}^{a]} + \bar{\xi}^{[a}\bar{\nabla}^{b]}h + \tfrac{1}{2}h\bar{\nabla}^{[a}\bar{\xi}^{b]}\right\} \\
&\alpha\left\{2\bar{R}Q^{ab}_{E} + 2(\bar{\nabla}_c\bar{R})\bar{\xi}^{[b}h^{a]c} + 4\bar{\xi}^{[a}\bar{\nabla}^{b]}R_L + 2R_L\bar{\nabla}^{[a}\bar{\xi}^{b]}\right\} \\
&\beta\Big\{\bar{\xi}^{[a}\bar{\nabla}^{b]}R_L + \bar{\xi}^{c}\bar{\nabla}^{[b}(R^{a]}{}_{c})_L + 2(R^{[b}{}_{c})_L(\bar{\nabla}^{a]}\bar{\xi}^{c}) + h_{cd}\bar{\xi}^{[b}\bar{\nabla}^{a]}\bar{R}^{cd} + 2h^{c[a}\bar{\xi}_d\bar{\nabla}_c\bar{R}^{b]d} + h\bar{\xi}_c\bar{\nabla}^{[b}\bar{R}^{a]c} \\
&+ 2(\bar{\nabla}_c\bar{\xi}_d)\bar{R}^{c[b}h^{a]d} + h\bar{R}^{c[b}(\bar{\nabla}^{a]}\bar{\xi}_c) + 2\bar{R}^{c[a}\bar{\xi}_d\bar{\nabla}_c h^{b]d} - 2\bar{R}^{c[a}\bar{\xi}^{d}\bar{\nabla}_d h^{b]}_c + \bar{\xi}^{[a}\bar{R}^{b]c}\bar{\nabla}_c h \\
&+ \bar{R}^{cd}\bar{\xi}^{[a}\bar{\nabla}^{b]}h_{cd} + 2\bar{\xi}^{d}\bar{R}_{cd}\bar{\nabla}^{[b}h^{a]c}\Big\}.
\end{aligned}
\tag{4.15}
$$

CHAPTER 5

THE GAUGE INVARIANCE

Let us start with an easier example of gauge invariance in electromagnetism. The transformations which takes the vector potential $A_b \rightarrow A_b + \partial_b \phi$, where ϕ is a scalar field, are called *gauge transformations*. The invariance of the electromagnetic field tensor $F_{ab} = \partial_b A_a - \partial_a A_b$ under such transformations is called *gauge invariance*. In short, the same physical field (in this case the electromagnetic field) can be described by different mathematical objects A_b, related among themselves by a certain set of transformations. The invariance of F_{ab} guarantees the invariance of charge defined in (2.18), which is an expected result since charge is a physical quantity.

For the gauge invariance of the gravitation we will follow[6]. The situation in gravity is more complicated than the one in electromagnetism. In the linearization, we have decomposed the metric as $g_{ab} = \bar{g}_{ab} + h_{ab}$, but have not specified the deviation uniquely. There exists some transformations which relates the background metric $\bar{g}_{ab} \in \mathscr{M}_b$ to the full metric $g_{ab} \in \mathscr{M}_f$, where $\mathscr{M}_b$ and $\mathscr{M}_f$ are the manifolds of the background spacetime and the full metric g_{ab} respectively. Assume that there exists a diffeomorphism $\phi : \mathscr{M}_b \rightarrow \mathscr{M}_f$ so that one can pull back g_{ab} as $(\phi^* g)_{ab}$. Next define the deviation as the difference

$$h_{ab} = (\phi^* g)_{ab} - \bar{g}_{ab}. \tag{5.1}$$

In this definition there is no restriction on the deviation to be small as described in the previous sections. In order to achieve that, let us define a vector field $\zeta(x)$ on $\mathscr{M}_b$ generating one parameter family of diffeomorphisms $\psi_\epsilon : \mathscr{M}_b \rightarrow \mathscr{M}_b$ such that for small ϵ the composition $(\phi \circ \psi_\epsilon)$ is small. Then we can define a family of deviations h_{ab} which are 'small" with parameter ϵ as

$$\begin{aligned} h^{\epsilon}_{ab} &= [(\phi \circ \psi_{\epsilon})^{*} g_{ab}] - \bar{g}_{ab} \\ &= [\psi^{*}_{\epsilon}(\phi^{*} g)]_{ab} - \bar{g}_{ab}. \end{aligned} \tag{5.2}$$

Using (5.1), one has

$$\begin{aligned} h^{\epsilon}_{ab} &= \psi^{*}_{\epsilon}(h + \bar{g})_{ab} - \bar{g}_{ab} \\ &= \psi^{*}_{\epsilon} h_{ab} + \psi^{*}_{\epsilon} \bar{g}_{ab} - \bar{g}_{ab}. \end{aligned} \tag{5.3}$$

For small ϵ, $\psi^{*}_{\epsilon} h_{ab} \approx h_{ab}$ at first order, and other terms can be written as

$$\begin{aligned} h^{\epsilon}_{ab} &= h_{ab} + \epsilon \left[\frac{\psi^{*}_{\epsilon}(\bar{g}_{ab}) - \bar{g}_{ab}}{\epsilon} \right] \\ &= h_{ab} + \epsilon \mathscr{L}_{\zeta(x)} \bar{g}_{ab} \\ &= h_{ab} + 2\bar{\nabla}_{(a}\zeta_{b)} \\ \delta_{\zeta} h_{ab} &= 2\bar{\nabla}_{(a}\zeta_{b)} \end{aligned} \tag{5.4}$$

where we have used the definition of Lie derivative given in appendix (A).

Having found how deviations behave under gauge transformations, we can discuss the gauge invariance of the charges defined by the ADT procedure. Charge is defined as

$$\begin{aligned} Q^{a}(\bar{\xi}) = \int_{\Sigma} d^{n-1}x \, \sqrt{-\bar{g}}\, (T^{ab})_{L} \bar{\xi}_{b} &= \int_{\Sigma} d^{n-1}x \, \sqrt{-\bar{g}}\, \bar{\nabla}_{b}(\mathcal{F}^{ab}) \\ &= \int_{\partial\Sigma} d^{n-2}y \, \sqrt{-\bar{g}^{\partial\Sigma}}\, \mathcal{F}^{ab} n_{b} \\ &= \int_{\partial\Sigma} dS_{\,b} \, \sqrt{-\bar{g}}\, \mathcal{F}^{ab}, \end{aligned} \tag{5.5}$$

and this has to be invariant under $h_{ab} \to h_{ab} + 2\bar{\nabla}_{(a}\zeta_{b)}$ in order to be physically acceptable. To check this we apply the transformation (5.4) on $(T^{ab})_{L}$. The invariance of which will guarantees the invariance of $\mathcal{F}^{ab}$ automatically. This approach is easier since one does not even have to find the charge, the linearized equations are enough.

Let us start with the linearized Einstein equations

$$\frac{1}{\kappa}(\mathscr{G}^{ab})_{L} = \frac{1}{\kappa}\Big[(R_{cd})_{L}\bar{g}^{ac}\bar{g}^{db} - \frac{1}{2}\bar{R}h^{ab} - \frac{1}{2}(R)_{L}\,\bar{g}^{ab} - \Lambda_{0}h^{ab} - h^{ac}\bar{\mathscr{G}}_{c}{}^{b} - h^{cb}\bar{\mathscr{G}}_{c}{}^{a}\Big]. \tag{5.6}$$

Now note that one does not have to consider last two terms since, for α, β one has

$$\alpha(A^{ab})_{L} = \alpha\Big[(A_{cd})_{L}\bar{g}^{ac}\bar{g}^{bd} - h^{ac}\bar{A}_{c}{}^{b} - h^{bc}\bar{A}_{c}{}^{a}\Big], \tag{5.7}$$

$$\beta(B^{ab})_{L} = \beta\Big[(B_{cd})_{L}\bar{g}^{ac}\bar{g}^{bd} - h^{ac}\bar{B}_{c}{}^{b} - h^{bc}\bar{B}_{c}{}^{a}\Big]. \tag{5.8}$$

Hence adding up (5.6), (5.7) and (5.8)

$$(T^{ab})_L = (T_{cd})_L \bar{g}^{ac}\bar{g}^{bd} - h^{ac}\bar{T}_c{}^b - h^{bc}\bar{T}_c{}^a, \tag{5.9}$$

where the last two terms are zero since $\bar{T}^a{}_c = 0$ (by assumption, the background satisfies the field equations). Moreover, $\bar{g}^{ac}\,\bar{g}^{bc}$ are invariant under gauge transformations, so it is enough for one to find how $(T_{cd})_L$ transforms. So consider

$$\frac{1}{\kappa}(\mathscr{G}_{cd})_L = \frac{1}{\kappa}\Big[(R_{cd})_L - \frac{1}{2}\bar{R}h_{cd} - \frac{1}{2}\,(R)_L\,\bar{g}_{cd} - \Lambda_0 h_{cd}\Big]. \tag{5.10}$$

Let us apply transformation term by term

$$\begin{aligned} R_L &= -\bar{\Box}h + \bar{\nabla}_a\bar{\nabla}^c h^a{}_c - \bar{R}_{ac}h^{ac}, \\ \delta_\zeta R_L &= -2\bar{\Box}\bar{\nabla}_a\zeta^a + \bar{\nabla}^a\bar{\nabla}^c\bar{\nabla}_a\zeta_c + \bar{\nabla}^a\bar{\Box}\zeta_a - 2\bar{R}^{ac}\bar{\nabla}_a\zeta_c. \end{aligned} \tag{5.11}$$

Now let us write the second term of (5.11) as

$$\begin{aligned} \bar{\nabla}^a\bar{\nabla}^c\bar{\nabla}_a\zeta_c &= \bar{\nabla}^a[\bar{\nabla}^c,\bar{\nabla}_a]\zeta_c + \bar{\Box}\bar{\nabla}^c\zeta_c \\ &= (\bar{\nabla}^a\bar{R}_a{}^e)\zeta_e + \bar{R}_a{}^e\bar{\nabla}^a\zeta_e + \bar{\Box}\bar{\nabla}^c\zeta_c \\ &= \bar{R}_a{}^e\bar{\nabla}^a\zeta_e + \zeta_e\frac{1}{2}\bar{\nabla}^e\bar{R} + \bar{\Box}\bar{\nabla}^c\zeta_c \end{aligned} \tag{5.12}$$

Likewise one gets the following for the third term

$$\begin{aligned} \bar{\nabla}^a\bar{\nabla}^c\bar{\nabla}_c\zeta_a &= [\bar{\nabla}^a,\bar{\nabla}^c](\bar{\nabla}_c\zeta_a) + \bar{\nabla}^c[\bar{\nabla}^a,\bar{\nabla}_c]\zeta_a + \bar{\Box}\bar{\nabla}^a\zeta_a \\ &= \frac{1}{2}(\bar{\nabla}^e\bar{R})\zeta_e + \bar{R}^e{}_c\bar{\nabla}^c\zeta_e + \bar{\Box}\bar{\nabla}^a\zeta_a. \end{aligned} \tag{5.13}$$

Inserting (5.12), (5.13) in (5.11) yields $\delta_\zeta R_L = (\bar{\nabla}^e\bar{R})\zeta_e$ which agrees with the calculation AdS background case [1] $\delta_\zeta R_L = 0$. For the linearized Ricci tensor $(R_{ab})_L$

$$\begin{aligned} (R_{cd})_L &= \frac{1}{2}\Big[\bar{\nabla}_a\bar{\nabla}_c h^a{}_d + \bar{\nabla}_a\bar{\nabla}_d h^a{}_c - \bar{\Box}h_{cd} - \bar{\nabla}_c\bar{\nabla}_d h\Big] \\ \delta_\zeta(R_{cd})_L &= \frac{1}{2}\Big[\underbrace{\bar{\nabla}_a\bar{\nabla}_c\bar{\nabla}^a\zeta_d}_{1} + \underbrace{\bar{\nabla}_a\bar{\nabla}_c\bar{\nabla}_d\zeta^a}_{2} + \underbrace{\bar{\nabla}_a\bar{\nabla}_d\bar{\nabla}^a\zeta_c}_{3} + \underbrace{\bar{\nabla}_a\bar{\nabla}_d\bar{\nabla}_c\zeta^a}_{4} \\ &\quad - \bar{\nabla}_a\bar{\nabla}^a\bar{\nabla}_c\zeta_d - \bar{\nabla}_a\bar{\nabla}^a\bar{\nabla}_d\zeta_c\Big] - \bar{\nabla}_c\bar{\nabla}_d\bar{\nabla}^a\zeta_a. \end{aligned} \tag{5.14}$$

Using commutators, one finds

$$1 \to \frac{1}{2}\bar{\nabla}_a\bar{\nabla}_c\bar{\nabla}^a\zeta_d = \frac{1}{2}\Big\{\bar{\nabla}_a[\bar{\nabla}_c,\bar{\nabla}^a]\zeta_d + \bar{\Box}(\bar{\nabla}_c\zeta_d)\Big\}$$

$$= \frac{1}{2}\Big\{(\bar{\nabla}^e\bar{R}_{cd})\zeta_e - (\bar{\nabla}_d\bar{R}^e{}_c)\zeta_e + \bar{R}_{cade}(\bar{\nabla}^a\zeta^e) + \bar{\Box}(\bar{\nabla}_c\zeta_d)\Big\}. \tag{5.15}$$

$$2 \to \frac{1}{2}\bar{\nabla}_a\bar{\nabla}_c\bar{\nabla}_d\zeta^a = \frac{1}{2}\Big\{[\bar{\nabla}_a,\bar{\nabla}_c]\bar{\nabla}_d\zeta^a + \bar{\nabla}_c[\bar{\nabla}_a,\bar{\nabla}_d]\zeta^a + \bar{\nabla}_c\bar{\nabla}_d\bar{\nabla}_a\zeta^a\Big\}$$

$$= \frac{1}{2}\Big\{\bar{R}_{acd}{}^e\bar{\nabla}_e\zeta^a + \bar{R}_{ce}(\bar{\nabla}_d\zeta^e) + (\bar{\nabla}_c\bar{R}_{de})\zeta^e + \bar{R}_{de}\bar{\nabla}_c\zeta^e + \bar{\nabla}_c\bar{\nabla}_d\bar{\nabla}_a\zeta^a\Big\}. \tag{5.16}$$

$$3 \to \frac{1}{2}\bar{\nabla}_a\bar{\nabla}_d\bar{\nabla}^a\zeta_c = \frac{1}{2}\Big\{\bar{\nabla}_a[\bar{\nabla}_d,\bar{\nabla}^a]\zeta_c + \bar{\Box}(\bar{\nabla}_d\zeta_c)\Big\}$$

$$= \frac{1}{2}\Big\{(\bar{\nabla}^e\bar{R}_{cd})\zeta_e - (\bar{\nabla}_c\bar{R}^e{}_d)\zeta_e + \bar{R}_{dace}(\bar{\nabla}^a\zeta^e) + \bar{\Box}(\bar{\nabla}_d\zeta_c)\Big\}. \tag{5.17}$$

$$4 \to \frac{1}{2}\bar{\nabla}_a\bar{\nabla}_d\bar{\nabla}_c\zeta^a = \frac{1}{2}\Big\{[\bar{\nabla}_a,\bar{\nabla}_d]\bar{\nabla}_c\zeta^a + \bar{\nabla}_d[\bar{\nabla}_a,\bar{\nabla}_c]\zeta^a + \bar{\nabla}_d\bar{\nabla}_c\bar{\nabla}_a\zeta^a\Big\}$$

$$= \frac{1}{2}\Big\{\bar{R}_{adc}{}^e\bar{\nabla}_e\zeta^a + (\bar{\nabla}_d\bar{R}_{ce})\zeta^e + \bar{R}_{ce}\bar{\nabla}_d\zeta^e + \bar{\nabla}_c\bar{\nabla}_d\bar{\nabla}_a\zeta^a\Big\}. \tag{5.18}$$

$$\tag{5.19}$$

Inserting all of these in (5.14), one gets

$$\delta_\zeta(R_{cd})_L = \zeta_e(\bar{\nabla}^e\bar{R}_{cd}) + \frac{1}{2}(\bar{\nabla}^a\zeta^e)\Big[\bar{R}_{cade} + \bar{R}_{ecda} + \bar{R}_{dace} + \bar{R}_{edca}\Big] + \bar{R}_{ce}(\bar{\nabla}_d\zeta^e) + \bar{R}_{de}(\bar{\nabla}_c\zeta^e) \tag{5.20}$$

After renaming of indices, the terms with the Riemann tensor give zero, yielding

$$\delta_\zeta(R_{cd})_L = \zeta_e(\bar{\nabla}^e\bar{R}_{cd}) + \bar{R}_{ce}(\bar{\nabla}_d\zeta^e) + \bar{R}_{de}(\bar{\nabla}_c\zeta^e). \tag{5.21}$$

Now taking $\bar{R}_{cd} \to \frac{2\Lambda}{2-D}\bar{g}_{cd}$ yields the AdS result. As for the full Einstein part, one finds $\delta_\zeta\frac{1}{\kappa}(\mathscr{G}_{cd})_L$ is

$$\frac{1}{\kappa}\delta_\zeta(\mathscr{G}_{cd})_L = \delta_\zeta(R_{cd})_L - \frac{1}{2}(\delta_\zeta R_L)\bar{g}_{cd} - \frac{1}{2}\Big(\bar{R}+\Lambda_0\Big)(\bar{\nabla}_c\zeta_d + \bar{\nabla}_d\zeta_c)$$

$$= \zeta_e(\bar{\nabla}^e\bar{R}_{cd}) + \bar{R}_{ce}(\bar{\nabla}_d\zeta^e) + \bar{R}_{de}(\bar{\nabla}_c\zeta^e) - \frac{1}{2}\Big(\bar{R}+\Lambda_0\Big)(\bar{\nabla}_c\zeta_d + \bar{\nabla}_d\zeta_c) - \frac{1}{2}\bar{g}_{cd}(\bar{\nabla}^e\bar{R})\zeta_e. \tag{5.22}$$

which can be written as

$$\frac{1}{\kappa}\delta_\zeta(\mathscr{G}_{cd})_L = \zeta^e\bar{\nabla}_e\left[\bar{R}_{cd} - \frac{1}{2}\bar{R}\bar{g}_{cd} - \Lambda_0\bar{g}_{cd}\right] + \left(\bar{R}_{ce} - \frac{1}{2}\bar{R}\bar{g}_{ce} - \Lambda_0\bar{g}_{ce}\right)(\bar{\nabla}_d\zeta^e) \tag{5.23}$$

$$+ \left(\bar{R}_{de} - \frac{1}{2}\bar{R}\bar{g}_{de} - \Lambda_0\bar{g}_{de}\right)(\bar{\nabla}_c\zeta^e)$$

$$= \zeta^e\bar{\nabla}_e(\bar{\mathscr{G}}_{cd}) + \bar{\mathscr{G}}_{ce}(\bar{\nabla}_d\zeta^e) + \bar{\mathscr{G}}_{de}(\bar{\nabla}_c\zeta^e). \tag{5.24}$$

Once again taking the AdS limit, one finds $\delta_\zeta(\mathscr{G}_{cd})_L = 0$ since the background solves the 'cosmological' Einstein equations by assumption. However in a general background $\bar{g}_{ab}$ it may be that $\bar{\mathscr{G}}_{cd} \neq 0$, so that one needs $\alpha(A_{ab})_L$ and $\alpha(B_{ab})_L$ to transform in the form of (5.24)

$$\alpha\delta_\zeta(A_{cd})_L = \zeta^e\bar{\nabla}_e(\bar{A}_{cd}) + \bar{A}_{ce}(\bar{\nabla}_d\zeta^e) + \bar{A}_{de}(\bar{\nabla}_c\zeta^e), \tag{5.25}$$

$$\beta\delta_\zeta(B_{cd})_L = \zeta^e\bar{\nabla}_e(\bar{B}_{cd}) + \bar{B}_{ce}(\bar{\nabla}_d\zeta^e) + \bar{B}_{de}(\bar{\nabla}_c\zeta^e), \tag{5.26}$$

so that $\frac{1}{\kappa}\delta_\zeta(\mathscr{G}_{cd})_L + \alpha\delta_\zeta(A_{cd})_L + \beta\delta_\zeta(B_{cd})_L$ reads

$$\delta_\zeta(T_{cd})_L = \zeta^e\bar{\nabla}_e\bar{T}_{cd} + \bar{T}_{ce}(\bar{\nabla}_d\zeta^e) + \bar{T}_{de}(\bar{\nabla}_c\zeta^e) = 0. \tag{5.27}$$

We will not perform explicit calculations of $\alpha\delta_\zeta(A_{cd})_L$ and $\beta\delta_\zeta(B_{cd})_L$ here. One can easily proceed first by calculating $\delta_\zeta(\Gamma^a{}_{bc})_L$ and $\delta_\zeta(R_{abcd})_L$, $\delta_\zeta(R^a{}_{bcd})_L$ which are

$$\begin{aligned}
\delta_\zeta(\Gamma^a{}_{bc})_L &= \zeta^e R_{ec}{}^a{}_b + \bar{\nabla}_c\bar{\nabla}_b\zeta^a, \\
\delta_\zeta(R^a{}_{bcd})_L &= -\bar{R}^a{}_{bde}\bar{\nabla}_c\zeta^e - \bar{\nabla}_c\bar{R}^a{}_{bde}\zeta^e + \bar{R}^a{}_{bce}\bar{\nabla}_d\zeta^e + \bar{\nabla}_d\bar{R}^a{}_{bce}\zeta^e + \bar{R}_b{}^e{}_{cd}\bar{\nabla}_e\zeta^a + \bar{R}^{ae}{}_{cd}\bar{\nabla}_b\zeta_e, \\
\delta_\zeta(R_{abcd})_L &= -\bar{R}_{abd}{}^e\bar{\nabla}_c\zeta_e - \bar{\nabla}_c\bar{R}_{abd}{}^e\zeta_e + \bar{R}_{abc}{}^e\bar{\nabla}_d\zeta_e + \bar{\nabla}_d\bar{R}_{abc}{}^e\zeta_e + \bar{R}_a{}^e{}_{cd}\bar{\nabla}_b\zeta_e - \bar{R}_{becd}\bar{\nabla}_a\zeta^e.
\end{aligned} \tag{5.28}$$

Inserting those in (3.11) and (3.12) yields (5.25) and (5.26) as expected. As a result, we've shown the charge defined from the field equations in (3.2) is gauge invariant. This result ensures the physical credibility of the charge.

