PRICING INFLATION INDEXED SWAPS USING AN EXTENDED HJM FRAMEWORK WITH JUMP PROCESS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF APPLIED MATHEMATICS
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

CEREN KARAHAN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
FINANCIAL MATHEMATICS

DECEMBER 2010

Approval of the thesis:

PRICING INFLATION INDEXED SWAPS USING AN EXTENDED HJM FRAMEWORK WITH JUMP PROCESS

submitted by **CEREN KARAHAN** in partial fulfillment of the requirements for the degree of **Master of Science in Department of Financial Mathematics, Middle East Technical University** by,

Prof. Dr. Ersan Akyıldız ________________
Director, Graduate School of **Applied Mathematics**

Assoc. Prof. Dr. Işıl Erol ________________
Head of Department, **Financial Mathematics**

Assoc. Prof. Dr. Azize Hayfavi ________________
Supervisor, **Department of Financial Mathematics, METU**

Examining Committee Members:

Assoc. Prof. Dr. Ömür Uğur ________________
Institute of Applied Mathematics, METU

Assoc. Prof. Dr. Azize Hayfavi ________________
Institute of Applied Mathematics, METU

Prof. Dr. Gerhard Wilhelm Weber ________________
Institute of Applied Mathematics, METU

Prof. Dr. Gül Ergün ________________
Department of Statistics, Hacettepe University

Assist. Prof. Dr. Yeliz Yolcu Okur ________________
Department of Mathematics and Computer Science, Çankaya University

Date: ________________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: CEREN KARAHAN

Signature :

ABSTRACT

PRICING INFLATION INDEXED SWAPS USING AN EXTENDED HJM FRAMEWORK WITH JUMP PROCESS

Karahan, Ceren
M.S., Department of Financial Mathematics
Supervisor : Assoc. Prof. Dr. Azize Hayfavi

December 2010, 57 pages

Inflation indexed instruments are designed to help protect investors against the changes in the general level of prices. So, they are frequently preferred by investors and they have become increasingly developing part of the market. In this study, firstly, the HJM model and foreign currency analogy used to price of inflation indexed instruments are investigated. Then, the HJM model is extended with finite number of Poisson process. Finally, under the extended HJM model, a pricing derivation of inflation indexed swaps, which are the most liquid ones among inflation indexed instruments in the market, is given.

Keywords: inflation indexed swap, HJM model, jump process, foreign currency analogy, inflation

ÖZ

ENFLASYONA ENDEKSLİ SWAPLARIN SIÇRAMA SÜRECİ İÇEREN GENİŞLETİLMİŞ HJM MODELİ KULLANILARAK FİYATLANDIRILMASI

Karahan, Ceren
Yüksek Lisans, Finansal Matematik Bölümü
Tez Yöneticisi : Doç. Dr. Azize Hayfavi

Aralık 2010, 57 sayfa

Enflasyona endeksli enstrümanlar yatırımcıları genel fiyat seviyesindeki değişimlere karşı korumaya yardımcı olmak için dizayn edilmiştir. Bu nedenle, yatırımcılar tarafından sıklıkla tercih edilmekte ve piyasanın hızla büyüyen bir parçası haline gelmektedir. Bu çalışmada, ilk olarak, enflasyona endeksli enstrümanların fiyatlanmasında kullanılan HJM model ve döviz analojisi incelenmiştir. Daha sonra, HJM model sonlu sayıda Poisson süreci eklenerek genişletilmiştir. Son olarak, genişletilmiş bu yeni HJM model altında, piyasada yer alan enflasyona endeksli enstrümanlar arasında likitidesi en fazla olan enflasyona endeksli swapların fiyatlandırılması yapılmıştır.

Anahtar Kelimeler: enflasyona endeksli swap, HJM model, sıçrama süreçleri, döviz analojisi, enflasyon

To my mother

ACKNOWLEDGMENTS

I would like to thank my supervisor Assoc. Prof. Dr. Azize Hayfavi for patiently guiding, motivating, and encouraging me throughout this study.

I am grateful to Prof. Dr. Gerhard Wilhelm Weber, Prof. Dr. Gül Ergün, Assoc. Prof. Dr. Ömür Uğur and Assist. Prof. Dr. Yeliz Yolcu Okur for their help during the correction of the thesis.

I also would like to thank to the members of the Institute of Applied Mathematics.

My greatest debts belongs to my mother Gülşen and my sister Seren. I am very grateful for their love, support and patient.

TABLE OF CONTENTS

LIST OF FIGURES

FIGURES

CHAPTER 1

INTRODUCTION

In the financial analysis, savers are interested in the real purchasing power of their savings. Since securities and other savings derivatives have been based on a nominal basis, new instruments have been offered to get future real value. Inflation indexed securities are among these new instruments. In fact, no other asset class is able to provide such a level of protection against the erosion of purchasing power (see [13]). Hence, these securities are getting more and more popular with this protection advantage. The main subject about inflation indexed securities is pricing. Foreign currency analogy has been used for modeling such as these securities. This analogy is based on Heath, Jarrow and Morton (HJM) [20] framework. So, it is important to learn HJM framework and foreign currency analogy to price inflation indexed securities.

The purposes of this thesis are to investigate importance of the inflation indexed securities, to review HJM framework and foreign currency analogy, to extend HJM framework with jumps under foreign currency analogy and to price inflation indexed swaps using this extended model.

The outline of this thesis is as follows. In Chapter 2, we present a review of the studies on the subject of inflation indexed securities. In Chapter 3, the mathematical preliminaries on stochastic calculus, jump process are given. Chapter 4 presents HJM framework and foreign currency analogy in detail that is introduced by Jarrow and Yıldırım. In Chapter 5, the extended HJM framework with jumps is introduced. In Chapter 6, derivation is given for prices inflation indexed swaps using the model in Chapter 5. Chapter 7 includes the conclusion part.

1.1 Inflation

In an economy, inflation is a rise in the general level of prices about a basket of goods and services over a period of time. If the prices decrease then it is called deflation. According to this basket components and their respective weights different inflation indexes can be defined. For example, the Consumer Price Index (CPI), the Producer Price Index (PPI) and the Gross Domestic Product (GDP) are the most common indexes. There are also product specific indexes like UK PRIX, which excludes mortgage interest payments and Euroland CPI ex-tobacco which excludes price changes on tobacco. Some indexes are connected with a category of the society like CPI-U for urban consumers in the USA and CPI for employees and workers in Italy.

The most common form of inflation in the financial market is the CPI. The CPI measures changes in the price level of consumer goods and services which are purchased by households to meet particular needs. Since following the price of whole goods and services in the market is difficult, the ones with more shares in the total of the consumer expenditures are taken into account. These specific goods and services and their weights are usually based upon expenditure data obtained from expenditure surveys for a sample of households.

So, the inflation rate is a measure of average change in prices across the economy over a specified period. In Turkey, it is published monthly by national statistic institutes. If the monthly rate is 2, it means that the prices would be 2 % higher than previous month. That is, a typical goods and services costing 100 TL last month would cost 102 TL this month.

If we have a look causes of the inflation, we can talk about three main reasons the first one is the level of monetary demand in the economy. In the case total demand is less than total supply; the price level will tend to rise, so inflation rises. The other factor is the level of costs. Increasing the cost of the row materials and productive effort raises inflation. The last factor is expectation of inflation. Increasing expectation affects wages and prices. Because today's wages can buy today's goods and services. The wages must rise to protect purchasing power. So, companies increase their costs because of the wages. This costing increase reflects to consumers such as high prices. Figure 1.1 shows monthly inflation rates between years 2009 and 2010.

Up to now we mentioned about definition and meaning of inflation and its causes. So, we

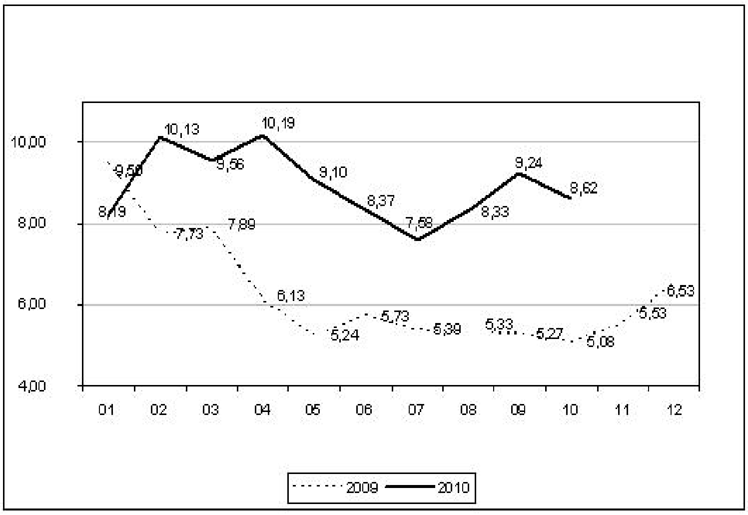

Figure 1.1: Annual Inflation Rate in Turkey

can say that inflation is one of the most important components of economy. Investors want to protect themselves from the purchasing power of money that changes with uncertainty future inflation. According to this, inflation indexed securities can be used as one of the most popular protection ways nowadays.

1.2 Inflation Indexed Securities

Cash flows of securities can base on an index in order to protect investors and issuers from the fluctuations in price of goods and services affected by that index. If that index is inflation, then these securities are called inflation indexed securities.

Although indexation theory has grown for the last two decades, its background started with the 18th century. Deacon, Derry and Mirfendereski [13] gives the history of indexation. In 1707, Bishop William Fleetwood did a study about purchasing power of money. A fellowship was established in 1450 with membership restricted with an annual income of less than £5. He examined changes in prices of corn, meat, drink, clothes between 1450 and 1700; and found a huge increase in the prices. At the end of the study, he showed that in 1707 an individual with a real annual income less than £5 would participate into the fellowship. In 1742 the State of Massachusetts issued bills indexed to the cost of silver on the London Exchange. When the silver price appreciated with great speed according to general price level, a group of commodities price were inserted indexation method. Then, a basket of goods and associated price index was defined for the first time. After the State of Massachusetts indexation definition, works on this subject increase rapidly. The first studies constructed on index to represent level of prices came from Sir George Shuckburgh Evelyn (1798), Joseph Lowe (1822) and Poulett Scope (1833). Stanly Jevons (1875) suggest that gold price be used by a price index. During the second half of the 20th century index debt become popular in the financial markets and several countries begin issuing indexed securities. In Turkey, the first inflation indexed bond was issued in 1994 based on Wholesale Price Index. After these bonds, new issued bonds were based on CPI.

The difference between conventional bonds and inflation indexed bonds is while Conventional bonds give fixed nominal return, inflation indexed bonds give real rate of return known in advance and return changes with the rate of the inflation. When conventional bonds are issued, real returns are not known because future inflation is not known. For example, a one-year conventional bond for 1000 TL pays out principal and nominal rate of return 5 %. This bond pays 1050 TL at maturity. Suppose inflation is 3% over the year, so it will cost 1030 TL to buy what 1000 TL buys at the beginning of the investment. Thus, 20 TL is the extra purchasing power at the maturity. Suppose inflation is 7 %, it will cost 1070 TL to buy what 1000 TL buys at the beginning of the investment. Thus, purchasing power decreases with inflation risk. However, if we buy an inflation indexed bond, we know that at maturity we will able to buy a certain number of baskets of goods and services.

Inflation indexed securities help issuers and investors to reduce their risk arising from inflation. Governments like issuer get some benefits from inflation indexed securities. By issuing indexed bonds eliminates the inflation risk premium which is one part of the yield of conventional nominal bonds. Since indexed bonds are free of inflation risk, it can be thought there is no need for the inflation risk premium component. So the governments obtain cheaper borrowing. At the same time, issuing inflation indexed securities give signal to the market about controlling inflation by government. And these derivatives help conducting asset-liability management more realistically. At the sight of investors, the certain real return can be attractive for investors who are risk averse and who want to protect their savings from being eroded by inflation.

There are some limitations of the benefits of inflation indexed securities. Since the indexes are not immediately available in the market, a lagged index must be used. For example, in the most countries CPI for a given month is published in the middle of the following month. So, it is more significant for the derivatives whose cash flows due in the second half of the publication month. On the other hand, different indexes give different measure of inflation. For example, while the GDP deflator is the best measure for the treasury, the CPI is the best measure for investors.

CHAPTER 2

LITERATURE REVIEW

In the 1990s, markets trading inflation indexed derivatives began to develop. So, the rise of inflation indexed markets has sparked some academic research activity.

Deacon, Derry and Mirfendereski [13] give one of the most detailed studies in the literature about inflation indexed securities. Their study includes the history of indexed bond markets county by country. Also, some fundamental information about indexed securities is given in detail.

Hughston [25] introduces a general theory for pricing and hedging of inflation indexed derivatives. He uses the foreign currency analogy with nominal, real discounted bonds and inflation index under such as domestic economy, foreign economy and exchange rate between these two economies, respectively.

Dodgson and Kainth [14] use two-process Hull-White model to price inflation indexed derivatives. They give the closed-form solutions with constant volatilities. Then, they extend their model with local stochastic volatility. Finally, they price inflation indexed caps and floors using Monte Carlo sampling with local volatility of inflation.

Stewart [36] uses Hull-White extended Vasicek model to price inflation indexed swaps, caps and floors. He gives the calibration of the inflation model and also gives the calibration of inflation indexed swaptions, caps and floors.

Jarrow and Yıldırım [27] use three-factor HJM model to price TIPS. They use the model with time series evolutions of the inflation index, nominal and real zero coupon bond price curves. They find that there is a negative correlation between real rate ad inflation rate. They give the hedging performance of the model. Finally, they give the pricing derivation of the inflation indexed call option.

Leung and Wu [29] give pricing derivation of inflation indexed derivatives under an extended HJM model which includes continuous compounding nominal and inflation forward rates. They introduce an extended market model for inflation rates and they give pricing formulas for this market model. They conclude with the calibration to the market data.

Mercurio [31] uses Jarrow-Yıldırım approach to model inflation. He gives pricing derivation of inflation indexed swaps and options using this model. Then, he introduces two market models and he gives pricing derivation of inflation indexed swaps and options using these market models. Finally, he tests the performance of the two market models by the calibration to the market data for inflation indexed swaps.

Beletski [2] presents the extended Vasicek model for inflation rate and a geometric Brownian motion for consumer price index. Under risk neutral measure pricing formulas of inflation indexed products are given. He also studies on the continuous time optimization problems.

Malvaez [30] gives pricing derivation of swaps and swaptions using Jarrow-Yıldırım model. He studies estimation of parameters for European swaptions using quasi-Monte Carlo simulation.

Belgrade, Benhamou and Koehler [3] give the relationship between zero coupon bond price and year on year swaps under no arbitrage assumption. Their model is driven only by the term structure of volatilities. They compute convexity adjustments for pricing of inflation indexed derivatives.

Haastrecht and Pelsser [19] use a multi-currency model to give the pricing of FX, inflation and stock options. They give their model with stochastic interest rate, stochastic volatility and with correlations. They derive pricing formulas for vanilla call/put options, forward starting options, inflation indexed swaps, caps and floors. Finally, they give an example of calibration to the market data.

Chiarella and Sklibosios [10] use Shirakawa framework to derive a stochastic dynamics system for instantaneous forward rates. They give bond pricing formulas. Finally, they give some numerical simulations.

Björk, Kabanov and Runggaldier [5] give the term structure of zero coupon bonds with interest rate that is modeled by Wiener and marked point process. They investigate the completeness and the uniqueness of a martingale measure for finite jump process.

Runggaldier [34] investigates pricing and hedging of contingent claims in financial markets which includes jump diffusion type models. He considers the stochastic volatility correlated with jump diffusions. He discusses completion of the market under jump diffusion setting. Finally, he gives hedging strategies both under incomplete and complete market condition.

Hinnerich [23] uses and extended HJM model to price inflation indexed swaps. Her model includes Wiener process and general marked point process for forward rates and inflation index. She also prices options on TIPS-bonds under the model driven by Wiener process. She introduces an inflation swap market model and gives pricing derivation of inflation indexed swaptions under this model.

Takadong [37] uses Levy distributions, macro economic factors, no arbitrage and pricing kernel models to improve the match between model prices and observed prices. He gives empirical study on market data of South Africa and America.

