

GEOMETRICAL PHASES AND MAGNETIC MONOPOLES

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ABSTRACT

GEOMETRICAL PHASES AND MAGNETIC MONOPOLES

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In this thesis, we study the subject of geometrical phases in detail by considering its various forms. We focus primarily on the relation between quantum geometrical phases and magnetic monopoles, and study how one can make use of the concepts of geometrical phases to define magnetic monopoles.

Keywords: geometrical phases, magnetic monopoles, gauge theories

ÖZ

GEOMETRİK FAZLAR VE MANYETİK TEK-KUTUPLAR

Değer, Sinan

Yüksek Lisans, Fizik Bölümü

Tez Yöneticisi : Doç. Dr. Bayram TEKİN

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Bu tez çalışmasında geometrik fazlar konusu ayrıntılı bir şekilde sunulmuştur. Bu sırada temel odak noktalarından bir tanesi kuantum geometrik fazların manyetik tek-kutuplarla olan ilişkisidir. Geometrik fazlar kullanılarak manyetik tek-kutup tanımının nasıl yapılabileceği incelenmiştir.

Anahtar Kelimeler: geometrik fazlar, manyetik tek-kutuplar, ayar kuramları

To my family

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CHAPTER 1

INTRODUCTION

I would like to begin with a *disclaimer*: I take it very seriously to note before beginning the present study that although all the material presented here is written in my own words and all of the calculations encountered are also done by myself, I claim no originality of the context. This thesis is presented as a review of a bundle of knowledge already available in physics literature. I cited all of the references used to prepare this thesis, along with some other material which serve as a useful guide in the context of my chosen topic. As stated although none of the material is original to this text, it is written through my interpretation of all the studied material, therefore ending in a unique review of concepts that are important to a discussion of geometrical phases and magnetic monopoles.

Though introduced into physics literature less than 30 years ago, it is no exaggeration to say that the topic of quantum geometrical phases had a profound and strong impact on almost all fields of modern physics. What we wish to do here is to take a stroll through the topic with some special emphasis on its aspects related to magnetic monopoles.

The basic work when it comes to geometrical phases is usually attributed to M.V. Berry, who has shown in a 1984 paper [1] that a system, when evolved through a circuit via the adiabatic change of environmental parameters, gains an extra phase factor other than the usual dynamical phase. In the very same paper, he stated that this phase is non-integrable (hence cannot be just set to 1), and showed that it has a geometrical nature. The discussion of Berry's paper [1] will be the focus of Chapter 2. This has set the topic in motion so as to say; following this paper were many other papers on the subject, some of which shall be the focus of our attention. Also, we refer the interested reader to some items here. First of all, there is a resource letter on the subject of geometrical phases which summarizes in detail the

literature available on the topic and lists them in categories [2]. There are text books that will be useful in the sense that they both contain Berry's phase and some necessary tools from quantum mechanics [3][4][5][6]. There are numerous papers that deal with Berry's phase from a different point of view, or use the results of Berry's paper on various problems; here we cite a few items of interest [7][8][9].

A short time after Berry's paper, B. Simon has shown that this phase corresponds to the holonomy in a line bundle, further strengthening the relation of this phase to geometry [11]. He has shown that the adiabatic theorem provides a natural way to define a connection for this line bundle, which as we shall see is of paramount importance for a discussion of geometrical phases. We will give brief review of this in Chapter 4. A short while later, Aharonov - Anandan has made a very important contribution [12]. In their version of the geometrical phase, the condition of an adiabatic evolution is lifted beautifully. They have shown that if the state under consideration was to traverse a closed circuit in the projective Hilbert space, then it is going to gain a geometrical phase in the Hilbert Space. And as stated, this evolution does not need to be done in an adiabatic manner. So it can be said that they generalized the result found by Berry. We shall discuss this important result in Chapter 3. For an application of this phase, we refer to [13].

The other main focus of this work is the subject of magnetic monopoles. As elusive as this particle is, there is no denying that modern theoretical physics is riddled with it. Even though it has its origins at much earlier times, perhaps the most important work that secured its position is by Dirac. In his paper [15], he showed that if a magnetic counterpart to electric charge exists, the quantization of charge follows naturally. But his consideration of the magnetic monopoles carries within a semi-infinite singularity, therefore making it a little cumbersome. There is a vast collection of work done on magnetic monopoles; our focus will be on its relation with geometrical phases though. Even within the pioneering work done by Berry [1], he has shown that for a certain problem, the geometrical phase is the flux of a magnetic monopole at the origin. Next, in a recent paper by Sonner and Tong [20], they used the non-Abelian version of the adiabatic geometrical phase to define a 't Hooft-Polyakov monopole. Unlike the Dirac monopole, the 't Hooft-Polyakov [21][22] monopole is without a singularity. It arises in a spontaneously broken gauge theory as a composite object. It is quite surprising that such a smooth configuration arises in the context of a Berry phase discussion.

Throughout the thesis, we set $\hbar = 1$. Vectors will be printed in bold, like **B** instead of B . Hats over vectors denote unit vectors, *i.e.* $\hat{B} = \frac{\mathbf{B}}{|\mathbf{B}|}$.

CHAPTER 2

BERRY'S GEOMETRICAL PHASE

The consideration to Berry's Phase [1] begins with the assumption that we have a quantum mechanical system that is not isolated. So the physics that defines our system should depend on a set of parameters, namely the environmental parameters $R = \{R_1, R_2, \dots, R_N\}$ with N being the dimension of this environmental space. Every R_i , $i = 1, 2, \dots, N$ corresponds to a particular configuration of the environment. We may think of the constituents of this set as the points of a manifold M , and that every such point is a different configuration of the environment. This manifold is then called the parameter space of our quantum system. It is plausible to pick our parameters to depend on time so as to give $R = R(t)$, describing an environment that evolves with time. And also, for the consideration of Berry's Phase, it is plausible to choose a three-dimensional parameter space, thereby making R a three-vector, $\mathbf{R}(t) = (R_1, R_2, R_3)$ [1][4][10]. Before moving further, we note that generalizing the following discussion to N parameters is not too difficult.

Being in this environment, the Hamiltonian will be described differently at each point of the background, *i.e.* it will be a function of these parameters; $H = H(\mathbf{R}(t))$, the time dependence coming from the fact that R changes with time. So the time evolution of a state ψ will be described by the time dependent version of Schrödinger's equation

$$i \frac{d\psi(t)}{dt} = H(\mathbf{R}(t))\psi(t) . \quad (2.1)$$

Though this equation gives the time evolution of our wave function, at any instance t , one can still use the time independent version

$$H(\mathbf{R})|n; \mathbf{R}\rangle = E_n(\mathbf{R})|n; \mathbf{R}\rangle . \quad (2.2)$$

Here, by making some assumptions, we shall obtain the problem that Berry considered [1]. The first assumption one needs to make is the one of single-valuedness of the observables. This assumption basically requires that if our evolving system happens to return to an initial point in the parameter space, then at any of such occurrences, the observables must be the same. For Berry's consideration, we have an evolution that is cyclic, *i.e.* the environmental parameters evolve in a closed path and return to their initial value after some period T . Therefore we should have [10]

$$C : R(0) \rightarrow R(t) \rightarrow R(T) = R(0) , \quad (2.3)$$

where C represents a closed curve in the parameter space. Since our Hamiltonian depends on the external parameter $\mathbf{R}(t)$, when we choose a periodic evolution, itself and its eigenvalues must be the same at $\mathbf{R}(0)$ and $\mathbf{R}(T)$,

$$H(\mathbf{R}(0)) = H(\mathbf{R}(T)) , \quad (2.4)$$

$$E_n(\mathbf{R}(0)) = E_n(\mathbf{R}(T)) . \quad (2.5)$$

Here, we note that $\mathbf{R}(T) = \mathbf{R}(0)$ does not imply that $|n; \mathbf{R}(T)\rangle = |n; \mathbf{R}(0)\rangle$. This follows from the fact that each parameter R correspond to a different patch in the manifold M . Since these vectors depend on R , and since we used multiple R 's to cover M , this forces that the eigenvectors be defined only locally. So when we start from $|n; \mathbf{R}(0)\rangle$, in general it is not sound to write the evolved state simply just as $|n; \mathbf{R}(t)\rangle$, because it requires that the state to evolve within the initial patch only, and that this patch does not change with time at all. But since we are working on a continuous space, the initial and evolved vectors are related via a simple transformation, namely a unitary transformation,

$$|n; \mathbf{R}(T)\rangle = e^{i\chi_n(R)} |n; \mathbf{R}(0)\rangle . \quad (2.6)$$

This is also true for different vectors like $|n; \mathbf{R}\rangle'$:

$$|n; \mathbf{R}\rangle' = e^{i\chi_n(R)} |n; \mathbf{R}\rangle , \quad (2.7)$$

i.e., the type of unitary transformation that connects the initial and evolved kets are also used to connect different kets in the Hilbert space for the current discussion. Here we emphasize two points. First, if vectors $\{|n; \mathbf{R}\rangle\}$ form a basis, the basis $\{|n; \mathbf{R}'\rangle\}$ obtained via the unitary phase transformation above is as valid as the first one. Next we note that for the sake of having continuity in the text, here we used R again as three-vectors. But the discussion holds as well for any arbitrary dimensional R for again any arbitrary dimensional manifold M , as long as some of the above assumptions are present (like there may be no degeneracy here). This transformation is also called a gauge transformation. Here the factor $e^{i\chi_n(R)}$ is an element of the group $U(1)$. In Chapter 5 we will see that in the presence of degeneracies, we will obtain a more complicated factor in the form of a non-Abelian phase factor. For the present discussion, we say that we have a $U(1)$ gauge symmetry.