CHAPTER 6

APPLICATIONS

In this chapter we will give the charges of various blackholes in NMG [10] in three dimensions. Namely, we will compute the charges of BTZ (Bañados-Teitelboim-Zanelli) [12], the "exotic" blackhole solution given in [2], and finally the Lifshitz blackhole in 3 dimensions [13]. The first two admit AdS as a background, which makes possible to use the charge definition given by [1]. However the last one is asymptotically Lifshitz which is no longer a constant curvature background and should be computed by the charge definition given (4.15) in this thesis. The charge of Lifshitz blackhole is also calculated by a different method in [3], and we will compare the two results in due time.

6.1 BTZ Blackhole

Let us start with the BTZ blackhole, which is a solution of the "cosmological" NMG theory whose action reads

$$S = \frac{1}{16\pi G}\int d^3x\,\sqrt{-g}\left[R - 2\lambda - \frac{1}{m^2}\left(R_{ab}R^{ab} - \frac{3}{8}R^2\right)\right], \tag{6.1}$$

with the associated field equations

$$R_{ab} - \frac{1}{2}Rg_{ab} + \lambda g_{ab} - \frac{1}{2m^2}K_{ab} = 0, \tag{6.2}$$

where

$$K_{ab} = 2\Box R_{ab} - \frac{1}{2}\nabla_a\nabla_b R - \frac{1}{2}\Box R g_{ab} + 4R_{acbd}R^{cd} - \frac{3}{2}RR_{ab} - R_{cd}R^{cd}g_{ab} + \frac{3}{8}R^2 g_{ab}. \tag{6.3}$$

The field equations above are satisfied by the BTZ blackhole metric [12] but we write here equation (4.48) of [3]

$$ds^2 = N^2dr^2 - N^{-2}dt^2 + r^2\left(d\phi + N_\phi dt\right)^2 , \tag{6.4}$$

where

$$N(r) \;=\; \left(-8GM + \frac{r^2}{\ell^2} + \frac{16G^2J^2}{r^2}\right)^{\frac{1}{2}} \;, \qquad N_\phi \;=\; -\frac{4GJ}{r^2} \;. \tag{6.5}$$

Inserting (6.4) into the field equations, one finds the relation between m and λ

$$m^2 = -\frac{1}{4l^2 - 4l^4\lambda}. \tag{6.6}$$

In our charge definition we have the field equations given as (3.2). Identifying our constants with the ones given in (6.1) and (6.2), one gets

$$\kappa = 16\pi G, \quad \Lambda_0 = -\lambda, \quad \beta = -\frac{1}{m^2} = \frac{4l^2 + 4l^4\lambda}{\kappa}, \quad \alpha = -\frac{3}{8}\beta. \tag{6.7}$$

In what follows we choose $\lambda = -\frac{1}{2l^2}$, then from (6.7) $\alpha = -\frac{3l^2}{4\kappa}$, $\beta = \frac{2l^2}{\kappa}$. With all the constants expressed in terms of κ, one can compute the charge using (4.15). Let us choose the background metric $\bar{g}_{ab}$ as $(M, J \to 0)$

$$d\bar{s}^2 = -\frac{r^2}{l^2}dt^2 + \frac{l^2}{r^2}dr^2 + r^2 d\phi^2. \tag{6.8}$$

The deviation part reads

$$ds_H^2 = 8GMdt^2 + \left(-\frac{l^2}{r^2} + \frac{1}{N(r)}\right)dr^2 - 8GJdtd\phi. \tag{6.9}$$

For the mass let us choose the timelike Killing vector $\bar{\xi}^a = (-1, 0, 0)$. Then the surface integral reads

$$M_{\text{BTZ}} = \int_0^{2\pi} \frac{8GM}{\kappa} d\phi = \frac{16\pi GM}{\kappa} = M, \tag{6.10}$$

where we used (6.7).

For the angular momentum we choose the Killing vector as $\bar{\xi}^a = (0, 0, 1)$ and find

$$J_{\text{BTZ}} = \int_0^{2\pi} \frac{8GJ}{\kappa} d\phi = \frac{16\pi GJ}{\kappa} = J. \tag{6.11}$$

Although we did not explicitly show the computations, in our calculation we did not even have to take the $r \to \infty$ limit to compute the mass or angular momentum. Also the contributions coming from the α and β terms were highly nontrivial, yet these pieces have added up to yield an elegant result.

In [3] the mass and the angular momentum of BTZ was found as

$$M_{\text{BTZ}} \;=\; M\left(\sigma + \frac{1}{2m^2\ell^2}\right) \;, \qquad J_{\text{BTZ}} \;=\; J\left(\sigma + \frac{1}{2m^2\ell^2}\right) \;, \tag{6.12}$$

where $\sigma = \pm 1$ controls the sign of the Einstein-Hilbert term. In our conventions $\sigma = +1$, which makes both of the charges vanish for $m^2 = -\frac{1}{2l^2}$. The results of [3] clearly do not agree with ours, which however looks rather natural.

6.2 The Blackhole of [2]

As a second example, let us consider the blackhole solution of NMG given by [2] (equation (3.9) there)

$$ds^2 = -\frac{4\rho^2}{l^2 f(\rho)}d\bar{t}^2 + f(\rho)\Big[q\bar{\varphi} - \frac{ql\ln|\rho/\rho_0|}{f(\rho)}d\bar{t}\Big]^2 + \frac{l^2 d\rho^2}{4\rho^2} \qquad (f(\rho) = 2\rho + ql^2\ln|\rho/\rho_0|). \tag{6.13}$$

This metric satisfies NMG provided $m^2 = \frac{1}{2l^2}$. Then from (6.7), $\lambda = -\frac{3}{2l^2} = -\Lambda_0, \beta = -\frac{2l^2}{\kappa}$ and $\alpha = \frac{3l^2}{4\kappa}$. For this metric the background is found by taking $q \to 0$.

$$d\bar{s}^2 = -\frac{2\rho}{l^2}dt^2 + \frac{l^2}{4\rho^2}d\rho^2 + 2\rho d\phi^2, \tag{6.14}$$

with the deviation part as

$$ds_H^2 = q\ln|\rho/\rho_0|\,d\bar{t}^2 - 2lq\ln|\rho/\rho_0|\,d\bar{t}\,d\phi + l^2 q\ln|\rho/\rho_0|\,d\phi^2. \tag{6.15}$$

With this decomposition, the charge definition (4.15) yields

$$M = \int_0^{2\pi}\frac{8q}{\kappa}d\bar{\varphi} = \frac{q}{G} \quad \text{with} \quad \bar{\xi}^a = (-1,0,0), \tag{6.16}$$

$$J = \int_0^{2\pi}\frac{8ql}{\kappa}d\bar{\varphi} = \frac{ql}{G} \quad \text{with} \quad \bar{\xi}^a = (0,0,1). \tag{6.17}$$

In [2], the mass and the angular momentum were found to be

$$M = \frac{2q}{G}, \quad J = \frac{2ql}{G}, \tag{6.18}$$

where the charge definition given by [1] is used to compute the mass. However we have found just half of (6.18) for the conserved charges. This is due to the choice of the κ term. In [2], $\kappa = 8\pi G$, the solid angle in 2-dimensions, whereas we followed (6.1), $\kappa = 16\pi G$. Therefore the results in agreement as they should be.

6.3 Lifshitz Blackhole

Finally, we will consider the Lifshitz blackhole solution of NMG in 3 dimensions given by [13]

$$ds^2 = -\frac{r^6}{\ell^6}\left(1 - \frac{M\ell^2}{r^2}\right)dt^2 + \frac{dr^2}{\left(\frac{r^2}{\ell^2} - M\right)} + \frac{r^2}{\ell^2}dx^2. \tag{6.19}$$

This metric solves the 'cosmological' NMG (6.2) with $\lambda = -\frac{13}{2\ell^2}$, $m^2 = -\frac{1}{2\ell^2}$, as discussed in detail in [13]. With these choices, our constants α and β read from (6.7)

$$\Lambda_0 = \frac{13}{2\ell^2},\ \beta = \frac{2\ell^2}{\kappa},\ \alpha = -\frac{3\ell^2}{4\kappa}. \tag{6.20}$$

Let us choose the background metric as the Lifshitz space obtained by taking $M \to 0$ in (6.19)

$$d\bar{s}^2 = -\frac{r^6}{\ell^6}dt^2 + \frac{\ell^2}{r^2}dr^2 + \frac{r^2}{\ell^2}dx^2. \tag{6.21}$$

Note the angular coordinate x is periodic with $2\pi\ell$, not 2π here. Then the deviation h_{ab} reads

$$ds_H^2 = \frac{Mr^4}{\ell^4}dt^2 + \frac{\ell^4 M}{r^2(r^2-\ell^2 M)}dr^2. \tag{6.22}$$

Inserting (6.22) and (6.21) in (4.15), one finds

$$M_{Lifshitz} = \int_0^{2\pi} \lim_{r\to\infty} \frac{M^2 r^2\left(-9\ell^4 M^2 + 24\ell^2 M r^2 - 7r^4\right)}{\ell\left(\ell^2 M - r^2\right)^3 \kappa} d\phi \tag{6.23}$$

$$= \int_0^{2\pi\ell} \frac{7M^2}{\ell\kappa}dx = \frac{7M^2}{8G}, \tag{6.24}$$

in which $\bar{\xi}^a = (-1, 0, 0)$ is used and in the last line, we have taken $\kappa = 16\pi G$ in accordance with the conventions given in [13].

The mass of the Lifshitz blackhole is also computed by [3] using the boundary stress tensor method. There, it is found to be

$$M_{Lifshitz} = \frac{M^2}{4G}. \tag{6.25}$$

Clearly, our result does not agree with the result found in [3].

CHAPTER 7

CONCLUSIONS

In this thesis we have defined the conserved charges of a quadratic curvature theory given by an action

$$I = \int d^D x \sqrt{-g} \left\{ \frac{1}{\kappa}(R + 2\Lambda_0) + \alpha R^2 + \beta R_{ab} R^{ab} \right\}. \tag{7.1}$$

The charge definition we have studied was first given by [8] for the "cosmological" Einstein theory. It was later extended to the quadratic curvature theories in constant curvature backgrounds by [5, 1]. In this thesis we have taken one more step and compute the charge in an arbitrary background, that admits at least one global (timelike) Killing vector. However, in order to define a conserved current with the ADT procedure, one has to weaken some of the assumptions made in the constant curvature background case. In the ADT procedure, it was assumed that the linearized field equations are covariantly conserved $\bar{\nabla}_a (T^{ab})_L = 0$. This is not satisfied for general backgrounds, so we have simply assumed that $\bar{\nabla}_a (\bar{\xi}_b (T^{ab})_L) = 0$ holds. If this condition is satisfied, then one can define a conserved current by rewriting $\bar{\xi}_b (T^{ab})_L$ as a divergence of a 2-form. After cumbersome calculations the corresponding Killing charges have been found, it has been shown that these reduce to their known AdS counterparts in the limit and that their gauge-invariance properly has been discussed.

As an application we have found the charges of various blackholes in NMG; namely, those of the BTZ [12], the blackhole given in [2], and finally the Lifshitz blackhole in three dimensions [13]. For BTZ, we have found $M_{BTZ} = M$, $J_{BTZ} = J$ which are convincing. Note that the result obtained by [3] using the boundary stress tensor method yields $M_{BTZ} = 0$, which clearly conflicts with ours. The second blackhole given by [2] has a certain finite mass calculated by the charge definition given by [1]. This was especially a good example to check the AdS limit of our definition. As expected, our results and the results of [2] the results were in agreement.

Finally, we have computed the charge of the Lifshitz blackhole in three dimensions. This example was a nontrivial one since it asymptotically admits a Lifshitz background and one can not use the charge definition given by [1]. The charge of the Lifshitz blackhole was calculated by [3] as $\frac{M^2}{4G}$, whereas we have found $\frac{7M^2}{8G}$ using our definition . Once again our results do not agree with [3] for the Lifshitz case . All these issues must be analyzed in detail.

As an extension to what we have done, one can also add the Gauss-Bonnet term $\gamma(R_{abcd}^2 - 4R_{ab}R^{ab}+R^2)$ to the action (7.1). Actually we have already calculated the contribution coming from Gauss-Bonnet term, however due to time limitations we could not present our findings in this thesis. With the Gauss-Bonnet contribution in hand, one can in principle find the charges of a family of Lifshitz blackholes in arbitrary dimensions [14].

For the future work, it remains to find a compact form of the β charge. Although we have written some of the terms compactly, we think the remaining terms can also be written in a nicer way. As a different problem, one might also consider extending the conserved Killing-Yano charge definition to "arbitrary" backgrounds, which was done for AdS backgrounds by [16, 17].

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APPENDIX A

KILLING VECTOR FIELDS

A.1 Killing equation

In this part of appendix we will give the definition of Killing vectors, derive the Killing equation and find the relation between the Killing vectors and the Riemann tensor. Some of the materials we used here follows from [7].

Since conserved charges are strongly related with the isometries of spacetime, we start by reviewing the definition of a Killing vector field. A diffeomorphism ϕ defined on a manifold $\mathscr{M}$, $\phi : \mathscr{M} \to \mathscr{M}$ is an *isometry*, if it carries the metric g onto itself, i.e $\phi_* \mathbf{g} = \mathbf{g}$ at every point on the manifold $\mathscr{M}$. If these one parameter group of diffeomorphisms ϕ_t are generated by a vector field ξ then they are called *Killing vector fields*. Using these, one can define the Lie derivative of a metric with respect to ξ as

$$\mathscr{L}_\xi \mathbf{g} = \lim_{t\to 0}(\mathbf{g} - \phi_* \mathbf{g}) = 0. \tag{A.1}$$

Here the right hand side is zero, because we restrict the diffeomorphisms to $\phi_* \mathbf{g} = \mathbf{g}$ at every point on the manifold $\mathscr{M}$. Using coordinate bases,

$$\mathscr{L}_\xi g_{ab} = \xi^c \partial_c g_{ab} + g_{cb} \partial_a \xi^c + g_{ca} \partial_b \xi^c = 0. \tag{A.2}$$

If there is no torsion, then one can replace the partial derivatives with the covariant ones

$$\mathscr{L}_\xi g_{ab} = \xi^c \nabla_c g_{ab} + g_{cb} \nabla_a \xi^c + g_{ca} \nabla_b \xi^c = 0. \tag{A.3}$$

By metric compatibility, the first term is zero. So one ends up with the Killing equation

$$\mathscr{L}_\xi g_{ab} = \nabla_b \xi_a + \nabla_b \xi_a = 0. \tag{A.4}$$

The vector fields satisfying this equation are called *Killing vector fields*. With this equation, one can determine the symmetries of a given spacetime manifold.

A.2 Relation with the Riemann tensor

Now let us derive some identities relating the Riemann tensor and the Killing vectors, which we will repeatedly use in our calculations. Starting from the Bianchi identity

$$R_{abcd} + R_{cabd} + R_{bcad} = 0, \tag{A.5}$$

multiplying this with ξ_d and writing the result in terms of commutators

$$\begin{aligned} R_{abcd}\xi^d + R_{cabd}\xi^d + R_{bcad}\xi^d &= 0, \\ [\nabla_a, \nabla_b]\xi_c + [\nabla_c, \nabla_a]\xi_b + [\nabla_b, \nabla_c]\xi_a &= 0, \\ \nabla_a\nabla_b\xi_c - \nabla_b\nabla_a\xi_c + \nabla_c\nabla_a\xi_b - \nabla_a\nabla_c\xi_b + \nabla_b\nabla_c\xi_a - \nabla_c\nabla_b\xi_a &= 0. \end{aligned} \tag{A.6}$$

Using the Killing equation (A.4). (A.6) reads

$$\begin{aligned} 2\nabla_a\nabla_b\xi_c + 2\nabla_c\nabla_a\xi_b - 2\nabla_b\nabla_a\xi_c = 0,& \\ [\nabla_a, \nabla_b]\xi_c = -\nabla_c\nabla_a\xi_b,& \\ R_{abce}\xi^e = -\nabla_c\nabla_a\xi_b.& \end{aligned} \tag{A.7}$$

This important identity can be contracted to get a relation between the Ricci tensor R_{ab} and ξ_b

$$R_{be}\xi^e = -\Box\xi_b, \quad \text{where } \Box \equiv \nabla_c\nabla^c. \tag{A.8}$$

Moreover, one can also show that $\mathscr{L}_\xi \mathbf{R} = 0 = \xi_a\nabla^a R$ which has been used extensively in our calculations. The derivation will be done in a way such that other useful identities will also show up.

Consider

$$\nabla_a\nabla_b\nabla_c\xi_d - \nabla_b\nabla_a\nabla_c\xi_d = [\nabla_a, \nabla_b]\nabla_c\xi_d. \tag{A.9}$$

Using (A.7) derived earlier, this can be written as

$$-\nabla_a\left[R_{cdb}{}^e\xi_e\right] + \nabla_b\left[R_{cda}{}^e\xi_e\right] = R_{abce}(\nabla^e\xi_d) + R_{abde}(\nabla_c\xi^e). \tag{A.10}$$

Now expanding derivatives, one finds

$$\xi_e\left(\nabla_b R_{cda}{}^e - \nabla_a R_{cdb}{}^e\right) = (\nabla_k\xi^e)\left(\delta^k_c R_{abde} - \delta^k_d R_{abce} + \delta^k_a R_{cdbe} - \delta^k_b R_{cdae}\right). \tag{A.11}$$

Contracting this in $a-d$ indices gives

$$-\xi^e\left(\nabla_b R_{ce} + \nabla^a R_{beca}\right) = R_{be}(\nabla_c \xi^e) + R_{abc}{}^e(\nabla_e \xi^a) + R_{cab}{}^e(\nabla^a \xi_e) + R_{ce}(\nabla_b \xi^e). \qquad \text{(A.12)}$$

Using the contracted Bianchi identity $\nabla_a R_{bec}{}^a = \nabla_e R_{bc} - \nabla_b R_{ec}$ and antisymmetry of third term yields

$$\mathscr{L}_\xi R_{bc} = \xi_e \nabla^e R_{bc} - R_{be}(\nabla^e \xi_c) - R_{ce}(\nabla^e \xi_b) = 0. \qquad \text{(A.13)}$$

This is another important identity used in our calculations. Finally contracting the free indices in (A.13) gives desired result

$$\mathscr{L}_\xi \mathbf{R} = \xi_e \nabla^e R = R_{be}(\nabla^e \xi^b) + R_{be}(\nabla^e \xi^b) = 0. \qquad \text{(A.14)}$$

We will make extensive use of (A.7), (A.10), (A.13) and (A.14) in what follows.