CHAPTER 3

PRELIMINARIES

This chapter gives definitions and basic rules that are used in this thesis. Stochastic process, jump process and change of numeraire sections are prepared using Shreve [35], Lamberton and Lapeyre [28], Björk [4], Brigo and Mercurio [8], Güney [18], Hinerrich [23], Cont and Tankov [11] and Runggaldier [34].

3.1 Fundamentals of Stochastic Process

Definition 3.1.1 *A continuous time stochastic process in a space E endowed with a σ-algebra $\mathcal{E}$ is a family X_t of random variables defined on a probability space (Ω, $\mathcal{A}$, $\mathbf{P}$) with values in a measurable space (E,$\mathcal{E}$).*

Definition 3.1.2 *Consider the probability space (Ω, $\mathcal{A}$, $\mathbf{P}$) a filtration $(\mathcal{F}_t)_{t\geq 0}$ is an increasing family of σ-algebras included in $\mathcal{A}$.*

Definition 3.1.3 *Let (Ω, $\mathcal{A}$, $\mathbf{P}$) be a probability space. A Brownian motion is a real valued continuous stochastic process $(X_t)_{t\geq 0}$ with independent and stationary increments.*

- *Continuity:* $\mathbf{P}$- *a.s. the map $s \longmapsto X_s(w)$ is continuous.*
- *Independent increments: If $s \leq t$, then $X_t - X_s$ is independent of $\mathcal{F}_s = \sigma\,(X_u, u \leq s)$.*
- *Stationary increments: If $s \leq t$, then $X_t - X_s$ and $X_{t-s} - X_0$ have the same probability law.*

Definition 3.1.4 *If is Brownian motion then $X_t - X_0$ is a normal random variable with mean rt and variance $\sigma^2 t$, where r and σ are constant real numbers.*

Definition 3.1.5 *On probability space $(\Omega, \mathcal{A}, \mathbf{P})$ a stochastic process $(X_t)_{t\geq 0}$ is an adapted adapted process to filtration $\mathcal{F}$ if for all t X_t is $\mathcal{F}_t$ - measurable.*

Definition 3.1.6 *An adapted sequence $(X_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ on this space martingale if for any $s \leq t$*

$$E[X_t \mid \mathcal{F}_s] \quad = \quad M_s.$$

Definition 3.1.7 *An adapted sequence $(X_t)_{t\geq 0}$ of random variables is predictable if, for all $t \geq 1$, X_t is $\mathcal{F}_{t-1}$ - measurable.*

Definition 3.1.8 .

- *If Y and Z are stochastic random variables and Z is $\mathcal{F}_{t-1}$ - measurable, then*

$$E[ZY \mid \mathcal{F}_t] \quad = \quad ZE[Y \mid \mathcal{F}_t].$$

- *If Y is stochastic random variable , and if $s < t$, then*

$$E[E[Y \mid \mathcal{F}_t] \mid \mathcal{F}_s] \quad = \quad E[Y \mid \mathcal{F}_s].$$

Definition 3.1.9 *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a filtered probability space and $(W_t)_{t\geq 0}$ an $\mathcal{F}$-Brownian motion. $(X_t)_{0\leq t\leq T}$ is an $\mathbf{R}$- valued Ito process if it can be written as $\mathbf{P}$ a.s. $\forall$ $t \leq T$,*

$$X_t = X_0 + \int_0^t K_s ds \int_0^t + H_s dW_s, \tag{3.1}$$

where

- *X_0 is $\mathcal{F}_0$- measurable.*
- *$(K_t)_{0\leq t\leq T}$ and $(H_t)_{0\leq t\leq T}$ are $\mathcal{F}_t$ adapted processes.*
- *$\int_0^t |K_s| ds < \infty$.*

- $\int_0^t H_s^2\, ds < \infty$ **P** *a.s.*.

Definition 3.1.10 *The quadratic variation of the Ito process is* $\langle X, X\rangle_t = \int_0^t H_s^2\, ds$.

Theorem 3.1.11 *Let* $(X_t)_{0\leq t\leq T}$ *be an Ito process satisfying (3.1) and f be twice continuously differentiable function, then*

$$f(X_t) \;=\; f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle X, X\rangle_s$$

where by definition $\langle X, X\rangle_t = \int_0^t H_s^2\, ds$ *and*

$$\int_0^t f'(X_s)\, dX_s \;=\; \int_0^t f'(X_s)K_s\, ds + \int_0^t f'(X_s)H_s\, dW(s).$$

Theorem 3.1.12 (Ito-Doeblin Formula) *Let* $(X_t)_{t\geq 0}$ *be an Ito process and f(t,x) be a function for which the partial derivatives,* $f_t(t,x)$, $f_x(t,x)$, $f_{xx}(t,x)$ *are defined and continuous. Then, for every* $T \geq 0$,

$$\begin{aligned} f(T, X_t) \;&=\; f(0, X_0) + \int_0^t f_t(t, X_t)\, dt + \int_0^t f_x(t, X_t)\, dX_t \\ &\quad + \frac{1}{2}\int_0^t f_{xx}(t, X_t)\, d\langle X, X\rangle_t. \end{aligned}$$

Proposition 3.1.1 (Ito-Integration by Parts Formula) *Let* $(X_t)_{t\geq 0}$ *and* $(Y_t)_{t\geq 0}$ *be two Ito processes such that*

$$X_t = X_0 + \int_0^t K_s\, ds + \int_0^t H_s\, dW_s$$

and

$$Y_t = Y_0 + \int_0^t M_s\, ds + \int_0^t N_s\, dW_s,$$

then

$$X_tY_t \;=\; X_0Y_0 + \int_0^t X_s\, dY_s + \int_0^t Y_s\, dX_s + \langle X, Y\rangle_t$$

with $\langle X, Y\rangle_t = \int_0^t H_sN_s\, ds$.

Definition 3.1.13 *Let* $(\Omega, \mathcal{F})$ *be probability space. Two probability measure* P *and* $\tilde{P}$ *on* $(\Omega, \mathcal{F})$ *are said to be equivalent if they agree, which sets in* $\mathcal{F}$ *have probability zero.*

Definition 3.1.14 *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let $\tilde{P}$ be another probability measure on $(\Omega, \mathcal{F})$ that is equivalent to P and let Z be almost surely positive random variable that relates P and $\tilde{P}$. Then Z is called the Radon-Nikodym derivative of $\tilde{P}$ with respect to P, and we write*

$$Z = \frac{d\tilde{P}}{dP}.$$

Definition 3.1.15 (Radon-Nikodym) *Let P and $\tilde{P}$ be equivalent probability measures defined on $(\Omega, \mathcal{F})$. There exists an almost surely positive random variable Z such that $E[Z] = 1$, and for every $A \in \mathcal{F}$ holds*

$$\tilde{P}(A) = \int_A Z(\omega) dP(\omega).$$

Theorem 3.1.16 (Girsanov Theorem) *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $(W_t)_{t\geq 0}$ be a Brownian motion for this space. Let $(\theta_t)_{t\geq 0}$ be an adapted process. Define*

$$\begin{aligned} Z_t &= \exp\left(-\int_0^t \theta_u \, dW_u - \frac{1}{2}\int_0^t \theta_u^2 \, du\right) \\ \tilde{W}_t &= W_t + \int_0^t \theta_u \, du \end{aligned}$$

and assume that

$$E \int_0^T \theta_u^2 Z_u^2 \, du < \infty.$$

Set $Z = Z(T)$. Then $EZ = 1$, and under probability measure $\tilde{P}$ the process $\tilde{W}_t$ is a Brownian motion.

Theorem 3.1.17 (Girsanov Theorem for multiple dimensions) *Let T be a fixed positive time and $\theta_u = (\theta(1), ..., \theta(d))$ be a d-dimensional adapted process. Define*

$$\begin{aligned} Z_t &= \exp\left(-\int_0^t \theta_u \, dW_u - \frac{1}{2}\int_0^t ||\theta_u||^2 \, du\right) \\ \tilde{W}_t &= W_t + \int_0^t \theta_u \, du \end{aligned}$$

and assume that

$$E \int_0^T ||\theta_u||^2 Z_u^2 \, du < \infty.$$

Set $Z = Z(T)$. Then $EZ = 1$, and under probability measure $\tilde{P}$ the process $\tilde{W}_t$ is a d-dimensional Brownian motion.

Definition 3.1.18 *$(X_t)_{0\leq t\leq T}$ is an Ito process if*

$$X_t \;=\; X_0 + \int_0^t K_s\, ds + \sum_{i=1}^{p} \int_0^t H_s^i\, dW_s^i$$

where

- *K_t and all the processes H_t^i are adapted to $\mathcal{F}_t$*
- *$\int_0^T |K_s|\, ds < \infty$,*
- *$\int_0^T (H_s^i)^{\,2}\, ds < \infty$.*

Proposition 3.1.2 *Let $(X_t^1, X_t^2, .., X_t^n)$ be n Ito processes*

$$X_t^i \;=\; X_0^i + \int_0^t K_s^i\, ds + \sum_{j=1}^{p} \int_0^t H_s^{i,j}\, dW_s^i$$

Then, if f is twice differentiable with respect to x and once differentiable with respect to t with continuous partial derivatives in (t,x),

$$\begin{aligned} f(t, X_t^1, .., X_t^n) \;&=\; f(0, X_0^1, .., X_0^n) + \int_0^t \frac{\partial f}{\partial s}(s, X_s^1, .., X_s^n) ds \\ &+ \sum_{i=1}^{n} \int_0^t \frac{\partial f}{\partial X_i}(s, X_s^1, .., X_s^n) dX_i \\ &+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t \frac{\partial^2 f}{\partial X_i \partial X_j}(s, X_s^1, .., X_s^n) d\langle X^i, X^j\rangle_s \end{aligned}$$

where

- $dX_s^i = K_s^i ds + \sum_{j=1}^{p} H_s^{i,j} dW_s^j$
- $d\langle X^i, X^j\rangle_s = \sum_{m=1}^{p} H_s^{i,m} H_s^{j,m} ds.$

Definition 3.1.19 *A T maturity zero-coupon bond is a contract that guarantees its holder the payment of one unit of currency at time T with no intermediate payments. The contract value at time $t < T$ is denoted by $P(t, T)$. Clearly, $P(t, T)$ is equal to 1 for all T.*

Definition 3.1.20 *The bank account/money market account process is defined by*

$$B_t = \exp \int_0^t r_s \, ds$$

with

$$\begin{aligned} dB_t &= r_t B_t \, dt \\ B_0 &= 1 \end{aligned}$$

Definition 3.1.21 *A discount factor $D(t, T)$ between to time instants at time t and T is the amount at time t that is equivalent to one unit of currency payable at time T and is given by*

$$D(t, T) = \frac{B(t)}{B(T)} = \exp(-\int_t^T r(s)ds).$$

Definition 3.1.22 *The simple compounded forward rate contracted at time t for expiry T and maturity S, $t < S < T$ is defined as*

$$L(t; S, T) = -\frac{P(t, T) - P(t, S)}{(T - S)P(t, T)}.$$

Definition 3.1.23 *The simple spot rate for $[T, S]$ is defined as*

$$L(S, T) = -\frac{P(S, T) - 1}{(T - S)P(S, T)}.$$

Definition 3.1.24 *The continuously compounded forward rate contracted at time t for expiry T and maturity S, $t < S < T$, is defined as*

$$R(t; S, T) = -\frac{\log P(t, T) - \log P(t, S)}{T - S}.$$

Definition 3.1.25 *The instantaneous forward rate which maturity T, contracted at time t, is defined as*

$$f(t, T) = \lim_{S \to T} L(t, T, S) = -\frac{\partial \ln P(t, T)}{\partial T}.$$

Definition 3.1.26 *The instantaneous short rate at time t is defined as*

$$r(t) = f(t, t).$$

Definition 3.1.27 *Any simply compounded forward rate spanning a time interval ending in T is a martingale under the T-forward measure for each* $0 \leq u \leq t \leq S \leq T$

$$E[F(t; S, T) \mid \mathcal{F}_t] = L(u; S, T).$$

Definition 3.1.28 *For* $t \leq s \leq T$

$$P(t, T) \quad = \quad P(t, s)\exp(-\int_s^T f(t, u)\, du)$$

and, in particular,

$$P(t, T) \quad = \quad \exp(-\int_t^T f(t, s)\, ds).$$

3.2 Fundamentals of Jump Process

Definition 3.2.1 *Let* $(\tau_i)_{i\geq 0}$ *be a sequence of independent exponential random variables with parameter* $\frac{1}{\lambda}$ *and* $T_n = \sum_{i=1}^{n} \tau_i$ *the process* $(N_t)_{t\geq 0}$ *defined by*

$$N_t = \sum_{n\geq 1} 1_{t\geq T_n}$$

is called a Poisson process with intensity λ*.*

Proposition 3.2.1 (Poisson Process) *Let* $(N_t)_{t\geq 0}$ *be a Poisson process.*

1. *For any* $t > 0$*,* (N_t) *is almost surely finite.*
2. *For any* ω*, the sample path ,* $t \longmapsto N_t(\omega)$ *is piecewise constant and increasing.*
3. *The sample paths* $t \longmapsto N_t$ *are right continuous with left limits (cadlag process).*
4. *For any* $t > 0$*,* $N_{t\_} = N_t$ *with probability 1.*
5. N_t *is continuous in probability,* $\forall t > 0$ *as* $s \to t$*,* $N_s \to N_t$*.*
6. *For any* $t > 0$*,* N_t *follows a Poisson distribution with parameter* λt*,* $\forall n \in \mathbf{N}$

$$P(N_t = n) = \frac{\exp(-\lambda t)(\lambda t)^n}{n!}$$

7. *The characteristic function of N_t is given by*

$$E(e^{iuN_t}) = \exp(\lambda t(e^{iu} - 1)),\ \forall u \in \mathbf{R}$$

8. *N_t has independent increments: for any $t_1 < t_2 < .. < t_n$; $N_{t_n} - N_{t_{n-1}}$, $N_{t_{n-1}} - N_{t_{n-2}}$,..., $N_{t_2} - N_{t_1}$, N_{t_1} are independent random variables.*

9. *The increments of N_t are stationary: for any $t > s$, $N_t - N_s$ has the same distribution of N_{t-s}.*

10. *N_t has the Markov property: $\forall t > s$*

$$E(f(N_t) \mid N_u, u \leq s) = E(f(N_t) \mid N_s).$$

Lemma 3.2.2 *Let X_t be a counting process with stationary independent increments. Then X_t is a Poisson process.*

Theorem 3.2.3 *Let N_t be a Poisson process with intensity λ. Then the compensated Poisson process*

$$M_t = N_t - \lambda t$$

is a martingale.

Proposition 3.2.2 *Let W_t be a Brownian motion and $M_t = N_t - \lambda t$ be a compensated Poisson process relative to the same filtration. Then*

$$[W, M]_t = 0.$$

Definition 3.2.4 *Let be given $E \subset \mathbb{R}^d$. A radon measure on $(E,\mathcal{B})$ is a measure μ such that for every compact measurable set $B \in \mathcal{B}$, $\mu(B) < \infty$.*

Definition 3.2.5 (Poisson Random Measure) *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and μ a Radon measure on $(E, \mathcal{E})$. A Poisson random measure on E with intensity measure μ is an integer valued random measure:*

$$M : \Omega \times \mathcal{E} \to \mathbb{N}$$

such that

1. *For all $w \in \Omega$, $M(w, .)$ is an integer valued radon measure on E: for any bounded measurable $A \in E$, $M(A) < \infty$ is an integer valued random variable.*

2. *For each measurable set $A \in E$ $M(., A) = M(A)$ is a Poisson random variable with parameter $\mu(A)$: $\forall k \in \mathbb{N}$*

$$P(M(A) = k) = \frac{\exp(-\mu(A))(\mu(A))^k}{k!}$$

3. *For disjoint measurable sets $A_1, ..., A_n \in \mathcal{E}$, the variables $M(A_1), ..., M(A_n)$ are independent.*

Theorem 3.2.6 *Let M be a Poisson random measure with intensity μ. Then compensated Poisson random measure*

$$\tilde{M} = M - \mu$$

is a martingale.

Proposition 3.2.3 *Let M be a Poison random measure on $E = [0, T] \times \mathbb{R}^d \setminus \{0\}$ with intensity μ with compensated random measure $\tilde{M} = M - \mu$ and $f : E \to \mathbb{R}^d$. Then the process*

$$\begin{aligned} X_t &= \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} f(s, y)\, \tilde{M}(ds, dy) \\ &= \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} f(s, y)\, M(ds, dy) - \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} f(s, y)\, \mu(ds, dy) \end{aligned}$$

is a martingale.

