THE EFFECT OF TEMPORAL AGGREGATION ON UNIVARIATE TIME SERIES ANALYSIS

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ABSTRACT

THE EFFECT OF TEMPORAL AGGREGATION ON UNIVARIATE TIME SERIES ANALYSIS

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Most of the time series are constructed by some kind of aggregation and temporal aggregation that can be defined as aggregation over consecutive time periods. Temporal aggregation takes an important role in time series analysis since the choice of time unit clearly influences the type of model and forecast results. A totally different time series model can be fitted on the same variable over different time periods. In this thesis, the effect of temporal aggregation on univariate time series models is studied by considering modeling and forecasting procedure via a simulation study and an application based on a southern oscillation data set. Simulation study shows how the model, mean square forecast error and estimated parameters change when temporally aggregated data is used for different orders of aggregation and sample sizes. Furthermore, the effect of temporal aggregation is also demonstrated through southern oscillation data set for different orders of aggregation. It is observed that the effect of temporal aggregation should be taken into account for data analysis since temporal aggregation can give rise to misleading results and inferences.

Keywords: Temporal Aggregation, ARMA models, Univariate Time Series Analysis

ÖZ

ZAMANSAL TOPLULAŞMANIN TEK DEĞİŞKENLİ ZAMAN SERİSİ ANALİZİNE ETKİSİ

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Zaman serilerinin çoğu çeşitli bir araya getirme yöntemleri ile oluşturulmaktadır ve zamansal toplulaştırma birbirini izleyen zaman dönemlerinin bir araya getirilmesi şeklinde tanımlanabilir. Zamansal toplulaştırma zaman serisi analizlerinde önemli bir rol üstlenmektedir çünkü zaman biriminin seçimi, modeli ve öngörü sonuçlarını etkilemektedir. Tamamen değişik bir model aynı değişkenin değişik zaman dönemleri için uygun olabilmektedir. Bu tezde, modelleme ve öngörü süreçleri göz önünde bulundurularak zamansal toplulaştırmanın tek değişkenli zaman serileri üstündeki etkisi benzetim çalışması ve güney salınımları veri setine dayanan uygulamayla çalışılmıştır. Benzetim çalışması, değişik toplulaştırma dereceleri ve örnek büyüklükleri için zamansal toplulaştırılmış veri kullanıldığında modelin, ortalama karesel öngörü hatasının ve tahmin edilen parametrelerin nasıl değiştiğini göstermiştir. Ayrıca, zamansal toplulaştırmanın değişik toplulaştırma dereceleri için etkileri güney salınımları veri seti kullanılarak da gösterilmiştir. Zamansal toplulaştırmanın yanlış sonuçlara ve çıkarımlara yol açması sebebiyle zamansal toplulaştırma etkisinin dikkate alınması gerektiği anlaşılmıştır.

Anahtar Kelimeler: Zamansal toplulaştırma, ARMA modelleri, Tek Değişkenli Zaman Serisi Analizi

To My Parents

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CHAPTER 1

INTRODUCTION

1.1 Temporal Aggregation and Related Literature Survey

A time series model is proposed in terms of basic time unit t. Although the model can be tested against observable data from a designed experiment in terms of the same time unit t , in some cases the time frequency of the observed data may not be the same as the assumed time unit t. Generally, observable data are obtained through aggregation and to receive meaningful results it is necessary to know the effect of aggregation on model structure, parameter estimation and forecasting.

A time series variable can be a flow variable or a stock variable. The values of the flow variable are obtained through temporal aggregation while the values of the stock variable are obtained through systematic sampling. Industrial production, gross domestic product, public deficit, financial returns can be given as examples of the flow variable and interest rate, unemployment rate, price of a commodity can be given as examples of the stock variable. In this study, flow variables and temporal aggregation will be considered. Let z_t be the equally spaced basic series and assume that the observed time series Z_T is the m-period non-overlapping aggregates of z_t defined as

$$Z_T = \sum_{j=0}^{m-1} z_{mT-j} = (1 + B + \cdots + B^{m-1}) z_{mT} ,$$

where B is the backshift operator, T is aggregate time unit and m is fixed order of aggregation. Z_T is called an aggregate series and z_t is called basic or nonaggregate series. The number of observations of Z_T is N=$[n/m]$, where n is the sample size of the basic series and m is the aggregation period.

An early example of research in the literature of aggregation is Quenouille (1958), where the effect of aggregation on stationary univariate series is discussed. Quenouille studied the stationary ARMA models with autoregressive order of p and moving average order not greater than p. Telser (1967) dealt with the structure of the aggregate sequence of an autoregressive system. Telser indicated that simple least squares estimates of the autoregressive coefficients are not consistent. He showed that it is possible to estimate the autoregressive coefficients consistently based on equally spaced sample of moving sums of basic data for non-overlapping discrete intervals. In his paper, he also reported that a purely AR(p) model transforms into a mixed ARMA(p,q) model where the roots of the AR polynomial of the aggregate series model are the m^{th} powers of the nonaggregate series AR polynomial. Amemiya and Wu (1972) also showed that if the original variable follows a p^{th} order autoregressive system, then mixed model for the aggregates follows a p^{th} order autoregressive system and qth order moving average system, where q is the largest integer satisfying $qm < (p+1)(m-1)+1$. This means that q is at most equal to p and if $m \geq p+1$, then $q = p$. Moreover, Amemiya and Wu proved that the moving average part was invertible. Brewer (1973) studied the effect of aggregation on the ARMA(p,q) model. He mentioned that nonaggregate ARMA(p,q) model can be transformed to an aggregate ARMA(p,r) model, where $r = [p+1+(q-p-1)/m]$ ($[x]$ is used to denote the integer part of x). Brewer also presented a generalization of the results obtained by Amemiya and Wu for ARMA models with exogenous variables (ARMAX models). Tiao (1972) was the first researcher who discussed the effects of aggregation on nonstationary univariate series. Tiao studied the aggregation effect on the integrated moving average models, IMA(d,q) and he showed when $m \rightarrow \infty$ the limiting model for the aggregates exists and equals to IMA(d,d) process, which is independent of p and q. Tiao and Wei (1976) considered the effect of temporal aggregation on the dynamic relationships between two discrete time series variables. Given the dynamic model

$$z_t = \begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} v(B)\tau(B) & \mu(B) \\ \tau(B) & 0 \end{bmatrix} \begin{bmatrix} a_t \\ e_t \end{bmatrix},$$

where $a_t \sim N(0, \sigma_a^2)$ and $e_t \sim N(0, \sigma_e^2)$ are independent, in terms of some basic time unit t, they obtained the corresponding model for the aggregate series. In their paper, it was shown that temporal aggregation can lead to a substantial loss in parameter estimation while the loss in prediction efficiency is much less severe. Wei (1978a) considered the effect of temporal aggregation on parameter estimation in a finite distributed lag model through the least squares procedure. The loss in efficiency due to aggregation was discussed and it was seen that the loss depends not only on the level of aggregation but also on the nature of input variable. Wei (1978b) studied aggregation effect on univariate multiplicative seasonal time series models. Wei revealed that for a stochastic time series model of order $(p, d, q) \times (P, D, Q)_s$, the corresponding model for the aggregates of m-component nonoverlapping sum is of order $(p, d, r) \times (P, D, Q)_S$ where $s = mS$ for some integer S and $r = \left[p + d + 1 + \frac{q-p-d-1}{m}\right]$. Also, it was mentioned that when $m \geq S$, aggregation reduces a seasonal model to a regular ARIMA model. Stram and Wei (1986b) dealt with the relationship of autocovariances between the nonaggregate and aggregate series. In their paper, the form of aggregate autocovariance function was computed and explained based on nonaggregate autocovariance function. Furthermore, Stram and Wei showed that mth order aggregate series of a stationary AR(p) process follows an ARMA(M,N) process where M is calculated by the help of set theory and $N = \left[p + 1 - \frac{(p+1)}{m}\right] - (p - M)$. Lütkepohl (1986) discussed forecasting aggregated vector ARMA processes. In his study, mean squared forecasting error was taken into account to determine the forecast accuracy. Drost and Nijman (1993) derived temporal aggregated data models by using ARMA models with symmetric GARCH errors. They pointed out that temporal aggregated data models are in the conditional heteroskedasticity of the GARCH form. Mamingi (1996) studied the effect of temporal aggregation on the Granger causality in error correction models by using Monte Carlo experiments. Marcellino (1999) investigated the temporal aggregated process when the nonaggregate series follows vector ARIMA process. In his study, the effects of temporal aggregation on a set of characteristics such as causality and cointegration are considered. Breitung and Swanson (2002) studied the temporal aggregation effect on Granger causality relations in VAR models based on large aggregation intervals. They mentioned several conditions that cover the informal content of error covariance matrices and the casual structure of VAR. Shellman (2004) dealt with how temporal aggregation affects the decisions on VAR parameter estimates, significance levels, Granger causality tests and impulse response functions based on event data. Silvestrini and Veredas (2008) presented an overview of temporal aggregation methods for univariate and multivariate time series models and they also gave some empirical applications. Teles, Wei and Hodgess (2008) investigated the effects of aggregate time series on the Dickey-Fuller test for a unit root. They determined that aggregate time series has an impact both on the empirical significance level and on the power of the test. Also, they presented critical point tables for the tests based on aggregate time series and showed their adequacy. In this thesis, the effects of temporal aggregation on univariate time series models will be studied. Through a simulation study the effect of temporal aggregation on real data will be discussed and comparison between the theoretical and the observed model fit will be presented.

1.2 Basic Time Series Concepts

A time series can be explained as an ordered sequence of observations where observations are taken at equally spaced intervals.

Time series analysis is a field in statistics and econometrics. While most of the statistical methods depend on the assumption that the observations are independent, in time series analysis the observations are dependent. Also, statistical methods are generally used for making comments about population based on a sample. However, in time series analysis it is almost impossible to have more than one observation at a given time point (Akgün, 2003).

Time series can be seen in variety of areas. Agriculture, economics, engineering, medical studies, meteorology and social studies are some areas where time series is observed and studied. For instance, in economics daily closing stock prices, weekly interest rates, quarterly sales, yearly earnings can be regarded as time series.

Time series analysis has two major objectives. The first one is to model the stochastic mechanism that generates the observed series, and the second one is to forecast future values based on history.

In order to model the stochastic mechanism that generates the observed series, a vital assumption which is called stationarity is needed. The basic idea of stationarity can be described as the probability laws governing the process which do not change with time (Akgün, 2003). In this thesis stationary series are considered.

A time series is said to be strictly stationary if the statistical properties of the time series are unaffected by a change of time origin. Moreover, a time series is said to be covariance stationary if the first and second order moments of the time series are unaffected by a change of time origin. A strict stationary process is always a covariance process while a covariance process is a strict stationary process if and only if the covariance process is normally distributed. In practice, it is enough to take covariance stationary processes and in this thesis study the term stationarity corresponds to covariance stationary.

By using the fact that covariance stationary process does not depend on time but rather depends on time intervals, the autocovariance function can be calculated easily. As a result of this, the autocorrelation function and the partial autocorrelation function, which have important roles to detect a time series model, can be found in a simple way. Autocorrelation function is useful for determining moving average orders, and partial autocorrelation function, which gives direct correlation between two observations of a time series, is useful for determining autoregressive orders.

Models for stationary series can be studied under the Autoregressive Moving Average (ARMA) models which are also called Box-Jenkins models. A Moving Average (MA) model of order q is represented as a linear combination of present and q past terms of white noise error terms. White noise error terms are independently and identically distributed random variables with constant mean and variance. Moreover, an Autoregressive (AR) model of order p is represented as a linear combination of p past values of itself plus a white noise error term. Lastly, an Autoregressive Moving Average (ARMA) model of order (p,q) is represented as a linear combination of present and q past terms of white noise error terms and p past values of itself. In thesis study, moving average, autoregressive and autoregressive moving average processes are explained in detail with their autocovariance, autocorrelation and partial autocorrelation functions.

Autoregressive moving average models are constructed based on basic time unit t. As mentioned in previous section, the time frequency of observed data may not be the same as the assumed time unit t. Generally, temporal aggregation is used for the observable data and it is important to know the effect of aggregation on model structure, parameter estimation and forecasting procedure.

1.3 The Relationship between the Basic Series and Aggregate Series

As stated before, there exists a relationship between the basic series z_t and the aggregate series Z_T. Firstly, let z_t be the equally spaced basic series and its dth difference $w_t = (1-B)^d z_t$ follows a covariance stationary process with zero mean. Also, define the aggregate series Z_T , whose dth difference is $U_T = (1-B)^d Z_T$, as $Z_T = I_{mT}$. So $I_t = (1 + B + \cdots + B^{m-1}) z_t$. Then

$$(1-B)Z_T = Z_T - Z_{T-1}$$

$$= I_{mT} - I_{m(T-1)}$$

$$= (1 - B^m) I_{mT} \ . \tag{1.3.1}$$

By using the Equation 1.3.1

$$U_T = (1 - B)^d Z_T$$

$$= (1 - B^m)^d I_{mT}$$

$$= [(1 + B + \cdots + B^{m-1})(1 - B)]^d (1 + B + \cdots + B^{m-1}) z_{mT}$$

$$= (1 + B + \cdots + B^{m-1})^{d+1} (1 - B)^d z_{mT}$$

$$= (1 + B + \cdots + B^{m-1})^{d+1} w_{mT} \ .$$

Since U_T is finite sum of covariance stationary process w_t, U_T is also a covariance stationary process. So, the basic stationarity assumption is not affected by aggregation (Wei, 2006, pp. 508).

Moreover, Stram and Wei (1986b) explained the relationship between the autocovariance functions of w_t and U_T by using the following equation

$$\gamma_U(k) = (1 + B + \cdots + B^{m-1})^{2(d+1)} \gamma_w [mk + (d+1)(m-1)] \ . \tag{1.3.2}$$

Equation 1.3.2 can be written in the matrix form like

$$\begin{bmatrix} \gamma_U(0) \\ \gamma_U(1) \\ \vdots \\ \gamma_U(k) \end{bmatrix} = A \begin{bmatrix} \gamma_w[-(d+1)(m-1)] \\ \gamma_w[-(d+1)(m-1)+1] \\ \vdots \\ \gamma_w(0) \\ \vdots \\ \gamma_w[mk + (d+1)(m-1)] \end{bmatrix}, \tag{1.3.3}$$

where A is the coefficient matrix which is equal to

$$\begin{bmatrix} C & 0_{mk} & \\ 0_m & C & 0_{m(k-1)} \\ \vdots & & \\ 0_{mk} & & C \end{bmatrix},$$

where 0_m and 0_{mk} denote the $1 \times m$ vector of zeros and $m \times k$ matrix of zeros respectively. C is a $1 \times [2(d+1)(m-1)+1]$ vector which shows the coefficients in the polynomial $(1 + B + \cdots + B^{m-1})^{2(d+1)}$ (Wei, 2006, pp. 509).

Since w_t is a stationary process, $\gamma_w(k) = \gamma_w(-k)$ for all k. Then Equation 1.3.3 can be reduced as

$$\begin{bmatrix} \gamma_U(0) \\ \gamma_U(1) \\ \vdots \\ \gamma_U(k) \end{bmatrix} = A_m^d \begin{bmatrix} \gamma_w(0) \\ \gamma_w(1) \\ \vdots \\ \gamma_w[mk + (d+1)(m-1)] \end{bmatrix},$$

where A_m^d is constructed by deleting the first $(d-1)(m-1)$ columns of the matrix A and adding these columns to the proper remaining columns of the matrix A (Wei, 2006, pp. 509).

1.4 Aims and Scope of the Study

The aim of this thesis is to use theoretical information related to temporally aggregated series in empirical studies and to check whether the theoretical inferences are valid in empirical studies or not. Also, we attempt to introduce the temporal aggregation issue to scientific researchers because we cannot see any study related to temporal aggregation in some of the countries, like Turkey. So, we want to show that practitioners should prefer to use nonaggregate series for data analysis to get reliable results.

In this chapter, the basic information about time series and temporal aggregation was given. Furthermore, the related literature survey about temporal aggregation was discussed for univariate and multivariate time series. As mentioned, our study will focus on univariate time series. The studies related to temporal aggregation of univariate series gave theoretical proofs about the aggregation affect. They showed the relationship between temporally aggregated series and basic series.

Also, they presented how the model changes when the series are temporally aggregated. However, in these studies generally theoretical proofs were given and a detailed application related to theory did not take a place. In this study, we will try to give an application related to theory with the help of a simulation study and an application based on a data set.

In Chapter 2, the autoregressive models will be discussed. The information about AR(1), AR(2) and the general AR(p) processes will be given by considering the autocovariance, autocorrelation and partial autocorrelation functions. Temporal aggregation of the AR(1), AR(2) and the general AR(p) processes will also be explained and the models belong to aggregated series will be shown theoretically.

Similarly, Chapter 3 will be about the moving average models. MA(1), MA(2) and the general MA(q) processes will be introduced and the properties of their autocovariance, autocorrelation and partial autocorrelation functions will be given. Temporal aggregation of MA(1), MA(2) and the general MA(q) models will be discussed by using the information in Section 1.3 and the theoretical results of temporal aggregation will be represented.

The autoregressive moving average models will be considered in Chapter 4. ARMA(1,1) and the general ARMA(p,q) processes will be examined like Chapter 2 and 3. The general theoretical aggregated models for ARMA(1,1) and the general ARMA(p,q) processes will be introduced and the theoretical model changes will be presented.

After the theoretical expressions, Chapter 5 will show the effects of temporal aggregation through a simulation study. The simulation study will be based on Teles, Wei and Hodgess (2008), where a simulation study shows the relative frequencies of the best empirical aggregate models for Z_T from an AR(1) process for different orders of aggregation and parameter values. They selected the best fitted model of Z_T based on Akaike's Information Criterion (Akaike, 1974) but they did not take into account the significance of the estimated parameters for aggregated models. Also, they are only interested in parameter values greater than or equal 0.9 for the basic AR(1) series since they test a unit root based on aggregate time series. Our simulation study in Chapter 5 will be an expanded form of the simulation study of the article. First of all, we will attach importance to significance of the parameters for aggregated series. The best fitted aggregate models will be selected by looking at the significance of estimated parameters and Akaike's information criterion. Also, for the basic series we will be interested in various positive and negative model parameters. In our simulation study for different aggregate orders and sample sizes the frequencies of best fitted aggregate models will be obtained from AR(1), AR(2), MA(1), MA(2) and ARMA(1,1), while Teles, Wei and Hodgess simulated 240 observations only from AR(1) process. Moreover, in Chapter 5 the tables related to mean square forecast errors and the estimated parameters of best fitted aggregated models will be given to observe the aggregation effect in detail.

Chapter 6 will focus on the impact of temporal aggregation based on a real life data set. A monthly data set related to southern oscillations, which are used for predicting the atmospheric and oceanic event El-Nino, between years 1955 and 1992 is selected. Firstly, data set will be analyzed without making any change and a model which can be thought as basic model will be fitted to this data. Then the series will be temporally aggregated by taking consecutive sums of original data series for m=3, m=6 and m=12, respectively. For each m value, temporally aggregated models will be constructed and the differences between them will be observed.

Finally, in Chapter 7 summary of the study and important findings related to simulation study and data application will be given.

CHAPTER 2

TEMPORAL AGGREGATION OF AUTOREGRESSIVE PROCESSES

In this chapter, we aim to show the effect of temporal aggregation on the autoregressive processes. For simplicity of the calculations, zero mean autoregressive processes will be considered.

A zero mean autoregressive process of order p can be shown as follows:

$$z_t = \emptyset_1 z_{t-1} + \cdots + \emptyset_p z_{t-p} + a_t \ ,$$

or

$$\emptyset_p(B) z_t = a_t \ ,$$

where

$\emptyset_p(B) = (1 - \emptyset_1 B - \cdots - \emptyset_p B^p)$ and $\{a_t\}$ is a zero mean white noise process with constant variance σ_a^2.

As seen the AR(p) model is in the inverted form and the summation is finite. So, it can be said that a finite autoregressive process is always invertible. If the roots of $\emptyset_p(B) = 0$ lie outside the unit circle, then the autoregressive process is stationary.

2.1 The First Order Autoregressive, AR(1) Process

A zero mean AR(1) process can be shown as follows:

$$z_t = \emptyset_1 z_{t-1} + a_t \ ,$$

or

$$(1 - \emptyset_1 B) z_t = a_t \ ,$$

where

$\{a_t\}$ is a zero mean white noise process with constant variance σ_a^2.

- Assuming the stationarity, the mean of z_t is equal to zero. The informal proof can be shown as follows:

$$E(z_t) = \emptyset_1 E(z_{t-1}) + E(a_t)$$

$$\mu = \emptyset_1 \mu + \ 0$$

$$(1 - \emptyset_1)\mu = 0.$$

If $\emptyset_1 = 1$, then z_t will be a random walk process which is not stationary. Since the stationarity condition is assumed, $\mu = 0$.

- Assuming the stationarity, the variance of z_t is equal to $\frac{\sigma_a^2}{1-\emptyset_1^2}$. It can be shown as follows:

$$Var(z_t) = Var(\emptyset_1 z_{t-1} + a_t)$$

$$= \emptyset_1^2 Var(z_{t-1}) + 2\emptyset_1 Cov(z_{t-1}, a_t) + Var(a_t)$$
$$= \emptyset_1^2 Var(z_t) + \sigma_a^2$$

$$= \frac{\sigma_a^2}{1 - \emptyset_1^2} \, .$$

2.1.1 The Autocovariance Function of AR(1) Process

$$Cov(z_t\ , z_{t-k}) = Cov(\emptyset_1 z_{t-1} + a_t\ , z_{t-k}).$$

Assuming the stationarity,

For k=0;

$$Cov(z_t, z_t) = \gamma_0 = Var(z_t) = Cov(\emptyset_1 z_{t-1} + a_t\ , z_t)$$

$$= \emptyset_1 Cov(z_{t-1}, z_t) + Cov\ (a_t, z_t)$$

$$= \emptyset_1 \gamma_1 + Cov(a_t, z_t)$$

$$Cov(a_t, z_t) = Cov(a_t, \emptyset_1 z_{t-1} + a_t)$$

$$= Cov(a_t, a_t) + \emptyset_1 Cov(z_{t-1}, a_t)$$

$$= \sigma_a^2.$$

Then,

$$Cov(z_t, z_t) = \gamma_0 = Var(z_t) = \emptyset_1 \gamma_1 + \sigma_a^2.$$

For k=1;

$$Cov(z_t, z_{t-1}) = \gamma_1 = Cov(\emptyset_1 z_{t-1} + a_t\ , z_{t-1})$$

$$= \emptyset_1 Cov(z_{t-1}, z_{t-1}) + Cov(a_t, z_{t-1})$$

$$Cov(z_t, z_{t-1}) = \gamma_1 = \emptyset_1 \gamma_0.$$

For k=2;

$$Cov(z_t, z_{t-2}) = \gamma_2 = Cov(\emptyset_1 z_{t-1} + a_t\ , z_{t-2})$$

$$= \emptyset_1 Cov(z_{t-1}, z_{t-2}) + Cov(a_t, z_{t-2})$$

$$Cov(z_t, z_{t-2}) = \gamma_2 = \emptyset_1 \gamma_1 = \emptyset_1^2 \gamma_0.$$

For k=3;

$$Cov(z_t, z_{t-3}) = \gamma_3 = Cov(\emptyset_1 z_{t-1} + a_t \, , z_{t-3})$$

$$= \emptyset_1 Cov(z_{t-1}, z_{t-3}) + Cov(a_t, z_{t-3})$$

$$Cov(z_t, z_{t-3}) = \gamma_3 = \emptyset_1 \gamma_2 = \emptyset_1^3 \gamma_0.$$

So, $\gamma_k = \begin{cases} \frac{\sigma^2}{1-\emptyset^2}, \; k = 0 \\ \\ \emptyset_1^k \gamma_0, \; k \geq 1 \end{cases}$.

2.1.2 The Autocorrelation Function of AR(1) Process

$$Corr(z_t \, , z_{t-k}) = \rho_k = \frac{Cov(z_t \, , z_{t-k})}{\sqrt{Var(z_t)}\sqrt{Var(z_{t-k})}} = \frac{\gamma_k}{\gamma_0}.$$

For k=1;

$$Corr(z_t, z_{t-1}) = \rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\emptyset_1 \gamma_0}{\gamma_0} = \emptyset_1.$$

For k=2;

$$Corr(z_t, z_{t-2}) = \rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{\emptyset_1^2 \gamma_0}{\gamma_0} = \emptyset_1^2.$$

For k=3;

$$Corr(z_t, z_{t-3}) = \rho_3 = \frac{\gamma_3}{\gamma_0} = \frac{\emptyset_1^3 \gamma_0}{\gamma_0} = \emptyset_1^3.$$

So for $k \geq 1$

$$\rho_k = \emptyset_1^k \; .$$

As seen, the autocorrelation function of AR(1) process decays exponentially or oscillating depending on the sign of the parameter $\emptyset_1$.

2.1.3 The Partial Autocorrelation Function of AR(1) Process

It is known that partial autocorrelation function gives the direct correlation and it is denoted by

$$\emptyset_{kk} = Corr(z_t , z_{t-k} | z_{t-1}, z_{t-2}, \dots , z_{t-(k-1)}) \,,$$

$$\emptyset_{11} = Corr(z_t , z_{t-1}) = \rho_1 \,,$$

$$\emptyset_{22} = Corr(z_t , z_{t-2} | z_{t-1}) \,,$$

$$\emptyset_{33} = Corr(z_t , z_{t-3} | z_{t-1}, z_{t-2}) \,,$$

$$\vdots$$

The partial autocorrelation function can be calculated by the help of Yule-Walker equations. Yule-Walker equations for AR(p) process are defined as

$$\rho_k = \emptyset_1 \rho_{k-1} + \emptyset_2 \rho_{k-2} + \cdots + \emptyset_p \rho_{k-p} \quad \text{for} \quad k \geq 1 \,, \qquad (2.1.3.1)$$

where $\emptyset_1, \emptyset_2, \dots , \emptyset_p$ correspond to partial autocorrelation and can be thought as $\emptyset_{p1}, \emptyset_{p2}, \dots , \emptyset_{pp}$, respectively.

Levinson and Durbin's recursive formula is an option for calculating the partial autocorrelation function (Durbin,1960). Levinson and Durbin's recursive formula is

$$\emptyset_{kk} = \begin{cases} \rho_1 & , \quad k = 1 \\ \dfrac{\rho_k - \sum_{j=1}^{k-1} \emptyset_{k-1,j} \rho_{k-j}}{1 - \sum_{j=1}^{k-1} \emptyset_{k-1,j} \rho_j} , & \quad k \geq 1 \end{cases} \,, \qquad (2.1.3.2)$$

where

$$\emptyset_{kj} = \begin{cases} \emptyset_{k-1,j} - \emptyset_{kk}\emptyset_{k-1,k-j}\,, & j = 1,2,3,\dots,k-1 \\ \\ \emptyset_{kk} & ,\ \ j \geq k \end{cases} .$$

By using Equation 2.1.3.2

$$\emptyset_{11} = \rho_1 = \emptyset_1\,,$$

$$\emptyset_{22} = \frac{\rho_2 - \emptyset_{11}\rho_1}{1 - \emptyset_{11}\rho_1}.$$

From Equation 2.1.3.1 for AR(1) process

$$\rho_2 = \emptyset_1\rho_1\,,$$

which is equivalent to

$$\rho_2 = \emptyset_{11}\rho_1.$$

Then,

$$\emptyset_{22} = \frac{\emptyset_{11}\rho_1 - \emptyset_{11}\rho_1}{1 - \emptyset_{11}\rho_1} = 0\,,$$

$$\emptyset_{33} = \frac{\rho_3 - (\emptyset_{21}\rho_2 + \emptyset_{22}\rho_1)}{1 - (\emptyset_{21}\rho_1 + \emptyset_{22}\rho_2)}\,,$$

$$\emptyset_{21} = \emptyset_{11} - \emptyset_{22}\emptyset_{11} = \emptyset_{11}.$$

From Equation 2.1.3.1 for AR(1) process

$$\rho_3 = \emptyset_1\rho_2\,,$$

which is equivalent to

$$\rho_3 = \emptyset_{11}\rho_2.$$

Then,

$$\emptyset_{33} = \frac{\emptyset_{11}\rho_1 - \emptyset_{11}\rho_1}{1 - \emptyset_{11}\rho_1} = 0.$$

So, $$\emptyset_{kk} = \begin{cases} \rho_1 = \emptyset_1, & k = 1 \\ 0, & k \geq 2 \end{cases}.$$

The partial autocorrelation function of AR(1) process cuts off after lag 1.

2.2 The Second Order Autoregressive, AR(2) Process

A zero mean AR(2) process can be shown as follows:

$$z_t = \emptyset_1 z_{t-1} + \emptyset_2 z_{t-2} + a_t \,,$$

or

$$(1 - \emptyset_1 B - \emptyset_2 B^2) z_t = a_t \,,$$

where $\{a_t\}$ is a zero mean white noise process with constant variance σ_a^2.

- Assuming the stationarity, the mean of z_t is equal to zero. The informal proof can be shown as follows:

$$E(z_t) = \emptyset_1 E(z_{t-1}) + \emptyset_2 E(z_{t-2}) + E(a_t)$$

$$\mu = \emptyset_1 \mu + \emptyset_2 \mu + 0$$

$$(1 - \emptyset_1 - \emptyset_2)\mu = 0.$$

If $\emptyset_1 + \emptyset_2 = 1$, then the stationarity condition $\emptyset_1 + \emptyset_2 < 1$ is not valid. Since the stationarity condition is assumed, $\mu = 0$.

2.2.1 The Autocovariance Function of AR(2) Process

$$Cov(z_t\ , z_{t-k}) = Cov(\emptyset_1 z_{t-1} + \emptyset_2 z_{t-2} + a_t\ , z_{t-k}).$$

Assuming the stationarity,

For k = 0;

$$Cov(z_t, z_t) = \gamma_0 = Var(z_t) = Cov(\emptyset_1 z_{t-1} + \emptyset_2 z_{t-2} + a_t\ , z_t)$$

$$= \emptyset_1 Cov(z_{t-1}, z_t) + \emptyset_2 Cov(z_{t-2}, z_t) + Cov(a_t, z_t)$$

$$Cov(z_t, z_t) = \gamma_0 = \emptyset_1 \gamma_1 + \emptyset_2 \gamma_2 + \ \sigma_a^2.$$

For k = 1;

$$Cov(z_t, z_{t-1}) = \gamma_1 = Cov(\emptyset_1 z_{t-1} + \emptyset_2 z_{t-2} + a_t\ , z_{t-1})$$

$$= \emptyset_1 Cov(z_{t-1}, z_{t-1}) + \emptyset_2 Cov(z_{t-2}, z_{t-1}) + Cov(a_t, z_{t-1})$$

$$Cov(z_t, z_{t-1}) = \gamma_1 = \emptyset_1 \gamma_0 + \emptyset_2 \gamma_1.$$

For k = 2;

$$Cov(z_t, z_{t-2}) = \gamma_2 = Cov(\emptyset_1 z_{t-1} + \emptyset_2 z_{t-2} + a_t\ , z_{t-2})$$

$$= \emptyset_1 Cov(z_{t-1}, z_{t-2}) + \emptyset_2 Cov(z_{t-2}, z_{t-2}) + Cov(a_t, z_{t-2})$$

$$Cov(z_t, z_{t-2}) = \gamma_2 = \emptyset_1 \gamma_1 + \emptyset_2 \gamma_0.$$

For k = 3;

$$Cov(z_t, z_{t-3}) = \gamma_3 = Cov(\emptyset_1 z_{t-1} + \emptyset_2 z_{t-2} + a_t\ , z_{t-3})$$

$$= \emptyset_1 Cov(z_{t-1}, z_{t-3}) + \emptyset_2 Cov(z_{t-2}, z_{t-3}) + Cov(a_t, z_{t-3})$$

$$Cov(z_t, z_{t-3}) = \gamma_3 = \emptyset_1 \gamma_2 + \emptyset_2 \gamma_1.$$

$$\text{So,}\quad \gamma_k = \begin{cases} \emptyset_1\gamma_1 + \emptyset_2\gamma_2 + \sigma_a^2, & k = 0 \\ \\ \emptyset_1\gamma_{k-1} + \emptyset_2\gamma_{k-2}, & k \geq 1 \end{cases} .$$

2.2.2 The Autocorrelation Function of AR(2) Process

$$Corr(z_t, z_{t-k}) = \rho_k = \frac{Cov(z_t, z_{t-k})}{\sqrt{Var(z_t)}\sqrt{Var(z_{t-k})}} = \frac{\gamma_k}{\gamma_0}.$$

For k=1;

$$Corr(z_t, z_{t-1}) = \rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\emptyset_1\gamma_0 + \emptyset_2\gamma_1}{\gamma_0} = \emptyset_1 + \emptyset_2\rho_1.$$

For k=2;

$$Corr(z_t, z_{t-2}) = \rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{\emptyset_1\gamma_1 + \emptyset_2\gamma_0}{\gamma_0} = \emptyset_1\rho_1 + \emptyset_2.$$

For k=3;

$$Corr(z_t, z_{t-3}) = \rho_3 = \frac{\gamma_3}{\gamma_0} = \frac{\emptyset_1\gamma_2 + \emptyset_2\gamma_1}{\gamma_0} = \emptyset_1\rho_2 + \emptyset_2\rho_1.$$

Thus,

$$\rho_k = \emptyset_1\rho_{k-1} + \emptyset_2\rho_{k-2} \qquad \text{for} \quad k \geq 1,$$

Above calculations show that the autocorrelation function of AR(2) process decays exponentially or oscillating depending on the sign and magnitude of the parameters $\emptyset_1$ and $\emptyset_2$.

2.2.3 The Partial Autocorrelation Function of AR(2) Process

By using Equation 2.1.3.2

$$\emptyset_{11} = \rho_1 = \frac{\emptyset_1}{1 - \emptyset_2},$$

$$\emptyset_{22} = \frac{\rho_2 - \emptyset_{11}\rho_1}{1 - \emptyset_{11}\rho_1} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} = \frac{\emptyset_2[(1 - \emptyset_2)^2 - \emptyset_1^2]}{(1 - \emptyset_2)^2 - \emptyset_1^2},$$

$$\emptyset_{33} = \frac{\rho_3 - (\emptyset_{21}\rho_2 + \emptyset_{22}\rho_1)}{1 - (\emptyset_{21}\rho_1 + \emptyset_{22}\rho_2)}.$$

From Equation 2.1.3.1 on page 15 for AR(2) process

$$\rho_3 = \emptyset_1\rho_2 + \emptyset_2\rho_1,$$

which is equivalent to

$$\rho_3 = \emptyset_{21}\rho_2 + \emptyset_{22}\rho_1.$$

Then,

$$\emptyset_{33} = \frac{(\emptyset_{21}\rho_2 + \emptyset_{22}\rho_1) - (\emptyset_{21}\rho_2 + \emptyset_{22}\rho_1)}{1 - (\emptyset_{21}\rho_1 + \emptyset_{22}\rho_2)} = 0,$$

$$\emptyset_{44} = \frac{\rho_4 - (\emptyset_{31}\rho_3 + \emptyset_{32}\rho_2 + \emptyset_{33}\rho_1)}{1 - (\emptyset_{31}\rho_1 + \emptyset_{32}\rho_2 + \emptyset_{33}\rho_3)},$$

$$\emptyset_{31} = \emptyset_{21} - \emptyset_{33}\emptyset_{22} = \emptyset_{21},$$

$$\emptyset_{32} = \emptyset_{22} - \emptyset_{33}\emptyset_{21} = \emptyset_{22}.$$

From Equation 2.1.3.1 on page 15 for AR(2) process

$$\rho_4 = \emptyset_1\rho_3 + \emptyset_2\rho_2,$$

which is equivalent to

$$\rho_4 = \emptyset_{21}\rho_3 + \emptyset_{22}\rho_2.$$

Then,

$$\emptyset_{44} = \frac{(\emptyset_{21}\rho_3 + \emptyset_{22}\rho_2) - (\emptyset_{21}\rho_3 + \emptyset_{22}\rho_2)}{1 - (\emptyset_{21}\rho_1 + \emptyset_{22}\rho_2)} = 0.$$

So, $\emptyset_{kk} = \begin{cases} \frac{\emptyset_1}{1-\emptyset_2} & , \ k=1 \\ \frac{\emptyset_2\left[(1-\emptyset_2)^2-\emptyset_1^2\right]}{(1-\emptyset_2)^2-\emptyset_1^2} & , \ k=2 \\ 0 & , \ k \geq 3 \end{cases}$.

The partial autocorrelation function of AR(2) process cuts off after lag 2.

2.3 The General p^{th} Order Autoregressive, AR(p) Process

A zero mean AR(p) model can be shown as follows:

$$z_t = \emptyset_1 z_{t-1} + \cdots + \emptyset_p z_{t-p} + a_t ,$$

or

$$\left(1 - \emptyset_1 B - \cdots - \emptyset_p B^p\right) z_t = a_t ,$$

where

$\{a_t\}$ is a zero mean white noise process with constant variance σ_a^2.

- Assuming the stationarity, the mean of z_t is equal to zero. The informal proof can be shown as follows:

$$E(z_t) = \emptyset_1 E(z_{t-1}) + \emptyset_2 E(z_{t-2}) + \cdots + \emptyset_p E\left(z_{t-p}\right) + E(a_t)$$

$$\mu = \emptyset_1 \mu + \emptyset_2 \mu + \cdots + \emptyset_p \mu + 0$$

$$\left(1 - \emptyset_1 - \emptyset_2 - \cdots - \emptyset_p\right)\mu = 0.$$

If $\emptyset_1 + \emptyset_2 + \cdots + \emptyset_p = 1$, then the stationarity condition $\emptyset_1 + \emptyset_2 + \cdots + \emptyset_p < 1$ is not valid. Since the stationarity condition is assumed, $\mu = 0$.

2.3.1 The Autocovariance Function of AR(p) Process

$$Cov(z_t\ ,z_{t-k}) = Cov\big(\emptyset_1 z_{t-1} + \cdots + \emptyset_p z_{t-p} + a_t\ ,z_{t-k}\big).$$

Assuming the stationarity,

For k = 0;

$$Cov(z_t, z_t) = \gamma_0 = Var(z_t) = Cov(\emptyset_1 z_{t-1} + \cdots + \emptyset_p z_{t-p} + a_t\ , z_t)$$

$$= \emptyset_1 Cov(z_{t-1}, z_t) + \cdots + \emptyset_p Cov\big(z_{t-p}, z_t\big) + Cov(a_t, z_t)$$

$$Cov(z_t, z_t) = \gamma_0 = \emptyset_1\gamma_1 + \cdots + \emptyset_p\gamma_p + \ \sigma_a^2.$$

For k = 1;

$$Cov(z_t, z_{t-1}) = \gamma_1 = Cov(\emptyset_1 z_{t-1} + \cdots + \emptyset_p z_{t-p} + a_t\ , z_{t-1})$$

$$= \emptyset_1 Cov(z_{t-1}, z_{t-1}) + \cdots + \emptyset_p Cov\big(z_{t-p}, z_{t-1}\big) + Cov(a_t, z_{t-1})$$

$$Cov(z_t, z_{t-1}) = \gamma_1 = \emptyset_1\gamma_0 + \cdots + \emptyset_p\gamma_{1-p}.$$

For k = 2;

$$Cov(z_t, z_{t-2}) = \gamma_2 = Cov(\emptyset_1 z_{t-1} + \cdots + \emptyset_p z_{t-p} + a_t\ , z_{t-2})$$

$$= \emptyset_1 Cov(z_{t-1}, z_{t-2}) + \cdots + \emptyset_p Cov\big(z_{t-p}, z_{t-2}\big) + Cov(a_t, z_{t-2})$$

$$Cov(z_t, z_{t-2}) = \gamma_2 = \emptyset_1\gamma_1 + \cdots + \emptyset_p\gamma_{2-p}.$$

For k = 3;

$$Cov(z_t, z_{t-3}) = \gamma_3 = Cov(\emptyset_1 z_{t-1} + \cdots + \emptyset_p z_{t-p} + a_t\ , z_{t-3})$$

$$= \emptyset_1 Cov(z_{t-1}, z_{t-3}) + \cdots + \emptyset_p Cov\big(z_{t-p}, z_{t-3}\big) + Cov(a_t, z_{t-2})$$

$$Cov(z_t, z_{t-3}) = \gamma_3 = \emptyset_1\gamma_2 + \cdots + \emptyset_p\gamma_{3-p}.$$

$$\text{So,}\quad \gamma_k = \begin{cases} \emptyset_1\gamma_1 + \cdots + \emptyset_p\gamma_p + \sigma_a^2, & k = 0 \\ \\ \emptyset_1\gamma_{k-1} + \cdots + \emptyset_p\gamma_{k-p}, & k \geq 1 \end{cases} .$$

2.3.2 The Autocorrelation Function of AR(p) Process

$$Corr(z_t\,, z_{t-k}\,) = \rho_k = \frac{Cov(z_t\,, z_{t-k})}{\sqrt{Var(z_t)}\,\sqrt{Var(z_{t-k})}} = \frac{\gamma_k}{\gamma_0}.$$

For k=1;

$$Corr(z_t, z_{t-1}) = \rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\emptyset_1\gamma_0 + \cdots + \emptyset_p\gamma_{1-p}}{\gamma_0} = \emptyset_1 + \cdots + \emptyset_p\rho_{1-p}.$$

For k=2;

$$Corr(z_t, z_{t-2}) = \rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{\emptyset_1\gamma_1 + \cdots + \emptyset_p\gamma_{2-p}}{\gamma_0} = \emptyset_1\rho_1 + \cdots + \emptyset_p\rho_{2-p}.$$

For k=3;

$$Corr(z_t, z_{t-3}) = \rho_3 = \frac{\gamma_3}{\gamma_0} = \frac{\emptyset_1\gamma_2 + \cdots + \emptyset_p\gamma_{3-p}}{\gamma_0} = \emptyset_1\rho_2 + \cdots + \emptyset_p\rho_{3-p}.$$

Hence,

$$\rho_k = \emptyset_1\rho_{k-1} + \cdots + \emptyset_p\rho_{k-p} \qquad \text{for} \quad k \geq 1,$$

The above calculations show that the autocorrelation function of AR(p) process decays exponentially or oscillating depending on the sign and magnitude of the parameters $\emptyset_1$, $\emptyset_2$, ... , $\emptyset_p$.