APPENDIX B

α CHARGE CALCULATION

Linearized $\alpha(A_{cd})_L\bar{g}^{ca}\bar{g}^{bd}$ in terms of h^{ab} calculated by CADABRA [11] reads

$$
\begin{aligned}
2\bar{\xi}_b\bar{g}^{ca}\bar{g}^{bd}\left\{\bar{R}_{cd}R_L+\bar{R}(R_{cd})_L\right\} = {} & \underbrace{2\,\bar{\xi}_b\bar{R}^{ab}\bar{\nabla}_c\bar{\nabla}^d h^c{}_d}_{II-1} - \underbrace{2\,\bar{\xi}_b\bar{R}^{ab}\bar{\nabla}_c\bar{\nabla}^c h}_{II-2} - \underbrace{2\,\bar{\xi}_b\bar{R}^{ab}\bar{R}_{cd}h^{cd}}_{II-3} \\
& + \underbrace{\bar{\xi}_b\bar{R}\bar{\nabla}_c\bar{\nabla}^a h^{cb}}_{II-4} + \underbrace{\bar{\xi}_b\bar{R}\bar{\nabla}_c\bar{\nabla}^b h^{ca}}_{II-5} - \underbrace{\bar{\xi}_b\bar{R}\bar{\nabla}_c\bar{\nabla}^c h^{ab}}_{II-6} - \underbrace{\bar{\xi}_b\bar{R}\bar{\nabla}^a\bar{\nabla}^b h}_{II-7},
\end{aligned}
$$

$$
-\frac{1}{2}\bar{\xi}_b\bar{g}^{ca}\bar{g}^{bd}\left\{2\bar{g}_{cd}\bar{R}R_L+h_{cd}\bar{R}^2\right\} = -\underbrace{\bar{\xi}^a\bar{R}\bar{\nabla}_c\bar{\nabla}^d h^c{}_d}_{III-1} + \underbrace{\bar{\xi}^a\bar{R}\bar{\nabla}_c\bar{\nabla}^c h}_{III-2} + \underbrace{\bar{\xi}^a\bar{R}\bar{R}_{cd}h^{cd}}_{III-3} - \underbrace{\bar{\xi}_b\frac{1}{2}\,h^{ab}\bar{R}^2}_{III-4},
$$

$$
\begin{aligned}
2\bar{\xi}_b\bar{g}^{ca}\bar{g}^{bd}\left\{\bar{g}_{cd}(\Box R)_L+h_{cd}\bar{\Box}\bar{R}\right\} = {} & \underbrace{2\,\bar{\xi}^a\bar{\nabla}_c\bar{\nabla}^c\bar{\nabla}^d\bar{\nabla}^e h_{de}}_{IV-1} - \underbrace{2\,\bar{\xi}^a\bar{\nabla}_c\bar{\nabla}^c\bar{\nabla}^d\bar{\nabla}_d h}_{IV-2} - \underbrace{4\,\bar{\xi}^a\bar{\nabla}_c\bar{R}^{de}\,\bar{\nabla}^c h_{de}}_{IV-3} \\
& - \underbrace{2\,\bar{\xi}^a\bar{R}^{cd}\bar{\nabla}_e\bar{\nabla}^e h_{cd}}_{IV-4} - \underbrace{2\,\bar{\xi}^a\bar{\nabla}^c\bar{R}\,\bar{\nabla}_d h^d{}_c}_{IV-5} + \underbrace{\bar{\xi}^a\bar{\nabla}^c\bar{R}\,\bar{\nabla}_c h}_{IV-6} \\
& - \underbrace{2\,\bar{\xi}^a\bar{\nabla}_c\bar{\nabla}^c\bar{R}^{de}\ h_{de}}_{IV-7} - \underbrace{2\,\bar{\xi}^a\bar{\nabla}_c\bar{\nabla}_d\bar{R}\ h^{dc}}_{IV-8} + \underbrace{2\,\bar{\xi}_b h^{ab}\bar{\nabla}^c\bar{\nabla}_c\bar{R}}_{IV-9},
\end{aligned}
$$

$$
\begin{aligned}
-2\bar{\xi}_b\bar{g}^{ca}\bar{g}^{bd}\left\{(\bar{\nabla}_c\bar{\nabla}_d R)_L\right\} = {} & -\underbrace{2\,\bar{\xi}_b\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^c\bar{\nabla}^d h_{cd}}_{V-1} + \underbrace{2\,\bar{\xi}_b\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^c\bar{\nabla}_c h}_{V-2} + \underbrace{2\,\bar{\xi}_b\bar{\nabla}^a\bar{R}^{cd}\,\bar{\nabla}^b h_{cd}}_{V-3} \\
& + \underbrace{2\,\bar{\xi}_b\bar{R}^{cd}\bar{\nabla}^a\bar{\nabla}^b h_{cd}}_{V-4} + \underbrace{\bar{\xi}_b\bar{\nabla}^c\bar{R}\,\bar{\nabla}^a h^b{}_c}_{V-5} + \underbrace{\bar{\xi}_b\bar{\nabla}^c\bar{R}\,\bar{\nabla}^b h^a{}_c}_{V-6} - \underbrace{\bar{\xi}_b\bar{\nabla}^c\bar{R}\,\bar{\nabla}_c h^{ba}}_{V-7} \\
& + \underbrace{2\,\bar{\xi}_b\bar{\nabla}^b\bar{R}^{cd}\,\bar{\nabla}^a h_{cd}}_{V-8} + \underbrace{2\,\bar{\xi}_b\bar{\nabla}^a\bar{\nabla}^b\bar{R}^{cd}\ h_{cd}}_{V-9}.
\end{aligned}
$$

Start with $IV-1$

$$
\begin{aligned}
2\bar{\xi}^a\bar{\nabla}_c\bar{\nabla}^c\bar{\nabla}^d\bar{\nabla}^e h_{de} &= 2\bar{\nabla}_c\left(\bar{\xi}^a\bar{\nabla}^c\bar{\nabla}^d\bar{\nabla}^e h_{de}\right) - 2(\bar{\nabla}^c\bar{\xi}^a)\bar{\nabla}_c\bar{\nabla}^d\bar{\nabla}^e h_{de} \\
&= 2\bar{\nabla}_c\left(\bar{\xi}^a\bar{\nabla}^c\bar{\nabla}^d\bar{\nabla}^e h_{de}\right) - 2\bar{\nabla}^c\left((\bar{\nabla}_c\bar{\xi}^a)\bar{\nabla}^d\bar{\nabla}^e h_{de}\right) + 2(\bar{\nabla}_c\bar{\nabla}^c\bar{\xi}^a)\bar{\nabla}^d\bar{\nabla}^e h_{de} \\
&= 2\bar{\nabla}_c\left(\bar{\xi}^a\bar{\nabla}^c\bar{\nabla}^d\bar{\nabla}^e h_{de}\right) - 2\bar{\nabla}^c\left((\bar{\nabla}_c\bar{\xi}^a)\bar{\nabla}^d\bar{\nabla}^e h_{de}\right) - \underbrace{2\bar{R}^{ab}\bar{\xi}_b\bar{\nabla}^d\bar{\nabla}^e h_{de}}_{Cancels\ II-1}. \qquad \text{(B.1)}
\end{aligned}
$$

As for $IV-2$

$$-2\bar{\xi}^a\bar{\nabla}_c\bar{\nabla}^c\bar{\nabla}^d\bar{\nabla}_d h = -2\bar{\nabla}_c\left((\bar{\nabla}^c\bar{\xi}^a)\bar{\nabla}^b\bar{\nabla}_b h + \bar{\xi}^a\bar{\nabla}^c\bar{\nabla}^b\bar{\nabla}_b h\right) + \underbrace{2\bar{R}^{ab}\bar{\xi}_b\bar{\nabla}^c\bar{\nabla}_c h}_{Cancels\ II-2}, \tag{B.2}$$

where we integrate by parts twice using the identity (A.8).

$IV-3+IV-4+IV-7$ can be written as

$$-4\,\bar{\xi}^a(\bar{\nabla}_c\bar{R}^{de})\,\bar{\nabla}^c h_{de} - 2\,\bar{\xi}^a\bar{R}^{cd}\bar{\nabla}_e\bar{\nabla}^e h_{cd} - 2\,\bar{\xi}^a\bar{\nabla}_c\bar{\nabla}^c\bar{R}^{de}\ h_{de} = -2\bar{\xi}^a\bar{\nabla}_c\bar{\nabla}^c(\bar{R}^{bd}h_{bd})$$

and again integrating by parts twice

$$\bar{\nabla}_c\left(2\bar{R}^{bd}h_{bd}(\bar{\nabla}^c\bar{\xi}^a) - 2\bar{\xi}^a\bar{\nabla}^c(\bar{R}^{bd}h_{bd})\right) + \underbrace{2\bar{R}^{ac}\bar{\xi}_c\bar{R}^{bd}h_{bd}}_{Cancels\ II-3}. \tag{B.3}$$

One has the following for $V-1+V-2$

$$\begin{aligned}-2\,\bar{\xi}_b\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^c\bar{\nabla}^d h_{cd} + 2\,\bar{\xi}_b\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^c\bar{\nabla}_c h &= -2\,\bar{\xi}_b\bar{\nabla}^b\bar{\nabla}^a\bar{\nabla}^c\bar{\nabla}^d h_{cd} + 2\,\bar{\xi}_b\bar{\nabla}^b\bar{\nabla}^a\bar{\nabla}^c\bar{\nabla}_c h\\ &= \bar{\nabla}_c\left(2\,\bar{\xi}^c\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}_b h - 2\,\bar{\xi}^c\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^d h_{bd}\right).\end{aligned} \tag{B.4}$$

Note that we have changed the order of the derivatives above since they are acting on a scalar. Here we also have used the antisymmetry of Killing equation $\bar{\nabla}^c\bar{\xi}_c = 0$.

From $V-3+V-4+V-8+V-9$

$$\begin{aligned}2\,\bar{\xi}_b\left[(\bar{\nabla}^a\bar{R}^{cd})\,\bar{\nabla}^b h_{cd} + 2\bar{R}^{cd}\bar{\nabla}^a\bar{\nabla}^b h_{cd} + 2(\bar{\nabla}^b\bar{R}^{cd})\,\bar{\nabla}^a h_{cd} + 2(\bar{\nabla}^a\bar{\nabla}^b\bar{R}^{cd}\,)\,h_{cd}\right] &= 2\bar{\xi}_b\bar{\nabla}^a\bar{\nabla}^b\left(\bar{R}_{cd}h^{cd}\right)\\ = 2\bar{\xi}_b\bar{\nabla}^b\bar{\nabla}^a\left(\bar{R}_{cd}h^{cd}\right) &= 2\bar{\nabla}_c\left(\bar{\xi}^c\bar{\nabla}^a(\bar{R}_{bd}h^{bd})\right).\end{aligned} \tag{B.5}$$

Likewise we have the following expression for $III-1+IV-5+IV-8$

$$-\bar{\xi}^a\bar{R}\bar{\nabla}_c\bar{\nabla}^d h^c{}_d - 2\,\bar{\xi}^a\bar{\nabla}^c\bar{R}\,\bar{\nabla}_d h^d{}_c - 2\,\bar{\xi}^a h^{dc}\bar{\nabla}_c\bar{\nabla}_d\bar{R} = -\bar{\xi}^a\bar{\nabla}_c\bar{\nabla}_d(\bar{R}h^{cd}) - \bar{\xi}^a h^{dc}\bar{\nabla}_c\bar{\nabla}_d\bar{R}. \tag{B.6}$$

Similarly $II-4+V-5$ gives the following

$$\bar{\xi}_b\left[\bar{R}\bar{\nabla}_c\bar{\nabla}^a h^{cb} + (\bar{\nabla}^c\bar{R})\bar{\nabla}^a h^b{}_c\right] = \bar{\xi}_b\bar{\nabla}_c(\bar{R}\bar{\nabla}^a h^{bc}) = \bar{\nabla}_c\left(\bar{\xi}_b\bar{R}\bar{\nabla}^a h^{bc}\right) - (\bar{\nabla}_c\bar{\xi}_b)\bar{R}\bar{\nabla}^a h^{bc} \tag{B.7}$$

Here the last term is zero from the antisymmetry of Killing equation (A.4). $II-5+V-6$ terms yield

$$\bar{\xi}_b\left[\bar{R}\bar{\nabla}_c\bar{\nabla}^b h^{ca} + (\bar{\nabla}_c\bar{R})\bar{\nabla}^b h^{ac}\right] = \bar{\xi}_b\bar{\nabla}_c(\bar{R}\bar{\nabla}^b h^{ac}) = \bar{\nabla}_c\left(\bar{\xi}_b\bar{R}\bar{\nabla}^b h^{ac}\right) - (\bar{\nabla}_c\bar{\xi}_b)\bar{R}\bar{\nabla}^b h^{ac}, \tag{B.8}$$

whereas the sum $II-6+V-7$ leads to

$$-\bar{\xi}_b\left[\bar{R}\bar{\nabla}_c\bar{\nabla}^c h^{ba}+(\bar{\nabla}_c\bar{R})\bar{\nabla}^c h^{ab}\right]=-\bar{\xi}_b\bar{\nabla}_c(\bar{R}\bar{\nabla}^c h^{ab})=-\bar{\nabla}_c\left(\bar{\xi}_b\bar{R}\bar{\nabla}^c h^{ab}\right)+(\bar{\nabla}_c\bar{\xi}_b)\bar{R}\bar{\nabla}^c h^{ab} \quad \text{(B.9)}$$

In a similar fashion, we find the following for the remaining parts:$III-2+IV-6$ gives

$$\bar{\xi}^a\bar{R}\bar{\nabla}_c\bar{\nabla}^c h+\bar{\xi}^a(\bar{\nabla}^c\bar{R})\bar{\nabla}_c h=\bar{\nabla}_c\left(\bar{\xi}^a\bar{R}\bar{\nabla}^c h\right)-(\bar{\nabla}_c\bar{\xi}^a)\bar{R}\bar{\nabla}^c h, \quad \text{(B.10)}$$

whereas $II-7$ leads to

$$-\bar{\xi}_b\bar{R}\bar{\nabla}^a\bar{\nabla}^b h=-\bar{\nabla}_c\left(\bar{R}\bar{\xi}^c\bar{\nabla}^a h\right)+\bar{\xi}^c\bar{\nabla}_c\bar{R}\bar{\nabla}^a h, \quad \text{(B.11)}$$

and the sum $IV-9+III-3+III-4$ amounts to

$$2\,\bar{\xi}_b h^{ab}\bar{\nabla}^c\bar{\nabla}_c\bar{R}+\bar{\xi}^a\bar{R}\bar{R}_{cd}h^{cd}-\frac{1}{2}\,\bar{\xi}_b h^{ab}\bar{R}^2. \quad \text{(B.12)}$$

As a result, summing up all the terms (B.1) to (B.12), on finds

$$\begin{aligned}
&\bar{\nabla}_c\Big\{4\bar{\xi}^{[a}\bar{\nabla}^{c]}\bar{\nabla}^d\bar{\nabla}^b h_{db}+2(\bar{\nabla}^{[a}\bar{\xi}^{c]})\bar{\nabla}^d\bar{\nabla}^b h_{db}+2(\bar{\nabla}^{[c}\bar{\xi}^{a]})\bar{\nabla}^b\bar{\nabla}_b h+4\bar{\xi}^{[c}\bar{\nabla}^{a]}\bar{\nabla}^b\bar{\nabla}_b h\\
&+2(\bar{\nabla}^{[c}\bar{\xi}^{a]})\bar{R}^{bd}h_{bd}+4\bar{\xi}^{[c}\bar{\nabla}^{a]}(\bar{R}^{bd}h_{bd})+2\bar{\xi}_b\bar{R}\bar{\nabla}^{[a}h^{c]b}+2\bar{\xi}^{[a}\bar{R}\bar{\nabla}^{c]}h\Big\}\\
&-\underbrace{\bar{\xi}^a\bar{\nabla}_c\bar{\nabla}_d(\bar{R}h^{cd})}_{\text{c}}-\underbrace{\bar{\xi}^a h^{dc}\bar{\nabla}_c\bar{\nabla}_d\bar{R}}_{OK-1}+\underbrace{\bar{\nabla}_c\left(\bar{\xi}_b\bar{R}\bar{\nabla}^b h^{ac}\right)}_{\text{a}}-\underbrace{(\bar{\nabla}_c\bar{\xi}_b)\bar{R}\bar{\nabla}^b h^{ac}}_{\text{b}}+\underbrace{(\bar{\nabla}_c\bar{\xi}_b)\bar{R}\bar{\nabla}^c h^{ab}}_{\text{d}} \qquad \text{(B.13)}\\
&-\underbrace{(\bar{\nabla}_c\bar{\xi}^a)\bar{R}\bar{\nabla}^c h}_{\text{e}}+\underbrace{\bar{\xi}^c\bar{\nabla}_c\bar{R}\bar{\nabla}^a h}_{\text{f}}+\underbrace{2\,\bar{\xi}_b h^{ab}\bar{\nabla}^c\bar{\nabla}_c\bar{R}}_{OK-3}+\underbrace{\bar{\xi}^a\bar{R}\bar{R}_{cd}h^{cd}}_{OK-4}-\underbrace{\frac{1}{2}\,\bar{\xi}_b h^{ab}\bar{R}^2}_{OK-2}.
\end{aligned}$$

Here we have denoted the on-shell terms we expect to get by 'OK'. Now we have some part of the charge and the terms that are not a total derivative must give the remaining part of the charge and the expected background field equations. Remember that we expect the background field equations to be in the form of (4.11).

Let us start by examining with the c term in equation (B.13)

$$\begin{aligned}
-\bar{\xi}^a\bar{\nabla}_c\bar{\nabla}_d(\bar{R}h^{cd})&=-\bar{\nabla}_c\left(\bar{\xi}^a\bar{\nabla}_d(\bar{R}h^{cd})\right)+(\bar{\nabla}_c\bar{\xi}^a)\bar{\nabla}_d(\bar{R}h^{cd})\\
&=\bar{\nabla}_c\left(\bar{\xi}^c\bar{\nabla}_d(\bar{R}h^{ad})-\bar{\xi}^a\bar{\nabla}_d(\bar{R}h^{cd})\right)-\bar{\xi}^c\bar{\nabla}_c\bar{\nabla}_d(\bar{R}h^{ad})+(\bar{\nabla}_c\bar{\xi}^a)\bar{\nabla}_d(\bar{R}h^{cd})\\
&=\bar{\nabla}_c\left(\bar{\xi}^c\bar{\nabla}_b(\bar{R}h^{ab})-\bar{\xi}^a\bar{\nabla}_b(\bar{R}h^{cb})\right)\\
&-\bar{\xi}^c\left(h^{ab}\bar{\nabla}_c\bar{\nabla}_b\bar{R}+(\bar{\nabla}_b\bar{R})\bar{\nabla}_c h^{ab}+(\bar{\nabla}_c\bar{R})\bar{\nabla}_b h^{ab}+\bar{R}\bar{\nabla}_b\bar{\nabla}_c h^{ab}\right)+(\bar{\nabla}_c\bar{\xi}^a)\bar{\nabla}_b(\bar{R}h^{cb})\\
&=\bar{\nabla}_c\left(\bar{\xi}^c\bar{\nabla}_b(\bar{R}h^{ab})-\bar{\xi}^a\bar{\nabla}_b(\bar{R}h^{cb})\right)-\bar{\xi}^c h^{ab}\bar{\nabla}_c\bar{\nabla}_b\bar{R}-\bar{\xi}^c(\bar{\nabla}_b\bar{R})\bar{\nabla}_c h^{ab}\\
&-\bar{\xi}^c(\bar{\nabla}_c\bar{R})\bar{\nabla}_b h^{ab}-\bar{\xi}^c\bar{R}\bar{\nabla}_b\bar{\nabla}_c h^{ab}+\bar{\nabla}_b\left((\bar{\nabla}_c\bar{\xi}^a)\bar{R}h^{cb}\right)-\bar{R}h^{bc}\bar{\nabla}_b\bar{\nabla}_c\bar{\xi}^a. \qquad \text{(B.14)}
\end{aligned}$$

Using this in equation (B.13), one finds that

$$\begin{aligned}
\text{a+b+c} &= \bar\nabla_c\left\{2\bar\xi^{[c}\bar\nabla_b(\bar R h^{a]b})\right\} - h^{ab}\bar\xi^c\bar\nabla_c\bar\nabla_b\bar R - (\bar\xi^c\bar\nabla_c\bar R)\bar\nabla_b h^{ab} + \bar R\bar\xi^c[\bar\nabla_b,\bar\nabla_c]h^{ab} \\
&+ \bar\nabla_c\left(\bar R h^{bc}\bar\nabla_b\bar\xi^a\right) - \bar R h^{bc}\bar\nabla_b\bar\nabla_c\bar\xi^a \\
&= \bar\nabla_c\left\{2\bar\xi^{[c}\bar\nabla_b(\bar R h^{a]b})\right\} + \bar\nabla_c\left(\bar R h^{bc}\bar\nabla_b\bar\xi^a - \bar R h^{ba}\bar\nabla_b\bar\xi^c\right) + \bar\nabla_c\left(\bar R h^{ba}\bar\nabla_b\bar\xi^c\right) \\
&- h^{ab}\bar\xi^c\bar\nabla_c\bar\nabla_b\bar R - (\bar\xi^c\bar\nabla_c\bar R)\bar\nabla_b h^{ab} + \bar R\bar\xi^c\left(\bar R_{bc}{}^a{}_d h^{db} + \bar R_{bc}{}^b{}_d h^{ad}\right) - \bar R h^{bc}\bar R^a{}_{cbd}\bar\xi^d \\
&= \bar\nabla_c\left\{2\bar\xi^{[c}\bar\nabla_b(\bar R h^{a]b}) + 2\bar R h^{b[c}\bar\nabla_b\bar\xi^{a]}\right\} + \bar\nabla_c\left(\bar R h^{ba}\bar\nabla_b\bar\xi^c\right) - h^{ab}\bar\xi^c\bar\nabla_c\bar\nabla_b\bar R - (\bar\xi^c\bar\nabla_c\bar R)\bar\nabla_b h^{ab} \\
&+ \bar\xi^d\bar R\bar R^a{}_{cbd}h^{bc} + \bar\xi^c\bar R h^{ab}\bar R_{bc} - \bar\xi^d\bar R\bar R^a{}_{cbd}h^{bc}.
\end{aligned} \tag{B.15}$$

The two terms with $\bar R^a{}_{cbd}$ cancel each other out, then adding term d from (B.13), one gets

$$\begin{aligned}
\text{a+b+c+d} &= \bar\nabla_c\left\{2\bar\xi^{[c}\bar\nabla_b(\bar R h^{a]b}) + 2\bar R h^{b[c}\bar\nabla_b\bar\xi^{a]}\right\} + (\bar\nabla_c\bar R)h^{ab}\bar\nabla_b\bar\xi^c + \bar R h^{ab}(\bar\nabla_c\bar\nabla_b\bar\xi^c) \\
&+ \bar R(\bar\nabla_c h^{ab})(\bar\nabla_b\bar\xi^c + \bar\nabla^c\bar\xi_b) - h^{ab}\bar\xi^c\bar\nabla_c\bar\nabla_b\bar R - (\bar\xi^c\bar\nabla_c\bar R)\bar\nabla_b h^{ab} + \bar R h^{ab}\bar R_{bc}\bar\xi^c.
\end{aligned} \tag{B.16}$$

Now using the Killing equation in (A.4) we have

$$\begin{aligned}
\text{a+b+c+d} &= \bar\nabla_c\left\{2\bar\xi^{[c}\bar\nabla_b(\bar R h^{a]b}) + 2\bar R h^{b[c}\bar\nabla_b\bar\xi^{a]}\right\} - h^{ab}\bar\xi^c\bar\nabla_c\bar\nabla_b\bar R + 2\bar R h^{ab}\bar R_{bc}\bar\xi^c \\
&+ (\bar\nabla_c\bar R)h^{ab}\bar\nabla_b\bar\xi^c - (\bar\xi^c\bar\nabla_c\bar R)\bar\nabla_b h^{ab} \\
&= \bar\nabla_c\left\{2\bar\xi^{[c}\bar\nabla_b(\bar R h^{a]b}) + 2\bar R h^{b[c}\bar\nabla_b\bar\xi^{a]}\right\} + 2\bar R h^{ab}\bar R_{bc}\bar\xi^c - h^{ab}\bar\xi^c\bar\nabla_c\bar\nabla_b\bar R \\
&- (\bar\xi^c\bar\nabla_c\bar R)\bar\nabla_b h^{ab} + h^{ab}\bar\nabla_b\left(\bar\xi^c\bar\nabla_c\bar R\right) - h^{ab}\bar\xi^c\bar\nabla_b\bar\nabla_c\bar R
\end{aligned} \tag{B.17}$$