Definition 3.2.7 (Marked Point Process) *A marked point process on $(\Omega, \mathcal{F}, \mathbf{P})$ is a sequence $(T_n, Y_n)_{n \geq 1}$, where*

1. *$(T_n)_{n \geq 1}$ is an increasing sequence of random times with $T_n \longrightarrow \infty$ almost surely as $n \longrightarrow \infty$.*

2. *Y_n is a sequence of random variables taking values in E.*

3. *The value of Y_n is $\mathcal{F}_{T_n}$ measurable.*

Definition 3.2.8 (Compound Poisson Process) *A compound Poisson process with intensity $\lambda > 0$ and jump size distribution f, is a stochastic Process Q_t defined as*

$$Q_t = \sum_{i=1}^{N_t} Y_i,$$

where jump sizes are independent identically distributed random variables with distribution f and N_t is a Poisson process with intensity λ, independent from $(Y_i)_{i\geq 1}$. The jumps in Q_t occur at the same time as the jumps in N_t.

Definition 3.2.9 (Levy Process) *A stochastic process X_t on $(\Omega, \mathcal{F}, \mathbf{P})$ with values in $\mathbb{R}^d$ such that $X_0 = 0$ is called a Levy process if it satisfies the following prosperities:*

1. *Independent increments: for every increasing sequence of times $t_0, ..., t_n$ the random variables $X_{t_0}, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$.*
2. *Stationary increments: the law of $X_{t+h} - X_t$ does not depend on t.*
3. *Stochastic continuity: for all $\epsilon > 0$, $\lim_{h\to 0} P(|X_{t+h} - X_t| \geq \epsilon) = 0$.*

Definition 3.2.10 *A probability distribution F on $\mathbb{R}^d$ is said to be infinitely divisible if for any integer $n \geq 2$, there exists n i.i.d random variables $Y_1, ..., Y_n$ such that $Y_1 + ... + Y_n$ has distribution F.*

Definition 3.2.11 (Levy Measure) *Let X_t be a Levy process on $\mathbb{R}^d$. The measure ν on $\mathbb{R}^d$ defined by:*

$$\nu(A) = E[number\ of\ \{t \in [0, 1] : \Delta X_t \neq 0, \Delta X_t \in A\}\ ] \quad (A \in \mathcal{B}(\mathbb{R}^d))$$

is called the Levy measure of X: $\nu(A)$ is expected number per unit time of jumps whose size belongs to A.

Proposition 3.2.4 *Let M be a random measure with intensity μ. Then the following formula holds for every measurable set B such that $\mu(B) < \infty$ and for all functions f such that $\int_B \exp(f(x))\mu(dx) < \infty$:*

$$E[\exp(\int_B f(x)M(dx))] = \exp(\int_B (\exp(f(x)) - 1)\mu(dx)).$$

Proposition 3.2.5 (Levy-Ito decomposition) *Let X_t be a Levy process on $\mathbb{R}^d$ and ν its Levy measure.*

1. *ν is a radon measure on $\mathbb{R}^d \setminus \{0\}$ and verifies:*

$$\int_{|x|\leq 1} |x|^2 \nu(dx) < \infty \qquad \int_{|x|\geq 1} \nu(dx) < \infty$$

2. *The Jump measure of X, denoted by μ, is a Poisson random measure on $[0, \infty[\times\mathbb{R}^d$ with intensity $\nu(dx)(dt)$.*

3.

$$X_t \;=\; \gamma_t + B_t + X_t^l + \lim_{\epsilon\to 0} \tilde{X}_t^\epsilon \tag{3.2}$$

where

$$X_t^l \;=\; \int_{|x|\geq 1} x\mu(ds, dx)$$

$$\tilde{X}_t^\epsilon \;=\; \int_{\epsilon\leq|x|<1} x(\mu(ds, dx) - \nu(dx)ds).$$

The terms in (3.2) are independent and convergence in the last term is almost sure and uniform in t on [0,T].

Definition 3.2.12 *Let X_t be a Levy process with the Poisson random measure μ on $E = [0, T] \times \mathbb{R}^d \setminus \{0\}$ and N_t Poisson process.*

$$X_t \;=\; X_0 + \int_0^t \alpha(s)ds + \int_0^t \beta(s)dW(s) + \int_0^t \int_E \gamma(s, y)\,\mu(ds, dy).$$

Given a $f \in C^{1,2}(\mathbb{R}^2)$, ito formula defined by

$$\begin{aligned} df(t, X_t) \;=\;& f_t(t, X_t)dt + f_X(t, X_t)\alpha(t)dt + \frac{1}{2} f_{XX}(t, X_t)\beta(t)^2 dt \\ &+ f_X(t, X_{t_-})\beta(t)dW(t) + \int_E [f(t, X_t + \gamma(t, y)) - f(t, X_{t_-})]\mu(dt, dy) \end{aligned}$$

3.3 Change of Measure

Definition 3.3.1 *A numeraire is any positive non-dividend paying asset.*

Theorem 3.3.2 *Let N be a positive non-dividend paying asset, either primary or derivative, in the multidimensional market. Then there exist a vector volatility process*

$$v(t) = (v_1(t), ..., v_d(t))$$

such that

$$dN(t) = R(t)N(t)dt + N(t)v(t)\, d\tilde{W}(t).$$

This equation is equivalent to each of the equations

$$d(D(t)N(t)) \quad = \quad D(t)N(t)v(t)\, d\tilde{W}(t)$$

$$D(t)N(t) \quad = \quad N(0)\exp(\int_0^t v(u)\, d\tilde{W}(u) - \frac{1}{2}\int_0^t \|v(u)\|^2\, du)$$

$$N(t) \quad = \quad N(0)\exp(\int_0^t v(u)\, d\tilde{W}(u) + \int_0^t (R(u) - \frac{1}{2}\|v(u)\|^2)\, du).$$

In the other words under the risk neutral measure every asset has a mean return equal to the interest rate.

According to multidimensional Girsanov theorem we can use the volatility vector of $N(t)$ to change the measure:

$$\tilde{W}_j^N(t) = -\int_0^t v_j(u)\, du + \tilde{W}_j(t),\ j = 1, ..., d$$

and a new probability measure:

$$\tilde{P}^N(A) = -\int_A D(t)N(t)d\tilde{P} \ \ for \ \ all \ \ A \in \mathcal{F}.$$

Theorem 3.3.3 *Let S(t) and N(t) be the prices of two assets denominated in a common currency and let* $\sigma(t) = (\sigma_1(t), .., \sigma_d(t))$ *and* $v(t) = (v_1(t), ..., v_d(t))$ *denote their respective volatility vector processes:*

$$d(D(t)S(t)) = D(t)S(t)\sigma(t)\, d\tilde{W}(t)$$

$$d(D(t)N(t)) = D(t)N(t)v(t)\, d\tilde{W}(t).$$

Take N(t) as a numeraire so that the price of $S(t)$ *becomes* $S^N(t) = \frac{S(t)}{N(t)}$*. Under the measure* $\tilde{P}^N$ *the process* $S^N(t)$ *is a martingale. Moreover*

$$dS^N(t) = S^N(t)(\sigma(t) - v(t))\, d\tilde{W}^N(t).$$

Proposition 3.3.1 *Assume that there exist a numeraire N and a probability measure Q^N, equivalent to the initial Q_0, such that the price of any traded asset X (with no-dividend payment) relative to N is a martingale under Q^N, i.e.,*

$$\frac{X_t}{N_t} = E^N(\frac{X_T}{N_T} \mid \mathcal{F}_t)$$

Let U be a arbitrary numeraire. Then exist a probability measure Q^U, equivalent to the initial Q_0, such that the price of any attainable claim Y normalized by U is a martingale under Q^U, i.e.,

$$\frac{Y_t}{U_t} = E^U(\frac{Y_T}{U_T} \mid \mathcal{F}_t).$$

Moreover the Radon-Nikodym derivative defining the measure Q^U is given by

$$\frac{dQ^U}{dQ^N} = \frac{U_T N_0}{U_0 N_T}.$$

Proposition 3.3.2 *Let Q^{IP} and Q^n are probability measures*

$$Z_T = \frac{Q^{T-IP}}{Q^n} \, on \, \mathcal{F}_t$$

where

$$Z_T = \frac{B^{IP}(t)}{B^n(t)} \frac{B^n(0)}{B^{IP}(0)}$$

then is Q^{IP} a martingale measure for the numeraire B^{IP}. Here $B^{IP}(t) = I(t)B^{IP}(t)$.

Proposition 3.3.3 *Let Q^r and Q^n are probability measures and $S^n(t)$ is arbitrage free process in the nominal economy*

$$Z_T = \frac{Q^r}{Q^n} \, on \, \mathcal{F}_t$$

where

$$Z_T = \frac{B^r(t)I(t)}{B^n(t)} \frac{B^n(0)}{B^r(0)I(0)}$$

then is Q^r a martingale measure for the numeraire $B^r(t)$ and $\frac{S^r(t)}{B^r(t)}$ is a Q^r martingale. Here, $S^n(t) = S^r(t)I(t)$.

Proposition 3.3.4 *Let $Q^{T,r}$ and Q^n are probability measures and $S^n(t)$ is arbitrage free process in the nominal economy*

$$Z_T = \frac{Q^r}{Q^n} \, on \, \mathcal{F}_t$$

where

$$Z_T = \frac{P^r(t,T)I(t)}{B^n(t)} \frac{B^n(0)}{P^r(0,T)I(0)}$$

then $Q^{T,r}$ is a martingale measure for the numeraire $P^r(t,T)$, and $\frac{S^r(t)}{P^r(t,T)}$ is a $Q^{T,r}$ martingale.

Theorem 3.3.4 (Multi Currency Change of Numeraire) *Let X(t) an exchange rate between domestic and foreign economies and N(t) is a numeraire in the domestic currency with martingale measure Q^N, M(t) is numeraire in the foreign currency with martingale measure Q^M. Then Radon-Nikodyn derivative looks as follows on $\mathcal{F}_T$*

$$\frac{dQ^N}{dQ^M} = \frac{N(T)}{X(T)M(T)} \frac{X(0)M(0)}{N(0)}.$$

Theorem 3.3.5 (Bayes Formula) *Let X_t be a stochastic process on $(\Omega, \mathcal{A}, \mathbf{P})$ and P and Q are probability measures on this space and $\mathcal{E}$ be σ- algebra with $\mathcal{E} \subseteq \mathcal{F}$. Since, the Radon-Nikodyn derivative is*

$$Z = \frac{dQ}{dP} \, on \, \mathcal{F}_t$$

The Bayes formula as follows

$$E^Q[X \mid \mathcal{E}_t] = \frac{E^P[Z X \mid \mathcal{E}_t]}{E^P[Z \mid \mathcal{E}_t]}.$$

CHAPTER 4

HEATH JARROW MORTON FRAMEWORK AND FOREIGN CURRENCY ANALOGY

In this chapter, we give Heath Jarrow Morton (HJM) framework and foreign currency analogy that is known as the fundamental of pricing inflation indexed derivatives.

4.1 Heath Jarrow Morton Framework

It is a framework to be used for modeling the dynamics of the instantaneous forward rates. The main insight of this framework is the relationship between the drift and volatility parameters of the forward rate dynamics for no arbitrage opportunity market.

HJM model represents the yield curve in terms of the forward rates, $f(t,T)$, which can be locked in at time t for borrowing at time $T \geq t$. For fixed t, the function $T \longmapsto f(t,T)$ is defined as the forward rate curve. The stochastic structure of the model is based on the evolution of the forward rate curve.

HJM model has a necessary and sufficient condition for a model driven by Brownian motion to be free of arbitrage. So, every Brownian motion driven model can be shown in HJM model with no arbitrage condition. In this chapter, we study HJM model with its drift conditions under no-arbitrage assumption and risk neutral probability measure.

4.1.1 No Arbitrage Condition

Assume a continuous trading economy with a trading interval $0 \leq T \leq \tau$ for a fixed $\tau > 0$. A probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with the probability measure P and filtration $\{\mathcal{F}_t \colon t \in [0,\tau]\}$.

$P(t, T)$ denotes the time t price of the T maturity bond for all $T \in [0, \tau]$ and $t \in [0, T]$. The forward rate, $f(t, T)$, satisfies the following equation:

$$f(t,T) \;=\; f(0,T) + \int_0^t \alpha(s,T)\,ds + \int_0^t \sigma(s,T)\,dW(s), \tag{4.1}$$

where $\{f(0,T)\colon\ T \in [0,\tau]\}$ is a fixed, nonrandom initial condition, $\alpha(t,T)$ and $\sigma(t,T)$ are adapted processes. The advantage of the initial condition is to provide a perfect fit between observed and theoretical bond at $t = 0$. The following relation between zero coupon bond prices and forward rates holds.

$$P(t,T) \;=\; \exp(-\int_t^T f(t,s)ds). \tag{4.2}$$

From the equation (4.1), we can reduce the dynamics of the bond prices given by the equation (4.2). Firstly, we compute the differential of $-\int_t^T f(t,s)ds$:

$$d(-\int_t^T f(t,s)ds) \;=\; f(t,t)dt - \int_t^T df(t,s)ds.$$

We know $r(t) = f(t,t)$ and

$$df(t,T) \;=\; \alpha(t,T)\,dt + \sigma(t,T)\,dW(t),\;\; 0 \le t \le T.$$

So, we get

$$d(-\int_t^T f(t,s)ds) \;=\; r(t)dt - \int_t^T [\alpha(t,s)\,dt + \sigma(t,s)\,dW(t)]ds.$$

Let define

$$\tilde{\alpha}(t,T) \;=\; \int_t^T \alpha(t,s)ds$$

$$\tilde{\sigma}(t,T) \;=\; \int_t^T \sigma(t,s)ds$$

The formula becomes

$$d(-\int_t^T f(t,s)\,ds) \;=\; r(t)dt - \tilde{\alpha}(t,T)\,dt - \tilde{\sigma}(t,T)\,dW(t).$$

We want to find the dynamics of the bond price given in the equation (4.2). Now, we use Ito-Doeblin formula according to $g(x) = e^x$, $g'(x) = e^x$, $g''(x) = e^x$ and

$$P(t,T) \;=\; g(-\int_t^T f(t,s)ds).$$

Then,

$$
\begin{aligned}
dP(t,T) &= g'(-\int_t^T f(t,s)\,ds)\,d(-\int_t^T f(t,s)ds) \\
&\quad +\frac{1}{2}g''(-\int_t^T f(t,s)\,ds)\,[\,d(-\int_t^T f(t,s)\,ds)\,]^2 \\
&= P(t,T)\,[\,r(t)dt - \tilde{\alpha}(t,T)\,dt - \tilde{\sigma}(t,T)\,dW(t)\,] \\
&\quad +\frac{1}{2}P(t,T)\tilde{\sigma}(t,T)^2 dt \\
&= P(t,T)([\,r(t) - \tilde{\alpha}(t,T) + \frac{1}{2}\tilde{\sigma}(t,T)^2\,]\,dt - \tilde{\sigma}(t,T)\,dW(t)). \qquad (4.3)
\end{aligned}
$$

After derivation of the bond price dynamics we can turn the drift condition derivation. The drift condition gives us the relationship between the drift $\alpha(t,T)$ and the volatility $\sigma(t,T)$ of the forward rate dynamics under no arbitrage assumption. There exits a probability measure $\hat{\mathrm{P}}$ under which discounted asset prices are martingale in the arbitrage free market. Give the discounted bond price, $\tilde{P}(t,T)$, with

$$
\tilde{P}(t,T) = P(t,T)\,exp(-\int_0^t r(s)\,ds),
$$

using integration by parts formula yields

$$
\begin{aligned}
d\tilde{P}(t,T) &= -P(t,T)\,exp(-\int_0^t r(s)\,ds)\,r(t)dt \\
&\quad +exp(-\int_0^t r(s)\,ds)\,dP(t,T) \\
&= -\tilde{P}(t,T)r(t)dt \\
&\quad +exp(-\int_0^t r(s)\,ds)\,P(t,T)\,[\,r(t) - \tilde{\alpha}(t,T) + \frac{1}{2}\tilde{\sigma}(t,T)^2\,]\,dt \\
&\quad -exp(-\int_0^t r(s)\,ds)\,P(t,T)\,\tilde{\sigma}(t,T)\,dW(t) \\
&= \tilde{P}(t,T)[-\tilde{\alpha}(t,T)\,dt + \frac{1}{2}\tilde{\sigma}(t,T)^2\;dt - \tilde{\sigma}(t,T)\,dW(t)].
\end{aligned}
$$

By Girsanov theorem, we change the probability measure to equivalent probability measure $\hat{\mathrm{P}}$ under which

$$
\hat{W}(t) = \int_0^t \lambda(s)ds + W(t),
$$

where $\lambda(s)$ is the market price of risk. So, discounted asset dynamics can be rewrite as

$$d\tilde{P}(t,T) \quad = \quad -\tilde{P}(t,T)\,\tilde{\sigma}(t,T)\,d\hat{W}(t).$$

For the market price of risk, $\lambda(s)$, we must solve the subsequent equation

$$[-\tilde{\alpha}(t,T)\,dt + \frac{1}{2}\tilde{\sigma}(t,T)^2\ \ dt - \tilde{\sigma}(t,T)\,dW(t) \quad = \quad -\tilde{\sigma}(t,T)\,[dW(t) + \lambda(t)\,dt].$$

or

$$[-\tilde{\alpha}(t,T) + \frac{1}{2}\tilde{\sigma}(t,T)^2\ ]\,dt \quad = \quad -\tilde{\sigma}(t,T)\,\lambda(t)\,dt$$

To solve $\lambda(t)$; $\tilde{\sigma}(t,T)$ and $\tilde{\alpha}(t,T)$ will be differentiated with respect to T;

$$\frac{\partial}{\partial T}\tilde{\alpha}(t,T) = \alpha(t,T), \quad \frac{\partial}{\partial T}\tilde{\sigma}(t,T) = \sigma(t,T)$$

Hence,

$$-\alpha(t,T) + \tilde{\sigma}(t,T)\sigma(t,T) \quad = \quad -\sigma(t,T)\,\lambda(t)$$

or

$$\alpha(t,T) \quad = \quad \sigma(t,T)\,[\,\tilde{\sigma}(t,T) + \lambda(t)\,].$$

So, we finally find the drift term of the forward rate dynamics can be written in terms of the volatility term under no arbitrage assumption.