We may use any linearly independent complete set of vectors to span our Hilbert space. One way to form such a set is to choose the eigenkets of the Hamiltonian; for every eigenket, our discrete and non-degenerate Hamiltonian will have a unique eigenvalue. The spectral resolution of H will be given by

$$H(R) = \sum_n E_n(R) |n; R\rangle \langle n; R|. \quad (2.8)$$

Next, we impose the condition of adiabaticity. Though the concept of adiabaticity appears on many diverse areas in physics in different forms, the main idea behind it in a nutshell is like the following. What one means by an adiabatic evolution is that this evolution takes places very slowly relative to a time which characterizes the given system. For example if one has a simple pendulum oscillating inside a box, an adiabatic evolution would not change its motion, and can be obtained by a change over the system like moving the box from one place to another, that takes in a very long time when compared to the frequency of oscillation of the pendulum [5]. For our purposes though, what we mean by an adiabatic evolution is one where the rate of change of the Hamiltonian is very close to zero. Also if, initially, our system begins from an eigenstate of a non-degenerate Hamiltonian, at any instant in our adiabatic evolution, it will remain an eigenstate of the instantaneous Hamiltonian. Furthermore, we assume that the state does not jump between states, meaning that if we started from the n^{th} eigenstate, our system shall remain an eigenstate of $H(R(t))$ with the same quantum number n for all times t . So for a cyclic and adiabatic evolution, and with making use of the single-valuedness of the observables, an eigenstate $|n; R(0)\rangle$ of $H(R(0))$, will remain in the n^{th} eigenstate at any t

during the evolution, apart from a phase factor which consists of the usual dynamical phase times a new phase, which thereafter came to be called as “Berry’s Phase”. Berry in his paper [1] proved that this phase is non-integrable and geometrical in nature. We next demonstrate how to obtain this phase factor. The conditions where the adiabatic approximation is valid are derived in Appendix A [10].

Here we consider the evolution of a state $|\psi(t)\rangle$. We have a Hamiltonian that depends on changing environmental parameters $H(\mathbf{R}(t))$, $\mathbf{R} = \{R_1, R_2, R_3\}$. Initially,

$$|\psi(0)\rangle = |n; \mathbf{R}(0)\rangle, \quad (2.9)$$

for n a normalized state. As was previously stated, our state evolves according to Schrödinger’s equation (2.1), and at any specific time t the time independent version of Schrödinger’s equation (2.2) holds. For a cyclic evolution, the path traversed between times $t = 0$ and $t = T$ will be a closed path in the parameter space. When evolved adiabatically, our system that was initially at $|n; \mathbf{R}(0)\rangle$ will evolve with H and hence be in state $|n; \mathbf{R}(t)\rangle$ at a later time t , accompanied by suitable phase factors.

Using this background, we make the following ansatz that our state ψ can be written as

$$|\psi(t)\rangle = \exp\left\{-i \int_0^t dt' E_n(\mathbf{R}(t'))\right\} \exp(i\gamma_n(t)) |n; \mathbf{R}(t)\rangle, \quad (2.10)$$

where γ_n is the phase factor that we set out to find. This ansatz uses the adiabatic theorem; it shows that our state will remain in the vicinity of the initial state, except for a phase factor. To find the rigorous form of γ_n , we begin by noting that $\psi(t)$ satisfies the Schrödinger’s equation. Substituting (2.10) to (2.1) we find

$$\frac{d}{dt}\gamma_n(t) = i\langle n; \mathbf{R}(t) | \frac{d}{dt} | n; \mathbf{R}(t) \rangle, \quad (2.11)$$

$$\frac{d}{dt}\gamma_n(t) = i\langle n; \mathbf{R}(t) | \nabla_R | n; \mathbf{R}(t) \rangle \cdot \frac{d}{dt} \mathbf{R}(t), \quad (2.12)$$

where the differential operator ∇_R is the gradient operator in the parameter space, *i.e.* $\nabla_R =$

$\hat{\mathbf{R}}_i \frac{\partial}{\partial R_i}$ where a sum over $i = 1, 2, 3$ for our three-dimensional consideration is implied. Integrating over the closed circuit \mathbf{C} gives γ_n :

$$\gamma_n(C) = i \oint_C \langle n; \mathbf{R}(t) | \nabla_R n; \mathbf{R}(t) \rangle \cdot d\mathbf{R} . \quad (2.13)$$

This is the geometrical phase change of ψ after an adiabatic and cyclic evolution that encircles the closed path \mathbf{C} in the parameter space. Here we note that as long as we require that our assumptions hold, our geometrical phase only depends on the path traversed \mathbf{C} , and nothing else. One can also show that γ_s is real, since our states are normalized (the dependence of $|n; \mathbf{R}(t)\rangle$ on $\mathbf{R}(t)$ will be abbreviated from now on):

$$\nabla_R \langle n | n \rangle = 0 , \quad (2.14)$$

$$\nabla_R \langle n | n \rangle = \langle \nabla_R n | n \rangle + \langle n | \nabla_R n \rangle = 0 , \quad (2.15)$$

thus

$$(\langle n | \nabla_R n \rangle)^* = -\langle n | \nabla_R n \rangle , \quad (2.16)$$

which shows that $\langle n | \nabla_R n \rangle$ is purely imaginary, thereby making γ_n real. Though (2.13) provides a form for the geometrical phase, its calculation is somewhat cumbersome. To look for a simpler form, one can use Stokes' theorem for the closed integral (2.13)

$$\gamma_n(C) = i \oint_C \langle n | \nabla_R n \rangle \cdot d\mathbf{R} = -\text{Im} \iint_S \nabla \times \langle n | \nabla n \rangle \cdot d\mathbf{S} , \quad (2.17)$$

where S is any surface bounded by the circuit C in our parameter space. Here, invoke:

$$\nabla \times [f(\mathbf{x}) \nabla g(\mathbf{x})] = (\nabla f(\mathbf{x})) \times (\nabla g(\mathbf{x})) , \quad (2.18)$$

to obtain:

$$\gamma_n(C) = -\text{Im} \iint_S d\mathbf{S} \cdot \langle \nabla n | \times | \nabla n \rangle . \quad (2.19)$$

Inserting a resolution of identity the above relation becomes

$$\gamma_n(C) = -\text{Im} \iint_S d\mathbf{S} \cdot \sum_{m \neq n} \langle \nabla n | m \rangle \times \langle m | \nabla n \rangle , \quad (2.20)$$

where the $m = n$ case is excluded since it brings the cross product of the vector with itself, hence it vanishes. To find these matrix elements, we differentiate the time-independent Schrödinger's equation with respect to R , then multiply by $\langle m |$ to get

$$\langle m | \nabla_R H | n \rangle + \langle m | H | \nabla_R n \rangle = E_n \langle m | \nabla_R n \rangle , \quad (2.21)$$

$$\langle m | \nabla_R n \rangle = \frac{\langle m | \nabla_R H | n \rangle}{E_n - E_m} . \quad (2.22)$$

Taking the complex conjugate yields the other term, thus we now have what later came to be known as the Berry's Phase:

$$\gamma_n(C) = - \iint_S d\mathbf{S} \cdot \mathbf{V}_n(R) , \quad (2.23)$$

where

$$\mathbf{V}_n(\mathbf{R}) = \text{Im} \sum_{m \neq n} \frac{\langle n | \nabla_R H | m \rangle \times \langle m | \nabla_R H | n \rangle}{(E_m(\mathbf{R}) - E_n(\mathbf{R}))^2} . \quad (2.24)$$

This was the central result of Berry's Paper [1]. Here follows several important notes on this result.

First of all, we see that our final result does not depend on $|\nabla n\rangle$ anymore. Next, we see that when the energy levels are degenerate, our result becomes divergent. Due to this reason, we assumed that our circuit \mathbf{C} does not pass over any degenerate configuration in the parameter space. The case for \mathbf{C} close to a degeneracy point is considered in the next section.

We now go a little back to discuss an item of importance that we encountered along the way. Remember that during the derivation we encountered the form:

$$\gamma_n(C) = i \oint_C \langle n | \nabla_R n \rangle \cdot d\mathbf{R} . \quad (2.25)$$

We call the integrand as the Berry vector potential

$$\mathbf{A}_n(\mathbf{R}) = i \langle n | \nabla_R | n \rangle , \quad (2.26)$$

and the one-form that follows naturally as the Berry connection one-form

$$A_n = i \langle n | d | n \rangle . \quad (2.27)$$

We will make use of these objects in the following chapters.

2.1 PHASE NEAR DEGENERACIES

Remember that in the derivation of the geometrical phase factor from the previous section we assumed that the eigenvalues of the Hamiltonian were non-degenerate. But our discussion here will show that the existence of degeneracies will be of great importance in understanding the nature of Berry's phase.

To analyze the degenerate case, we consider a problem where we have a spin inside a magnetic field [1][4]. Here, the adiabatic change of the magnetic field will cause our state, which will be realized through "a spin eigenstate", to traverse a path in the parameter space. Here we see that the environmental parameter is the magnetic field, and R is formed by the components of this magnetic field. Changing this field adiabatically causes a non-integrable phase to the spin-state. Here we have to emphasize that the path traversed adiabatically in the B -space, again does not pass over any point where there are degeneracies. But this time, we have a degenerate Hamiltonian, and we will solve our problem close to a degeneracy point.

So suppose that the n^{th} energy eigenstate is degenerate with the m^{th} eigenstate at a point R_d of the parameter space. Without any loss of generality we can make this degeneracy point the origin of our parameter space. We again note that the circuit traversed that is formed by the adiabatic variation of the magnetic field does not pass through this configuration. Again, we use a three-vector R , namely $\mathbf{B}(t)$. Let us choose our spin to point in the positive z -direction. If we change the direction this magnetic field slowly without altering its magnitude, the spin component along the magnetic field will take values of $S_z = n$, where $n = -s, \dots, +s$. The Hamiltonian is

$$H(\mathbf{B}) = -g\mu\mathbf{S} \cdot \mathbf{B} , \quad (2.28)$$

and here we note that $g\mu\mathbf{S}$ is the magnetic moment. The corresponding energy eigenvalues are given as

$$E_n = -g\mu n B . \quad (2.29)$$

We see that for $B = 0$, there is a $2s+1$ -fold degeneracy coming from the values of n . Due to our choice, $B = 0$ corresponds to the origin of our parameter space. Note that since we are not changing the magnitude of B , the path obtained through the variation of the direction of B will reside on a 2-sphere of radius B . To calculate the geometrical phase gained during a complete cycle, γ_n , we will use (2.23), \mathbf{V}_n will be calculated using (2.24), which gives

$$\mathbf{V}_n(\mathbf{B}) = \text{Im} \sum_{m \neq n} \frac{\langle n; \mathbf{B} | \mathbf{S} | m; \mathbf{B} \rangle \times \langle m; \mathbf{B} | \mathbf{S} | n; \mathbf{B} \rangle}{B^2(m-n)^2}, \quad (2.30)$$

for $\nabla_{\mathbf{B}} H(\mathbf{B}) = -g\mu\mathbf{S}$ and $(E_m - E_n)^2 = B^2(m-n)^2$ from (2.29). For calculational simplification, we can make the z -axes of the spin- and B-spaces to coincide. Though simple, the calculation of components of \mathbf{V}_n require some cumbersome calculations. Only the relevant parts will be provided here. After using the ladder operators $S_{\pm} = S_x \pm iS_y$ a few times, we see that only the terms $m = n \pm 1$ survive, and that the \hat{x} and \hat{y} components of \mathbf{V}_n are zero. The final component is

$$\begin{aligned} \mathbf{V}_n(\mathbf{B})_{\hat{z}} = \frac{1}{B^2} \text{Im} \{ & \langle n | S_x | n+1 \rangle \langle n+1 | S_y | n \rangle - \langle n | S_y | n+1 \rangle \langle n+1 | S_x | n \rangle \\ & + \langle n | S_x | n-1 \rangle \langle n-1 | S_y | n \rangle - \langle n | S_y | n-1 \rangle \langle n-1 | S_x | n \rangle \} \end{aligned} \quad (2.31)$$