For the trace part, one has

$$\begin{aligned}
\text{e} &= -(\bar\nabla_c\bar\xi^a)\bar R\bar\nabla^c h = \bar\nabla_c\left(h\bar R\bar\nabla^a\bar\xi^c\right) - h(\bar\nabla^a\bar\xi^c)\bar\nabla_c\bar R - h\bar R\bar R^{ab}\bar\xi_b \\
&= \bar\nabla_c\left(h\bar R\bar\nabla^{[a}\bar\xi^{c]}\right) - h\bar\nabla^a\left(\bar\xi^c\bar\nabla_c\bar R\right) + h\bar\xi^c\bar\nabla^a\bar\nabla_c\bar R - h\bar R\bar R^{ab}\bar\xi_b.
\end{aligned} \tag{B.18}$$

Finally adding the f term to (B.18), one gets

$$\text{e+f} = \bar\nabla_c\left(h\bar R\bar\nabla^{[a}\bar\xi^{c]}\right) - h\bar\nabla^a\left(\bar\xi^c\bar\nabla_c\bar R\right) + h\bar\xi^c\bar\nabla^a\bar\nabla_c\bar R - h\bar R\bar R^{ab}\bar\xi_b + \bar\xi^c\bar\nabla_c\bar R\bar\nabla^a h. \tag{B.19}$$

Using (B.17) and (B.19) in, (B.13) one finds

$$\begin{aligned}
&\bar\nabla_b\Big\{4\bar\xi^{[a}\bar\nabla^{b]}\bar\nabla_c\bar\nabla_d h^{cd} + 2(\bar\nabla^{[a}\bar\xi^{b]})\bar\nabla_c\bar\nabla_d h^{cd} + 2\bar\nabla_c\bar\nabla^c h(\bar\nabla^{[b}\bar\xi^{a]}) + 4\bar\xi^{[b}\bar\nabla^{a]}\bar\nabla_c\bar\nabla^c h + 2(\bar R_{cd}h^{cd})\bar\nabla^{[b}\bar\xi^{a]} \\
&+ 4\bar\xi^{[b}\bar\nabla^{a]}\left(\bar R_{cd}h^{cd}\right) + 2\bar\xi_c\bar R\bar\nabla^{[a}h^{b]c} + 2\bar R\bar\xi^{[a}\bar\nabla^{b]}h + 2\bar\xi^{[b}\bar\nabla_c\left(\bar R h^{a]c}\right) + 2\bar R h^{c[b}\bar\nabla_c\bar\xi^{a]} + h\bar R\bar\nabla^{[a}\bar\xi^{b]}\Big\} \\
&\underline{+h^{ab}\bar\nabla_b\left(\bar\xi^c\bar\nabla_c\bar R\right) - (\bar\xi_c\bar\nabla^c\bar R)\bar\nabla_b h^{ab} - h\bar\nabla^a\left(\bar\xi_c\bar\nabla^c\bar R\right) + (\bar\xi_c\bar\nabla^c\bar R)\bar\nabla^a h} \\
&+ h^{ab}\bar\xi^c\bar A_{bc} + \frac{1}{2}\bar\xi^a h_{cd}\bar A^{cd} - \frac{1}{2}\bar\xi_b h\bar A^{ab},
\end{aligned} \tag{B.20}$$

where one should recall that $A_{ab} = 2\bar{R}\bar{R}_{ab} - \frac{1}{2}\bar{g}_{ab}\bar{R}^2 + 2\bar{g}_{ab}\bar{\nabla}_c\bar{\nabla}^c\bar{R} - 2\bar{\nabla}_a\bar{\nabla}_b\bar{R}$. The underlined terms are all zero by the identity (A.14) we have derived earlier. Notice that the background terms are just in the form we want! To easily compare things with AdS case ([1] equation 31), (B.20) can be written compactly as

$$\alpha\bar{\nabla}_b\Big[2\bar{R}Q_E^{ab} + 2(\bar{\nabla}_c\bar{R})\bar{\xi}^{[b}h^{a]c} + 4\bar{\xi}^{[a}\bar{\nabla}^{b]}R_L + 2R_L\bar{\nabla}^{[a}\bar{\xi}^{b]}\Big] \tag{B.21}$$

where $Q_E^{ab} \equiv \bar{\xi}_c\bar{\nabla}^{[a}h^{b]c} + \bar{\xi}^{[b}\bar{\nabla}_c h^{a]c} + h^{c[b}\bar{\nabla}_c\bar{\xi}^{a]} + \bar{\xi}^{[a}\bar{\nabla}^{b]}h + \frac{1}{2}h\bar{\nabla}^{[a}\bar{\xi}^{b]}$ is the contribution charge coming from Einstein equation we have calculated in Chapter 4.

APPENDIX C

β CHARGE CALCULATION

Linearized $\beta(B_{cd})_L\bar{g}^{ca}\bar{g}^{bd}$ in terms of h^{ab} calculated by CADABRA [11] reads

$$\begin{aligned}\frac{1}{2}\bar{\xi}_b\bar{g}^{ab}(\Box R)_L =& \underbrace{\frac{1}{2}\,\bar{\nabla}^b\bar{\nabla}_b\bar{\nabla}^c\bar{\nabla}^d h_{cd}\;\bar{\xi}^a}_{\text{E}} - \underbrace{\frac{1}{2}\,\bar{\nabla}^b\bar{\nabla}_b\bar{\nabla}^c\bar{\nabla}_c h\;\bar{\xi}^a}_{\text{H-1}} - \underbrace{\bar{\nabla}^b\bar{R}^{cd}\,\bar{\nabla}_b h_{cd}\,\bar{\xi}^a}_{\text{25}} - \underbrace{\frac{1}{2}\,\bar{R}^{bc}\bar{\nabla}^d\bar{\nabla}_d h_{bc}\;\bar{\xi}^a}_{\text{F}} \\ &- \underbrace{\frac{1}{2}\,\bar{\nabla}^b\bar{R}\,\bar{\nabla}^c h_{bc}\,\bar{\xi}^a}_{\text{21}} + \underbrace{\frac{1}{4}\,\bar{\nabla}^b\bar{R}\,\bar{\nabla}_b h\,\bar{\xi}^a}_{\text{H-2}} - \underbrace{\frac{1}{2}\,\bar{\nabla}^b\bar{\nabla}_b\bar{R}^{cd}\;\bar{\xi}^a h_{cd}}_{\text{24}} - \underbrace{\frac{1}{2}\,\bar{\nabla}^b\bar{\nabla}^c\bar{R}\;\bar{\xi}^a h_{bc}}_{\text{OK-5}} + \underbrace{\frac{1}{2}\,\bar{\nabla}^b\bar{\nabla}_b\bar{R}\;\bar{\xi}^c h^a{}_c}_{\text{OK-3}},\end{aligned}$$

$$\begin{aligned}-\bar{\xi}_b\bar{g}^{ac}\bar{g}^{bd}(\bar{\nabla}_c\bar{\nabla}_d R)_L =& -\underbrace{\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^c\bar{\nabla}^d h_{cd}\;\bar{\xi}_b}_{\text{D}} + \underbrace{\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^c\bar{\nabla}_c h\;\bar{\xi}_b}_{\text{H-3}} + \underbrace{\bar{\nabla}^a\bar{R}^{bc}\;\bar{\nabla}^d h_{bc}\,\bar{\xi}_d}_{\text{27}} + \underbrace{\bar{R}^{bc}\bar{\nabla}^a\bar{\nabla}^d h_{bc}\;\bar{\xi}_d}_{\text{29}} \\ &+ \underbrace{\frac{1}{2}\,\bar{\nabla}^b\bar{R}\,\bar{\nabla}^a h_b{}^c\,\bar{\xi}_c}_{\text{15}} + \underbrace{\frac{1}{2}\,\bar{\nabla}^b\bar{R}\,\bar{\nabla}^c h^a{}_b\,\bar{\xi}_c}_{\text{19}} - \underbrace{\frac{1}{2}\,\bar{\nabla}^b\bar{R}\,\bar{\nabla}_b h^{ac}\,\bar{\xi}_c}_{\text{13}} + \underbrace{\bar{\nabla}^b\bar{R}^{cd}\,\bar{\nabla}^a h_{cd}\,\bar{\xi}_b}_{\text{30}} + \underbrace{\bar{\nabla}^a\bar{\nabla}^b\bar{R}^{cd}\;\bar{\xi}_b h_{cd}}_{\text{26}},\end{aligned}$$

$$\begin{aligned}2\bar{\xi}_b\bar{g}^{ac}\bar{g}^{bd}(R_{cfdk}R^{fk})_L =& \underbrace{2\,\bar{R}^{bc}\bar{\nabla}^d\bar{\nabla}_b h^a{}_c\bar{\xi}_d}_{\text{18}} - \underbrace{\bar{R}^{bc}\bar{\nabla}^d\bar{\nabla}^a h_{bc}\;\bar{\xi}_d}_{\text{28}} - \underbrace{\bar{R}^{bc}\bar{\nabla}_b\bar{\nabla}_c h^{ad}\;\bar{\xi}_d}_{\text{14}} - \underbrace{\bar{R}^{bc}\bar{\nabla}_b\bar{\nabla}^d h^a{}_c\;\bar{\xi}_d}_{\text{17}} \\ &+ \underbrace{\bar{R}^{bc}\bar{\nabla}_b\bar{\nabla}^a h_c{}^d\;\bar{\xi}_d}_{\text{16}} + \underbrace{2\,\bar{R}^{bc}\bar{R}_b{}^d{}_c{}^e\bar{\xi}_d h^a{}_e}_{\text{OK-2}} - \underbrace{\bar{R}^{abcd}\bar{\nabla}^e\bar{\nabla}_b h_{ce}\;\bar{\xi}_d}_{\text{7}} - \underbrace{\bar{R}^{abcd}\bar{\nabla}^e\bar{\nabla}_c h_{be}\;\bar{\xi}_d}_{\text{8}} + \underbrace{\bar{R}^{abcd}\bar{\nabla}^e\bar{\nabla}_e h_{bc}\;\bar{\xi}_d}_{\text{9}} \\ &+ \underbrace{\bar{R}^{abcd}\bar{\nabla}_b\bar{\nabla}_c h\;\bar{\xi}_d}_{\text{H-4}} + \underbrace{2\,\bar{R}^{bc}\bar{R}^{ad}{}_b{}^e\bar{\xi}_e h_{cd} - 2\,\bar{R}^{bc}\bar{R}^a{}_b{}^{de}\bar{\xi}_d h_{ce}}_{\text{35}},\end{aligned}$$

$$\begin{aligned}-\frac{1}{2}\bar{\xi}_b\bar{g}^{ab}(R_{cd}R^{cd})_L =& -\underbrace{\bar{R}^{bc}\bar{\nabla}^d\bar{\nabla}_b h_{cd}\;\bar{\xi}^a}_{\text{22}} + \underbrace{\frac{1}{2}\,\bar{R}^{bc}\bar{\nabla}^d\bar{\nabla}_d h_{bc}\;\bar{\xi}^a}_{\text{F}} + \underbrace{\frac{1}{2}\,\bar{R}^{bc}\bar{\nabla}_b\bar{\nabla}_c h\;\bar{\xi}^a}_{\text{H-5}} \\ &+ \underbrace{\bar{R}^{bc}\bar{R}_b{}^d\bar{\xi}^a h_{cd}}_{\text{23}} - \underbrace{\frac{1}{2}\,\bar{R}^{bc}\bar{R}_{bc}\bar{\xi}^d h^a{}_d}_{\text{OK-4}},\end{aligned}$$

$$
\begin{aligned}
\bar{\xi}_b \bar{g}^{ac}\bar{g}^{bd}(\Box R_{cd})_L = {} & \underbrace{\frac{1}{2}\,\bar{\nabla}^b\bar{\nabla}_b\bar{\nabla}^c\bar{\nabla}^a h_c{}^d\;\bar{\xi}_d}_{\text{A}} + \underbrace{\frac{1}{2}\,\bar{\nabla}^b\bar{\nabla}_b\bar{\nabla}^c\bar{\nabla}^d h^a{}_c\;\bar{\xi}_d}_{\text{C}} - \underbrace{\frac{1}{2}\,\bar{\nabla}^b\bar{\nabla}_b\bar{\nabla}^c\bar{\nabla}_c h^{ad}\;\bar{\xi}_d}_{\text{B}} \\
& - \underbrace{\frac{1}{2}\,\bar{\nabla}^b\bar{\nabla}_b\bar{\nabla}^a\bar{\nabla}^c h\;\bar{\xi}_c}_{\text{H-6}} - \underbrace{\bar{\nabla}^b\bar{R}^{cd}\,\bar{\nabla}_b h^a{}_c\,\bar{\xi}_d}_{31} - \underbrace{\frac{1}{2}\,\bar{R}^{bc}\bar{\nabla}^d\bar{\nabla}_d h^a{}_b\,\bar{\xi}_c}_{1} - \underbrace{\bar{\nabla}^b\bar{R}^{cd}\,\bar{\nabla}^a h_{bc}\,\bar{\xi}_d}_{10} - \underbrace{\frac{1}{2}\,\bar{R}^{bc}\bar{\nabla}^d\bar{\nabla}^a h_{bd}\;\bar{\xi}_c}_{3} \\
& + \underbrace{\bar{\nabla}^b\bar{R}^{cd}\,\bar{\nabla}_c h^a{}_b\,\bar{\xi}_d}_{34} + \underbrace{\frac{1}{2}\,\bar{R}^{bc}\bar{\nabla}^d\bar{\nabla}_b h^a{}_d\,\bar{\xi}_c}_{2} - \underbrace{\bar{\nabla}^b\bar{R}^{ac}\,\bar{\nabla}_b h_c{}^d\,\bar{\xi}_d}_{11} - \underbrace{\frac{1}{2}\,\bar{R}^{ab}\bar{\nabla}^c\bar{\nabla}_c h_b{}^d\;\bar{\xi}_d}_{5} - \underbrace{\bar{\nabla}^b\bar{R}^{ac}\,\bar{\nabla}^d h_{bc}\,\bar{\xi}_d}_{20} \\
& - \underbrace{\frac{1}{2}\,\bar{R}^{ab}\bar{\nabla}^c\bar{\nabla}^d h_{bc}\;\bar{\xi}_d}_{6} + \underbrace{\bar{\nabla}^b\bar{R}^{ac}\,\bar{\nabla}_c h_b{}^d\,\bar{\xi}_d}_{12} + \underbrace{\frac{1}{2}\,\bar{R}^{ab}\bar{\nabla}^c\bar{\nabla}_b h_c{}^d\,\bar{\xi}_d}_{4} - \underbrace{\bar{\nabla}^b\bar{R}^{ac}\,\bar{\nabla}^d h_{bd}\,\bar{\xi}_c}_{32} \\
& + \underbrace{\frac{1}{2}\,\bar{\nabla}^b\bar{R}^{ac}\,\bar{\nabla}_b h\,\bar{\xi}_c}_{\text{H-7}} - \underbrace{\bar{\nabla}^b\bar{\nabla}^c\bar{R}^{ad}\;\bar{\xi}_d h_{bc}}_{33}.
\end{aligned}
$$

where we have denoted the on-shell terms we expect to get by “OK” as before. First we will start with the trace parts $\sum_{i=1}^{7} H - i$ because these can be written as charge plus on-shell terms. Start with $H-3$

$$
\begin{aligned}
\bar{\xi}_b\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^c\bar{\nabla}_c h &= \frac{1}{2}\bar{\xi}_b\bar{\nabla}^b\bar{\nabla}^a\bar{\nabla}^c\bar{\nabla}_c h + \frac{1}{2}\bar{\xi}_b\bar{\nabla}^a[\bar{\nabla}^b,\bar{\nabla}^c]\bar{\nabla}_c h + \frac{1}{2}\bar{\xi}_b\bar{\nabla}^a\bar{\nabla}^c\bar{\nabla}^b\bar{\nabla}_c h \\
&= \frac{1}{2}\bar{\xi}_c\bar{\nabla}^c\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}_b h + \frac{1}{2}\bar{\xi}_b[\bar{\nabla}^a,\bar{\nabla}^c]\bar{\nabla}_c\bar{\nabla}^b h + \frac{1}{2}\bar{\xi}_b\bar{\nabla}^c\bar{\nabla}^a\bar{\nabla}_c\bar{\nabla}^b h + \frac{1}{2}\bar{\xi}_b\bar{\nabla}^a[\bar{\nabla}^b,\bar{\nabla}^c]\bar{\nabla}_c h \\
&= \frac{1}{2}\bar{\nabla}_c\left(\bar{\xi}^c\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}_b h\right) + \frac{1}{2}\bar{\nabla}_c\left(\bar{\xi}^b\bar{\nabla}^a\bar{\nabla}^c\bar{\nabla}_b h\right) - \frac{1}{2}(\bar{\nabla}_c\bar{\xi}^b)\bar{\nabla}^a\bar{\nabla}^c\bar{\nabla}_b h \\
&\quad + \frac{1}{2}\bar{\xi}_b\bar{\nabla}^a[\bar{\nabla}^b,\bar{\nabla}^c]\bar{\nabla}_c h + \frac{1}{2}\bar{\xi}_b[\bar{\nabla}^a,\bar{\nabla}^c]\bar{\nabla}_c\bar{\nabla}^b h, \qquad \text{(C.1)}
\end{aligned}
$$

then $H-1$

$$
-\frac{1}{2}\bar{\xi}^a\bar{\nabla}^b\bar{\nabla}_b\bar{\nabla}^c\bar{\nabla}_c h = -\frac{1}{2}\bar{\nabla}_c\left(\bar{\xi}^a\bar{\nabla}^c\bar{\nabla}^b\bar{\nabla}_b h\right) + \frac{1}{2}(\bar{\nabla}_c\bar{\xi}^a)\bar{\nabla}^c\bar{\nabla}^b\bar{\nabla}_b h. \qquad \text{(C.2)}
$$

Likewise for $H-6$

$$
-\frac{1}{2}\bar{\xi}_c\bar{\nabla}^b\bar{\nabla}_b\bar{\nabla}^a\bar{\nabla}^c h = -\frac{1}{2}\bar{\nabla}_c\left(\bar{\xi}^b\bar{\nabla}^c\bar{\nabla}^a\bar{\nabla}_b h\right) + \frac{1}{2}(\bar{\nabla}_c\bar{\xi}^b)\bar{\nabla}^c\bar{\nabla}^a\bar{\nabla}_b h. \qquad \text{(C.3)}
$$

The sum of $H-3+H-1+H-6$ leads to

$$
\begin{aligned}
H-3+H-1+H-6 &= \frac{1}{2}\bar{\nabla}_c\left(\bar{\xi}^c\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}_b h - \bar{\xi}^a\bar{\nabla}^c\bar{\nabla}^b\bar{\nabla}_b h\right) \\
&\quad + \frac{1}{2}\bar{\nabla}_c\left(\bar{\xi}^b\bar{\nabla}^a\bar{\nabla}^c\bar{\nabla}_b h - \bar{\xi}^b\bar{\nabla}^c\bar{\nabla}^a\bar{\nabla}_b h\right) \\
&\quad + \frac{1}{2}(\bar{\nabla}_c\bar{\xi}^a)\bar{\nabla}^c\bar{\nabla}^b\bar{\nabla}_b h + \frac{1}{2}(\bar{\nabla}_c\bar{\xi}_b)[\bar{\nabla}^c,\bar{\nabla}^a]\bar{\nabla}^b h \\
&\quad + \frac{1}{2}\bar{\xi}^b[\bar{\nabla}^a,\bar{\nabla}^c]\bar{\nabla}_c\bar{\nabla}_b h + \frac{1}{2}\bar{\xi}_b\bar{\nabla}^a[\bar{\nabla}^b,\bar{\nabla}^c]\bar{\nabla}_c h.
\end{aligned} \qquad \text{(C.4)}
$$

In order to get a charge, integrate by parts the third term above, then add and subtract anti-symmetric piece

$$
\begin{aligned}
H-3+H-1+H-6 = &\frac{1}{2}\bar{\nabla}_c\left(F_1^{ac}+F_2^{ac}\right)+\frac{1}{2}\bar{\nabla}_c\left((\bar{\nabla}_b\bar{\xi}^a)\bar{\nabla}^c\bar{\nabla}^b h-(\bar{\nabla}_b\bar{\xi}^c)\bar{\nabla}^a\bar{\nabla}^b h\right)\\
&+\frac{1}{2}(\bar{\nabla}_c\bar{\nabla}_b\bar{\xi}^c)\bar{\nabla}^a\bar{\nabla}^b h+\underbrace{\frac{1}{2}(\bar{\nabla}_b\bar{\xi}^c)\bar{\nabla}_c\bar{\nabla}^a\bar{\nabla}^b h}_{1}+\frac{1}{2}(\bar{\nabla}_b\bar{\xi}^a)[\bar{\nabla}^b,\bar{\nabla}^c]\bar{\nabla}_c h\\
&\underbrace{+\frac{1}{2}(\bar{\nabla}_c\bar{\xi}_b)[\bar{\nabla}^c,\bar{\nabla}^a]\bar{\nabla}^b h}_{2}+\frac{1}{2}\bar{\xi}^b[\bar{\nabla}^a,\bar{\nabla}^c]\bar{\nabla}_c\bar{\nabla}_b h+\frac{1}{2}\bar{\xi}_b\bar{\nabla}^a[\bar{\nabla}^b,\bar{\nabla}^c]\bar{\nabla}_c h\\
&-\frac{1}{2}(\bar{\nabla}_c\bar{\nabla}_b\bar{\xi}^a)\bar{\nabla}^c\bar{\nabla}^b h,
\end{aligned}
\tag{C.5}
$$

where $F_1^{ac}\equiv 2\bar{\xi}^{[c}\bar{\nabla}^{a]}\bar{\nabla}^b\bar{\nabla}_b h$, $F_2^{ac}\equiv 2\bar{\xi}^b\bar{\nabla}^{[a}\bar{\nabla}^{c]}\bar{\nabla}_b h$, $F_3^{ac}\equiv 2(\bar{\nabla}_b\bar{\xi}^{[a})\bar{\nabla}^{c]}\bar{\nabla}^b h$. As one can verify the sum $1+2=\frac{1}{2}(\bar{\nabla}_b\bar{\xi}^c)\bar{\nabla}_c\bar{\nabla}^a\bar{\nabla}^b h+\frac{1}{2}(\bar{\nabla}_c\bar{\xi}_b)[\bar{\nabla}^c,\bar{\nabla}^a]\bar{\nabla}^b h=0$. Using this in (C.5) reads

$$
\begin{aligned}
H-3+H-1+H-6 = &\frac{1}{2}\bar{\nabla}_c\left(F_1^{ac}+F_2^{ac}+F_3^{ac}\right)+\underbrace{\frac{1}{2}(\bar{\nabla}_c\bar{\nabla}_b\bar{\xi}^c)\bar{\nabla}^a\bar{\nabla}^b h}_{3}\\
&+\underbrace{\frac{1}{2}(\bar{\nabla}_b\bar{\xi}^a)[\bar{\nabla}^b,\bar{\nabla}^c]\bar{\nabla}_c h}_{4}+\underbrace{\frac{1}{2}\bar{\xi}^b[\bar{\nabla}^a,\bar{\nabla}^c]\bar{\nabla}_c\bar{\nabla}_b h}_{5}\\
&+\underbrace{\frac{1}{2}\bar{\xi}_b\bar{\nabla}^a[\bar{\nabla}^b,\bar{\nabla}^c]\bar{\nabla}_c h}_{6}-\underbrace{\frac{1}{2}(\bar{\nabla}_c\bar{\nabla}_b\bar{\xi}^a)\bar{\nabla}^c\bar{\nabla}^b h}_{11}.
\end{aligned}
\tag{C.6}
$$