2.3.3 The Partial Autocorrelation Function of AR(p) Process

As stated at the Equation 2.1.3.1 on page 15 the Yule-Walker equations for AR(p) process is

$$\rho_k = \emptyset_1 \rho_{k-1} + \cdots + \emptyset_p \rho_{k-p} \qquad \text{for } k \geq 1,$$

It is logical to think like

- $\emptyset_1$ corresponds to $\emptyset_{p1}$
- $\emptyset_2$ corresponds to $\emptyset_{p2}$

  $\vdots$
- $\emptyset_p$ corresponds to $\emptyset_{pp}$

Then,

$$\rho_1 = \emptyset_{11} + \emptyset_{22}\rho_1 + \cdots + \emptyset_{pp}\rho_{1-p} \ ,$$

$$\rho_2 = \emptyset_{11}\rho_1 + \emptyset_{22} + \cdots + \emptyset_{pp}\rho_{2-p} \ ,$$

$$\rho_3 = \emptyset_{11}\rho_2 + \emptyset_{22}\rho_1 + \cdots + \emptyset_{pp}\rho_{3-p} \ ,$$

$$\vdots$$

$$\rho_p = \emptyset_{11}\rho_{p-1} + \emptyset_{22}\rho_{p-2} + \cdots + \emptyset_{pp} \ ,$$

$$\vdots$$

$$\rho_{p+j} = \emptyset_{11}\rho_{p+j-1} + \emptyset_{22}\rho_{p+j-2} + \cdots + \emptyset_{pp}\rho_j .$$

It is understood that ρ_k can be expressed in terms of $\emptyset_{11}, \emptyset_{22}, \ldots, \emptyset_{pp}$ when $k \geq p$. Therefore, for $k > p$

$$\emptyset_{kk} = 0.$$

The partial autocorrelation function of AR(p) process cuts off after lag p.

2.4 Temporal Aggregation of AR(1) Process

Suppose that the basic series follows a zero mean AR(1) model

$$(1-\emptyset_1 B)z_t = a_t \ ,$$

where $\{a_t\}$ is a zero mean white noise process with constant variance σ_a^2.

Temporal aggregation of AR(1) process will be explained by looking at m=3 case. The aggregated series with aggregation period is

$$Z_T = (1+B+B^2)z_{3T} \ ,$$

where m=3 and d=0.

Letting $\emptyset_p(B) = \prod_{j=1}^{p}(1-\delta_j \ B)$ and multiplying $\prod_{j=1}^{p}\left[\frac{\left(1-\delta_j^m B^m\right)}{(1-\delta_j B)}\right]\left[\frac{(1-B^m)^{d+1}}{(1-B)^{d+1}}\right]$ on both sides of nonaggregate series z_t :

$$\frac{(1-\delta_1^3 B^3)(1-B^3)}{(1-\delta_1 B)(1-B)}(1-\emptyset_1 B)z_t = \frac{(1-\delta_1^3 B^3)(1-B^3)}{(1-\delta_1 B)(1-B)}a_t$$

$$(1-\delta_1^3 B^3)(1+B+B^2)z_t = (1+\delta_1 B+\delta_1^2 B^2)(1+B+B^2)a_t$$

$(1+B+B^2-\delta_1^3 B^3-\delta_1^3 B^4-\delta_1^3 B^5)z_t = [1+(1+\delta_1)B+(1+\delta_1+\delta_1^2)B^2+(\delta_1+\delta_1^2)B^3+\delta_1^2 B^4]a_t.$

Substitute t for $3T$

$(1+B+B^2-\delta_1^3 B^3-\delta_1^3 B^4-\delta_1^3 B^5)z_{3T} = [1+(1+\delta_1)B+(1+\delta_1+\delta_1^2)B^2+(\delta_1+\delta_1^2)B^3+\delta_1^2 B^4]a_{3T}$

$z_{3T}+z_{3T-1}+z_{3T-2}-\delta_1^3 z_{3T-3}-\delta_1^3 z_{3T-4}-\delta_1^3 z_{3T-5} = a_{3T}+(1+\delta_1)a_{3T-1}+(1+\delta_1+\delta_1^2)a_{3T-2}+(\delta_1+\delta_1^2)a_{3T-3}+\delta_1^2 a_{3T-4}$

$$Z_T-\delta_1^3 Z_{T-1} = a_{3T}+(1+\delta_1)a_{3T-1}+(1+\delta_1+\delta_1^2)a_{3T-2}+(\delta_1+\delta_1^2)a_{3T-3}+\delta_1^2 a_{3T-4}$$

$$(1-\delta_1^3 B)Z_T = a_{3T} + (1+\delta_1)a_{3(T-\frac{1}{3})} + (1+\delta_1+\delta_1^2)a_{3(T-\frac{2}{3})} + (\delta_1 + \delta_1^2)a_{3(T-1)} + \delta_1^2 a_{3(T-\frac{4}{3})} .$$

Say $(1-\delta_1^3 B)Z_T = X_{3T}$.

It is obvious that $Cov(X_{3T}, X_{3(T-K)})$ will be equal to zero if K is greater than the integer part of $\frac{4}{3}$ which is equal to 1 (Amemiya and Wu, 1972).

Consequently, the aggregated series of an AR(1) model follows an ARMA(1,1) model when m=3.

$$(1-\delta_1^3 B)Z_T = (1-\beta_1 B)A_T ,$$

where $\{A_T\}$ is a zero mean white noise process with constant variance σ_A^2.

The parameters β_1 and σ_A^2 of the aggregate series Z_T are functions of $\emptyset_1$ and σ_a^2. Also, it is useful to state that the root of AR polynomial of the aggregate series, Z_T is the third power of the root of AR polynomial of the basic series z_t (Telser, 1967).

2.5 Temporal Aggregation of AR(2) Process

Suppose that the basic series follows a zero mean AR(2) model

$$(1-\emptyset_1 B-\emptyset_2 B^2)z_t = a_t ,$$

where $\{a_t\}$ is a zero mean white noise process with constant variance σ_a^2.

Temporal aggregation of AR(2) process will be explained by looking at m=3 case. The aggregated series with aggregation period is

$$Z_T = (1+B+B^2)z_{3T} ,$$

where m=3 and d=0.

Letting $\emptyset_p(B) = \prod_{j=1}^{p}(1-\delta_j B)$ and multiplying $\prod_{j=1}^{p}\left[\frac{\left(1-\delta_j^m B^m\right)}{(1-\delta_j B)}\right]\left[\frac{(1-B^m)^{d+1}}{(1-B)^{d+1}}\right]$ on both sides of basic series z_t :

$$\frac{(1-\delta_1^3B^3)(1-\delta_2^3B^3)}{(1-\delta_1B)(1-\delta_2B)}\frac{(1-B^3)}{(1-B)}(1-\emptyset_1B-\emptyset_2B^2)z_t = \frac{(1-\delta_1^3B^3)(1-\delta_2^3B^3)}{(1-\delta_1B)(1-\delta_2B)}\frac{(1-B^3)}{(1-B)}a_t$$

$$(1-\delta_1^3B^3)(1-\delta_2^3B^3)(1+B+B^2)z_t = (1+\delta_1B+\delta_1^2B^2)(1+\delta_2B+\delta_2^2B^2)(1+B+B^2)a_t$$

$$[1-(\delta_1^3+\delta_2^3)B^3+\delta_1^3\delta_2^3B^6](1+B+B^2)z_t = [1+(\delta_1+\delta_2)B+(\delta_1^2+\delta_1\delta_2+\delta_2^2)B^2+(\delta_1\delta_2^2+\delta_1^2\delta_2)B^3+\delta_1^2\delta_2^2B^4](1+B+B^2)a_t$$

$$[1+B+B^2-(\delta_1^3+\delta_2^3)B^3-(\delta_1^3+\delta_2^3)B^4-(\delta_1^3+\delta_2^3)B^5+\delta_1^3\delta_2^3B^6+\delta_1^3\delta_2^3B^7+\delta_1^3\delta_2^3B^8]z_t = [1+(1+\delta_1+\delta_2)B+(1+\delta_1+\delta_2+\delta_1^2+\delta_1\delta_2+\delta_2^2)B^2+(\delta_1+\delta_2+\delta_1^2+\delta_1\delta_2+\delta_2^2+\delta_1\delta_2^2+\delta_1^2\delta_2)B^3+(\delta_1^2+\delta_1\delta_2+\delta_2^2+\delta_1\delta_2^2+\delta_1^2\delta_2+\delta_1^2\delta_2^2)B^4+(\delta_1\delta_2^2+\delta_1^2\delta_2+\delta_1^2\delta_2^2)B^5+\delta_1^2\delta_2^2B^6]a_t.$$

Substitute t for $3T$

$$[1+B+B^2-(\delta_1^3+\delta_2^3)B^3-(\delta_1^3+\delta_2^3)B^4-(\delta_1^3+\delta_2^3)B^5+\delta_1^3\delta_2^3B^6+\delta_1^3\delta_2^3\delta_1^3\delta_2^3B^7+\delta_1^3\delta_2^3B^8]z_{3T} = [1+(1+\delta_1+\delta_2)B+(1+\delta_1+\delta_2+\delta_1^2+\delta_1\delta_2+\delta_2^2)B^2+(\delta_1+\delta_2+\delta_1^2+\delta_1\delta_2+\delta_2^2+\delta_1\delta_2^2+\delta_1^2\delta_2)B^3+(\delta_1^2+\delta_1\delta_2+\delta_2^2+\delta_1\delta_2^2+\delta_1^2\delta_2+\delta_1^2\delta_2^2)B^4+(\delta_1\delta_2^2+\delta_1^2\delta_2+\delta_1^2\delta_2^2)B^5+\delta_1^2\delta_2^2B^6]a_{3T}$$

$$Z_T-(\delta_1^3+\delta_2^3)Z_{T-1}+\delta_1^3\delta_2^3Z_{T-2} = a_{3T}+(1+\delta_1+\delta_2)a_{3T-1}+(1+\delta_1+\delta_2+\delta_1^2+\delta_1\delta_2+\delta_2^2)a_{3T-2}+(\delta_1+\delta_2+\delta_1^2+\delta_1\delta_2+\delta_2^2+\delta_1\delta_2^2+\delta_1^2\delta_2)a_{3T-3}+(\delta_1^2+\delta_1\delta_2+\delta_2^2+\delta_1\delta_2^2+\delta_1^2\delta_2+\delta_1^2\delta_2^2)a_{3T-4}+(\delta_1\delta_2^2+\delta_1^2\delta_2+\delta_1^2\delta_2^2)a_{3T-5}+\delta_1^2\delta_2^2a_{3T-6}$$

$$(1-\delta_1^3B)(1-\delta_2^3B)Z_T = a_{3T}+(1+\delta_1+\delta_2)a_{3(T-\frac{1}{3})}+(1+\delta_1+\delta_2+\delta_1^2+\delta_1\delta_2+\delta_2^2)a_{3(T-\frac{2}{3})}+(\delta_1+\delta_2+\delta_1^2+\delta_1\delta_2+\delta_2^2+\delta_1\delta_2^2+\delta_1^2\delta_2)a_{3(T-1)}+(\delta_1^2+\delta_1\delta_2+\delta_2^2+\delta_1\delta_2^2+\delta_1^2\delta_2+\delta_1^2\delta_2^2)a_{3(T-\frac{4}{3})}+(\delta_1\delta_2^2+\delta_1^2\delta_2+\delta_1^2\delta_2^2)a_{3(T-\frac{5}{3})}+\delta_1^2\delta_2^2a_{3(T-2)}.$$

Say $(1-\delta_1^3B)(1-\delta_2^3B)Z_T = X_{3T}$.

It is obvious that $Cov(X_{3T}, X_{3(T-K)})$ will be equal to zero if K is greater than 2 (Amemiya and Wu, 1972).

Consequently, the aggregated series of an AR(2) model follows an ARMA(2,2) model when m=3.

$$(1-\delta_1^3 B)(1-\delta_2^3 B)Z_T = (1-\beta_1 B-\beta_2 B^2)A_T\ ,$$

where $\{A_T\}$ is a zero mean white noise process with constant variance σ_A^2.

The parameters β_1, β_2 and σ_A^2 of the aggregate series Z_T are functions of $\emptyset_1$, $\emptyset_2$ and σ_a^2. Also, it is useful to state that the roots of AR polynomial of the aggregate series Z_T are the third power of the root of AR polynomial of the basic series z_t (Telser, 1967).

2.6 Temporal Aggregation of AR(p) Process

Suppose that the basic series follows a zero mean AR(p) model

$$(1-\emptyset_1 B-\cdots-\emptyset_p B^p)z_t = a_t\ ,$$

where $\{a_t\}$ is a zero mean white noise process with constant variance σ_a^2.

The aggregated series with aggregation period is

$$Z_T = (1+B+B^2)z_{3T}\ ,$$

where m=3 and d=0.

Letting $\emptyset_p(B) = \prod_{j=1}^{p}(1-\delta_j\ B)$ and multiplying $\prod_{j=1}^{p}\left[\frac{(1-\delta_j^m B^m)}{(1-\delta_j B)}\right]\left[\frac{(1-B^m)^{d+1}}{(1-B)^{d+1}}\right]$ on both sides of basic series z_t :

$$\frac{(1-\delta_1^3 B^3)(1-\delta_2^3 B^3)\ldots(1-\delta_p^3 B^3)}{(1-\delta_1 B)(1-\delta_2 B)\ldots(1-\delta_p B)}\frac{(1-B^3)}{(1-B)}(1-\emptyset_1 B-\cdots-\emptyset_p B^p)z_t$$
$$=\frac{(1-\delta_1^3 B^3)(1-\delta_2^3 B^3)\ldots(1-\delta_p^3 B^3)}{(1-\delta_1 B)(1-\delta_2 B)\ldots(1-\delta_p B)}\ \frac{(1-B^3)}{(1-B)}a_t$$

$$(1-\delta_1^3B^3)(1-\delta_2^3B^3)\dots\left(1-\delta_p^3B^3\right)(1+B+B^2)z_t=(1+\delta_1B+\delta_1^2B^2)(1+\delta_2B+\delta_2^2B^2)\dots\left(1+\delta_pB+\delta_p^2B^2\right)(1+B+B^2)a_t.$$

Substitute t for $3T$

$$(1-\delta_1^3B^3)(1-\delta_2^3B^3)\dots\left(1-\delta_p^3B^3\right)(1+B+B^2)z_{3T}=(1+\delta_1B+\delta_1^2B^2)(1+\delta_2B+\delta_2^2B^2)\dots\left(1+\delta_pB+\delta_p^2B^2\right)(1+B+B^2)a_{3T}$$

$$(1-\delta_1^3B^3)(1-\delta_2^3B^3)\dots\left(1-\delta_p^3B^3\right)(1+B+B^2)z_{3T}=(1-\delta_1^3B)(1-\delta_2^3B)\dots\left(1-\delta_p^3B\right)Z_T.$$

Say $(1-\delta_1^3B)(1-\delta_2^3B)\dots\left(1-\delta_p^3B\right)Z_T=X_{3T}$

$X_{3T}=(1+\delta_1B+\delta_1^2B^2)(1+\delta_2B+\delta_2^2B^2)\dots\left(1+\delta_pB+\delta_p^2B^2\right)(1+B+B^2)a_{3T}.$

After several calculations it can be easily seen that $Cov\left(X_{3T},X_{3(T-K)}\right)=0$ if K is greater than the integer part of $\frac{2(p+1)}{3}$ (Amemiya and Wu, 1972).

Consequently, the aggregated series of an AR(p) model follows an ARMA(p,$\left[\frac{2(p+1)}{3}\right]$) model when m=3.

$$(1-\delta_1^3B)(1-\delta_2^3B)\dots\left(1-\delta_p^3B\right)Z_T = \left(1-\beta_1B-\beta_2B^2-\cdots-\beta_{\left[\frac{2(p+1)}{3}\right]}B^{\left[\frac{2(p+1)}{3}\right]}\right)A_T\,,$$

where $\{A_T\}$ is a zero mean white noise process with constant variance σ_A^2.

The parameters $\beta_1,\beta_2\,,\dots,\beta_{\left[\frac{2(p+1)}{3}\right]}$ and σ_A^2 of the aggregate series Z_T are functions of $\emptyset_1$, $\emptyset_2$,…,$\emptyset_p$ and σ_a^2. Also, it is useful to state that the roots of AR polynomial of the aggregate series Z_T are the third power of the roots of AR polynomial of the nonaggregate series z_t (Telser, 1967).

For general m^{th} order aggregate is

$$Z_T=(1+B+B^2)z_{mT}\,,$$

where d=0.

Letting $\emptyset_p(B) = \prod_{j=1}^{p}(1-\delta_j\ B)$ and multiplying $\prod_{j=1}^{p}\left[\frac{(1-\delta_j^m B^m)}{(1-\delta_j B)}\right]\left[\frac{(1-B^m)^{d+1}}{(1-B)^{d+1}}\right]$ on both sides of basic series z_t :

$$\frac{(1-\delta_1^m B^m)(1-\delta_2^m B^m)\ldots(1-\delta_p^m B^m)}{(1-\delta_1 B)(1-\delta_2 B)\ldots(1-\delta_p B)}\frac{(1-B^m)}{(1-B)}\left(1-\emptyset_1 B-\cdots-\emptyset_p B^p\right)z_t$$
$$=\frac{(1-\delta_1^m B^m)(1-\delta_2^m B^m)\ldots(1-\delta_p^m B^m)}{(1-\delta_1 B)(1-\delta_2 B)\ldots(1-\delta_p B)}\frac{(1-B^m)}{(1-B)}a_t\ .$$

When the above equation is written in the explicit form, it is found that the m^{th} aggregate of an AR(p) model is an ARMA(p, $\left[\frac{(m-1)(p+1)}{m}\right]$) (Amemiya and Wu, 1972).

$$\prod_{j=1}^{p}(1-\delta_j^m B)Z_T=\left(1-\beta_1 B-\beta_2 B^2-\cdots-\beta_{\left[\frac{(m-1)(p+1)}{m}\right]}B^{\left[\frac{(m-1)(p+1)}{m}\right]}\right)A_T\ ,$$

where $\{A_T\}$ is a zero mean white noise process with constant variance σ_A^2.

The parameters β_1, β_2 , …, $\beta_{\left[\frac{(m-1)(p+1)}{m}\right]}$ and σ_A^2 of the aggregate series Z_T are functions of $\emptyset_1$, $\emptyset_2$,…,$\emptyset_p$and σ_a^2. Also, it is useful to state that the roots of AR polynomial of the aggregate series Z_T are the m^{th} power of the roots of AR polynomial of the nonaggregate series z_t (Telser, 1967).

CHAPTER 3

TEMPORAL AGGREGATION OF MOVING AVERAGE PROCESSES

In this chapter, the effect of temporal aggregation on moving average processes will be discussed. For simplicity of the calculations, zero mean moving average processes will be considered.

A moving average process of order q can be shown as follows:

$$z_t = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} \ ,$$

or

$$z_t = \theta(B) a_t \ ,$$

where

$\theta(B) = (1 - \theta_1 B - \cdots - \theta_q B^q)$ and $\{a_t\}$ is a zero mean white noise process with constant variance σ_a^2.

As it can be seen, the MA(q) model is in the random shock form and the summation is finite. So, it can be said that a finite moving average process is always stationary. If the roots of $\theta(B) = 0$ lie outside the unit circle then the moving average process is invertible.

3.1 The First Order Moving Average, MA(1) Process

A zero mean MA(1) process can be shown as follows:

$$z_t = a_t - \theta_1 a_{t-1} ,$$

or

$$z_t = (1 - \theta_1 B) a_t ,$$

where $\{a_t\}$ is a zero mean white noise process with constant variance σ_a^2.

- The mean of z_t is equal to zero since $\{a_t\}$ is a zero mean white noise process.
$$E(z_t) = 0 .$$

- The variance of z_t is equal to $(1 + \theta_1^2)\sigma_a^2$. It can be shown as follows:
$$Var(z_t) = Var(a_t - \theta_1 a_{t-1})$$
$$= Var(a_t) + \theta_1^2 Var(a_{t-1}) - 2\theta_1 Cov(a_t , a_{t-1})$$
$$= \sigma_a^2 + \theta_1^2 \sigma_a^2 = (1 + \theta_1^2)\sigma_a^2 .$$

3.1.1 The Autocovariance Function of MA(1) Process

$$Cov(z_t , z_{t-k}) = Cov(a_t - \theta_1 a_{t-1} , a_{t-k} - \theta_1 a_{t-k-1}).$$

For k=0 ;

$$Cov(z_t , z_t) = \gamma_0 = Var(z_t) = Cov(a_t - \theta_1 a_{t-1} , a_t - \theta_1 a_{t-1})$$

$$= Cov(a_t , a_t) - 2\theta_1 Cov(a_t , a_{t-1}) + \theta_1^2 Cov(a_{t-1}, a_{t-1})$$

$$Cov(z_t , z_t) = \gamma_0 = \sigma_a^2 + \theta_1^2 \sigma_a^2 = (1 + \theta_1^2)\sigma_a^2.$$

For k=1 ;

$$Cov\ (z_t , z_{t-1}) = \gamma_1 = Cov(a_t - \theta_1 a_{t-1} , a_{t-1} - \theta_1 a_{t-2})$$

$$= Cov(a_t, a_{t-1}) - \theta_1 Cov(a_t, a_{t-2}) - \theta_1 Cov(a_{t-1}, a_{t-1}) + \theta_1^2 Cov(a_{t-1}, a_{t-2})$$

$$Cov\ (z_t\ , z_{t-1}) = \gamma_1 = -\theta_1 \sigma_a^2.$$

For k=2 ;

$$Cov(z_t\ , z_{t-2}) = \gamma_2 = Cov(a_t - \theta_1 a_{t-1}\ , a_{t-2} - \theta_1 a_{t-3})$$

$$= Cov(a_t, a_{t-2}) - \theta_1 Cov(a_t, a_{t-3}) - \theta_1 Cov(a_{t-1}, a_{t-2}) + \theta_1^2 Cov(a_{t-1}, a_{t-3})$$

$$Cov(z_t\ , z_{t-2}) = \gamma_2 = 0.$$

For k=3 ;

$$Cov(z_t\ , z_{t-3}) = \gamma_3 = Cov(a_t - \theta_1 a_{t-1}\ , a_{t-3} - \theta_1 a_{t-4})$$

$$= Cov(a_t, a_{t-3}) - \theta_1 Cov(a_t, a_{t-4}) - \theta_1 Cov(a_{t-1}, a_{t-3}) + \theta_1^2 Cov(a_{t-1}, a_{t-4})$$

$$Cov(z_t\ , z_{t-3}) = \gamma_3 = 0.$$

So, $$\gamma_k = \begin{cases} (1+\theta_1^2)\sigma_a^2\ , & k = 0 \\ -\theta_1 \sigma_a^2 & ,\ k = 1 \\ 0 & ,\ k > 1 \end{cases} .$$

3.1.2 The Autocorrelation Function of MA(1) Process

$$Corr(z_t\ , z_{t-k}\ ) = \rho_k = \frac{Cov(z_t\ , z_{t-k})}{\sqrt{Var(z_t)}\ \sqrt{Var(z_{t-k})}} = \frac{\gamma_k}{\gamma_0}.$$

For k=1;

$$Corr(z_t, z_{t-1}) = \rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{-\theta_1 \sigma_a^2}{(1+\theta_1^2)\sigma_a^2} = \frac{-\theta_1}{(1+\theta_1^2)}.$$

For k=2;

$$Corr(z_t, z_{t-2}) = \rho_2 = \frac{\gamma_2}{\gamma_0} = 0.$$

For k=3;

$$Corr(z_t, z_{t-3}) = \rho_3 = \frac{\gamma_3}{\gamma_0} = 0.$$

So, $\rho_k = \begin{cases} \frac{-\theta_1}{(1+\theta_1^2)}, & k = 1 \\ 0, & k \geq 2 \end{cases}$.

Above calculations show that the autocorrelation function of MA(1) process cuts off after lag 1.

3.1.3 The Partial Autocorrelation Function of MA(1) Process

By using Equation 2.1.3.2 on page 15

$$\emptyset_{11} = \rho_1 = \frac{-\theta_1}{1+\theta_1^2} = \frac{-\theta_1(1-\theta_1^2)}{1-\theta_1^4} ,$$

$$\emptyset_{22} = \frac{-\rho_1^2}{1-\rho_1^2} = \frac{-\theta_1^2}{1+\theta_1^2+\theta_1^4} = \frac{-\theta_1^2(1-\theta_1^2)}{1-\theta_1^6},$$

$$\emptyset_{33} = \frac{\rho_1^3}{1-2\rho_1^2} = \frac{-\theta_1^3}{1+\theta_1^2+\theta_1^4+\theta_1^6} = \frac{-\theta_1^3(1-\theta_1^2)}{1-\theta_1^8}.$$

For $k \geq 1$;

$$\emptyset_{kk} = \frac{-\theta_1^k(1-\theta_1^2)}{1-\theta_1^{2(k+1)}}.$$

As it can be seen, the partial autocorrelation function of MA(1) process does not cut off after lag 1. It shows exponential or oscillating decay depending on the parameter

θ_1. If the sign of θ_1 is positive, then the partial autocovariance function shows exponential decay and if the sign of θ_1 is negative, then it shows oscillating decay.

3.2 The Second Order Moving Average, MA(2) Process

A zero mean MA(2) process can be shown as follows:

$$z_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} ,$$

or

$$z_t = (1 - \theta_1 B - \theta_2 B^2) a_t ,$$

where $\{a_t\}$ is a zero mean white noise process with constant variance σ_a^2.

- The mean of z_t is equal to zero since $\{a_t\}$ is a zero mean white noise process.

$$E(z_t) = 0 .$$

3.2.1 The Autocovariance Function of MA(2) Process

$$Cov(z_t , z_{t-k}) = Cov(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} , a_{t-k} - \theta_1 a_{t-k-1} - \theta_2 a_{t-k-2}).$$

For k=0 ;

$Cov(z_t , z_t) = \gamma_0 = Var(z_t) = Cov(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} , a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2})$

$= Cov(a_t, a_t) - 2\theta_1 Cov(a_t, a_{t-1}) - 2\theta_2 Cov(a_t, a_{t-2}) + \theta_1^2 Cov(a_{t-1}, a_{t-1}) + 2\theta_1\theta_2 Cov(a_{t-1}, a_{t-2}) + \theta_2^2 Cov(a_{t-2}, a_{t-2})$

$$Cov(z_t , z_t) = \gamma_0 = \sigma_a^2 + \theta_1^2\sigma_a^2 + \theta_2^2\sigma_a^2 = (1 + \theta_1^2 + \theta_2^2)\sigma_a^2.$$

For k=1 ;

$$Cov(z_t , z_{t-1}) = \gamma_1 = Cov(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} , a_{t-1} - \theta_1 a_{t-2} - \theta_2 a_{t-3})$$

$= Cov(a_t, a_{t-1}) - \theta_1 Cov(a_t, a_{t-2}) - \theta_2 Cov(a_t, a_{t-3}) - \theta_1 Cov(a_{t-1}, a_{t-1}) + (\theta_1^2 - \theta_2) Cov(a_{t-1}, a_{t-2}) + \theta_1\theta_2 Cov(a_{t-1}, a_{t-3}) + \theta_1\theta_2 Cov(a_{t-2}, a_{t-2}) + \theta_2^2 Cov(a_{t-2}, a_{t-3})$

$$Cov(z_t, z_{t-1}) = \gamma_1 = -\theta_1\sigma_a^2 + \theta_1\theta_2\sigma_a^2 = -\theta_1(1-\theta_2)\sigma_a^2.$$

For k=2 ;

$$Cov(z_t, z_{t-2}) = \gamma_2 = Cov(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}\, ,\, a_{t-2} - \theta_1 a_{t-3} - \theta_2 a_{t-4})$$

$= Cov(a_t, a_{t-2}) - \theta_1 Cov(a_t, a_{t-3}) - \theta_2 Cov(a_t, a_{t-4}) - \theta_1 Cov(a_{t-1}, a_{t-2}) + \theta_1^2 Cov(a_{t-1}, a_{t-3}) + \theta_1\theta_2 Cov(a_{t-1}, a_{t-4}) - \theta_2 Cov(a_{t-2}, a_{t-2}) + \theta_1\theta_2 Cov(a_{t-2}, a_{t-3}) + \theta_2^2 Cov(a_{t-2}, a_{t-4})$

$$Cov(z_t, z_{t-2}) = \gamma_2 = -\theta_2\sigma_a^2.$$

For k=3;

$$Cov(z_t, z_{t-3}) = \gamma_3 = Cov(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}\, ,\, a_{t-3} - \theta_1 a_{t-4} - \theta_2 a_{t-5})$$

$= Cov(a_t, a_{t-3}) - \theta_1 Cov(a_t, a_{t-4}) - \theta_2 Cov(a_t, a_{t-5}) - \theta_1 Cov(a_{t-1}, a_{t-3}) + \theta_1^2 Cov(a_{t-1}, a_{t-4}) + \theta_1\theta_2 Cov(a_{t-1}, a_{t-5}) - \theta_2 Cov(a_{t-2}, a_{t-3}) + \theta_1\theta_2 Cov(a_{t-2}, a_{t-4}) + \theta_2^2 Cov(a_{t-2}, a_{t-5})$

$$Cov(z_t, z_{t-3}) = \gamma_3 = 0.$$

For k=4;

$$Cov(z_t, z_{t-4}) = \gamma_4 = Cov(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}\, ,\, a_{t-4} - \theta_1 a_{t-5} - \theta_2 a_{t-6})$$

$= Cov(a_t, a_{t-4}) - \theta_1 Cov(a_t, a_{t-5}) - \theta_2 Cov(a_t, a_{t-6}) - \theta_1 Cov(a_{t-1}, a_{t-4}) + \theta_1^2 Cov(a_{t-1}, a_{t-5}) + \theta_1\theta_2 Cov(a_{t-1}, a_{t-6}) - \theta_2 Cov(a_{t-2}, a_{t-4}) + \theta_1\theta_2 Cov(a_{t-2}, a_{t-5}) + \theta_2^2 Cov(a_{t-2}, a_{t-6})$

$$Cov(z_t, z_{t-4}) = \gamma_4 = 0.$$

$$
\text{So, } \gamma_k = \begin{cases} (1+\theta_1^2+\theta_2^2)\sigma_a^2, & k=0 \\ -\theta_1(1-\theta_2)\sigma_a^2, & k=1 \\ -\theta_2\sigma_a^2, & k=2 \\ 0, & k>2 \end{cases}.
$$

3.2.2 The Autocorrelation Function of MA(2) Process

$$Corr(z_t, z_{t-k}) = \rho_k = \frac{Cov(z_t, z_{t-k})}{\sqrt{Var(z_t)}\sqrt{Var(z_{t-k})}} = \frac{\gamma_k}{\gamma_0}.$$

For k=1;

$$Corr(z_t, z_{t-1}) = \rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{-\theta_1(1-\theta_2)\sigma_a^2}{(1+\theta_1^2+\theta_2^2)\sigma_a^2} = \frac{-\theta_1(1-\theta_2)}{1+\theta_1^2+\theta_2^2}.$$

For k=2;

$$Corr(z_t, z_{t-2}) = \rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{-\theta_2\sigma_a^2}{(1+\theta_1^2+\theta_2^2)\sigma_a^2} = \frac{-\theta_2}{1+\theta_1^2+\theta_2^2}.$$

For k=3;

$$Corr(z_t, z_{t-3}) = \rho_3 = \frac{\gamma_3}{\gamma_0} = 0.$$

For k=4;

$$Corr(z_t, z_{t-4}) = \rho_4 = \frac{\gamma_4}{\gamma_0} = 0.$$

So, $\rho_k = \begin{cases} \frac{-\theta_1(1-\theta_2)}{1+\theta_1^2+\theta_2^2}, & k = 1 \\ \frac{-\theta_2}{1+\theta_1^2+\theta_2^2}, & k = 2 \\ 0, & k > 2 \end{cases}$.

As it can be seen, the autocorrelation function of MA(2) process cuts off after lag 2.

3.2.3 The Partial Autocorrelation Function of MA(2) Process

By the help of Equation 2.1.3.2 on page 15 the partial autocorrelation function of MA(2) process is found as

$$\emptyset_{11} = \rho_1 ,$$

$$\emptyset_{22} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} ,$$

$$\emptyset_{33} = \frac{\rho_1^3 - \rho_1\rho_2(2 - \rho_2)}{1 - \rho_2^2 - 2\rho_1^2(1 - \rho_2)} ,$$

$$\vdots$$

As seen, the partial autocorrelation function of MA(2) process does not cut off after lag 2. It shows exponential or oscillating decay depending on the sign and magnitude of θ_1 and θ_2.

3.3 The General q^{th} Order Moving Average, MA(q) Process

A zero mean MA(q) model can be shown as follows:

$$z_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q} ,$$

or

$$z_t = \left(1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q\right) a_t \ ,$$

where $\{a_t\}$ is a zero mean white noise process with constant variance σ_a^2.

- The mean of z_t is equal to zero since $\{a_t\}$ is a zero mean white noise process.

$$E(z_t) = 0.$$

3.3.1 The Autocovariance Function of MA(q) Process

$$Cov(z_t, z_{t-k}) = Cov\big(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q},\ a_{t-k} - \theta_1 a_{t-k-1} - \theta_2 a_{t-k-2} - \cdots - \theta_q a_{t-k-q}\big).$$

For k=0;

$Cov(z_t, z_t) = Var(z_t) = \gamma_0 = Cov(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q},\ a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q})$

$$= Cov(a_t, a_t) + \theta_1^2 Cov(a_{t-1}, a_{t-1}) + \theta_2^2 Cov(a_{t-2}, a_{t-2}) + \cdots + \theta_q^2 Cov\big(a_{t-q}, a_{t-q}\big)$$

$$Cov(z_t, z_t) = Var(z_t) = \gamma_0 = \sigma_a^2 + \theta_1^2 \sigma_a^2 + \theta_2^2 \sigma_a^2 + \cdots + \theta_q^2 \sigma_a^2$$
$$= \left(1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2\right) \sigma_a^2.$$

For k=1;

$Cov(z_t, z_{t-1}) = \gamma_1 = Cov(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q},\ a_{t-1} - \theta_1 a_{t-2} - \theta_2 a_{t-3} - \cdots - \theta_q a_{t-1-q})$

$= -\theta_1 Cov(a_{t-1}, a_{t-1}) + \theta_1\theta_2 Cov(a_{t-2}, a_{t-2}) + \theta_2\theta_3 Cov(a_{t-3}, a_{t-3}) + \cdots + \theta_{q-1}\theta_q Cov(a_{t-1-q}, a_{t-1-q})$

$$Cov(z_t, z_{t-1}) = \gamma_1 = -\theta_1 \sigma_a^2 + \theta_1\theta_2 \sigma_a^2 + \theta_2\theta_3 \sigma_a^2 + \cdots + \theta_{q-1}\theta_q \sigma_a^2$$

$$= \left(-\theta_1 + \theta_1\theta_2 + \theta_2\theta_3 + \cdots + \theta_{q-1}\theta_q\right) \sigma_a^2.$$

For k=2;

$Cov(z_t, z_{t-2}) = \gamma_2 = Cov(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q},\ a_{t-2} - \theta_1 a_{t-3} - \theta_2 a_{t-4} - \cdots - \theta_q a_{t-2-q})$

$= -\theta_2 Cov(a_{t-2}, a_{t-2}) + \theta_1\theta_3 Cov(a_{t-3}, a_{t-3}) + \theta_2\theta_4 Cov(a_{t-4}, a_{t-4}) + \cdots + \theta_{q-2}\theta_q Cov(a_{t-2-q}, a_{t-2-q})$

$$Cov(z_t, z_{t-2}) = \gamma_2 = -\theta_2\sigma_a^2 + \theta_1\theta_3\sigma_a^2 + \theta_2\theta_4\sigma_a^2 + \cdots + \theta_{q-2}\theta_q\sigma_a^2$$

$$= \left(-\theta_2 + \theta_1\theta_3 + \theta_2\theta_4 + \cdots + \theta_{q-2}\theta_q\right)\sigma_a^2.$$

For k=3;

$Cov(z_t, z_{t-3}) = \gamma_3 = Cov(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q},\ a_{t-3} - \theta_1 a_{t-4} - \theta_2 a_{t-5} - \cdots - \theta_q a_{t-3-q})$

$= -\theta_3 Cov(a_{t-3}, a_{t-3}) + \theta_1\theta_4 Cov(a_{t-4}, a_{t-4}) + \theta_2\theta_5 Cov(a_{t-5}, a_{t-5}) + \cdots + \theta_{q-3}\theta_q Cov(a_{t-3-q}, a_{t-3-q})$

$$Cov(z_t, z_{t-3}) = \gamma_3 = -\theta_3\sigma_a^2 + \theta_1\theta_4\sigma_a^2 + \theta_2\theta_5\sigma_a^2 + \cdots + \theta_{q-3}\theta_q\sigma_a^2$$

$$= \left(-\theta_3 + \theta_1\theta_4 + \theta_2\theta_5 + \cdots + \theta_{q-3}\theta_q\right)\sigma_a^2.$$

For k=q;

$Cov\left(z, z_{t-q}\right) = \gamma_q = Cov(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q},\ a_{t-q} - \theta_1 a_{t-q-1} - \theta_2 a_{t-q-2} - \cdots - \theta_q a_{t-2q})$

$$Cov\left(z, z_{t-q}\right) = \gamma_q = -\theta_q\sigma_a^2.$$

For k=q+1;

$Cov\left(z_t, z_{t-q-1}\right) = \gamma_{q+1} = Cov(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q},\ a_{t-q-1} - \theta_1 a_{t-q-2} - \theta_2 a_{t-q-3} - \cdots - \theta_q a_{t-2q-1})$

$$Cov\left(z_t, z_{t-q-1}\right) = \gamma_{q+1} = 0.$$

So, $\gamma_k = \begin{cases} \sigma_a^2 \sum_{j=0}^{q-k} \theta_j \, \theta_{j+k}, & k = 0,1,2,\dots,q \\ 0 & , \; k > q \end{cases}$.

where θ_0 is defined to be -1.

3.3.2 The Autocorrelation Function of MA(q) Process

$$Corr(z_t, z_{t-k}) = \rho_k = \frac{Cov(z_t, z_{t-k})}{\sqrt{Var(z_t)}\sqrt{Var(z_{t-k})}} = \frac{\gamma_k}{\gamma_0}.$$

So, $\rho_k = \begin{cases} \frac{\sum_{j=0}^{q-k} \theta_j \theta_{j+k}}{1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2}, & k = 1,2,\dots,q \\ 0 & , \; k > q \end{cases}$.

where θ_0 is defined to be -1.

As it can be seen the autocorrelation function of MA(q) process cuts off after lag q.

3.3.3 The Partial Autocorrelation Function Of MA(q) Process

It can be easily understood that the partial autocorrelation function of MA(q) process does not cut off after lag q by looking at the partial autocorrelation functions of MA(1) and MA(2) processes. The partial autocorrelation function shows exponential or oscillating decay depending on the sign and magnitude of parameters $\theta_1, \theta_2, \dots, \theta_q$.

3.4 Temporal Aggregation of MA(1) Process

Suppose that the basic series follows a zero mean MA(1) model

$$z_t = (1 - \theta_1 B)a_t ,$$

where $\{a_t\}$ is a zero mean white noise process with constant variance σ_a^2.

As stated

$$\gamma_k = \begin{cases} (1+\theta_1^2)\sigma_a^2 , & k = 0 \\ -\theta_1 \sigma_a^2 , & k = 1 \\ 0 , & k > 1 \end{cases} .$$

Temporal aggregation of MA(1) process will be explained by looking at m=3 case. The aggregated series with aggregation period is

$$Z_T = (1 + B + B^2) z_{3T}$$

$$Z_T = (1 + B + B^2)(a_{3T} - \theta_1 a_{3T-1})$$

$$Z_T = a_{3T} - \theta_1 a_{3T-1} + a_{3T-1} - \theta_1 a_{3T-2} + a_{3T-2} - \theta_1 a_{3T-3}$$

$$Z_T = a_{3T} + (1 - \theta_1)a_{3T-1} + (1 - \theta_1)a_{3T-2} - \theta_1 a_{3T-3} ,$$

and the autocovariance function for Z_T is

$$Cov(Z_T, Z_{T-k}) =$$
$$Cov\big(a_{3T} + (1-\theta_1)a_{3T-1} + (1-\theta_1)a_{3T-2} - \theta_1 a_{3T-3} \ , a_{3(T-k)} + (1 -$$
$$\theta_1)a_{3(T-k)-1} + (1-\theta_1)a_{3(T-k)-2} - \theta_1 a_{3(T-k)-3}\big).$$

It can be easily seen that the autocovariance function is equal to zero when k>1. Since the autocovariance function of Z_T cuts off after lag 1, the aggregated series of an MA(1) model follows an MA(1) model when m=3.

Also, the third aggregate of an MA(1) model can be found from Equation 1.3.2 on page 7

$$\gamma_U(k) = (1 + B + \cdots + B^{m-1})^{2(d+1)}\gamma_w[mk + (d+1)(m-1)]\,,$$

where m=3 and d=0

$$\gamma_Z(k) = (1 + B + B^2)^2\gamma_z(3k+2)$$

$$\gamma_Z(k) = (1 + 2B + B^2 + 2B^2 + 2B^3 + B^4)\gamma_z(3k+2)$$

$$\gamma_Z(k) = (1 + 2B + 3B^2 + 2B^3 + B^4)\gamma_z(3k+2)$$

$$\begin{bmatrix}\gamma_Z(0)\\ \gamma_Z(1)\\ \gamma_Z(2)\\ \gamma_Z(3)\end{bmatrix} = \begin{bmatrix}1&2&3&2&1&0&0&0&0&0&0&0&0&0\\0&0&0&1&2&3&2&1&0&0&0&0&0&0\\0&0&0&0&0&0&1&2&3&2&1&0&0&0\\0&0&0&0&0&0&0&0&0&1&2&3&2&1\end{bmatrix}\begin{bmatrix}\gamma_z(-2)\\ \gamma_z(-1)\\ \gamma_z(0)\\ \gamma_z(1)\\ \vdots\\ \gamma_z(11)\end{bmatrix}$$

$$\begin{bmatrix}\gamma_Z(0)\\ \gamma_Z(1)\\ \gamma_Z(2)\\ \gamma_Z(3)\end{bmatrix} = \begin{bmatrix}3&4&2&0&0&0&0&0&0&0&0&0\\0&1&2&3&2&1&0&0&0&0&0&0\\0&0&0&0&1&2&3&2&1&0&0&0\\0&0&0&0&0&0&0&1&2&3&2&1\end{bmatrix}\begin{bmatrix}\gamma_z(0)\\ \gamma_z(1)\\ \vdots\\ \gamma_z(11)\end{bmatrix}$$

$$\begin{bmatrix}\gamma_Z(0)\\ \gamma_Z(1)\\ \gamma_Z(2)\\ \gamma_Z(3)\end{bmatrix} = \begin{bmatrix}3\gamma_z(0) + 4\gamma_z(1)\\ \gamma_z(1)\\ 0\\ 0\end{bmatrix}.$$

As seen from the above equation the third aggregate of an MA(1) model is an MA(1) model.

$$Z_T = (1 - \Theta_1 B)A_T\,,$$

where $\{A_T\}$ is a zero mean white noise process with constant variance σ_A^2.

$$\gamma_Z(0) = (1 + \Theta_1^2)\sigma_A^2 = 3(1 + \theta_1^2)\sigma_a^2 + 4(-\theta_1)\sigma_a^2$$

$$\gamma_Z(1) = -\Theta_1\sigma_A^2 = -\theta_1\sigma_a^2.$$

The parameters Θ_1 and σ_A^2 of the aggregate model Z_T can be found as a function of θ_1 and σ_a^2 by solving

$$\frac{1+\Theta_1^2}{-\Theta_1} = \frac{3(1+\theta_1^2)+4(-\theta_1)}{-\theta_1},$$

and

$$\sigma_A^2 = \frac{-\theta_1\sigma_a^2}{-\Theta_1}.$$

3.5 Temporal Aggregation of MA(2) Process

Suppose that the basic series follows a zero mean MA(2) model

$$z_t = (1-\theta_1 B - \theta_2 B^2)a_t ,$$

where $\{a_t\}$ is a zero mean white noise process with constant variance σ_a^2.