Here, make use of the following relations to simplify the above relation

$$\langle n \pm 1 | S_x | n \rangle = \frac{1}{2} [s(s+1) - n(n \pm 1)]^{\frac{1}{2}}, \quad (2.32)$$

$$\langle n \pm 1 | S_y | n \rangle = \mp \frac{1}{2} i [s(s+1) - n(n \pm 1)]^{\frac{1}{2}}. \quad (2.33)$$

Substituting these into (2.31) yields

$$V_n(\mathbf{B})_{\hat{z}} = \frac{1}{B^2} \text{Im}(in), \quad (2.34)$$

$$V_n(\mathbf{B})_{\hat{z}} = \frac{n}{B^2}. \quad (2.35)$$

Now reverting back to a general coordinate configuration of spin- and B-spaces,

$$\mathbf{V}_n(\mathbf{B}) = \frac{n}{B^2} \mathbf{B}, \quad (2.36)$$

so γ_n is given as

$$\gamma_n(C) = -n \iint_{S(C)} \frac{\hat{\mathbf{B}}}{B^2} \cdot d\mathbf{B}. \quad (2.37)$$

Close consideration reveals that this result has the same form as the expression of solid angle Ω subtended by a curve C seen from the origin:

$$\Omega \equiv - \int_A \frac{\hat{\mathbf{R}}}{R^2} \cdot d\mathbf{A} , \quad (2.38)$$

where \mathbf{R} is a vector from the origin to the surface enclosed by the loop. Thus [1][4]

$$\gamma_n(C) = -n \iint_{S(C)} d\Omega = -n\Omega(C) , \quad (2.39)$$

where $\Omega(C)$ is the solid angle our closed circuit subtends at the degeneracy point. There is a very important conclusion here; as can be seen from (2.39), the phase γ_n that our state gains is dependent on nothing else but the quantum number n and the solid angle Ω . Therefore what we have is an object that has to do purely with the geometry of the system, and as long as the evolution is done in an adiabatic manner, does not depend on anything else (e.g. the magnetic field or the properties of spin etc.) This is the reason for the labeling “geometrical” to our phase.

Here we discuss another object of great importance for our consideration. Remembering the form of the vector-valued function:

$$\mathbf{V}_n(\mathbf{B}) = n \frac{\mathbf{B}}{B^3} . \quad (2.40)$$

Using this vector valued function we see that γ_n calculated via (2.37) is nothing but the flux of the magnetic field of a magnetic monopole through a surface S bounded by C .

We discuss the properties of this magnetic monopole later in Chapter 6. Although what we encountered here is a Dirac monopole, the discussion of its properties will not be given along the line of his work [15], but along the line of a modern version of it, namely the Wu-Yang picture [16][17]. But in any case, the form of (2.40) already suggests that, in the space of the B-field, there is a singularity at the origin (at $\mathbf{B} = 0$ in our discussion above) and we have a $\frac{1}{B^2}$ type “magnetic” field reminiscent of the Dirac monopole. The idea that one has a magnetic monopole in the space of magnetic fields (not the real space) is somewhat confusing but remarkable.

CHAPTER 3

THE AHARONOV-ANANDAN GEOMETRICAL PHASE

We now proceed to the version of geometrical phase introduced by Aharonov and Anandan in their 1987 paper [12]. They showed that in order for a system to gain a geometrical phase, an adiabatic change in the parameter space is not needed. A general cyclic evolution in the projective Hilbert space \mathcal{P} will result in a non-integrable geometrical phase to our state in the Hilbert space \mathcal{H} . Furthermore, their result contains Berry's result as the adiabatic limit.

We begin by picking up a normalized state $\psi(t) \in \mathcal{H}$. It is a postulate of Quantum Mechanics that the time evolution of this state is given by the Schrödinger's equation (2.1). After a complete cycle, comparing the initial and final states we have

$$|\psi(T)\rangle = e^{i\phi}|\psi(0)\rangle, \quad (3.1)$$

where $\phi \in \mathbb{R}$, is the object of consideration. Now we define a projection map $\Pi : \mathcal{H} \rightarrow \mathcal{P}$ in the usual sense that

$$\Pi(|\psi\rangle) = \{|\psi'\rangle : |\psi'\rangle = c|\psi\rangle, c \in \mathbb{C}\}, \quad (3.2)$$

with $|\psi'\rangle$ being the projection of $|\psi\rangle \in \mathcal{H}$ in \mathcal{P} .

$|\psi(t)\rangle$ defines a curve $\mathbf{C} : [0, T] \rightarrow \mathcal{H}$. The projection of this curve corresponds to a closed curve in \mathcal{P} , *i.e.* $\tilde{C} \equiv \Pi(C)$ is a closed curve, as we shall show shortly.

Here we define $|\tilde{\psi}(t)\rangle = e^{-if(t)}|\psi(t)\rangle$ such that $f(T) - f(0) = \phi$. So,

$$|\tilde{\psi}(T)\rangle = e^{-if(T)}|\psi(T)\rangle = e^{-if(T)}e^{i\phi}|\psi(0)\rangle. \quad (3.3)$$

Substitute $|\psi(0)\rangle = e^{if(0)}|\tilde{\psi}(0)\rangle$:

$$|\tilde{\psi}(T)\rangle = e^{i\phi}e^{-i(f(T)-f(0))}|\tilde{\psi}(0)\rangle, \quad (3.4)$$

$$|\tilde{\psi}(T)\rangle = |\tilde{\psi}(0)\rangle , \quad (3.5)$$

which is a useful relation for $|\tilde{\psi}(T)\rangle \in \mathcal{P}$. That shows that the curve we have in the projective Hilbert space \mathcal{P} , \tilde{C} , is a closed curve. Now insert $|\psi(t)\rangle = e^{if(t)}|\tilde{\psi}(t)\rangle$ into (2.1)

$$H(t)e^{if(t)}|\tilde{\psi}(t)\rangle = i \left(i \frac{df}{dt} e^{if(t)}|\tilde{\psi}(t)\rangle + e^{if(t)} \frac{d}{dt} |\tilde{\psi}(t)\rangle \right) . \quad (3.6)$$

Take the inner product with $\langle\psi(t)| = \langle\tilde{\psi}(t)|e^{-if(t)}$

$$\langle\psi(t)|H(t)|\psi(t)\rangle = -\frac{df}{dt} + i\langle\tilde{\psi}(t)|\frac{d}{dt}|\tilde{\psi}(t)\rangle . \quad (3.7)$$

Thus

$$-\frac{df}{dt} = \langle\psi(t)|H(t)|\psi(t)\rangle - i\langle\tilde{\psi}(t)|\frac{d}{dt}|\tilde{\psi}(t)\rangle . \quad (3.8)$$

Now we can get rid of the arbitrary factor f by integrating from 0 to T ,

$$-\int_0^T \left(\frac{df}{dt} \right) dt = f(T) - f(0) = \phi , \quad (3.9)$$

$$\phi = \int_0^T \langle\psi(t)|H(t)|\psi(t)\rangle dt + \int_0^T \langle\tilde{\psi}|i\left(\frac{d}{dt}|\tilde{\psi}\rangle\right) dt . \quad (3.10)$$

The first integrand can be recognized as the dynamical part of the phase ϕ , the second one, which we shall label from now on as β , is the phase which we will call as the ‘‘Aharonov-Anandan Phase’’ (AA Phase) [12].

$$\beta = i \int_0^T \langle\tilde{\psi}|\left(\frac{d}{dt}|\tilde{\psi}\rangle\right) dt . \quad (3.11)$$

When written as:

$$\beta = i \int_0^T \langle\tilde{\psi}|d|\tilde{\psi}\rangle , \quad (3.12)$$

we see that this phase factor is independent of the parameter t and is uniquely defined up to $2\pi n, n \in \mathbb{Z}$. Furthermore, it does not depend on H or any other factor. Thus this new factor depends on the geometry of the path \tilde{C} in the projective Hilbert space only, hence rightfully deserving the labeling ‘‘geometrical phase’’.

We define the one-form inside (3.12):

$$A = i\langle\tilde{\psi}|d|\tilde{\psi}\rangle, \quad (3.13)$$

as the Aharonov-Anandan connection one-form [10][12]. We shall make use of this object in the following chapters.

Now we demonstrate that in the adiabatic limit, the more general AA Phase has the same form as Berry's phase [12]. To do so, we use a slowly varying Hamiltonian and remember that for any specific t ,

$$H(t)|n(t)\rangle = E_n(t)|n(t)\rangle, \quad (3.14)$$

holds. We also make use of a complete set of kets $\{|n(t)\rangle\}$ as basis, and using it our state can most generally be defined as

$$|\psi(t)\rangle = \sum_n a_n(t) \exp\left(-i \int E_n dt\right) |n(t)\rangle, \quad (3.15)$$

This satisfies (2.1), substituting and then multiplying by $\langle m(t)|$, $m \neq n$ yields (with an obvious suppression of t dependencies):

$$\dot{a}_m = - \sum_n a_n \exp\left\{i \int (E_m - E_n) dt\right\} \langle m|\dot{n}\rangle, \quad (3.16)$$

where $|\dot{n}\rangle = \frac{d}{dt}|n\rangle$; the above relation by separating the $n = m$ term becomes

$$\dot{a}_m = -a_m \langle m|\dot{m}\rangle - \sum_{n \neq m} a_n \exp\left\{i \int (E_m - E_n) dt\right\} \langle m|\dot{n}\rangle. \quad (3.17)$$

We can find a more suitable form for $\langle m|\dot{n}\rangle$ by taking the time derivative of (3.14)

$$\left(\frac{d}{dt}H\right)|n\rangle + H|\dot{n}\rangle = \dot{E}_n|n\rangle + E_n|\dot{n}\rangle, \quad (3.18)$$

again suppressing the t -dependency. Multiply by $\langle m|$:

$$\langle m|\dot{n}\rangle = \frac{\langle m|\frac{d}{dt}H|n\rangle}{E_n - E_m}. \quad (3.19)$$

Now substitute this form to (3.17)

$$\dot{a}_m = -a_m \langle m|\dot{m} \rangle - \sum_{n \neq m} a_n \exp \left(i \int (E_m - E_n) dt \right) \frac{\langle m|\frac{d}{dt} H|n \rangle}{E_n - E_m} . \quad (3.20)$$

which is a very suitable form for our purpose. Here, we invoke (A.11), which is the condition of adiabacity and makes the second term vanish for the problem at hand. Thus within the adiabatic approximation, the above relation is simply equal to:

$$\dot{a}_m + a_m \langle m|\dot{m} \rangle = 0 , \quad (3.21)$$

$$a_m(t) = e^{-\int \langle m|\dot{m} \rangle dt} a_m(0) , \quad (3.22)$$

and by our formulation, $a_m = e^{i\gamma_m}$, note the similarity between the above relation and (2.11).