Theorem 4.1.1 (HJM Drift Condition) *A term structure model for a zero coupon bond prices of all maturities* $[0,\tau]$ *and driven by a Brownian motion does not admit arbitrage if there exist a process* $\lambda(t)$ *such that*

$$\alpha(t,T) \quad = \quad \sigma(t,T)\,[\,\tilde{\sigma}(t,T) + \lambda(t)\,].$$

Here $\sigma(t,T)$ *and* $\alpha(t,T)$ *are drift and diffusion, respectively of the forward rate*

$$df(t,T) \quad = \quad \alpha(t,T)\,dt + \sigma(t,T)\,dW(t),$$

where $\int_t^T \sigma(t,s)ds = \tilde{\sigma}(t,T)$ *and* $\lambda(t)$ *is the market price of risk.*

4.1.2 Under Risk Neutral Measure

Under risk neutral measure, the local rate of return is equal to the short rate. In equation (4.3),

$$r(t) - \tilde{\alpha}(t,T) + \frac{1}{2}\tilde{\sigma}(t,T)^2 \quad = \quad r(t).$$

Hence, we get

$$\tilde{\alpha}(t,T) \quad = \quad \frac{1}{2}\tilde{\sigma}(t,T)^2$$

using definition of $\tilde{\alpha}, \tilde{\sigma}$ and differentiation with respect to T, which gives

$$\alpha(t,T) \quad = \quad \sigma(t,T)\;\tilde{\sigma}(t,T).$$

Hence, we get the drift term of the forward the forward rate under risk neutral measure. The following theorem gives the term structure evolution under this measure.

Theorem 4.1.2 (Term-structure evolution under risk neutral measure) *In a term structure model satisfying the HJM no arbitrage condition of Theorem (4.1.1), the forward rates evolve according to the equation*

$$df(t,T) \quad = \quad \sigma(t,T)\;\tilde{\sigma}(t,T)dt + \sigma(t,T)\,d\hat{W}(t)$$

and the zero coupon bond prices evolve according to the equation

$$dP(t,T) \quad = \quad r(t)P(t,T)dt - \tilde{\sigma}(t,T)P(t,T)\,d\hat{W}(t) \tag{4.4}$$

where $\hat{W}(t)$ is a Brownian motion under a risk neutral measure $\hat{\mathrm{P}}$. Here, $r(t) = f(t,t)$ is the short rate. The discounted bond prices satisfy

$$d\tilde{P}(t,T) \quad = \quad -\tilde{P}(t,T)\,\tilde{\sigma}(t,T)\,d\hat{W}(t).$$

The solution to the stochastic differential equation (4.4) is

$$\begin{aligned} P(t,T) &= P(0,T)exp\{\int_0^t r(u)\,du - \int_0^t \tilde{\sigma}(u,T)\,d\hat{W}(u) - \frac{1}{2}\int_0^T \tilde{\sigma}(u,T)^2\,du\} \\ &= \frac{P(0,T)}{B(t)}exp\{-\int_0^t \tilde{\sigma}(u,T)\,d\hat{W}(u) - \frac{1}{2}\int_0^T \tilde{\sigma}(u,T)^2\,du\}. \end{aligned}$$

As we mentioned before, every Brownian motion driven model can be shown in HJM model with no arbitrage condition. To better understand this, we will give an example with the one factor Hull-White model. The interest rate dynamics of the model are of the form

$$dr(t) = (a(t) - b(t)r)dt + \sigma(t,T)\, d\hat{W}(t),$$

where $\hat{W}(t)$ is a Brownian motion under a risk neutral measure and $a(t)$, $b(t)$, $\sigma(t)$ are nonrandom positive functions. The zero coupon bond price in this model is given by

$$P(t,T) = e^{-r(t)C(t.T)-A(t,T)},$$

where

$$C(t,T) = \int_t^T exp(-\int_t^s b(u)\, du)\, ds,$$

$$A(t,T) = \int_t^T (a(s)C(s,T) - \frac{1}{2}\sigma^2(s)C^2(s,T))\, ds$$

By using forward rate definition we get

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T} = r(t)\frac{\partial}{\partial T}C(t,T) + \frac{\partial}{\partial T}A(t,T)$$

and with $C'(t,T)$, $A'(t,T)$ denoting derivatives with respect to t, we get the forward rate differential:

$$\begin{aligned} df(t,T) &= \frac{\partial}{\partial T}C(t,T)dr(t) + r(t)\frac{\partial}{\partial T}C'(t,T)dt + \frac{\partial}{\partial T}A'(t,T)dt \\ &= \frac{\partial}{\partial T}C(t,T)\{(a(t) - b(t)r)dt + \sigma(t)\, d\hat{W}(t)\} \\ &\quad + r(t)\frac{\partial}{\partial T}C'(t,T)dt + \frac{\partial}{\partial T}A'(t,T)dt \\ &= [\frac{\partial}{\partial T}C(t,T)\{(a(t) - b(t)r) + r(t)\frac{\partial}{\partial T}C'(t,T) + \frac{\partial}{\partial T}A'(t,T)]dt \\ &\quad + \frac{\partial}{\partial T}C(t,T)\sigma(t)\, d\hat{W}(t) \end{aligned} \tag{4.5}$$

Hence the HJM no arbitrage condition becomes,

$$\frac{\partial}{\partial T}C(t,T)(a(t) - b(t)r) + r(t)\frac{\partial}{\partial T}C'(t,T) + \frac{\partial}{\partial T}A'(t,T)$$

$$
\begin{aligned}
&= \frac{\partial}{\partial T}C(t,T)\sigma(t)\int_t^T \frac{\partial}{\partial u}C(t,u)\sigma(u)\,du \\
&= \frac{\partial}{\partial T}C(t,T)\sigma(t)[C(t,T)-C(t,t)]\sigma(t) \\
&= [\frac{\partial}{\partial T}C(t,T)]C(t,T)\sigma(t)^2.
\end{aligned}
$$

4.2 Foreign Currency Analogy

For modeling inflation, there are two types of approaches. the first one is the *econometrics models*. These type of models forecast the inflation rate using time series data. The second one is *option pricing based models*. These type of models are used to price inflation indexed derivatives. The most well-know application of pricing inflation indexed derivatives is the study of Jarrow and Yıldırım. They used foreign currency analogy.

In foreign currency analogy, nominal dollars correspond to domestic currency, real dollars correspond to foreign currency and inflation index correspond to exchange rate between two economies under no-arbitrage assumption. In this section, we will give basic rules of foreign currency analogy that is based on Jarrow and Yıldırım model.

<u>Definitions and Notations:</u>

- $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space with filtration $\{\mathcal{F}_t : t \in [0,T]\}$ and objective probability $\mathbf{P}$.
- Nominal is denoted by n, real is denoted by r and inflation is denoted by I. There are three Brownian motions nominal, real and inflation $(W^n(t), W^r(t), W^I(t))$ with correlations

$$
\begin{aligned}
dW^n(t)dW^r(t) &= \rho_{nr}dt \\
dW^n(t)dW^I(t) &= \rho_{nI}dt \\
dW^r(t)dW^I(t) &= \rho_{rI}dt
\end{aligned}
$$

- $P^r(t,T)$ is the price of a real zero-coupon bond at time t with maturity T.
- $P^n(t,T)$ is the price of a nominal zero-coupon bond at time t with maturity T.

- I(t) is the inflation index at time t with dynamics;

$$\frac{dI(t)}{I(t)} = \mu^I(t)dt + \sigma^I(t)dW^I(t)$$

- $f^i(t,T)$ is the forward rates at time t for date T, for $i = r, n$, with

$$df^i(t,T) = \alpha^i(t,T)dt + \sigma^i(t,T)dW^i(t)$$

- $r^i(t)$ is the spot rate at time t with $r^i(t) = f^i(t,t)$

- $B^i(t)$ is the money market account

- The price in dollars of a real zero-coupon bond is denoted by $P^{TIPS}(t,T)$, where

$$P^{TIPS}(t,T) = I(t)P^r(t,T). \tag{4.6}$$

Under these definitions, an example to better understand the foreign currency analogy will be given below.
If $I(t)$ is the price of a Hamburger, then we have the followings:

A nominal bond:

- pays out 1 dollar at maturity

- $P^n(t,T)$ is the price of a nominal bond in dollar

A hamburger indexed bond:

- pays out a dollar amount that is enough to buy 1 Hamburger at maturity

- $P^{TIPS}(t,T)$ is the price of a Hamburger-inflation protected bond in dollar

A real bond:

- Pays out 1 Hamburger at maturity

- $P^r(t,T)$ is the price of a real bond in Hamburgers

According to these definitions and notations, Jarrow and Yıldırım model continues with two important propositions. The first one gives the drift conditions under no-arbitrage assumption.

Proposition 4.2.1 *In the arbitrage free market* $\frac{P^n(t,T)}{B^n(t)}$, $\frac{I(t)P^r(t,T)}{B^n(t)}$ *and* $\frac{I(t)B^r(t)}{B^n(t)}$ *are martingales if and only if the following conditions hold.*

$$\alpha^n(t,T) = \sigma^n(t,T)(\int_t^T \sigma^n(t,s)ds - h^n(t)) \tag{4.7}$$

$$\alpha^r(t,T) = \sigma^r(t,T)(\int_t^T \sigma^r(t,s)ds - \sigma^I(t)\rho_{rI} - h^r(t)) \tag{4.8}$$

$$\mu^I(t) = r^n(t) - r^r(t) - \sigma^I(t)h^I(t) \tag{4.9}$$

where

$$\tilde{W}^n(t) = W^n(t) - \int_0^t h^n(s)d(s)$$
$$\tilde{W}^r(t) = W^r(t) - \int_0^t h^r(s)d(s)$$
$$\tilde{W}^I(t) = W^I(t) - \int_0^t h^I(s)d(s).$$

Proof. Let $\frac{P^n(t,T)}{B^n(t)}$ be martingale under $\tilde{P}$ measure. In the previous section we have already found the dynamics of $P^n(t,T)$ as follows

$$dP^n(t,T) \quad = \quad P^n(t,T)[\ r^n(t) - \tilde{\alpha^n}(t,T) + \frac{1}{2}\tilde{\sigma^n}(t,T)^2\ ]\ dt - \tilde{\sigma^n}(t,T)\, dW^n(t). \tag{4.10}$$

The money market account $B^n(t)$ has the dynamics

$$dB^n(t) = B^n(t)r^n(t)dt$$

Then,

$$\begin{aligned} d(\frac{P^n(t,T)}{B^n(t)}) &= d((P^n(t,T)B^n(t)^{-1}) \\ &= dP^n(t,T)B^n(t)^{-1} + P^n(t,T)dB^n(t)^{-1} + d\langle P^n, (B^n)^{-1}\rangle_t \\ &= \frac{P^n(t,T)}{B^n(t)}[(-\tilde{\alpha}^n(t,T) + \frac{1}{2}\tilde{\sigma}^n(t,T)^2)dt - \tilde{\sigma}^n(t,T)dW^n(t)]. \end{aligned}$$

To change the measure from objective P measure to martingale $\tilde{P}$ measure, we use $h^n(t)$ as the market price of risk as follows

$$d\tilde{W}^n(t) = dW^n(t) - h^n(t)dt.$$

Then the new dynamics is equal to

$$d(\frac{P^n(t,T)}{B^n(t)}) \quad = \quad -\frac{P^n(t,T)}{B^n(t)}\tilde{\sigma}^n(t,T)d\tilde{W}^n(t).$$

Hence the subsequent equation must be satisfied

$$(-\tilde{\alpha}^n(t,T) + \frac{1}{2}\tilde{\sigma}^n(t,T)^2)dt - \tilde{\sigma}^n(t,T)dW^n(t) \quad = \quad -\tilde{\sigma}^n(t,T)d\tilde{W}^n(t).$$

Since

$$\tilde{\alpha}^n(t,T) \quad = \quad \int_t^T \alpha^n(t,s)ds$$

$$\tilde{\sigma}^n(t,T) \quad = \quad \int_t^T \sigma^n(t,s)ds$$

and taking the derivative with respect to T gives the result

$$\alpha^n(t,T) = \sigma^n(t,T)(\int_t^T \sigma^n(t,s)ds - h^n(t)).$$

Let $\frac{I(t)B^r(t)}{B^n(t)}$ be martingale under $\tilde{P}$ measure. Since

$$d(B^r(t)B^n(t)^{-1}) = (B^r(t)B^n(t)^{-1})[r^r(t) - r^n(t)]dt.$$

Under the objective measure P, we get

$$\begin{aligned} d(\frac{I(t)B^r(t)}{B^n(t)}) \quad &= \quad dI(t)B^r(t)B^n(t)^{-1} + I(t)d(B^r(t)B^n(t)^{-1}) + d\langle I, B^r(B^n)^{-1}\rangle_t \\ &= \quad \frac{I(t)B^r(t)}{B^n(t)}[(\mu^I(t) + r^r(t) - r^n(t))dt + \sigma^I(t)dW^I(t)]. \end{aligned}$$

To change the measure from objective P measure to martingale $\tilde{P}$ measure, we use $h^I(t)$ as the market price of risk as follows

$$d\tilde{W}^I(t) = dW^I(t) - h^I(t)dt.$$

Then the new dynamics is equal to

$$d(\frac{I(t)B^r(t)}{B^n(t)}) \quad = \quad \frac{I(t)B^r(t)}{B^n(t)}\sigma^I(t)d\tilde{W}^I(t).$$

Hence the subsequent equation must be satisfied:

$$(\mu^I(t) + r^r(t) - r^n(t))dt + \sigma^I(t)dW^I(t) = \sigma^I(t)d\tilde{W}^I(t).$$