As stated

$$\gamma_k = \begin{cases} (1+\theta_1^2+\theta_2^2)\sigma_a^2, & k=0 \\ -\theta_1(1-\theta_2)\sigma_a^2, & k=1 \\ -\theta_2\sigma_a^2, & k=2 \\ 0, & k>2 \end{cases}.$$

Temporal aggregation of MA(2) process will be explained by looking at m=3 case. The aggregated series with aggregation period is

$$Z_T = (1+B+B^2)z_{3T}$$

$$Z_T = (1+B+B^2)(a_{3T} - \theta_1 a_{3T-1} - \theta_2 a_{3T-2})$$

$$Z_T = a_{3T} - \theta_1 a_{3T-1} - \theta_2 a_{3T-2} + a_{3T-1} - \theta_1 a_{3T-2} - \theta_2 a_{3T-3} + a_{3T-2} - \theta_1 a_{3T-3} - \theta_2 a_{3T-4}$$

$$Z_T = a_{3T} + (1-\theta_1)a_{3T-1} + (1-\theta_1-\theta_2)a_{3T-2} - (\theta_1+\theta_2)a_{3T-3} - \theta_2 a_{3T-4}\,,$$

and the autocovariance function for Z_T is

$$Cov(Z_T, Z_{T-k}) = Cov\big(a_{3T} + (1-\theta_1)a_{3T-1} + (1-\theta_1-\theta_2)a_{3T-2} - (\theta_1 + \theta_2)a_{3T-3} - \theta_2 a_{3T-4}\ ,\ a_{3(T-k)} + (1-\theta_1)a_{3(T-k)-1} + (1-\theta_1-\theta_2)a_{3(T-k)-2} - (\theta_1+\theta_2)a_{3(T-k)-3} - \theta_2 a_{3(T-k)-4}\big).$$

It can be easily seen that the autocovariance function is equal to zero when $k > \frac{4}{3}$. Since T can take integer values, the autocovariance function is equal to zero when k>1. So, the aggregated series of an MA(2) model follows an MA(1) model when m=3.

Also, the third aggregate of an MA(2) model can be found from the Equation 1.3.2 on page 7

$$\gamma_U(k) = (1 + B + \cdots + B^{m-1})^{2(d+1)}\gamma_w[mk + (d+1)(m-1)]\,,$$

where d=0 and m=3

$$\gamma_Z(k) = (1 + B + B^2)^2\gamma_z(3k+2)$$

$$\gamma_Z(k) = (1 + 2B + B^2 + 2B^2 + 2B^3 + B^4)\gamma_z(3k+2)$$

$$\gamma_Z(k) = (1 + 2B + 3B^2 + 2B^3 + B^4)\gamma_z(3k+2)$$

$$\begin{bmatrix}\gamma_Z(0)\\ \gamma_Z(1)\\ \gamma_Z(2)\\ \gamma_Z(3)\end{bmatrix} = \begin{bmatrix}1&2&3&2&1&0&0&0&0&0&0&0&0&0\\0&0&0&1&2&3&2&1&0&0&0&0&0&0\\0&0&0&0&0&0&1&2&3&2&1&0&0&0\\0&0&0&0&0&0&0&0&0&1&2&3&2&1\end{bmatrix}\begin{bmatrix}\gamma_z(-2)\\ \gamma_z(-1)\\ \gamma_z(0)\\ \gamma_z(1)\\ \vdots\\ \gamma_z(11)\end{bmatrix}$$

$$\begin{bmatrix}\gamma_Z(0)\\ \gamma_Z(1)\\ \gamma_Z(2)\\ \gamma_Z(3)\end{bmatrix} = \begin{bmatrix}3&4&2&0&0&0&0&0&0&0&0&0\\0&1&2&3&2&1&0&0&0&0&0&0\\0&0&0&0&1&2&3&2&1&0&0&0\\0&0&0&0&0&0&0&1&2&3&2&1\end{bmatrix}\begin{bmatrix}\gamma_z(0)\\ \gamma_z(1)\\ \vdots\\ \gamma_z(11)\end{bmatrix}$$

$$\begin{bmatrix} \gamma_Z(0) \\ \gamma_Z(1) \\ \gamma_Z(2) \\ \gamma_Z(3) \end{bmatrix} = \begin{bmatrix} 3\gamma_z(0) + 4\gamma_z(1) + 2\gamma_z(2) \\ \gamma_z(1) + 2\gamma_z(2) \\ 0 \\ 0 \end{bmatrix}.$$

As seen from the above equation the aggregated series of an MA(2) model follows an MA(1) model when m=3.

$$Z_T = (1 - \Theta_1 B)A_T$$

where $\{A_T\}$ is a zero mean white noise process with constant variance σ_A^2.

$$\gamma_Z(0) = (1 + \Theta_1^2)\sigma_A^2 = 3(1 + \theta_1^2 + \theta_2^2)\sigma_a^2 + 4(-\theta_1 + \theta_1\theta_2)\sigma_a^2 + 2(-\theta_2)\sigma_a^2$$

$$\gamma_Z(1) = -\Theta_1\sigma_A^2 = (-\theta_1 + \theta_1\theta_2)\sigma_a^2 + 2(-\theta_2)\sigma_a^2.$$

The parameters Θ_1 and σ_A^2 of the aggregate model Z_T can be found as a function of θ_1 and σ_a^2 by solving

$$\frac{1 + \Theta_1^2}{-\Theta_1} = \frac{3(1 + \theta_1^2 + \theta_2^2) + 4(-\theta_1 + \theta_1\theta_2) + 2(-\theta_2)}{(-\theta_1 + \theta_1\theta_2) + 2(-\theta_2)},$$

and

$$\sigma_A^2 = \frac{(-\theta_1 + \theta_1\theta_2)\sigma_a^2 + 2(-\theta_2)\sigma_a^2}{-\Theta_1}.$$

3.6 Temporal Aggregation of MA(q) Process

Suppose that the basic series follows a zero mean MA(q) model

$$z_t = \left(1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q\right)a_t\ ,$$

where $\{a_t\}$ is a zero mean white noise process with constant variance σ_a^2.

As stated

$$\gamma_k = \begin{cases} \sigma_a^2 \sum_{j=0}^{q-k} \theta_j \, \theta_{j+k}, & k = 0,1,2,\dots,q \\ 0 & , \; k > q \end{cases} .$$

The aggregated series with aggregation period is

$$Z_T = (1 + B + B^2) z_{3T}$$

$$Z_T = (1 + B + B^2)\left(a_{3T} - \theta_1 a_{3T-1} - \theta_2 a_{3T-2} - \dots - \theta_q a_{3T-q}\right)$$

$$Z_T = a_{3T} - \theta_1 a_{3T-1} - \theta_2 a_{3T-2} - \dots - \theta_q a_{3T-q} + a_{3T-1} - \theta_1 a_{3T-2} - \theta_2 a_{3T-3} - \dots - \theta_q a_{3T-q-1} + a_{3T-2} - \theta_1 a_{3T-3} - \theta_2 a_{3T-4} - \dots - \theta_q a_{3T-q-2}$$

$$Z_T = a_{3T} + (1 - \theta_1) a_{3T-1} + (1 - \theta_1 - \theta_2) a_{3T-2} + \dots - \theta_q a_{3T-q-2} ,$$

and the autocovariance function for Z_T is

$$Cov(Z_T, Z_{T-k}) = Cov\left(a_{3T} + (1-\theta_1)a_{3T-1} + (1-\theta_1-\theta_2)a_{3T-2} + \dots - \theta_q a_{3T-q-2} , a_{3(T-k)} + (1-\theta_1)a_{3(T-k)-1} + (1-\theta_1-\theta_2)a_{3(T-k)-2} + \dots - \theta_q a_{3(T-k)-q-2}\right) .$$

It is seen that the autocovariance function is equal to zero when k is greater than the integer part of $\frac{q+2}{3}$. So, the aggregated series of an MA(1) model follows an MA(N_0) model where N_0 is less than or equal to $\frac{q+2}{3}$ and m=3.

Also, from the Equation 1.3.2 on page 7

$$\gamma_U(k) = (1 + B + \dots + B^{m-1})^{2(d+1)} \gamma_w [mk + (d+1)(m-1)] ,$$

it can be seen that $\gamma_U(k)$ is the linear transformation of $\gamma_w(j)$ where j is between $mk - (d+1)(m-1)$ and $mk + (d+1)(m-1)$.

So $\gamma_U(l)$ is a weighted sum of $\gamma_w(k)$ where

$$ml - (d+1)(m-1) \le k \le ml + (d+1)(m-1) .$$

Since $\gamma_w(k) = 0$ for $|k| > q$, $\gamma_U(l) = 0$ for $l > q^* = d + 1 + \frac{(q-d-1)}{m}$.

The nonzero autocovariances of U_T are $\gamma_U(0)$, $\gamma_U(1)$,…, $\gamma_U(q^*)$ which means the autocorrelation function of U_T cuts off after lag q^*. Then it can be said that U_T follows an IMA(d,N_0) process (Wei, 2006, pp. 511)

$$U_T = (1-B)^d Z_T = \left(1 - \Theta_1 B - \Theta_2 B^2 \ldots - \Theta_{N_0} B^{N_0}\right) A_T,$$

where $\{A_T\}$ is a zero mean white noise process with constant variance σ_A^2 and

$$N_0 \leq q^* = \left[d + 1 + \frac{(q-d-1)}{m}\right].$$

In this thesis work, stationary time series models are considered that is d can be thought as zero. Then the mth order aggregate model of MA(q) process will follow an MA(N_0) process

$$Z_T = \left(1 - \Theta_1 B - \Theta_2 B^2 \ldots - \Theta_{N_0} B^{N_0}\right) A_T ,$$

where $\{A_T\}$ is a zero mean white noise process with constant variance σ_A^2 and

$$N_0 \leq q^* = \left[1 + \frac{(q-1)}{m}\right].$$

The parameters Θ_i's and σ_A^2 of the mth order aggregate model Z_T are the functions of θ_i's and σ_a^2.

CHAPTER 4

TEMPORAL AGGREGATION OF AUTOREGRESSIVE MOVING AVERAGE PROCESSES

In this chapter, the effect of temporal aggregation on ARMA processes will be explained. For simplicity of the calculations, zero mean autoregressive moving average processes will be considered.

A zero mean autoregressive moving average process of order p for autoregressive part and q for moving average part can be shown as follows:

$$z_t = \emptyset_1 z_{t-1} + \cdots + \emptyset_p z_{t-p} + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} ,$$

or

$$\emptyset(B)\, z_t = \theta(B) a_t ,$$

where $\emptyset(B) = (1 - \emptyset_1 B - \cdots - \emptyset_p B^p)$, $\theta(B) = (1 - \theta_1 B - \cdots - \theta_q B^q)$ and $\{a_t\}$ is a zero mean white noise process with constant variance σ_a^2.

As seen the ARMA(p,q) model is neither in random shock form nor in inverted form. For stationarity the roots of t $\emptyset_p(B) = 0$ must lie outside the unit circle and for invertibility the roots of $\theta_p(B) = 0$ must lie outside the unit circle.

4.1 The ARMA(1,1) Process

A zero mean ARMA(1,1) process can be shown as follows:

$$z_t = \emptyset_1 z_{t-1} + a_t - \theta_1 a_{t-1} ,$$

or

$$(1 - \emptyset_1 B) z_t = (1 - \theta_1 B) a_t ,$$

where $\{a_t\}$ is a zero mean white noise process with constant variance σ_a^2.

- Assuming the stationarity, the mean of z_t is equal to zero. The informal proof can be shown as follows:

$$E(z_t) = \emptyset_1 E(z_{t-1}) + E(a_t) - \theta_1 E(a_{t-1})$$

$$\mu = \emptyset_1 \mu + 0$$

$$(1 - \emptyset_1)\mu = 0.$$

If $\emptyset_1 = 1$, then z_t is not stationary. Since the stationarity condition is assumed, $\mu = 0$.

- Assuming the stationarity, the variance of z_t is equal to $\frac{1+\theta_1^2-2\emptyset_1\theta_1}{1-\emptyset_1^2}\sigma_a^2$. It can be shown as follows:

$$Var(z_t) = Var(\emptyset_1 z_{t-1} + a_t - \theta_1 a_{t-1})$$

$$= Var(\emptyset_1 z_{t-1} + a_t) - 2Cov(\emptyset_1 z_{t-1} + a_t , \theta_1 a_{t-1}) + \theta_1^2 Var(a_{t-1})$$

$$= \emptyset_1^2 Var(z_{t-1}) + \sigma_a^2 - 2\emptyset_1\theta_1\sigma_a^2 + \theta_1^2\sigma_a^2$$

$$= \frac{1 - 2\emptyset_1\theta_1 + \theta_1^2}{1 - \emptyset_1^2}\sigma_a^2 .$$

4.1.1 The Autocovariance Function of ARMA(1,1) Process

$$Cov(z_t , z_{t-k}) = Cov(\emptyset_1 z_{t-1} + a_t - \theta_1 a_{t-1} , z_{t-k}).$$

Assuming the stationarity,

For k=0;

$$Cov(z_t, z_t) = \gamma_0 = Var(z_t) = Cov(\emptyset_1 z_{t-1} + a_t - \theta_1 a_{t-1}\,, z_t)$$

$$= \emptyset_1 Cov(z_{t-1}, z_t) + Cov(a_t, z_t) - \theta_1 Cov(a_{t-1}, z_t)$$

$$= \emptyset_1 \gamma_1 + \sigma_a^2 - \theta_1 Cov(a_{t-1}, \emptyset_1 z_{t-1} + a_t - \theta_1 a_{t-1})$$

$$= \emptyset_1 \gamma_1 + \sigma_a^2 - \theta_1 \emptyset_1 \sigma_a^2 + \theta_1^2 \sigma_a^2$$

$$Cov(z_t, z_t) = \gamma_0 = \emptyset_1 \gamma_1 + (1 - \theta_1 \emptyset_1 + \theta_1^2)\sigma_a^2.$$

For k=1;

$$Cov(z_t, z_{t-1}) = \gamma_1 = Cov(\emptyset_1 z_{t-1} + a_t - \theta_1 a_{t-1}\,, z_{t-1})$$

$$= \emptyset_1 Cov(z_{t-1}, z_{t-1}) + Cov(a_t, z_{t-1}) - \theta_1 Cov(a_{t-1}, z_{t-1})$$

$$Cov(z_t, z_{t-1}) = \gamma_1 = \emptyset_1 \gamma_0 - \theta_1 \sigma_a^2.$$

For k=2;

$$Cov(z_t, z_{t-2}) = \gamma_2 = Cov(\emptyset_1 z_{t-1} + a_t - \theta_1 a_{t-1}\,, z_{t-2})$$

$$= \emptyset_1 Cov(z_{t-1}, z_{t-2}) + Cov(a_t, z_{t-2}) - \theta_1 Cov(a_{t-1}, z_{t-2})$$

$$Cov(z_t, z_{t-2}) = \gamma_2 = \emptyset_1 \gamma_1.$$

For k=3;

$$Cov(z_t, z_{t-3}) = \gamma_3 = Cov(\emptyset_1 z_{t-1} + a_t - \theta_1 a_{t-1}\,, z_{t-3})$$

$$= \emptyset_1 Cov(z_{t-1}, z_{t-3}) + Cov(a_t, z_{t-3}) - \theta_1 Cov(a_{t-1}, z_{t-3})$$

$$Cov(z_t, z_{t-3}) = \gamma_3 = \emptyset_1 \gamma_2.$$

So, $\gamma_k = \begin{cases} \frac{1-2\emptyset_1\theta_1+\theta_1^2}{1-\emptyset_1^2}\sigma_a^2, & k = 0 \\ \frac{(\emptyset_1-\theta_1)(1-\emptyset_1\theta_1)}{1-\emptyset_1^2}\sigma_a^2, & k = 1 \\ \emptyset_1\gamma_{k-1}, & k > 1 \end{cases}$.

4.1.2 The Autocorrelation Function of ARMA(1,1) Process

$$Corr(z_t\,, z_{t-k}) = \rho_k = \frac{Cov(z_t\,, z_{t-k})}{\sqrt{Var(z_t)}\,\sqrt{Var(z_{t-k})}} = \frac{\gamma_k}{\gamma_0}.$$

For k=1;

$$Corr(z_t, z_{t-1}) = \rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{(\emptyset_1 - \theta_1)(1 - \emptyset_1\theta_1)}{1 - 2\emptyset_1\theta_1 + \theta_1^2}.$$

For k=2;

$$Corr(z_t, z_{t-2}) = \rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{\emptyset_1\gamma_1}{\gamma_0} = \emptyset_1\rho_1.$$

For k=3;

$$Corr(z_t, z_{t-3}) = \rho_3 = \frac{\gamma_3}{\gamma_0} = \frac{\emptyset_1\gamma_2}{\gamma_0} = \emptyset_1\rho_2.$$

So, $\rho_k = \begin{cases} \frac{(\emptyset_1-\theta_1)(1-\emptyset_1\theta_1)}{1-2\emptyset_1\theta_1+\theta_1^2}, & k = 1 \\ \emptyset_1\rho_{k-1}, & k > 1 \end{cases}$.

As it can be seen, the autocorrelation function of ARMA(1,1) process reflects the properties of AR(1) process autocorrelation function and MA(1) process autocorrelation function. The MA(1) effect can be observed by looking at ρ_1 and the AR(1) effect can be observed by looking at $\rho_1, \rho_2, \ldots$. The autocorrelation function of ARMA(1,1) process shows exponential or oscillating decay after lag one depending on the sign and magnitude of $\emptyset_1$.

4.1.3 The Partial Autocorrelation Function of ARMA(1,1) Process

The calculation of the partial autocorrelation function of ARMA(1,1) process needs complicated calculations and giving the general form of the partial autocorrelation function of ARMA(1,1) process is difficult. Still, the shape of the partial autocorrelation function of ARMA(1,1) process can be predicted since it reflects the properties of AR(1) process partial autocorrelation function and MA(1) process partial autocorrelation function. Since the partial autocorrelation functions of AR(1) and MA(1) processes are known from previous sections, it can be said that the AR(1) effect is observed at lag one and MA(1) effect is observed after lag one. So, the partial autocorrelation function of ARMA(1,1) process shows exponential or oscillating decay after lag one depending on the sign and magnitude of θ_1 similar to the autocorrelation function of ARMA(1,1) process.

4.2 The General ARMA(p,q) Process

A zero mean ARMA(p,q) model can be shown as follows:

$$z_t = \emptyset_1 z_{t-1} + \cdots + \emptyset_p z_{t-p} + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} \ ,$$

or

$$\left(1 - \emptyset_1 B - \cdots - \emptyset_p B^p\right) z_t = \left(1 - \theta_1 B - \cdots - \theta_q B^q\right) a_t \ ,$$

where $\{a_t\}$ is a zero mean white noise process with constant variance σ_a^2.

- Assuming the stationarity, the mean of z_t is equal to zero. The informal proof can be shown as follows:

$$E(z_t) = \emptyset_1 E(z_{t-1}) + \cdots + \emptyset_p E(z_{t-p}) + E(a_t) - \theta_1 E(a_{t-1}) - \cdots - \theta_q E(a_{t-q})$$

$$\mu = \emptyset_1 \mu + \cdots + \emptyset_p \mu + 0$$

$$(1 - \emptyset_1 - \cdots - \emptyset_p)\mu = 0.$$

If $\emptyset_1 + \cdots + \emptyset_p = 1$, then z_t is not stationary. Since the stationarity condition is assumed, $\mu = 0$.

4.2.1 The Autocovariance Function of ARMA(p,q) Process

$$Cov(z_t, z_{t-k}) = Cov(\emptyset_1 z_{t-1} + \cdots + \emptyset_p z_{t-p} + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}, z_{t-k}).$$

Assuming the stationarity,

$$Cov(z_t, z_{t-k}) = \emptyset_1 Cov(z_{t-1}, z_{t-k}) + \cdots + \emptyset_p Cov(z_{t-p}, z_{t-k}) + Cov(a_t, z_{t-k}) - \theta_1 Cov(a_{t-1}, z_{t-k}) - \cdots - \theta_q Cov(a_{t-q}, z_{t-k}).$$

For k=0 ;

$$Cov(z_t, z_t) = Var(z_t) = \gamma_0$$

$$= \emptyset_1 Cov(z_{t-1}, z_t) + \cdots + \emptyset_p Cov(z_{t-p}, z_t) + Cov(a_t, z_t) - \theta_1 Cov(a_{t-1}, z_t) - \cdots - \theta_q Cov(a_{t-q}, z_t)$$

$$= \emptyset_1 \gamma_1 + \cdots + \emptyset_p \gamma_p + \sigma_a^2 - \theta_1 \emptyset_1 \sigma_a^2 + \theta_1^2 \sigma_a^2 - \cdots - \theta_p \emptyset_p \sigma_a^2 + \theta_p^2 \sigma_a^2 + \theta_{p+1}^2 \sigma_a^2 + \cdots + \theta_q^2 \sigma_a^2$$

$$Cov(z_t, z_t) = Var(z_t) = \gamma_0 = (\emptyset_1 \gamma_1 + \cdots + \emptyset_p \gamma_p) + (1 - \theta_1 \emptyset_1 + \theta_1^2 - \cdots - \theta_p \emptyset_p + \theta_p^2 + \theta_{p+1}^2 + \cdots + \theta_q^2)\sigma_a^2.$$

For k=1;

$$Cov(z_t, z_{t-1}) = \gamma_1$$

$$= \emptyset_1 Cov(z_{t-1}, z_{t-1}) + \cdots + \emptyset_p Cov(z_{t-p}, z_{t-1}) + Cov(a_t, z_{t-1}) - \theta_1 Cov(a_{t-1}, z_{t-1}) - \cdots - \theta_q Cov(a_{t-q}, z_{t-1})$$

$$= \emptyset_1 \gamma_0 + \cdots + \emptyset_p \gamma_{1-p} - \theta_1 \sigma_a^2 - \cdots + \theta_q \theta_{q-1} \sigma_a^2$$

$$Cov(z_t, z_{t-1}) = \gamma_1 = (\emptyset_1 \gamma_0 + \cdots + \emptyset_p \gamma_{1-p}) + (-\theta_1 - \cdots + \theta_q \theta_{q-1})\sigma_a^2.$$

For k=2;

$$Cov(z_t, z_{t-2}) = \gamma_2$$

$$= \emptyset_1 Cov(z_{t-1}, z_{t-2}) + \cdots + \emptyset_p Cov(z_{t-p}, z_{t-2}) + Cov(a_t, z_{t-2}) - \theta_1 Cov(a_{t-1}, z_{t-2}) - \cdots - \theta_q Cov(a_{t-q}, z_{t-2})$$

$$= \emptyset_1 \gamma_1 + \cdots + \emptyset_p \gamma_{2-p} - \theta_2 \sigma_a^2 - \cdots + \theta_q \theta_{q-2} \sigma_a^2$$

$$Cov(z_t, z_{t-2}) = \gamma_2 = (\emptyset_1 \gamma_0 + \cdots + \emptyset_p \gamma_{2-p}) + (-\theta_2 - \cdots + \theta_q \theta_{q-2})\sigma_a^2.$$

For k=q;

$$Cov(z_t, z_{t-q}) = \gamma_q$$

$$= \emptyset_1 Cov(z_{t-1}, z_{t-q}) + \cdots + \emptyset_p Cov(z_{t-p}, z_{t-q}) + Cov(a_t, z_{t-q}) - \theta_1 Cov(a_{t-1}, z_{t-q}) - \cdots - \theta_q Cov(a_{t-q}, z_{t-q})$$

$$Cov(z_t, z_{t-q}) = \gamma_q = \emptyset_1 \gamma_{q-1} + \cdots + \emptyset_p \gamma_{q-p} - \theta_q \sigma_a^2.$$

For k= q+1;

$$Cov(z_t, z_{t-q-1}) = \gamma_{q+1}$$

$$= \emptyset_1 Cov(z_{t-1}, z_{t-q-1}) + \cdots + \emptyset_p Cov(z_{t-p}, z_{t-q-1}) + Cov(a_t, z_{t-q-1}) - \theta_1 Cov(a_{t-1}, z_{t-q-1}) - \cdots - \theta_q Cov(a_{t-q}, z_{t-q-1})$$

$$Cov(z_t, z_{t-q-1}) = \gamma_{q+1} = \emptyset_1 \gamma_q + \cdots + \emptyset_p \gamma_{q+1-p}.$$

For k=q+2;

$$Cov(z_t, z_{t-q-2}) = \gamma_{q+2}$$

$$= \emptyset_1 Cov(z_{t-1}, z_{t-q-2}) + \cdots + \emptyset_p Cov(z_{t-p}, z_{t-q-2}) + Cov(a_t, z_{t-q-2}) - \theta_1 Cov(a_{t-1}, z_{t-q-2}) - \cdots - \theta_q Cov(a_{t-q}, z_{t-q-2})$$

$$Cov(z_t, z_{t-q-2}) = \gamma_{q+2} = \emptyset_1 \gamma_{q+1} + \cdots + \emptyset_p \gamma_{q+2-p}.$$

So ,

$$\gamma_k = \begin{cases} (\emptyset_1\gamma_1 + \cdots + \emptyset_p\gamma_p) + (1 - \theta_1\emptyset_1 + \theta_1^2 - \cdots - \theta_p\emptyset_p + \theta_p^2 + \theta_{p+1}^2 + \ldots + \theta_q^2)\sigma_a^2 & , \ k = 0 \\ (\emptyset_1\gamma_{k-1} + \cdots + \emptyset_p\gamma_{k-p}) + (-\theta_k - \cdots + \theta_q\theta_{q-k})\sigma_a^2 & , \ k = 1,2,\ldots,q \\ \emptyset_1\gamma_{k-1} + \cdots + \emptyset_p\gamma_{k-p} & , \ k \geq q+1 \end{cases}.$$

4.2.2 The Autocorrelation Function of ARMA(p,q) Process

$$Corr(z_t , z_{t-k}) = \rho_k = \frac{Cov(z_t , z_{t-k})}{\sqrt{Var(z_t)}\sqrt{Var(z_{t-k})}} = \frac{\gamma_k}{\gamma_0}.$$

By looking at the autocovariance function of ARMA(p,q) process it can be concluded that

$$\rho_k = \begin{cases} \dfrac{(\emptyset_1\gamma_{k-1} + \cdots + \emptyset_p\gamma_{k-p}) + (-\theta_k - \cdots + \theta_q\theta_{q-k})}{(\emptyset_1\gamma_1 + \cdots + \emptyset_p\gamma_p) + (1 - \theta_1\emptyset_1 + \theta_1^2 - \cdots - \theta_p\emptyset_p + \theta_p^2 + \theta_{p+1}^2 + \cdots + \theta_q^2)} & , k = 1,2,\ldots,q \\ \emptyset_1\rho_{k-1} + \cdots + \emptyset_p\rho_{k-p} & , \ k \geq q+1 \end{cases}.$$

The autocorrelation function of ARMA(p,q) process reflects the properties of AR(p) process autocorrelation function and MA(q) process autocorrelation function. The MA(q) effect can be observed by looking at $\rho_1, \rho_2, \ldots , \rho_q$ and the AR(p) effect can be observed by looking at $\rho_{q+1}, \rho_{q+2}, \ldots$ .The autocorrelation function of ARMA(p,q) process shows exponential or oscillating decay after lag q depending on the sign and magnitude of $\emptyset_1, \emptyset_2, \ldots, \emptyset_p$.

4.2.3 The Partial Autocorrelation Function of ARMA(p,q) Process

Giving the general form of the partial autocorrelation function of ARMA(p,q) process is difficult like ARMA(1,1) process since it needs complicated calculations. Still, the shape of the partial autocorrelation function of ARMA(p,q) process can be predicted since it reflects the properties of AR(p) process partial autocorrelation function and MA(q) process partial autocorrelation function. Since the partial autocorrelation functions of AR(p) and MA(q) processes are known from previous sections, it can be said that the AR(p) effect is observed at lags 1,2,…,p and MA(q) effect is observed after lag p+1. So, the partial autocorrelation function of ARMA(p,q) process shows exponential or oscillating decay after lag p depending on the sign and magnitude of $\theta_1, \theta_2, \ldots, \theta_q$.

4.3 Temporal Aggregation of ARMA(1,1) Process

Suppose that the basic series follows a zero mean ARMA(1,1) model

$$z_t = \emptyset_1 z_{t-1} + a_t - \theta_1 a_{t-1} \,,$$

or

$$(1 - \emptyset_1 B) z_t = (1 - \theta_1 B) a_t \,,$$

where

- $\{a_t\}$ is a zero mean white noise process with constant variance σ_a^2.
- The polynomials $\emptyset_p(B)$ and $\theta_q(B)$ have no roots in common and the roots of $\emptyset_p(B)$ and $\theta_q(B)$ are outside the unit circle.
- The model has no hidden periodicity of order m. A model has hidden periodicity of order m if $\delta_i \neq \delta_j$ (the roots of $\emptyset_p(B)$) but $\delta_i^m = \delta_j^m$.

Temporal aggregation of ARMA(1,1) process is explained by looking at m=3 case. The model for its third order aggregate is

$$Z_T = (1 + B + B^2)z_{3T}$$

where m=3 and d=0.

Letting $\emptyset_p(B) = \prod_{j=1}^{p}(1 - \delta_j B)$ and multiplying $\prod_{j=1}^{p}\left[\frac{(1-\delta_j^m B^m)}{(1-\delta_j B)}\right]\left[\frac{(1-B^m)^{d+1}}{(1-B)^{d+1}}\right]$ on both sides of basic series z_t :

$$\frac{(1-\delta_1^3 B^3)(1-B^3)}{(1-\delta_1 B)(1-B)}(1-\emptyset_1 B)z_t = \frac{(1-\delta_1^3 B^3)(1-B^3)}{(1-\delta_1 B)(1-B)}(1-\theta_1 B)a_t$$

$$(1-\delta_1^3 B^3)(1+B+B^2)z_t = (1+\delta_1 B+\delta_1^2 B^2)(1+B+B^2)(1-\theta_1 B)a_t$$

$$(1+B+B^2-\delta_1^3 B^3-\delta_1^3 B^4-\delta_1^3 B^5)z_t = [1+(1+\delta_1)B+(1+\delta_1+\delta_1^2)B^2+ (\delta_1+\delta_1^2)B^3+\delta_1^2 B^4](1-\theta_1 B)a_t$$

$$(1+B+B^2-\delta_1^3 B^3-\delta_1^3 B^4-\delta_1^3 B^5)z_t = [1+(1+\delta_1-\theta_1)B+(1+\delta_1+\delta_1^2- \theta_1-\theta_1\delta_1)B^2+(\delta_1+\delta_1^2-\theta_1-\theta_1\delta_1-\theta_1\delta_1^2)B^3+(\delta_1^2-\theta_1\delta_1-\theta_1\delta_1^2)B^4- \theta_1\delta_1^2 B^5]a_t.$$

Substitute t for $3T$

$$(1+B+B^2-\delta_1^3 B^3-\delta_1^3 B^4-\delta_1^3 B^5)z_{3T} = [1+(1+\delta_1-\theta_1)B+(1+\delta_1+ \delta_1^2-\theta_1-\theta_1\delta_1)B^2+(\delta_1+\delta_1^2-\theta_1-\theta_1\delta_1-\theta_1\delta_1^2)B^3+(\delta_1^2-\theta_1\delta_1-\theta_1\delta_1^2)B^4- \theta_1\delta_1^2 B^5]a_{3T}$$

$$z_{3T}+z_{3T-1}+z_{3T-2}-\delta_1^3 z_{3T-3}-\delta_1^3 z_{3T-4}-\delta_1^3 z_{3T-5} = a_{3T}+(1+\delta_1- \theta_1)a_{3T-1}+(1+\delta_1+\delta_1^2-\theta_1-\theta_1\delta_1)a_{3T-2}+(\delta_1+\delta_1^2-\theta_1-\theta_1\delta_1- \theta_1\delta_1^2)a_{3T-3}+(\delta_1^2-\theta_1\delta_1-\theta_1\delta_1^2)a_{3T-4}-\theta_1\delta_1^2 a_{3T-5}$$

$$Z_T-\delta_1^3 Z_{T-1} = a_{3T}+(1+\delta_1-\theta_1)a_{3(T-\frac{1}{3})}+(1+\delta_1+\delta_1^2-\theta_1-\theta_1\delta_1)a_{3(T-\frac{2}{3})}+ (\delta_1+\delta_1^2-\theta_1-\theta_1\delta_1-\theta_1\delta_1^2)a_{3(T-1)}+(\delta_1^2-\theta_1\delta_1-\theta_1\delta_1^2)a_{3(T-\frac{4}{3})}-\theta_1\delta_1^2 a_{3\left(T-\frac{5}{3}\right)}.$$

Say $(1-\delta_1^3 B)Z_T = X_{3T}$.

It is obvious that $Cov(X_{3T}, X_{3(T-K)})$ will be equal to zero if K is greater than the integer part of $\frac{5}{3}$ which is equal to 1 (Brewer, 1973).

Consequently, the aggregated series of an ARMA(1,1) model follows an ARMA(1,1) model when m=3.

$$(1 - \delta_1^3 B)Z_T = (1 - \beta_1 B)A_T$$

where $\{A_T\}$ is a zero mean white noise process with constant variance σ_A^2.

The parameters β_1 and σ_A^2 of the aggregate series Z_T are functions of $\emptyset_1$, θ_1 and σ_a^2. Also it is useful to state that the root of AR polynomial of the aggregate series Z_T is the third power of the root of AR polynomial of the nonaggregate series z_t .

4.4 Temporal Aggregation of ARMA(p,q) Process

Suppose that the basic series follows a zero mean ARMA(p,q) model

$$z_t = \emptyset_1 z_{t-1} + \cdots + \emptyset_p z_{t-p} + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} ,$$

or

$$\left(1 - \emptyset_1 B - \cdots - \emptyset_p B^p\right) z_t = \left(1 - \theta_1 B - \cdots - \theta_q B^q\right) a_t$$

where

- $\{a_t\}$ is a zero mean white noise process with constant variance σ_a^2.
- The polynomials $\emptyset_p(B)$ and $\theta_q(B)$ have no roots in common and the roots of $\emptyset_p(B)$ and $\theta_q(B)$ are outside the unit circle.
- The model has no hidden periodicity of order m. A model has hidden periodicity of order m if $\delta_i \neq \delta_j$ (the roots of $\emptyset_p(B)$) but $\delta_i^m = \delta_j^m$.

The model for its third order aggregate is

$$Z_T = (1 + B + B^2) z_{3T}$$

where m=3 and d=0.

Letting $\emptyset_p(B)=\prod_{j=1}^{p}(1-\delta_j\ B)$ and multiplying $\prod_{j=1}^{p}\left[\frac{\left(1-\delta_j^m B^m\right)}{(1-\delta_j B)}\right]\left[\frac{(1-B^m)^{d+1}}{(1-B)^{d+1}}\right]$ on both sides of nonaggregate series z_t :

$$\frac{(1-\delta_1^3B^3)(1-\delta_2^3B^3)\dots(1-\delta_p^3B^3)}{(1-\delta_1B)(1-\delta_2B)\dots(1-\delta_pB)}\frac{(1-B^3)}{(1-B)}(1-\emptyset_1B-\dots-\emptyset_pB^p)z_t$$
$$=\frac{(1-\delta_1^3B^3)(1-\delta_2^3B^3)\dots(1-\delta_p^3B^3)}{(1-\delta_1B)(1-\delta_2B)\dots(1-\delta_pB)}\frac{(1-B^3)}{(1-B)}(1-\theta_1B-\dots-\theta_qB^q)a_t$$

$$(1-\delta_1^3B^3)(1-\delta_2^3B^3)\dots(1-\delta_p^3B^3)(1+B+B^2)z_t=(1+\delta_1B+\delta_1^2B^2)(1+\delta_2B+\delta_2^2B^2)\dots(1+\delta_pB+\delta_p^2B^2)(1+B+B^2)(1-\theta_1B-\dots-\theta_qB^q)a_t.$$

Substitute t for $3T$

$$(1-\delta_1^3B^3)(1-\delta_2^3B^3)\dots(1-\delta_p^3B^3)(1+B+B^2)z_{3T}=(1+\delta_1B+\delta_1^2B^2)(1+\delta_2B+\delta_2^2B^2)\dots(1+\delta_pB+\delta_p^2B^2)(1+B+B^2)(1-\theta_1B-\dots-\theta_qB^q)a_{3T}$$

$$(1-\delta_1^3B^3)(1-\delta_2^3B^3)\dots(1-\delta_p^3B^3)(1+B+B^2)z_{3T}=(1-\delta_1^3B)(1-\delta_2^3B)\dots(1-\delta_p^3B)Z_T.$$

Say $(1-\delta_1^3B)(1-\delta_2^3B)\dots(1-\delta_p^3B)Z_T=X_{3T}$,

$$X_{3T}=(1+\delta_1B+\delta_1^2B^2)(1+\delta_2B+\delta_2^2B^2)\dots(1+\delta_pB+\delta_p^2B^2)(1+B+B^2)(1-\theta_1B-\dots-\theta_qB^q)a_{3T}\ .$$

After several calculations it can be easily seen that $Cov\left(X_{3T},X_{3(T-K)}\right)=0$ if K is greater than the integer part of $\frac{2(p+1)+q}{3}$ (Brewer, 1973).

Consequently, the aggregated series of an ARMA(p,q) model follows an ARMA(p,$\left[\frac{2(p+1)+q}{3}\right]$) model when m=3.

$$(1-\delta_1^3B)(1-\delta_2^3B)\dots(1-\delta_p^3B)Z_T$$
$$=\left(1-\beta_1B-\beta_2B^2-\dots-\beta_{\left[\frac{2(p+1)+q}{3}\right]}B^{\left[\frac{2(p+1)+q}{3}\right]}\right)A_T\ ,$$

where $\{A_T\}$ is a zero mean white noise process with constant variance σ_A^2.

The parameters β_1, β_2 , …, $\beta_{\left[\frac{2(p+1)+q}{3}\right]}$ and σ_A^2 of the aggregate series Z_T are functions of $\emptyset_1$, $\emptyset_2$,…,$\emptyset_p$ and $\theta_1, \theta_2, \dots, \theta_q$ and σ_a^2. Also, it is useful to state that the roots of AR polynomial of the aggregate series Z_T are the third power of the roots of AR polynomial of the basic series z_t .

For general m^{th} order aggregate is

$$Z_T = (1 + B + B^2) z_{mT} \ ,$$

where d=0.

Letting $\emptyset_p(B) = \prod_{j=1}^{p}(1 - \delta_j B)$ and multiplying $\prod_{j=1}^{p}\left[\frac{\left(1-\delta_j^m B^m\right)}{(1-\delta_j B)}\right]\left[\frac{(1-B^m)^{d+1}}{(1-B)^{d+1}}\right]$ on both sides of nonaggregate series z_t :

$$\frac{(1-\delta_1^m B^m)(1-\delta_2^m B^m)\dots(1-\delta_p^m B^m)}{(1-\delta_1 B)(1-\delta_2 B)\dots(1-\delta_p B)} \frac{(1-B^m)}{(1-B)}\left(1 - \emptyset_1 B - \dots - \emptyset_p B^p\right) z_t =$$
$$\frac{(1-\delta_1^m B^m)(1-\delta_2^m B^m)\dots(1-\delta_p^m B^m)}{(1-\delta_1 B)(1-\delta_2 B)\dots(1-\delta_p B)} \frac{(1-B^m)}{(1-B)}\left(1 - \theta_1 B - \dots - \theta_q B^q\right) a_t.$$

When the above equation is written in the explicit form, it is found that the m^{th} aggregate of an ARMA(p,q) model is an ARMA(p, $\left[p + 1 + \frac{q-p-1}{m}\right]$) (Brewer, 1973).

$$\prod_{j=1}^{p}(1 - \delta_j^m B) Z_T = \left(1 - \beta_1 B - \beta_2 B^2 - \dots - \beta_{\left[p+1+\frac{q-p-1}{m}\right]} B^{\left[p+1+\frac{q-p-1}{m}\right]}\right) A_T \ ,$$

where $\{A_T\}$ is a zero mean white noise process with constant variance σ_A^2.

The parameters β_1, β_2 , …, $\beta_{\left[p+1+\frac{q-p-1}{m}\right]}$ and σ_A^2 of the aggregate series Z_T are functions of $\emptyset_1$, $\emptyset_2$,…,$\emptyset_p$ and $\theta_1, \theta_2, \dots, \theta_q$ and σ_a^2 . Also, it is useful to state that the roots of AR polynomial of the aggregate series Z_T are the m^{th} power of the roots of AR polynomial of the basic series z_t .

CHAPTER 5

SIMULATION STUDY

The aim of this chapter is to show how the model, mean square forecast errors and estimated parameters change when the aggregate series is used instead of basic series. These changes will be illustrated by simulation studies.

Simulation studies for 1000 replications are conducted by using a computer program R 2.10.0 and the results of simulation studies will be summarized in five sections. In these sections, the effect of aggregation will be explained for the basic series belongs to AR(1), AR(2), MA(1), MA(2) and ARMA(1,1) models, respectively. In each section, simulation results will be given according to basic series' parameter value(s), order of aggregation and sample size. For the basic series' parameter value, both positive and negative parameter values will be considered by taking into account stationarity. Also, the parameter values will be selected in a wide range to see whether the magnitude of basic model parameter affects the temporal aggregation or not. For instance, parameter values -0.9 and 0.9 for AR(1) process would be useful to understand how the results change if the basic model parameter is close to unit root. Three, six and twelve will be used for the order of aggregation because in real life generally data are aggregated as quarterly, semi annually and annually. It is expected to receive worse results as order of aggregation increases. For the sample size, n=120, n=300 and n=900 will be evaluated and it is expected to have better results as sample size increases.

Mainly, there will be three tables for each section. First table shows the frequencies of best fitted models according to Akaike's Information Criterion for the aggregate series. The aggregate model will be determined by considering the significance of parameters and for instance an AR(2) model like $Z_T = \beta_1 Z_{T-1} + \beta_2 Z_{T-2} + A_T$ will be selected by looking at the significance of the β_2 parameter and the significance of β_1 parameter will not be taken into consideration for selecting the aggregate model ($\alpha = 0.05$ level will be used for significance). If two or more models will be fitted for the aggregate series, the best model will be chosen according to Akaike's Information Criterion. The aggregate model that have smallest Akaike's Information Criterion will be chosen as best fitted aggregate model. By looking at this table, it can be observed that how the model shifts, if the aggregate series is used for orders of aggregation and sample sizes stated above. The frequently selected best aggregate models will be shown in bold face. (Also, WN stands for white noise process.)