From the last result it directly follows that

$$\gamma_m = i \int_0^t \langle m|\dot{m} \rangle dt , \quad (3.23)$$

which is also obtained quite simply from (2.11). We note that this phase factor, when we have $|\tilde{\psi}(t)\rangle \simeq |m(t)\rangle$ within the adiabatic approximation, has the same form as (3.11). What is important here is that the phase β in (3.11) was not found within the adiabatic approximation, that result was exact. Another thing of importance is the fact that while deriving β , we did not impose the condition that our state starts from an eigenstate of the Hamiltonian. Therefore the obtained result shows that by choosing an adiabatically evolving H , one can obtain Berry's phase as a limiting case of the more general AA phase, (3.11) [12].

CHAPTER 4

HOLONOMY AND GEOMETRICAL PHASES

In this chapter, we will observe the geometrical nature of the phase factor more thoroughly with the help of some important objects from differential geometry. A discussion of a selection of these, including introductory discussions of fibre bundles and holonomy, can be found in Appendix C.

We will first begin by Barry Simon's interpretation of Berry's phase [11]. Next, we will use the fibre bundle techniques within the discussion of the Aharonov-Anandan phase [10][12]. Both discussions will be based on [10].

4.1 SIMON'S INTERPRETATION OF BERRY'S PHASE

We start by briefly reminding the background at hand. The background that we used while considering Berry's phase will be intact, i.e we are going to have an adiabatically evolving Hamiltonian that depends on a set of environmental parameters, $R = \{R_1, R_2, \dots, R_N\}$ where N is the dimension of the parameter space. We also assume that $H(R(t))$ and its eigenvalues are non-degenerate, and even though we have a time dependent Hamiltonian, there are no level crossings between states. Hence if our system is initially at the n th eigenstate, due to adiabatic approximation it will remain in this state except for phase factors, and if it is at energy state $E_n(R(0))$ initially, it will not jump to energy state $E_m(R(t))$ for $n \neq m$ at a later time t . We also require the evolution to be cyclic and that the observables are single valued. An adiabatic change of the external physical parameter in the environment space will cause a phase of geometrical origin in the Hilbert space, \mathcal{H} .

Simon, in his paper [11], found that Berry's phase, is the holonomy in a line bundle, and the

necessary connection is provided by the adiabatic theorem. Although by following a different method, this will be what we shall show in this section [10].

The discussion starts with the construction of the necessary bundle, which is a complex line bundle that we shall denote as L^n . It is complex since the geometrical phase to our state will be realized in the Hilbert space \mathcal{H} , and we need $L^n \subset \mathcal{H}$. Again the $U(1)$ symmetry presents itself, we choose $U(1)$ as both our structure group and our standard fibre. We need a manifold M , which will be constructed as was mentioned in Chapter 2; *i.e.* the environmental parameters R_i will correspond to different points in this manifold. Also we need our map $\Pi : L^n \rightarrow M$, so our line bundle can be denoted as: $(L^n, M, \Pi, U(1))$

By taking into consideration the nature of our problem, the fibres of the bundle L^n can be defined as

$$L_x^n \equiv \{\psi \in \mathcal{H} : |\psi\rangle = c|n; R\rangle, c \in \mathbb{C}\}, \quad (4.1)$$

representing the fibre at point $x \in L^n$, the line bundle.

Next, we try to find the transition functions $g_{\alpha\beta}$. To do so, we follow a similar procedure that let us make the general definition of the transition functions in the first place. We have already mentioned that the state kets are only defined up to a phase transformation. Since each environmental parameter correspond to a specific situation of the environment, the kets $|n; R\rangle$ that correspond to them will only be defined locally. And as was given in the definition of fibre bundles, we use a covering set of open neighborhoods $\{U_\alpha\}$ of our base manifold, which we denoted as M here. So corresponding to different open neighborhoods, there will be different single-valued eigenvectors. Assume two open neighborhoods U_α and U_β , and related with these are the eigenvectors $|n; R, \alpha\rangle$ and $|n; R, \beta\rangle$ respectively. We discussed that since both satisfy (2.2), they must be related through a phase only, as in

$$|n; R, \alpha\rangle = e^{i\chi_{\alpha\beta}(R)}|n; R, \beta\rangle, \quad \chi_{\alpha\beta}(R) \in \mathbb{R}. \quad (4.2)$$

There is a very nice trick that lets one to define the transition functions. When we consider a covering $\{U_\alpha\}$ of the base space M , we obtain all the phase factors that relate the eigenvectors. And since the fibre is defined using these eigenvectors (times a complex number), what relates them should also tell us what relates the elements of fibres. In other words, the phase factor that we used to relate the eigenvectors also tell us how to glue the fibres together, which is the

definition of the transition functions. Therefore

$$g_{\alpha\beta} \equiv e^{i\chi_{\alpha\beta}} , \quad (4.3)$$

and we notice that $e^{i\chi_{\alpha\beta}} \in U(1)$, which is also appropriate, since we know that the transition functions form the structure group which we use as $U(1)$ here.

Now that we have the transition functions, we can find the associated principal bundle to L^n . The method to achieve this was outlined in Appendix C. We need to keep the transition functions intact, but we have to change the fibres L_x^n of L^n by the group $U(1)$, which in turn corresponds to only picking states in \mathcal{H} that are normalized. So our new bundle (which is a principal bundle) is $(\lambda^n, M, \Pi, U(1))$, with the fibres given as

$$\lambda_p^n \equiv \{\psi \in \mathcal{H} : |\psi\rangle = e^{i\alpha}|n; R\rangle, \alpha \in [0, 2\pi)\} , \quad (4.4)$$

for p denoting the fibre point. We note that this form is very suitable for our purposes, it looks very much like (2.8). Next we need to impose the condition of a cyclic motion, which is to say that we need to pick a closed curve in M , which we can define as

$$C : [0, T] \rightarrow R(t) \in M \quad \text{together with the condition} \quad R(0) = R(T) , \quad (4.5)$$

since we are considering the single-valued case, we know that when $R(0) = R(T)$, the defining physical situations are exactly the same. Now we can obtain Berry's result within the background we just formed. In Berry's problem, the initial state is $|\psi(0)\rangle = |n; R(0)\rangle$, and certainly this resides within the fibre $\lambda_{R(0)}^n$ with $\alpha = 0$. It was shown in Section 2.1 that this state, after traversing a cyclic path adiabatically, will return to the initial ray and will be in the same eigenstate apart from phase factors. Due to the adiabatic approximation, the horizontal lift of the path R in M will be mapped to a single fibre λ^n in the total space. The local nature of these kets enable us to define them using local sections. (2.7) from Chapter 2 is just in the form of the relation that we used to relate two local sections (C.18) [10]

$$s_1(x) \rightarrow s_2(x) = s_1(x) \cdot g(x), \quad \forall x \in U_\alpha, g(x) \in G , \quad (4.6)$$

where $g(x) \in U(1)$ for our problem. As stated, the adiabatic approximation forces the horizontal lift into a single fibre. Using the local section, we can easily define the corresponding

local connection one-form. When we obtain the local connection one form, it is easy to relate this to the geometrical phase. Since we have a principal bundle, this also has the structure of a group. And within this group, we can find the element which relates the initial and end points of this path, and in Section C.1, we defined this to be the holonomy. For an Abelian connection, the holonomy (C.19) is defined to be

$$\exp\left(i \oint_C A_n\right). \quad (4.7)$$

See that the path-ordering is not required here. Remember that we found that the Berry connection (2.27) transforms like an Abelian object in (B.6), we also found that the geometrical phase is given exactly like the form above in (2.25). Hence we conclude that the geometrical phase found by Berry is exactly the holonomy on a complex line bundle:

$$\exp(i\gamma_n(C)) = \exp\left(i \oint_C A_n\right). \quad (4.8)$$

4.2 AHARONOV-ANANDAN PHASE ON A FIBRE BUNDLE

In this section, we are going to obtain the results of Chapter 3 using the methodology of fibre bundles. We quickly remember the background that enabled us to define the Aharonov-Anandan phase [12]. We had a cyclic evolution hence a closed curve in the projective Hilbert space that did not necessarily correspond to a closed path in the Hilbert space. The evolution need not be adiabatic, nor our initial state necessarily is an eigenstate of the Hamiltonian.

When trying to obtain this phase using holonomy and fibre bundles, we immediately remember the concept of a horizontal lift. Remember that even if we pick a closed path in the base space, its horizontal lift which resides in the total space, is not necessarily a closed curve. So if we choose the base space to be the projective Hilbert space \mathcal{P} or $\mathbb{C}P^\infty$, the total space to be a subspace of the Hilbert space \mathcal{H} , choose the surjection Π as the natural map between \mathcal{H} and \mathcal{P} , and choose $U(1)$ as our structure group, we will indeed obtain the structure necessary to realize the Aharonov-Anandan phase on a fibre bundle [10].

Using these, we start forming the Aharonov-Anandan (AA) principal fibre bundle, η . We can define the projective Hilbert \mathcal{P} or $\mathbb{C}P^{N-1}$ (where $N = \dim \mathcal{H}$) space as the space that

the equivalence classes of rays of \mathcal{H} generate. This is to say that $\mathbb{C}P^{N-1} = \mathcal{H} / \sim$ with the equivalence relation “ \sim ” being defined as:

$$\psi \sim \psi' \quad \text{iff} \quad |\psi'\rangle = c|\psi\rangle, \quad c \in \mathbb{C}, c \neq 0, \quad (4.9)$$

where, just as in (3.2), ψ' and $\psi \in \mathcal{H}$. We define our projection map as the one used in again (3.2), $\Pi : \mathcal{H} \rightarrow \mathcal{P}$. Here, note that we can generalize any of the following to the case where we have an infinite dimensional Hilbert space; the projective Hilbert space then is $\mathbb{C}P^\infty$.