So, it gives the result

$$\mu^I(t) = r^n(t) - r^r(t) - \sigma^I(t)h^I(t).$$

Let $\frac{I(t)P^r(t,T)}{B^n(t)}$ be martingale under $\tilde{P}$ measure. We know that we have already found the dynamics of $P^r(t,T)$ in the previous section

$$dP^r(t,T) = P^r(t,T)[r^r(t) - \tilde{\alpha}^r(t,T) + \frac{1}{2}\tilde{\sigma}^r(t,T)^2]dt - P^r(t,T)\tilde{\sigma}^r(t,T)dW^r(t)$$

so that by the integration by parts formula we obtain

$$\begin{aligned} d(I(t)P^r(t,T)) &= dI(t)P^r(t,T) + dP^r(t,T)I(t) + d\langle I, P^r\rangle_t \\ &= I(t)P^r(t,T)[\ \mu^I(t) + r^r(t) - \tilde{\alpha}^r(t,T) + \frac{1}{2}\tilde{\sigma}^r(t,T)^2 \\ &\quad -\sigma^I(t)\tilde{\sigma}^r(t,T)\rho_{rI}\ ]dt \\ &\quad -I(t)P^r(t,T)[\ \tilde{\sigma}^r(t,T)dW^r(t) - \sigma^I(t)dW^I(t)\ ]. \end{aligned} \tag{4.11}$$

and, then,

$$\begin{aligned} d(\frac{I(t)P^r(t,T)}{B^n(t)}) &= \frac{I(t)P^r(t,T)}{B^n(t)}[\mu^I(t) + r^r(t) - \tilde{\alpha}^r(t,T) - r^n(t) \\ &\quad -\sigma^I(t)\tilde{\sigma}^r(t,T)\rho_{rI} + \frac{1}{2}\tilde{\sigma}^r(t,T)^2]dt \\ &\quad -\frac{I(t)P^r(t,T)}{B^n(t)}[\tilde{\sigma}^r(t,T)dW^r(t) - \sigma^I(t)dW^I(t)] \end{aligned}$$

By using (4.9), we finally get:

$$\begin{aligned} d(\frac{I(t)P^r(t,T)}{B^n(t)}) &= \frac{I(t)P^r(t,T)}{B^n(t)}[-\tilde{\alpha}^r(t,T) - \sigma^I(t)h^I(t) \\ &\quad +\frac{1}{2}\tilde{\sigma}^r(t,T)^2 - \sigma^I(t)\tilde{\sigma}^r(t,T)\rho_{rI}]dt \\ &\quad +\frac{I(t)P^r(t,T)}{B^n(t)}[-\tilde{\sigma}^r(t,T)dW^r(t) + \sigma^I(t)dW^I(t)]. \end{aligned}$$

To change the measure from objective P measure to martingale $\tilde{P}$ measure, we use $h^r(t)$ and $h^I(t)$ as the market price of risks as follows

$$d\tilde{W}^r(t) = dW^r(t) - h^r(t)dt$$

$$d\tilde{W}^I(t) = dW^I(t) - h^I(t)dt.$$

Then the new dynamics is equal to

$$d(\frac{I(t)P^r(t,T)}{B^n(t)}) = \frac{I(t)P^r(t,T)}{B^n(t)}(-\tilde{\sigma}^r(t,T)d\tilde{W}^r(t) + \sigma^I(t)d\tilde{W}^I(t)).$$

Hence, the subsequent equation must be satisfied

$$[-\tilde{\alpha}^r(t,T) - \sigma^I(t)h^I(t) + \frac{1}{2}\tilde{\sigma}^r(t,T)^2 - \sigma^I(t)\tilde{\sigma}^r(t,T)\rho_{rI}]dt - \tilde{\sigma}^r(t,T)dW^r(t) + \sigma^I(t)dW^I(t)$$

$$= -\tilde{\sigma}^r(t,T)d\tilde{W}^r(t) + \sigma^I(t)d\tilde{W}^I(t)$$

Using the definitions of $\tilde{\alpha}$ and $\tilde{\sigma}$, and taking derivative with respect to T we get the result

$$\alpha^r(t,T) = \sigma^r(t,T)(\int_t^T \sigma^r(t,s)ds - \sigma^I(t)\rho_{rI} - h^r(t)).$$

■

Proposition 4.2.2 *Under risk neutral measure the following dynamics are satisfied:*

$$\textit{1)} \quad df^n(t,T) = \sigma^n(t,T)\int_t^T \sigma^n(t,s)ds + \sigma^n(t,T)d\tilde{W}^n(t) \tag{4.12}$$

$$\textit{2)} \quad df^r(t,T) = \sigma^r(t,T)(\int_t^T \sigma^r(t,s)ds - \rho_{rI}\sigma^I(t))dt + \sigma^r(t,T)d\tilde{W}^r(t) \tag{4.13}$$

$$\textit{3)} \quad \frac{dI(t)}{I(t)} = [r^n(t) - r^r(t)]\,dt + \sigma^I(t)d\tilde{W}^I(t) \tag{4.14}$$

$$\textit{4)} \quad \frac{dP^n(t,T)}{P^n(t,T)} = r^n(t)dt - (\int_t^T \sigma^n(t,s)ds)d\tilde{W}^n(t) \tag{4.15}$$

$$\textit{5)} \quad \frac{dP^{TIPS}(t,T)}{P^{TIPS}(t,T)} = r^n(t)dt + \sigma^I(t)d\tilde{W}^I(t) - (\int_t^T \sigma^r(t,s)ds)d\tilde{W}^r(t) \tag{4.16}$$

$$\textit{6)} \quad \frac{dP^r(t,T)}{P^r(t,T)} = [r^r(t) - \rho_{rI}\sigma^I(t)\int_t^T \sigma^r(t,s)ds]dt - (\int_t^T \sigma^r(t,s)ds)d\tilde{W}^r(t) \tag{4.17}$$

Proof. We know

$$df^n(t,T) = \alpha^n(t,T)dt + \sigma^n(t,T)dW^n(t)$$

$$df^r(t,T) = \alpha^r(t,T)dt + \sigma^r(t,T)dW^r(t)$$

$$\frac{dI(t)}{I(t)} = \mu^I dt + \sigma^I(t)dW^I(t)$$

Substituting α^n, α^r and μ^I which are found in the previous proposition into these equations and then using

$$d\tilde{W}(t) = dW(t)$$

since under risk neutral measure the local rate of return is equal to short rate, finally we will get the equations in (1), (2) and (3).

The dynamics $P^n(t,T)$ which is found in the previous proposition

$$dP^n(t,T) = P^n(t,T)[(\, r^n(t) - \tilde{\alpha}^n(t,T) + \frac{1}{2}\tilde{\sigma}^n(t,T)^2\,)\, dt - \tilde{\sigma}(t,T)\, dW(t)]. \quad (4.18)$$

Integrating subsequent equation from t to T

$$\alpha^n(t,T) = \sigma^n(t,T)(\int_t^T \sigma^n(t,s)ds - h^n(t))$$

using the definitions $\tilde{\alpha}^n, \tilde{\sigma}^n$ and $h^n(t) = 0$ yields

$$\tilde{\alpha}^n(t,T) = \frac{1}{2}\tilde{\sigma}^n(t,T)^2$$

and we know from the risk neutral measure: $d\tilde{W}(t) = dW(t)$.

These two equations give (4).

The dynamics $P^r(t,T)$ which is found in the previous proposition

$$dP^r(t,T) = P^r(t,T)[r^r(t) - \tilde{\alpha}^r(t,T) + \frac{1}{2}\tilde{\sigma}^r(t,T)^2]dt - P^r(t,T)\tilde{\sigma}^r(t,T)dW^r(t)$$

Integrating subsequent equation from t to T gives

$$\alpha^r(t,T) = \sigma^r(t,T)(\int_t^T \sigma^r(t,s)ds - \sigma^I(t)\rho_{rI} - h^r(t))$$

and using the definitions $\tilde{\alpha}^n, \tilde{\sigma}^n$ and $h^r(t) = 0$ gives

$$\tilde{\alpha}^r(t,T) = \frac{1}{2}\tilde{\sigma}^r(t,T)^2 - \tilde{\sigma}^r(t,T)\sigma^I(t)\rho_{rI}.$$

From the risk neutral measure we know $d\tilde{W}(t) = dW(t)$.

These two equations give (6).

The last dynamics, $P^{TIPS}(t,T)$, can be found using $P^{TIPS}(t,T) = P^r(t,T)I(t)$ with (3) and (6).

By integration by parts formula, we obtain

$$\begin{aligned}
\frac{dP^{TIPS}(t,T)}{P^{TIPS}(t,T)} &= [(r^n(t) - r^r(t))dt + \sigma^I(t)d\tilde{W}^I(t)] \\
&\quad + [(r^r(t) - \rho_{rI}\sigma^I(t)\tilde{\sigma}^r(t,T))dt - \tilde{\sigma}^r(t,T)d\tilde{W}^r(t)] \\
&\quad - \sigma^I(t)\tilde{\sigma}^r(t,T)\rho_{rI}dt \\
&= r^n(t)dt + \sigma^I(t)d\tilde{W}^I(t) - (\int_t^T \sigma^r(t,s)ds)d\tilde{W}^r(t).
\end{aligned}$$

■

CHAPTER 5

THE EXTENDED HEATH JARROW MORTON FRAMEWORK

Models included jump processes provide more realistic representation of price dynamics and more flexibility in modeling. In this section, we give structure of the HJM model allowing for jumps. Dynamics of nominal bonds, real bonds, inflation indexed bonds and inflation are given under the martingale measure.

Assumption 5.1 *Let* (Ω, $\mathcal{F}$, P) *be a probability space where P is the objective probability measure. The filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *is generated by both Wiener process* $W(t)$ *and a Poisson random measure* $\mu(dt,dz)$ *on* $\mathbb{R}_+ \times Z$*,* $Z \in \mathfrak{B}(\mathbb{R})$*, with compensator* $\lambda(dt,dz) = \nu(dz)dt$*.*

Assumption 5.2 *Assume that there exists a market with bonds and inflation indexed bonds for all maturities* $T > 0$*. In that market,* $P^n(t,T)$ *and* $P^{IP}(t,T)$ *are differentiable with respect to* T*.*

In this chapter, we denote $P^n(t,T)$ as the price in dollar at time t of a nominal zero coupon bond which pays one dollar at the maturity T. We write $I(t)$ for the consumer price index, $P^{IP}(t,T)$ for the price in dollar at time t of a contract which pays dollar value of one CPI unit at time T, $P^r(t,T)$ for the price of an inflation indexed zero coupon bond. Furthermore, we define $P^r(t,T)$ by

$$P^r(t,T) \quad = \quad \frac{P^{IP}(t,T)}{I(t)}.$$

We can say that $P^r(t,T)$ is the price in CPI baskets of a real bond which pays one CPI basket at time T.

Assumption 5.3 *Under the objective probability measure P the dynamics of forward rates* *and inflation are given by*

$$df^i(t,T) = \alpha^i(t,T)\,dt + \beta^i(t,T)\,dW^P(t) + \int_Z \gamma^i(t,z,T)\,\mu(dt,dz) \quad , i = n, r \tag{5.1}$$

$$dI(t) = a^I(t)I(t)\,dt + b^I(t)I(t)\,dW^P(t) + I(t_-)\int_Z c^I(t,z)\,\mu(dt,dz) \tag{5.2}$$

where $\alpha^i, a^i, \beta^i, b^I, \sigma^I, \beta^I$ *are adapted processes.*

Assumption 5.4 *Assume that the market is arbitrage free.*

This assumption implies that there exists an equivalent martingale measure Q^n such that,

$$\frac{P^n(t,T)}{B^n(t)}, \frac{P^{IP}(t,T)}{B^n(t)} \;\; are \;\; Q^n martingales.$$

Proposition 5.1 *If* $f^n(t,T)$, $f^r(t,T)$ *and* $I(t)$ *satisfies Assumption (5.3) then under nominal martingale measure* Q^n *the following dynamics hold:*

$$\frac{dI(t)}{I(t_-)} = [r^n(t) - r^r(t)]\,dt + b^I(t)\,dW(t) + \int_Z c^I(t,z)\,\tilde{\mu}(dt,dz) \tag{5.3}$$

$$\frac{dP^n(t,T)}{P^n(t_-,T)} = r^n(t)\,dt + \sigma^n(t,T)\,dW(t) + \int_Z \delta^n(t,z,T)\,\tilde{\mu}(dt,dz) \tag{5.4}$$

$$\frac{dP^{IP}(t,T)}{P^{IP}(t_-,T)} = r^n(t)\,dt + \sigma^{IP}(t,T)\,dW(t) + \int_Z \delta^{IP}(t,z,T)\,\tilde{\mu}(dt,dz) \tag{5.5}$$

$$\frac{dP^r(t,T)}{P^r(t_-,T)} = a^r(t,T)\,dt + \sigma^r(t,T)\,dW(t) + \int_Z \delta^r(t,z,T)\,\tilde{\mu}(dt,dz) \tag{5.6}$$

where

$$\begin{aligned}
\sigma^i(t,T) &= -\int_t^T \beta^i(t,u)\,du \quad , for \;\; i = n, r \\
\sigma^{IP}(t,T) &= b^I(t) + \sigma^r(t,T) \\
\delta^i(t,z,T) &= \exp(-\int_t^T \gamma^i(t,z,u)\,du) - 1 \\
\delta^{IP}(t,z,T) &= \delta^r(t,z,T) + c^I(t,z) + \delta^r(t,z,T)c^I(t,z) \\
a^r(t,T) &= r^r(t) - b^I(t)\sigma^r(t,T) - \int_Z \delta^r(t,z,T)c^I(t,z)\,\nu^P(dz)
\end{aligned}$$

Proof. We know the zero coupon bond price is defined by

$$P^i(t,T) = \exp(-\int_t^T f^i(t,u)du)$$

and say $X(t,T) = -\int_t^T f^i(t,u)du$, so $P^i(t,T) = \exp(X(t,T))$ and we also know the forward rate dynamics from Assumption (5.3) and integrating equation (5.1), we have

$$f^i(t,u) = f^i(0,u) + \int_0^t \alpha^i(s,u)\,ds + \int_0^t \beta^i(s,u)\,dW^P(s) + \int_0^t \int_Z \gamma^i(s,z,u)\,\mu(ds,dz)$$

By using relation between instantaneous interest rate and forward rate, $r^i(t) = f^i(t,t)$, we get

$$r^i(t) = f^i(0,t) + \int_0^t \alpha^i(s,t)\,ds + \int_0^t \beta^i(s,t)\,dW^P(s) + \int_0^t \int_Z \gamma^i(s,z,t)\,\mu(ds,dz).$$

Hence, $-\int_t^T f^i(t,u)du$ can be found using Fubini theorem and some integral splitting as follows:

$$\begin{aligned}
X(t,T) &= -\int_t^T f^i(0,u)\,du - \int_t^T \int_0^t \alpha^i(s,u)\,ds\,du - \int_t^T \int_0^t \beta^i(s,u)\,dW^P(s)\,du \\
&\quad - \int_t^T \int_0^t \int_Z \gamma^i(s,z,u)\,\mu(ds,dz)\,du \\
&= -\int_0^T f^i(0,u)\,du + \int_0^t f^i(0,u)\,du - \int_0^t \int_s^T \alpha^i(s,u)\,du\,ds \\
&\quad + \int_0^t \int_s^t \alpha^i(s,u)\,du\,ds - \int_0^t \int_s^T \beta^i(s,u)\,du\,dW^P(s) \\
&\quad + \int_0^t \int_s^t \beta^i(s,u)\,du\,dW^P(s) - \int_0^t \int_s^T \int_Z \gamma^i(s,z,u)\,du\,\mu(ds,dz) \\
&\quad + \int_0^t \int_s^t \int_Z \gamma^i(s,z,u)\,du\,\mu(ds,dz) \\
&= -\int_0^T f^i(0,u)\,du - \int_0^t \int_s^T \alpha^i(s,u)\,du\,ds - \int_0^t \int_s^T \beta^i(s,u)\,du\,dW^P(s) \\
&\quad - \int_0^t \int_s^T \int_Z \gamma^i(s,z,u)\,du\,\mu(ds,dz) + \int_0^t f^i(0,u)\,du \\
&\quad + \int_0^t \int_0^u \alpha^i(s,u)\,ds\,du + \int_0^t \int_0^u \beta^i(s,u)\,dW^P(s)\,du \\
&\quad + \int_0^t \int_0^u \int_Z \gamma^i(s,z,u)\,\mu(ds,dz)\,du.
\end{aligned}$$

Since the four last terms give $\int_0^t r^i(s)\,ds$, we can rewrite the above equation as follows

$$\begin{aligned} X(t,T) &= X(0,T) + \int_0^t r^i(s)\,ds - \int_0^t \int_s^T \alpha^i(s,u)\,du\,ds \\ &\quad - \int_0^t \int_s^T \beta^i(s,u)\,du\,dW^P(s) - \int_0^t \int_s^T \int_Z \gamma^i(s,z,u)\,du\,\mu(ds,dz) \end{aligned}$$