Rewrite 4 and 5 in (C.6) as

$$
\begin{aligned}
5 &= \frac{1}{2}\bar{\xi}^b[\bar{\nabla}^a,\bar{\nabla}^c]\bar{\nabla}_c\bar{\nabla}_b h=-\frac{1}{2}\bar{\xi}^b\bar{R}^{ac}\bar{\nabla}_c\bar{\nabla}_b h-\frac{1}{2}\bar{R}^{abcd}(\bar{\nabla}_b\bar{\nabla}_c h)\bar{\xi}_d\\
4 &= \frac{1}{2}(\bar{\nabla}_b\bar{\xi}^a)[\bar{\nabla}^b,\bar{\nabla}^c]\bar{\nabla}_c h=-\frac{1}{2}(\bar{\nabla}_b\bar{\xi}^a)\bar{R}^{bc}\bar{\nabla}_c h.
\end{aligned}
\tag{C.7}
$$

Using 3 + 6 one can generate one more charge as

$$
\begin{aligned}
3+6 &= -\frac{1}{2}\bar{\xi}_b(\bar{\nabla}^a\bar{R}^{bc})\bar{\nabla}_c h\\
&= -\frac{1}{2}\bar{\nabla}_c\left(h\bar{\xi}_b\bar{\nabla}^a\bar{R}^{bc}\right)+\frac{1}{2}h\bar{\xi}_b\bar{\nabla}_c\bar{\nabla}^a\bar{R}^{bc}\\
&= -\frac{1}{2}\bar{\nabla}_c\left(h\bar{\xi}_b\bar{\nabla}^a\bar{R}^{bc}\right)+\frac{1}{2}h\bar{\xi}_b[\bar{\nabla}_c,\bar{\nabla}^a]\bar{R}^{bc}+\frac{1}{2}h\bar{\xi}_b\bar{\nabla}^a\bar{\nabla}_c\bar{R}^{bc}\\
&= \frac{1}{2}\bar{\nabla}_c\left(h\bar{\xi}_b\bar{\nabla}^c\bar{R}^{ba}-h\bar{\xi}_b\bar{\nabla}^a\bar{R}^{bc}\right)-\frac{1}{2}(\bar{\nabla}_c h)\bar{\xi}_b\bar{\nabla}^c\bar{R}^{ab}-\frac{1}{2}h(\bar{\nabla}_c\bar{\xi}_b)\bar{\nabla}^c\bar{R}^{ab}-\frac{1}{2}h\bar{\xi}_b\bar{\nabla}_c\bar{\nabla}^c\bar{R}^{ab}\\
&+\frac{1}{4}h\bar{\xi}_b\bar{\nabla}^a\bar{\nabla}^b\bar{R}+\frac{1}{2}h\bar{\xi}_b[\bar{\nabla}_c,\bar{\nabla}^a]\bar{R}^{bc}\\
&= \frac{1}{2}\bar{\nabla}_c\left(F_4^{ac}\right)-\frac{1}{2}(\bar{\nabla}_c h)\bar{\xi}_b\bar{\nabla}^c\bar{R}^{ab}-\frac{1}{2}h(\bar{\nabla}_c\bar{\xi}_b)\bar{\nabla}^c\bar{R}^{ab}-\underbrace{\frac{1}{2}h\bar{\xi}_b\bar{\nabla}_c\bar{\nabla}^c\bar{R}^{ab}}_{OK-1}\\
&+\underbrace{\frac{1}{4}h\bar{\xi}_b\bar{\nabla}^a\bar{\nabla}^b\bar{R}}_{\frac{1}{2}OK-2}+\frac{1}{2}h\bar{\xi}_b[\bar{\nabla}_c,\bar{\nabla}^a]\bar{R}^{bc}.
\end{aligned}
\tag{C.8}
$$

Notice that, the second term above cancels with $H-7$ in linearized equations. Using this and (C.7), (C.6) reads

$$H-3+H-1+H-6+H-7=\frac{1}{2}\bar{\nabla}_c\left(F_1^{ac}+F_2^{ac}+F_3^{ac}\right)\underbrace{-\frac{1}{2}h\bar{\xi}_b\bar{\nabla}_c\bar{\nabla}^c\bar{R}^{ab}}_{OK-1}+\underbrace{\frac{1}{4}h\bar{\xi}_b\bar{\nabla}^a\bar{\nabla}^b\bar{R}}_{\frac{1}{2}OK-2}$$
$$\underbrace{-\frac{1}{2}h(\bar{\nabla}_c\bar{\xi}_b)\bar{\nabla}^c\bar{R}^{ab}}_{8}+\underbrace{\frac{1}{2}h\bar{\xi}_b[\bar{\nabla}_c,\bar{\nabla}^a]\bar{R}^{bc}}_{7}\underbrace{-\frac{1}{2}(\bar{\nabla}_b\bar{\xi}^a)\bar{R}^{bc}\bar{\nabla}_c h}_{9}$$
$$\underbrace{-\frac{1}{2}\bar{\xi}^b\bar{R}^{ac}\bar{\nabla}_c\bar{\nabla}_b h}_{10}-\bar{R}^{abcd}(\bar{\nabla}_b\bar{\nabla}_c h)\bar{\xi}_d, \tag{C.9}$$

here the last term cancels with $H-4$. Expanding the commutator in 7

$$7=\frac{1}{2}h\bar{\xi}_b[\bar{\nabla}_c,\bar{\nabla}^a]\bar{R}^{bc}=-\frac{1}{2}h\bar{\xi}_b\bar{R}^{acbd}\bar{R}_{cd}+\frac{1}{2}h\bar{\xi}_c\bar{R}^{ab}\bar{R}^c{}_b. \tag{C.10}$$

Take a charge from 8

$$8=-\frac{1}{2}h(\bar{\nabla}_c\bar{\xi}_b)\bar{\nabla}^c\bar{R}^{ab}=-\frac{1}{2}\bar{\nabla}_c\left(h\bar{R}^{ab}(\bar{\nabla}_b\bar{\xi}^c)\right)-\frac{1}{2}\bar{R}^{ab}(\bar{\nabla}_c h)(\bar{\nabla}_b\bar{\xi}^c)-\frac{1}{2}\bar{R}^{ab}h\bar{R}_{bc}\bar{\xi}^c, \tag{C.11}$$

and the antisymmetric part of from 9

$$9=-\frac{1}{2}(\bar{\nabla}_b\bar{\xi}^a)\bar{R}^{bc}\bar{\nabla}_c h=-\frac{1}{2}\bar{\nabla}_c\left(h\bar{R}^{bc}(\bar{\nabla}_b\bar{\xi}^a)\right)+\frac{1}{4}h(\bar{\nabla}^b\bar{R})\bar{\nabla}_b\bar{\xi}^a+\frac{1}{2}h\bar{R}^{abcd}\bar{R}_{bc}\bar{\xi}_d. \tag{C.12}$$

The sum $7+8+9$ leads to

$$7+8+9=\frac{1}{2}\bar{\nabla}_c\left(F_5^{ac}\right)-h\bar{\xi}_b\bar{R}^{acbd}\bar{R}_{cd}-\frac{1}{2}\bar{R}^{ab}(\bar{\nabla}_c h)\bar{\nabla}_b\bar{\xi}^c+\frac{1}{4}h(\bar{\nabla}^b\bar{R})\bar{\nabla}_b\bar{\xi}^a, \tag{C.13}$$

where $F_5^{ac}\equiv 2h(\bar{\nabla}_b\bar{\xi}^{[c})\bar{R}^{a]b}$. One last charge can be generated from 10

$$10=-\frac{1}{2}\bar{\xi}^b\bar{R}^{ac}\bar{\nabla}_c\bar{\nabla}_b h=-\frac{1}{2}\bar{\nabla}_c\left(\bar{\xi}^c\bar{R}^{ab}\bar{\nabla}_b h\right)+\frac{1}{2}\bar{\xi}^c\bar{\nabla}_c\bar{R}^{ab}\bar{\nabla}_b h$$
$$=-\frac{1}{2}\bar{\nabla}_c\left(\bar{\xi}^c\bar{R}^{ab}\bar{\nabla}_b h\right)+\frac{1}{2}(\bar{\nabla}_b h)(\bar{R}^{ac}\bar{\nabla}_c\bar{\xi}^b)+\frac{1}{2}(\bar{\nabla}_b h)(\bar{R}^{bc}\bar{\nabla}_c\bar{\xi}^a) \tag{C.14}$$
$$=\frac{1}{2}\bar{\nabla}_c\left(\bar{\xi}^a\bar{R}^{bc}\bar{\nabla}_b h-\bar{\xi}^c\bar{R}^{ba}\bar{\nabla}_b h\right)-\frac{1}{4}\bar{\xi}^a(\bar{\nabla}^b\bar{R})\bar{\nabla}_b h-\frac{1}{2}\bar{\xi}^a\bar{R}^{bc}\bar{\nabla}_b\bar{\nabla}_c h$$
$$+\frac{1}{2}\bar{\xi}^c\bar{\nabla}_c\bar{R}^{ab}\bar{\nabla}_b h, \tag{C.15}$$

in (C.14) we used the identity (A.13). With the help of (C.13), (C.15), the sum $7+8+9+10$ can be written as

$$7+8+9+10=\frac{1}{2}\bar{\nabla}_c\left(F_5^{ac}+F_6^{ac}\right)\underbrace{-h\bar{\xi}_b\bar{R}^{acbd}\bar{R}_{cd}}_{OK-3}-\frac{1}{4}\bar{\xi}^a(\bar{\nabla}^b\bar{R})\bar{\nabla}_b h$$
$$-\frac{1}{2}\bar{\xi}^a\bar{R}^{bc}\bar{\nabla}_c\bar{\nabla}_b h+\frac{1}{4}h(\bar{\nabla}^b\bar{R})\bar{\nabla}_b\bar{\xi}^a, \tag{C.16}$$

where $F^{ac}_{6} \equiv 2\bar{\xi}^{[a}\bar{R}^{c]b}\bar{\nabla}_b h$. The third and the fourth terms cancel with $H-2$, $H-5$ respectively. Therefore instead of (C.9) one can write

$$\sum_{i=1}^{7} H-i = \frac{1}{2}\bar{\nabla}_c\left(\sum_{i=1}^{6} F^{ac}_{\text{i}}\right) + \frac{1}{4}h(\bar{\nabla}^b\bar{R})(\bar{\nabla}_b\bar{\xi}^a) - \underbrace{\frac{1}{2}h\bar{\xi}_b\bar{\nabla}_c\bar{\nabla}^c\bar{R}^{ab}}_{OK-1} + \underbrace{\frac{1}{4}h\bar{\xi}_b\bar{\nabla}^a\bar{\nabla}^b\bar{R}}_{\frac{1}{2}OK-2} - \underbrace{h\bar{\xi}_b\bar{R}^{acbd}\bar{R}_{cd}}_{OK-3}. \tag{C.17}$$

One must have the divergence plus, the barred equation of motions in the form $-\frac{1}{2}h\bar{\xi}_b\bar{B}^{ab}$ where $\bar{B}_{ab} = 2\bar{R}_{acbd}\bar{R}^{cd} - \bar{\nabla}_a\bar{\nabla}_b\bar{R} - \frac{1}{2}\bar{g}_{ab}\bar{R}_{cd}\bar{R}^{cd} + \bar{\nabla}_c\bar{\nabla}^c\bar{R}_{ab} + \frac{1}{2}\bar{\nabla}_c\bar{\nabla}^c\bar{R}$. Then the second term on right hand side must be equal to $\frac{1}{2}OK-2 = \frac{1}{4}\bar{\xi}_b h\bar{\nabla}^a\bar{\nabla}^b\bar{R}$

$$\frac{1}{4}h(\bar{\nabla}^b\bar{R})(\bar{\nabla}_b\bar{\xi}^a) = -\frac{1}{4}\bar{\nabla}^a\left(\bar{\xi}_b h\bar{\nabla}^b\bar{R}\right) + \frac{1}{4}h\bar{\xi}_b\bar{\nabla}^a\bar{\nabla}^b\bar{R} + \frac{1}{4}\bar{\xi}_b h(\bar{\nabla}^b\bar{R})(\bar{\nabla}^a h). \tag{C.18}$$

First two terms are zero by the identity (A.14), so we have expected result. The trace part of the β charge can be written as

$$\begin{aligned}\frac{1}{2}\bar{\nabla}_b\left(\sum_{i=1}^{6} F^{ab}_{\text{i}}\right) =&\bar{\nabla}_b\Big[h\bar{\xi}_c\bar{\nabla}^{[b}R^{a]c} + \bar{\xi}^{[a}R^{b]c}\bar{\nabla}_c h + hR^{c[a}\bar{\nabla}_c\bar{\xi}^{b]} \\ &+ \bar{\xi}^{[b}\bar{\nabla}^{a]}\bar{\nabla}_c\bar{\nabla}^c h + \bar{\xi}^c\bar{\nabla}^{[a}\bar{\nabla}^{b]}\bar{\nabla}_c h + \bar{\nabla}_c\bar{\xi}^{[a}\bar{\nabla}^{b]}\bar{\nabla}^c h\Big].\end{aligned} \tag{C.19}$$

Now let us proceed to the remaining terms with 4 derivatives. Start with A

$$A = \frac{1}{2}\,\bar{\xi}_d\bar{\nabla}^b\bar{\nabla}_b\bar{\nabla}^c\bar{\nabla}^a h_c{}^d \;= \frac{1}{2}\,\bar{\xi}_d(\bar{\nabla}^c\bar{\nabla}_c\bar{\nabla}^b\bar{\nabla}^a h_b{}^d \;- \bar{\nabla}^b\bar{\nabla}^a\bar{\nabla}^c\bar{\nabla}_c h_b{}^d) + \frac{1}{2}\,\bar{\xi}_d\bar{\nabla}^c\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}_b h_c{}^d\;,$$

where we add and subtract term above. As for B

$$\begin{aligned}B = \frac{1}{2}\,\bar{\nabla}^b\bar{\nabla}_b\bar{\nabla}^c\bar{\nabla}_c h^{ad}\;\bar{\xi}_d &= -\frac{1}{2}\,\bar{\nabla}_c\left(\bar{\xi}_d\bar{\nabla}^c\bar{\nabla}^b\bar{\nabla}_b h^{ad}\right) + \frac{1}{2}\,(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^c\bar{\nabla}^b\bar{\nabla}_b h^{ad} \\ &= \frac{1}{2}\,\bar{\nabla}_c\left(\bar{\xi}_d\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}_b h^{cd} - \bar{\xi}_d\bar{\nabla}^c\bar{\nabla}^b\bar{\nabla}_b h^{ad}\right) \\ &- \frac{1}{2}\,\bar{\nabla}_c\left(\bar{\xi}_d\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}_b h^{cd}\right) + \frac{1}{2}\,(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^c\bar{\nabla}^b\bar{\nabla}_b h^{ad}.\end{aligned} \tag{C.20}$$

First two terms of equation (C.20) can be written as $Q^{ac}_{1} \equiv 2\bar{\xi}_d\bar{\nabla}^{[a}\bar{\nabla}_c\bar{\nabla}^c h^{b]d}$. The sum $A+B$ leads to

$$A + B = \frac{1}{2}\bar{\nabla}_c Q^{ac}_{1} + \frac{1}{2}(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^c\bar{\nabla}_b\bar{\nabla}^b h^{ad} + \frac{1}{2}\bar{\xi}^d(\bar{\nabla}_c\bar{\nabla}^c\bar{\nabla}^b\bar{\nabla}^a - \bar{\nabla}^b\bar{\nabla}^a\bar{\nabla}_c\bar{\nabla}^c)h_{bd}. \tag{C.21}$$

Similarly C gives the following

$$\begin{aligned}C = \frac{1}{2}\,\bar{\xi}_d\bar{\nabla}^b\bar{\nabla}_b\bar{\nabla}^c\bar{\nabla}^d h^a{}_c &= \frac{1}{2}\bar{\nabla}_c\left(\bar{\xi}_d\bar{\nabla}^c\bar{\nabla}_b\bar{\nabla}^d h^{ab}\right) - \frac{1}{2}(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^c\bar{\nabla}_b\bar{\nabla}^d h^{ab} \\ &= \frac{1}{2}\bar{\nabla}_c\left(\bar{\xi}_d\bar{\nabla}^c\bar{\nabla}_b\bar{\nabla}^d h^{ab} - \bar{\xi}_d\bar{\nabla}^a\bar{\nabla}_b\bar{\nabla}^d h^{cb}\right) \\ &+ \frac{1}{2}\bar{\nabla}_c\left(\bar{\xi}_d\bar{\nabla}^a\bar{\nabla}_b\bar{\nabla}^d h^{cb}\right) - \frac{1}{2}(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^c\bar{\nabla}_b\bar{\nabla}^d h^{ab} \\ &= \frac{1}{2}\bar{\nabla}_c Q^{ac}_{2} + \frac{1}{2}\bar{\nabla}_c\left(\bar{\xi}_d\bar{\nabla}^a\bar{\nabla}_b\bar{\nabla}^d h^{cb}\right) - \frac{1}{2}(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^c\bar{\nabla}_b\bar{\nabla}^d h^{ab},\end{aligned} \tag{C.22}$$

where $Q_2^{ac} \equiv 2\bar{\xi}_d \bar{\nabla}^{[c}\bar{\nabla}_b\bar{\nabla}^d h^{a]b}$. Take the half of D

$$\frac{1}{2}D = -\frac{1}{2}\bar{\xi}_b\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^c\bar{\nabla}^d h_{cd} = -\frac{1}{2}\bar{\xi}_d\bar{\nabla}^a\bar{\nabla}^d\bar{\nabla}^c\bar{\nabla}^b h_{cb} = -\frac{1}{2}\bar{\xi}_d\bar{\nabla}^d\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^c h_{bc}$$
$$= \frac{1}{2}\bar{\xi}_d(\bar{\nabla}^c\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^d h_{bc} - \bar{\nabla}^d\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^c h_{bc}) - \frac{1}{2}\bar{\xi}_d\bar{\nabla}^c\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^d h_{bc}, \tag{C.23}$$

and the sum $\mathrm{C} + \frac{1}{2}\,\mathrm{D}$

$$\mathrm{C} + \tfrac{1}{2}\,\mathrm{D} = \frac{1}{2}\bar{\nabla}_c Q_2^{ac} + \frac{1}{2}(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^a\bar{\nabla}_b\bar{\nabla}^d h^{cb} - \frac{1}{2}(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^c\bar{\nabla}_b\bar{\nabla}^d h^{ab}$$
$$+ \frac{1}{2}\bar{\xi}_d(\bar{\nabla}^c\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^d h_{bc} - \bar{\nabla}^d\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^c h_{bc}). \tag{C.24}$$