Second table shows the mean square forecast errors of the best fitted models for aggregated series and the basic series. As it is known, mean square forecast error is used for measuring the difference between the observed value of the data and the forecasted value. So, it is expected to have lower mean square forecast error values for basic series compared to aggregate series for sample sizes and orders of aggregation stated above. In order to calculate the mean square forecast error values last fifteen observations will be used for basic series and last five observations will be used for aggregate series. The mean square forecast errors listed in the tables will be the average values by taking into account the frequencies which take place at frequency tables. If the frequency of best fitted aggregate model is zero, then the mean square forecast error belong to this aggregate model will not be shown and it will be colored with black.

There exists some forecast accuracy measures like mean square error, absolute error, mean absolute percentage error. In our simulation study, using mean square forecast error as a measure of forecast accuracy is appropriate since Chen and Yang (2004) stated that mean square forecast error performs better if the error terms are normally distributed. Also, Chen and Yang (2004) discussed the problems related with mean absolute percentage error. Moreover, Janacek and Swift (1993) reveals that mean square error is an useful forecast accuracy measure for the prediction error.

Lastly, the third table illustrates the parameter change due to temporal aggregation. The estimated parameters of best fitted models for aggregate series can be seen from the third table. Also, the estimated parameter values listed in the tables will be the average values by taking into account the frequencies which take place at frequency tables. Again, like mean square forecast error table, if the frequency of best fitted aggregate model is zero, then the estimated parameter value(s) belong to this aggregate model will not be shown and black color will be used for the coverage.

5.1 Simulation Results for AR(1) Model

As stated before, the aggregate model from an AR(1) process is an ARMA(1,1) model when m=3, m=6 and m=12. However, this theoretical aggregate model can change depending on the basic model parameter, order of aggregation and the sample size.

Table 5.1.1 Frequencies of Best Fitted Aggregate Models for AR(1)

	Φ= - 0.9						**Φ= - 0.5**					
	AR(1)	**ARMA(1,1)**	**AR(2)**	**AR(3)**	**AR(4)**	**WN**	**AR(1)**	**ARMA(1,1)**	**AR(2)**	**AR(3)**	**AR(4)**	**WN**
m=3-n=120	**0.801**	0.068	0.047	0.037	0.045	0.002	0.193	0.322	0.056	0.043	0.048	**0.338**
m=3-n=300	**0.860**	0.043	0.029	0.035	0.033	0.000	**0.396**	0.239	0.057	0.036	0.036	0.236
m=3-n=900	**0.879**	0.035	0.019	0.034	0.033	0.000	**0.736**	0.075	0.053	0.047	0.023	0.066
m=6-n=120	0.082	0.152	0.116	0.077	0.139	**0.434**	0.091	0.186	0.105	0.075	0.134	**0.409**
m=6-n=300	0.067	**0.405**	0.069	0.051	0.053	0.355	0.074	0.379	0.044	0.047	0.042	**0.414**
m=6-n=900	0.139	**0.403**	0.097	0.058	0.042	0.261	0.147	0.315	0.052	0.040	0.033	**0.413**
m=12-n=120	0.001	0.000	0.003	0.289	**0.706**	0.001	0.000	0.000	0.003	0.282	**0.715**	0.000
m=12-n=300	0.085	0.272	0.100	0.077	0.096	**0.370**	0.071	0.244	0.088	0.056	0.096	**0.445**
m=12-n=900	0.150	0.329	0.066	0.041	0.044	**0.370**	0.069	0.342	0.052	0.034	0.045	**0.458**
	Φ= - 0.1						**Φ= 0.1**					
	AR(1)	**ARMA(1,1)**	**AR(2)**	**AR(3)**	**AR(4)**	**WN**	**AR(1)**	**ARMA(1,1)**	**AR(2)**	**AR(3)**	**AR(4)**	**WN**
m=3-n=120	0.062	0.363	0.064	0.054	0.055	**0.402**	0.043	0.359	0.053	0.038	0.065	**0.442**
m=3-n=300	0.049	0.379	0.040	0.047	0.044	**0.441**	0.042	0.339	0.036	0.044	0.044	**0.495**
m=3-n=900	0.067	0.330	0.042	0.048	0.041	**0.472**	0.072	0.279	0.045	0.048	0.030	**0.526**
m=6-n=120	0.064	0.178	0.094	0.077	0.136	**0.451**	0.043	0.169	0.091	0.091	0.131	**0.475**
m=6-n=300	0.060	0.363	0.064	0.051	0.036	**0.426**	0.034	0.363	0.054	0.048	0.048	**0.453**
m=6-n=900	0.049	0.350	0.040	0.042	0.041	**0.478**	0.046	0.363	0.035	0.045	0.038	**0.473**
m=12-n=120	0.000	0.000	0.006	0.299	**0.695**	0.000	0.000	0.000	0.001	0.272	**0.727**	0.000
m=12-n=300	0.050	0.242	0.090	0.061	0.096	**0.461**	0.043	0.276	0.079	0.068	0.087	**0.447**
m=12-n=900	0.047	0.350	0.044	0.034	0.046	**0.479**	0.051	0.339	0.054	0.028	0.053	**0.475**
	Φ= 0.5						**Φ= 0.9**					
	AR(1)	**ARMA(1,1)**	**AR(2)**	**AR(3)**	**AR(4)**	**WN**	**AR(1)**	**ARMA(1,1)**	**AR(2)**	**AR(3)**	**AR(4)**	**WN**
m=3-n=120	0.051	0.358	0.047	0.049	0.055	**0.440**	**0.607**	0.100	0.184	0.047	0.062	0.000
m=3-n=300	**0.596**	0.087	0.072	0.033	0.043	0.169	**0.400**	0.261	0.248	0.045	0.046	0.000
m=3-n=900	**0.772**	0.019	0.132	0.049	0.028	0.000	0.099	**0.448**	0.344	0.055	0.054	0.000
m=6-n=120	0.044	0.144	0.113	0.064	0.140	**0.495**	**0.403**	0.036	0.174	0.056	0.101	0.230
m=6-n=300	0.090	0.288	0.049	0.052	0.041	**0.480**	**0.610**	0.068	0.208	0.052	0.056	0.006
m=6-n=900	0.309	0.180	0.046	0.054	0.036	**0.375**	**0.354**	0.294	0.291	0.034	0.027	0.000
m=12-n=120	0.000	0.000	0.004	0.242	**0.754**	0.000	0.000	0.000	0.003	0.283	**0.714**	0.000
m=12-n=300	0.044	0.271	0.093	0.057	0.116	**0.419**	0.303	0.077	0.137	0.080	0.091	**0.312**
m=12-n=900	0.054	0.326	0.037	0.042	0.041	**0.500**	**0.683**	0.021	0.189	0.037	0.065	0.005

Simulation results of Table 5.5.1 reveals that the frequently selected best fitted aggregated model is generally AR(1) or white noise in empirical studies. Here, in this simulation study white noise model is selected as best fitted aggregate model when all of the aggregate model parameters belong to AR(1), ARMA(1,1), AR(2), AR(3) and AR(4) are insignificant. For the parameter values -0.1 and 0.1, the white noise is frequently selected and for other parameter values AR(1) is chosen for m=3 case. As it is expected when the order of aggregation increases, aggregate model shifts from AR(1) to white noise. It can be useful to state that AR(2) and AR(3) are not frequently selected as best fitted aggregated model for all conditions. Also, it is expected to have more clear results for large sample sizes but the results do not show big changes when sample size increases except m=12 and n=120 case.

Table 5.1.2 Mean Square Forecast Errors of Best Fitted Aggregate Models

for AR(1)

	Φ= - 0.9							Φ= - 0.5						
	AR(1)	ARMA(1,1)	AR(2)	AR(3)	AR(4)	WN	BASIC	AR(1)	ARMA(1,1)	AR(2)	AR(3)	AR(4)	WN	BASIC
m=3-n=120	4.155	5.181	4.442	4.021	4.950	5.421	**3.919**	1.959	2.106	2.432	2.149	2.831	2.114	**1.306**
m=3-n=300	4.430	5.387	4.951	5.461	4.480		**3.967**	2.177	2.036	2.014	2.064	1.975	2.065	**1.333**
m=3-n=900	4.372	3.448	4.306	3.371	5.171		**3.964**	1.982	1.985	1.760	2.004	2.262	1.824	**1.289**
m=6-n=120	3.061	3.078	3.909	3.986	4.225	**2.914**	4.035	3.268	**2.924**	3.733	3.428	4.081	3.042	4.050
m=6-n=300	**2.876**	3.027	3.221	3.433	3.500	2.968	3.908	3.282	3.284	3.826	3.758	3.392	3.360	**1.284**
m=6-n=900	2.937	3.030	2.988	3.264	**2.581**	2.845	3.772	3.701	3.325	3.329	3.907	3.510	3.297	**1.327**
m=12-n=120	5.689		6.850	14.449	26.216	1.302	**3.853**			14.921	17.254	30.076		**1.306**
m=12-n=300	5.273	5.642	6.619	7.889	8.106	5.729	**3.960**	7.242	6.197	6.669	7.446	8.120	5.974	**1.331**
m=12-n=900	5.097	5.267	5.376	5.694	5.080	5.241	**3.854**	5.508	5.739	5.122	6.683	7.346	6.072	**1.300**
	Φ= - 0.1							Φ= 0.1						
	AR(1)	ARMA(1,1)	AR(2)	AR(3)	AR(4)	WN	BASIC	AR(1)	ARMA(1,1)	AR(2)	AR(3)	AR(4)	WN	BASIC
m=3-n=120	3.218	2.850	2.745	3.094	3.501	2.799	**1.028**	2.902	3.714	3.890	4.328	4.401	3.511	**1.011**
m=3-n=300	2.706	2.658	2.958	3.230	2.757	2.823	**1.004**	3.501	3.481	3.137	3.603	3.639	3.648	**1.012**
m=3-n=900	2.475	2.553	2.567	2.784	2.727	2.727	**0.999**	3.400	3.336	3.061	3.752	3.407	3.603	**1.015**
m=6-n=120	6.590	6.137	7.432	6.211	7.595	5.446	**1.021**	9.785	8.083	8.895	9.101	9.940	7.261	**1.016**
m=6-n=300	5.970	5.503	6.002	4.806	5.235	5.087	**1.015**	6.843	7.625	7.546	8.153	7.477	7.058	**0.995**
m=6-n=900	5.160	5.290	5.384	5.196	5.547	5.070	**1.007**	7.803	7.300	7.340	8.187	8.788	7.080	**1.012**
m=12-n=120			18.130	25.991	48.969		**1.020**			19.323	36.649	57.850		**1.017**
m=12-n=300	11.579	10.789	11.231	16.446	14.131	10.648	**1.012**	17.439	15.234	19.652	19.672	19.958	15.188	**1.027**
m=12-n=900	10.812	11.154	8.569	8.719	10.287	10.462	**1.002**	13.856	14.640	14.742	14.392	14.867	14.587	**0.994**
	Φ= 0.5							Φ= 0.9						
	AR(1)	ARMA(1,1)	AR(2)	AR(3)	AR(4)	WN	BASIC	AR(1)	ARMA(1,1)	AR(2)	AR(3)	AR(4)	WN	BASIC
m=3-n=120	4.098	3.488	4.630	3.417	4.343	3.439	**0.987**	34.950	33.194	40.612	38.515	42.068		**4.144**
m=3-n=300	6.889	7.236	7.005	8.680	7.346	6.834	**1.269**	32.857	29.691	33.216	36.962	42.038		**3.833**
m=3-n=900	7.246	7.482	7.292	8.246	7.318		**1.303**	35.805	32.216	31.132	27.845	32.356		**3.917**
m=6-n=120	23.066	20.580	23.009	21.232	26.021	21.592	**1.356**	159.993	172.144	161.160	197.000	218.843	184.378	**4.244**
m=6-n=300	18.324	19.160	20.297	20.796	22.813	18.972	**1.294**	142.605	153.766	170.230	181.483	167.921	163.840	**3.837**
m=6-n=900	19.131	19.627	15.867	18.621	19.433	19.276	**1.277**	133.795	143.266	145.827	162.593	130.971		**3.819**
m=12-n=120			54.963	116.069	178.538		**1.347**			1064.350	1698.157	1626.587		**4.166**
m=12-n=300	52.466	47.827	50.393	72.499	65.972	46.770	**1.297**	506.431	585.181	598.829	640.505	766.749	537.375	**3.980**
m=12-n=900	42.807	42.172	47.025	44.764	45.870	45.907	**1.309**	518.742	648.060	506.477	586.816	500.474	849.205	**3.841**

When the results of Table 5.1.2 are analyzed, it is clearly seen that basic model has lower mean square forecast error than best fitted aggregated models' mean square forecast errors as expected. Moreover, it is observed that the mean square forecast errors for the best fitted aggregate models increase as the order of aggregation increases. For $\emptyset$ = -0.9 and m=6 case, there exists a problem that minimum mean square error belongs to white noise for n=120, AR(1) for n=300 and AR(4) for n=900 . This problem may be occurred because of the basic model parameter value which is very close to the negative unit root that creates nonstationarity. In general, it can be said that mean square forecast errors show an increase when the series is temporally aggregated.

In Table 5.1.3, we present the mean of estimated parameters for best fitted aggregate models. For ARMA(1,1) case, the first box indicates the autoregressive parameter and the second box indicates the moving average parameter. Estimated parameter for best fitted aggregate AR(1) model is not always the same as the basic model parameter as expected. So, temporal aggregation changes the estimated parameter for the basic series from an AR(1) process. Also as stated in previous chapters, the roots of AR polynomial of the best fitted aggregate ARMA(1,1) model are theoretically expected to be the mth power of roots of AR polynomial of the basic model. However, this theoretical result cannot be observed from Table 5.1.3 for the best fitted aggregate ARMA(1,1) model.

5.2 Simulation Results for AR(2) Model

Theoretical aggregate model for AR(2) process is ARMA(2,2) model for m=3, m=6 and m=12. The simulation studies will also consider AR(1), ARMA(1,1), AR(2) and white noise processes for the aggregate model since theoretical results may change in empirical studies.

Basic series are modeled for various parameter values and stationarity conditions are taken into consideration. The stationarity conditions for AR(2) model can be summarized as $\emptyset_1 + \emptyset_2 < 1$, $\emptyset_2 - \emptyset_1 < 1$ and $-1 < \emptyset_2 < 1$.

Table 5.1.3 Estimated Parameter Values of Best Fitted Aggregate Models for AR(1)

	Φ= - 0.9											
	AR(1)	ARMA(1,1)		AR(2)		AR(3)			AR(4)			
m=3-n=120	-0.7	-0.6	-0.3	-0.8	-0.2	-0.8	-0.1	-0.1	-0.7	0.0	-0.1	-0.3
m=3-n=300	-0.7	-0.6	-0.2	-0.8	-0.1	-0.7	0.0	0.0	-0.8	0.0	-0.1	-0.1
m=3-n=900	-0.7	-0.7	-0.1	-0.7	0.0	-0.7	0.0	0.0	-0.7	0.0	0.0	-0.1
m=6-n=120	-0.5	0.1	-0.3	-0.1	-0.5	-0.2	-0.2	-0.4	-0.1	-0.3	-0.1	-0.5
m=6-n=300	-0.3	0.3	-0.5	-0.1	-0.3	-0.1	-0.1	-0.3	-0.1	-0.1	-0.1	-0.3
m=6-n=900	-0.2	0.4	-0.6	-0.1	-0.2	-0.1	0.0	-0.2	0.0	-0.1	0.0	-0.2
m=12-n=120	-0.8	[redacted]		-0.8	-0.9	-1.0	-1.0	-1.0	-0.4	-0.7	-0.4	-0.7
m=12-n=300	-0.4	0.3	-0.6	-0.2	-0.5	-0.2	-0.2	-0.3	-0.2	-0.2	-0.1	-0.4
m=12-n=900	-0.3	0.4	-0.6	-0.1	-0.3	-0.1	0.0	-0.1	0.0	0.0	0.0	-0.2
	Φ= - 0.5											
	AR(1)	ARMA(1,1)		AR(2)		AR(3)			AR(4)			
m=3-n=120	-0.4	0.3	-0.6	-0.2	-0.3	-0.1	0.0	-0.1	-0.1	0.0	0.0	-0.2
m=3-n=300	-0.3	0.3	-0.4	-0.1	-0.1	-0.2	0.0	0.0	-0.1	0.0	0.0	0.0
m=3-n=900	-0.2	0.1	-0.2	-0.2	-0.1	-0.2	0.0	0.0	-0.2	0.0	0.0	0.0
m=6-n=120	-0.5	0.1	-0.2	-0.1	-0.5	-0.1	-0.2	-0.5	-0.2	-0.2	-0.1	-0.6
m=6-n=300	-0.3	0.3	-0.4	-0.1	-0.2	-0.1	0.0	0.0	-0.1	-0.1	-0.1	-0.3
m=6-n=900	-0.2	0.1	-0.2	-0.1	-0.1	0.0	0.0	0.0	-0.1	0.0	0.0	-0.1
m=12-n=120	[redacted]			-1.5	-1.0	-0.8	-0.8	-1.0	-0.4	-0.7	-0.4	-0.7
m=12-n=300	-0.4	0.1	-0.3	-0.1	-0.4	-0.1	-0.1	-0.2	-0.1	-0.2	-0.1	-0.5
m=12-n=900	-0.2	0.2	-0.3	0.0	-0.1	0.0	0.0	-0.1	0.0	0.0	0.0	-0.1
	Φ= - 0.1											
	AR(1)	ARMA(1,1)		AR(2)		AR(3)			AR(4)			
m=3-n=120	-0.3	0.1	-0.3	0.0	-0.2	0.0	0.0	-0.2	0.0	-0.1	0.0	-0.3
m=3-n=300	-0.2	0.1	-0.2	0.0	-0.1	0.0	0.0	0.0	0.0	0.0	0.0	-0.1
m=3-n=900	-0.1	0.1	-0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
m=6-n=120	-0.2	0.0	-0.2	-0.1	-0.5	-0.1	-0.1	-0.4	-0.1	-0.1	-0.1	-0.5
m=6-n=300	-0.2	0.1	-0.2	0.0	-0.2	0.0	0.0	-0.2	0.0	0.0	0.0	-0.2
m=6-n=900	-0.1	0.1	-0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.1
m=12-n=120	[redacted]			-1.1	-1.0	-0.8	-0.8	-1.0	-0.3	-0.7	-0.4	-0.7
m=12-n=300	-0.3	0.2	-0.3	-0.1	-0.4	-0.1	-0.1	-0.2	0.0	-0.1	0.0	-0.5
m=12-n=900	-0.1	0.2	-0.2	0.0	-0.1	0.0	0.0	0.0	0.0	0.0	0.0	-0.2
	Φ= 0.1											
	AR(1)	ARMA(1,1)		AR(2)		AR(3)			AR(4)			
m=3-n=120	0.1	0.1	-0.2	0.0	-0.3	0.0	0.0	-0.2	0.0	0.0	0.0	-0.3
m=3-n=300	0.1	0.1	-0.1	0.0	-0.1	0.0	0.0	-0.1	0.0	0.0	0.0	-0.2
m=3-n=900	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
m=6-n=120	-0.3	0.0	-0.1	-0.1	-0.5	-0.1	-0.1	-0.3	-0.1	-0.2	-0.1	-0.5
m=6-n=300	0.0	0.1	-0.2	0.0	-0.2	0.0	0.0	-0.2	0.0	0.0	0.0	-0.1
m=6-n=900	0.1	0.0	0.0	0.0	0.0	0.0	0.0	-0.1	0.0	0.0	0.0	-0.1
m=12-n=120	[redacted]			-0.5	-1.0	-0.8	-0.9	-1.0	-0.3	-0.6	-0.3	-0.7
m=12-n=300	-0.2	0.2	-0.3	-0.1	-0.4	-0.1	-0.1	-0.2	-0.1	-0.1	-0.1	-0.5
m=12-n=900	0.0	0.0	-0.1	0.0	-0.1	0.0	0.0	0.0	0.0	0.0	0.0	-0.1
	Φ= 0.5											
	AR(1)	ARMA(1,1)		AR(2)		AR(3)			AR(4)			
m=3-n=120	0.1	0.2	-0.3	0.0	-0.3	0.0	0.0	-0.2	0.0	-0.1	0.0	-0.3
m=3-n=300	0.3	-0.3	0.5	0.3	-0.2	0.3	0.0	-0.1	0.3	0.0	0.0	-0.1
m=3-n=900	0.3	-0.3	0.6	0.3	-0.1	0.3	0.0	0.0	0.3	0.0	0.0	-0.1
m=6-n=120	0.3	0.2	-0.3	0.0	-0.5	0.0	0.0	-0.4	0.0	-0.2	0.0	-0.6
m=6-n=300	0.3	0.0	0.0	0.1	-0.3	0.1	0.0	-0.2	0.0	0.0	0.0	-0.2
m=6-n=900	0.2	-0.2	0.3	0.1	-0.1	0.1	0.0	0.0	0.1	0.0	0.0	0.0
m=12-n=120	[redacted]			-0.9	-1.0	-0.7	-0.7	-1.0	-0.3	-0.6	-0.3	-0.7
m=12-n=300	-0.1	0.2	-0.4	0.0	-0.4	0.0	0.0	-0.2	0.0	-0.1	-0.1	-0.4
m=12-n=900	0.2	0.1	-0.1	0.0	-0.1	0.0	0.0	0.0	0.0	0.0	0.0	-0.2
	Φ= 0.9											
	AR(1)	ARMA(1,1)		AR(2)		AR(3)			AR(4)			
m=3-n=120	0.7	0.6	0.6	1.0	-0.4	0.8	-0.2	-0.1	0.8	-0.2	0.1	-0.3
m=3-n=300	0.8	0.6	0.4	1.0	-0.3	0.9	-0.2	0.1	0.9	-0.1	0.1	-0.2
m=3-n=900	0.8	0.7	0.3	1.0	-0.2	0.9	-0.3	0.1	0.9	-0.2	0.1	0.0
m=6-n=120	0.6	0.3	0.5	0.7	-0.6	0.5	-0.2	-0.2	0.5	-0.3	0.2	-0.6
m=6-n=300	0.6	0.5	0.5	0.8	-0.4	0.7	-0.2	0.0	0.7	-0.2	0.1	-0.2
m=6-n=900	0.6	0.5	0.4	0.8	-0.2	0.8	-0.2	0.2	0.8	-0.1	0.0	-0.2
m=12-n=120	[redacted]			-0.3	-1.0	-0.6	-0.6	-1.0	-0.1	-0.6	-0.3	-0.6
m=12-n=300	0.5	-0.2	0.3	0.5	-0.5	0.3	-0.1	-0.2	0.4	-0.3	0.1	-0.5
m=12-n=900	0.4	-0.3	0.7	0.6	-0.3	0.5	-0.1	0.0	0.5	-0.2	0.1	-0.2

Table 5.2.1 Frequencies of Best Fitted Aggregate Models for AR(2)

	Φ1= 0.1 - Φ2= 0.8					Φ1= 0.1 - Φ2= 0.1				
	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN
m=3-n=120	**0.788**	0.021	0.090	0.097	0.004	0.061	0.280	0.054	**0.375**	0.230
m=3-n=300	**0.641**	0.014	0.239	0.106	0.000	0.132	0.199	0.059	**0.409**	0.201
m=3-n=900	0.277	0.000	**0.577**	0.146	0.000	**0.350**	0.144	0.037	0.349	0.120
m=6-n=120	**0.518**	0.045	0.177	0.139	0.121	0.051	0.178	0.116	0.210	**0.445**
m=6-n=300	**0.605**	0.139	0.207	0.049	0.000	0.036	0.307	0.058	**0.388**	0.211
m=6-n=900	0.326	**0.337**	0.295	0.042	0.000	0.056	0.254	0.041	**0.484**	0.165
m=12-n=120	0.039	0.000	**0.535**	0.003	0.423	0.080	0.005	**0.481**	0.003	0.431
m=12-n=300	**0.576**	0.035	0.176	0.109	0.104	0.050	0.301	0.088	0.255	**0.306**
m=12-n=900	**0.550**	0.038	0.182	0.123	0.107	0.031	0.309	0.038	**0.449**	0.173
	Φ1= 0.3 - Φ2= 0.5					**Φ1= - 0.1 - Φ2= - 0.8**				
	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN
m=3-n=120	**0.780**	0.048	0.085	0.077	0.010	0.035	0.242	**0.479**	0.202	0.042
m=3-n=300	**0.847**	0.051	0.068	0.034	0.000	0.005	0.121	**0.605**	0.268	0.001
m=3-n=900	**0.730**	0.081	0.157	0.032	0.000	0.000	0.036	**0.559**	0.405	0.000
m=6-n=120	**0.349**	0.042	0.181	0.167	0.261	**0.495**	0.092	0.112	0.082	0.219
m=6-n=300	**0.686**	0.049	0.170	0.082	0.013	**0.720**	0.073	0.083	0.114	0.010
m=6-n=900	**0.579**	0.142	0.251	0.028	0.000	**0.863**	0.029	0.078	0.030	0.000
m=12-n=120	0.044	0.000	**0.558**	0.000	0.398	0.125	0.006	0.503	0.001	**0.365**
m=12-n=300	0.212	0.112	0.138	0.221	**0.317**	0.103	0.279	0.110	0.221	**0.287**
m=12-n=900	**0.652**	0.046	0.120	0.142	0.040	0.119	0.311	0.080	**0.330**	0.160
	Φ1= - 0.1 - Φ2= - 0.1					**Φ1= - 0.3 - Φ2= - 0.5**				
	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN
m=3-n=120	0.084	0.332	0.055	**0.340**	0.189	0.135	**0.367**	0.094	0.276	0.128
m=3-n=300	0.092	0.302	0.038	**0.407**	0.161	0.242	**0.269**	0.152	0.253	0.084
m=3-n=900	0.209	0.199	0.042	**0.403**	0.147	0.315	0.208	**0.329**	0.136	0.012
m=6-n=120	0.075	0.172	0.100	0.204	**0.449**	0.159	0.136	0.123	0.168	**0.414**
m=6-n=300	0.048	0.333	0.047	**0.403**	0.169	0.213	0.286	0.061	**0.294**	0.146
m=6-n=900	0.076	0.271	0.044	**0.466**	0.143	**0.522**	0.127	0.055	0.229	0.067
m=12-n=120	0.091	0.003	**0.514**	0.000	0.392	0.099	0.004	**0.501**	0.001	0.395
m=12-n=300	0.062	0.279	0.086	0.238	**0.335**	0.106	0.275	0.088	0.220	**0.311**
m=12-n=900	0.048	0.315	0.039	**0.429**	0.169	0.129	0.279	0.046	**0.393**	0.153
	Φ1= - 0.1 - Φ2= 0.8					**Φ1= 0.3 - Φ2= - 0.5**				
	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN
m=3-n=120	0.149	0.084	**0.552**	0.113	0.102	**0.384**	0.224	0.074	0.202	0.116
m=3-n=300	0.026	0.008	**0.892**	0.074	0.000	**0.664**	0.070	0.073	0.155	0.038
m=3-n=900	0.000	0.000	**0.778**	0.222	0.000	**0.826**	0.006	0.090	0.078	0.000
m=6-n=120	0.265	0.054	0.158	0.195	**0.328**	0.098	0.169	0.112	0.184	**0.437**
m=6-n=300	**0.669**	0.034	0.140	0.120	0.037	0.115	0.294	0.063	**0.351**	0.177
m=6-n=900	**0.682**	0.052	0.234	0.032	0.000	0.234	0.189	0.055	**0.372**	0.150
m=12-n=120	0.046	0.002	**0.512**	0.000	0.440	0.108	0.003	**0.474**	0.000	0.415
m=12-n=300	0.152	0.136	0.110	0.267	**0.335**	0.093	0.249	0.104	0.245	**0.309**
m=12-n=900	**0.535**	0.071	0.106	0.209	0.079	0.078	0.308	0.038	**0.400**	0.176

The results of Table 5.2.1 indicate that generally theoretical aggregate model ARMA(2,2) is the frequently selected best fitted aggregated model for $\emptyset_1$=0.1 - $\emptyset_2$ = 0.1 and $\emptyset_1$= -0.1 - $\emptyset_2$ = -0.1 when m =3 and m =6. AR (2) is the frequently selected best fitted aggregated model for $\emptyset_1$= -0.1 - $\emptyset_2$ = 0.8 and $\emptyset_1$= -0.1 - $\emptyset_2$ = -0.8 when m =3. Furthermore, frequently selected best fitted aggregated model for m =12 and n =120 case is AR(2) independent of the parameter values. This shows that reliable results cannot be obtained for higher orders of aggregation and small sample sizes. Also, white noise is frequently selected best fitted aggregated model for small sample sizes and high orders of aggregation. In this section, white noise is selected if all of the best fitted aggregate models listed in Table 5.2.1 have insignificant parameters like previous section.

All of the results of Table 5.2.2 show that the average of mean square forecast errors belong to basic model smaller than best fitted aggregate models' mean square forecast errors. Again, it is observed that mean square forecast error increases as the orders of aggregation increase. Especially, for m=12 case, the best fitted aggregated models have larger mean square forecast errors. It is obvious that aggregated models have worse forecast values compared to the forecasted values from the basic model. Also the forecast values get worsen as the order of aggregation increases.

For Table 5.2.3, when the best fitted aggregated model is ARMA(1,1) the first box shows the estimated parameter for the autoregressive part and the second box shows the estimated parameter for the moving average part. Similarly, when the best fitted aggregated model is ARMA(2,2), the first two box show the estimated parameter for the autoregressive part and the others show the estimated parameter for the moving average part. It is clearly seen that when the best fitted aggregated model is AR(2) the estimated parameters mostly different than the basic model parameters from Table 5.2.3. Also as stated in previously, the roots of AR polynomial of the best fitted aggregate ARMA(2,2) model are theoretically expected to be the mth power of roots of AR polynomial of the basic model. However, this theoretical result cannot be observed from Table 5.2.3 for the best fitted aggregate ARMA(2,2) model.

Table 5.2.2 Mean Square Forecast Errors of Best Fitted Aggregate Models

for AR(2)

	Φ1= 0.1 - Φ2= 0.8						Φ1= 0.1 - Φ2= 0.1					
	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	BASIC	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	BASIC
m=3-n=120	16.627	16.581	16.907	21.209	14.708	**2.521**	3.251	4.214	4.227	4.585	3.890	**1.057**
m=3-n=300	16.180	13.604	14.861	15.513	[redacted]	**2.466**	3.711	3.546	4.080	4.033	3.868	**1.010**
m=3-n=900	14.602	[redacted]	14.406	15.985	[redacted]	**2.388**	3.580	3.433	4.197	3.872	4.068	**1.005**
m=6-n=120	87.284	106.205	99.189	147.001	89.426	**2.658**	9.368	9.237	11.503	11.779	8.848	**1.044**
m=6-n=300	70.857	62.984	80.187	93.895	[redacted]	**2.426**	7.981	9.186	8.092	8.812	8.975	**1.011**
m=6-n=900	69.238	68.892	71.458	73.299	[redacted]	**2.373**	8.452	8.636	7.712	8.525	8.537	**1.034**
m=12-n=120	616.142	[redacted]	659.424	465.736	449.407	**2.510**	27.213	24.287	36.372	30.671	22.074	**1.035**
m=12-n=300	344.415	427.803	403.966	512.303	405.056	**2.452**	20.463	19.719	23.570	24.245	18.620	**1.029**
m=12-n=900	320.990	352.315	356.154	433.323	320.905	**2.316**	19.577	19.158	18.603	19.417	17.624	**1.008**
	Φ1= 0.3 - Φ2= 0.5						Φ1= - 0.1 - Φ2= - 0.8					
	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	BASIC	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	BASIC
m=3-n=120	12.541	15.889	15.422	15.740	13.429	**1.885**	2.981	3.342	3.375	3.291	3.463	**2.271**
m=3-n=300	11.572	12.303	13.242	11.327	[redacted]	**1.771**	5.841	3.347	3.268	3.360	**0.502**	2.214
m=3-n=900	11.883	11.675	9.755	12.177	[redacted]	**1.776**	[redacted]	3.244	3.068	2.959	[redacted]	**2.137**
m=6-n=120	54.312	61.800	60.731	69.156	57.450	**1.827**	5.432	6.267	6.618	7.402	5.409	**2.144**
m=6-n=300	46.662	50.025	48.803	62.258	60.099	**1.784**	5.174	5.020	5.189	6.978	5.541	**2.127**
m=6-n=900	47.193	43.561	45.064	44.351	[redacted]	**1.815**	5.511	6.018	4.804	6.370	[redacted]	**2.223**
m=12-n=120	225.358	[redacted]	355.459	[redacted]	223.768	**1.918**	8.026	5.552	10.853	12.016	6.487	**2.231**
m=12-n=300	173.188	177.185	191.983	218.663	169.351	**1.875**	6.236	5.653	6.826	6.755	5.554	**2.194**
m=12-n=900	150.098	172.543	155.325	169.015	157.284	**1.790**	5.303	5.467	5.539	6.180	5.117	**2.181**
	Φ1= - 0.1 - Φ2= - 0.1						Φ1= - 0.3 - Φ2= - 0.5					
	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	BASIC	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	BASIC
m=3-n=120	2.696	2.671	2.738	2.943	2.442	**1.022**	1.853	1.916	2.041	2.053	1.934	**1.316**
m=3-n=300	2.394	2.705	2.443	2.775	2.474	**1.027**	1.810	2.033	1.899	2.073	2.012	**1.344**
m=3-n=900	2.370	2.598	2.892	2.687	2.505	**1.029**	1.860	1.868	1.855	1.714	1.637	**1.313**
m=6-n=120	5.788	4.889	5.880	6.322	4.719	**1.026**	3.384	3.625	4.122	3.929	3.118	**1.335**
m=6-n=300	5.130	4.604	6.085	5.017	4.740	**1.019**	3.155	2.737	3.495	3.283	2.780	**1.323**
m=6-n=900	4.996	4.792	4.640	4.902	4.849	**1.046**	2.950	2.925	3.148	3.196	2.924	**1.309**
m=12-n=120	12.319	7.603	18.351	[redacted]	10.118	**1.028**	7.094	4.989	9.160	15.966	5.305	**1.309**
m=12-n=300	10.670	9.322	9.843	10.672	9.144	**1.011**	5.380	5.086	5.868	6.279	4.962	**1.324**
m=12-n=900	9.212	9.171	7.646	9.434	9.109	**1.024**	4.928	4.982	5.091	4.986	4.975	**1.305**
	Φ1= - 0.1 - Φ2= 0.8						Φ1= 0.3 - Φ2= - 0.5					
	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	BASIC	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	BASIC
m=3-n=120	8.973	10.063	9.408	10.796	10.813	**2.449**	4.232	4.386	4.448	4.688	3.855	**1.311**
m=3-n=300	9.273	6.792	8.926	8.450	[redacted]	**2.396**	4.063	3.581	4.326	4.284	4.274	**1.367**
m=3-n=900	[redacted]	[redacted]	8.779	8.005	[redacted]	**2.387**	4.048	3.086	3.861	3.938	[redacted]	**1.338**
m=6-n=120	27.972	26.596	31.324	40.372	28.135	**2.486**	7.198	6.359	7.417	7.430	5.980	**1.377**
m=6-n=300	23.556	28.984	25.076	33.664	25.857	**2.376**	5.909	6.275	5.346	6.100	6.015	**1.329**
m=6-n=900	24.827	27.348	23.793	31.516	[redacted]	**2.416**	5.789	5.761	4.781	5.931	5.773	**1.300**
m=12-n=120	109.040	170.075	155.249	[redacted]	95.307	**2.567**	12.742	13.951	20.956	[redacted]	11.996	**1.311**
m=12-n=300	86.620	86.966	108.288	98.093	82.831	**2.431**	11.761	11.304	11.950	11.914	10.201	**1.328**
m=12-n=900	81.203	69.314	79.810	84.674	84.929	**2.319**	12.267	10.144	9.482	10.045	10.420	**1.299**

Table 5.2.3 Estimated Parameter Values of Best Fitted Aggregate Models for AR(2)

	Φ1= 0.1 - Φ2= 0.8								
	AR(1)	ARMA(1,1)		AR(2)		ARMA(2,2)			
m=3-n=120	0.7	0.7	0.0	0.5	0.3	1.0	-0.4	-0.4	0.8
m=3-n=300	0.8	0.8	-0.1	0.6	0.3	0.6	0.0	0.1	0.4
m=3-n=900	0.8	[redacted]		0.7	0.2	0.3	0.3	0.3	0.2
m=6-n=120	0.7	0.5	0.8	0.9	-0.6	1.2	-0.8	-1.0	0.9
m=6-n=300	0.7	0.6	0.5	1.0	-0.4	0.9	-0.4	-0.2	0.5
m=6-n=900	0.8	0.6	0.3	1.0	-0.2	0.4	0.2	0.6	0.2
m=12-n=120	-0.5	[redacted]		0.1	-0.9	1.6	-1.0	-1.3	0.8
m=12-n=300	0.6	0.2	0.6	0.8	-0.5	0.9	-0.8	-0.8	0.9
m=12-n=900	0.6	0.2	0.6	0.8	-0.5	0.9	-0.8	-0.7	0.9
	Φ1= 0.1 - Φ2= 0.1								
	AR(1)	ARMA(1,1)		AR(2)		ARMA(2,2)			
m=3-n=120	0.3	0.0	-0.1	0.0	-0.3	0.1	-0.7	-0.1	0.9
m=3-n=300	0.2	0.0	0.0	0.1	-0.2	0.0	-0.8	0.1	0.8
m=3-n=900	0.2	-0.2	0.3	0.1	0.0	0.0	-0.8	0.0	0.8
m=6-n=120	-0.2	0.0	-0.1	0.0	-0.5	0.5	-0.7	-0.8	0.9
m=6-n=300	0.1	0.1	-0.1	0.1	-0.2	0.0	-0.7	0.0	0.9
m=6-n=900	0.2	0.0	0.1	0.0	0.0	0.0	-0.8	0.1	0.9
m=12-n=120	-0.8	-1.0	1.0	-0.4	-0.9	0.3	-1.0	-1.5	0.8
m=12-n=300	-0.1	0.2	-0.3	0.0	-0.4	0.1	-0.7	-0.1	0.9
m=12-n=900	0.0	0.2	-0.3	0.0	-0.1	0.1	-0.7	-0.1	0.8
	Φ1= 0.3 - Φ2= 0.5								
	AR(1)	ARMA(1,1)		AR(2)		ARMA(2,2)			
m=3-n=120	0.6	0.6	0.4	0.9	-0.4	0.9	-0.6	-0.4	0.8
m=3-n=300	0.7	0.6	0.3	0.9	-0.2	0.6	-0.2	0.1	0.4
m=3-n=900	0.7	0.6	0.2	0.8	-0.1	0.1	0.3	0.7	0.2
m=6-n=120	0.6	0.0	0.2	0.6	-0.6	0.9	-0.9	-0.9	0.9
m=6-n=300	0.6	0.2	0.5	0.7	-0.4	0.6	-0.6	-0.1	0.8
m=6-n=900	0.6	0.4	0.3	0.7	-0.2	0.8	-0.3	-0.2	0.0
m=12-n=120	-0.6	[redacted]		-0.1	-0.9	[redacted]			
m=12-n=300	0.5	-0.1	0.3	0.4	-0.5	0.6	-0.8	-0.5	1.0
m=12-n=900	0.4	-0.1	0.5	0.4	-0.3	0.2	-0.7	0.1	0.9
	Φ1= - 0.1 - Φ2= - 0.8								
	AR(1)	ARMA(1,1)		AR(2)		ARMA(2,2)			
m=3-n=120	-0.4	0.4	-0.9	-0.2	-0.5	0.4	-0.8	-0.8	0.8
m=3-n=300	-0.3	0.5	-0.9	-0.2	-0.4	0.5	-0.7	-0.8	0.6
m=3-n=900	[redacted]	0.4	-0.8	-0.2	-0.4	0.3	-0.6	-0.7	0.4
m=6-n=120	-0.6	-0.4	-0.3	-0.7	-0.5	-0.3	-0.7	-0.2	0.8
m=6-n=300	-0.5	0.0	-0.6	-0.7	-0.3	-0.5	-0.5	0.1	0.6
m=6-n=900	-0.5	-0.4	-0.2	-0.6	-0.2	-0.8	-0.4	0.4	0.3
m=12-n=120	-0.8	-1.0	1.0	-0.5	-0.9	-1.5	-1.0	-1.6	0.9
m=12-n=300	-0.5	0.2	-0.5	-0.2	-0.5	0.1	-0.7	-0.4	0.9
m=12-n=900	-0.3	0.5	-0.7	-0.1	-0.3	0.0	-0.6	-0.1	0.7

Table 5.2.3 (Continued)

	Φ1= - 0.1 - Φ2= - 0.1								
	AR(1)	ARMA(1,1)		AR(2)		ARMA(2,2)			
m=3-n=120	-0.3	0.2	-0.4	-0.1	-0.3	-0.1	-0.6	0.0	0.8
m=3-n=300	-0.2	0.2	-0.3	-0.1	-0.2	-0.1	-0.7	0.1	0.8
m=3-n=900	-0.2	0.3	-0.3	-0.1	-0.1	0.0	-0.7	-0.1	0.8
m=6-n=120	-0.4	0.1	-0.3	-0.1	-0.5	0.4	-0.7	-0.8	0.9
m=6-n=300	-0.3	0.2	-0.3	0.0	-0.1	0.0	-0.7	0.0	0.8
m=6-n=900	-0.2	0.2	-0.3	0.0	-0.1	0.0	-0.8	-0.1	0.8
m=12-n=120	-0.8	-1.0	1.0	-0.4	-0.9	[redacted]			
m=12-n=300	-0.3	0.1	-0.3	-0.1	-0.4	0.1	-0.7	-0.3	0.9
m=12-n=900	-0.2	0.2	-0.3	0.0	-0.1	0.0	-0.7	0.0	0.8
	Φ1= - 0.3 - Φ2= - 0.5								
	AR(1)	ARMA(1,1)		AR(2)		ARMA(2,2)			
m=3-n=120	-0.4	0.4	-0.8	-0.3	-0.4	0.0	-0.7	-0.2	0.8
m=3-n=300	-0.3	0.5	-0.8	-0.2	-0.3	0.2	-0.7	-0.3	0.7
m=3-n=900	-0.2	0.5	-0.7	-0.2	-0.2	0.5	-0.6	-0.6	0.6
m=6-n=120	-0.5	0.1	-0.3	-0.3	-0.5	0.2	-0.7	-0.7	0.9
m=6-n=300	-0.4	0.4	-0.7	-0.2	-0.3	0.0	-0.6	-0.1	0.8
m=6-n=900	-0.2	0.2	-0.4	-0.2	-0.2	0.1	-0.7	-0.2	0.7
m=12-n=120	-0.8	-1.0	1.0	-0.5	-0.9	1.4	-1.0	-1.5	0.9
m=12-n=300	-0.5	0.2	-0.4	-0.2	-0.4	0.0	-0.7	-0.3	0.9
m=12-n=900	-0.3	0.3	-0.4	-0.1	-0.2	-0.1	-0.7	0.0	0.8
	Φ1= - 0.1 - Φ2= 0.8								
	AR(1)	ARMA(1,1)		AR(2)		ARMA(2,2)			
m=3-n=120	0.5	-0.4	0.4	0.0	0.5	0.0	-0.5	0.3	0.9
m=3-n=300	0.5	-0.7	0.7	0.1	0.4	-0.2	0.2	0.6	0.4
m=3-n=900	[redacted]			0.1	0.5	-0.3	0.4	0.6	0.2
m=6-n=120	0.6	-0.1	0.2	0.5	-0.6	0.9	-0.8	-1.0	0.9
m=6-n=300	0.5	0.0	0.6	0.6	-0.4	0.5	-0.7	-0.2	0.8
m=6-n=900	0.5	0.3	0.3	0.6	-0.2	0.4	-0.4	0.0	0.3
m=12-n=120	-0.6	-1.0	1.0	-0.1	-0.9	[redacted]			
m=12-n=300	0.5	-0.1	0.1	0.3	-0.5	0.4	-0.8	-0.4	1.0
m=12-n=900	0.3	-0.3	0.6	0.4	-0.3	0.0	-0.7	0.2	0.9
	Φ1= 0.3 - Φ2= - 0.5								
	AR(1)	ARMA(1,1)		AR(2)		ARMA(2,2)			
m=3-n=120	-0.4	0.3	-0.6	-0.4	-0.4	-0.2	-0.6	0.0	0.8
m=3-n=300	-0.3	0.3	-0.6	-0.4	-0.2	0.0	-0.6	-0.2	0.7
m=3-n=900	-0.3	0.0	-0.2	-0.3	-0.1	-0.3	-0.5	-0.1	0.5
m=6-n=120	-0.5	0.0	-0.1	-0.1	-0.5	0.1	-0.7	-0.5	0.9
m=6-n=300	-0.3	0.3	-0.5	-0.1	-0.2	0.0	-0.6	-0.1	0.8
m=6-n=900	-0.2	0.3	-0.5	-0.1	-0.2	-0.1	-0.7	0.0	0.8
m=12-n=120	-0.8	-1.0	1.0	-0.4	-0.9	[redacted]			
m=12-n=300	-0.4	0.2	-0.4	-0.1	-0.4	0.1	-0.8	-0.3	1.0
m=12-n=900	-0.3	0.2	-0.4	-0.1	-0.1	0.0	-0.7	0.0	0.8

5.3 Simulation Results for MA(1) Model

As mentioned, the temporally aggregated model for MA(1) model is an MA(1) model for m =3, m=6 and m=12. The simulation studies also take into consideration MA(2) model and white noise for the aggregated series.