Let define

$$A^i(t,T) = -\int_t^T \alpha^i(t,u)\,du, \tag{5.7}$$

$$\sigma^i(t,T) = -\int_t^T \beta^i(t,u)\,du, \tag{5.8}$$

$$D^i(t,z,T) = -\int_t^T \gamma^i(t,z,u)\,du, \tag{5.9}$$

and using definitions (5.7), (5.8), (5.9) and $df^i(t,T)$ is found by

$$dX(t,T) = [r^i(t) + A^i(t,T)]\,dt + \sigma^i(t,T)\,dW^P(t) + \int_Z D^i(t,z,T)\,\mu(dt,dz).$$

Using Ito formula with $P^i(t,T) = g(X(t,T))$ we will get the the zero coupon bond price dynamics under objective probability P

$$\begin{aligned} \frac{dP^i(t,T)}{P^i(t_-,T)} &= (r^i(t) + A^i(t,T) + \frac{1}{2}\sigma^i(t,T)^2)\,dt + \sigma^i(t,T)\,dW^P(t) \\ &\quad + \int_Z \delta^i(t,z,T)\,\mu(dt,dz), \end{aligned} \tag{5.10}$$

where

$$\delta^i(t,z,T) = \exp((D^i(t,z,T)) - 1 = \exp(-\int_t^T \gamma^i(t,z,u)\,du) - 1. \tag{5.11}$$

We know the dynamics of $I(t)$ from Assumption (5.3) and $P^r(t,T)$ from equation (5.10) for $i = r$, to find $P^{IP}(t,T)$ we will use $P^{IP}(t,T) = I(t)P^r(t,T)$ with integration by parts formula. At the end of the calculation, we will get the below equation,

$$\begin{aligned} dP^{IP}(t,T) &= P^r(t,T)dI(t) + I(t)dP^r(t,T) + d < P^r, I >_t \\ \frac{dP^{IP}(t,T)}{P^{IP}(t_-,T)} &= [r^r(t) + A^r(t,T) + \frac{1}{2}\sigma^r(t,T)^2 + a^I(t) \\ &\quad + \sigma^r(t,T)b^I(t)]\,dt + (\sigma^r(t,T) + b^I(t))\,dW^P(t) \\ &\quad + \int_Z \delta^{IP}(t,z,T)\,\mu(dt,dz) \end{aligned}$$

where

$$\begin{aligned}\delta^{IP}(t,z,T) &= \delta^r(t,z,T) + c^I(t,z) + c^I(t,z)\delta^r(t,z,T),\\ \sigma^{IP}(t,T) &= \sigma^r(t,T) + b^I(t).\end{aligned}$$

Up to now, we find the dynamics of $P^n(t,T)$, $P^r(t,T)$, $P^{IP}(t,T)$ under objective probability measure P. But here we need these dynamics under martingale measure Q^n. So the next step is changing measure from P to Q^n. By using Girsanov theorem for jump processes intensity measure of Poisson process under new measure, $\lambda(dt,dz) = \theta(t,z)\lambda^P(dt,dz)$, will be get and $dW^P(t) = h(t)dt + dW(t)$ where W is Q^n-Wiener process. Furthermore we will use $\tilde{\mu}(dt,dz) = \mu(dt,dz) - \lambda(dt,dz)$. Hence the dynamics of $P^n(t,T)$, $P^r(t,T)$, $P^{IP}(t,T)$ under Q^n are given by,

$$\begin{aligned}\frac{dI(t)}{I(t_-)} &= [a^I(t) + b^I(t)h(t)]\,dt + b^I(t)\,dW(t)\\ &\quad + \int_Z c^I(t,z)\theta(t,z)\,\lambda^P(dt,dz) + \int_Z c^I(t,z)\,\tilde{\mu}(dt,dz), \qquad (5.12)\\ \frac{dP^n(t,T)}{P^n(t_-,T)} &= [r^n(t) + A^n(t,T) + \frac{1}{2}\sigma^n(t,T)^2 + \sigma^n(t,T)h(t)]\,dt + \sigma^n(t,T)\,dW(t)\\ &\quad + \int_Z \delta^n(t,z,T)\theta(t,z)\,\lambda^P(dt,dz) + \int_Z \delta^n(t,z,T)\,\tilde{\mu}(dt,dz), \qquad (5.13)\\ \frac{dP^{IP}(t,T)}{P^{IP}(t_-,T)} &= [r^r(t) + A^r(t,T) + \frac{1}{2}\sigma^r(t,T)^2 + a^I(t) + \sigma^r(t,T)b^I(t)\\ &\quad + \sigma^r(t,T)h(t) + b^I(t)h(t)]\,dt + \sigma^{IP}(t,T)\,dW(t)\\ &\quad + \int_Z \delta^{IP}(t,z,T)\theta(t,z)\,\lambda^P(dt,dz) + \int_Z \delta^{IP}(t,z,T)\,\tilde{\mu}(dt,dz). \qquad (5.14)\end{aligned}$$

Since $B^n(t)$ is the risk neutral numeraire $\frac{P^n(t,T)}{B^n(t)}$ and $\frac{P^{IP}(t,T)}{B^n(t)}$ are Q^n martingales, the drift terms of $P^n(t,T)$ and $P^{IP}(t,T)$ must be equal to the nominal short rate:

$$\begin{aligned}r^n(t) &= r^n(t) + A^n(t,T) + \frac{1}{2}\sigma^n(t,T)^2 + \sigma^n(t,T)h(t)\\ &\quad + \int_Z \delta^n(t,z,T)\theta(t,z)\,\nu^P(dz) \qquad (5.15)\end{aligned}$$

and

$$\begin{aligned}r^n(t) &= r^r(t) + A^r(t,T) + \frac{1}{2}\sigma^r(t,T)^2 + a^I(t) + \sigma^r(t,T)b^I(t)\\ &\quad + \sigma^r(t,T)h(t) + b^I(t)h(t) + \int_Z \delta^{IP}(t,z,T)\theta(t,z)\,\nu^P(dz). \qquad (5.16)\end{aligned}$$

Hence we find the dynamics of $P^n(t,T)$ and $P^{IP}(t,T)$ follows

$$\frac{dP^n(t,T)}{P^n(t_-,T)} = r^n(t)\,dt + \sigma^n(t,T)\,dW(t) + \int_Z \delta^n(t,z,T)\,\tilde{\mu}(dt,dz),$$

$$\frac{dP^{IP}(t,T)}{P^{IP}(t_-,T)} = r^n(t)\,dt + \sigma^{IP}(t,T)\,dW(t) + \int_Z \delta^{IP}(t,z,T)\,\tilde{\mu}(dt,dz).$$

From equations (5.15), (5.16) and using $\delta^{IP}(t,z,T) = \delta^r(t,z,T) + c^I(t,z) + c^I(t,z)\delta^r(t,z,T)$ we get three drift conditions:

$$A^n(t,T) = -\frac{1}{2}\sigma^n(t,T)^2 - \sigma^n(t,T)h(t) - \int_Z \delta^n(t,z,T)\theta(t,z)\,\nu^P(dz)$$

$$\begin{aligned} A^r(t,T) &= -\frac{1}{2}\sigma^r(t,T)^2 - \sigma^r(t,T)b^I(t) - \sigma^r(t,T)h(t) \\ &\quad - \int_Z \delta^r(t,z,T)(1 + c^I(t,z))\theta(t,z)\,\nu^P(dz) \end{aligned}$$

$$a^I(t) = r^n(t) - r^r(t) - b^I(t)h(t) - \int_Z c^I(t,z)\theta(t,z)\,\nu^P(dz).$$

By using (5.12) and the drift condition of $a^I(t)$ we get the dynamics of $I(t)$ as follows

$$\frac{dI(t)}{I(t_-)} = [r^n(t) - r^r(t)]\,dt + b^I(t)\,dW(t) + \int_Z c^I(t,z)\,\tilde{\mu}(dt,dz). \tag{5.17}$$

Finally, we will find the dynamics of $P^r(t,T)$ using $P^r(t,T) = \frac{P^{IP}(t,T)}{I(t)}$. Here, we use the dynamics of $P^{IP}(t,T)$ and $I(t)$ which are found in the previous steps to calculate $d(\frac{P^{IP}(t,T)}{I(t)})$:

$$\begin{aligned} d(\frac{P^{IP}(t,T)}{I(t)}) &= \frac{P^{IP}(t,T)}{I(t)}[r^n(t) - r^n(t) + r^r(t) - \sigma^{IP}(t,T)b^I(t) + b^I(t)^2]dt \\ &\quad + \frac{P^{IP}(t,T)}{I(t)} \int_Z \frac{c^I(t,z)(\delta^{IP}(t,z,T) - c^I(t,z))}{1 + c^I(t,z)} \nu^P(dz)dt \\ &\quad + \frac{P^{IP}(t,T)}{I(t)}[\sigma^{IP}(t,T) - b^I(t)]dW(t) \\ &\quad + \frac{P^{IP}(t_-,T)}{I(t_-)} \int_Z \frac{\delta^{IP}(t,z,T) - c^I(t,z)}{1 + c^I(t,z)}\,\tilde{\mu}(dt,dz) \end{aligned}$$

and, more simply,

$$\begin{aligned} \frac{dP^r(t,T)}{P^r(t_-,T)} &= a^r(t,T) + \sigma^r(t,T)\,dW(t) \\ &\quad + \int_Z \delta^r(t,z,T)\,\tilde{\mu}(dt,dz) \end{aligned}$$

where

$$a^r(t,T) = r^r(t) - \sigma^r(t,T)b^I(t) - \int_Z \delta^r(t,z,T)c^I(t,z)\,\nu^P(dz).$$

So, we finally get the last dynamics $P^r(t,T)$ as follows:

$$\frac{dP^r(t,T)}{P^r(t_-,T)} = a^r(t,T)\,dt + \sigma^r(t,T)\,dW(t) + \int_Z \delta^r(t,z,T)\,\tilde{\mu}(dt,dz). \qquad (5.18)$$

■

Proposition 5.2 *The drift conditions that has to be satisfied in order to be free of arbitrage market are:*

$$\alpha^n(t,T) = \beta^n(t,T)\Big(\int_t^T \beta^r(t,u)\,du - h(t)\Big) + \int_Z (1+\delta^n(t,z,T))\gamma^n(t,z,T)\theta(t,z)\,\nu^P(dz).$$

$$\alpha^r(t,T) = \beta^r(t,T)\Big(\int_t^T \beta^r(t,u)\,du - b^I(t) - h(t)\Big) + \int_Z (1+c^I(t,z))(1+\delta^r(t,z,T))\gamma^r(t,z,T)\theta(t,z)\,\nu^P(dz)$$

$$a^I(t) = r^n(t) - r^r(t) - b^I(t)h(t) - \int_Z c^I(t,z)\theta(t,z)\,\nu^P(dz).$$

Proof. We know the discounted asset price $\frac{P^n(t,T)}{B^n(t)}$ is martingale under Q^n measure. Proposition (5.1) gives $\frac{dP^n(t,T)}{P^n(t_-,T)}$ under this measure:

$$\frac{dP^n(t,T)}{P^n(t_-,T)} = [r^n(t) + A^n(t,T) + \frac{1}{2}\sigma^n(t,T)^2 + \sigma^n(t,T)h(t)]\,dt + \sigma^n(t,T)\,dW(t) + \int_Z \delta^n(t,z,T)\theta(t,z)\,\lambda^P(dt,dz) + \int_Z \delta^n(t,z,T)\,\tilde{\mu}(dt,dz)$$

The nominal money market account has the following dynamics

$$dB^n(t) = \exp\Big(\int_0^t r^n(t)\,dt\Big).$$

By the integration by parts formula, we receive

$$\begin{aligned}
d(\frac{P^n(t,T)}{B^n(t)}) &= d(P^n(t,T)B^n(t)^{-1}) \\
&= dP^n(t,T)B^n(t)^{-1} + P^n(t,T)dB^n(t)^{-1} + d\langle P^n,(B^n)^{-1}\rangle_t \\
&= \frac{P^n(t,T)}{B^n(t)}[A^n(t,T) + \frac{1}{2}\sigma^n(t,T)^2 + \sigma^n(t,T)h(t)]\,dt \\
&\quad + \frac{P^n(t,T)}{B^n(t)}\sigma^n(t,T)\,dW(t) \\
&\quad + \frac{P^n(t,T)}{B^n(t)}\int_Z \delta^n(t,z,T)\theta(t,z)\,\lambda^P(dt,dz) \\
&\quad + \frac{P^n(t_-,T)}{B^n(t_-)}\int_Z \delta^n(t,z,T)\,\tilde{\mu}(dt,dz).
\end{aligned}$$

Since the discounted asset prices are martingale, we get

$$A^n(t,T) = -\frac{1}{2}\sigma^n(t,T)^2 - \sigma^n(t,T)h(t) - \int_Z \delta^n(t,z,T)\theta(t,z)\,\nu^P(dz)$$

Differentiating both sides with respect to T and using the definitions given before, we get

$$\begin{aligned}
A^n(t,T) &= -\int_t^T \alpha^n(t,u)\,du \\
\sigma^n(t,T) &= -\int_t^T \beta^n(t,u)\,du \\
\delta^n(t,z,T) &= \exp(-\int_t^T \gamma^n(t,z,u)\,du) - 1.
\end{aligned}$$

Finally, we get the first drift condition of $f^n(t,T)$:

$$\begin{aligned}
\alpha^n(t,T) &= \beta^n(t,T)(\int_t^T \beta^r(t,s)\,ds - h(t)) \\
&\quad + \int_Z (1 + \delta^n(t,z,T))\gamma^n(t,z,T)\theta(t,z)\,\nu^P(dz).
\end{aligned}$$

For the second drift condition that belongs to $f^r(t,T)$, we will the use martingale property of discounted asset price $\frac{P^{IP}(t,T)}{B^n(t)}$ under the measure Q^n. Proposition (5.1) gives $\frac{dP^{IP}(t,T)}{P^{IP}(t_-,T)}$ under this measure.

$$\begin{aligned}
\frac{dP^{IP}(t,T)}{P^{IP}(t_-,T)} &= [r^r(t) + A^r(t,T) + \frac{1}{2}\sigma^r(t,T)^2 + a^I(t) + \sigma^r(t,T)b^I(t,T) \\
&\quad +\sigma^r(t,T)h(t) + b^I(t)h(t)]\,dt + \sigma^{IP}(t,T)\,dW(t) \\
&\quad + \int_Z \delta^{IP}(t,z,T)\theta(t,z)\,\lambda^P(dt,dz) + \int_Z \delta^{IP}(t,z,T)\,\tilde{\mu}(dt,dz).
\end{aligned}$$

Using integration by parts formula, we obtain

$$\begin{aligned}
d(P^{IP}(t,T))B^n(t)^{-1}) &= d(P^{IP}(t,T)B^n(t)^{-1} + dB_n(t)^{-1}P^{IP}(t,T) \\
&= \frac{P^{IP}(t,T)}{B^n(t)}[r^r(t) - r^n(t) + A^r(t,T) + \frac{1}{2}\sigma^r(t,T)^2 + a^I(t) \\
&\quad +\sigma^r(t,T)b^I(t,T) + \sigma^r(t,T)h(t) + b^I(t)h(t)]\,dt \\
&\quad +\frac{P^{IP}(t,T)}{B^n(t)}\sigma^{IP}(t,T)\,dW(t) \\
&\quad +\frac{P^{IP}(t,T)}{B^n(t)}\int_Z \delta^{IP}(t,z,T)\theta(t,z)\,\lambda^P(dt,dz) \\
&\quad +\frac{P^{IP}(t_-,T)}{B^n(t_-)}\int_Z \delta^{IP}(t,z,T)\,\tilde{\mu}(dt,dz).
\end{aligned}$$

Using martingale property of discounted asset prices and $\delta^{IP}(t,z,T) = \delta^r(t,z,T) + c^I(t,z) + c^I(t,z)\delta^r(t,z,T)$, separate drift term of $d(P^{IP}(t,T)B^n(t)^{-1})$ into two equations as follows

$$\begin{aligned}
A^r(t,T) &= -\frac{1}{2}\sigma^r(t,T)^2 - \sigma^r(t,T)b^I(t) - \sigma^r(t,T)h(t) \\
&\quad - \int_Z \delta^r(t,z,T)(1 + c^I(t,z))\theta(t,z)\,\nu^P(dz)
\end{aligned}$$

and

$$a^I(t) = r^n(t) - r^r(t) - b^I(t)h(t) - \int_Z c^I(t,z)\theta(t,z)\,\nu^P(dz).$$

The latter one gives directly the drift condition of $I(t)$. For the drift condition of $f^r(t,T)$, differentiating both sides of the former one with respect to T and using the definitions $A^r(t,T)$,

$\sigma^r(t,T)$, $\delta^r(t,z,T)$ given before as follows:

$$\begin{aligned} A^r(t,T) &= -\int_t^T \alpha^r(t,u)\,du \\ \sigma^r(t,T) &= -\int_t^T \beta^r(t,u)\,du \\ \delta^r(t,z,T) &= \exp(-\int_t^T \gamma^r(t,z,u)\,du) - 1. \end{aligned}$$

Then we get

$$\begin{aligned} \alpha^r(t,T) &= \beta^r(t,T)(\int_t^T \beta^r(t,u)\,du - b^I(t) - h(t)) \\ &\quad + \int_Z (1 + c^I(t,z))(1 + \delta^r(t,z,T))\gamma^r(t,z,T)\theta(t,z)\,\nu^P(dz). \end{aligned}$$

■

CHAPTER 6

PRICING INFLATION INDEXED SWAPS WITH THE EXTENDED HEATH JARROW MORTON FRAMEWORK

In this chapter, we will give pricing derivation of zero coupon and year-on-year inflation indexed swaps in the extended HJM framework built in the previous chapter. The main aim under pricing inflation indexed swaps is that these are the most popular and commonly traded derivatives among inflation indexed derivatives in the market.