Summing up the other half of D, with E

$$\mathrm{E} + \tfrac{1}{2}\,\mathrm{D} = \frac{1}{2}\bar{\nabla}_c\left(\bar{\xi}^a\bar{\nabla}^c\bar{\nabla}^b\bar{\nabla}^d h_{bd} - \bar{\xi}^a\bar{\nabla}^c\bar{\nabla}^b\bar{\nabla}^d h_{bd}\right) - \frac{1}{2}(\bar{\nabla}_c\bar{\xi}^a)\bar{\nabla}^c\bar{\nabla}^b\bar{\nabla}^d h_{bd}$$
$$= \frac{1}{2}\bar{\nabla}_c Q_3^{ac} - \frac{1}{2}(\bar{\nabla}_c\bar{\xi}^a)\bar{\nabla}^c\bar{\nabla}^b\bar{\nabla}^d h_{bd}, \tag{C.25}$$

where $Q_3^{ac} \equiv 2\bar{\xi}^{[a}\bar{\nabla}^{c]}\bar{\nabla}^b\bar{\nabla}^d h_{db}$. Turn back to equation (C.21) and write it as

$$A + B = \frac{1}{2}\bar{\nabla}_c Q_1^{ac} + \frac{1}{2}(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^c\bar{\nabla}_b\bar{\nabla}^b h^{ad} + \frac{1}{2}\bar{\xi}^d(\bar{\nabla}_c\bar{\nabla}^c\bar{\nabla}^b\bar{\nabla}^a - \bar{\nabla}_c\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^c)h_{bd}$$
$$+ \frac{1}{2}\bar{\xi}^d(\bar{\nabla}_c\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^c - \bar{\nabla}^b\bar{\nabla}^a\bar{\nabla}_c\bar{\nabla}^c)h_{bd}$$
$$= \frac{1}{2}\bar{\nabla}_c Q_1^{ac} + \frac{1}{2}\bar{\nabla}_c\left(\bar{\xi}^d\bar{\nabla}^c\bar{\nabla}^b\bar{\nabla}^a h_{bd} - \bar{\xi}^d\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^c h_{bd}\right) + \frac{1}{2}(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^c\bar{\nabla}_b\bar{\nabla}^b h^{ad}$$
$$- \frac{1}{2}(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^c\bar{\nabla}_b\bar{\nabla}^a h^{bd} + \frac{1}{2}(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^a\bar{\nabla}_b\bar{\nabla}^c h^{bd} + \frac{1}{2}\bar{\xi}^d\left(\bar{\nabla}_c\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^c - \bar{\nabla}^b\bar{\nabla}^a\bar{\nabla}_c\bar{\nabla}^c\right)h_{bd}$$
$$= \frac{1}{2}\bar{\nabla}_c Q_1^{ac} + \frac{1}{2}\bar{\nabla}_c Q_4^{ac} + \frac{1}{2}(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^c\bar{\nabla}_b\bar{\nabla}^b h^{ad}$$
$$- \frac{1}{2}(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^c\bar{\nabla}_b\bar{\nabla}^a h^{bd} + \frac{1}{2}(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^a\bar{\nabla}_b\bar{\nabla}^c h^{bd} + \frac{1}{2}\bar{\xi}^d\left(\bar{\nabla}_c\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^c - \bar{\nabla}^b\bar{\nabla}^a\bar{\nabla}_c\bar{\nabla}^c\right)h_{bd}, \tag{C.26}$$

where $Q_4^{ac} \equiv 2\bar{\xi}^d\bar{\nabla}^{[c}\bar{\nabla}^b\bar{\nabla}^{a]}h_{db}$. As a result the sum of (C.24), (C.25), (C.26) will be

$$A + B + C + D + E = \frac{1}{2}\bar{\nabla}_c\left(\sum_{i=1}^{4} Q_i^{ac}\right) + \underbrace{\frac{1}{2}\bar{\xi}^d\left(\bar{\nabla}_c\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^c - \bar{\nabla}^b\bar{\nabla}^a\bar{\nabla}_c\bar{\nabla}^c\right)h_{bd}}_{J}$$
$$+ \underbrace{\frac{1}{2}\bar{\xi}_d(\bar{\nabla}^c\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^d h_{bc} - \bar{\nabla}^d\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^c h_{bc})}_{K} + \underbrace{\frac{1}{2}(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^c\bar{\nabla}_b\bar{\nabla}^b h^{ad}}_{F}$$
$$\underbrace{- \frac{1}{2}(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^c\bar{\nabla}_b\bar{\nabla}^a h^{bd}}_{H} \underbrace{- \frac{1}{2}(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^c\bar{\nabla}_b\bar{\nabla}^d h^{ab}}_{G} \underbrace{- \frac{1}{2}(\bar{\nabla}_c\bar{\xi}^a)\bar{\nabla}^c\bar{\nabla}_b\bar{\nabla}_d h^{bd}}_{I}. \tag{C.27}$$

From equation (C.27), F can be written as

$$F = \frac{1}{2}(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^c\bar{\nabla}_b\bar{\nabla}^b h^{ad} = \frac{1}{2}\bar{\nabla}_c\left((\bar{\nabla}^c\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^b h^{ad}\right) + \frac{1}{2}\bar{R}_{bc}\bar{\xi}^c\bar{\nabla}_d\bar{\nabla}^d h^{ab}. \tag{C.28}$$

The last term of F cancels with 1 from the linearized equations. By adding and subtracting the antisymmetric term the sum $F+1$ reads

$$\begin{aligned} F+1 &= \frac{1}{2}\bar{\nabla}_c\left((\bar{\nabla}^c\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^b h^{ad} - (\bar{\nabla}^a\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^b h^{cd}\right) + \frac{1}{2}\bar{\nabla}_c\left((\bar{\nabla}^a\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^b h^{cd}\right) \\ &= \frac{1}{2}\bar{\nabla}_c Q_5^{ac} + \frac{1}{2}\bar{\nabla}_c\left((\bar{\nabla}^a\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^b h^{cd}\right), \end{aligned} \tag{C.29}$$

where $Q_5^{ac} \equiv 2(\bar{\nabla}^{[c}\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^b h^{a]d}$. Integrating by parts G amounts to

$$G = -\frac{1}{2}(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^c\bar{\nabla}_b\bar{\nabla}^d h^{ab} = -\frac{1}{2}\bar{\nabla}_c\left((\bar{\nabla}^c\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^d h^{ab}\right) - \frac{1}{2}\bar{R}^{bc}\bar{\nabla}^d\bar{\nabla}_b h^a{}_d\bar{\xi}_c. \tag{C.30}$$

Similarly $G+2$ gives the following

$$\begin{aligned} G+2 &= \frac{1}{2}\bar{\nabla}_c\left((\bar{\nabla}^a\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^d h^{bc} - (\bar{\nabla}^c\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^d h^{ba}\right) - \frac{1}{2}\bar{\nabla}_c\left((\bar{\nabla}^a\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^d h^{cb}\right) \\ &= \frac{1}{2}\bar{\nabla}_c Q_6^{ac} - \frac{1}{2}\bar{\nabla}_c\left((\bar{\nabla}^a\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^d h^{cb}\right), \end{aligned} \tag{C.31}$$

where $Q_6^{ac} \equiv 2(\bar{\nabla}^{[a}\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^d h^{bc]}$. As for H

$$\begin{aligned} H &= -\frac{1}{2}(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^c\bar{\nabla}_b\bar{\nabla}^a h^{bd} \\ &= -\frac{1}{2}\bar{\nabla}_c\left((\bar{\nabla}^c\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^a h^{bd}\right) - \frac{1}{2}\bar{R}_{bc}\bar{\xi}^c\bar{\nabla}_d\bar{\nabla}^a h^{bd} \\ &= \frac{1}{2}\bar{\nabla}_c\left((\bar{\nabla}^a\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^c h^{bd} - (\bar{\nabla}^c\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^a h^{bd}\right) \\ &\quad - \frac{1}{2}\bar{\nabla}_c\left((\bar{\nabla}^a\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^c h^{bd}\right) - \frac{1}{2}\bar{R}_{bc}\bar{\xi}^c\bar{\nabla}_d\bar{\nabla}^a h^{bd} \\ &= \frac{1}{2}\bar{\nabla}_c Q_7^{ac} - \frac{1}{2}\bar{\nabla}_c\left((\bar{\nabla}^a\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^c h^{bd}\right) - \frac{1}{2}\bar{R}_{bc}\bar{\xi}^c\bar{\nabla}_d\bar{\nabla}^a h^{bd}. \end{aligned} \tag{C.32}$$

Using (C.29), (C.31), (C.32) the sum $F+G+H+J+I+1+2$ reads

$$\begin{aligned} F+G+H+J+I+1+2 = \frac{1}{2}\bar{\nabla}_c\left(\sum_{i=5}^{7} Q_i^{ac}\right) &+ \underbrace{\frac{1}{2}\bar{\nabla}_c\left((\bar{\nabla}^a\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^b h^{cd}\right)}_{a} \underbrace{- \frac{1}{2}\bar{\nabla}_c\left((\bar{\nabla}^a\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^d h^{cb}\right)}_{b} \\ &\underbrace{- \frac{1}{2}\bar{\nabla}_c\left((\bar{\nabla}^a\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^c h^{bd}\right)}_{c} - \frac{1}{2}\bar{R}_{bc}\bar{\xi}^c\bar{\nabla}_d\bar{\nabla}^a h^{bd} \underbrace{- \frac{1}{2}(\bar{\nabla}_c\bar{\xi}^a)\bar{\nabla}^c\bar{\nabla}_b\bar{\nabla}_d h^{bd}}_{I}. \end{aligned} \tag{C.33}$$

Let us expand the derivatives in a, b c using the identity (A.13)

$$\begin{aligned} a &= \frac{1}{2}(\bar{\nabla}_c\bar{\nabla}^a\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^b h^{cd} + \frac{1}{2}(\bar{\nabla}^a\bar{\xi}_d)\,\bar{\nabla}_c\bar{\nabla}_b\bar{\nabla}^b h^{cd} = -\frac{1}{2}\bar{R}^{abcd}\bar{\xi}_d\bar{\nabla}_e\bar{\nabla}^e h_{bc} + \frac{1}{2}(\bar{\nabla}^a\bar{\xi}_d)\,\bar{\nabla}_c\bar{\nabla}_b\bar{\nabla}^b h^{cd}, \\ b &= -\frac{1}{2}(\bar{\nabla}_c\bar{\nabla}^a\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^d h^{cb} - \frac{1}{2}(\bar{\nabla}^a\bar{\xi}_d)\,\bar{\nabla}_c\bar{\nabla}_b\bar{\nabla}^d h^{cb} = \frac{1}{2}\bar{R}^{abcd}\bar{\xi}_d\bar{\nabla}_e\bar{\nabla}_b h^e{}_c - \frac{1}{2}(\bar{\nabla}^a\bar{\xi}_d)\,\bar{\nabla}_c\bar{\nabla}_b\bar{\nabla}^d h^{cb}, \\ c &= -\frac{1}{2}(\bar{\nabla}_c\bar{\nabla}^a\bar{\xi}_d)\bar{\nabla}_b\bar{\nabla}^c h^{db} - \frac{1}{2}(\bar{\nabla}^a\bar{\xi}_d)\,\bar{\nabla}_c\bar{\nabla}_b\bar{\nabla}^c h^{db} = \frac{1}{2}\bar{R}^{abcd}\bar{\xi}_d\bar{\nabla}_e\bar{\nabla}_c h^e{}_b - \frac{1}{2}(\bar{\nabla}^a\bar{\xi}_d)\,\bar{\nabla}_c\bar{\nabla}_b\bar{\nabla}^c h^{db}. \end{aligned}$$

inserting these in (C.33)

$$F+G+H+J+I+1+2=\frac{1}{2}\bar{\nabla}_c\left(\sum_{i=5}^{7}Q_i^{ac}\right)+\frac{1}{2}\bar{R}^{abcd}\bar{\xi}_d(\underbrace{\bar{\nabla}_e\bar{\nabla}_c h^e{}_b}_{ii}+\underbrace{\bar{\nabla}_e\bar{\nabla}_b h^e{}_c}_{iii}-\underbrace{\bar{\nabla}_e\bar{\nabla}^e h_{cb}}_{iv})$$
$$+\frac{1}{2}(\bar{\nabla}^a\bar{\xi}_d)\left([\bar{\nabla}^d,\bar{\nabla}^b]\bar{\nabla}^c+\bar{\nabla}^b[\bar{\nabla}^d,\bar{\nabla}^c]\right)h_{bc}$$
$$+\frac{1}{2}(\bar{\nabla}^a\bar{\xi}_d)\left([\bar{\nabla}_b,\bar{\nabla}_c]\bar{\nabla}^c h^{bd}\right)-\frac{1}{2}\bar{R}_{bc}\bar{\xi}^c\bar{\nabla}_d\bar{\nabla}^a h^{bd}. \tag{C.34}$$

Notice the last term in (C.34) is the same as 3 in linearized equations. Let us proceed with the term J in equation (C.27)

$$J=\frac{1}{2}\bar{\xi}^d\left(\bar{\nabla}_c\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^c-\bar{\nabla}^b\bar{\nabla}^a\bar{\nabla}_c\bar{\nabla}^c\right)h_{bd}=\frac{1}{2}\bar{\xi}^d[\bar{\nabla}_c,\bar{\nabla}^a]\bar{\nabla}^b\bar{\nabla}^c h_{bd}+\frac{1}{2}\bar{\xi}^d\bar{\nabla}^a\bar{\nabla}_c\bar{\nabla}^b\bar{\nabla}^c h_{bd}$$
$$+\frac{1}{2}\bar{\xi}^d[\bar{\nabla}^a,\bar{\nabla}^b]\bar{\nabla}_c\bar{\nabla}^c h_{bd}-\frac{1}{2}\bar{\xi}^d\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}_c\bar{\nabla}^c h_{bd}$$
$$=\frac{1}{2}\bar{R}^{ab}\bar{\xi}^d\bar{\nabla}^c\bar{\nabla}_b h_{cd}+\underbrace{\frac{1}{2}\bar{R}^{abcd}\bar{\xi}_d\bar{\nabla}^e\bar{\nabla}_b h_{ec}}_{iii}-\frac{1}{2}\bar{R}^{ab}\bar{\xi}^d\bar{\nabla}^c\bar{\nabla}_c h_{bd}$$
$$-\underbrace{\frac{1}{2}\bar{R}^{abcd}\bar{\xi}_d\bar{\nabla}^e\bar{\nabla}_e h_{bc}}_{iv}+\frac{1}{2}\bar{\xi}^d\bar{\nabla}^a\left([\bar{\nabla}_c,\bar{\nabla}^b]\bar{\nabla}^c h_{bd}\right). \tag{C.35}$$

In (C.35) the first and the third terms are realized 4 and 5 from the linearized equations. Also the second and the fourth are same as the terms in (C.34). As for K

$$K=\frac{1}{2}\bar{\xi}_d(\bar{\nabla}^c\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^d h_{bc}-\bar{\nabla}^d\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^c h_{bc})=\frac{1}{2}\bar{\xi}_d[\bar{\nabla}^c,\bar{\nabla}^a]\bar{\nabla}^b\bar{\nabla}^d h_{bc}+\frac{1}{2}\bar{\xi}_d\bar{\nabla}^a\bar{\nabla}^b\bar{\nabla}^c\bar{\nabla}^d h_{bc}$$
$$-\frac{1}{2}\bar{\xi}_d\bar{\nabla}^a\bar{\nabla}^d\bar{\nabla}^b\bar{\nabla}^c h_{bc}$$
$$=\underbrace{\frac{1}{2}\bar{R}^{abcd}\bar{\xi}_d\bar{\nabla}^e\bar{\nabla}_c h_{eb}}_{ii}+\frac{1}{2}\bar{R}^{ab}\bar{\xi}_d\bar{\nabla}^c\bar{\nabla}^d h_{bc}$$
$$+\frac{1}{2}\bar{\xi}_d\bar{\nabla}^a\left([\bar{\nabla}^b,\bar{\nabla}^d]\bar{\nabla}^c+\bar{\nabla}^b[\bar{\nabla}^c,\bar{\nabla}^d]\right)h_{bc}, \tag{C.36}$$

where the second term cancels with 6 in the linearized equations. Using (C.36), (C.35), (C.34) and (C.27) the sum $A + B + C + D + E + \sum_{i=1}^{6} i$ leads to

$$\begin{aligned} A + B + C + D + E + \sum_{i=1}^{6} i = \frac{1}{2}\bar{\nabla}_c\left(\sum_{i=1}^{7} Q_i^{ac}\right) + \bar{R}^{abcd}\bar{\xi}_d(\bar{\nabla}_e\bar{\nabla}_c h^e{}_b + \bar{\nabla}_e\bar{\nabla}_b h^e{}_c - \bar{\nabla}_e\bar{\nabla}^e h_{cb}) \\ - \underbrace{\bar{R}_{bc}\bar{\xi}^c\bar{\nabla}_d\bar{\nabla}^a h^{bd}}_{e} + \underbrace{\bar{R}^{ba}\bar{\xi}^d\bar{\nabla}^c\bar{\nabla}_b h_{cd}}_{g} - \underbrace{\bar{R}^{ba}\bar{\xi}^d\bar{\nabla}_c\bar{\nabla}^c h_{bd}}_{f} \\ + \underbrace{\frac{1}{2}(\bar{\nabla}^a\bar{\xi}_d)\left([\bar{\nabla}_b, \bar{\nabla}_c]\bar{\nabla}^c h^{bd}\right) - \frac{1}{2}\bar{\xi}^d\bar{\nabla}^a\left([\bar{\nabla}^b, \bar{\nabla}_c]\bar{\nabla}^c h_{bd}\right)}_{d} \\ \underbrace{+\frac{1}{2}(\bar{\nabla}^a\bar{\xi}_d)\left([\bar{\nabla}^d, \bar{\nabla}^b]\bar{\nabla}^c + \bar{\nabla}^b[\bar{\nabla}^d, \bar{\nabla}^c]\right)h_{bc}}_{d} \\ \underbrace{-\frac{1}{2}\bar{\xi}_d\bar{\nabla}^a\left([\bar{\nabla}^d, \bar{\nabla}^b]\bar{\nabla}^c + \bar{\nabla}^b[\bar{\nabla}^d, \bar{\nabla}^c]\right)h_{bc}}_{d}. \end{aligned} \tag{C.37}$$

From the linearized equations again one can realize the second term above, cancels with the sum $7+8+9$. In order to simplify the calculation define $X^d \equiv [\bar{\nabla}_b, \bar{\nabla}_c]\bar{\nabla}^c h^{bd} + [\bar{\nabla}^d, \bar{\nabla}^b]\bar{\nabla}^c h_{bc} + \bar{\nabla}^b[\bar{\nabla}^d, \bar{\nabla}^c]h_{bc}$ then, $d = \frac{1}{2}(\bar{\nabla}^a\bar{\xi}_d)X^d - \frac{1}{2}\bar{\xi}_d\bar{\nabla}^a X^d$. Expanding the commutators and using the contracted Bianchi identity X^d reads

$$X^d = h_{bc}(\bar{\nabla}^d\bar{R}^{bc} - \bar{\nabla}^c\bar{R}^{bd} - \bar{\nabla}^b\bar{R}^{dc}) - 2\bar{R}^{bd}\bar{\nabla}^c h_{bc}. \tag{C.38}$$

Using $10,\ 11,\ 12$ from linearized equations, $e,\ f,\ g$ in (C.37) reads

$$\begin{aligned} e + 10 &= -\bar{\xi}^d\bar{\nabla}_c\left(\bar{R}_{bd}\bar{\nabla}^a h^{bc}\right), \\ f + 11 &= -\bar{\xi}^d\bar{\nabla}_c\left(\bar{R}^{ab}\bar{\nabla}^c h_{bd}\right), \\ g + 12 &= \bar{\xi}_d\bar{\nabla}_c\left(\bar{R}^{ab}\bar{\nabla}_b h^{dc}\right). \end{aligned} \tag{C.39}$$

Then equations (C.38) and (C.39), (C.37) amounts to

$$\begin{aligned} A + B + C + D + E + \sum_{i=1}^{12} i = \frac{1}{2}\bar{\nabla}_c\left(\sum_{i=1}^{7} Q_i^{ac}\right) + \bar{\xi}^d\bar{\nabla}_c\left(\underbrace{\bar{R}^{ab}\bar{\nabla}_b h_d{}^c}_{j} - \underbrace{\bar{R}^{ab}\bar{\nabla}^c h_{bd}}_{h} - \underbrace{\bar{R}_{bd}\bar{\nabla}^a h^{bc}}_{l}\right) \\ + \frac{1}{2}(\bar{\nabla}^a\bar{\xi}_c)\Big[h_{bd}(\bar{\nabla}^c\bar{R}^{bd} - \underbrace{\bar{\nabla}^d\bar{R}^{bc} - \bar{\nabla}^b\bar{R}^{dc}}_{n}) - \underbrace{\bar{R}^{bc}\bar{\nabla}^d h_{bd} - \bar{R}^{cd}\bar{\nabla}^b h_{bd}}_{k}\Big] \\ - \frac{1}{2}\bar{\xi}_c\bar{\nabla}^a\Big[h_{bd}(\bar{\nabla}^c\bar{R}^{bd} - \underbrace{\bar{\nabla}^d\bar{R}^{bc} - \bar{\nabla}^b\bar{R}^{dc}}_{p}) - \underbrace{\bar{R}^{bc}\bar{\nabla}^d h_{bd} - \bar{R}^{cd}\bar{\nabla}^b h_{bd}}_{m}\Big]. \end{aligned} \tag{C.40}$$

One can generate charge from j and write the remaining using 13, 14 from the linearized equations

$$\bar{\xi}_d \bar{\nabla}_c \left(\bar{R}^{ab} \bar{\nabla}_b h^{dc} \right) = \bar{\nabla}_c \left(\bar{\xi}_d \bar{R}^{ab} \bar{\nabla}_b h^{cd} - \bar{\xi}_d \bar{R}^{bc} \bar{\nabla}_b h^{ad} \right) + (\bar{\nabla}_c \bar{\xi}_d) \bar{R}^{bc} \bar{\nabla}_b h^{ad}$$
$$+ \bar{\xi}_d \left(\frac{1}{2} \bar{\nabla}^b \bar{R} \right) \bar{\nabla}_b h^{ad} + \bar{\xi}_d \bar{R}^{bc} \bar{\nabla}_c \bar{\nabla}_b h^{ad},$$
$$j + 13 + 14 = \bar{\nabla}_c Q_8^{ac} + (\bar{\nabla}_c \bar{\xi}_d) \bar{R}^{bc} \bar{\nabla}_b h^{ad}, \tag{C.41}$$

where $Q_8^{ac} \equiv 2\bar{\xi}_d \bar{R}^{b[a} \bar{\nabla}_b h^{c]d}$. In a similar fashion h with 15, 16

$$h + 15 + 16 = \bar{\nabla}_c Q_9^{ac} + (\bar{\nabla}_c \bar{\xi}^d)(\bar{R}^{ab} \bar{\nabla}^c h_{bd} - \bar{R}^{bc} \bar{\nabla}^a h_{bd}), \tag{C.42}$$

where $Q_9^{ac} \equiv 2\bar{\xi}^d \bar{R}^{b[c} \bar{\nabla}^{a]} h_{bd}$.