Remember that in Chapter 3, we chose our states $\psi(t) \in \mathcal{H}$ as normalized. To compensate for that here, we choose our base space $S(\mathcal{H})$ as the subspace of \mathcal{H} that contains only normalized states,

$$S(\mathcal{H}) \equiv \{\psi \in \mathcal{H} : \langle \psi | \psi \rangle = 1\}. \quad (4.10)$$

Now we define the fibres. Following the above equation, the fibres are defined as the normalized rays in \mathcal{H}

$$\eta_\kappa \equiv \{\psi \in \mathcal{H} : |\psi\rangle\langle\psi| = \kappa, \langle\psi|\psi\rangle = 1\}, \quad (4.11)$$

where $\kappa \in \mathcal{P}$ is defined as $\Pi(|\psi\rangle) = |\psi\rangle\langle\psi| = \kappa$. This says that the projection map Π maps the states $\psi \in \mathcal{H}$, the total space, to the base space \mathcal{P} in the manner shown. This also shows that the individual fibres are in the form of the group $U(1)$ or the unit circle S^1 , which is to be expected from the fibres of a principal bundle whose structure group is also $U(1)$. Thus the total space of our principal bundle η is made up of all the unit vectors in \mathcal{H} , so as a manifold it can be defined as the unit sphere $S^\infty \subset \mathcal{H}$. If we consider finite dimensional \mathcal{H} , S^∞ becomes S^{2N-1} , and we will use this as the total space of our principal bundle. If we gather what we have obtained so far, the AA principal bundle is defined as [10]

$$\eta : (S^{2N-1}, \mathbb{C}P^{N-1}, \Pi, U(1)). \quad (4.12)$$

For our problem, the choice of $c = e^{i\alpha}$, $\alpha \in [0, 2\pi)$ in (4.9) is very convenient in the sense that it yields the form that was considered in Chapter 3.

As in Chapter 3, (3.5), pick a closed loop in the base space $\mathbb{C}P^{N-1}$

$$\tilde{C} : t \in [0, T] \rightarrow \kappa(t) \in \mathcal{H}, \quad \kappa(0) = \kappa(T) . \quad (4.13)$$

The horizontal lift of this loop is a curve in the total space, and is denoted as C , with the property that $\Pi C = \tilde{C}$, just like in Chapter 3. Due to our choice $c = e^{i\alpha}$, $\alpha \in [0, 2\pi)$, and using the equivalence relation interpretation, we have a relation of the form (4.9) between fibres. This is just the form (C.18) for the AA principal bundle case, with the structure group again being $U(1)$. So using these sections, we can again form the local connection one form, which is what we already found in Chapter 3, the Aharonov-Anandan connection one-form (3.13). Again corresponding to this $U(1)$ connection one form, the geometrical phase is defined as the holonomy (C.19)

$$\exp(i\gamma_n(C)) = \exp\left(i \oint_C A_n\right) , \quad (4.14)$$

again without the path-ordering, where A_n is (3.13)

$$A = i\langle \tilde{\psi} | d | \tilde{\psi} \rangle , \quad (4.15)$$

with $\tilde{\psi}$ corresponding to κ here.

CHAPTER 5

THE NON-ABELIAN GEOMETRICAL PHASE

For the discussion until now, even though the spectrum was diverse, we had some basic assumptions. While proceeding along the path, we have dropped some of them to the stature of a “special case”, like the adiabatic approximation. For the purposes of the non-Abelian phase, we will give up on one of the remaining, which is the non-degeneracy of the eigenvalues of the Hamiltonian. The concept of the non-Abelian phase was introduced by Wilczek and Zee [14]. Our discussion here follows that of [10]. We will only consider the adiabatic version of the non-Abelian geometrical phase.

5.1 THE ADIABATIC NON-ABELIAN PHASE

Take the eigenvalue $E_n(R)$ of the Hamiltonian $H(R)$ to be N -fold degenerate. Also impose that while evolving, N does not change, and that again there are no level crossings. So if initially the eigenvalue is the N -fold degenerate $E_n(R_0)$, for some $R_0 \in M$ (where M is a differentiable manifold), we will not obtain $E_m(R)$ with $n \neq m$ for any $R \in M$, due to the adiabatic approximation. Finally we require that the degeneracy subspaces $\mathcal{H}_n(R)$ and $\mathcal{H}_m(R)$ corresponding to $E_n(R)$ and $E_m(R)$ respectively do not intersect, hence removing any complication that will follow about what happens at $\mathcal{H}_n(R) \cap \mathcal{H}_m(R)$, which might both correspond to $E_n(R)$ or $E_m(R)$ [10].

There are many similarities between the degenerate and non-degenerate case as we shall see shortly. Since we have N -fold degeneracy here, we are going to include another term in definition of the states that represents the degeneracy, as in

$$|n; R\rangle \rightarrow |n, a; R\rangle \quad a = 1, 2, \dots, N. \quad (5.1)$$

For the cyclic and adiabatic case, our state $|n, a; R(0)\rangle \in \mathcal{H}_n(R(0))$ will start as an eigenstate of the initial Hamiltonian $H(R(0))$, and at any other time $t \in [0, T]$, will be an eigenstate of the Hamiltonian again, together with $|n, a; R(t)\rangle \in \mathcal{H}_n(R(t))$. And they should satisfy

$$H(R)|n, a; R\rangle = E_n(R)|n, a; R\rangle, \quad \forall a = 1, 2, \dots, N. \quad (5.2)$$

We may choose them to be orthonormal. When we do so, a set of orthonormal vectors cover the degeneracy subspace for any n . This set of eigenvectors also allow us to make a definition of the degeneracy subspace as the space that they span

$$\mathcal{H}_n(R) \equiv \text{Span}\{|n, a; R\rangle, a = 1, 2, \dots, N\}. \quad (5.3)$$

Degeneracy eigenspaces are formed by the eigenvectors that correspond to the energies carrying that degeneracy. Note that these vectors, much like the non-degenerate case, are defined locally, on an open neighborhood within $\mathcal{H}_n(R)$ which in turn is a space within M . The inner product of two kets from separate degeneracy spaces is equal to zero, more generally we have

$$\langle n, a; R | m, b; R \rangle = \delta_{nm} \delta_{ab}. \quad (5.4)$$

Again similar to the previous discussion, here we have that any two eigenvectors belonging to the same degeneracy eigenspace should be related to each other by a unitary transformation,

$$|n, a; R\rangle' = \sum_{b=1}^N |n, b; R\rangle U^{ba}(R), \quad (5.5)$$

where $U^{ba}(R)$ are the elements of the $N \times N$ unitary matrix $U(R)$. We can also form an orthonormal basis using the primed eigenvectors, and they will also represent the same physics. This apparently is the generalization of the non-degenerate case.

With some important generalizations having been done, we can now show what our problem looks like for the non-Abelian case. As was stated, we wish to investigate the case of a cyclic or periodic adiabatic change in the parameter space, so again we have time-dependent environmental parameters $R(t)$ within the aforementioned manifold M . Our Hamiltonian thus depends on these parameters, and due to the cyclic evolution, at the period T , we have $H(R(0)) = H(R(T))$, and our state, which is an eigenstate of the initial Hamiltonian, will undergo a cyclic evolution. If we label our initial state $|\psi(0)\rangle = |n, a; R(0)\rangle$ where $|n, a; R(0)\rangle \in \mathcal{H}_n(R(0))$, after our assumptions, we can say that at any time t , $|n, a; R(t)\rangle \in \mathcal{H}_n(R(t))$. And after a period, it

should return to the initial degeneracy subspace $\mathcal{H}_n(R(0))$. And this requires that the initial and final (after one period) states be related to each other by a unitary transformation

$$|\psi(T)\rangle = U|\psi(0)\rangle, \quad (5.6)$$

where U here is an $N \times N$ unitary matrix, which is an element of the unitary group $U(N)$. As in Chapter 2, after a period, our state will gain a dynamical and a geometrical phase. Here, U is the generalization of the total phase factor (which is an element of $U(1)$) of (2.10).

Bearing in mind the assumptions above, we define our initial state and an evolved state at a later time t as

$$|\psi(0)\rangle = \sum_{a=1}^N c_n^a(0) |n, a; R(0)\rangle, \quad (5.7)$$

$$|\psi(t)\rangle = \sum_{a=1}^N c_n^a(t) |n, a; R(t)\rangle. \quad (5.8)$$

The second equation above is the generalization of the ansatz we made in Chapter 2, via (2.10). To find out more about U explicitly, we plug that equation to Schrödinger's equation (2.1), then multiplying the resultant with $\langle n, b; R(t) |$ yields

$$\frac{d}{dt} c_n^b(t) + \sum_{a=1}^N \left[iE_n(R(t)) \delta_{ab} + \langle n, b; R(t) | \frac{d}{dt} | n, a; R(t) \rangle \right] c_n^a(t) = 0, \quad (5.9)$$

integrating this relation gives

$$c_n^b(t) = \sum_{a=1}^N \left[\mathcal{T} \exp \int_0^t \left(-iE_n(R(t')) \mathbf{1}^N dt' + iA_n^N(R(t')) \right) \right]^{ba} c_n^a(0), \quad (5.10)$$

where $\mathbf{1}^N$ is the N -dimensional identity matrix, and \mathcal{T} is the time-ordering operator. We also introduced a very important object here, the $[A_n^N]^{ba}(R(t'))$ in the integral are the ba^{th} entries of the $N \times N$ matrix A_n^N :

$$\begin{aligned} [A_n^N]^{ba}(R(t')) &\equiv i \langle n, b; R(t') | \frac{d}{dt'} | n, a; R(t') \rangle dt' \\ &= i \langle n, b; R | d | n, a; R \rangle. \end{aligned} \quad (5.11)$$

One can show that the matrix A_n^N is Hermitian,

$$A_n^{N\dagger} = A_n^N, \quad (5.12)$$

and that under the gauge transformation (5.5), it transforms as

$$A_n^N \rightarrow A_n^N = U^{-1}(R) \cdot A_n^N \cdot U(R) + iU^{-1}(R) \cdot dU(R), \quad (5.13)$$

where $U(R)$ is the matrix that is formed by the matrix elements in transformation (5.5), the dependency on R follows from them being defined only locally. We note that this is exactly the gauge transformation of a local connection one-form, found in Appendix C. The importance of this matrix is that it is the non-Abelian version of the Berry connection one-form that was found in Chapter 2.