Table 5.3.1 Frequencies of Best Fitted Aggregate Models for MA(1)

	θ= -0.9			θ= -0.5			θ= -0.1		
	MA(1)	**MA(2)**	**WN**	**MA(1)**	**MA(2)**	**WN**	**MA(1)**	**MA(2)**	**WN**
m=3-n=120	**0.909**	0.089	0.002	**0.502**	0.146	0.352	0.080	0.174	**0.746**
m=3-n=300	**0.891**	0.107	0.002	**0.840**	0.062	0.098	0.088	0.056	**0.856**
m=3-n=900	**0.949**	0.051	0.000	**0.944**	0.056	0.000	0.109	0.067	**0.824**
m=6-n=120	**0.725**	0.194	0.081	0.330	0.224	**0.446**	0.225	0.254	**0.521**
m=6-n=300	**0.914**	0.084	0.002	0.321	0.152	**0.527**	0.089	0.106	**0.805**
m=6-n=900	**0.927**	0.073	0.000	**0.699**	0.069	0.232	0.076	0.060	**0.864**
m=12-n=120	0.246	0.000	**0.754**	0.133	0.000	**0.867**	0.117	0.000	**0.883**
m=12-n=300	**0.787**	0.155	0.058	0.209	0.246	**0.545**	0.132	0.272	**0.596**
m=12-n=900	**0.901**	0.099	0.000	0.210	0.084	**0.706**	0.057	0.092	**0.851**
	θ= 0.1			**θ= 0.5**			**θ= 0.9**		
	MA(1)	**MA(2)**	**WN**	**MA(1)**	**MA(2)**	**WN**	**MA(1)**	**MA(2)**	**WN**
m=3-n=120	0.080	0.180	**0.740**	0.103	0.143	**0.754**	0.090	0.146	**0.764**
m=3-n=300	0.054	0.067	**0.879**	0.113	0.073	**0.814**	0.153	0.061	**0.786**
m=3-n=900	0.094	0.067	**0.839**	0.296	0.053	**0.651**	0.373	0.058	**0.569**
m=6-n=120	0.206	0.253	**0.541**	0.205	0.268	**0.527**	0.192	0.294	**0.514**
m=6-n=300	0.078	0.116	**0.806**	0.085	0.110	**0.805**	0.080	0.115	**0.805**
m=6-n=900	0.064	0.052	**0.884**	0.073	0.055	**0.872**	0.071	0.067	**0.862**
m=12-n=120	0.127	0.000	**0.873**	0.099	0.000	**0.901**	0.102	0.000	**0.898**
m=12-n=300	0.147	0.252	**0.601**	0.134	0.242	**0.624**	0.156	0.278	**0.566**
m=12-n=900	0.060	0.091	**0.849**	0.063	0.073	**0.864**	0.074	0.082	**0.844**

The results of Table 5.3.1 show that generally frequently selected best fitted aggregate model is white noise. White noise is frequently selected best fitted aggregate model when θ is equal to -0.1, 0.1, 0.5 and 0.9 for m=3, m=6 and m=12.

Again in this section white noise is selected if all of the parameter values belong to MA(1) and MA(2) models are insignificant. MA(1) is frequently selected best fitted aggregate model when θ is equal to -0.9 for m=3 and m=6. Also, MA(1) is frequently selected best fitted aggregate model when θ is equal to -0.5 and m=3. It is clearly seen that MA(2) model is not frequently selected best fitted aggregate model for all conditions. In summary, empirical studies show that aggregate model for MA(1) is white noise for positive basic model parameter and the frequency of aggregate white noise model increases as the negative basic model parameter increases.

Table 5.3.2 Mean Square Forecast Errors of Best Fitted Aggregate Models for MA(1)

	θ= - 0.9				θ= - 0.5			
	MA(1)	**MA(2)**	**WN**	**BASIC**	**MA(1)**	**MA(2)**	**WN**	**BASIC**
m=3-n=120	1.790	1.794	2.102	**1.749**	1.741	1.839	1.759	**1.216**
m=3-n=300	1.660	**1.582**	3.964	1.793	1.762	1.533	1.843	**1.255**
m=3-n=900	1.665	1.714		1.756	1.755	1.650		**1.233**
m=6-n=120	**1.775**	1.932	2.063	1.785	2.905	3.104	2.720	**1.242**
m=6-n=300	1.732	1.810	**1.031**	1.734	2.555	2.659	2.507	**1.213**
m=6-n=900	**1.725**	1.877		1.752	2.477	2.114	2.689	**1.236**
m=12-n=120	2.019		2.131	**1.765**	4.119		4.553	**1.229**
m=12-n=300	1.776	1.827	1.890	**1.733**	4.580	4.499	4.145	**1.254**
m=12-n=900	1.836	1.808		**1.737**	4.273	4.110	4.165	**1.252**
	θ= - 0.1				**θ= 0.1**			
	MA(1)	**MA(2)**	**WN**	**BASIC**	**MA(1)**	**MA(2)**	**WN**	**BASIC**
m=3-n=120	2.453	2.915	2.648	**0.996**	4.025	4.494	3.500	**1.035**
m=3-n=300	2.568	2.637	2.563	**1.002**	4.027	3.839	3.434	**1.028**
m=3-n=900	2.784	2.833	2.611	**0.990**	3.387	3.505	3.317	**1.011**
m=6-n=120	5.943	6.548	5.458	**1.013**	8.694	8.028	7.673	**1.011**
m=6-n=300	6.135	5.421	5.242	**0.999**	7.301	7.292	7.326	**1.018**
m=6-n=900	5.729	6.217	5.217	**1.013**	6.442	7.356	7.093	**1.003**
m=12-n=120	11.492		12.026	**1.021**	16.545		17.256	**1.022**
m=12-n=300	10.569	11.403	10.414	**1.001**	16.300	16.660	14.435	**0.986**
m=12-n=900	10.669	9.876	10.352	**1.028**	14.676	16.431	14.302	**0.993**
	θ= 0.5				**θ= 0.9**			
	MA(1)	**MA(2)**	**WN**	**BASIC**	**MA(1)**	**MA(2)**	**WN**	**BASIC**
m=3-n=120	5.053	7.083	5.949	**1.262**	9.310	9.450	9.128	**1.785**
m=3-n=300	6.020	5.604	5.868	**1.255**	9.268	10.983	8.940	**1.773**
m=3-n=900	5.895	4.901	5.688	**1.238**	8.943	9.138	9.160	**1.738**
m=6-n=120	14.728	15.322	13.065	**1.249**	22.230	24.138	21.939	**1.794**
m=6-n=300	14.304	13.489	12.777	**1.220**	20.506	23.599	20.748	**1.804**
m=6-n=900	11.662	13.829	12.082	**1.199**	19.078	18.844	19.853	**1.753**
m=12-n=120	29.921		30.641	**1.265**	52.315		49.515	**1.774**
m=12-n=300	31.222	31.925	26.908	**1.244**	52.774	49.579	43.431	**1.744**
m=12-n=900	21.880	28.729	26.612	**1.254**	44.255	40.964	42.359	**1.763**

The simulation results which take place at Table 5.3.2 reveals that mean square forecast errors of basic model are less than mean square forecast errors of best fitted aggregated models. There is a problem when θ = -0.9 which similar to the problem that exists in section 5.1. Some mean square forecast errors of the best aggregated models are less than the mean square forecast errors of basic series. Again, this problem might arise from the parameter value which is very close to negative unit root. Furthermore, as expected, mean square forecast errors increase, when the order of aggregation increases.

Table 5.3.3 Estimated Parameter Values of Best Fitted Aggregate Models for MA(1)

	θ= -0.9			θ= -0.5			θ= -0.1		
	MA(1)	**MA(2)**		**MA(1)**	**MA(2)**		**MA(1)**	**MA(2)**	
m=3-n=120	-0.9	-1.0	0.1	-0.5	-0.4	-0.3	-0.3	-0.2	-0.2
m=3-n=300	-0.9	-1.0	0.1	-0.4	-0.3	0.0	-0.2	0.0	-0.1
m=3-n=900	-0.8	-0.8	0.0	-0.3	-0.3	0.0	-0.1	0.0	0.0
m=6-n=120	-1.0	-1.3	0.5	-0.9	-0.2	-0.3	-0.7	-0.1	-0.4
m=6-n=300	-0.9	-0.8	-0.1	-0.4	-0.3	-0.2	-0.2	-0.1	-0.2
m=6-n=900	-0.8	-0.8	-0.1	-0.3	-0.2	0.0	-0.1	0.0	0.0
m=12-n=120	-1.0			-1.0			-1.0		
m=12-n=300	-0.9	-0.8	0.1	-0.7	-0.2	-0.3	-0.4	-0.1	-0.2
m=12-n=900	-0.8	-0.7	-0.1	-0.3	-0.1	-0.1	-0.2	0.0	-0.1
	θ= 0.1			**θ= 0.5**			**θ= 0.9**		
	MA(1)	**MA(2)**		**MA(1)**	**MA(2)**		**MA(1)**	**MA(2)**	
m=3-n=120	0.0	-0.1	-0.2	0.3	-0.1	-0.2	0.3	-0.1	-0.2
m=3-n=300	0.0	0.0	-0.1	0.2	0.0	0.0	0.2	0.1	-0.1
m=3-n=900	0.1	0.0	0.0	0.2	0.1	0.0	0.2	0.1	0.0
m=6-n=120	-0.6	-0.1	-0.3	-0.5	0.0	-0.2	-0.5	0.0	-0.3
m=6-n=300	0.0	-0.1	-0.2	0.1	0.0	-0.1	0.1	-0.1	-0.2
m=6-n=900	0.0	0.0	0.0	0.1	0.0	0.0	0.1	0.0	-0.1
m=12-n=120	-1.0			-1.0			-1.0		
m=12-n=300	-0.4	-0.1	-0.3	-0.3	-0.1	-0.3	-0.3	-0.1	-0.3
m=12-n=900	-0.1	0.0	-0.1	0.0	0.0	-0.1	0.1	0.0	-0.1

As seen from Table 5.3.3 estimated parameters of best fitted aggregated models change depend on the sample size and the order of aggregation. For m=12 and n=120 case, the estimated MA(1) parameter value of aggregate series is very close to -1 which can be thought as a sign of noninvertibility. Again, it is clearly seen that the results of small sample sizes and high order of aggregation are not reliable. When MA(1) is the best fitted aggregated model, the estimated parameter is not the same as parameter value of basic series. So, temporal aggregation plays a role in parameter estimation.

5.4 Simulation Results for MA(2) Model

The theoretical aggregate model of MA(2) process is an MA(1) model. Also, in simulation studies MA(2) model and white noise are taken into account similar to section 5.3.

Table 5.4.1 Frequencies of Best Fitted Aggregate Models for MA(2)

	θ1= 0.1 - θ2= 0.8			θ1= 0.1 - θ2= 0.1			θ1= 0.3 - θ2= 0.5			θ1= 0.3 - θ2= - 0.5		
	MA(1)	MA(2)	WN	MA(1)	MA(2)	WN	MA(1)	MA(2)	WN	MA(1)	MA(2)	WN
m=3-n=120	0.278	0.121	**0.601**	0.098	0.146	**0.756**	0.238	0.138	**0.624**	0.360	0.159	**0.481**
m=3-n=300	**0.633**	0.075	0.292	0.125	0.070	**0.805**	**0.524**	0.069	0.407	**0.671**	0.068	0.261
m=3-n=900	**0.938**	0.058	0.004	0.255	0.061	**0.684**	**0.915**	0.057	0.028	**0.938**	0.057	0.005
m=6-n=120	0.183	0.284	**0.533**	0.196	0.290	**0.514**	0.200	0.274	**0.526**	0.280	0.255	**0.465**
m=6-n=300	0.116	0.112	**0.772**	0.076	0.128	**0.796**	0.097	0.104	**0.799**	0.219	0.130	**0.651**
m=6-n=900	0.177	0.059	**0.764**	0.065	0.052	**0.883**	0.158	0.058	**0.784**	**0.500**	0.060	0.440
m=12-n=120	0.110	0.000	**0.890**	0.110	0.000	**0.890**	0.088	0.000	**0.912**	0.110	0.000	**0.890**
m=12-n=300	0.141	0.282	**0.577**	0.138	0.261	**0.601**	0.121	0.271	**0.608**	0.187	0.266	**0.547**
m=12-n=900	0.060	0.095	**0.845**	0.074	0.094	**0.832**	0.069	0.073	**0.858**	0.161	0.078	**0.761**
	θ1= - 0.1 - θ2= - 0.8			θ1= -0.1 - θ2= - 0.1			θ1= - 0.3 - θ2= - 0.5			θ1= - 0.1 - θ2= 0.8		
	MA(1)	MA(2)	WN	MA(1)	MA(2)	WN	MA(1)	MA(2)	WN	MA(1)	MA(2)	WN
m=3-n=120	**0.904**	0.096	0.000	0.148	0.176	**0.676**	**0.900**	0.093	0.007	0.294	0.138	**0.568**
m=3-n=300	**0.916**	0.084	0.000	0.246	0.061	**0.693**	**0.924**	0.076	0.000	**0.651**	0.068	0.281
m=3-n=900	**0.943**	0.057	0.000	**0.528**	0.053	0.419	**0.939**	0.061	0.000	**0.932**	0.063	0.005
m=6-n=120	**0.737**	0.187	0.076	0.255	0.229	**0.516**	**0.711**	0.165	0.124	0.175	0.264	**0.561**
m=6-n=300	**0.916**	0.082	0.002	0.111	0.133	**0.756**	**0.871**	0.118	0.011	0.103	0.120	**0.777**
m=6-n=900	**0.920**	0.080	0.000	0.132	0.060	**0.808**	**0.975**	0.025	0.000	0.196	0.066	**0.738**
m=12-n=120	0.232	0.000	**0.768**	0.137	0.000	**0.863**	0.227	0.000	**0.773**	0.113	0.000	**0.887**
m=12-n=300	**0.818**	0.140	0.042	0.154	0.241	**0.605**	**0.682**	0.179	0.139	0.131	0.261	**0.608**
m=12-n=900	**0.908**	0.092	0.000	0.090	0.064	**0.846**	**0.908**	0.087	0.005	0.081	0.090	**0.829**

The results of Table 5.4.1 indicate that the aggregate model of MA(2) process is an MA(1) model or white noise in empirical studies. Most of the frequently selected best fitted aggregate models are white noise. Especially when θ_1=0.1 - θ_2=0.1, all of the frequently selected best fitted aggregate models are white noise. Moreover, the frequently selected best fitted aggregate model is white noise for m=12 and n=120 case independent of basic model parameter values. In this simulation study, white noise is selected as best fitted aggregate model when all of the parameters belong to MA(1) and MA(2) models are insignificant. MA(1) is frequently selected best fitted aggregate model for smaller order of aggregation but when θ_1= - 0.1 - θ_2= -0.8 and θ_1= - 0.3 - θ_2= -0.5, MA(1) is frequently selected best fitted aggregate model for all conditions except m=12 and n=120 case.

When the results of Table 5.4.2 are analyzed, again it is understood that the mean square forecast errors of basic model are less than the mean square forecast errors of best fitted aggregate models. Furthermore, as expected the mean square forecast errors of best fitted aggregate models increase when the order of aggregation increases. The simulation results of Table 5.4.2 verify our claim that aggregate models have worse forecast values than basic models.

As seen from Table 5.4.3, estimated parameters for best fitted aggregated models change when the sample size and order of aggregation change. Specifically, for m=12 and n=120 case the estimated parameter of best fitted aggregate model MA(1) is very close to -1 and this satisfies aggregate model might have invertibility problems. Also, the basic model parameters and best fitted aggregate model MA(2) estimated parameters are not consistent most of the conditions. So, again it can be said that temporal aggregation affects the parameter estimation.

Table 5.4.2 Mean Square Forecast Errors of Best Fitted Aggregate Models for MA(2)

	θ1= 0.1 - θ2= 0.8				θ1= 0.1 - θ2= 0.1			
	MA(1)	**MA(2)**	**WN**	**BASIC**	**MA(1)**	**MA(2)**	**WN**	**BASIC**
m=3-n=120	7.615	8.054	7.677	**1.594**	4.032	4.325	3.856	**1.046**
m=3-n=300	7.565	6.828	7.406	**1.583**	3.729	3.701	3.803	**1.025**
m=3-n=900	7.268	7.326	8.116	**1.559**	3.672	3.894	3.778	**1.022**
m=6-n=120	22.073	22.778	19.958	**1.618**	9.242	10.246	8.797	**1.023**
m=6-n=300	21.067	18.362	18.851	**1.561**	9.478	9.619	7.944	**1.029**
m=6-n=900	19.456	18.573	18.758	**1.607**	8.400	7.874	8.197	**1.026**
m=12-n=120	52.322		47.196	**1.595**	22.240		20.335	**1.049**
m=12-n=300	48.017	50.878	40.109	**1.567**	20.957	18.973	16.959	**1.014**
m=12-n=900	42.117	46.156	41.894	**1.577**	19.109	18.835	16.623	**1.041**
	θ1= 0.3 - θ2= 0.5				**θ1= - 0.1 - θ2= - 0.8**			
	MA(1)	**MA(2)**	**WN**	**BASIC**	**MA(1)**	**MA(2)**	**WN**	**BASIC**
m=3-n=120	7.697	7.384	6.709	**1.316**	2.981	3.159		**1.597**
m=3-n=300	6.728	7.047	6.991	**1.327**	3.023	3.122		**1.572**
m=3-n=900	6.692	7.012	8.896	**1.304**	2.880	3.431		**1.580**
m=6-n=120	18.251	20.909	17.307	**1.338**	2.949	3.391	3.412	**1.530**
m=6-n=300	16.655	20.891	17.039	**1.302**	2.984	3.042	4.425	**1.564**
m=6-n=900	15.205	16.656	16.773	**1.310**	3.031	3.018		**1.571**
m=12-n=120	45.485		45.213	**1.346**	3.330		3.570	**1.569**
m=12-n=300	41.981	43.409	36.066	**1.286**	3.118	3.420	3.502	**1.610**
m=12-n=900	33.222	35.602	37.556	**1.330**	3.088	3.191		**1.533**
	θ1= - 0.1 - θ2= - 0.1				**θ1= - 0.3 - θ2= - 0.5**			
	MA(1)	**MA(2)**	**WN**	**BASIC**	**MA(1)**	**MA(2)**	**WN**	**BASIC**
m=3-n=120	2.686	2.934	2.500	**1.018**	2.224	2.233	2.240	**1.304**
m=3-n=300	2.436	2.713	2.535	**1.024**	2.276	2.188		**1.330**
m=3-n=900	2.596	2.612	2.531	**1.036**	2.288	2.309		**1.316**
m=6-n=120	5.467	5.598	4.716	**1.039**	2.514	2.468	2.689	**1.308**
m=6-n=300	5.256	4.673	4.505	**1.014**	2.527	2.748	1.888	**1.325**
m=6-n=900	4.208	5.670	4.337	**1.021**	2.414	2.305		**1.290**
m=12-n=120	10.094		9.929	**1.035**	2.830		3.241	**1.305**
m=12-n=300	9.441	10.037	8.499	**1.012**	2.632	3.216	2.745	**1.313**
m=12-n=900	8.624	8.050	8.558	**1.031**	2.712	2.966	2.334	**1.317**
	θ1= 0.3 - θ2= - 0.5				**θ1= - 0.1 - θ2= 0.8**			
	MA(1)	**MA(2)**	**WN**	**BASIC**	**MA(1)**	**MA(2)**	**WN**	**BASIC**
m=3-n=120	3.715	4.113	3.454	**1.305**	5.486	7.045	5.963	**1.559**
m=3-n=300	3.667	3.934	3.611	**1.306**	5.892	5.425	5.888	**1.573**
m=3-n=900	3.633	3.715	2.668	**1.280**	5.855	5.149	3.525	**1.547**
m=6-n=120	6.547	6.858	5.892	**1.333**	16.381	18.634	15.609	**1.618**
m=6-n=300	6.026	6.085	5.963	**1.287**	15.540	16.419	15.776	**1.610**
m=6-n=900	5.363	5.223	5.309	**1.286**	15.136	14.124	15.023	**1.558**
m=12-n=120	11.968		10.890	**1.311**	34.358		38.701	**1.574**
m=12-n=300	9.641	11.308	9.701	**1.315**	36.851	36.576	33.681	**1.564**
m=12-n=900	10.339	9.574	9.745	**1.316**	34.089	36.236	32.795	**1.561**

Table 5.4.3 Estimated Parameter Values of Best Fitted Aggregate Models for MA(2)

	θ1= 0.1 - θ2= 0.8			θ1= 0.1 - θ2= 0.1			θ1= 0.3 - θ2= 0.5			θ1= 0.3 - θ2= - 0.5		
	MA(1)	MA(2)		MA(1)	MA(2)		MA(1)	MA(2)		MA(1)	MA(2)	
m=3-n=120	0.4	0.1	-0.2	0.2	-0.1	-0.2	0.4	0.0	-0.2	-0.5	-0.4	-0.2
m=3-n=300	0.3	0.2	-0.1	0.2	0.0	-0.1	0.3	0.2	-0.1	-0.3	-0.3	-0.1
m=3-n=900	0.3	0.3	0.0	0.1	0.0	0.0	0.2	0.2	0.0	-0.3	-0.3	0.0
m=6-n=120	-0.3	0.0	-0.3	-0.5	-0.1	-0.3	-0.4	-0.1	-0.3	-0.8	-0.1	-0.2
m=6-n=300	0.3	0.0	-0.2	0.1	-0.1	-0.2	0.2	0.0	-0.1	-0.4	-0.2	-0.2
m=6-n=900	0.2	0.0	-0.1	0.1	0.0	-0.1	0.2	0.0	-0.1	-0.2	-0.2	0.0
m=12-n=120	-1.0			-1.0			-0.9			-1.0		
m=12-n=300	-0.3	-0.1	-0.2	-0.4	-0.1	-0.3	-0.3	-0.1	-0.3	-0.6	-0.2	-0.3
m=12-n=900	0.1	0.0	-0.1	0.0	0.0	0.0	0.1	0.0	-0.1	-0.3	-0.1	-0.1
	θ1= - 0.1 - θ2= - 0.8			θ1= -0.1 - θ2= - 0.1			θ1= - 0.3 - θ2= - 0.5			θ1= - 0.1 - θ2= 0.8		
	MA(1)	MA(2)		MA(1)	MA(2)		MA(1)	MA(2)		MA(1)	MA(2)	
m=3-n=120	-1.0	-1.1	0.2	-0.5	-0.3	-0.2	-0.9	-0.8	-0.1	0.4	0.1	-0.3
m=3-n=300	-0.9	-0.9	-0.1	-0.3	-0.1	-0.1	-0.8	-0.7	-0.1	0.3	0.2	0.0
m=3-n=900	-0.9	-0.9	0.0	-0.2	-0.1	0.0	-0.7	-0.7	0.0	0.3	0.3	0.0
m=6-n=120	-1.0	-1.3	0.6	-0.8	0.0	-0.2	-1.0	-0.9	0.2	-0.3	0.0	-0.4
m=6-n=300	-0.9	-0.9	-0.1	-0.3	-0.2	-0.2	-0.7	-0.7	-0.1	0.3	0.0	-0.2
m=6-n=900	-0.9	-0.8	0.0	-0.2	0.0	0.0	-0.7	-0.6	0.0	0.2	0.1	0.0
m=12-n=120	-1.0			-1.0			-1.0			-1.0		
m=12-n=300	-1.0	-1.1	0.3	-0.5	-0.1	-0.3	-0.9	-0.5	-0.1	-0.1	-0.1	-0.3
m=12-n=900	-0.8	-0.7	-0.1	-0.2	-0.1	-0.2	-0.6	-0.6	-0.1	0.1	0.0	-0.1

5.5 Simulation Results for ARMA(1,1)

As it is stated before, when the basic series follows an ARMA(1,1) process then the aggregated model is also an ARMA(1,1) model theoretically for m = 3, m = 6 and m = 12. In the simulation studies, for aggregate series AR(1), AR(2), ARMA(2,2) models and white noise are also considered. Again, in this section white noise is selected as best fitted aggregate model if all of the parameters belong to AR(1), AR(2) and ARMA(2,2) are insignificant.

Table 5.5.1 Frequencies of Best Fitted Aggregate Models for ARMA(1,1)

	Φ= 0.1 - θ= 0.8					Φ = 0.1 - θ= 0.1				
	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN
m=3-n=120	0.071	0.265	0.067	**0.349**	0.248	0.049	0.319	0.055	**0.356**	0.221
m=3-n=300	0.145	0.210	0.053	**0.394**	0.198	0.053	0.263	0.038	**0.461**	0.185
m=3-n=900	**0.418**	0.126	0.043	0.297	0.116	0.120	0.194	0.045	**0.463**	0.178
m=6-n=120	0.059	0.164	0.106	0.206	**0.465**	0.070	0.171	0.129	0.175	**0.455**
m=6-n=300	0.048	0.297	0.048	**0.408**	0.199	0.037	0.304	0.052	**0.424**	0.183
m=6-n=900	0.057	0.238	0.054	**0.459**	0.192	0.045	0.278	0.033	**0.471**	0.173
m=12-n=120	0.100	0.003	**0.475**	0.001	0.421	0.096	0.001	**0.489**	0.001	0.413
m=12-n=300	0.058	0.278	0.088	0.245	**0.331**	0.048	0.272	0.085	0.256	**0.339**
m=12-n=900	0.038	0.296	0.041	**0.431**	0.194	0.033	0.287	0.054	**0.429**	0.197
	Φ= 0.3 - θ= 0.5					Φ= - 0.1 - θ= - 0.8				
	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN
m=3-n=120	0.124	0.235	0.073	**0.338**	0.230	**0.418**	0.091	0.415	0.060	0.016
m=3-n=300	**0.345**	0.129	0.085	0.296	0.145	0.109	0.104	**0.735**	0.052	0.000
m=3-n=900	**0.673**	0.063	0.085	0.149	0.030	0.001	0.050	**0.904**	0.045	0.000
m=6-n=120	0.053	0.147	0.102	0.216	**0.482**	**0.354**	0.046	0.231	0.101	0.268
m=6-n=300	0.045	0.320	0.040	**0.402**	0.193	**0.400**	0.139	0.383	0.069	0.009
m=6-n=900	0.115	0.205	0.055	**0.462**	0.163	0.091	0.078	**0.775**	0.056	0.000
m=12-n=120	0.073	0.001	**0.487**	0.003	0.436	0.141	0.002	**0.535**	0.003	0.319
m=12-n=300	0.051	0.277	0.120	0.224	**0.328**	**0.393**	0.101	0.197	0.118	0.191
m=12-n=900	0.049	0.285	0.047	**0.443**	0.176	**0.439**	0.139	0.330	0.086	0.006
	Φ= - 0.1 - θ= - 0.1					Φ= - 0.3 - θ= - 0.5				
	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN
m=3-n=120	0.075	**0.363**	0.042	0.343	0.177	**0.459**	0.197	0.147	0.136	0.061
m=3-n=300	0.096	0.272	0.040	**0.437**	0.155	**0.505**	0.102	0.299	0.092	0.002
m=3-n=900	0.168	0.220	0.038	**0.438**	0.136	0.189	0.039	**0.733**	0.039	0.000
m=6-n=120	0.069	0.196	0.081	0.212	**0.442**	0.209	0.102	0.151	0.153	**0.385**
m=6-n=300	0.048	0.343	0.046	**0.381**	0.182	**0.354**	0.272	0.077	0.193	0.104
m=6-n=900	0.060	0.262	0.040	**0.471**	0.167	**0.603**	0.090	0.181	0.115	0.011
m=12-n=120	0.097	0.006	**0.449**	0.001	0.447	0.127	0.002	**0.499**	0.000	0.372
m=12-n=300	0.059	0.285	0.086	0.226	**0.344**	0.152	0.252	0.109	0.177	**0.310**
m=12-n=900	0.034	0.307	0.034	**0.453**	0.172	**0.300**	0.212	0.076	0.280	0.132
	Φ= - 0.1 - θ= 0.8					Φ= 0.3 - θ= - 0.5				
	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN
m=3-n=120	0.041	0.333	0.058	**0.353**	0.215	0.137	**0.342**	0.054	0.313	0.154
m=3-n=300	0.062	0.245	0.041	**0.450**	0.202	0.262	0.264	0.053	**0.290**	0.131
m=3-n=900	0.171	0.185	0.050	**0.435**	0.159	**0.568**	0.114	0.073	0.190	0.055
m=6-n=120	0.060	0.194	0.083	0.202	**0.461**	0.100	0.168	0.114	0.164	**0.454**
m=6-n=300	0.037	0.297	0.056	**0.412**	0.198	0.099	0.336	0.044	**0.346**	0.175
m=6-n=900	0.053	0.248	0.035	**0.481**	0.183	0.166	0.236	0.040	**0.405**	0.153
m=12-n=120	0.085	0.001	**0.501**	0.000	0.413	0.097	0.003	**0.471**	0.001	0.428
m=12-n=300	0.055	0.286	0.082	0.235	**0.342**	0.061	0.290	0.083	0.257	**0.309**
m=12-n=900	0.037	0.298	0.055	**0.433**	0.177	0.054	0.283	0.040	**0.433**	0.190

Table 5.5.2 Mean Square Forecast Errors of Best Fitted Aggregate Models for ARMA(1,1)

	Φ= 0.1 - θ= 0.8						Φ= 0.1 - θ= 0.1					
	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	BASIC	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	BASIC
m=3-n=120	10.153	11.012	11.620	11.543	10.165	**1.849**	4.277	4.180	4.477	4.869	4.033	**1.061**
m=3-n=300	10.420	9.541	10.240	10.486	10.406	**1.807**	4.042	4.020	4.366	4.568	4.163	**1.059**
m=3-n=900	9.644	9.924	9.165	9.831	9.477	**1.767**	4.056	3.901	4.085	3.987	4.031	**1.054**
m=6-n=120	29.096	24.039	30.927	30.885	23.644	**1.787**	9.734	8.878	11.452	12.362	9.005	**1.056**
m=6-n=300	22.461	21.555	24.369	25.128	22.133	**1.818**	8.271	8.974	9.212	9.741	7.906	**1.047**
m=6-n=900	24.651	20.766	23.636	21.915	20.834	**1.767**	9.184	8.736	7.869	9.102	9.098	**1.048**
m=12-n=120	69.029	45.818	102.120	72.870	55.543	**1.807**	27.157	43.701	36.589	86.511	20.157	**1.049**
m=12-n=300	57.600	50.063	58.033	62.963	46.081	**1.789**	19.306	19.555	22.654	22.031	19.246	**1.042**
m=12-n=900	43.585	48.783	53.395	50.061	46.089	**1.748**	16.249	18.581	17.046	18.964	17.623	**1.025**
	Φ= 0.3 - θ= 0.5						Φ= - 0.1 - θ= - 0.8					
	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	BASIC	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	BASIC
m=3-n=120	11.134	11.077	12.549	11.905	11.905	**1.750**	1.738	1.756	1.760	1.730	**1.428**	1.767
m=3-n=300	9.972	10.247	9.251	10.249	9.868	**1.675**	1.846	1.712	1.646	**1.601**	[redacted]	1.762
m=3-n=900	9.549	9.755	8.473	10.315	9.9013	**1.633**	2.480	1.617	1.588	**1.538**	[redacted]	1.752
m=6-n=120	29.555	27.211	31.901	33.475	25.783	**1.689**	2.034	2.060	2.126	2.280	2.011	**1.720**
m=6-n=300	23.843	24.229	28.958	27.734	22.688	**1.665**	1.887	1.893	1.808	1.894	2.718	**1.728**
m=6-n=900	24.545	22.809	25.666	25.138	23.358	**1.639**	1.934	1.790	**1.703**	1.731	[redacted]	1.740
m=12-n=120	75.140	96.563	100.709	183.701	62.596	**1.695**	2.813	2.265	3.870	3.556	2.268	**1.787**
m=12-n=300	53.334	56.118	60.899	66.045	50.741	**1.671**	2.014	1.939	2.291	2.632	1.987	**1.792**
m=12-n=900	53.271	51.966	53.501	51.998	51.596	**1.663**	1.921	2.089	2.043	2.093	1.883	**1.799**
	Φ= - 0.1 - θ= - 0.1						Φ= - 0.3 - θ= - 0.5					
	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	BASIC	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	BASIC
m=3-n=120	2.521	2.607	2.715	2.818	2.029	**1.038**	1.641	1.770	1.808	1.895	1.674	**1.638**
m=3-n=300	2.391	2.357	2.171	2.503	2.515	**1.035**	1.655	1.741	1.699	1.545	**1.206**	1.659
m=3-n=900	2.345	2.494	2.359	2.344	2.345	**1.036**	1.666	1.662	**1.617**	1.800	[redacted]	1.639
m=6-n=120	4.925	4.756	5.823	6.018	5.067	**1.056**	2.374	2.321	2.772	3.248	2.064	**1.654**
m=6-n=300	5.200	4.608	3.880	4.843	4.383	**1.012**	2.067	2.175	2.223	2.455	2.252	**1.673**
m=6-n=900	3.731	4.458	3.899	4.852	4.740	**1.035**	2.094	1.983	2.042	2.083	2.544	**1.694**
m=12-n=120	12.472	15.370	16.102	86.755	10.365	**1.052**	3.874	5.318	6.483	[redacted]	3.433	**1.613**
m=12-n=300	8.940	9.517	9.774	10.570	9.227	**1.034**	3.267	3.250	3.079	3.754	3.240	**1.631**
m=12-n=900	9.985	8.918	9.235	9.304	8.088	**1.037**	3.151	2.991	3.387	3.443	2.817	**1.609**
	Φ= - 0.1 - θ= 0.8						Φ= 0.3 - θ= - 0.5					
	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	BASIC	AR(1)	ARMA(1,1)	AR(2)	ARMA(2,2)	WN	BASIC
m=3-n=120	7.681	7.428	7.217	8.130	7.623	**1.482**	2.221	2.497	2.741	2.598	2.098	**1.054**
m=3-n=300	6.293	7.222	6.602	7.726	6.317	**1.451**	2.276	2.377	2.409	2.432	2.192	**1.022**
m=3-n=900	7.293	7.261	6.774	7.221	6.478	**1.471**	2.368	2.303	2.397	2.236	2.293	**1.056**
m=6-n=120	20.063	15.523	23.363	21.860	16.739	**1.493**	4.283	4.203	5.644	5.210	4.012	**1.068**
m=6-n=300	16.146	15.599	17.077	17.641	15.776	**1.494**	4.087	3.983	4.370	4.052	4.052	**1.036**
m=6-n=900	13.902	15.436	14.744	14.511	15.021	**1.429**	4.367	3.706	4.713	3.888	3.818	**1.059**
m=12-n=120	48.305	50.872	58.530	[redacted]	36.578	**1.529**	10.968	6.896	13.691	5.238	8.282	**1.069**
m=12-n=300	33.099	34.324	41.393	41.649	32.078	**1.478**	9.038	7.643	9.028	8.218	7.737	**1.047**
m=12-n=900	31.405	32.480	33.567	34.035	31.775	**1.461**	7.315	6.999	6.410	7.842	7.102	**1.039**

Table 5.5.3 Estimated Parameter Values of Best Fitted Aggregate Models for ARMA(1,1)

	Φ= 0.1 - θ= 0.8								
	AR(1)	ARMA(1,1)		AR(2)		ARMA(2,2)			
m=3-n=120	0.3	0.0	-0.1	0.0	-0.3	0.1	-0.7	-0.1	0.9
m=3-n=300	0.3	-0.1	0.1	0.1	-0.2	0.1	-0.7	0.0	0.8
m=3-n=900	0.2	-0.2	0.3	0.1	-0.1	0.0	-0.7	0.1	0.8
m=6-n=120	-0.2	0.1	-0.2	-0.1	-0.5	0.5	-0.7	-0.7	0.9
m=6-n=300	0.2	0.1	-0.2	0.0	-0.2	0.0	-0.7	0.0	0.8
m=6-n=900	0.1	0.0	0.0	0.0	-0.1	0.0	-0.8	0.1	0.8
m=12-n=120	-0.8	-1.0	1.0	-0.4	-0.9	-1.8	-1.0	-1.5	1.0
m=12-n=300	-0.1	0.1	-0.3	-0.1	-0.3	0.3	-0.8	-0.5	1.0
m=12-n=900	0.0	0.0	-0.1	0.1	-0.1	0.0	-0.7	0.1	0.8
	Φ= 0.1 - θ= 0.1								
	AR(1)	ARMA(1,1)		AR(2)		ARMA(2,2)			
m=3-n=120	0.2	0.1	-0.2	0.1	-0.3	0.1	-0.7	-0.1	0.9
m=3-n=300	0.2	0.0	0.0	0.0	-0.1	-0.1	-0.7	0.1	0.8
m=3-n=900	0.1	0.0	0.1	0.0	-0.1	0.1	-0.8	0.0	0.8
m=6-n=120	-0.2	0.0	-0.1	-0.1	-0.5	0.4	-0.7	-0.8	0.9
m=6-n=300	0.0	0.2	-0.3	0.0	-0.2	0.0	-0.7	0.0	0.8
m=6-n=900	0.1	0.1	-0.1	0.0	-0.1	0.1	-0.8	-0.1	0.9
m=12-n=120	-0.8	-1.0	1.0	-0.4	-0.9	-2.0	-1.0	-1.5	0.8
m=12-n=300	-0.2	0.1	-0.3	-0.1	-0.4	0.2	-0.8	-0.3	0.9
m=12-n=900	0.0	0.1	-0.1	0.0	-0.1	0.1	-0.7	0.0	0.8
	Φ= 0.3 - θ= 0.5								
	AR(1)	ARMA(1,1)		AR(2)		ARMA(2,2)			
m=3-n=120	0.4	0.0	0.0	0.2	-0.3	0.1	-0.7	0.0	0.9
m=3-n=300	0.3	-0.2	0.3	0.2	-0.2	0.0	-0.8	0.1	0.8
m=3-n=900	0.2	-0.5	0.6	0.2	-0.1	-0.1	-0.7	0.2	0.7
m=6-n=120	0.1	0.0	-0.1	0.0	-0.5	0.4	-0.8	-0.7	0.9
m=6-n=300	0.3	0.1	-0.1	0.1	-0.2	0.1	-0.7	0.0	0.8
m=6-n=900	0.2	-0.1	0.2	0.1	-0.1	0.0	-0.8	0.1	0.8
m=12-n=120	-0.7	-1.0	1.0	-0.4	-0.9	0.0	-0.3	-0.4	0.9
m=12-n=300	-0.1	0.1	-0.3	-0.1	-0.4	0.2	-0.7	-0.3	0.9
m=12-n=900	0.2	0.1	-0.1	0.0	-0.2	0.1	-0.7	-0.1	0.9
	Φ= - 0.1 - θ= - 0.8								
	AR(1)	ARMA(1,1)		AR(2)		ARMA(2,2)			
m=3-n=120	-0.5	0.3	-0.9	-0.6	-0.4	0.1	-0.3	-0.9	0.4
m=3-n=300	-0.5	0.2	-0.9	-0.6	-0.3	0.2	-0.1	-0.9	0.2
m=3-n=900	-0.5	0.0	-0.7	-0.6	-0.3	0.0	0.0	-0.6	-0.1
m=6-n=120	-0.6	-0.2	-0.5	-0.5	-0.6	0.1	-0.6	-1.0	0.7
m=6-n=300	-0.5	0.4	-1.0	-0.6	-0.4	0.0	-0.3	-0.6	0.4
m=6-n=900	-0.5	0.2	-0.8	-0.6	-0.3	0.2	-0.1	-0.8	0.2
m=12-n=120	-0.8	-1.0	1.0	-0.7	-0.9	-1.0	-1.0	-1.6	1.0
m=12-n=300	-0.5	0.1	-0.6	-0.5	-0.5	0.0	-0.6	-0.6	0.7
m=12-n=900	-0.4	0.4	-0.9	-0.5	-0.3	0.3	-0.4	-0.7	0.5

Table 5.5.3 (Continued)

	Φ= - 0.1 - θ= - 0.1								
	AR(1)	ARMA(1,1)		AR(2)		ARMA(2,2)			
m=3-n=120	-0.3	0.2	-0.4	-0.1	-0.3	0.0	-0.7	0.0	0.8
m=3-n=300	-0.2	0.1	-0.2	-0.1	-0.1	0.0	-0.7	-0.1	0.8
m=3-n=900	-0.1	0.2	-0.3	0.0	0.0	0.0	-0.8	0.0	0.8
m=6-n=120	-0.4	0.1	-0.3	-0.1	-0.5	0.3	-0.7	-0.7	0.9
m=6-n=300	-0.2	0.2	-0.3	0.0	-0.2	0.0	-0.7	0.0	0.9
m=6-n=900	-0.2	0.1	-0.2	0.0	-0.1	0.0	-0.8	0.0	0.8
m=12-n=120	-0.8	-1.0	1.0	-0.5	-0.9	-2.0	-1.0	0.0	-1.0
m=12-n=300	-0.4	0.1	-0.3	-0.1	-0.4	0.1	-0.7	-0.3	0.9
m=12-n=900	-0.2	0.2	-0.3	0.0	-0.2	0.0	-0.7	0.0	0.8
	Φ= - 0.3 - θ= - 0.5								
	AR(1)	ARMA(1,1)		AR(2)		ARMA(2,2)			
m=3-n=120	-0.5	0.4	-0.8	-0.5	-0.4	0.0	-0.5	-0.4	0.6
m=3-n=300	-0.4	0.4	-0.8	-0.5	-0.3	0.3	-0.3	-0.6	0.4
m=3-n=900	-0.4	0.2	-0.6	-0.4	-0.2	0.4	-0.1	-0.8	0.3
m=6-n=120	-0.6	-0.1	-0.2	-0.4	-0.6	0.1	-0.7	-0.6	0.9
m=6-n=300	-0.4	0.4	-0.8	-0.3	-0.4	0.0	-0.6	-0.2	0.7
m=6-n=900	-0.3	0.5	-0.8	-0.4	-0.2	0.1	-0.5	-0.3	0.6
m=12-n=120	-0.8	-1.0	1.0	-0.6	-0.9	[redacted]			
m=12-n=300	-0.5	0.1	-0.4	-0.3	-0.5	-0.1	-0.7	-0.1	0.9
m=12-n=900	-0.3	0.4	-0.6	-0.2	-0.3	-0.1	-0.7	-0.1	0.8
	Φ= - 0.1 - θ= 0.8								
	AR(1)	ARMA(1,1)		AR(2)		ARMA(2,2)			
m=3-n=120	0.2	0.1	-0.1	0.0	-0.2	0.0	-0.7	0.1	0.9
m=3-n=300	0.2	0.0	0.0	0.0	-0.1	0.0	-0.8	0.0	0.8
m=3-n=900	0.1	-0.1	0.1	0.0	0.0	0.0	-0.8	0.1	0.8
m=6-n=120	-0.2	-0.1	0.0	-0.1	-0.5	0.4	-0.8	-0.6	0.9
m=6-n=300	0.1	0.1	-0.2	0.0	-0.2	0.0	-0.7	0.0	0.9
m=6-n=900	0.1	0.1	-0.1	0.0	-0.1	0.0	-0.8	0.1	0.8
m=12-n=120	-0.8	-1.0	1.0	-0.4	-0.9	[redacted]			
m=12-n=300	-0.3	0.1	-0.3	0.0	-0.4	0.3	-0.7	-0.5	0.9
m=12-n=900	0.0	0.1	-0.2	0.0	-0.1	0.0	-0.7	0.0	0.8
	Φ= 0.3 - θ= - 0.5								
	AR(1)	ARMA(1,1)		AR(2)		ARMA(2,2)			
m=3-n=120	-0.4	0.3	-0.5	-0.2	-0.3	-0.1	-0.6	0.0	0.8
m=3-n=300	-0.3	0.4	-0.5	-0.1	-0.2	0.0	-0.7	-0.1	0.8
m=3-n=900	-0.2	0.3	-0.4	-0.2	-0.1	0.1	-0.7	-0.2	0.7
m=6-n=120	-0.5	0.0	-0.2	-0.2	-0.5	0.3	-0.8	-0.6	0.9
m=6-n=300	-0.3	0.2	-0.4	-0.1	-0.3	0.0	-0.6	-0.1	0.8
m=6-n=900	-0.2	0.3	-0.4	-0.1	-0.1	0.0	-0.7	-0.1	0.7
m=12-n=120	-0.8	-1.0	1.0	-0.5	-0.9	-1.1	-1.0	0.0	-1.0
m=12-n=300	-0.4	0.2	-0.4	-0.1	-0.4	0.1	-0.7	-0.2	0.9
m=12-n=900	-0.2	0.2	-0.3	-0.1	-0.2	0.1	-0.7	-0.1	0.8

The results of Table 5.5.1 indicate that theoretical aggregate model ARMA(1,1) is frequently best fitted aggregate model only when Φ=0.3 - θ= - 0.1 and Φ= 0.3 - θ= - 0.1 for m= 3 and n = 120 case. ARMA(2,2) is generally chosen as frequently selected best fitted aggregate model and the frequency of best fitted aggregate ARMA(2,2) model is close to the frequency of aggregate ARMA(1,1) model. AR(1) is also frequently selected as the best aggregate model especially for Φ= - 0.3- θ= - 0.5 and Φ= - 0.1- θ= - 0.8 . Moreover, it is clearly seen that for m=12 and n = 120 case frequently best selected model is AR(2) without depending on the basic model parameters. As expected, when the sample size decreases and the order of aggregation increases, frequently selected best fitted aggregate model becomes white noise. Similar to previous sections, it is understood that temporal aggregation is effective on model selection and causes model shifts from basic model.