Then from the symmetry there must be a charge that has $\bar{\xi}^d \bar{R}^{bc} \bar{\nabla}_d h^a{}_b$ term in it. Such a term can be generated as follows: Write $17 = -\bar{R}^{bc}(\bar{\nabla}_b \bar{\nabla}^d h^a{}_c)\bar{\xi}_d$ as

$$17 = -2\bar{R}^{bc} \bar{\xi}^d (\bar{\nabla}_b \bar{\nabla}_d h^a{}_c) + \bar{R}^{bc} \bar{\xi}^d (\bar{\nabla}_b \bar{\nabla}_d h^a{}_c),$$

and note that $18 = 2\bar{R}^{bc} \bar{\xi}^d (\bar{\nabla}_d \bar{\nabla}_b h^a{}_c)$. Then the sum $17 + 18$ yields

$$17 + 18 = 2\bar{R}^{bc} \bar{\xi}^d [\bar{\nabla}_d, \bar{\nabla}_b] h^a{}_c + \bar{R}^{bc} \bar{\xi}^d (\bar{\nabla}_b \bar{\nabla}_d h^a{}_c). \tag{C.43}$$

Moreover adding $19 = \frac{1}{2}(\bar{\nabla}^b \bar{R})(\bar{\nabla}^d h^a{}_b)\bar{\xi}_d$ to (C.43) amounts to

$$17 + 18 + 19 = 2\bar{R}^{bc} \bar{\xi}^d [\bar{\nabla}_d, \bar{\nabla}_b] h^a{}_c + \bar{\xi}^d \bar{\nabla}_c \left(\bar{R}^{bc} \bar{\nabla}_d h^a{}_b \right)$$
$$= 2\bar{R}^{bc} \bar{\xi}^d [\bar{\nabla}_d, \bar{\nabla}_b] h^a{}_c + \bar{\nabla}_c \left(\bar{\xi}^d \bar{R}^{bc} \bar{\nabla}_d h^a{}_b - \bar{\xi}^d \bar{R}^{ba} \bar{\nabla}_d h^c{}_b \right)$$
$$- (\bar{\nabla}_c \bar{\xi}_d) \bar{R}^{bc} \bar{\nabla}^d h^a{}_b + \bar{\nabla}_c \left(\bar{\xi}^d \bar{R}^{ba} \bar{\nabla}_d h^c{}_b \right)$$
$$= \bar{\nabla}_c Q_{10}^{ac} + 2\bar{R}^{bc} \bar{\xi}^d [\bar{\nabla}_d, \bar{\nabla}_b] h^a{}_c - (\bar{\nabla}_c \bar{\xi}_d) \bar{R}^{bc} \bar{\nabla}^d h^a{}_b$$
$$+ (\bar{\nabla}_c \bar{\xi}_d) \bar{R}^{ab} \bar{\nabla}^d h_b{}^c + \bar{\xi}_d (\bar{\nabla}_c \bar{R}^{ab}) \bar{\nabla}_d h_b{}^c + \bar{\xi}_d \bar{R}^{ab} \bar{\nabla}_c \bar{\nabla}^d h_b{}^c, \tag{C.44}$$

where $Q_{10}^{ac} \equiv 2\bar{\xi}_d \bar{R}^{b[c} \bar{\nabla}^d h^{a]}{}_b$. Likewise add 20 to (C.44)

$$17 + 18 + 19 + 20 = \bar{\nabla}_c Q_{10}^{ac} - (\bar{\nabla}_c \bar{\xi}_d)\left[\bar{R}^{ab} \bar{\nabla}^c h_b{}^d + \bar{R}^{bc} \bar{\nabla}^d h^a{}_b \right] + 2\bar{R}^{bc} \bar{\xi}^d [\bar{\nabla}_d, \bar{\nabla}_b] h^a{}_c$$
$$+ \bar{\xi}_d \bar{R}^{ab} [\bar{\nabla}_c, \bar{\nabla}^d] h_b{}^c + \bar{\xi}_c \bar{R}^{ab} \bar{\nabla}^c \bar{\nabla}_d h_b{}^d. \tag{C.45}$$

The sum (C.41), (C.42), (C.45) can be written more compactly as

$$j + h + \sum_{i=13}^{20} i = \bar{\nabla}_c \left(\sum_{i=8}^{10} Q_i^{ac} \right) + (\bar{\nabla}_c \bar{\xi}_d)\left[\bar{R}^{bc} \bar{\nabla}_b h^{ad} - \bar{R}^{ab} \bar{\nabla}^c h_b{}^d - \bar{R}^{bc} \bar{\nabla}^d h^a{}_b \right]$$
$$+2\bar{R}^{bc} \bar{\xi}^d [\bar{\nabla}_d, \bar{\nabla}_b] h^a{}_c + \bar{\xi}_d \bar{R}^{ab} [\bar{\nabla}_c, \bar{\nabla}^d] h_b{}^c + \underbrace{\bar{\xi}_c \bar{R}^{ab} \bar{\nabla}^c \bar{\nabla}_d h_b{}^d}_{\pi}. \tag{C.46}$$

Turning back to (C.40) and take $k = (\bar{\nabla}_c \bar{\xi}^a)\bar{R}^{bc}\bar{\nabla}^d h_{bd}$, which can be put inside the charge as $Q_{11}^{ac} = 2\bar{\xi}^{[a}\bar{R}^{c]b}\bar{\nabla}^d h_{bd}$. Derivative of this;

$$\begin{aligned}\bar{\nabla}_c Q_{11}^{ac} &= (\bar{\nabla}_c \bar{\xi}^a)\bar{R}^{bc}\bar{\nabla}^d h_{bd} + \bar{\xi}^a(\frac{1}{2}\bar{\nabla}^b \bar{R})\bar{\nabla}^d h_{bd} + \bar{\xi}^a \bar{R}^{bc}\bar{\nabla}_c \bar{\nabla}^d h_{bd} \\ &- (\bar{\xi}^c \bar{\nabla}_c \bar{R}^{ab})\bar{\nabla}^d h_{bd} - \bar{\xi}^c \bar{R}^{ab}\bar{\nabla}_c \bar{\nabla}^d h_{bd},\end{aligned} \tag{C.47}$$

using the identity (A.13) on fourth term in (C.47) yields

$$\begin{aligned}\bar{\nabla}_c Q_{11}^{ac} &= (\bar{\nabla}_c \bar{\xi}^a)\bar{R}^{bc}\bar{\nabla}^d h_{bd} + \bar{\xi}^a(\frac{1}{2}\bar{\nabla}^b \bar{R})\bar{\nabla}^d h_{bd} + \bar{\nabla}_c \bar{\nabla}^d h_{bd}(\bar{\xi}^a \bar{R}^{bc} - \bar{\xi}^c \bar{R}^{ab}) \\ &- (\bar{R}^{ac}\bar{\nabla}_c \bar{\xi}^b)\bar{\nabla}^d h_{bd} - (\bar{R}^{bc}\bar{\nabla}_c \bar{\xi}^a)\bar{\nabla}^d h_{bd} \\ &= \bar{\xi}^a(\frac{1}{2}\bar{\nabla}^b \bar{R})\bar{\nabla}^d h_{bd} + \bar{\nabla}_c \bar{\nabla}^d h_{bd}(\bar{\xi}^a \bar{R}^{bc} - \bar{\xi}^c \bar{R}^{ab}) - (\bar{R}^{ac}\bar{\nabla}_c \bar{\xi}^b)\bar{\nabla}^d h_{bd}.\end{aligned} \tag{C.48}$$

In order to generate the first two terms in (C.48), consider 21, 22, 23 from the linearized equations and π from (C.46). First rewrite 22 as

$$\begin{aligned}-\bar{\xi}^a \bar{R}^{bc}\bar{\nabla}^d \bar{\nabla}_b h_{dc} &= -\bar{\xi}^a \bar{R}^{bc}[\bar{\nabla}^d, \bar{\nabla}_b]h_{dc} - \bar{\xi}^a \bar{R}^{bc}\bar{\nabla}_c \bar{\nabla}^d h_{db} \\ &= -\bar{\xi}^a \bar{R}^{bc}\bar{\nabla}_c \bar{\nabla}^d h_{db} + \bar{\xi}^a h^{cd}\bar{R}_{ecfd}\bar{R}^{ef} - \bar{\xi}^a \bar{R}^{bc}\bar{R}_b{}^d h_{cd}.\end{aligned}$$

Then the sum 21 + 22 + 23 + π yields

$$\begin{aligned}21 + 22 + 23 + \pi &= -\bar{\xi}^a \frac{1}{2}(\bar{\nabla}^b \bar{R})\bar{\nabla}^d h_{bd} + \bar{\xi}^a h^{cd}\bar{R}_{ecfd}\bar{R}^{ef} - \bar{\xi}^a \bar{R}^{bc}\bar{\nabla}_c \bar{\nabla}^d h_{db} + \bar{\xi}_c \bar{R}^{ab}\bar{\nabla}^c \bar{\nabla}_d h_b{}^d \\ &= \bar{\xi}^a h^{cd}\bar{R}_{cedf}\bar{R}^{ef} - \bar{R}^{ac}(\bar{\nabla}_c \bar{\xi}^b)\bar{\nabla}^d h_{bd} + \bar{\nabla}_c Q_{11}^{ac}.\end{aligned} \tag{C.49}$$

Using (C.49), (C.46) can be written as

$$\begin{aligned}j + h + \sum_{i=13}^{23} i &= \bar{\nabla}_c \left(\sum_{i=8}^{11} Q_i^{ac}\right) + (\bar{\nabla}_c \bar{\xi}_d)\left[\bar{R}^{bc}\bar{\nabla}_b h^{ad} - \bar{R}^{ab}\bar{\nabla}^c h_b{}^d - \bar{R}^{bc}\bar{\nabla}^d h^a{}_b - \bar{R}^{ac}\bar{\nabla}_b h^{bd}\right] \\ &+ \underbrace{\bar{\xi}^a h^{cd}\bar{R}_{cedf}\bar{R}^{ef}}_{OK-1} + 2\bar{R}^{bc}\bar{\xi}^d[\bar{\nabla}_d, \bar{\nabla}_b]h^a{}_c + \bar{\xi}_d \bar{R}^{ab}[\bar{\nabla}_c, \bar{\nabla}^d]h_b{}^c.\end{aligned} \tag{C.50}$$

Next turn back to the equation (C.40) and examine the terms we didn't, namely m, n, p, l, k. Start with l

$$\begin{aligned}l = -\bar{\xi}^d \bar{\nabla}_c \left(\bar{R}_{bd}\bar{\nabla}^a h^{bc}\right) &= -\bar{\xi}^d(\bar{\nabla}_c \bar{R}_{bd})\bar{\nabla}^a h^{bc} - \bar{\xi}^d \bar{R}_{bd}\bar{\nabla}_c \bar{\nabla}^a h^{bc} \\ &= -\bar{\xi}^d(\bar{\nabla}_c \bar{R}_{bd})\bar{\nabla}^a h^{bc} - \bar{\xi}^d \bar{R}_{bd}[\bar{\nabla}_c, \bar{\nabla}^a]h^{bc} - \bar{\xi}^d \bar{R}_{bd}\bar{\nabla}^a \bar{\nabla}_c h^{bc} \\ &= -\bar{\xi}^d(\bar{\nabla}_c \bar{R}_{bd})\bar{\nabla}^a h^{bc} - \bar{\xi}_c \bar{R}^{bc}[\bar{\nabla}^d, \bar{\nabla}^a]h_{bd} - \bar{\nabla}^a \left(\bar{R}^{bc}\bar{\xi}_c \bar{\nabla}^d h_{bd}\right) \\ &+ \bar{\xi}_c(\bar{\nabla}^a \bar{R}^{bc})\bar{\nabla}^d h_{bd} + \bar{R}^{bc}(\bar{\nabla}^a \bar{\xi}_c)\bar{\nabla}^d h_{bd}.\end{aligned} \tag{C.51}$$

Likewise m reads

$$m = \bar{\xi}_c \bar{\nabla}^a \left(\bar{R}^{cd} \bar{\nabla}^b h_{bd}\right) = \bar{\nabla}^a (\bar{\xi}_c \bar{R}^{cd} \bar{\nabla}^b h_{bd}) - (\bar{\nabla}^a \bar{\xi}_c) \bar{R}^{bc} \bar{\nabla}^d h_{bd}. \tag{C.52}$$

The sum of (C.51),(C.52) is then

$$l + m = -\bar{\xi}^d (\bar{\nabla}_c \bar{R}_{bd}) \bar{\nabla}^a h^{bc} + \bar{\xi}_d \bar{R}^{bd} [\bar{\nabla}^a, \bar{\nabla}^c] h_{bc} + \bar{\xi}_c (\bar{\nabla}^a \bar{R}^{cd}) \bar{\nabla}^b h_{bd}. \tag{C.53}$$

Similarly k and n reads

$$\begin{aligned} k &= -(\bar{\nabla}^a \bar{\xi}_c) \bar{R}^{bc} \bar{\nabla}^d h_{bd} = \bar{\nabla}_c \left[(\bar{\nabla}_d \bar{\xi}^a) \bar{R}^{bd} h_b{}^c\right] - h_b{}^c (\bar{\nabla}_c \bar{\nabla}_d \bar{\xi}^a) \bar{R}^{bd} - h_b{}^c (\bar{\nabla}_d \bar{\xi}^a) \bar{\nabla}_c \bar{R}^{bd} \\ k + n &= \bar{\nabla}_c \left[(\bar{\nabla}_d \bar{\xi}^a) \bar{R}^{bd} h_b{}^c\right] - h_b{}^c (\bar{\nabla}_c \bar{\nabla}_d \bar{\xi}^a) \bar{R}^{bd}, \end{aligned} \tag{C.54}$$

whereas p leads to

$$p = \bar{\xi}_c \bar{\nabla}^a \left(h_{bd} \bar{\nabla}^d \bar{R}^{bc}\right) = \bar{\xi}_d (\bar{\nabla}^a h_{bc}) \bar{\nabla}^c \bar{R}^{bd} + \bar{\xi}_d h_{bc} \bar{\nabla}^a \bar{\nabla}^c \bar{R}^{bd}. \tag{C.55}$$

The sum (C.55),(C.53) reads

$$\begin{aligned} l + m + p &= \bar{\xi}_d \bar{R}^{bd} [\bar{\nabla}^a, \bar{\nabla}^c] h_{bc} + \bar{\xi}_c (\bar{\nabla}^a \bar{R}^{cd}) \bar{\nabla}^b h_{bd} + \bar{\xi}_d h_{bc} \bar{\nabla}^a \bar{\nabla}^c \bar{R}^{bd} \\ &= \bar{\xi}_d \bar{R}^{bd} [\bar{\nabla}^a, \bar{\nabla}^c] h_{bc} + h_{bc} \bar{\xi}_d [\bar{\nabla}^a, \bar{\nabla}^c] \bar{R}^{bd} + \bar{\xi}_d \bar{\nabla}^c \left(h_{bc} \bar{\nabla}^a \bar{R}^{bd}\right) \\ &= \bar{\xi}_d [\bar{\nabla}^a, \bar{\nabla}^c] \left(\bar{R}^{bd} h_{bc}\right) + \bar{\xi}_d \bar{\nabla}^c \left(h_{bc} \bar{\nabla}^a \bar{R}^{bd}\right). \end{aligned} \tag{C.56}$$

In the second line in (C.56) we used

$$\bar{\xi}_d [\bar{\nabla}^a, \bar{\nabla}^c] \left(\bar{R}^{bd} h_{bc}\right) = \bar{\xi}_d \left([\bar{\nabla}^a, \bar{\nabla}^c] \bar{R}^{bd}\right) h_{bc} + \bar{\xi}_d \left([\bar{\nabla}^a, \bar{\nabla}^c] h_{bc}\right) \bar{R}^{bd}. \tag{C.57}$$

Using (C.56), (C.54) and (C.50), in (C.40) one finds

$$\begin{aligned} A + B + C + D + E + \sum_{i=1}^{23} i &= \frac{1}{2} \bar{\nabla}_c \left(\sum_{i=1}^{7} Q_i^{ac}\right) + \bar{\nabla}_c \left(\sum_{i=8}^{11} Q_i^{ac}\right) + \bar{\xi}^a h^{cd} \bar{R}_{cedf} \bar{R}^{ef} \\ &+ (\bar{\nabla}_c \bar{\xi}_d)(\underbrace{\bar{R}^{bc} \bar{\nabla}_b h^{ad}}_{t} - \underbrace{\bar{R}^{bc} \bar{\nabla}^a h_b{}^d}_{r} - \underbrace{\bar{R}^{bc} \bar{\nabla}^d h^a{}_b}_{x} - \underbrace{\bar{R}^{ac} \bar{\nabla}_b h^{bd}}_{u}) \\ &+ \underbrace{\frac{1}{2} (\bar{\nabla}^a \bar{\xi}_c) h_{bd} \bar{\nabla}^c \bar{R}^{bd} - \frac{1}{2} \bar{\xi}_c \bar{\nabla}^a \left(h_{bd} \bar{\nabla}^c \bar{R}^{bd}\right)}_{q} \\ &+ 2 \bar{R}^{bc} \bar{\xi}^d [\bar{\nabla}_d, \bar{\nabla}_b] h^a{}_c + \bar{\xi}_d \bar{R}^{ab} [\bar{\nabla}_c, \bar{\nabla}^d] h_b{}^c \\ &+ \bar{\xi}_d [\bar{\nabla}^a, \bar{\nabla}^c] \left(\bar{R}^{bd} h_{bc}\right) + \underbrace{\bar{\nabla}_c \left((\bar{\nabla}_d \bar{\xi}^a) \bar{R}^{bd} h_b{}^c\right)}_{z} \\ &- h_b{}^c (\bar{\nabla}_c \bar{\nabla}_d \bar{\xi}^a) \bar{R}^{bd} + \underbrace{\bar{\xi}_d \bar{\nabla}^c \left(h_{bc} \bar{\nabla}^a \bar{R}^{bd}\right)}_{s}. \end{aligned} \tag{C.58}$$

As for q in (C.58)

$$\begin{aligned} q &= \frac{1}{2}(\bar\nabla^a\bar\xi_c)h_{bd}\bar\nabla^c\bar R^{bd} - \frac{1}{2}\bar\xi_c\bar\nabla^a\left(h_{bd}\bar\nabla^c\bar R^{bd}\right) \\ &= -\frac{1}{2}\bar\nabla^a\left(h_{bd}\bar\xi_c\bar\nabla^c\bar R^{bd}\right) + (\bar\nabla^a\bar\xi_c)h_{bd}\bar\nabla^c\bar R^{bd} \\ &= -\frac{1}{2}\bar\nabla^a\left(h_{bd}(\bar R^{bc}\bar\nabla_c\bar\xi^d + \bar R^{dc}\bar\nabla_c\bar\xi^b)\right) - (\bar\nabla_c\bar\xi^a)h_{bd}\bar\nabla^c\bar R^{bd} \\ &= -\bar\nabla^a\left(h_{bd}\bar R^{bc}(\bar\nabla_c\bar\xi^d)\right) - \bar\nabla_c\left(\bar\xi^a h_{bd}\bar\nabla^c\bar R^{bd}\right) \\ &+ \bar\xi^a h_{bd}\bar\nabla^c\bar\nabla_c\bar R^{bd} + \bar\xi^a(\bar\nabla_c h_{bd})\bar\nabla^c\bar R^{bd}. \end{aligned} \tag{C.59}$$

Using 24 and 25 from the linearized equations (C.59) reads

$$q + 24 + 25 = -\bar\nabla^a\left(h_{bd}\bar R^{bc}(\bar\nabla_c\bar\xi^d)\right) - \bar\nabla_c\left(\bar\xi^a h_{bd}\bar\nabla^c\bar R^{bd}\right) + \underbrace{\frac{1}{2}\bar\xi^a h_{bd}\bar\nabla^c\bar\nabla_c\bar R^{bd}}_{OK-6}. \tag{C.60}$$

In order to get a charge, write the sum 27 + 26 as

$$27 + 26 = \bar\nabla^a\bar R^{bc}\,\bar\nabla^d h_{bc}\,\bar\xi_d + \bar\nabla^a\bar\nabla^b\bar R^{cd}\;\bar\xi_b h_{cd} = \bar\nabla_c\left(\bar\xi^c h_{bd}\bar\nabla^a\bar R^{bd}\right) + \bar\xi_c h_{bd}[\bar\nabla^a,\bar\nabla^c]\bar R^{bd}. \tag{C.61}$$

Then the sum of (C.60), (C.61) reads

$$q + 24 + 25 + 26 + 27 = \bar\nabla_c Q^{ac}_{12} + \bar\xi_c h_{bd}[\bar\nabla^a,\bar\nabla^c]\bar R^{bd} + \frac{1}{2}\bar\xi^a h_{bd}\bar\nabla^c\bar\nabla_c\bar R^{bd} - \bar\nabla^a\left(h_{bd}\bar R^{bc}(\bar\nabla_c\bar\xi^d)\right), \tag{C.62}$$

where $Q^{ac}_{12} \equiv 2\bar\xi^{[c}h_{bd}\bar\nabla^{a]}\bar R^{bd}$. Moreover the sum of 28 + 29 can be written as

$$28 + 29 = -\bar R^{bc}\bar\nabla^d\bar\nabla^a h_{bc}\;\bar\xi_d + \bar R^{bc}\bar\nabla^a\bar\nabla^d h_{bc}\;\bar\xi_d = \bar R^{bd}\bar\xi_c[\bar\nabla^a,\bar\nabla^c]h_{bd}, \tag{C.63}$$

(C.63) and the commutator in (C.62) will sum up to zero

$$[\bar\nabla^a,\bar\nabla^c](\bar R^{bd}h_{bd}) = \bar\xi_c h_{bd}[\bar\nabla^a,\bar\nabla^c]\bar R^{bd} + \bar R^{bd}\bar\xi_c[\bar\nabla^a,\bar\nabla^c]h_{bd} = 0.$$

As a result the sum (C.62), (C.63) compactly written as

$$q + \sum_{i=24}^{29} = Q^{ac}_{12} + \frac{1}{2}\bar\xi^a h_{bd}\bar\nabla^c\bar\nabla_c\bar R^{bd} - \bar\nabla^a\left(h_{bd}\bar R^{bc}(\bar\nabla_c\bar\xi^d)\right). \tag{C.64}$$

Let us expand the derivative in the last term in (C.64)

$$q + \sum_{i=24}^{29} = \bar\nabla_c Q^{ac}_{12} + \frac{1}{2}\bar\xi^a h_{bd}\bar\nabla^c\bar\nabla_c\bar R^{bd} - (\bar\nabla^a h_{bd})\bar R^{bc}(\bar\nabla_c\bar\xi^d) - h_{bd}(\bar\nabla^a\bar R^{bc})(\bar\nabla_c\bar\xi^d) - h_{bd}\bar R^{bc}(\bar\nabla^a\bar\nabla_c\bar\xi^d), \tag{C.65}$$

the third term above is same as r in (C.58). Also by using identity (A.13) 30 from the linearized equations can be written as

$$30 = \bar{\nabla}^b \bar{R}^{cd}\, \bar{\nabla}^a h_{cd}\, \bar{\xi}_b = 2(\bar{\nabla}^a h_{bd})\bar{R}^{bc}(\bar{\nabla}_c \bar{\xi}^d).$$

The sum q, 30 and (C.65) reads

$$q + r + \sum_{i=24}^{30} = \bar{\nabla}_c Q_{12}^{ac} + \frac{1}{2}\bar{\xi}^a h_{bd}\bar{\nabla}^c\bar{\nabla}_c\bar{R}^{bd} - h_{bd}(\bar{\nabla}^a\bar{R}^{bc})(\bar{\nabla}_c\bar{\xi}^d) - h_{bd}\bar{R}^{bc}(\bar{\nabla}^a\bar{\nabla}_c\bar{\xi}^d). \qquad \text{(C.66)}$$

In order to get one more charge and on shell term take 31

$$31 = -\bar{\nabla}^b \bar{R}^{cd}\, \bar{\nabla}_b h^a{}_c\, \bar{\xi}_d = -\bar{\nabla}_c\left(\bar{\xi}^d h^{ab}\bar{\nabla}^c\bar{R}_{bd}\right) + h^{ab}(\bar{\nabla}_c\bar{\xi}_d)(\bar{\nabla}^c\bar{R}_b{}^d) + \underbrace{h^{ab}\bar{\xi}^d\bar{\nabla}_c\bar{\nabla}^c\bar{R}_{bd}}_{OK-7}, \qquad \text{(C.67)}$$

antisymmetric part follows from s

$$s = \bar{\xi}_d\bar{\nabla}^c\left(h_{bc}\bar{\nabla}^a\bar{R}^{bd}\right) = \bar{\nabla}_c\left(\bar{\xi}_d h_b{}^c\bar{\nabla}^a\bar{R}^{bd}\right) - (\bar{\nabla}^c\bar{\xi}_d)h_{bc}\bar{\nabla}^a\bar{R}^{bd}. \qquad \text{(C.68)}$$

As a result we have

$$31 + s = \bar{\nabla}_c Q_{13}^{ac} + h^{ab}\bar{\xi}^d\bar{\nabla}_c\bar{\nabla}^c\bar{R}_{bd} + (\bar{\nabla}_c\bar{\xi}^d)\left(h^{ab}\bar{\nabla}^c\bar{R}_{bd} + h^b{}_d\bar{\nabla}^a\bar{R}_b{}^c\right) \qquad \text{(C.69)}$$

where $Q_{13}^{ac} \equiv 2\bar{\xi}^d h^{b[c}\bar{\nabla}^{a]}\bar{R}_{bd}$.