To reach the non-Abelian form of Berry's adiabatic phase, we first bring (5.10) to a more suitable form. First, we see that the time-ordering operator does not affect the first integral, the eigenenergies. Due to the degeneracy in energies, we did not put a degeneracy index on E_n , since in this degeneracy subspace, all the states $|n, a; R\rangle$ correspond to the same energy E_n . Due to these reasons, we can take that integral out of the sum over a . Next, just like we did in the last line of (5.11), the time dependence of the second integrand can be suppressed, therefore ending in an integral over R . Also the time-ordering operator \mathcal{T} needs to be replaced by the path-ordering operator \mathcal{P} , since the resultant objects do not commute also over the R parameters. What we obtain once we apply these is

$$c_n^b(t) = e^{-i \int_0^t E_n(R(t')) dt'} \sum_a \left[\mathcal{P} \exp \left(i \int_{R(0)}^{R(t)} A_n^N(R) \right) \right]^{ba} c_n^a(0), \quad (5.14)$$

which gives a useful form of the coefficients $c_n^b(t)$ as we shall soon be able to appreciate. Substituting it to (5.8) we gather:

$$|\psi(t)\rangle = \sum_{a,b=1}^N e^{-i \int_0^t E_n(R(t')) dt'} \left[\mathcal{P} \exp \left(i \int_{R(0)}^{R(t)} A_n^N(R) \right) \right]^{ba} c_n^a(0) |n, b; R(0)\rangle, \quad (5.15)$$

Due to the cyclic and single-valued nature of our evolution, we know that at the period T we have $R(0) = R(T)$. So when we consider the above equation after a complete cycle, we get

$$|\psi(T)\rangle = \sum_{a,b=1}^N e^{-i \int_0^T E_n(R(t')) dt'} \left[\mathcal{P} \exp \left(i \oint_C A_n^N \right) \right]^{ba} c_n^a(0) |n, b; R(0)\rangle . \quad (5.16)$$

This sum over two degeneracy parameters seems cumbersome, we can get rid of one of them in the following manner. Remember that in (5.9), we took the initial state to be as the superposition of the eigenstates of the initial Hamiltonian $H(R(0))$. But if we assume that initially, our state is at one of these eigenstates, say,

$$|\psi(0)\rangle = |n, a; R(0)\rangle , \quad (5.17)$$

which is to say that $c_n^a(0) = 1$ and all the other c coefficients are equal to zero, then we can take one of the sums out of the equation. Denote the state (5.17) as $|\psi^a(0)\rangle$; at a later time t we have (with the sum over a dropped since we have only one coefficient which itself is equal to 1):

$$|\psi^a(t)\rangle = \sum_{b=1}^N e^{-i \int_0^t E_n(R(t')) dt'} \left[\mathcal{P} \exp \left(i \int_{R(0)}^{R(t)} A_n^N(R) \right) \right]^{ba} |n, b; R(0)\rangle . \quad (5.18)$$

At the period $t = T$, or $R(t) = R(T) = R(0)$,

$$|\psi^a(T)\rangle = \sum_{b=1}^N e^{-i \int_0^T E_n(R(t')) dt'} \left[\mathcal{P} \exp \left(i \oint_C A_n^N \right) \right]^{ba} |n, b; R(0)\rangle , \quad (5.19)$$

which is the result for $n = a$, when we start from an eigenvalue of the initial Hamiltonian. To obtain the general form, first we define a frame for our degeneracy subspace. Our set of eigenvectors, $|n, a; R\rangle$, $a = 1, \dots, N$ define a frame $s_n(R)$ for the degeneracy subspace

$$s_n(R) = (|n, 1; R\rangle, \dots, |n, N; R\rangle) , \quad (5.20)$$

and due to transformation (5.5), the frames should be transformed to each other via the unitary transformation:

$$s_n(R) \rightarrow s'_n(R) = s_n(R)U(R) , \quad (5.21)$$

where $U(R)$ is a unitary matrix made up of elements present in (5.5). In order to generalize (5.19), we choose our initial state as:

$$\psi_n(0) = s_n(R(0)) , \quad (5.22)$$

and we make use of the fact that by (5.19), we know how each element in this frame behaves under an adiabatic time evolution. So the time evolved version of $\psi_n(0)$, or $\psi_n(t)$, should be given in terms of the time evolved versions of the elements of that frame

$$\psi_n(t) = \left[e^{-i \int_0^t E_n(R(t')) dt'} \mathcal{P} \exp \left(i \int_{R(0)}^{R(t)} A_n^N(R) \right) \right] \cdot \psi_n(0) , \quad (5.23)$$

at the period T [10],

$$\psi_n(T) = \left[e^{-i \int_0^T E_n(R(t')) dt'} \mathcal{P} \exp \left(i \oint_C A_n^N \right) \right] \cdot \psi_n(0) . \quad (5.24)$$

We can't help but notice the similarity of this equation to (2.10). Indeed we immediately see that in the above result, the first exponential is the usual dynamical phase factor that follows from the time evolution of a system that has a time-dependent Hamiltonian. And the second exponential is nothing but the non-Abelian generalization of Berry's Phase. It is only convenient that for the non-Abelian Berry's phase, the term in the closed integral is the non-Abelian generalization of Berry connection one-form. In the Abelian case, the geometrical phase is given as

$$\exp(i\gamma_n(C)) = \exp \left(i \oint_C A_n \right) . \quad (5.25)$$

Thus it follows that for the non-Abelian case, we have

$$\exp(i\gamma_n(C)) = \mathcal{P} \exp \left(i \oint_C A_n^N \right) , \quad (5.26)$$

where the \mathcal{P} follows from the non-commutative nature of the unitary matrices in the exponential [10].

CHAPTER 6

GEOMETRICAL PHASES AND MAGNETIC MONOPOLES

In this section we proceed to investigate the relationship between geometrical phases and magnetic monopoles. First, a discussion of the Wu-Yang monopole [16][17] will be given, which serves as an introduction to this chapter. The subject of magnetic monopoles is studied on detail in [18], one can also refer to [19] for a concise discussion of said subject. Later on, following [20], we will see that one can use the ideas of non-Abelian Berry phase to obtain a 't Hooft-Polyakov monopole, which is a topological magnetic charge that is smooth throughout its configuration [21][22].

6.1 WU-YANG MONOPOLE

In his 1931 paper [15], Dirac studied a couple of the greatest mysteries in physics, namely the magnetic monopoles and the quantization of charge. He successfully showed that when a magnetic counterpart to the electric charge exists, both charges do indeed get quantized. With all its triumph, his definition was not without some problems of its own. To exemplify, his definition of the magnetic monopole comes necessarily with a singularity along a semi-infinite axis, which makes the situation a lot more cumbersome. This is one of the main reasons that we are going to study Dirac's monopole from a modern point of view, namely the point of view of Wu-Yang.

Actually, the existence of a singularity is necessary for our purposes here as we shall observe. The reason to this is that the vector potential \mathbf{A} corresponding to the magnetic field of a monopole can not actually be singularity free [17]. The proof to this is as follows. We have a magnetic monopole of strength g at the origin of the usual sphere which has a radius r . We

discuss the problem of the magnetic flux of this monopole at the origin through the surface of the sphere, or S^2 . Assume a line \mathbf{C} parallel the equator on S^2 that divides the surface to two parts a and b , not necessarily equally. The magnetic flux in terms of the vector potential can be found via

$$\Phi = \int \mathbf{B} \cdot d\mathbf{s} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\mathbf{l} , \quad (6.1)$$

where we made use of the Stokes' theorem before the last equality. The line integral is carried over the parallel on the sphere. Now we assume that we have a vector potential that is singularity free along the whole configuration, we can find the contribution of the portions to the total flux. The contribution from a and b are

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \Phi_a , \quad (6.2)$$

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \Phi_b , \quad (6.3)$$

respectively. Since we assumed that there exists one singularity free vector potential, the same integrand is used for both calculation. Subtracting these gives us the total flux through the total surface

$$\Phi_a - \Phi_b = 0 , \quad (6.4)$$

which can not be the case, since we have a magnetic monopole present. The total flux was to be found as $\Phi = cg$, where c is a constant and g is the magnetic charge. This contradiction arises from the fact that we may not define a single vector potential for this configuration that is singularity free. Hence we conclude that the singularities exist necessarily. But since the magnetic field of the configuration is well-defined, thus although mathematically present, the singularities should not cause any physical problems. To understand the situation better, we approach the problem from a different angle [16][17].

We begin by attempting to cover S^2 using charts, we know that we need at least two of them. The two covers will have an intersection, and for simplicity, we may choose it to be a great circle. Note that the calculations here will be done using the spherical polar coordinates. Call one of the hemispheres as a and the other as b , just as in the above discussion. And as

observed, we may not use a single singularity free vector potential. So here, we choose two, one for each hemisphere, and we require them to be singularity free in the hemispheres that they are defined in. And we are going to use the following particular component distribution for the vector potentials [16][17]; for a ,

$$(A_a)_r = (A_a)_\theta = 0, \quad (A_a)_\phi = \frac{g}{r \sin \theta} (1 - \cos \theta), \quad (6.5)$$

for b ,

$$(A_b)_r = (A_b)_\theta = 0, \quad (A_b)_\phi = -\frac{g}{r \sin \theta} (1 + \cos \theta), \quad (6.6)$$

so that they only have the azimuthal component different than zero. There are two very important aspects of this choice; the first being that neither has any singularities in their respective regions of definition. The second is that, taking the curl of these, we easily see that they give the magnetic field of a monopole at the origin. The magnetic field defined this way provides two different values for the field at the overlap; and thus they should be equal to each other there

$$\nabla \times \mathbf{A}_a = \nabla \times \mathbf{A}_b, \quad (6.7)$$

$$\nabla \times (\mathbf{A}_a - \mathbf{A}_b) = 0 \rightarrow \mathbf{A}_a = \mathbf{A}_b + \nabla \lambda, \quad (6.8)$$

for $\lambda = 2g\phi$, where g is the magnetic charge. We see that the two vector potentials are different by the gradient of a scalar function only. Now we consider an electron of mass m and charge e in the field of this monopole. For the two regions a and b , there will be separate wave functions, ψ_a and ψ_b respectively. And corresponding to the two different vector potentials, the Schrödinger's equation will be given as

$$\frac{1}{2m} (\mathbf{p} - e\mathbf{A}_a)^2 \psi_a = E \psi_a, \quad (6.9)$$

$$\frac{1}{2m} (\mathbf{p} - e\mathbf{A}_b)^2 \psi_b = E \psi_b, \quad (6.10)$$

in regions a and b respectively. The fact that the vector potentials in the two regions are different only by a gradient requires these two wave functions to be related through a phase transformation,

$$\psi_a = e^{ie\lambda} \psi_b = e^{i(2eg)\phi} \psi_b . \quad (6.11)$$

Then, we again consider the situation at the overlapping region, the great circle. There, the wave functions must be single-valued. Along the great circle or the equator, the azimuthal angle changes from 0 to 2π . Plugging these values into the above equation we find at $\phi = 0$:

$$\psi_a = \psi_b , \quad (6.12)$$

and at $\phi = 2\pi$:

$$\psi_a = e^{2\pi i(2eg)} \psi_b . \quad (6.13)$$

From the last equation it follows that

$$4\pi eg = 2\pi n, \quad n \in \mathbb{Z} , \quad (6.14)$$

or

$$2eg = n, \quad n \in \mathbb{Z} , \quad (6.15)$$

which is the Dirac quantization condition [15].