The simulation results of Table 5.5.2 again points out that basic model have smaller mean square forecast error than aggregate model mean square forecast errors. There exists a problem similar to previous sections 5.1 and 5.3. For Φ= -0.1 – θ = -0.8 and Φ= -0.3 – θ = -0.5 some of the mean square forecast errors of best fitted aggregated models are larger than mean square forecast errors of the basic series. Again, this problem may arise from the parameter values of the basic model. Also, it is easily observed that mean square forecast errors of the best fitted aggregated models increase as the order of aggregation increases.

The mean of estimated parameters for the best fitted aggregate models can be seen from Table 5.5.3. It can be clearly seen that estimated parameter values changes according to aggregation level and sample size. The worst results are obtained for m=12 and n =120 cases, most of the estimated parameters are very close to -1 or 1 which can be thought as an indicator of nonstationarity or noninvertibility. Furthermore, estimated parameters of the best fitted aggregated model ARMA(1,1) are not consistent with the parameters of the basic series for most of the conditions. Also, as stated before, the roots of AR polynomial of the best fitted aggregate ARMA(1,1) model are the mth power of the roots of AR polynomial of the basic model. However, this cannot be observed from Table 5.5.3. It can be concluded that temporal aggregation of ARMA(1,1) model affects the parameter estimation.

CHAPTER 6

APPLICATION

In this chapter, a real life data set will be used in order to show the effects of temporal aggregation. The data set is taken from Engineering Statistics Handbook web page (http://www.itl.nist.gov/div898/handbook/pmc/section4/pmc4412.htm, last visited on August 2010) . It is consisted of monthly observations between dates 1955 and 1992 and the data set is about the southern oscillation. Southern oscillation can be explained as the difference between Tahiti and the Darwin Islands according to barometric pressure. Southern oscillations are used for predicting El-Nino which is an atmospheric and oceanic event. Repeated southern oscillation values less than -1 can be thought as an indicator of El-Nino.

This chapter has four sections which analyze the data when m=0, m=3, m=6 and m=12, respectively. In each section, a model will be identified for the data set after the diagnostic checks for the residuals and based on this model, the estimated parameters and mean square forecast errors will be determined. For basic series last fifteen observations and for aggregated series last five observations will be used for calculating mean square errors. The data analyses are conducted by the help of computer programs R 2.10.0 and E-Views 6.

It is useful to give brief information about the residual tests used in diagnostic checks. Ljung-Box Test, Jarque-Bera Test and Breusch-Pagan Test will be used in data analysis.

- **Ljung-Box Test:** It is also known as modified Box-Pierce Test and used for detecting serial autocorrelation between residuals. The test hypothesis is

$$H_0: \rho_1 = \rho_2 = \cdots = \rho_K ,$$

with the test statistic

$$Q = n(n+2)\sum_{k=1}^{K}\frac{\hat{\rho}_k^2}{n-k},$$

where K is the maximum lag length, n is the number of observations and $\hat{\rho}_k$ is the sample autocorrelation at lag k.

Q statistic follows the $\chi^2(K-p-q)$ (For a detailed discussion see Ljung and Box, 1978).

- **Jarque-Bera Test:** It is used for testing the normality of error terms. The null hypothesis is the normality of residuals. Jarque-Bera tests whether the coefficient of skewness and coefficient of excess kurtosis are jointly equal to zero.

Jarque-Bera test statistic is

$$JB = n\left[\frac{\beta_1^2}{6} + \frac{(\beta_2 - 3)^2}{24}\right],$$

where β_1 is the skewness and β_2 is the kurtosis.

JB statistic follows the $\chi^2(2)$ (For detailed discussion see Jarque and Bera, 1981).

- **Breusch-Pagan Test:** It is used to detect heteroskedasticity. The null hypothesis is the homoskedasticity of the residuals. Breush-Pagan test fits a linear regression model to the residuals and rejects if too much variance is explained by the explanatory variables of the artificial regression model (Zeileis and Hothorn, 2002).

  The Breusch-Pagan test statistic is

  $$LM = nR^2_{artificial} \ ,$$

  LM statistic follows the $\chi^2(r)$ where r is the number of regressors without the constant term in the artificial regression model. (For detailed discussion Breusch and Pagan, 1979)

6.1 Data Analysis for Basic Series

As stated before, the data set consists of monthly observations between 1955 and 1992 which means that there are 456 observations. So, data set will be analyzed for 441 observations since last fifteen observations will be used for calculating mean square errors for the fitted and observed values .

First of all, it is needed to look at the time series plot of the data to make a visual inspection for stationarity. If time series plot satisfies stationarity, the Augmented Dickey Fuller test (Said and Dickey, 1984) will be conducted to be sure about the stationarity.

Then, autocorrelation and partial autocorrelation function plots will be analyzed for making an estimation about the orders of the ARIMA model.

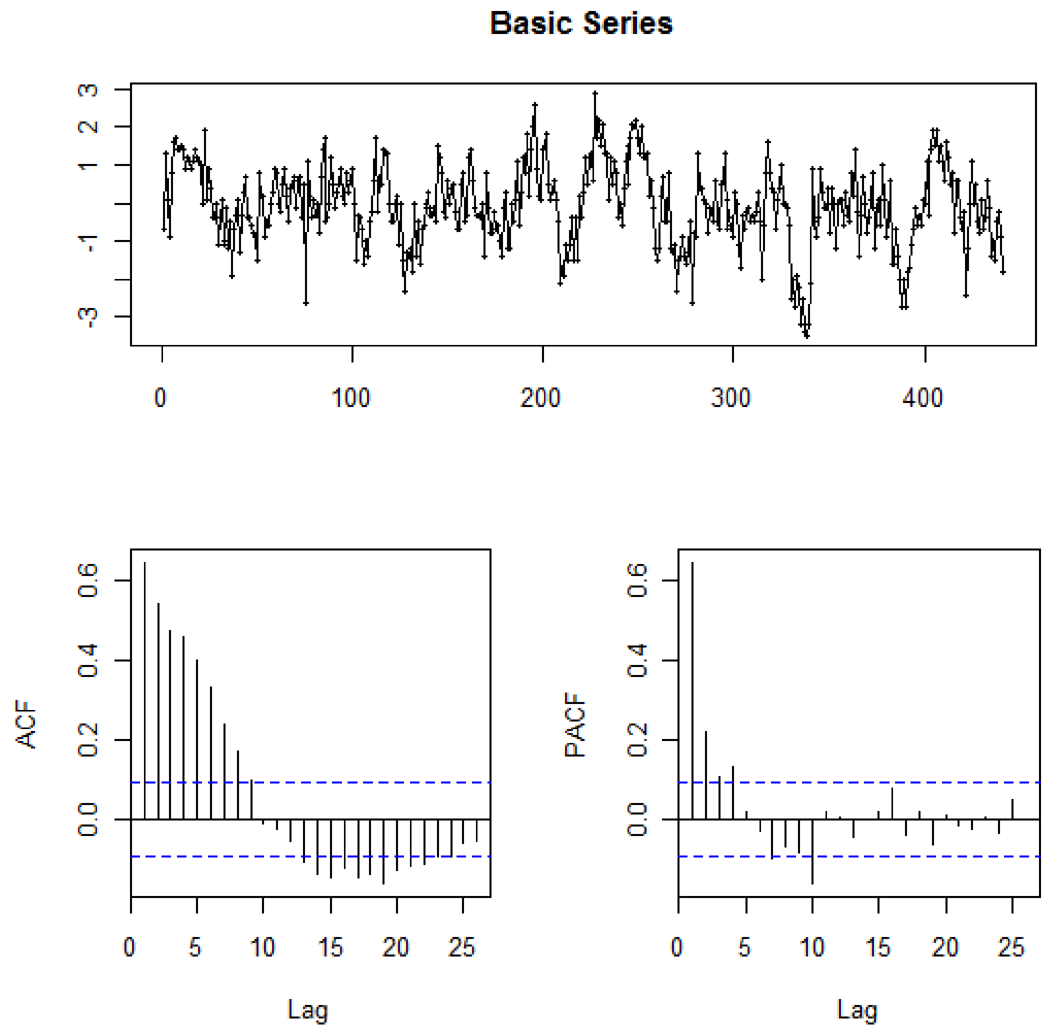

Figure 6.1.1 Time series, ACF and PACF Plots for the Basic Series

The time series plot in the Figure 6.1.1 is an indicator for stationarity because the mean and the dispersion seem constant and no extreme values are seen. Also, there is no need to check the trend and we do not suspect from seasonality by looking at this time series plot. The seasonality graph of E-Views 6 can be used for checking seasonality of the basic series.

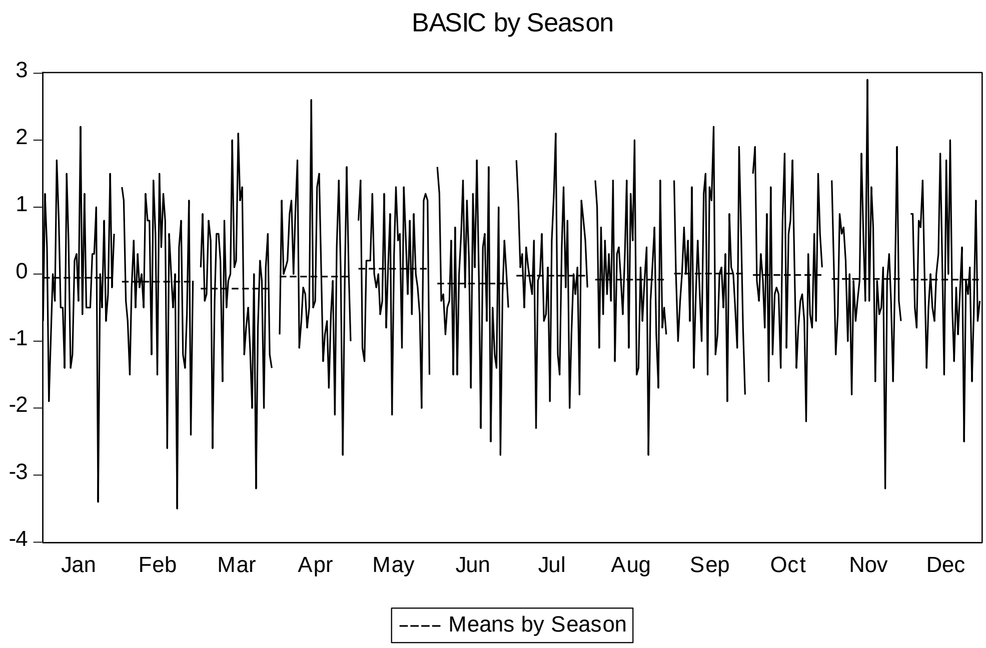

Figure 6.1.2 Seasonality Graph for the Basic Series

As seen, means of the months are very close to each other which means that seasonality will not be taken into consideration in the model identification part. Stationarity must be also tested by a statistical analysis and Augmented Dickey Fuller test is used to make a decision about the stationarity by the help of R 2.10.0.

Table 6.1.1 Augmented Dickey Fuller Test for the Basic Series

```
data:  basicseries
Dickey-Fuller = -5.4496, Lag order = 7, p-value = 0.01
alternative hypothesis: stationary
```

Table 6.1.1 reveals that the series is stationary since the p-value is smaller than the alpha value which is equal to 0.05. Since all the conditions satisfy stationarity, the autocorrelation and partial autocorrelation plots which take place at Figure 6.1.1 can be analyzed. The autocorrelation function shows oscillating decay and the partial autocorrelation function cuts off after lag four. So, AR(4) model will be fitted.

Table 6.1.2 AR(4) Model for the Basic Series

	ar1	ar2	ar3	ar4
coef	4.667535e-01	0.14416005	0.04177965	0.133049855
s.e.	4.736679e-02	0.05258676	0.05254270	0.047552558
t ratio	9.854023e+00	2.74137519	0.79515607	2.797953701
p-value	6.585397e-23	0.00611826	0.42652272	0.005142748

The intercept parameter removed from the model since its p-value is larger than the alpha value 0.05. The stepwise ARIMA procedure is not applied to be consistent with Chapter 5. In simulation studies we considered only the p-value of ar4 parameter and ar3 parameter will be in the model although its p-value is larger than 0.05. Before forecasting, diagnostic check will be useful to get more reliable results.

Figure 6.1.3 indicates that there exists no problem about the diagnostic checks. Standardized residuals do not show a specific pattern and residuals seem uncorrelated. There exists no significant spike at ACF of Residuals except lag zero and p-values for Ljung-Box statistics are larger than 0.05 which means that error terms are uncorrelated.

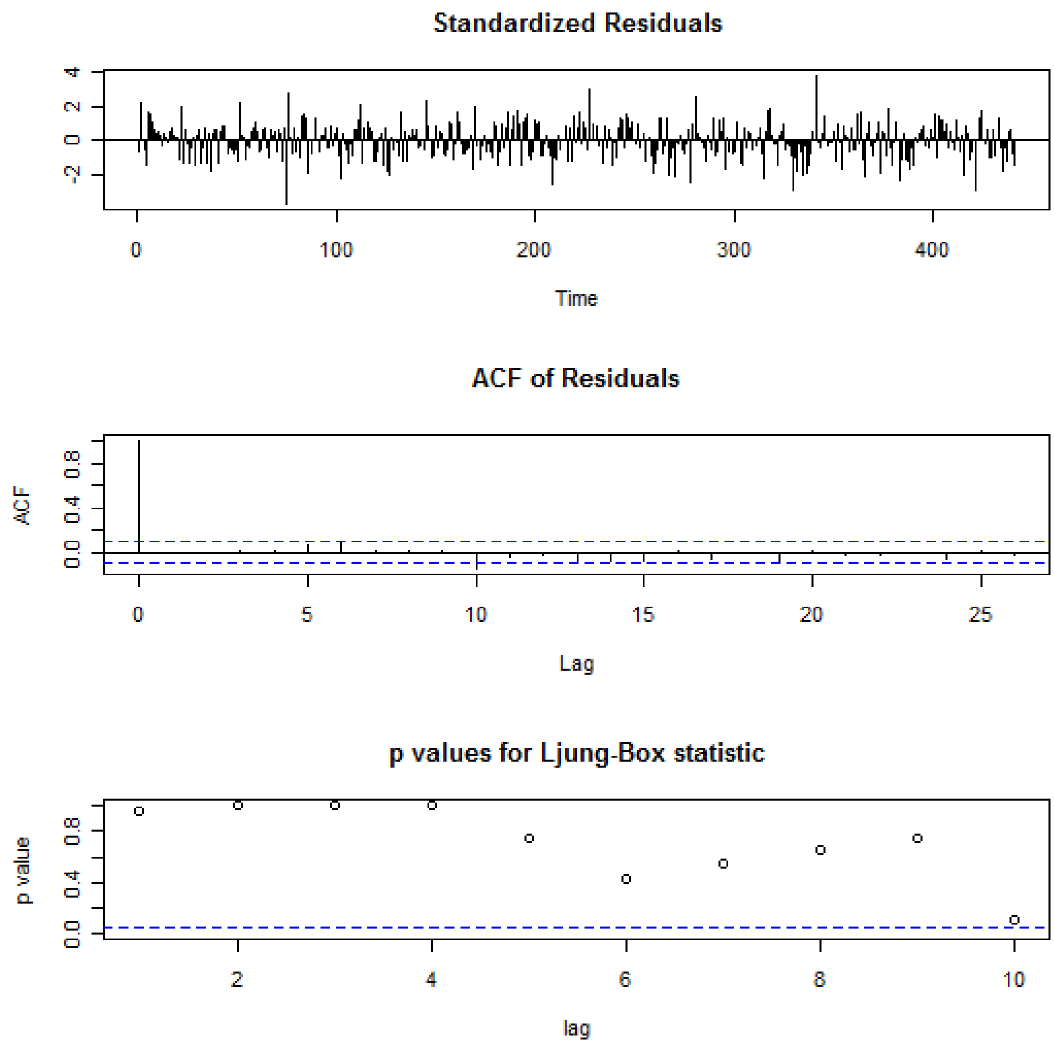

Figure 6.1.3 Diagnostic Check for the Basic Series

Table 6.1.3 Jarque-Bera Test Results for the Basic Series

```
        Jarque Bera Test

data:  fitbasic$residuals
X-squared = 3.9565, df = 2, p-value = 0.1383
```

Table 6.1.3 shows that residuals are distributed normally since p-value is relatively large and there is not enough evidence to reject the null hypothesis. The normality assumption is also tested by Shapiro-Wilk normality test and the p -value is found as 0.5483. So, the null hypothesis of Shapiro-Wilk test , which is similar to the null hypothesis of Jarque-Bera test, cannot be rejected.

Table 6.1.4 Breusch-Pagan Test Results for the Basic Series

```
        studentized Breusch-Pagan test

data:  basicseries.model
BP = 3.1225, df = 4, p-value = 0.5375
```

Table 6.1.4 indicates that residuals have constant variance since we cannot reject the null hypothesis by considering the p-value in the Table 6.1.4.

There exists no problem for diagnostic checks so forecasting can be done based on AR(4) model.

Table 6.1.5 Forecasted Values for the Basic Series

```
Time Series:
Start = 442
End = 456
Frequency = 1
 [1] -1.0447811 -0.8113549 -0.7242667 -0.7381593 -0.6218548 -0.5348762
 [7] -0.4665056 -0.4190437 -0.3679262 -0.3227958 -0.2832823 -0.2498828
[13] -0.2199105 -0.1934505 -0.1701267
```

The mean square error is computed by using the forecasting values and the original data set values. After making calculations the mean square forecast error is found as 1.516959. This value will be used in the following sections to make comparison.

6.2 Data Analysis for Aggregate Series when m=3

In this section, basic series is temporally aggregated for m =3 and the data set is consists of quarterly observations between 1955 and 1992. The quarterly observations are obtained by summing consecutive three observations of the basic data set. The last five observations used for calculating mean square error and so 147 observations are used for analysis. The same procedure applied in Section 6.1 will be used for this section.

Figure 6.2.1 shows the time series, autocorrelation function and partial autocorrelation functions. Time series plot is important to decide on stationarity. The time series plot in Figure 6.2.1 shows that series is stationary since mean and dispersion seems constant.

There is no need to suspect from seasonality since basic series does not have seasonality effect and this can be understood by looking at Figure 6.2.2.

By looking at Table 6.2.1, it can be concluded that aggregated series is stationary. Since we do not have any problem about stationarity, the autocorrelation and partial autocorrelation plots in Figure 6.2.1 can be analyzed. Since autocorrelation function plot shows oscillating decay after lag two and partial function plot shows oscillating decay after lag four, ARMA(4,2) model can be fitted to the aggregated series. Furthermore, theoretically it is expected that aggregated series would follow an ARMA(4,3) model when m=3 since basic series follows an AR(4) process.

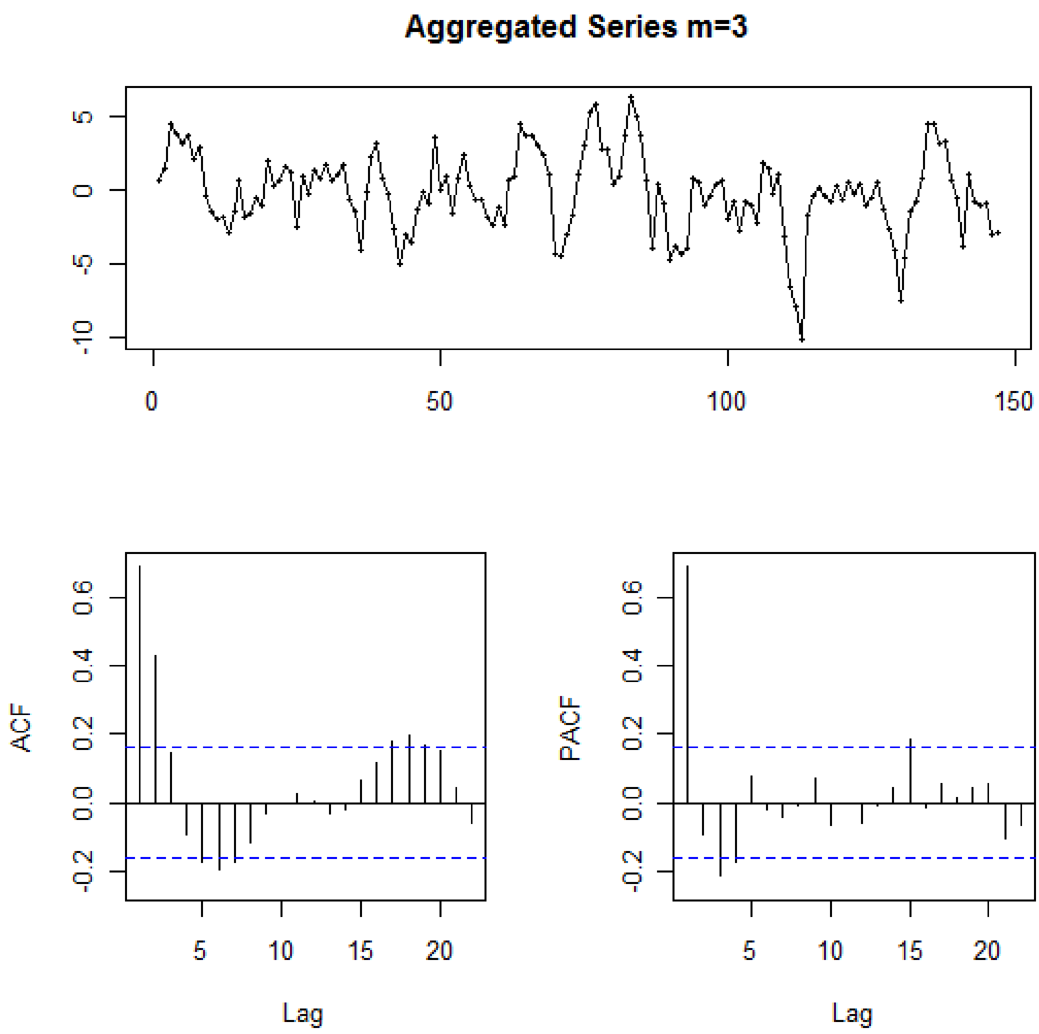

Figure 6.2.1 Time series, ACF and PACF Plots for the Aggregate Series when m=3

Table 6.2.1 Augmented Dickey Fuller Test for the Aggregate Series when m=3

```
data:  quarterseries
Dickey-Fuller = -4.9825, Lag order = 5, p-value = 0.01
alternative hypothesis: stationary
```

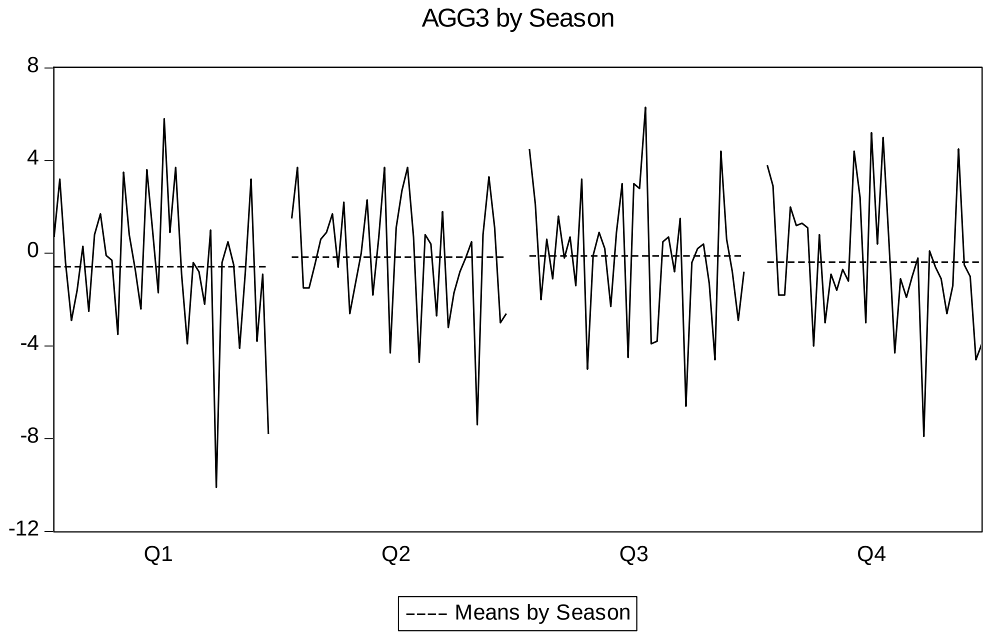

Figure 6.2.2 Seasonality Graph for the Aggregate Series when m=3

Table 6.2.2 AR(4) Model for the Aggregate Series when m=3

	ar1	ar2	ar3	ar4
coef	6.983297e-01	0.08363729	-0.08826732	-0.17186272
s.e.	8.077261e-02	0.09946719	0.09965800	0.08156201
t ratio	8.645624e+00	0.84085304	-0.88570230	-2.10714176
p-value	5.351195e-18	0.40043027	0.37577797	0.03510529

Table 6.2.2 shows the results of AR(4) model since the moving average part and intercept found insignificant. The ar2 and ar3 are also seen insignificant but as stated since ar4 is significant, we do not drop them from the model.

Figure 6.2.3 shows that there exists no problem about the diagnostic checks. Standardized residuals do not show a specific pattern and residuals seem uncorrelated. There exists no significant spike at ACF of Residuals except lag zero and p-values for Ljung-Box statistics are larger than 0.05 which means that error terms are uncorrelated.

Table 6.2.3 Jarque-Bera Test Results for the Aggregate Series when m =3

```
        Jarque Bera Test

data:  fitquarter$residuals
X-squared = 0.3473, df = 2, p-value = 0.8406
```

Table 6.2.3 reveals that the null hypothesis of Jarque-Bera test cannot be rejected. So, residuals have normal distribution. Also, the normality assumption is satisfied by Shapiro-Wilk normality test. The p - value belongs to Shapiro-Wilk normality test is 0.5129 which indicates that residuals are normally distributed.

Table 6.2.4 Breusch-Pagan Test Results for the Aggregate Series when m =3

```
        studentized Breusch-Pagan test

data:  quarterseries.model
BP = 2.0449, df = 4, p-value = 0.7275
```

Table 6.2.4 illustrates that there exists no heteroskedasticity problem for the aggregated series. Since p-value is relatively large, the null hypothesis cannot be rejected. So, residuals have constant variance.

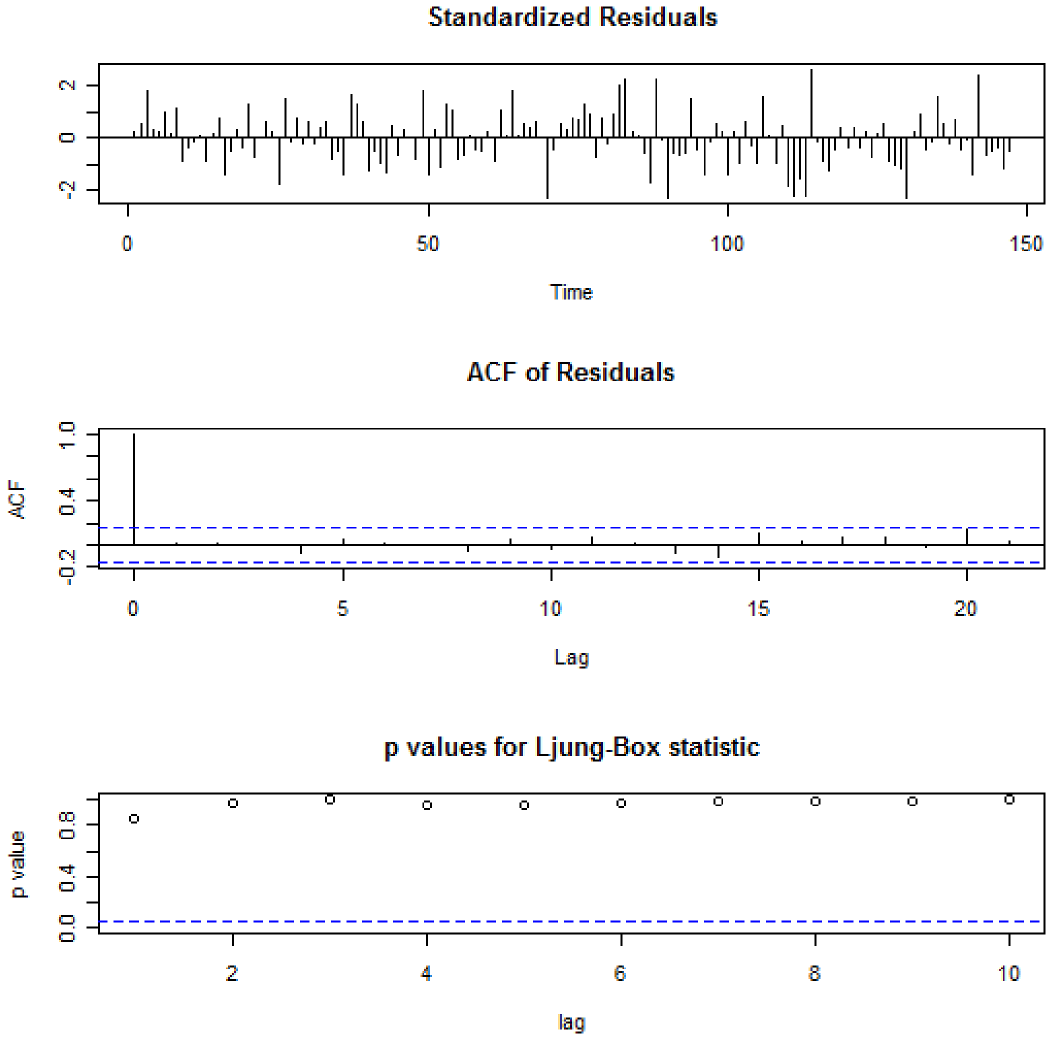

Figure 6.2.3 Diagnostic Check for the Aggregate Series when m=3

Since all of the diagnostic checks are satisfied, forecast values based on this model can be obtained.

Table 6.2.5 Forecasted Values for the Aggregate Series when m=3

```
Time Series:
Start = 148
End = 152
Frequency = 1
[1] -2.0247646 -1.2370229 -0.2616322  0.3909557  0.7083040
```

The mean square forecast error is computed as 15.56546 which is larger than the mean square forecast error that corresponds to basic series.

6.3 Data Analysis for Aggregate Series when m=6

In this section, the basic series is aggregated for m=6 by summing consecutive six observations of the basic series and the series will be consisted of semi annually observations between 1955 and 1992. There will be 76 observations and last five observations are used for calculating mean square error. So, data will be analyzed by using 71 observations. Again, the same procedure applied in previous sections will be conducted.

Since the mean and dispersion seems constant, time series plot in Figure 6.3.1 indicates that series is stationary. Also, the seasonality effect cannot be observed and this can be also understood by looking at Figure 6.3.2.

Table 6.3.1 also shows that the series is stationary. The null hypothesis is rejected since p-value is smaller than the alpha level 0.05.

Since stationarity condition is satisfied, the autocorrelation and partial autocorrelation can be examined. The autocorrelation function plot shows oscillating decay after lag one and the partial autocorrelation function plot shows oscillating decay after lag two. So, ARMA(2,1) can be fitted for the aggregated series.

Furthermore, theoretical aggregate for the series is an ARMA(4,4) when m=6 because the basic series follows an AR(4) process.

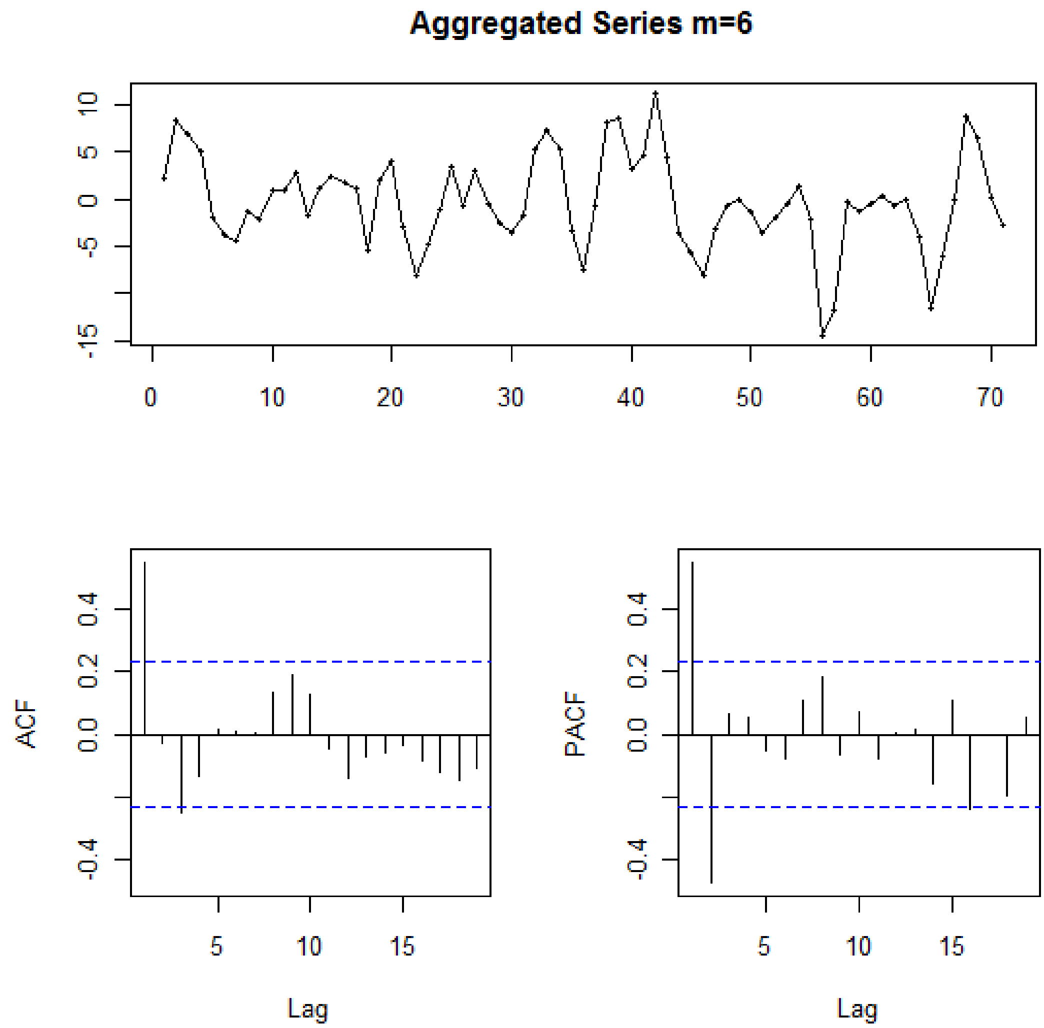

Figure 6.3.1 Time series, ACF and PACF Plots for the Aggregate Series when m=6

Table 6.3.1 Augmented Dickey Fuller Test for the Aggregate Series when m=6

```
data:  semiannualseries
Dickey-Fuller = -3.6287, Lag order = 4, p-value = 0.03727
alternative hypothesis: stationary
```

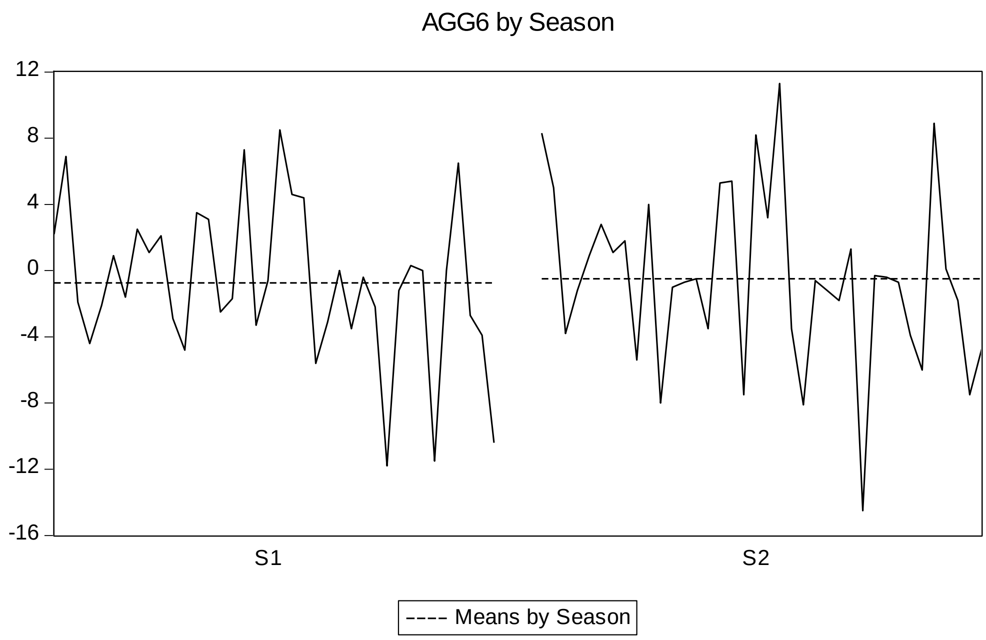

Figure 6.3.2 Seasonality Graph for the Aggregate Series when m=6

Table 6.3.2 AR(2) Model for the Aggregate Series when m=6

	ar1	ar2
coef	8.134751e-01	-4.804830e-01
s.e.	1.037003e-01	1.044393e-01
t ratio	7.844478e+00	-4.600596e+00
p-value	4.347572e-15	4.212842e-06

The results for AR(2) model is obtained because moving average part and intercept are insignificant at alpha 0.05 level. Figure 6.3.3 shows the results of diagnostic checks based on AR(2) model. As seen, standardized residuals do not have a specific pattern and residuals seem to be uncorrelated. The ACF of Residuals has no significant spike except lag zero and the p-values for Ljung-Box statistics are larger than alpha level 0.05 which means that residuals are uncorrelated.

Table 6.3.3 Jarque-Bera Test Results for the Aggregate Series when m =6

```
        Jarque Bera Test

data:  fitsemiannual$residuals
X-squared = 4.2266, df = 2, p-value = 0.1208
```

The null hypothesis of Jarque-Bera test cannot be rejected at 0.05 alpha level and it is concluded that residuals have normal distribution. Moreover, the normality assumption satisfied by Shapiro-Wilk normality test . The p-value belongs to Shapiro-Wilk test is 0.2343 which indicates that residuals are normally distributed.

Table 6.3.4 Breusch-Pagan Test Results for the Aggregate Series when m =6

```
        studentized Breusch-Pagan test

data:  semiannual.model
BP = 0.029, df = 2, p-value = 0.9856
```

It can be said that residuals have constant variance by looking at Table 6.3.4 since the p-value is relatively large. All the diagnostic check conditions are satisfied so forecasting can be done based on AR(2) model.