Another charge can be obtained from the sum 32 + 33

$$\begin{aligned} 32 + 33 &= -\bar{\nabla}^b\bar{R}^{ac}\,\bar{\nabla}^d h_{bd}\,\bar{\xi}_c - \bar{\nabla}^b\bar{\nabla}^c\bar{R}^{ad}\,\bar{\xi}_d h_{bc} = -\bar{\xi}_d\bar{\nabla}_c\left(h^{bc}\bar{\nabla}_b\bar{R}^{ad}\right) \\ &= -\bar{\nabla}_c\left(\bar{\xi}_d h^{bc}\bar{\nabla}_b\bar{R}^{ad}\right) + (\bar{\nabla}_c\bar{\xi}_d)h^{bc}\bar{\nabla}_b\bar{R}^{ad}. \end{aligned} \qquad \text{(C.70)}$$

This can be put in the charge as $Q_{14}^{ac} \equiv 2\bar{\xi}_d h^{b[a}\bar{\nabla}_b\bar{R}^{c]d}$ by adding and subtracting the antisymmetric piece, then (C.70) can be written as

$$32 + 33 = \bar{\nabla}_c Q_{14}^{ac} - \bar{\xi}_d(\bar{\nabla}_c h^{ab})\bar{\nabla}_b\bar{R}^{cd} + (\bar{\nabla}_c\bar{\xi}_d)h^{bc}(\bar{\nabla}_b\bar{R}^{ad}) - \bar{\xi}_d h^{ab}[\bar{\nabla}_c,\bar{\nabla}_b]\bar{R}^{cd} - \bar{\xi}_d h^{ab}\bar{\nabla}_b\bar{\nabla}_c\bar{R}^{cd}. \qquad \text{(C.71)}$$

The second term above is the minus of 34 so the sum 34 and (C.71)

$$32 + 33 + 34 = \bar{\nabla}_c Q_{14}^{ac} + (\bar{\nabla}_c\bar{\xi}_d)h^{bc}(\bar{\nabla}_b\bar{R}^{ad}) - \bar{\xi}_d h^{ab}[\bar{\nabla}_c,\bar{\nabla}_b]\bar{R}^{cd} - \frac{1}{2}\bar{\xi}_d h^{ab}\bar{\nabla}_b\bar{\nabla}^d\bar{R} \qquad \text{(C.72)}$$

Using (C.72) and (C.69), (C.66) can be extended as

$$\begin{aligned} q + r + s + \sum_{i=24}^{34} i = \bar{\nabla}_c\left(\sum_{i=12}^{14} Q_i^{ac}\right) &+ \underbrace{\frac{1}{2}\bar{\xi}^a h_{bd}\bar{\nabla}^c\bar{\nabla}_c\bar{R}^{bd}}_{OK-6} + \underbrace{h^{ab}\bar{\xi}_d\bar{\nabla}_c\bar{\nabla}^c\bar{R}_{bd}}_{OK-7} - \underbrace{\frac{1}{2}\bar{\xi}_d h^{ab}\bar{\nabla}_b\bar{\nabla}^d\bar{R}}_{OK-8} \\ &+ \bar{\xi}_d h^{ab}[\bar{\nabla}_b,\bar{\nabla}_c]\bar{R}^{cd} - h_{bd}\bar{R}^{bc}(\bar{\nabla}^a\bar{\nabla}_c\bar{\xi}^d) + (\bar{\nabla}_c\bar{\xi}^d)\left(h^{ab}\bar{\nabla}^c\bar{R}_{bd} + h^{bc}\bar{\nabla}_b\bar{R}^a{}_d\right). \end{aligned} \qquad \text{(C.73)}$$

Let us return to the remaining terms in (C.58), namely t, u, x, y, z. Start with t

$$(\bar{\nabla}_c\bar{\xi}_d)\bar{R}^{bc}\bar{\nabla}_b h^{ad} = \bar{\nabla}_c\left((\bar{\nabla}_b\bar{\xi}_d)h^{ad}\bar{R}^{bc}\right) - h^{ad}\bar{R}^{bc}\bar{\nabla}_c\bar{\nabla}_b\bar{\xi}_d + \frac{1}{2}h^{ab}(\bar{\nabla}_b\bar{\xi}_c)(\bar{\nabla}^c\bar{R}), \tag{C.74}$$

last term in (C.74) can be written as

$$\frac{1}{2}h^{ab}(\bar{\nabla}_b\bar{\xi}_c)\bar{\nabla}^c\bar{R} = \frac{1}{2}\bar{\nabla}_b\left(h^{ab}\bar{\xi}_c\bar{\nabla}^c\bar{R}\right) - \frac{1}{2}h^{ab}\bar{\xi}_c\bar{\nabla}_b\bar{\nabla}^c\bar{R} - \frac{1}{2}(\bar{\nabla}_b h^{ab})\bar{\xi}_c\bar{\nabla}^c\bar{R}.$$

The first and the third terms are zero by identity (A.14) then can be written (C.74) as

$$t = \bar{\nabla}_c\left((\bar{\nabla}_b\bar{\xi}_d)h^{ad}\bar{R}^{bc}\right) - h^{ad}\bar{R}^{bc}\bar{\nabla}_c\bar{\nabla}_b\bar{\xi}_d - \frac{1}{2}h^{ab}\bar{\xi}^c\bar{\nabla}_b\bar{\nabla}_c\bar{R}. \tag{C.75}$$

For the antisymmetric part, write u as,

$$u = -(\bar{\nabla}_c\bar{\xi}_d)\bar{R}^{ac}\bar{\nabla}_b h^{bd} = -\bar{\nabla}_c\left((\bar{\nabla}_b\bar{\xi}_d)\bar{R}^{ab}h^{cd}\right) + h^{cd}\bar{R}^{ab}\bar{\nabla}_c\bar{\nabla}_b\bar{\xi}_d + h^{cd}(\bar{\nabla}_b\bar{\xi}_d)\bar{\nabla}_c\bar{R}^{ab}. \tag{C.76}$$

then the sum $t+u$ leads to

$$t + u = \bar{\nabla}_c Q^{ac}_{15} \underbrace{- \frac{1}{2}h^{ab}\bar{\xi}^c\bar{\nabla}_b\bar{\nabla}_c\bar{R}}_{OK-8} + (\bar{\nabla}_c\bar{\nabla}_b\bar{\xi}_d)\left(h^{cd}\bar{R}^{ab} - h^{ad}\bar{R}^{bc}\right) + h^{bd}(\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}_b\bar{R}^{ac} \tag{C.77}$$

where $Q^{ac}_{15} \equiv 2(\bar{\nabla}_b\bar{\xi}_d)h^{d[a}\bar{R}^{c]b}$. Add (C.77) to (C.73)

$$\begin{aligned} q + r + s + t + u + \sum_{i=24}^{34} i = {} & \bar{\nabla}_c\left(\sum_{i=12}^{15} Q^{ac}_i\right) + \underbrace{\frac{1}{2}\bar{\xi}^a h_{bd}\bar{\nabla}^c\bar{\nabla}_c\bar{R}^{bd}}_{OK-6} + \underbrace{h^{ab}\bar{\xi}^d\bar{\nabla}_c\bar{\nabla}^c\bar{R}_{bd}}_{OK-7} \underbrace{- \bar{\xi}_d h^{ab}\bar{\nabla}_b\bar{\nabla}^d\bar{R}}_{OK-8} \\ & + \bar{\xi}_d h^{ab}[\bar{\nabla}_b, \bar{\nabla}_c]\bar{R}^{cd} - h_{bd}\bar{R}^{bc}(\bar{\nabla}^a\bar{\nabla}_c\bar{\xi}^d) \\ & + (\bar{\nabla}_c\bar{\nabla}_b\bar{\xi}_d)\left(h^{cd}\bar{R}^{ab} - h^{ad}\bar{R}^{bc}\right) + \underbrace{(\bar{\nabla}_c\bar{\xi}^d)h^{ab}\bar{\nabla}^c\bar{R}_{bd}}_{y}. \end{aligned} \tag{C.78}$$

One can generate final part of charge from the remaining x, y and z

$$\begin{aligned} x + y &= (\bar{\nabla}_c\bar{\xi}_d)\bar{\nabla}^c\left(h^a{}_b\bar{R}^{bd}\right) = \bar{\nabla}_c\left((\bar{\nabla}^c\bar{\xi}_d)h^a{}_b\bar{R}^{bd}\right) - (\bar{\nabla}_c\bar{\nabla}^c\bar{\xi}_d)h^a{}_b\bar{R}^{bd}, \\ z &= \bar{\nabla}_c\left((\bar{\nabla}_d\bar{\xi}^a)\bar{R}^{bd}h_b{}^c\right) = -\bar{\nabla}_c\left((\bar{\nabla}^a\bar{\xi}_d)h_b{}^c\bar{R}^{bd}\right), \\ x + y + z &= \bar{\nabla}_c Q^{ac}_{16} - (\bar{\nabla}_c\bar{\nabla}^c\bar{\xi}_d)h^a{}_b\bar{R}^{bd}, \end{aligned} \tag{C.79}$$

where $Q^{ac}_{16} \equiv 2(\bar{\nabla}^{[c}\bar{\xi}^d)h^{a]b}\bar{R}_{bd}$.

Now we can gather all the terms by using (C.58), (C.78) and (C.79) as;

$$\begin{aligned} Everything = {} & \frac{1}{2}\bar{\nabla}_c\left(\sum_{i=1}^{7} Q^{ac}_i\right) + \bar{\nabla}_c\left(\sum_{i=8}^{16} Q^{ac}_i\right) + (OK - terms) \\ & + R^a + \underbrace{2\,\bar{R}^{bc}\bar{R}^{ad}{}_b{}^e\bar{\xi}_e h_{cd} - 2\,\bar{R}^{bc}\bar{R}^a{}_b{}^{de}\bar{\xi}_d h_{ce}}_{35}, \end{aligned} \tag{C.80}$$

where

$$
\begin{aligned}
R^a \equiv\ & 2\bar{R}^{bc}\bar{\xi}^d[\bar{\nabla}_d, \bar{\nabla}_b]h^a{}_c + \bar{R}^{ab}\bar{\xi}^d[\bar{\nabla}_c, \bar{\nabla}_d]h_b{}^c + \bar{\xi}_d[\bar{\nabla}^a, \bar{\nabla}^c](\bar{R}^{bd}h_{bc}) \\
& - (\bar{\nabla}_c\bar{\nabla}_d\bar{\xi}^a)h_b{}^c\bar{R}^{bd} + h^{ab}\bar{\xi}_d[\bar{\nabla}_b, \bar{\nabla}_c]\bar{R}^{cd} + h_{bc}\bar{R}^{bd}(\bar{\nabla}^a\bar{\nabla}^c\bar{\xi}_d) \\
& - (\bar{\nabla}_c\bar{\nabla}^c\bar{\xi}_d)h^a{}_b\bar{R}^{bd} + (\bar{\nabla}_c\bar{\nabla}_b\bar{\xi}_d)(h^{cd}\bar{R}^{ab} - h^{ad}\bar{R}^{bc}), \qquad \text{(C.81)}
\end{aligned}
$$

$$
\begin{aligned}
OK - terms \equiv\ & h^{ab}\bar{\xi}^c\left(2\bar{R}_{bdce}\bar{R}^{de} - \bar{\nabla}_b\bar{\nabla}_c\bar{R} + \bar{\nabla}_d\bar{\nabla}^d\bar{R}_{bc}\right) + h^{ab}\bar{\xi}_b\left(\frac{1}{2}\bar{\nabla}_d\bar{\nabla}^d\bar{R} - \frac{1}{2}\bar{R}_{de}\bar{R}^{de}\right) \\
& + \bar{\xi}^a h^{cd}\left(\bar{R}_{cedf}\bar{R}^{ef} - \frac{1}{2}\bar{\nabla}_c\bar{\nabla}_d\bar{R} + \frac{1}{2}\bar{\nabla}^e\bar{\nabla}_e\bar{R}_{cd}\right). \qquad \text{(C.82)}
\end{aligned}
$$

Since we must have the charge plus the background equations in the form $h^{ab}\bar{\xi}^c\bar{B}_{bc}+\frac{1}{2}\bar{\xi}^a h^{cd}\bar{B}_{cd}$. Which is equal to $OK - terms$, where $\bar{B}_{ab} = 2\bar{R}_{acbd}\bar{R}^{cd} - \bar{\nabla}_a\bar{\nabla}_b\bar{R} - \frac{1}{2}\bar{g}_{ab}\bar{R}_{cd}\bar{R}^{cd} + \bar{\nabla}_c\bar{\nabla}^c\bar{R}_{ab} + \frac{1}{2}\bar{\nabla}_c\bar{\nabla}^c\bar{R}$. The other remaining terms $R^a + \underbrace{2\,\bar{R}^{bc}\bar{R}^{ad}{}_b{}^e\bar{\xi}_e h_{cd} - 2\,\bar{R}^{bc}\bar{R}^a{}_b{}^{de}\bar{\xi}_d h_{ce}}_{35}$ must be zero.

In order to show that write the sixth term in R^a as

$$
h_{bc}\bar{R}^{bd}(\bar{\nabla}^a\bar{\nabla}^c\bar{\xi}_d) = h_{bc}\bar{R}^{bd}[\bar{\nabla}^a, \bar{\nabla}^c]\bar{\xi}_d + h_{bc}\bar{R}^{bd}(\bar{\nabla}^c\bar{\nabla}^a\bar{\xi}_d). \qquad \text{(C.83)}
$$

Likewise the sum of the third, the fourth, the sixth term reads,

$$
\bar{\xi}_d[\bar{\nabla}^a, \bar{\nabla}^c](\bar{R}^{bd}h_{bc}) - (\bar{\nabla}_c\bar{\nabla}_d\bar{\xi}^a)h_b{}^c\bar{R}^{bd} + h_{bc}\bar{R}^{bd}[\bar{\nabla}^a, \bar{\nabla}^c]\bar{\xi}_d + h_{bc}\bar{R}^{bd}(\bar{\nabla}^c\bar{\nabla}^a\bar{\xi}_d),
$$

which can be written as

$$
[\bar{\nabla}^a, \bar{\nabla}^c](\bar{\xi}_d\bar{R}^{bd}h_{bc}) + 2h_{bc}\bar{R}^{bd}\bar{\nabla}^c\bar{\nabla}^a\bar{\xi}_d = -\bar{R}^{ac}h_{bc}\bar{R}^{bd}\bar{\xi}_d - 2\bar{R}^{bd}\bar{R}^a{}_d{}^{ce}h_{bc}\bar{\xi}_e. \qquad \text{(C.84)}
$$

Insert (C.84) in R^a and convert the commutators to the Riemann tensor

$$
\begin{aligned}
R^a =\ & \underbrace{-\bar{R}^{ac}h_{bc}\bar{R}^{bd}\bar{\xi}_d}_{1} - 2\bar{R}^{bd}\bar{R}^a{}_d{}^{ce}h_{bc}\bar{\xi}_e + 2\bar{R}^{bc}\bar{\xi}^d(\bar{R}_{db}{}^a{}_e h^e{}_c + \underbrace{\bar{R}_{dbc}{}^e h^a{}_e}_{4}) \\
& + \bar{R}^{ab}\bar{\xi}^d(\underbrace{\bar{R}_{cdb}{}^e h_e{}^c}_{2} + \underbrace{\bar{R}_{cd}{}^c{}_e h_b{}^e}_{1}) + h^{ab}\bar{\xi}_d(\underbrace{\bar{R}_{bc}{}^c{}_e\bar{R}^{ed}}_{3} + \underbrace{\bar{R}_{bc}{}^d{}_e\bar{R}^{ce}}_{4}) \\
& - \underbrace{\bar{R}_{dc}{}^{ce}\bar{\xi}_e h^a{}_b\bar{R}^{bd}}_{3} + \bar{R}_{bdce}\bar{\xi}^e(\underbrace{h^{cd}\bar{R}^{ab}}_{2} - \underbrace{h^{ad}\bar{R}^{bc}}_{4}). \qquad \text{(C.85)}
\end{aligned}
$$

Terms 1, 2, 3, 4 cancel each other and the remaining terms are

$$
-2\bar{R}^{bd}\bar{R}^a{}_d{}^{ce}h_{bc}\bar{\xi}_e + 2\bar{R}^{bc}\bar{\xi}^d\bar{R}_{db}{}^a{}_e h^e{}_c = 2\bar{R}^{bc}\bar{R}^a{}_b{}^{de}\bar{\xi}_d h_{ce} - 2\bar{R}^{bc}\bar{R}^{ad}{}_b{}^e\bar{\xi}_e h_{cd}, \qquad \text{(C.86)}
$$

and 35

$$
2\,\bar{R}^{bc}\bar{R}^{ad}{}_b{}^e\bar{\xi}_e h_{cd} - 2\,\bar{R}^{bc}\bar{R}^a{}_b{}^{de}\bar{\xi}_d h_{ce}.
$$

As a result $R^a + 35 = 0$ what we expect.

Finally β charge is

$$\bar{\nabla}_b\Big[\bar{\xi}_d\bar{\nabla}^{[a}\bar{\nabla}_c\bar{\nabla}^c h^{b]d} + \bar{\xi}_d\bar{\nabla}^{[b}\bar{\nabla}_c\bar{\nabla}^d h^{a]c} + \bar{\xi}^{[a}\bar{\nabla}^{b]}\bar{\nabla}^c\bar{\nabla}^d h_{cd} + \bar{\xi}^d\bar{\nabla}^{[b}\bar{\nabla}^c\bar{\nabla}^{a]}h_{cd} + \bar{\nabla}^{[b}\bar{\xi}_d\bar{\nabla}_c\bar{\nabla}^c h^{a]d}$$
$$+ \bar{\nabla}^{[a}\bar{\xi}_d\bar{\nabla}_c\bar{\nabla}^d h^{b]c} + \bar{\nabla}^{[a}\bar{\xi}_d\bar{\nabla}_c\bar{\nabla}^{b]}h^{cd} + 2\bar{\xi}_d R^{c[a}\bar{\nabla}_c h^{b]d} + 2\bar{\xi}^d R^{c[b}\bar{\nabla}^{a]}h_{cd} + 2\bar{\xi}^d R^{c[b}\bar{\nabla}_d h^{a]}_c$$
$$+ 2\bar{\xi}^{[b}R^{ca]}\bar{\nabla}^d h_{cd} + 2h_{cd}\bar{\xi}^{[b}\bar{\nabla}^{a]}R^{cd} + 2\bar{\xi}^d h^{c[b}\bar{\nabla}^{a]}R_{cd} + 2\bar{\xi}_d h^{c[a}\bar{\nabla}_c R^{b]d} + 2\bar{\nabla}_c\bar{\xi}_d h^{d[a}R^{b]c}$$
$$+ 2\bar{\nabla}^d\bar{\xi}^{[a}h^{b]c}R_{dc} + h\bar{\xi}_c\bar{\nabla}^{[b}R^{a]c} + \bar{\xi}^{[a}R^{b]c}\bar{\nabla}_c h + hR^{c[a}\bar{\nabla}_c\bar{\xi}^{b]}$$
$$+ \bar{\xi}^{[b}\bar{\nabla}^{a]}\bar{\nabla}_c\bar{\nabla}^c h + \bar{\xi}^c\bar{\nabla}^{[a}\bar{\nabla}^{b]}\bar{\nabla}_c h + \bar{\nabla}_c\bar{\xi}^{[a}\bar{\nabla}^{b]}\bar{\nabla}^c h\Big]. \tag{C.87}$$