6.2 NON-ABELIAN GEOMETRICAL PHASE AND 'T HOOFT-POLYAKOV MONOPOLE

We now have all the inventory we need to proceed to the problem whether it is possible to define the 't Hooft-Polyakov monopole [21][22] as a Berry connection. It has been shown in [20] that this is actually possible, using the non-Abelian version of Berry's Phase. In this section we analyze their result.

Remember that in Section 2.1, it was shown that for the problem of a spinning particle in a

magnetic field, with Hamiltonian

$$H(\mathbf{B}) = -g\mu\mathbf{S} \cdot \mathbf{B} , \quad (6.16)$$

we can obtain a Dirac type monopole in the parameter or the magnetic field space.

The discussion begins by constructing a Hamiltonian that has a doubly degenerate ground state. We have a neutral, spin 1/2 particle moving on S^2 , and like in Section 2.1, the external parameter is a magnetic field. For this problem, the magnetic field vector \mathbf{B} is a constant, and its variation over the sphere is given as $\mathbf{B} \cos \theta$. The Hamiltonian in the required form is given as [20]

$$H = -\frac{1}{2m} \Delta \mathbf{I}_2 - \mathbf{B} \cdot \boldsymbol{\sigma} \cos \theta , \quad (6.17)$$

where \mathbf{I}_2 is the 2×2 unit matrix, and Δ is the Laplacian on \mathbf{S}^2 with unit radius

$$\Delta = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} . \quad (6.18)$$

Remember that we have a spin 1/2 particle. The spin-up $|\uparrow\rangle$ and spin-down $|\downarrow\rangle$ states are defined as the normalized eigenvectors of $\mathbf{B} \cdot \boldsymbol{\sigma}$ with eigenvalues $+B$ and $-B$ respectively, for $|\mathbf{B}| = B$. As was mentioned, we need a Hamiltonian that has two ground states. By construction, we see that this Hamiltonian is invariant under the application of

$$\mathbf{B} \rightarrow -\mathbf{B}, \quad \theta \rightarrow \pi - \theta , \quad (6.19)$$

i.e., our Hamiltonian has a \mathbf{Z}_2 symmetry. That will provide us with the necessary doubly degenerate ground state. Taking \mathbf{B} to $-\mathbf{B}$ corresponds to exchanging spin-up and spin-down states.

For $\mathbf{B} \neq 0$, the spin-up state $|\uparrow\rangle$ is localized near $\theta = 0$ and the spin-down state $|\downarrow\rangle$ near $\theta = \pi$. We can see that via the following logic: In (6.17), the first term has to do with the spatial components, where the second term is related to spin. If we pick $\boldsymbol{\sigma} = \sigma_z$ and

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ,$$

$$|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} ,$$

then we find that $\mathbf{B} \cdot \boldsymbol{\sigma} \cos \theta$ is given as

$$\begin{pmatrix} -B \cos \theta & 0 \\ 0 & B \cos \theta \end{pmatrix},$$

showing that when this operator is applied on spin states, for $|\uparrow\rangle$ and $|\downarrow\rangle$ to yield minimum energy result $-B$, $\theta = 0$ and $\theta = \pi$ respectively. And since in our problem, the relative positions of \mathbf{B} and $\boldsymbol{\sigma}$ is arbitrary, we can generalize these angle requirements to arbitrary $\boldsymbol{\sigma}$.

When $\mathbf{B} = 0$, we end up with the simple Hamiltonian:

$$H = -\frac{1}{2m} \Delta I_2. \quad (6.20)$$

Therefore the ground states will not be localized, they will just be in a uniform superposition state on the sphere. Most generally, we can write the normalized ground states in such a form [20]

$$|1\rangle = \psi(\cos \theta; B) |\uparrow\rangle, \quad |2\rangle = \psi(-\cos \theta; B) |\downarrow\rangle, \quad (6.21)$$

where the ψ represent the spatial wave function depending on B and $\cos \theta$ (ϕ does not appear here on the grounds that it is a cyclic coordinate), and the arrows represent the spin states. Make $\cos \theta = x$, the Laplacian after this becomes

$$\Delta = (1 - x^2) \frac{\partial^2}{\partial x^2} - 2x \frac{\partial}{\partial x} + \frac{1}{1 - x^2} \frac{\partial^2}{\partial \phi^2}. \quad (6.22)$$

With the properly changed Hamiltonian, we can thus write the Schrödinger's equation for the spatial wave function as

$$-\frac{1}{2m}(1 - x^2)\psi'' + \frac{1}{m}x\psi' - Bx\psi = E_0\psi, \quad (6.23)$$

where E_0 is the ground state energy, and $\psi' = \frac{d\psi}{dx}$; note that we use a full derivative here since the spatial wave function does not depend on ϕ .

To find the Berry phase for this degenerate system, we need to use the non-Abelian version of the geometrical phase. Initially, the system is at one of the ground states. Then through the

adiabatic variation of the magnetic field over a closed path in the parameter space, our state will gain a geometrical phase. Using states $|1\rangle$ and $|2\rangle$, we can find the matrix elements of the non-Abelian Berry connection, defined in (5.11). For easier computation, we will take the derivative over $\mathbf{B} = \mathbf{R}$, so the entries of the connection are given as

$$(\mathbf{A}_n)^{ba} = i\langle b | \frac{\partial}{\partial \mathbf{B}} | a \rangle , \quad (6.24)$$

where $a, b = 1, 2$. For the diagonal entries, we just have the U(1) Abelian connection (2.26)

$$\mathbf{A}_n = i\langle n | \frac{\partial}{\partial \mathbf{B}} | n \rangle . \quad (6.25)$$

So their calculation gives the Dirac monopole case

$$(\mathbf{A}_n)^{11} = \langle 1 | \frac{\partial}{\partial \mathbf{B}} | 1 \rangle = \mathbf{A}_{Dirac} , \quad (6.26)$$

$$(\mathbf{A}_n)^{22} = \langle 2 | \frac{\partial}{\partial \mathbf{B}} | 2 \rangle = -\mathbf{A}_{Dirac} . \quad (6.27)$$

The off-diagonal elements are [20]

$$(\mathbf{A}_n)^{21} = \langle 2 | \frac{\partial}{\partial \mathbf{B}} | 1 \rangle = f(B) \langle \downarrow | \frac{\partial}{\partial \mathbf{B}} | \uparrow \rangle , \quad (6.28)$$

$$(\mathbf{A}_n)^{12} = \langle 1 | \frac{\partial}{\partial \mathbf{B}} | 2 \rangle = f(B) \langle \uparrow | \frac{\partial}{\partial \mathbf{B}} | \downarrow \rangle , \quad (6.29)$$

where the term that we took out of the sandwich $f(B)$ is:

$$f(B) = 2\pi \int_0^\pi \sin \theta d\theta \psi^*(-\cos \theta; B) \psi(\cos \theta; B) , \quad (6.30)$$

We notice that the calculation of $f(B)$ requires the explicit form of the spatial wave function ψ . The explicit form of ψ follows directly from the solution of (6.23).

Now that we have all the entries of \mathbf{A}_n , we can write down its explicit form. First, we note that since the diagonal elements were found to be in the form of a Dirac Monopole, inevitably there are singularities present. The Dirac string will show in their calculations. But one can

transform these to a gauge where the singularities disappear. After this transformation, one obtains the non-Abelian connection as [20]

$$A_\mu = \frac{1}{2B^2} \epsilon_{\mu\nu\rho} B_\nu \sigma^\rho [1 - f(B)] , \quad (6.31)$$

which is the connection of the smooth 't Hooft-Polyakov monopole.

Since we could not solve (6.23), we were unable to obtain $f(B)$ explicitly. Here, assuming that we have a θ -dependent potential in the Hamiltonian (6.17) in the following specific form:

$$V(\theta) = \frac{1}{2} m B^2 \sin^2 \theta , \quad (6.32)$$

we obtain (6.23) in a form that we can solve. For the ground state energy $E_0 = 0$, the exact wave function is found as

$$\psi(\cos \theta; B) = \left(\frac{Bm}{2\pi \sinh(2Bm)} \right)^{\frac{1}{2}} e^{Bm \cos \theta} . \quad (6.33)$$

Now that we have the explicit form of the ground state wave functions, we can plug them into (6.30) to obtain $f(B)$ that appears in (6.31) as

$$f(B) = \frac{2Bm}{\sinh(2Bm)} . \quad (6.34)$$

Therefore we obtain the exact 't Hooft-Polyakov monopole solution

$$A_\mu = \frac{1}{2B^2} \epsilon_{\mu\nu\rho} B_\nu \sigma^\rho \left[1 - \frac{2Bm}{\sinh(2Bm)} \right] . \quad (6.35)$$

In the B-space, this solution does not have any singularities, since as $B \rightarrow 0$, $A_\mu \rightarrow 0$ and as $B \rightarrow \infty$, A_μ becomes a Dirac monopole. This specific exact solution is known as the Bogomol'nyi, Prasad, Sommerfield (BPS) solution [23][24] because of the particular choice of the potential (6.32) which rendered the Schrödinger equation solvable.

CHAPTER 7

CONCLUSION

It is an interesting phenomenon that an important object, such as the geometrical phase which presents itself on a very wide spectrum of physical situations, could have been overlooked for a significant time period of modern physics. Starting from Berry's initial discussion [1], the geometrical phase went on to take many different forms. We tried to review some of them within this thesis. In Chapter 3, the methodology for general cyclic evolutions were discussed through the work of Aharonov and Anandan [12]. It was shown that corresponding to an arbitrary cyclic evolution in the projective Hilbert space, there occurs a geometrical phase to the wave function in the Hilbert space. And that wave function need not be an eigenstate of the Hamiltonian, neither the evolution needs to be adiabatic. Next, we showed in Chapter 5 that it is possible to relate the concept of geometrical phases to some very important objects from differential geometry, fibre bundles and holonomy [11][10].