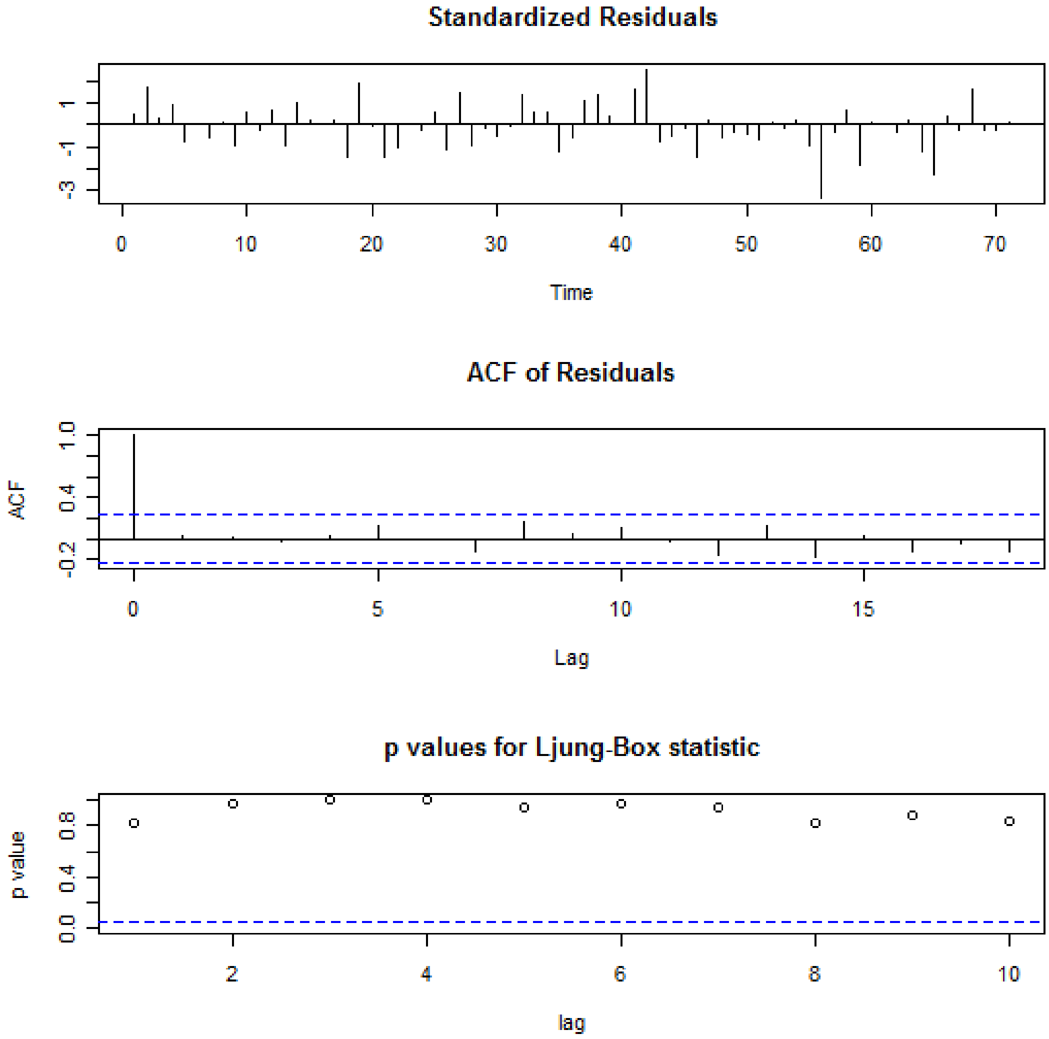

Figure 6.3.3 Diagnostic Check for the Aggregate Series when m=6

Table 6.3.5 Forecasted Values for the Aggregate Series when m=6

```
Time Series:
Start = 72
End = 76
Frequency = 1
[1] -2.2444310 -0.5284846  0.6485018  0.7814679  0.3241106
```

The calculated mean square forecast error is 45.64593 which is larger than the mean square forecast errors of basic series and aggregate series when m=3.

6.4 Data Analysis for Aggregated Series when m=12

The basic series is aggregated for m=12 by summing twelve consecutive observations of the basic series and the data set is consisted of annually observations between 1955 and 1992. Since last five observations are used for mean square error, 33 observations will be used for the data analysis.

Stationarity condition is satisfied visually since by looking at the time series plot in Figure 6.4.1 it can be concluded that mean and dispersion seems to be constant. Moreover, no seasonal effect is observed from the time series plot as expected because the basic series is aggregated for m =12. Since the observations are annually, seasonal graph cannot be obtained from E-Views 6. In order to be sure about stationarity, Augmented Dickey Fuller test is conducted and the results can be seen from Table 6.4.1.

Table 6.4.1 Augmented Dickey Fuller Test for the Aggregate Series when m=12

```
data:  annualseries
Dickey-Fuller = -4.3164, Lag order = 1, p-value = 0.01
alternative hypothesis: stationary
```

Table 6.4.1 points out that the series is stationary because p-value is relatively small. Since stationarity condition is also satisfied by Augmented Dickey Fuller test, the autocorrelation and partial autocorrelation function plots can be examined. All the lags are within critical bands for both of the plots. This means that white noise will be fitted for aggregate series when m=12. Although theoretically aggregated series should follow an ARMA(4,4) model for m=12, white noise will be used for estimation.

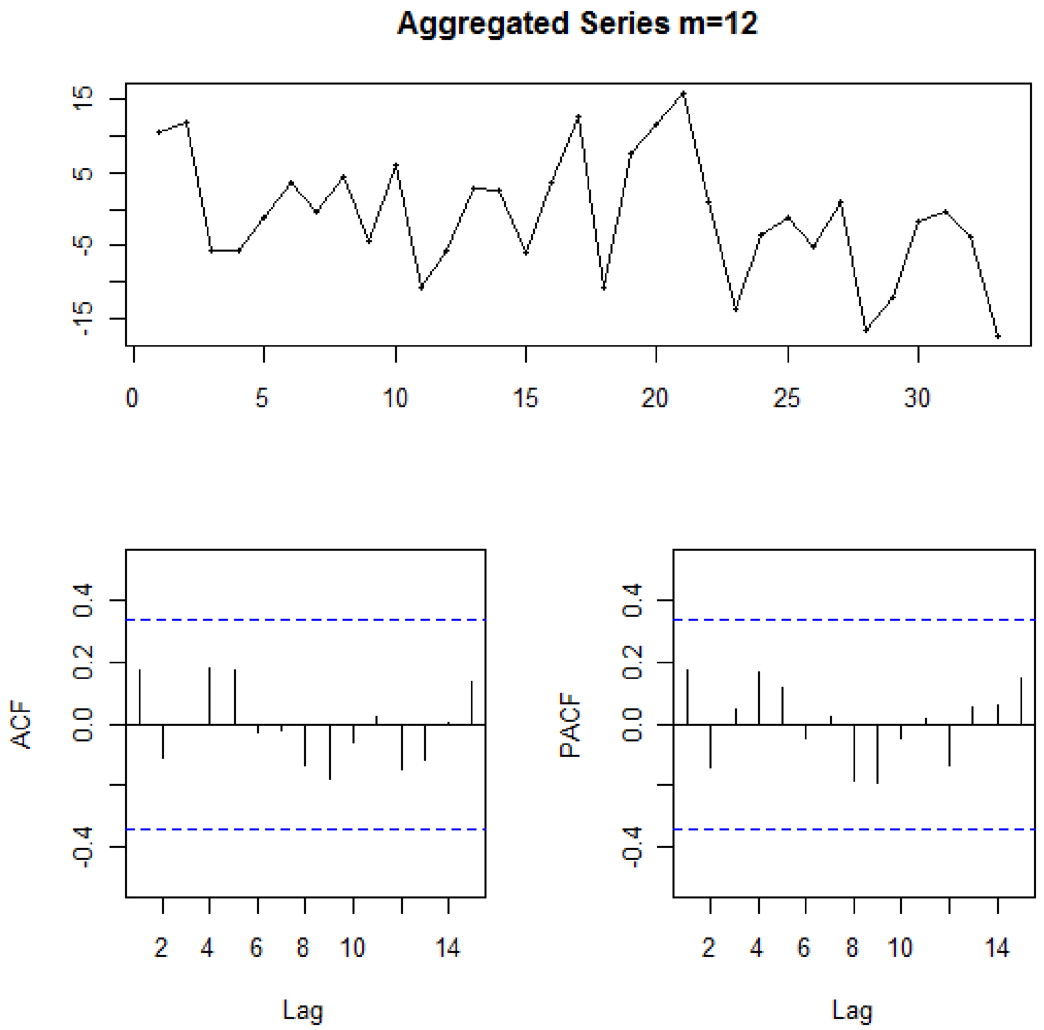

Figure 6.4.1 Time series, ACF and PACF Plots for the Aggregate Series when m=12

Table 6.4.2 White Noise for the Aggregate Series when m=12

	intercept
coef	-0.9606061
s.e.	1.4527280
t ratio	-0.6612429
p-value	0.5084566

Table 6.4.2 shows that intercept parameter is insignificant since the p-value is relatively large. The white noise model will be conducted without intercept and the model without intercept is used for diagnostic checks and forecasting.

Figure 6.4.3 shows that there exists no problem with diagnostic checks. The standardized residuals do not have an obvious pattern. Also, ACF of Residuals and p values for Ljung-Box statistic reveals that residuals are uncorrelated.

Table 6.4.3 Jarque-Bera Test Results for the Aggregate Series when m =12

```
        Jarque Bera Test

data:  fitannual$resid
X-squared = 0.3831, df = 2, p-value = 0.8257
```

As seen from Table 6.4.3, the residuals distributed normally since p-value is relatively large. Also, the p-value of Shapiro-Wilk normality test equals to 0.7983 which indicates that residuals have normal distribution.

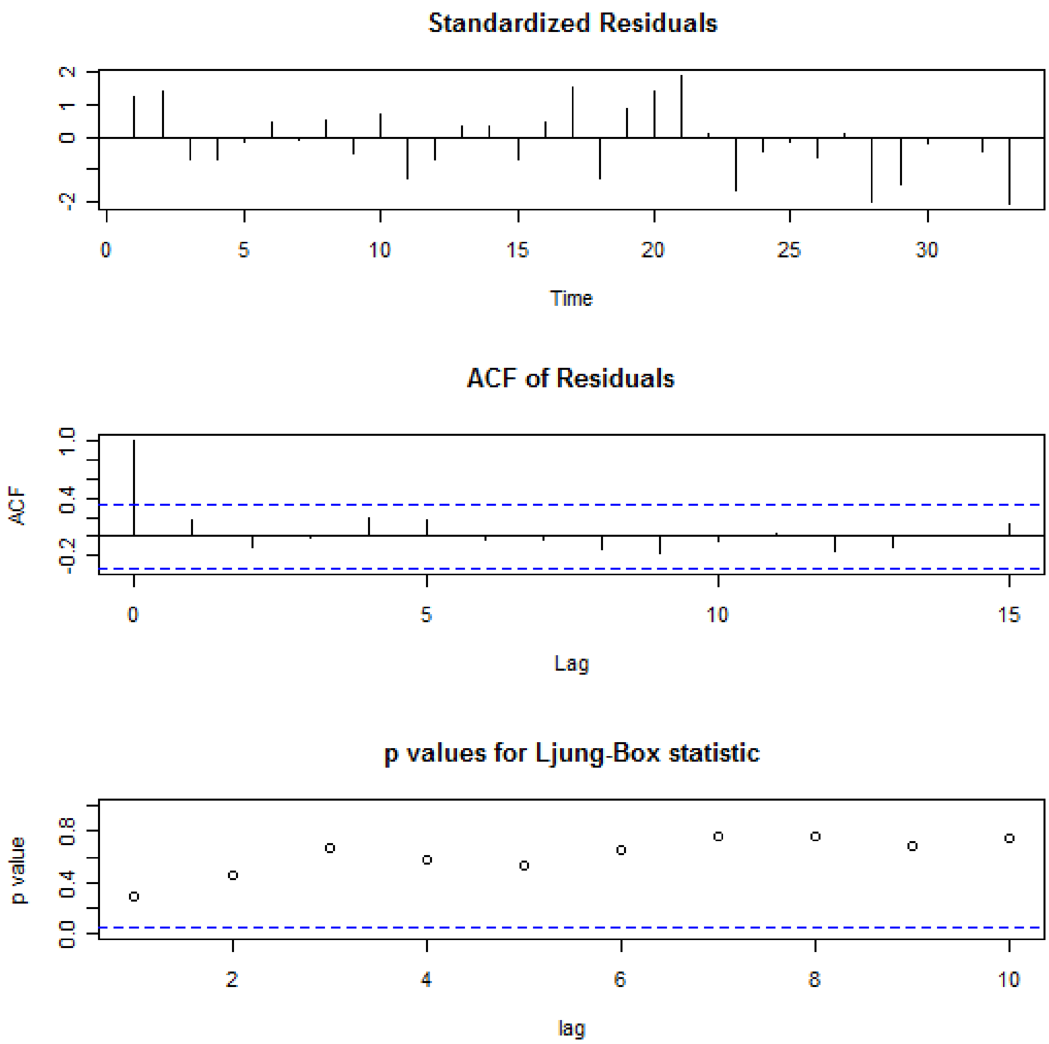

Figure 6.4.2 Diagnostic Check for the Aggregate Series when m=12

Table 6.4.4 Breusch-Pagan Test Results for the Aggregate Series when m =12

```
        studentized Breusch-Pagan test

data:  annual.model
BP = 0.1615, df = 1, p-value = 0.6878
```

It can be understood that residuals have constant variance by considering the p-value at Table 6.4.4. All of the diagnostic check assumptions are satisfied and forecasting can be done for finding mean square forecast error.

Table 6.4.5 Forecasted Values for the Aggregated Series when m=12

```
Time Series:
Start = 34
End = 38
Frequency = 1
[1] 0 0 0 0 0
```

All the forecasted values are zero because our model is white noise without intercept.

The mean square error is computed as 100.198 which is the largest mean square error value among basic series and aggregated series when m=3 and m=6.

6.5 The Summary of Data Analysis

The data set is about the southern oscillation which shows the difference between Tahiti and the Darwin Islands according to barometric pressure. The data set consists of monthly observations between 1955 and 1992 which means that data set has 456 observations. The data set firstly analyzed by using 456 observations. Firstly, the model was identified by using 451 observations and the last fifteen observations were used for calculating mean square forecast error. An AR(4) model is fitted for the basic series. The diagnostic checks which test the uncorrelatedness, normality and homoskedasticity of residuals did not reveal a problem for residuals and so the basic AR(4) model was used for forecasting and mean square forecast error was obtained for the basic series. Then, the data set was temporally aggregated by summing consecutive three observations of the basic series. So, the new data are consisted of quarterly observations between 1955 and 1992 and it has 152 observations. Last five observations were used for the mean square forecast error and the data analysis was conducted according to 147 observations. Theoretically an ARMA (4,3) model is expected for the quarterly data set but again an AR(4) model which satisfied all diagnostic checks for residuals was fitted for the aggregate series. Although the fitted model is the same as the fitted model of the basic series, the estimated parameters and mean square forecast error were changed. The aggregate model for quarterly series has worse forecast values compared to the forecast values of basic series since the quarterly aggregated series has larger mean square forecast error. The data series was also aggregated by summing consecutive six observations which means that the data set with monthly observations is converted to a data set with semi annually observations. The new aggregate series has 76 observations and again last five observations were used for calculating mean square forecast error. The data analysis was conducted based on 71 observations and an AR(2) model which satisfied all the diagnostic checks was fitted to the series. Again, it was understood that theoretical results are not valid since theoretically an ARMA(4,4) model is expected for this data set. As seen the fitted model was changed and the mean square forecast error based on forecasted values from AR(2) model is larger than both of the mean square forecast errors of basic series and the quarterly aggregated series. Lastly, the basic series was temporally aggregated by summing consecutive twelve observations so the new data set has 38 yearly observations. Again 33 observations were used for model fitting and last five observations were used for mean square forecast error. It was seen that the new model for the yearly aggregated series was white noise although theoretically an ARMA(4,4) model is expected. Since a white noise model without an intercept was fitted all of the forecasted values were equal to zero. The yearly aggregated series has the largest mean square forecast error among all the series analyzed. In conclusion, it is seen that the mean square forecast error increases as the order of aggregation increase. Also, it is seen that theoretically introduced temporally aggregate models are not valid for this data set. However, it is necessary to state that and the aggregate series for all orders of aggregation were stationary with no seasonality effect as theoretically expected since the basic series was stationary with no seasonality.

CHAPTER 7

CONCLUSION

Temporal aggregation can be defined as aggregation over consecutive time periods. In time series analysis, temporal aggregation is important since a totally different model can be fitted which gives worse forecasts than the forecasts obtained by the basic series for observations of the same variable over time periods. Temporal aggregation is preferred because simpler models which have more clear comments can be obtained by temporal aggregation. However, a scientist should be aware of the effects of temporal aggregation on the data set. In some countries, we cannot see a study about the temporal aggregation and our aim is to introduce the temporal aggregation concept and its effects to scientific researchers. Our study can be a reference for further research about the importance of using nonaggregate series.

The study focuses on the effects of temporal aggregation on univariate time series and discusses how the model, parameter estimates and mean square forecast errors change, when temporally aggregated data are used instead of basic series. In order to understand the temporal aggregation effect theoretically, the univariate time series models are introduced with their autocorrelation and partial autocorrelation functions. The summary of how the autocorrelation and partial autocorrelation behave for the univariate time series can be seen from Table 7.1.

Based on the autocovariance function of univariate series, it is found that how the model changes when the series is temporally aggregated. The calculations for finding the temporal aggregate models for autoregressive, moving average and autoregressive moving average processes discussed through this study.

Also, the summary of how the model changes theoretically when the series is aggregated can be seen from Table 7.2 (m is the order of aggregation and $[x]$ is used to denote the integer part of x)

Table 7.1 The ACF and PACF Patterns for Univariate Time series

Process	**ACF**	**PACF**
AR(p)	Shows exponential or oscillating decay	Cuts-off after lag p
MA(q)	Cuts-off after lag q	Shows exponential or oscillating decay
ARMA(p, q)	Decays exponentially or oscillatory after lag q	Shows exponential or oscillating decay p

Table 7.2 Temporal Aggregate Models for Univariate Time series

Basic Model	**Temporal Aggregate Model**
AR(p)	ARMA(p, $\left[\frac{(m-1)(p+1)}{m}\right]$)
MA(q)	MA($\left[1+\frac{(q-1)}{m}\right]$)
ARMA(p, q)	ARMA(p, $\left[p+1+\frac{q-p-1}{m}\right]$

After theoretical expressions through a simulation study, the effect of temporal aggregation on real data is discussed and comparison between the theoretical and the observed model fit are presented. Simulation study is based on the article of Teles, Wei and Hodges (2008).

They developed a simulation study which shows the frequencies of best fitted aggregated models based on Akaike's information criteria for basic AR(1) model for a near unit root case. They gave the results for n=240 and basic model parameters close to unit root when m = 4, m = 6 and m = 12. The simulation study in this thesis is more comprehensive, because the frequencies of the best fitted aggregate models not only based on Akaike's information criterion. The significance of the aggregate model parameters are taken into consideration and if two or more models fitted to the aggregate series, model which has smaller Akaike's information criterion is selected. The frequency tables for the best fitted aggregate series are presented for basic AR(1), AR(2), MA(1), MA(2) and ARMA(1,1). The basic model parameter values are not only the values close to unit root, they are selected in a wide range. Also, the orders of aggregation are taken as three, six and twelve because in real life the data sets are generally aggregated quarterly, semi annually and annually. The simulation studies are conducted for the sample sizes 120, 300 and 900 to see how sample size affects the temporal aggregation. Furthermore, mean square forecast error and parameter estimate tables are given apart from frequency tables. Mean square forecast error tables show the average mean square forecast error values based on the frequencies. Similarly, parameter estimate tables show the average parameter estimate values based on the frequencies. Basically simulation results reveal that

- Frequently the best fitted model for the aggregate series is generally different from the basic model and different models are fitted depending on the sample size and order of aggregation. The change of the fitted models can be seen more apparently for large orders of aggregation and small sample sizes.

- The mean square forecast errors belong to best fitted aggregate models are larger than mean square forecast error belong to basic series. This indicates that forecasted values for the aggregate series worse than the forecasted values for the basic series. Also, it is seen that as order of aggregation increases the mean square forecast errors increase.

- The parameter estimates show that different parameters are estimated for different orders of aggregation and sample sizes. The theoretical results for the parameter estimation which take place Chapter 2, 3 and 4 cannot be observed in the simulation study.

The detail information about the simulation study can be seen from Chapter 5.

Also, a real life data set is analyzed to see the effect of temporal aggregation on model structure, parameter estimation and forecasting. A data set about the southern oscillation is selected. The data set is consisted of 456 monthly observations between the years 1955 and 1992. Firstly, data is analyzed without making any aggregation. An AR(4) model which satisfies all the diagnostic checks for residuals is fitted by using 441 observations and last fifteen observations are used for calculating mean square forecast error. Then, the data set is aggregated by summing consecutive three observations and the new data set consists of quarterly observations between years 1955 and 1992. Although theoretically an ARMA(4,3) model is expected for the data series, an AR(4) model which satisfies all residual diagnostic checks is fitted with 147 observations and last five observations are used for calculating mean square forecast error. The fitted model is the same as with the fitted model of the basic data set but the parameter estimates and mean square forecast error change. The mean square forecast error is larger than the mean square forecast error of basic series. The data set is also aggregated by summing six consecutive observations of the basic data set and the new data set is consisted of semi annually observations between years 1955 and 1992. Although an ARMA(4,4) model is expected for the data set, an AR(2) model which satisfies all the diagnostic checks for residuals is fitted by using 71 observations and last five observations are used for mean square forecast error. The mean square forecast error is larger than the mean square forecast errors of basic and quarterly aggregate series. Finally, the data set is aggregated by summing consecutive twelve observations and the new data set is consisted of annually 38 observations. Theoretically, an ARMA(4,4) model is expected for the data set but the fitted model is white noise.

Since a white noise model without an intercept is fitted, all of the five forecasted values are equal to zero and the mean square forecast error belongs to annually aggregated data set is the largest mean square forecast error. Also by data analysis, it is seen that as the order of aggregation increases, the mean square forecast error increases. Furthermore, it is understood that theoretical results may not be always valid for the application. The details about the data analysis can be seen from Chapter 6.

Finally, temporal aggregation subject needs more investigation since the studies related to this subject are limited. Especially, in some countries there exists no studies about temporal aggregation and we hope this study can be a reference about temporal aggregation. This thesis study focuses on the effects of temporal aggregation on univariate time series and in the future the effects of temporal aggregation on multivariate time series can be investigated. Also, in order to prevent the information loss occurred because of aggregation, disaggregation methods can be studied for both univariate and multivariate time series.

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APPENDIX A

R CODES FOR AR(1) SIMULATION

```
cat("\n","Enter order of aggregation,sample size,phi value","\n")
pp<-scan(n=3)
oa<-pp[1]
size<-pp[2]
par<-pp[3]
cwp <- function (object){
#
# cwp <--> ``coefficients with p-values"
#
    coef <- coef(object)
    if (length(coef) > 0) {
        mask <- object$mask
        sdev <- sqrt(diag(vcov(object)))
        t.rat <- rep(NA, length(mask))
        t.rat[mask] <- coef[mask]/sdev
        pt <- 2 * pnorm(-abs(t.rat))
        setmp <- rep(NA, length(mask))
        setmp[mask] <- sdev
        sum <- rbind(coef, setmp, t.rat, pt)
        dimnames(sum) <- list(c("coef", "s.e.", "t ratio", "p-value"),
            names(coef))
        return(sum)
    } else return(NA)

}
count1<-function(object){            #function for AR(1) model#
fit1.pred<-predict(object,n.ahead=5)
error1=0
for(m in 1:5){
error1=error1+(simag.ar1[(size/oa)-5+m]-fit1.pred$pred[m])^2
}
meanerror1=error1/5
```

```
c[1]=c[1]+cwp(object)[1,1]
x<-c(1,meanerror1,c[1])
return(x)
}

count2<-function(object){          # function for ARMA(1,1) model #
fit2.pred<-predict(object,n.ahead=5)
error2=0
for(m in 1:5){
error2=error2+(simag.ar1[(size/oa)-5+m]-fit2.pred$pred[m])^2
}
meanerror2=error2/5
c[2]=c[2]+cwp(object)[1,1]
c[3]=c[3]+cwp(object)[1,2]
y<-c(1,meanerror2,c[2],c[3])
return(y)
}
count3<-function(object){           # function for AR(2) model #
fit3.pred<-predict(object,n.ahead=5)
error3=0
for(m in 1:5){
error3=error3+(simag.ar1[(size/oa)-5+m]-fit3.pred$pred[m])^2
}
meanerror3=error3/5
if(cwp(object)[4,1]<0.05) c[4]=c[4]+cwp(object)[1,1]
c[5]=c[5]+cwp(object)[1,2]
z<-c(1,meanerror3,c[4],c[5])
return(z)
}
count4<-function(object){    # function for AR(3) model #
fit4.pred<-predict(object,n.ahead=5)
error4=0
for(m in 1:5){
error4=error4+(simag.ar1[(size/oa)-5+m]-fit4.pred$pred[m])^2
}
meanerror4=error4/5
if(cwp(object)[4,1]<0.05) c[6]=c[6]+cwp(object)[1,1]
if(cwp(object)[4,2]<0.05) c[7]=c[7]+cwp(object)[1,2]
c[8]=c[8]+cwp(object)[1,3]
w<-c(1,meanerror4,c[6],c[7],c[8])
return(w)
}
```

```
count5<-function(object){      # function for AR(4) model #
fit5.pred<-predict(fit5,n.ahead=5)
error5=0
for(m in 1:5){
error5=error5+(simag.ar1[(size/oa)-5+m]-fit5.pred$pred[m])^2
}
meanerror5=error5/5
if(cwp(object)[4,1]<0.05) c[9]=c[9]+cwp(object)[1,1]
if(cwp(object)[4,2]<0.05) c[10]=c[10]+cwp(object)[1,2]
if(cwp(object)[4,3]<0.05) c[11]=c[11]+cwp(object)[1,3]
c[12]=c[12]+cwp(object)[1,4]
t<-c(1,meanerror5,c[9],c[10],c[11],c[12])
return(t)
}
c<-mat.or.vec(12,1) ; x2<-mat.or.vec(3,1) ; y2<-mat.or.vec(4,1) ; z2<-
mat.or.vec(4,1) ; w2<-mat.or.vec(5,1) ; t2<-mat.or.vec(6,1)
f<-0;g<-0;summeanerror0<-0;summeanerror6<-0;q<-1000;a<-0
ssim.ar1<-c();ssimag.ar1<-c()
for(j in 1:3){
while(q>0){
for(i in 1:q){
sim.ar1<-arima.sim(list(ar=c(par)),n=size)  # AR(1) basic series #
for(j in 1:size-15){
ssim.ar1[j]=sim.ar1[j]
}
fit0<-arima(ssim.ar1,order=c(1,0,0)) # AR(1) basic fitted model #
a<-a+1
fit0.pred<-predict(fit0,n.ahead=15)
error0=0
for(m in 1:15){
error0=error0+(sim.ar1[size-15+m]-fit0.pred$pred[m])^2
}
meanerror0=error0/15
summeanerror0=summeanerror0+meanerror0
k<-matrix(sim.ar1,nrow=oa)
simag.ar1<-apply(k,2,sum)   # aggregate series#
for(j in 1:(size/oa)-5){
ssimag.ar1[j]=simag.ar1[j]
}
fit1<-try(arima(ssimag.ar1,order=c(1,0,0)),TRUE)  # AR(1) aggregate fitted model #
fit2<-try(arima(ssimag.ar1,order=c(1,0,1)),TRUE) # ARMA(1,1) agg. fitted model #
fit3<-try(arima(ssimag.ar1,order=c(2,0,0)),TRUE) # AR(2) aggregate fitted model #
```

```
fit4<-try(arima(ssimag.ar1,order=c(3,0,0)),TRUE) # AR(3) aggregate fitted model #
fit5<-try(arima(ssimag.ar1,order=c(4,0,0)),TRUE) # AR(4) aggregate fitted model #
fit6<-try(arima(ssimag.ar1,order=c(0,0,0)),TRUE) # WN aggregate fitted model #
if(!inherits(fit1,"try-error") && !inherits(fit2,"try-error") && !inherits(fit3,"try-
error") && !inherits(fit4,"try-error") && !inherits(fit5,"try-error") &&
!inherits(fit6,"try-error")){
aic<-c(fit1$aic,fit2$aic,fit3$aic,fit4$aic,fit5$aic)
if(!is.na(cwp(fit1)[4,1]) && !is.na(cwp(fit2)[4,1]) && !is.na(cwp(fit2)[4,2]) &&
!is.na(cwp(fit3)[4,1]) && !is.na(cwp(fit3)[4,2]) && !is.na(cwp(fit4)[4,1]) &&
!is.na(cwp(fit4)[4,2]) && !is.na(cwp(fit4)[4,3]) && !is.na(cwp(fit5)[4,1]) &&
!is.na(cwp(fit5)[4,2]) && !is.na(cwp(fit5)[4,3]) && !is.na(cwp(fit5)[4,4])){

if(cwp(fit1)[4,1]<0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,3]<0.05 && cwp(fit5)[4,4]<0.05){# all models#
if(min(aic) == fit1$aic ) x2=x2+count1(fit1)
if(min(aic) == fit2$aic ) y2=y2+count2(fit2)
if(min(aic) == fit3$aic ) z2=z2+count3(fit3)
if(min(aic) == fit4$aic ) w2=w2+count4(fit4)
if(min(aic) == fit5$aic ) t2=t2+count5(fit5)
}else
if(cwp(fit1)[4,1]<0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,3]<0.05 && cwp(fit5)[4,4]>0.05){#not AR(4)#
if(fit1$aic < fit2$aic && fit1$aic < fit3$aic && fit1$aic < fit4$aic)
x2=x2+count1(fit1)
if(fit2$aic < fit1$aic && fit2$aic < fit3$aic && fit2$aic < fit4$aic)
y2=y2+count2(fit2)
if(fit3$aic < fit1$aic && fit3$aic < fit2$aic && fit3$aic < fit4$aic)
z2=z2+count3(fit3)
if(fit4$aic < fit1$aic && fit4$aic < fit2$aic && fit4$aic < fit3$aic)
w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]<0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,3]>0.05 && cwp(fit5)[4,4]<0.05){#not AR(3)#
if(fit1$aic < fit2$aic && fit1$aic < fit3$aic && fit1$aic < fit5$aic)
x2=x2+count1(fit1)
if(fit2$aic < fit1$aic && fit2$aic < fit3$aic && fit2$aic < fit5$aic)
y2=y2+count2(fit2)
if(fit3$aic < fit1$aic && fit3$aic < fit2$aic && fit3$aic < fit5$aic)
z2=z2+count3(fit3)
if(fit5$aic < fit1$aic && fit5$aic < fit2$aic && fit5$aic < fit3$aic)
t2=t2+count5(fit5)
}else
if(cwp(fit1)[4,1]>0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,3]<0.05 && cwp(fit5)[4,4]<0.05){#not AR(1)#
```

```
if(fit2$aic<fit3$aic && fit2$aic<fit4$aic && fit2$aic<fit5$aic) y2=y2+count2(fit2)
if(fit3$aic<fit2$aic && fit3$aic<fit4$aic && fit3$aic<fit5$aic) z2=z2+count3(fit3)
if(fit4$aic<fit2$aic && fit4$aic<fit3$aic && fit4$aic<fit5$aic) w2=w2+count4(fit4)
if(fit5$aic<fit2$aic && fit5$aic<fit3$aic && fit5$aic<fit4$aic) t2=t2+count5(fit5)
}else
if(cwp(fit1)[4,1]<0.05 && (cwp(fit2)[4,1] > 0.05 || cwp(fit2)[4,2] > 0.05) &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,3]<0.05 && cwp(fit5)[4,4]<0.05){#not
ARMA(1,1) #
if(fit1$aic < fit3$aic && fit1$aic < fit4$aic && fit1$aic<fit5$aic)
x2=x2+count1(fit1)
if(fit3$aic < fit1$aic && fit3$aic < fit4$aic && fit3$aic<fit5$aic)
z2=z2+count3(fit3)
if(fit4$aic < fit1$aic && fit4$aic < fit3$aic && fit4$aic<fit5$aic)
w2=w2+count4(fit4)
if(fit5$aic < fit1$aic && fit5$aic < fit3$aic && fit5$aic<fit4$aic) t2=t2+count5(fit5)
}else

if(cwp(fit1)[4,1]<0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,3]<0.05 && cwp(fit5)[4,4]<0.05){# not AR(2)#
if(fit1$aic<fit2$aic && fit1$aic<fit4$aic && fit1$aic<fit5$aic) x2=x2+count1(fit1)
if(fit2$aic<fit1$aic && fit2$aic<fit4$aic && fit2$aic<fit5$aic) y2=y2+count2(fit2)
if(fit4$aic<fit1$aic && fit4$aic<fit2$aic && fit4$aic<fit5$aic) w2=w2+count4(fit4)
if(fit5$aic<fit1$aic && fit5$aic<fit2$aic && fit5$aic<fit4$aic) t2=t2+count5(fit5)
}else
if(cwp(fit1)[4,1]<0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,3]>0.05 && cwp(fit5)[4,4]>0.05){#not AR(3),
AR(4)#
if(fit1$aic<fit2$aic && fit1$aic<fit3$aic) x2=x2+count1(fit1)
if(fit2$aic<fit1$aic && fit2$aic<fit3$aic) y2=y2+count2(fit2)
if(fit3$aic<fit1$aic && fit3$aic<fit2$aic) z2=z2+count3(fit3)
}else
if(cwp(fit1)[4,1]<0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,3]<0.05 && cwp(fit5)[4,4]>0.05){ ){#not
AR(2), AR(4)#
if(fit1$aic<fit2$aic && fit1$aic<fit4$aic) x2=x2+count1(fit1)
if(fit2$aic<fit1$aic && fit2$aic<fit4$aic) y2=y2+count2(fit2)
if(fit4$aic<fit1$aic && fit4$aic<fit2$aic) w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]<0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,3]>0.05 && cwp(fit5)[4,4]<0.05){ ){#not
AR(2), AR(3)#
if(fit1$aic<fit2$aic && fit1$aic<fit5$aic) x2=x2+count1(fit1)
if(fit2$aic<fit1$aic && fit2$aic<fit5$aic) y2=y2+count2(fit2)
if(fit5$aic<fit1$aic && fit5$aic<fit2$aic) t2=t2+count5(fit5)
```

```
}else
if(cwp(fit1)[4,1]>0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,3]<0.05 && cwp(fit5)[4,4]>0.05){ ){#not
AR(1), AR(4)#
if(fit2$aic<fit3$aic && fit2$aic<fit4$aic) x2=x2+count1(fit1)
if(fit3$aic<fit2$aic && fit3$aic<fit4$aic) z2=z2+count3(fit3)
if(fit4$aic<fit2$aic && fit4$aic<fit3$aic) w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]>0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,3]>0.05 && cwp(fit5)[4,4]<0.05){ ){#not
AR(1), AR(3)#
if(fit2$aic<fit3$aic && fit2$aic<fit5$aic) y2=y2+count2(fit2)
if(fit3$aic<fit2$aic && fit3$aic<fit5$aic) z2=z2+count3(fit3)
if(fit5$aic<fit2$aic && fit5$aic<fit3$aic) t2=t2+count5(fit5)
}else
if(cwp(fit1)[4,1]<0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,3]<0.05 && cwp(fit5)[4,4]>0.05){ ){ #not
ARMA(1,1), AR(4)#
if(fit1$aic<fit3$aic && fit1$aic<fit4$aic) x2=x2+count1(fit1)
if(fit3$aic<fit1$aic && fit3$aic<fit4$aic) z2=z2+count3(fit3)
if(fit4$aic<fit1$aic && fit4$aic<fit3$aic) w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]<0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,3]>0.05 && cwp(fit5)[4,4]<0.05){ #not
ARMA(1,1), AR(3)#
if(fit1$aic<fit3$aic && fit1$aic<fit5$aic) x2=x2+count1(fit1)
if(fit3$aic<fit1$aic && fit3$aic<fit5$aic) z2=z2+count3(fit3)
if(fit5$aic<fit1$aic && fit5$aic<fit3$aic) t2=t2+count5(fit5)
}else
if(cwp(fit1)[4,1]>0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,3]<0.05 && cwp(fit5)[4,4]<0.05){ #not AR(1),
AR(3)#
if(fit2$aic<fit4$aic && fit2$aic<fit5$aic) y2=y2+count2(fit2)
if(fit4$aic<fit2$aic && fit4$aic<fit5$aic) w2=w2+count4(fit4)
if(fit5$aic<fit2$aic && fit5$aic<fit4$aic) t2=t2+count5(fit5)
}else
if(cwp(fit1)[4,1]>0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,3]<0.05 && cwp(fit5)[4,4]<0.05){ #not AR(1),
ARMA(1,1)#
if(fit3$aic<fit4$aic && fit3$aic<fit5$aic) z2=z2+count3(fit3)
if(fit4$aic<fit3$aic && fit4$aic<fit5$aic) w2=w2+count4(fit4)
if(fit5$aic<fit3$aic && fit5$aic<fit4$aic) t2=t2+count5(fit5)
}else
```

```
if(cwp(fit1)[4,1]<0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,3]<0.05 && cwp(fit5)[4,4]<0.05){
if(fit1$aic<fit4$aic && fit1$aic<fit5$aic) x2=x2+count1(fit1)
if(fit4$aic<fit1$aic && fit4$aic<fit5$aic) w2=w2+count4(fit4)
if(fit5$aic<fit1$aic && fit5$aic<fit4$aic) t2=t2+count5(fit5)
}else
if(cwp(fit1)[4,1]<0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,3]>0.05 && cwp(fit5)[4,4]>0.05){ # AR(1),
ARMA (1,1) #
if(fit1$aic<fit2$aic) x2=x2+count1(fit1)
if(fit2$aic<fit1$aic) y2=y2+count2(fit2)
}else
if(cwp(fit1)[4,1]<0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,3]>0.05 && cwp(fit5)[4,4]>0.05){# AR(1),
AR(2) #
if(fit1$aic<fit3$aic) x2=x2+count1(fit1)
if(fit3$aic<fit1$aic) z2=z2+count3(fit3)
}else
if(cwp(fit1)[4,1]<0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,3]<0.05 && cwp(fit5)[4,4]>0.05){ ){# AR(1),
AR(3) #
if(fit1$aic<fit4$aic) x2=x2+count1(fit1)
if(fit4$aic<fit1$aic) w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]<0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,3]>0.05 && cwp(fit5)[4,4]<0.05){ ){# AR(1),
AR(4) #
if(fit1$aic<fit5$aic) x2=x2+count1(fit1)
if(fit5$aic<fit1$aic) t2=t2+count5(fit5)
}else
if(cwp(fit1)[4,1]>0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,3]>0.05 && cwp(fit5)[4,4]>0.05){ #
ARMA(1,1), AR(2) #
if(fit2$aic<fit3$aic) y2=y2+count2(fit2)
if(fit3$aic<fit2$aic) z2=z2+count3(fit3)
}else
if(cwp(fit1)[4,1]>0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,3]<0.05 && cwp(fit5)[4,4]>0.05){
#ARMA(1,1) , AR(3) #
if(fit2$aic<fit4$aic) y2=y2+count2(fit2)
if(fit4$aic<fit2$aic) w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]>0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,3]>0.05 && cwp(fit5)[4,4]<0.05){ ){
```

```
# ARMA(1,1), AR(4) #
if(fit2$aic<fit5$aic) y2=y2+count2(fit2)
if(fit5$aic<fit2$aic) t2=t2+count5(fit5)
}else
if(cwp(fit1)[4,1]>0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,3]<0.05 && cwp(fit5)[4,4]>0.05){# AR(2) ,
AR(3) #
if(fit3$aic<fit4$aic) z2=z2+count3(fit3)
if(fit4$aic<fit3$aic) w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]>0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,3]>0.05 && cwp(fit5)[4,4]<0.05){# AR(2),
AR(4) #
if(fit3$aic<fit5$aic) z2=z2+count3(fit3)
if(fit5$aic<fit3$aic) t2=t2+count5(fit5)
}else
if(cwp(fit1)[4,1]>0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,3]<0.05 && cwp(fit5)[4,4]<0.05){ # AR(3),
AR(4) #
if(fit4$aic<fit5$aic) w2=w2+count4(fit4)
if(fit5$aic<fit4$aic) t2=t2+count5(fit5)
}else
if(cwp(fit1)[4,1]<0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,3]>0.05 && cwp(fit5)[4,4]>0.05){ # AR(1) #
x2=x2+count1(fit1)
}else
if(cwp(fit1)[4,1]>0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,3]>0.05 && cwp(fit5)[4,4]>0.05){ #
ARMA(1,1) #
y2=y2+count2(fit2)
}else
if(cwp(fit1)[4,1]>0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,3]>0.05 && cwp(fit5)[4,4]>0.05){ # AR(2) #
z2=z2+count3(fit3)
}else
if(cwp(fit1)[4,1]>0.05 &&(cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,3]<0.05 && cwp(fit5)[4,4]>0.05){ # AR(3) #
w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]>0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,3]>0.05 && cwp(fit5)[4,4]<0.05){ # AR(4) #
t2=t2+count5(fit5)
}else
```

```
if(cwp(fit1)[4,1]>0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,3]>0.05 && cwp(fit5)[4,4]>0.05){ # WN#
f=f+1
fit6.pred<-predict(fit6,n.ahead=5)
error6=0
for(m in 1:5){
error6=error6+(simag.ar1[(size/oa)-5+m]-fit6.pred$pred[m])^2
}
meanerror6=error6/5
summeanerror6=summeanerror6+meanerror6
}else
g=g+1
}
}
}
q=1000-(x2[1]+y2[1]+z2[1]+w2[1]+t2[1]+f+g)
}
}
f1=x2[1] # frequency of aggregate AR(1) #
f2=y2[1] # frequency of aggregate ARMA(1,1) #
f3=z2[1] # frequency of aggregate AR(2) #
f4=w2[1] # frequency of aggregate AR(3) #
f5=t2[1] # frequency of aggregate AR(4) #
f6=f #frequency of WN #
f7=g # check whether g is equal to zero or not #

meanmeanerror0<-summeanerror0/a # MSE of basic model#
meanmeanerror1<-x2[2]/x2[1] # MSE of aggregate AR(1) model #
meanmeanerror2<-y2[2]/y2[1] # MSE of aggregate ARMA(1,1) model #
meanmeanerror3<-z2[2]/z2[1]  # MSE of aggregate AR(2) model #
meanmeanerror4<-w2[2]/w2[1] # MSE of aggregate AR(3) model #
meanmeanerror5<-t2[2]/t2[1] # MSE of aggregate AR(4) model #
meanmeanerror6<-summeanerror6/f # MSE of aggregate WN model #

coef1 <- x2[3]/x2[1] # coefficient of AR(1) model #
coef2 <- y2[3]/y2[1] ; coef3 <- y2[4]/y2[1] # coefficients of ARMA(1,1) model #
coef4 <- z2[3]/z2[1] ; coef5 <- z2[4]/z2[1] # coefficients of AR(2) model #
coef6 <- w2[3]/w2[1] ; coef7 <- w2[4]/w2[1] ; coef8 <- w2[5]/w2[1] # coefficients
of AR(3) model #
coef9 <- t2[3]/t2[1]; coef10 <- t2[4]/t2[1]; coef11 <- t2[5]/t2[1]; coef12 <- t2[6]/t2[1]
# coefficients of AR(4) model #
```