A swap is an agreement that on each payment date Party A pays out Party B a floating rate over a predefined period, while Party B pays out Party A a fixed rate. Payment dates start with T_1 and continue with $T_2, ..., T_M$ where T_0 is the start date which has no payment. In the inflation indexed derivatives market, the floating rate is the inflation rate which is the percentage return of the CPI index over a period of a time.

6.1 Pricing of Zero Coupon Inflation Indexed Swaps

In a zero coupon inflation indexed swap (ZCIIS) one party pays a fixed rate and receives inflation rate over the time interval $[T_0, T_M]$ with only one payment at time T_M. Let K be the contract fixed rate and N be the nominal value then the fixed amount is

$$N((1+K)^{T_M - T_0} - 1)$$

and the floating amount is

$$N(\frac{I(T_M)}{I(T_0)} - 1).$$

If $ZCIIS\,(t, T_M, I(0), N)$ denotes the value of the zero coupon inflation indexed swap at time t and $t \in [T_0, T_M]$, then

$$ZCIIS\,(t, T_M, I(0), N) \;=\; N\,E_n^{T_M}[\exp(-\int_t^{T_M} r^n(s)\,ds)\,(\frac{I(T_M)}{I(T_0)} - (1+K)^{T_M-T_0}) \mid F_t]$$

where $E_n^{T_M}(. \mid F_t)$ denotes the conditional expectation with respect to nominal T-forward measure and the expectation based on the information at time t. Notice that this pricing valuation is for the party who pays out fixed amount and receives floating amount. When we rewrite the equation noticing the definition of $P^n(t, T_M)$ we get

$$\begin{aligned} ZCIIS\,(t, T_M, I(0), N) \;&=\; N\,P^n(t, T_M)\,E_n^{T_M}[\frac{I(T_M)}{I(T_0)} \mid F_t] \\ &\quad -N\,P^n(t, T_M)\,(1+K)^{T_M-T_0}. \end{aligned} \tag{6.1}$$

Since $I(T_0)$ is F_t measurable and

$$\frac{I(t)P^r(t, T)}{P^n(t, T)} \;\; is\;\; a\;\; Q^{n,T}\;\; martingale$$

the conditional expectation in equation (6.1) becomes

$$\begin{aligned} E_n^{T_M}[\frac{I(T_M)}{I(T_0)} \mid F_t] \;&=\; \frac{1}{I(T_0)}\,E_n^{T_M}[\frac{I(T_M)P^r(T_M, T_M)}{P^n(T_M, T_M)} \mid F_t] \\ &=\; \frac{1}{I(T_0)}\,\frac{I(t)P^r(t, T_M)}{P^n(t, T_M)}. \end{aligned}$$

Hence the value of ZCIIS at time t is

$$ZCIIS\,(t, T_M, I(0), N) \;=\; N\,\frac{I(t)}{I(T_0)}P^r(t, T_M) - N\,P^n(t, T_M)\,(1+K)^{T_M-T_0}. \tag{6.2}$$

If we take $t = T_0$, that is, swap is initially traded equation (6.2) becomes

$$ZCIIS\,(T_0, T_M, I(0), N) \;=\; N\,P^r(T_0, T_M) - N\,P^n(T_0, T_M)\,(1+K)^{T_M-T_0}. \tag{6.3}$$

The obtained price is not based on any specific assumptions about asset dynamics, but only it depends on the absence of arbitrage. Hence, it is model independent we note take that there is also a simple replicating argument that proves the equation (6.2):

- At time T_0 buy $\frac{1}{I(T_0)}$ inflation indexed bond with maturity T_M.
- At time T_M receive the dollar value of $\frac{1}{I(T_0)}$ CPI units. This value will be $\frac{I(T_M)}{I(T_0)}$.
- At time T_0 find the price of $\frac{1}{I(T_0)}$ inflation indexed bond. This price will be $P^r(T_0, T_M)$.

6.2 Pricing of Year-on-Year Inflation Indexed Swaps

In a year-on-year inflation indexed swap(YYIIS) one party pays the fixed amount and receives the inflation rate at each payment time $T_{p+1}, T_{p+2}, ..., T_M$ for the interval $[T_p, T_M]$ where T_p is the start date of the agreement. So, for each period $[T_i, T_{i+1}]$, $i = p, ..., M-1$, the fixed amount is

$$N\,\tau_i\,K$$

and the floating amount is

$$N\,\tau_i\,(\frac{I(T_i)}{I(T_{i-1})} - 1)$$

where τ_i is the year fraction such as $\tau_i = T_i - T_{i-1}$. Let $YYIIS\,(t, T_{i+1}, T_i, N)$ denote the value of YYIIS at time t then the pricing formula can be written by following expression

$$\begin{aligned} YYIIS\,(t, T_{i+1}, T_i, N) &= \sum_{i=p}^{M-1} N\,\tau_{i+1}\,E_n^{T_{i+1}}[\exp(-\int_t^{T_{i+1}} r^n(s)\,ds)\,(\frac{I(T_{i+1})}{I(T_i)} - (1+K))\mid F_t] \\ &= \sum_{i=p}^{M-1} N\,\tau_{i+1}\,P^n(t, T_{i+1})\,E_n^{T_{i+1}}[\frac{I(T_{i+1})}{I(T_i)}\mid F_t] \\ &\quad - \sum_{i=p}^{M-1} N\,\tau_{i+1}\,P^n(t, T_{i+1})\,(1+K). \end{aligned} \tag{6.4}$$

In the above equation we must find the value of the conditional expectation to get the price of YYIIS. We know

$$\frac{I(t)P^r(t, T_{i+1})}{P^n(t, T_{i+1})} \;\; is \;\; a \;\; Q^{n,T_{i+1}} \;\; martingale$$

and $I(T_i)$ is F_{T_i} measurable. Using these measurable and martingale properties, the expectation becomes

$$\begin{aligned} E_n^{T_{i+1}}[\frac{I(T_{i+1})}{I(T_i)}\mid F_t] &= E_n^{T_{i+1}}[\frac{1}{I(T_i)}\,E_n^{T_{i+1}}[I(T_{i+1})\mid F_{T_i}]\mid F_t] \\ &= E_n^{T_{i+1}}[\frac{1}{I(T_i)}\,E_n^{T_{i+1}}[\frac{I(T_{i+1})P^r(T_{i+1}, T_{i+1})}{P^n(T_{i+1}, T_{i+1})}\mid F_{T_i}]\mid F_t] \\ &= E_n^{T_{i+1}}[\frac{P^r(T_i, T_{i+1})}{P^n(T_i, T_{i+1})}\mid F_t]. \end{aligned} \tag{6.5}$$

In equation (6.5), the numeraire is $P^n(T_i, T_{Ti+1})$ which is a martingale under Q^{n,T_i}. So, changing the measure from $Q^{n,T_{i+1}}$ to Q^{n,T_i} enables us to compute the conditional expectation. The Radon-Nikodym derivative for the measure changing is

$$\begin{aligned} Z_t^{n,T_{i+1}/n,T_i} &= \frac{dQ^{n,T_{i+1}}}{dQ^{n,T_i}} \mid_t \\ &= \frac{P^n(t,T_{i+1})}{P^n(t,T_i)} \frac{P^n(0,T_i)}{P^n(0,T_{i+1})}. \end{aligned}$$

By using Bayes formula, we obtain

$$\begin{aligned} E_n^{T_{i+1}}[\frac{P^r(T_i,T_{i+1})}{P^n(T_i,T_{i+1})} \mid F_t] &= \frac{E_n^{T_i}[\frac{P^r(T_i,T_{i+1})}{P^n(T_i,T_{i+1})} Z_{T_i}^{n,T_{i+1}/n,T_i} \mid F_t]}{Z_t^{n,T_{i+1}/n,T_i}} \\ &= \frac{E_n^{T_i}[\frac{P^r(T_i,T_{i+1})}{P^n(T_i,T_{i+1})} \frac{P^n(T_i,T_{i+1})}{P^n(T_i,T_i)} \frac{P^n(0,T_i)}{P^n(0,T_{i+1})} \mid F_t]}{\frac{P^n(t,T_{i+1})}{P^n(t,T_i)} \frac{P^n(0,T_i)}{P^n(0,T_{i+1})}} \\ &= \frac{P^n(t,T_i)}{P^n(t,T_{i+1})} E_n^{T_i}[P^r(T_i,T_{i+1}) \mid F_t]. \qquad (6.6) \end{aligned}$$

Hence, the equation (6.5) reduces the equation (6.6). The conditional expectation in equation (6.6) is the nominal price of the real zero coupon bond price $P^r(T_i, T_{i+1})$ at time T_i. If real rates were deterministic, then this expectation would reduce the

$$E_n^{T_i}[P^r(T_i,T_{i+1}) \mid F_t] = P^r(T_i,T_{i+1})$$

However, we assumed real rates are stochastic. It shows us unlike the model independency of ZCIIS, YYIIS is model dependent. So, to calculate the conditional expectation in equation (6.6), we need to use the model that was built in previous chapter. In this step we need to change the measure from the nominal Q^{n,T_i} to the real Q^{r,T_i} forward measure. The Radon-Nikodym derivative of this changing is

$$\begin{aligned} Z_t^{n,T_i/r,T_i} &= \frac{dQ^{n,T_i}}{dQ^{r,T_i}} \mid_t \\ &= \frac{P^n(t,T_i)}{P^r(t,T_i)\, I(t)} \frac{P^r(0,T_i)I(0)}{P^n(0,T_i)}. \qquad (6.7) \end{aligned}$$

By using Bayes formula, we get

$$E_n^{T_i}[P^r(T_i,T_{i+1}) \mid F_t] = \frac{E_r^{T_i}[\frac{P^r(T_i,T_{i+1})}{P^r(T_i,T_i)} Z_{T_i}^{n,T_i/r,T_i} \mid F_t]}{Z_t^{n,T_i/r,T_i}}. \qquad (6.8)$$

When the above conditional expectation is calculated we will use the dynamics of I, P^n, P^r defined in the previous chapter:

$$\begin{aligned}
\frac{dI(t)}{I(t_-)} &= [r^n(t) - r^r(t)]\,dt + b^I(t)\,dW(t) + \int_Z c^I(t,z)\,\tilde{\mu}(dt,dz) \\
\frac{dP^n(t,T)}{P^n(t_-,T)} &= r^n(t)\,dt + \sigma^n(t,T)\,dW(t) + \int_Z \delta^n(t,z,T)\,\tilde{\mu}(dt,dz) \\
\frac{dP^r(t,T)}{P^r(t_-,T)} &= a^{r,T}(t)\,dt + \sigma^r(t,T)\,dW(t) + \int_Z \delta^r(t,z,T)\,\tilde{\mu}(dt,dz).
\end{aligned}$$

Firstly, our aim is to find the dynamics of $\frac{P^n(t,T_i)}{P^r(t,T_i)I(t)}$ by using above dynamics. By integration by parts formula for jump processes

$$\begin{aligned}
d(\frac{P^r(t,T_i)I(t)}{P^r(t_-,T_i)I(t_-)}) &= [r^n(t) - r^r(t) + \sigma^r(t,T_i) + b^I(t)\sigma^r(t,T_i)]\,dt + \int_Z \delta^r(t,z,T_i)c^I(t,z)\,\lambda(dt,dz) \\
&\quad + (\sigma^r(t,T_i) + b^I(t))\,dW(t) \\
&\quad + \int_Z (\delta^r(t,z,T_i) + c^I(t,z) + \delta^r(t,z,T_i)c^I(t,z))\,\tilde{\mu}(dt,dz).
\end{aligned}$$

We find $P^r(t,T_i)I(t)$ and we already know $P^n(t,T_i)$, then we can find $\frac{P^n(t,T_i)}{P^r(t,T_i)I(t)}$ and obtain

$$\begin{aligned}
d(\frac{P^n(t,T_i)}{P^r(t,T_i)I(t)}) &= \frac{P^n(t,T_i)}{P^r(t,T_i)I(t)}[r^n(t) - r^r(t) - \sigma^r(t,T_i) - b^I(t)\sigma^r(t,T_i) \\
&\quad - \int_Z \delta^r(t,z,T_i)c^I(t,z)\,\lambda(dt,dz) \\
&\quad -(b^I(t) + \sigma^r(t,T_i))\sigma^n(t,T_i) + (b^I(t) + \sigma^r(t,T_i))^2]\,dt + \\
&\quad - \frac{P^n(t,T_i)}{P^r(t,T_i)I(t)} \int_Z \frac{(\delta^{IP}(t,z,T_i))\,(\delta^n(t,z,T_i) - \delta^{IP}(t,z,T_i))}{(1 + \delta^{IP}(t,z,T_i))}\,\lambda(dt,dz) \\
&\quad + \frac{P^n(t,T_i)}{P^r(t,T_i)I(t)}(\sigma^n(t,T_i) - b^I(t) - \sigma^r(t,T_i))\,dW(t) \\
&\quad + \frac{P^n(t_-,T_i)}{P^r(t_-,T_i)I(t_-)} \int_Z \frac{(\delta^n(t,z,T_i) - \delta^{IP}(t,z,T_i))}{(1 + \delta^{IP}(t,z,T_i))}\,\tilde{\mu}(dt,dz).
\end{aligned}$$

Since $X(t) = \frac{P^n(t,T_i)}{P^r(t,T_i)I(t)}$, at the same time we get dynamics of $\frac{Z_t^{n,T_i/r,T_i}}{Z_{t_-}^{n,T_i/r,T_i}}$ under Q^{n,T_i}.

$$\frac{dZ_t^{n,T_i/r,T_i}}{Z_{t_-}^{n,T_i/r,T_i}} = \frac{dX(t)}{X(t_-)}. \tag{6.9}$$

We know $Z_t^{n,T_i/r,T_i}$ is a martingale under Q^{r,T_i}, so changing measure from Q^{n,T_i} to Q^{r,T_i} change only the drift term not the others. Hence the dynamics of $Z_t^{n,T_i/r,T_i}$ under Q^{r,T_i} is given by

$$\begin{aligned}
d(\frac{Z_t^{n,T_i/r,T_i}}{Z_{t_-}^{n,T_i/r,T_i}}) &= (\sigma^n(t,T_i) - b^I(t) - \sigma^r(t,T_i))\, dW^{r,T_i}(t) \\
&\quad + \int_Z \frac{(\delta^n(t,z,T_i) - \delta^{IP}(t,z,T_i))}{(1+\delta^{IP}(t,z,T_i))}\, \tilde{\mu}^{r,T_i}(dt,dz).
\end{aligned}$$