APPENDİX B

R CODES FOR AR(2) SIMULATION

```
cat("\n","Enter order of aggregation,sample size,phi values","\n")
pp<-scan(n=4)

oa<-pp[1]
size<-pp[2]
par1<-pp[3]
par2<-pp[4]

cwp <- function (object){
#
# cwp <--> ``coefficients with p-values"
#
    coef <- coef(object)
    if (length(coef) > 0) {
        mask <- object$mask
        sdev <- sqrt(diag(vcov(object)))
        t.rat <- rep(NA, length(mask))
        t.rat[mask] <- coef[mask]/sdev
        pt <- 2 * pnorm(-abs(t.rat))
        setmp <- rep(NA, length(mask))
        setmp[mask] <- sdev
        sum <- rbind(coef, setmp, t.rat, pt)
        dimnames(sum) <- list(c("coef", "s.e.", "t ratio", "p-value"),
            names(coef))
        return(sum)
    } else return(NA)
}
count1<-function(object){ # function for aggregate AR(1) model #
fit1.pred<-predict(object,n.ahead=5)
error1=0
for(m in 1:5){
error1=error1+(simag.ar1[((size/oa)-5)+m]-fit1.pred$pred[m])^2
```

```
}
meanerror1=error1/5
c[1]=c[1]+cwp(object)[1,1]
x<-c(1,meanerror1,c[1])
return(x)
}
count2<-function(object){ # function for  aggregate ARMA(1,1) model #
fit2.pred<-predict(object,n.ahead=5)
error2=0
for(m in 1:5){
error2=error2+(simag.ar1[((size/oa)-5)+m]-fit2.pred$pred[m])^2
}
meanerror2=error2/5
c[2]=c[2]+cwp(object)[1,1]
c[3]=c[3]+cwp(object)[1,2]
y<-c(1,meanerror2,c[2],c[3])
return(y)
}
count3<-function(object){ # function for aggregate AR(2) model #
fit3.pred<-predict(object,n.ahead=5)
error3=0
for(m in 1:5){
error3=error3+(simag.ar1[((size/oa)-5)+m]-fit3.pred$pred[m])^2
}
meanerror3=error3/5
if(cwp(object)[4,1]<0.05) c[4]=c[4]+cwp(object)[1,1]
c[5]=c[5]+cwp(object)[1,2]
z<-c(1,meanerror3,c[4],c[5])
return(z)
}
count4<-function(object){ # function for ARMA(2,2) model #
fit4.pred<-predict(object,n.ahead=5)
error4=0
for(m in 1:5){
error4=error4+(simag.ar1[((size/oa)-5)+m]-fit4.pred$pred[m])^2
}
meanerror4=error4/5
if(cwp(object)[4,1]<0.05) c[6]=c[6]+cwp(object)[1,1]
if(cwp(object)[4,3]<0.05) c[8]=c[8]+cwp(object)[1,3]
c[7]=c[7]+cwp(object)[1,2]
c[9]=c[9]+cwp(object)[1,4]
w<-c(1,meanerror4,c[6],c[7],c[8],c[9])
```

```
return(w)
}
c<-mat.or.vec(9,1) ; x2<-mat.or.vec(3,1) ; y2<-mat.or.vec(4,1) ; z2<-mat.or.vec(4,1)
; w2<-mat.or.vec(6,1)
f<-0;g<-0;summeanerror0<-0;summeanerror6<-0;q<-1000;a<-0
ssim.ar1<-c();ssimag.ar1<-c()
for(j in 1:3){
while(q>0){
for(i in 1:q){
sim.ar1<-arima.sim(list(ar=c(par1,par2)),n=size) # AR(2) basic series #
for(j in 1:(size-15)){
ssim.ar1[j]=sim.ar1[j]
}
fit0<-arima(ssim.ar1,order=c(2,0,0)) # AR(2) basic fitted model #
a<-a+1
fit0.pred<-predict(fit0,n.ahead=15)
error0=0
for(m in 1:15){
error0=error0+(sim.ar1[(size-15)+m]-fit0.pred$pred[m])^2
}
meanerror0=error0/15
summeanerror0=summeanerror0+meanerror0
k<-matrix(sim.ar1,nrow=oa)
simag.ar1<-apply(k,2,sum) # aggregated series #
for(j in 1:((size/oa)-5)){
ssimag.ar1[j]=simag.ar1[j]
}
fit1<-try(arima(ssimag.ar1,order=c(1,0,0)),TRUE) # Aggregate AR(1) fit #
fit2<-try(arima(ssimag.ar1,order=c(1,0,1)),TRUE) # Aggregate ARMA(1,1) fit #
fit3<-try(arima(ssimag.ar1,order=c(2,0,0)),TRUE) # Aggregate AR(2) fit #
fit4<-try(arima(ssimag.ar1,order=c(2,0,2)),TRUE) # Aggregate ARMA(2,2) fit #
fit6<-try(arima(ssimag.ar1,order=c(0,0,0)),TRUE) # Aggregate WN fit #

if(!inherits(fit1,"try-error") && !inherits(fit2,"try-error") && !inherits(fit3,"try-
error") && !inherits(fit4,"try-error") && !inherits(fit6,"try-error")){
aic<-c(fit1$aic,fit2$aic,fit3$aic,fit4$aic)
if(!is.na(cwp(fit1)[4,1]) && !is.na(cwp(fit2)[4,1]) && !is.na(cwp(fit2)[4,2]) &&
!is.na(cwp(fit3)[4,1]) && !is.na(cwp(fit3)[4,2]) && !is.na(cwp(fit4)[4,1]) &&
!is.na(cwp(fit4)[4,2]) && !is.na(cwp(fit4)[4,3]) && !is.na(cwp(fit4)[4,4])){

if(cwp(fit1)[4,1]<0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,2]<0.05 && cwp(fit4)[4,4]<0.05){ #all models#
if(min(aic) == fit1$aic ) x2=x2+count1(fit1)
```

```
if(min(aic) == fit2$aic ) y2=y2+count2(fit2)
if(min(aic) == fit3$aic ) z2=z2+count3(fit3)
if(min(aic) == fit4$aic ) w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]<0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]<0.05 && (cwp(fit4)[4,2]>0.05 || cwp(fit4)[4,4]>0.05)){ # not
ARMA(2,2) #
if(fit1$aic < fit2$aic && fit1$aic < fit3$aic) x2=x2+count1(fit1)
if(fit2$aic < fit1$aic && fit2$aic < fit3$aic) y2=y2+count2(fit2)
if(fit3$aic < fit1$aic && fit3$aic < fit2$aic) z2=z2+count3(fit3)
}else
if(cwp(fit1)[4,1]<0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,2]<0.05 && cwp(fit4)[4,4]<0.05){ #not
ARMA(1,1) #
if(fit1$aic < fit3$aic && fit1$aic < fit4$aic) x2=x2+count1(fit1)
if(fit3$aic < fit1$aic && fit3$aic < fit4$aic) z2=z2+count3(fit3)
if(fit4$aic < fit1$aic && fit4$aic < fit3$aic) w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]>0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,2]<0.05 && cwp(fit4)[4,4]<0.05){ #not AR(1)#
if(fit2$aic<fit3$aic && fit2$aic<fit4$aic) y2=y2+count2(fit2)
if(fit3$aic<fit2$aic && fit3$aic<fit4$aic) z2=z2+count3(fit3)
if(fit4$aic<fit2$aic && fit4$aic<fit3$aic) w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]<0.05 && cwp(fit2)[4,1] < 0.05 && cwp(fit2)[4,2] <0.05 &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,2]<0.05 && cwp(fit4)[4,4]<0.05){ #not AR(2)#
if(fit1$aic < fit2$aic && fit1$aic < fit4$aic) x2=x2+count1(fit1)
if(fit2$aic < fit1$aic && fit2$aic < fit4$aic) y2=y2+count2(fit2)
if(fit4$aic < fit1$aic && fit4$aic < fit2$aic) w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]<0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]>0.05 && (cwp(fit4)[4,2]>0.05 || cwp(fit4)[4,4]>0.05)){# AR(1),
ARMA(1,1) #
if(fit1$aic<fit2$aic) x2=x2+count1(fit1)
if(fit2$aic<fit1$aic) y2=y2+count2(fit2)
}else
if(cwp(fit1)[4,1]>0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]<0.05 && (cwp(fit4)[4,2]>0.05 || cwp(fit4)[4,4]>0.05)){#ARMA(1,1) ,
AR(2) #
if(fit2$aic<fit3$aic) y2=y2+count2(fit2)
if(fit3$aic<fit2$aic) z2=z2+count3(fit3)
}else
if(cwp(fit1)[4,1]>0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,2]<0.05 && cwp(fit4)[4,4]<0.05){
```

```
# AR(2) , ARMA (2,2) #
if(fit3$aic<fit4$aic) z2=z2+count3(fit3)
if(fit4$aic<fit3$aic) w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]<0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]<0.05 && (cwp(fit4)[4,2]>0.05 || cwp(fit4)[4,4]>0.05)){ # AR(1) ,
AR(2) #
if(fit1$aic<fit3$aic) x2=x2+count1(fit1)
if(fit3$aic<fit1$aic) z2=z2+count3(fit3)
}else
if(cwp(fit1)[4,1]<0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,2]<0.05 && cwp(fit4)[4,4]<0.05){ # AR(1),
ARMA(1,1) #
if(fit1$aic<fit4$aic) x2=x2+count1(fit1)
if(fit4$aic<fit1$aic) w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]>0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,2]<0.05 && cwp(fit4)[4,4]<0.05){ #
ARMA(1,1) , ARMA (2,2) #
if(fit2$aic<fit4$aic) y2=y2+count2(fit2)
if(fit4$aic<fit2$aic) w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]<0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]>0.05 && (cwp(fit4)[4,2]>0.05 || cwp(fit4)[4,4]>0.05)){ # AR(1) #
x2=x2+count1(fit1)
}else
if(cwp(fit1)[4,1]>0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]>0.05 && (cwp(fit4)[4,2]>0.05 || cwp(fit4)[4,4]>0.05)){#ARMA(1,1)#
y2=y2+count2(fit2)
}else
if(cwp(fit1)[4,1]>0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]<0.05 && (cwp(fit4)[4,2]>0.05 || cwp(fit4)[4,4]>0.05)){ # AR(2) #
z2=z2+count3(fit3)
}else
if(cwp(fit1)[4,1]>0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]>0.05 &&cwp(fit4)[4,2]<0.05 &&cwp(fit4)[4,4]<0.05){#ARMA(2,2)#
w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]>0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]>0.05 && (cwp(fit4)[4,2]>0.05 || cwp(fit4)[4,4]>0.05)){ #WN#
f=f+1
fit6.pred<-predict(fit6,n.ahead=5)
error6=0
for(m in 1:5){
```

```
error6=error6+(simag.ar1[((size/oa)-5)+m]-fit6.pred$pred[m])^2
}
meanerror6=error6/5
summeanerror6=summeanerror6+meanerror6
}else
g=g+1
}
}
}
q=1000-(x2[1]+y2[1]+z2[1]+w2[1]+f+g)
}
}


f1=x2[1]     # frequency of aggregate AR(1) model #
f2=y2[1]     # frequency of aggregate ARMA(1,1) model #
f3=z2[1]    # frequency of aggregate AR(2) model #
f4=w2[1]   # frequency of aggregate ARMA(1,1) model #
f6=f      # frequency of aggregate WN model #
f7=g     # to control whether g is equal to zero or not #

meanmeanerror0<-summeanerror0/a  # MSE of aggregate AR(1) model #
meanmeanerror1<-x2[2]/x2[1]          # MSE of aggregate ARMA(1,1) model #
meanmeanerror2<-y2[2]/y2[1]          # MSE of aggregate AR(2) model #
meanmeanerror3<-z2[2]/z2[1]          # MSE of aggregate ARMA(1,1) model #
meanmeanerror4<-w2[2]/w2[1]        # MSE of aggregate WN model #
meanmeanerror6<-summeanerror6/f  # MSE of aggregate WN model #

coef1 <- x2[3]/x2[1] # coefficient of aggregate AR(1) model #
coef2 <- y2[3]/y2[1] ; coef3 <- y2[4]/y2[1] # coefficients of aggregate ARMA(1,1)
model #
coef4 <- z2[3]/z2[1] ; coef5 <- z2[4]/z2[1] #coefficients of aggregate AR(2) model #
coef6 <- w2[3]/w2[1] ; coef7 <- w2[4]/w2[1] ; coef8 <- w2[5]/w2[1] ; coef9 <-
w2[6]/w2[1]    # coefficients of aggregate ARMA(2,2) model #
```

APPENDIX C

R CODES FOR MA(1) SIMULATON

```
cat("\n","Enter order of aggregation,sample size,theta value","\n")
pp<-scan(n=3)

oa<-pp[1]
size<-pp[2]
par<-pp[3]

cwp <- function (object){
#
# cwp <--> ``coefficients with p-values"
#
    coef <- coef(object)
    if (length(coef) > 0) {
        mask <- object$mask
        sdev <- sqrt(diag(vcov(object)))
        t.rat <- rep(NA, length(mask))
        t.rat[mask] <- coef[mask]/sdev
        pt <- 2 * pnorm(-abs(t.rat))
        setmp <- rep(NA, length(mask))
        setmp[mask] <- sdev
        sum <- rbind(coef, setmp, t.rat, pt)
        dimnames(sum) <- list(c("coef", "s.e.", "t ratio", "p-value"),
            names(coef))
        return(sum)
    } else return(NA)
}

count1<-function(object){ # function for aggregate MA(1) model #
fit1.pred<-predict(object,n.ahead=5)
error1=0
for(m in 1:5){
error1=error1+(simag.ma1[((size/oa)-5)+m]-fit1.pred$pred[m])^2
```

```
}
meanerror1=error1/5
c[1]=c[1]+cwp(object)[1,1]
x<-c(1,meanerror1,c[1])
return(x)
}
count2<-function(object){ # function for aggregate MA(2) model #
fit2.pred<-predict(object,n.ahead=5)
error2=0
for(m in 1:5){
error2=error2+(simag.ma1[((size/oa)-5)+m]-fit2.pred$pred[m])^2
}
meanerror2=error2/5
if(cwp(object)[4,1]<0.05) c[2]=c[2]+cwp(object)[1,1]
c[3]=c[3]+cwp(object)[1,2]
y<-c(1,meanerror2,c[2],c[3])
return(y)
}
c<-mat.or.vec(3,1) ; x2<-mat.or.vec(3,1) ; y2<-mat.or.vec(4,1)
f<-0;g<-0;summeanerror0<-0;summeanerror3<-0;q<-1000;a<-0
ssim.ma1<-c();ssimag.ma1<-c()

for(j in 1:3){
while(q>0){
for(i in 1:q){
sim.ma1<-arima.sim(list(ma=c(par)),n=size) # MA(1) basic series #
for(j in 1:(size-15)){
ssim.ma1[j]=sim.ma1[j]
}
fit0<-arima(ssim.ma1,order=c(0,0,1)) # MA(1) basic fitted model #
a<-a+1
fit0.pred<-predict(fit0,n.ahead=15)
error0=0
for(m in 1:15){
error0=error0+(sim.ma1[(size-15)+m]-fit0.pred$pred[m])^2
}
meanerror0=error0/15
summeanerror0=summeanerror0+meanerror0
k<-matrix(sim.ma1,nrow=oa)
simag.ma1<-apply(k,2,sum) # aggregate series #
for(j in 1:((size/oa)-5)){
ssimag.ma1[j]=simag.ma1[j]
```

```
}
fit1<-try(arima(ssimag.ma1,order=c(0,0,1)),TRUE) # aggregate MA(1) fit #
fit2<-try(arima(ssimag.ma1,order=c(0,0,2)),TRUE) # aggregate MA (2) fit #
fit3<-try(arima(ssimag.ma1,order=c(0,0,0)),TRUE) # aggregate WN fit #

if(!inherits(fit1,"try-error") && !inherits(fit2,"try-error") && !inherits(fit3,"try-
error")){
aic<-c(fit1$aic,fit2$aic)
if(!is.na(cwp(fit1)[4,1]) && !is.na(cwp(fit2)[4,1]) && !is.na(cwp(fit2)[4,2])){

if(cwp(fit1)[4,1]<0.05 && cwp(fit2)[4,2]<0.05){ # all models #
if(min(aic) == fit1$aic ) x2=x2+count1(fit1)
if(min(aic) == fit2$aic ) y2=y2+count2(fit2)
}else
if(cwp(fit1)[4,1]<0.05 && cwp(fit2)[4,2]>0.05){ # MA(1) #
x2=x2+count1(fit1)
}else
if(cwp(fit1)[4,1]>0.05 && cwp(fit2)[4,2]<0.05){ # MA(2) #
y2=y2+count2(fit2)
}else
if(cwp(fit1)[4,1]>0.05 && cwp(fit2)[4,2]>0.05){ # WN #
f=f+1
fit3.pred<-predict(fit3,n.ahead=5)
error3=0
for(m in 1:5){
error3=error3+(simag.ma1[((size/oa)-5)+m]-fit3.pred$pred[m])^2
}
meanerror3=error3/5
summeanerror3=summeanerror3+meanerror3
}else
g=g+1
}
}
}
q=1000-(x2[1]+y2[1]+f+g)
}
}
```

```
f1=x2[1]  # frequency of aggregate MA(1) model #
f2=y2[1]  # frequency of aggregate MA(2) model #
f6=f  # frequency of WN model #
f7=g # to control whether g is equal to zero or not #


meanmeanerror0<-summeanerror0/a # MSE of basic model #
meanmeanerror1<-x2[2]/x2[1]  #  MSE of aggregate MA(1) model #
meanmeanerror2<-y2[2]/y2[1]  # MSE of aggregate MA(2) model #
meanmeanerror3<-summeanerror3/f   #MSE of WN model #


coef1 <- x2[3]/x2[1] # coefficient of aggregate AR(1) model #
coef2 <- y2[3]/y2[1] ; coef3 <- y2[4]/y2[1] #coefficients of aggregate AR(2) model #
```

APPENDIX D

R CODES FOR MA(2) SIMULATION

```
cat("\n","Enter order of aggregation,sample size,theta values","\n")
pp<-scan(n=4)

oa<-pp[1]
size<-pp[2]
par1<-pp[3]
par2<-pp[4]

cwp <- function (object){
#
# cwp <--> ``coefficients with p-values"
#
    coef <- coef(object)
    if (length(coef) > 0) {
        mask <- object$mask
        sdev <- sqrt(diag(vcov(object)))
        t.rat <- rep(NA, length(mask))
        t.rat[mask] <- coef[mask]/sdev
        pt <- 2 * pnorm(-abs(t.rat))
        setmp <- rep(NA, length(mask))
        setmp[mask] <- sdev
        sum <- rbind(coef, setmp, t.rat, pt)
        dimnames(sum) <- list(c("coef", "s.e.", "t ratio", "p-value"),
            names(coef))
        return(sum)
    } else return(NA)
}

count1<-function(object){ # fuction for aggregate MA(1) model #
fit1.pred<-predict(object,n.ahead=5)
error1=0
for(m in 1:5){
```

```
error1=error1+(simag.ma1[((size/oa)-5)+m]-fit1.pred$pred[m])^2}
meanerror1=error1/5
c[1]=c[1]+cwp(object)[1,1]
x<-c(1,meanerror1,c[1])
return(x)
}
count2<-function(object){ # function for aggregate MA(2) model #
fit2.pred<-predict(object,n.ahead=5)
error2=0
for(m in 1:5){
error2=error2+(simag.ma1[((size/oa)-5)+m]-fit2.pred$pred[m])^2
}
meanerror2=error2/5
if(cwp(object)[4,1]<0.05) c[2]=c[2]+cwp(object)[1,1]
c[3]=c[3]+cwp(object)[1,2]
y<-c(1,meanerror2,c[2],c[3])
return(y)
}
c<-mat.or.vec(3,1) ; x2<-mat.or.vec(3,1) ; y2<-mat.or.vec(4,1)
f<-0;g<-0;summeanerror0<-0;summeanerror3<-0;q<-1000;a<-0
ssim.ma1<-c();ssimag.ma1<-c()
for(j in 1:3){
while(q>0){
for(i in 1:q){
sim.ma1<-arima.sim(list(ma=c(par1,par2)),n=size) # basic MA(2) series #
for(j in 1:(size-15)){
ssim.ma1[j]=sim.ma1[j]
}
fit0<-arima(ssim.ma1,order=c(0,0,2)) # MA(2) basic fitted model #
a<-a+1
fit0.pred<-predict(fit0,n.ahead=15)
error0=0
for(m in 1:15){
error0=error0+(sim.ma1[(size-15)+m]-fit0.pred$pred[m])^2
}
meanerror0=error0/15
summeanerror0=summeanerror0+meanerror0
k<-matrix(sim.ma1,nrow=oa)
simag.ma1<-apply(k,2,sum) # aggregate series #
for(j in 1:((size/oa)-5)){
ssimag.ma1[j]=simag.ma1[j]
}
```

```
fit1<-try(arima(ssimag.ma1,order=c(0,0,1)),TRUE) # aggregate MA(1) fit #
fit2<-try(arima(ssimag.ma1,order=c(0,0,2)),TRUE) # aggregate MA(2) fit #
fit3<-try(arima(ssimag.ma1,order=c(0,0,0)),TRUE) # aggregate WN fit #

if(!inherits(fit1,"try-error") && !inherits(fit2,"try-error") && !inherits(fit3,"try-
error")){
aic<-c(fit1$aic,fit2$aic)
if(!is.na(cwp(fit1)[4,1]) && !is.na(cwp(fit2)[4,1]) && !is.na(cwp(fit2)[4,2])){

if(cwp(fit1)[4,1]<0.05 && cwp(fit2)[4,2]<0.05){ # MA(1), MA(2) #
if(min(aic) == fit1$aic ) x2=x2+count1(fit1)
if(min(aic) == fit2$aic ) y2=y2+count2(fit2)
}else
if(cwp(fit1)[4,1]<0.05 && cwp(fit2)[4,2]>0.05){ # MA(1) #
x2=x2+count1(fit1)
}else
if(cwp(fit1)[4,1]>0.05 && cwp(fit2)[4,2]<0.05){ # MA(2) #
y2=y2+count2(fit2)
}else
if(cwp(fit1)[4,1]>0.05 && cwp(fit2)[4,2]>0.05){ # WN #
f=f+1
fit3.pred<-predict(fit3,n.ahead=5)
error3=0
for(m in 1:5){
error3=error3+(simag.ma1[((size/oa)-5)+m]-fit3.pred$pred[m])^2
}
meanerror3=error3/5
summeanerror3=summeanerror3+meanerror3
}else
g=g+1

}
}
}
q=1000-(x2[1]+y2[1]+f+g)
}
}


f1=x2[1]  # frequency of aggregate MA(1) model #
f2=y2[1]  # frequency of aggregate MA(2) model #
f6=f  # frequency of aggregate WN model #
```

```
f7=g # to control whether g is equal to zero or not #

meanmeanerror0<-summeanerror0/a # MSE of basic model #
meanmeanerror1<-x2[2]/x2[1]     # MSE of aggregate MA(1) model #
meanmeanerror2<-y2[2]/y2[1]     # MSE of aggregate MA(2) model #
meanmeanerror3<-summeanerror3/f  #MSE of aggregate WN model #


coef1 <- x2[3]/x2[1] # coefficient of aggregate MA(1) model #
coef2 <- y2[3]/y2[1] ; coef3 <- y2[4]/y2[1]    #coefficients of aggregate MA(2)
model #
```

APPENDIX E

R CODES FOR ARMA(1,1) SIMULATION

```
cat("\n", "Enter order of aggregation, sample size, phi values","\n")
pp<-scan(n=4)

oa<-pp[1]
size<-pp[2]
par1<-pp[3]
par2<-pp[4]

cwp <- function (object){
#
# cwp <--> ``coefficients with p-values"
#
    coef <- coef(object)
    if (length(coef) > 0) {
        mask <- object$mask
        sdev <- sqrt(diag(vcov(object)))
        t.rat <- rep(NA, length(mask))
        t.rat[mask] <- coef[mask]/sdev
        pt <- 2 * pnorm(-abs(t.rat))
        setmp <- rep(NA, length(mask))
        setmp[mask] <- sdev
        sum <- rbind(coef, setmp, t.rat, pt)
        dimnames(sum) <- list(c("coef", "s.e.", "t ratio", "p-value"),
            names(coef))
        return(sum)
    } else return(NA)
}
count1<-function(object){ # function for AR(1) model #
fit1.pred<-predict(object,n.ahead=5)
error1=0
for(m in 1:5)
```

```
error1=error1+(simag.ar1[((size/oa)-5)+m]-fit1.pred$pred[m])^2
}
meanerror1=error1/5
c[1]=c[1]+cwp(object)[1,1]
x<-c(1,meanerror1,c[1])
return(x)
}
count2<-function(object){ # function for ARMA(1,1) model #
fit2.pred<-predict(object,n.ahead=5)
error2=0
for(m in 1:5){
error2=error2+(simag.ar1[((size/oa)-5)+m]-fit2.pred$pred[m])^2
}
meanerror2=error2/5
c[2]=c[2]+cwp(object)[1,1]
c[3]=c[3]+cwp(object)[1,2]
y<-c(1,meanerror2,c[2],c[3])
return(y)
}
count3<-function(object){ # function for AR(2) model #
fit3.pred<-predict(object,n.ahead=5)
error3=0
for(m in 1:5){
error3=error3+(simag.ar1[((size/oa)-5)+m]-fit3.pred$pred[m])^2
}
meanerror3=error3/5
if(cwp(object)[4,1]<0.05) c[4]=c[4]+cwp(object)[1,1]
c[5]=c[5]+cwp(object)[1,2]
z<-c(1,meanerror3,c[4],c[5])
return(z)
}
count4<-function(object){ # function for ARMA (2,2) model #
fit4.pred<-predict(object,n.ahead=5)
error4=0
for(m in 1:5){
error4=error4+(simag.ar1[((size/oa)-5)+m]-fit4.pred$pred[m])^2
}
meanerror4=error4/5
if(cwp(object)[4,1]<0.05) c[6]=c[6]+cwp(object)[1,1]
if(cwp(object)[4,3]<0.05) c[8]=c[8]+cwp(object)[1,3]
c[7]=c[7]+cwp(object)[1,2]
c[9]=c[9]+cwp(object)[1,4]
```

```
w<-c(1,meanerror4,c[6],c[7],c[8],c[9])
return(w)
}
c<-mat.or.vec(9,1) ; x2<-mat.or.vec(3,1) ; y2<-mat.or.vec(4,1) ; z2<-mat.or.vec(4,1)
; w2<-mat.or.vec(6,1)
f<-0;g<-0;summeanerror0<-0;summeanerror6<-0;q<-1000;a<-0
ssim.ar1<-c();ssimag.ar1<-c()
for(j in 1:3){
while(q>0){
for(i in 1:q){
sim.ar1<-arima.sim(list(ar=c(par1),ma=c(par2)),n=size) # ARMA (1,1) basic series #
for(j in 1:(size-15)){
ssim.ar1[j]=sim.ar1[j]
}
fit0<-arima(ssim.ar1,order=c(1,0,1)) # ARMA(1,1) basic model fit #
a<-a+1
fit0.pred<-predict(fit0,n.ahead=15)
error0=0
for(m in 1:15){
error0=error0+(sim.ar1[(size-15)+m]-fit0.pred$pred[m])^2
}
meanerror0=error0/15
summeanerror0=summeanerror0+meanerror0
k<-matrix(sim.ar1,nrow=oa)
simag.ar1<-apply(k,2,sum) # aggregate series #
for(j in 1:((size/oa)-5)){
ssimag.ar1[j]=simag.ar1[j]
}
fit1<-try(arima(ssimag.ar1,order=c(1,0,0)),TRUE) # AR(1) aggregate fit #
fit2<-try(arima(ssimag.ar1,order=c(1,0,1)),TRUE) # ARMA(1,1) aggregate fit #
fit3<-try(arima(ssimag.ar1,order=c(2,0,0)),TRUE) # AR(2) aggregate fit #
fit4<-try(arima(ssimag.ar1,order=c(2,0,2)),TRUE) # ARMA(2,2) aggregate fit #
fit6<-try(arima(ssimag.ar1,order=c(0,0,0)),TRUE) # WN aggregate fit #

if(!inherits(fit1,"try-error") && !inherits(fit2,"try-error") && !inherits(fit3,"try-
error") && !inherits(fit4,"try-error") && !inherits(fit6,"try-error")){
aic<-c(fit1$aic,fit2$aic,fit3$aic,fit4$aic)
if(!is.na(cwp(fit1)[4,1]) && !is.na(cwp(fit2)[4,1]) && !is.na(cwp(fit2)[4,2]) &&
!is.na(cwp(fit3)[4,1]) && !is.na(cwp(fit3)[4,2]) && !is.na(cwp(fit4)[4,1]) &&
!is.na(cwp(fit4)[4,2]) && !is.na(cwp(fit4)[4,3]) && !is.na(cwp(fit4)[4,4])){

if(cwp(fit1)[4,1]<0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,2]<0.05 && cwp(fit4)[4,4]<0.05){#all models#
```

```
if(min(aic) == fit1$aic ) x2=x2+count1(fit1)
if(min(aic) == fit2$aic ) y2=y2+count2(fit2)
if(min(aic) == fit3$aic ) z2=z2+count3(fit3)
if(min(aic) == fit4$aic ) w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]<0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]<0.05 && (cwp(fit4)[4,2]>0.05 || cwp(fit4)[4,4]>0.05)){#not
ARMA(2,2)#
if(fit1$aic < fit2$aic && fit1$aic < fit3$aic) x2=x2+count1(fit1)
if(fit2$aic < fit1$aic && fit2$aic < fit3$aic) y2=y2+count2(fit2)
if(fit3$aic < fit1$aic && fit3$aic < fit2$aic) z2=z2+count3(fit3)
}else
if(cwp(fit1)[4,1]<0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,2]<0.05 && cwp(fit4)[4,4]<0.05){# not
ARMA(1,1) #
if(fit1$aic < fit3$aic && fit1$aic < fit4$aic) x2=x2+count1(fit1)
if(fit3$aic < fit1$aic && fit3$aic < fit4$aic) z2=z2+count3(fit3)
if(fit4$aic < fit1$aic && fit4$aic < fit3$aic) w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]>0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,2]<0.05 && cwp(fit4)[4,4]<0.05){#not AR(1)#
if(fit2$aic<fit3$aic && fit2$aic<fit4$aic) y2=y2+count2(fit2)
if(fit3$aic<fit2$aic && fit3$aic<fit4$aic) z2=z2+count3(fit3)
if(fit4$aic<fit2$aic && fit4$aic<fit3$aic) w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]<0.05 && cwp(fit2)[4,1] < 0.05 && cwp(fit2)[4,2] <0.05 &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,2]<0.05 && cwp(fit4)[4,4]<0.05){#not AR(2)#
if(fit1$aic < fit2$aic && fit1$aic < fit4$aic) x2=x2+count1(fit1)
if(fit2$aic < fit1$aic && fit2$aic < fit4$aic) y2=y2+count2(fit2)
if(fit4$aic < fit1$aic && fit4$aic < fit2$aic) w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]<0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]>0.05 && (cwp(fit4)[4,2]>0.05 || cwp(fit4)[4,4]>0.05)){# AR(1),
ARMA(1,1) #
if(fit1$aic<fit2$aic) x2=x2+count1(fit1)
if(fit2$aic<fit1$aic) y2=y2+count2(fit2)
}else
if(cwp(fit1)[4,1]>0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]<0.05 && (cwp(fit4)[4,2]>0.05 || cwp(fit4)[4,4]>0.05)){# ARMA(1,1),
AR(2) #
if(fit2$aic<fit3$aic) y2=y2+count2(fit2)
if(fit3$aic<fit2$aic) z2=z2+count3(fit3)
}else
```

```
if(cwp(fit1)[4,1]>0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]<0.05 && cwp(fit4)[4,2]<0.05 && cwp(fit4)[4,4]<0.05){# AR(2),
ARMA(2,2) #
if(fit3$aic<fit4$aic) z2=z2+count3(fit3)
if(fit4$aic<fit3$aic) w2=w2+count4(fit4)
}else

if(cwp(fit1)[4,1]<0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]<0.05 && (cwp(fit4)[4,2]>0.05 || cwp(fit4)[4,4]>0.05)){# AR(1),
AR(2) #
if(fit1$aic<fit3$aic) x2=x2+count1(fit1)
if(fit3$aic<fit1$aic) z2=z2+count3(fit3)
}else
if(cwp(fit1)[4,1]<0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,2]<0.05 && cwp(fit4)[4,4]<0.05){ # AR(1),
ARMA(2,2) #
if(fit1$aic<fit4$aic) x2=x2+count1(fit1)
if(fit4$aic<fit1$aic) w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]>0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,2]<0.05 && cwp(fit4)[4,4]<0.05){
#ARMA(1,1) , ARMA(2,2) #
if(fit2$aic<fit4$aic) y2=y2+count2(fit2)
if(fit4$aic<fit2$aic) w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]<0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]>0.05 && (cwp(fit4)[4,2]>0.05 || cwp(fit4)[4,4]>0.05)){ #AR(1)#
x2=x2+count1(fit1)
}else
if(cwp(fit1)[4,1]>0.05 && cwp(fit2)[4,1]<0.05 && cwp(fit2)[4,2]<0.05 &&
cwp(fit3)[4,2]>0.05 && (cwp(fit4)[4,2]>0.05 || cwp(fit4)[4,4]>0.05)){#ARMA(1,1)#
y2=y2+count2(fit2)
}else
if(cwp(fit1)[4,1]>0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]<0.05 && (cwp(fit4)[4,2]>0.05 || cwp(fit4)[4,4]>0.05)){# AR(2) #
z2=z2+count3(fit3)
}else
if(cwp(fit1)[4,1]>0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]>0.05 && cwp(fit4)[4,2]<0.05 && cwp(fit4)[4,4]<0.05){#
ARMA(2,2) #
w2=w2+count4(fit4)
}else
if(cwp(fit1)[4,1]>0.05 && (cwp(fit2)[4,1]>0.05 || cwp(fit2)[4,2]>0.05) &&
cwp(fit3)[4,2]>0.05 && (cwp(fit4)[4,2]>0.05 || cwp(fit4)[4,4]>0.05)){ # WN #
```

```
f=f+1
fit6.pred<-predict(fit6,n.ahead=5)
error6=0
for(m in 1:5){
error6=error6+(simag.ar1[((size/oa)-5)+m]-fit6.pred$pred[m])^2
}
meanerror6=error6/5
summeanerror6=summeanerror6+meanerror6
}else
g=g+1
}
}
}
q=1000-(x2[1]+y2[1]+z2[1]+w2[1]+f+g)
}
}


f1=x2[1]  # frequency of aggregate AR(1) model #
f2=y2[1]  # frequency of aggregate ARMA(1,1) model #
f3=z2[1]  # frequency of aggregate AR(2) model #
f4=w2[1]  # frequency of aggregate ARMA(2,2) model #
f6=f    # frequency of WN model #
f7=g  # to control whether g is equal to zero or not #


meanmeanerror0<-summeanerror0/a # MSE of basic model #
meanmeanerror1<-x2[2]/x2[1]  # MSE of aggregate AR(1) model #
meanmeanerror2<-y2[2]/y2[1]  # MSE of aggregate ARMA(1,1) model #
meanmeanerror3<-z2[2]/z2[1] # MSE of aggregate AR(2) model #
meanmeanerror4<-w2[2]/w2[1]  # MSE of aggregate ARMA(2,2) model #
meanmeanerror6<-summeanerror6/f #MSE of aggregate WN model #

coef1 <- x2[3]/x2[1] # coefficient of AR(1) model #
coef2 <- y2[3]/y2[1] ; coef3 <- y2[4]/y2[1] # coefficients of ARMA(1,1) model #
coef4 <- z2[3]/z2[1] ; coef5 <- z2[4]/z2[1] # coefficients of AR(2) model #
coef6 <- w2[3]/w2[1] ; coef7 <- w2[4]/w2[1] ; coef8 <- w2[5]/w2[1] ; coef9 <-
w2[6]/w2[1]   # coefficients of ARMA(2,2) model #
```

APPENDIX F

R CODES FOR BASIC SERIES ANALYSIS

```
> basic<-read.table("basic.txt")
> library(tseries)
> library(forecast)
> library(lmtest)
> basicseries<-data.frame(basic)[,1]
> tsdisplay(basicseries)
> adf.test(basicseries)
> fitbasic<-arima(basicseries,order=c(4,0,0))
> cwp(fitbasic)
> fitbasic<-arima(basicseries,order=c(4,0,0),include.mean=FALSE)
> cwp(fitbasic)
> tsdiag(fitbasic)
> jarque.bera.test(fitbasic$residuals)
>shapiro.test(fitbasic$residuals)
> basicseries1<-embed(basicseries,5)[,1]
> basicseries2<-embed(basicseries,5)[,2]
> basicseries3<-embed(basicseries,5)[,3]
> basicseries4<-embed(basicseries,5)[,4]
> basicseriess<-basicseries[1:437]
> basicseries.model<-
basicseriess~basicseries1+basicseries2+basicseries3+basicseries4
> var1.model<-
~I(basicseries1^2)+I(basicseries2^2)+I(basicseries3^2)+I(basicseries4^2)
> bptest(basicseries.model,var1.model)
> predict.basicseries<-predict(fitbasic,n.ahead=15)
> predict.basicseries
> x<-c(-1.5,-0.8,-2.3,-3.4,-1.4,-3,-1.4,0,-1.2,-0.8,0,0,-1.9,-0.9,-1.1)
> error=0
> for(i in 1:15){
+ error=error+(x[i]-predict.basicseries$pred[i])^2
+ }
> meanerror=error/15
> meanerror
```

APPENDIX G

R CODES FOR AGGREGATE SERIES WHEN m=3

```
> quarter<-read.table("quarter.txt")
> library(tseries) ; library(forecast) ; library(lmtest)
> quarterseries<-data.frame(quarter)[,1]
> tsdisplay(quarterseries) ; adf.test(quarterseries)
> fitquarter<-arima(quarterseries,order=c(4,0,2))
> cwp(fitquarter)
> fitquarter<-arima(quarterseries,order=c(4,0,2),include.mean=FALSE)
> cwp(fitquarter)
> fitquarter<-arima(quarterseries,order=c(4,0,1))
> cwp(fitquarter)
> fitquarter<-arima(quarterseries,order=c(4,0,1),include.mean=FALSE)
> cwp(fitquarter)
> fitquarter<-arima(quarterseries,order=c(4,0,0))
> cwp(fitquarter)
> fitquarter<-arima(quarterseries,order=c(4,0,0),include.mean=FALSE)
> cwp(fitquarter)
> tsdiag(fitquarter)
> jarque.bera.test(fitquarter$residuals)
>shapiro.test(fitquarter$residuals)
> quarterseries1<-embed(quarterseries,5)[,1]
> quarterseries2<-embed(quarterseries,5)[,2]
> quarterseries3<-embed(quarterseries,5)[,3]
> quarterseries4<-embed(quarterseries,5)[,4]
> quarterseriess<-quarterseries[1:143]
> quarterseries.model<-
quarterseriess~quarterseries1+quarterseries2+quarterseries3+quarterseries4
> var2.model<-
~I(quarterseries1^2)+I(quarterseries2^2)+I(quarterseries3^2)+I(quarterseries4^2)
> bptest(quarterseries.model,var2.model)
> predict.quarterseries<-predict(fitquarter,n.ahead=5)
> predict.quarterseries
> error<-0
> y<-c(-4.6,-7.8,-2.6,-0.8,-3.9)
> for(i in 1:5){
+ error=error+(y[i]-predict.quarterseries$pred[i])^2
+ }
> meanerror=error/5
```

APPENDIX H

R CODES FOR AGGREFATE SERIES WHEN m=6

```
> semiannual<-read.table("semiannual.txt")
> library(tseries) ; library(forecast) ; library(lmtest)
> semiannualseries<-data.frame(semiannual)[,1]
> tsdisplay(semiannualseries) ; adf.test(semiannualseries)
> fitsemiannual<-arima(semiannualseries,order=c(2,0,1))
> cwp(fitsemiannual)
> fitsemiannual<-arima(semiannualseries,order=c(2,0,1),include.mean=0)
> cwp(fitsemiannual)
> fitsemiannual<-arima(semiannualseries,order=c(2,0,0))
> cwp(fitsemiannual)
> fitsemiannual<-arima(semiannualseries,order=c(2,0,0),include.mean=0)
> cwp(fitsemiannual)
> tsdiag(fitsemiannual)
> jarque.bera.test(fitsemiannual$residuals) ; shapiro.test(fitsemiannual$residuals)
> semiannualseries1<-embed(semiannualseries,3)[,1]
> semiannualseries2<-embed(semiannualseries,3)[,2]
> semiannualseriess<-semiannualseries[1:69]
> semiannual.model<-semiannualseriess~semiannualseries1+semiannualseries2
> var3.model<-~I(semiannualseries1^2)+I(semiannualseries2^2)
> bptest(semiannual.model,var3.model)
> predict.semiannualseries<-predict(fitsemiannual,n.ahead=5)
> predict.semiannualseries
> error<-0
> z<-c(-1.8,-3.9,-7.5,-10.4,-4.7)
> for(i in 1:5){
+ error=error+(z[i]-predict.semiannualseries$pred[i])^2
+ }
> meanerror=error/5
> meanerror
```

APPENDIX I

R CODES FOR AGGREGATE SERIES WHEN m=12

```
> annual<-read.table("annual.txt")
> library(tseries)
> library(forecast)
> library(lmtest)
> annualseries<-data.frame(annual)[,1]
> tsdisplay(annualseries)
> adf.test(annualseries,k=1)
> fitannual<-arima(annualseries,order=c(0,0,0))
> cwp(fitannual)
> fitannual<-arima(annualseries,order=c(0,0,0),include.mean=FALSE)
> tsdiag(fitannual)
> jarque.bera.test(fitannual$residuals)
> shapiro.test(fitannual$residuals)
> annualseriess<-annualseries[1:32]
> annual.model<-annualseriess ~ annualseriess
> fitannualresiduals2<-embed(fitannual$residuals,2)[,1]
> var4.model<-~I(fitannualresiduals2^2)
> bptest(annual.model,var4.model)
> predict.annualseries<-predict(fitannual,n.ahead=5)
> predict.annualseries
> error=0
> w<-c(8.9,6.6,-4.5,-11.4,-15.1)
> for(i in 1:5){
+ error=error+(w[i]-predict.annualseries$pred[i])^2
+ }
> meanerror=error/5
> meanerror
```