On the other hand, $P^r(T_i, T_{i+1}) = \frac{P^r(t,T_{i+1})}{P^r(t,T_i)}$ is also martingale under Q^{r,T_i} ; and its dynamics is given by

$$\begin{aligned}
d\frac{P^r(t,T_{i+1})}{P^r(t,T_i)} &= \frac{P^r(t,T_{i+1})}{P^r(t,T_i)}(\sigma^r(t,T_{i+1}) - \sigma^r(t,T_i))dW^{r,T_i}(t) \\
&\quad + \frac{P^r(t_-,T_{i+1})}{P^r(t_-,T_i)} \int_Z \frac{(\delta^r(t,z,T_{i+1}) - \delta^r(t,z,T_i))}{(1+\delta^r(t,z,T_i))}\, \tilde{\mu}^{r,T_i}(dt,dz)
\end{aligned}$$

Hence, we get the dynamics of $Z_t^{n,T_i/r,T_i}$ and $\frac{P^r(t,T_{i+1})}{P^r(t,T_i)}$. By the integration by parts formula, we obtain

$$\begin{aligned}
d(\frac{P^r(t,T_{i+1})}{P^r(t,T_i)} Z_t^{n,T_i/r,T_i}) &= \frac{P^r(t,T_{i+1})}{P^r(t,T_i)} Z_t^{n,T_i/r,T_i}[(\sigma^r(t,T_{i+1}) - \sigma^r(t,T_i)) \\
&\quad (\sigma^n(t,T_i) - b^I(t) - \sigma^r(t,T_i))dt + \int_Z (mn)\, \lambda^{r,T_i}(dt,dz)] \\
&\quad + \frac{P^r(t,T_{i+1})}{P^r(t,T_i)} Z_t^{n,T_i/r,T_i}[\sigma^n(t,T_i) + \sigma^r(t,T_{i+1}) \\
&\quad -2\sigma^r(t,T_i) - b^I(t)]dW^{r,T_i}(t) \\
&\quad + \frac{P^r(t_-,T_{i+1})}{P^r(t_-,T_i)} Z_{t_-}^{n,T_i/r,T_i} \int_Z (m+n+mn)\, \tilde{\mu}^{r,T_i}(dt,dz)
\end{aligned} \tag{6.10}$$

where

$$m = \frac{(\delta^r(t,z,T_{i+1}) - \delta^r(t,z,T_i))}{(1+\delta^r(t,z,T_i))} \;\; and \;\; n = \frac{(\delta^n(t,z,T_i) - \delta^{IP}(t,z,T_i))}{(1+\delta^{IP}(t,z,T_i))}.$$

Let

$$S(t) = \frac{P^r(t,T_{i+1})}{P^r(t,T_i)} Z_t^{n,T_i/r,T_i}$$

and $Y(t) = lnS(t)$; then, the solution of the this stochastic equation is

$$
\begin{aligned}
S(T_i) &= S(t)\exp[\int_t^{T_i}(\sigma^r(s,T_{i+1}) - \sigma^r(s,T_i))(\sigma^n(s,T_i) - b^I(s) - \sigma^r(s,T_i))ds \\
&\quad - \int_t^{T_i}\int_Z (m+n)\lambda^{r,T_i}(ds,dz) \\
&\quad + \int_t^{T_i}(\sigma^n(s,T_i) + \sigma^r(s,T_{i+1}) - 2\sigma^r(s,T_i) - b^I(s))dW^{r,T_i}(s) \\
&\quad - \int_t^{T_i}\frac{1}{2}(\sigma^n(s,T_i) + \sigma^r(s,T_{i+1}) - 2\sigma^r(s,T_i) - b^I(s))^2 ds \\
&\quad + \int_t^{T_i}\int_Z \ln(1+m+n+mn)\mu^{r,T_i}(ds,dz)].
\end{aligned}
$$

Here, we assume that the coefficients are deterministic to ensure closed-form solution. From moment generating function for Brownian motion

$$
\begin{aligned}
&E_r^{T_i}[\exp\int_t^{T_i}(\sigma^n(s,T_i) + \sigma^r(s,T_{i+1}) - 2\sigma^r(s,T_i) - b^I(s))dW^{r,T_i}(s)] \\
&= \exp(\frac{1}{2}\int_t^{T_i}(\sigma^n(s,T_i) + \sigma^r(s,T_{i+1}) - 2\sigma^r(s,T_i) - b^I(s))^2 ds,
\end{aligned}
$$

and from exponential formula for Poisson random measure

$$
E_r^{T_i}[\exp(\int_t^{T_i}\int_Z \ln(1+m+n+mn)d\mu^{r,T_i}(ds,dz))] = \exp(\int_t^{T_i}\int_Z (m+n+mn)\lambda^{r,T_i}(ds,dz)
$$

we finally get

$$
\begin{aligned}
E_r^{T_i}[S(T_i)] &= S(t)\exp(\int_t^{T_i}(\sigma^r(s,T_{i+1}) - \sigma^r(s,T_i))(\sigma^n(s,T_i) - b^I(s) - \sigma^r(t,T_i))ds \\
&\quad + \int_t^{T_i}\int_Z (mn)\lambda^{r,T_i}(ds,dz)).
\end{aligned}
$$

Hence, if we go back to the conditional expectation in the equation (6.8), we will get

$$
E_r^{T_i}[\frac{P^r(T_i,T_{i+1})}{P^r(T_i,T_i)} Z_{T_i}^{n,T_i/r,T_i} \mid F_t] = \frac{P^r(t,T_{i+1})}{P^r(t,T_i)} Z_t^{n,T_i/r,T_i} e^{C(t,T_i,T_{i+1})} \tag{6.11}
$$

where

$$
\begin{aligned}
C(t,T_i,T_{i+1}) &= \exp(\int_t^{T_i}(\sigma^r(s,T_{i+1}) - \sigma^r(s,T_i))(\sigma^n(s,T_i) - \sigma^r(s,T_i) - b^I(s))\,d(s) \\
&\quad + \int_Z \frac{(\delta^r(s,z,T_{i+1}) - \delta^r(s,z,T_i))}{(1+\delta^r(s,z,T_i))}\frac{(\delta^n(s,z,T_i) - \delta^{IP}(s,z,T_i))}{(1+\delta^{IP}(s,z,T_i))}\lambda^{r,T_i}(ds,dz).
\end{aligned}
$$

If we insert equation (6.11) into the equation (6.6), then we get

$$E_n^{T_i}[P^r(T_i, T_{i+1}) \mid F_t] \;=\; \frac{P^r(t, T_{i+1})}{P^r(t, T_i)}\, e^{C(t,T_i,T_{i+1})}. \qquad (6.12)$$

Finally, using equations (6.5), (6.6), (6.12), we obtain the expectation which is mentioned firstly in the beginning of the section

$$E_n^{T_{i+1}}[(\frac{I(T_{i+1})}{I(T_i)}) \mid F_t] \;=\; \frac{P^n(t, T_i)}{P^n(t, T_{i+1})}\, \frac{P^r(t, T_{i+1})}{P^r(t, T_i)}\, e^{C(t,T_i,T_{i+1})}. \qquad (6.13)$$

The general formula of the YYIIS valuation is given at the beginning of this chapter in equation (6.4). Inserting result (6.13) into this valuation formula, we get

$$\begin{aligned} YYIIS(t, T_{i+1}, T_i, N) &= \sum_{i=p}^{M-1} N\, \tau_{i+1}\, P^n(t, T_i)\, \frac{P^r(t, T_{i+1})}{P^r(t, T_i)}\, e^{C(t,T_i,T_{i+1})} \\ &\quad - \sum_{i=p}^{M-1} N\, \tau_{i+1}\, P^n(t, T_{i+1})\,(1 + K). \qquad (6.14) \end{aligned}$$

The forward swap rate which is value of K for which the price of the swap is zero can be shown when $YYIIS(t, T_{i+1}, T_i, N) = 0$ with the notation $R_p^M(t)$,

$$R_p^M(t) \;=\; \frac{\sum_{i=p}^{M-1} N\, \tau_{i+1}\, P^n(t, T_i)\, \frac{P^r(t,T_{i+1})}{P^r(t,T_i)}\, e^{C(t,T_i,T_{i+1})} - \sum_{i=p}^{M-1} N\, \tau_{i+1}\, P^n(t, T_{i+1})}{\sum_{i=p}^{M-1} N\, \tau_{i+1}\, P^n(t, T_{i+1})}.$$

Here, the pricing formula depends on real and nominal bond prices and their volatilities. We can rewrite this formula that depends on the inflation indexed bonds and their volatilities instead of real bonds and their volatilities. It enables to strip real zero coupon bond prices from inflation indexed bonds. Using the definition of inflation indexed bonds $P^{IP}(t, T) = I(t)P^r(t, T)$, if we product the left hand side of the pricing equation (6.14) by $\frac{I(t)}{I(t)}$, then we get new pricing formula and forward swap rate

$$\begin{aligned} YYIIS(t, T_{i+1}, T_i, N) &= \sum_{i=p}^{M-1} N\, \tau_{i+1}\, P^n(t, T_i)\, \frac{P^{IP}(t, T_{i+1})}{P^{IP}(t, T_i)}\, e^{C(t,T_i,T_{i+1})} \\ &\quad - \sum_{i=p}^{M-1} N\, \tau_{i+1}\, P^n(t, T_{i+1})\,(1 + K) \end{aligned}$$

and

$$R_p^M(t) \;=\; \frac{\sum_{i=p}^{M-1} N\, \tau_{i+1}\, P^n(t, T_i)\, \frac{P^{IP}(t,T_{i+1})}{P^{IP}(t,T_i)}\, e^{C(t,T_i,T_{i+1})} - \sum_{i=p}^{M-1} N\, \tau_{i+1}\, P^n(t, T_{i+1})}{\sum_{i=p}^{M-1} N\, \tau_{i+1}\, P^n(t, T_{i+1})}.$$

CHAPTER 7

CONCLUSION

Inflation indexed securities are designed to help protect both issuers and investors from changes in the general level of prices in the real economy. These instruments have become increasingly popular in the last two decades. Hence, also pricing of these instruments has became important. There is a group of studies on pricing. If we look these studies, we can see that they use HJM model and foreign currency analogy in principle. In our study, we add jump process to these principle models. Because, the reality of financial markets can be given by models with the jump process than models based on only Brownian motion. In the real world, we know that asset price processes have jumps and taking them into consideration gives more accurate results in the studies.

In this thesis, firstly we give structure of inflation with its causes, history of indexation, definitions and advantages of inflation indexed securities. Then, we introduce HJM framework which models instantaneous forward rates under no-arbitrage condition. For pricing of indexed securities we have to know foreign currency analogy which is based on HJM framework. So, we give foreign currency analogy in detail. And then, we present a HJM framework allowing for jumps. In these extended model, instantaneous forward rates and inflation index are allowed to be driven by both Wiener and Poisson process. Since in the real world asset prices have jumps, inserting jump process into the pricing models gives more realistic results. Finally, under this extended HJM model, we give a pricing derivation of inflation indexed swaps which are the most commonly used in the financial markets.

Our further studies will continue with pricing inflation indexed options, swaptions and the other new derivatives. Stochastic volatility of price process dynamics can be taken into consideration. By calibration to the market data, the performance of our models can be tested.

REFERENCES

[1] E. Balaban, *Türkiye'de enflasyona endeksli kamu menkul kıymetlerinin çıkarılmasına ilişkin bir çalışma*, TCMB Araştırma Genel Müdürlüğü, 97/2, 1997.

[2] T. Beletski, *Inflation linked products and optimal invesment with macro derivatives*, Master of Science Thesis, 2006.

[3] N. Belgrade, E. Benhamou and E. Koehler, *A market model for inflation*, CDC IXIS, 2004.

[4] T. Björk, *Arbitrage Theory in Continuous Time*, Oxford University Press, 1998.

[5] T. Björk, Y. Kabanov and W. Runggaldier, *Bond markets where prices are driven by a general marked point process*, Working Paper, Stockholm School of Economics, 67, 1996.

[6] T. Björk, Y. Kabanov and W. Runggaldier, *Bond market structure in the presence of marked point processes*, Mathematical Finance, 7(2): 211-239, 1997.

[7] A. Brace, D. Gatarek and M. Musiela, *The market model of interest rate dynamics*, Mathematical Finance, 7(2): 127-155, 1997.

[8] D. Brigo, F. Mercurio, *Interest Rate Models: Theory and Practice with Smile, Inflation and Credit*, Springer-Finance, Second Edition, 2006.

[9] M. Chamon, P. Mauro, *Pricing Growth-Indexed Bonds* IMF Working Paper, WP/05/216, 2005.

[10] C. Chiarella, C. N. Sklibosios, *A class of jump-diffusion bond pricing models within the HJM-Framework*, Asia-Pacific Financial Markets, 10: 87-127, 2003.

[11] R. Cont, P. Tankov, *Financial Modelling with Jump Processes*, Word Scientific, 2000.

[12] S. R. Das, S. Foresi, *Exact solutions for bond and option prices with systematic jump risk*, Review of Derivatives Research, 1: 7 -24, 1996.

[13] M. Deacon, A. Derry and D. Mirfendereski. Inflation-Indexed Securities, Bonds, Swaps and other Derivatives. Wiley-Finance, 2004.

[14] M. Dodgson, D. Kainth, *Inflation-linked derivatives*, Royal Bank of Scotland Risk Training Course, Market Risk Group, 2006.

[15] E. Eberlein, *Jump-type Levy processes*, University of Freiburg, 2009.

[16] J. A. Garcia, A. V. Rixtel, *Inflation-linked bonds from a central bank respective*, Bank of Spain, 0705, 2007.

[17] P. Glasserman, S. G. Kou, *The term structure of simple forward rates with jump risk*, Mathematical Finance, 13(3): 383-410 , 2003.

[18] İ. E. Güney, *A market model for pricing inflation indexed bonds with jumps incorporation*, Master of Science Thesis, METU, 2008.

[19] A. Haastrecht, A. Pelsser, *Generic pricing of FX, inflation and stock options under stochastic interest rates and stochastic volatility*, Working Paper, 2008.

[20] D. Heath, R. Jarrow and A. Morton, *Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation*, Econometrica, 60(1): 77-105, 1992.

[21] M. Henrard, *TIPS Options in the Jarrow-Yıldırım model*, Munich Personel RePEc Archive, 1423, 2006.

[22] S. L. Heston, *A closed-form solution for optins with stochastic volatility with applications to bond and currency options*, The Review of Financial Studies, 6 (2): 327-343, 1993.

[23] M. Hinnerich, *On the pricing of swaps and swaptions* Licentiate Thesis in Department of Finance, Stocholm School of Economics, 2006.

[24] T. S. Y. Ho, S. Lee, *Term structure movements and the pricing of interest rate contingent claims*, The Journal of Finance, 41: 1011-1029, 1986.

[25] L. P. Hughston, *Inflation derivatives*, Working Paper, Merrill Lynch, 1998.

[26] R. A. Jarrow, D. B. Madan, *Option pricing using the term structure of interest rates to hedge systematic discontinuities in asset returns*, Working Paper, Cornell University, 1991.

[27] R. Jarrow, Y. Yıldırım, *Pricing Treasury inflation protected securities and related derivatives using an HJM model, Journal of Financial and Quantitative Analysis*, 38(2): 337-357,2003.

[28] D. Lamberton, B. Lapeyre, *Introduction to Stochastic Calculus Applied to Finance*, Chapman & Hall/CRC, 2000.

[29] K. S. Leung, L. Wu, *Inflation Derivatives: HJM framework and market models*, Master of Science Thesis, The Chinese University of Hong Kong, 2008.

[30] L. Malvaez, *Valuation of Infation-Indexed Derivatives with three factor model*, Master Thesis, 2005.

[31] F. Mercurio, *Pricing inflation-indexed derivatives*, Quantitative Finance, 5(3): 289-302, 2005.

[32] F. Mercurio, N. Moreni, *Pricing inflation-indexed option with stochastic volatility*, August 2005.

[33] B. Oksendal, *Stochastic Differential Equations: An Introduction with Applications*, Springer, 1998.

[34] W. Runggaldier, *Jump-diffusion models*, Lecture Notes University of Padova, 2002.

[35] S. Shreve, *Stochastic Calculus for Finance II* , Springer, 2004.

[36] A. Stewart, *Pricing inflation-indexed derivatives using the extended Vasicek model of Hull and White*, Master Thesis in Mathematical Finance, Exeter College, University of Oxford, 2007.

[37] T. Z. Takadong, *Pricing models for inflation linked derivatives in an illiquid market*, Master of Science Thesis, 2008.

[38] E. Tekmen, *Enflasyona endeksli tahviller ve Türkiye uygulaması*, Hazine Müsteşarlığı Kamu Finansmanı Genel Müdürlüğü Uzmanlık Tezi, 2005.