Another topic of interest to us was the relation between the geometrical phase and the magnetic monopoles. It was shown that starting even from the most basic approach to geometrical physics, one can relate the concept of geometrical phases to a very important topic in physics, the magnetic monopole. The Abelian phase of Berry was shown to yield monopole-like solutions which live on the parameter space. Then, following [20], when one considers the non-Abelian generalization of Berry's phase [14][10], it is indeed possible to construct a simple degenerate quantum mechanical system where a 't Hooft-Polyakov monopole [21][22] presents itself. These results are both very interesting and profound.

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Appendix A

VALIDITY OF THE ADIABATIC APPROXIMATION

Though a definition of adiabacity was made in Section 2.1, when to make this approximation is suitable was not shown explicitly. Here we show the conditions that lets one use the adiabacity condition without causing any problems related to physics [10].

We begin by choosing an orthogonal basis $\{|n; R(t)\rangle\}$, $\langle m; R(t)|n; R(t)\rangle = \delta_{mn}$. Hence our state can be defined as

$$\psi(t) = \sum_m c_m(t) |m; R(t)\rangle . \quad (\text{A.1})$$

Remember that Berry chooses this initial state to be an eigenfunction of the initial Hamiltonian. And remember it was stated that if we have a system that evolves adiabatically, it is going to stay at close vicinity to its initial state (call it n) at any time t . This means that there will be no crossing between $\{|n; R(t)\rangle\}$, which is to say that all $c_m(t)$, except for the one corresponding to the initial state n , will be equal to zero,

$$\psi(0) = |n; R(0)\rangle , \quad (\text{A.2})$$

$$\psi(t) = c_n(t) |n; R(t)\rangle , \quad (\text{A.3})$$

with the appropriate choice of $c_n(0) = 1$. This state has to satisfy the Schrödinger's equation; substituting yields

$$i(\dot{c}_n(t)) + ic_n(t) \frac{d}{dt} |n; R(t)\rangle = c_n(t) E_n(R(t)) |n; R(t)\rangle , \quad (\text{A.4})$$

$$[\dot{c}_n(t) + iE_n(R(t))c_n(t)] |n; R(t)\rangle = -c_n(t) \frac{d}{dt} |n; R(t)\rangle . \quad (\text{A.5})$$

Multiplying by $\langle m; R(t)|$ shows that

$$\langle m; R(t)| \frac{d}{dt} |n; R(t)\rangle = 0 , \quad (\text{A.6})$$

where $m \neq n$. This expression is the necessary and sufficient condition for the validity of the adiabatic approximation.

But as stated in Chapter 2, adiabaticity has to do with the change of the Hamiltonian in time. So what one would like to obtain is an expression that shows the validity of the adiabatic condition that contains time derivatives of the Hamiltonian. And in order to obtain such a form, we need to start by differentiating the time-independent Schrödinger Equation (2.2):

$$dH(R)|n; R\rangle + H(R)d|n; R\rangle = dE_n(R)|n; R\rangle + E_n(R)d|n; R\rangle . \quad (\text{A.7})$$

Note that the time dependence of the basis vectors are suppressed, since we know that at any particular instance t , (2.2) is satisfied. Multiplying by $\langle m; R|$, $m \neq n$ gives, with abbreviating the dependence on R ,

$$\langle m|dH|n\rangle + E_m\langle m|d|n\rangle = E_n\langle m|d|n\rangle , \quad (\text{A.8})$$

$$\langle m|d|n\rangle = \frac{\langle m|dH|n\rangle}{E_n - E_m} . \quad (\text{A.9})$$

The time evolving version is obtained by replacing differential operators with time derivatives and R by $R(t)$ everywhere

$$\langle m; R(t)| \frac{d}{dt} |n; R(t)\rangle = \frac{\langle m; R(t)| \frac{d}{dt} H(R(t)) |n; R(t)\rangle}{E_n(R(t)) - E_m(R(t))} , \quad (\text{A.10})$$

and from the necessary condition (A.6) that was previously obtained, we find the expression for the validity of adiabatic approximation which contains time derivatives of the Hamiltonian

$$\frac{\langle m; R(t)| \frac{d}{dt} H(R(t)) |n; R(t)\rangle}{E_n(R(t)) - E_m(R(t))} = 0, \forall m \neq n . \quad (\text{A.11})$$

One can use the adiabaticity approximation if we have a Hamiltonian that oscillates small enough that the left hand side of this expression can be taken as zero [10].

Appendix B

THE NON-INTEGRABLE NATURE OF BERRY'S PHASE

One of the most important properties of Berry's Phase is definitely its non-integrable nature. This means that we can not get rid of it by mere transformations.

In Chapter 2, we have stated that the basis kets are defined only up to a phase factor [10];

$$|n; R\rangle' = e^{i\chi_n(R)}|n; R\rangle, \quad (\text{B.1})$$

from (2.2). When we compare this with the form of (2.6), it seems as if one can just set this extra phase factor to unity by a phase transformation. Historically, Fock showed that this in fact is the case [10], and so one does not need to introduce this extra factor that is defined up to a choice of gauge, and hence can easily be transformed away. The problem with this approach was, Fock did not consider cyclic evolutions. Now we show that when one considers cyclic evolutions, hence the state returns to its initial condition after some time, the geometrical phase is there to stay.

We begin by checking how the gauge transformation (B.1) affects some objects that we made use of before. Remember (2.26)

$$\mathbf{A}^n(R) = i\langle n; R | \nabla | n; R \rangle. \quad (\text{B.2})$$

Under the gauge transformation (B.1) this transforms as

$$\mathbf{A}^n(R) \rightarrow \mathbf{A}'^n(R) = i\langle n; R | \nabla' | n; R \rangle' \quad (\text{B.3})$$

$$= i\langle n; R | e^{-i\chi_n(R)} \nabla e^{i\chi_n(R)} | n; R \rangle \quad (\text{B.4})$$

$$= i\langle n; R | e^{-i\chi_n(R)} [(\nabla e^{i\chi_n(R)}) | n; R \rangle + e^{i\chi_n(R)} \nabla | n; R \rangle] , \quad (\text{B.5})$$

which gives

$$\mathbf{A}'^n(R) = \mathbf{A}^n(R) - \nabla \chi(R). \quad (\text{B.6})$$

Then we can use this to find γ_n

$$\gamma_n \rightarrow \gamma'_n = \int_{R(0)}^{R(t)} \mathbf{A}^m(R) dR \quad (\text{B.7})$$

$$= \gamma_n - \chi_n(R(t)) + \chi_n(R(0)) . \quad (\text{B.8})$$

So while transforming, both \mathbf{A}^n and γ_n receive extra factors.

The basis $\{|n; R\rangle'\}$ is just as acceptable as $\{|n; R\rangle\}$, one can use both with equal validity. So we can write the adiabatic evolution of the state ψ using the basis $\{|n; R\rangle'\}$

$$|\psi(t)\rangle = \exp\left\{-i \int_0^t dt' E_n R(t')\right\} \exp(i\gamma'_n(t)) |n; R(t)\rangle' , \quad (\text{B.9})$$

and due to the gauge transformation we have

$$\exp(i\gamma'_n(t)) |n; R\rangle' = \exp(i\gamma'_n(t)) e^{i\chi_n(R)} |n; R\rangle . \quad (\text{B.10})$$

Therefore when the arbitrary phase factor χ_n equals $-\gamma_n$, our state is given as

$$|\psi(t)\rangle = \exp\left\{-i \int_0^t dt' E_n R(t')\right\} |n; R\rangle , \quad (\text{B.11})$$

since the gauge transformation gives one the freedom to just make the extra phase factor γ_n go to unity at will with an appropriate choice of the $\chi_n(R)$ factor. The only remaining term is the usual dynamical phase, coming from the time evolution of ψ . This result gives the impression that one can just proceed with doing calculations without caring too much about the extra phase γ_n since it can be transformed away by a simple change of bases. However, in what follows we show that a gauge transformation will not be able to get rid of that factor when we consider a cyclic evolution.

What led us to the previous conclusion was our freedom on the choice of the factor $\chi_n(R)$. For a single-valued and cyclic evolution defined by (2.3), we should necessarily have

$$e^{i\chi_n(R(T))} = e^{i\chi_n(R(0))} , \quad (\text{B.12})$$

which restricts the value of this factor to modulo an integer multiple 2π

$$\chi_n(R(T)) = \chi_n(R(0)) + 2m\pi, \quad m \in \mathbb{Z} . \quad (\text{B.13})$$

Substituting this result to (B.8) we obtain

$$\gamma'_n = \gamma_n - 2m\pi, \quad m \in \mathbb{Z}, \quad (\text{B.14})$$

which shows that we have a phase factor that is invariant under the gauge transformation (B.1) [10].

Appendix C

A SHORT INTRODUCTION TO FIBRE BUNDLES

Though a vast topic by itself, here we present a short appendix about fibre bundles. Hopefully, it will prove satisfactory for our purposes. The topic of fibre bundles is a useful tool for physicists. Especially principal fibre bundles have proven invaluable within the framework of gauge theories. We note again that what is presented here is merely an introduction to the subject, and therefore not in the format an advanced maths book will have. This appendix also serves for the purpose of setting our notation for chapters making use of the fibre bundles. For a more detailed discussion on fibre bundles, we refer the interested reader to [25][27] which are text books on differential geometry. Also [10] and [26] provide a concise consideration of the topic.

For defining a fibre bundle, we make use of the following objects:

1. A topological space called the *total space*, denoted as E ,
2. A topological space called the *base space*, denoted as X ,
3. A surjective map Π between the total and base spaces, called the *projection*, $\Pi : E \rightarrow X$,

Next we need the definition of a “fibre“. The inverse of the projection maps elements of X to fibres F_x . It is plausible to define another topological space F , called the *standard fibre*, which all the F_x are homeomorphic copies of.

The other object that we shall need is the *structure group* G . The constituents of this group will be presented shortly.

More so than often, one works with a total space E that does not have a trivial topology. To have a better picture of the situation at hand, it is handy a tool to define another space, a so

called “locally trivialized” space, which can be defined as a product $U_\alpha \times F$. To define it, first we form a set of open covers of X , $\{U_\alpha\}$. Also require homeomorphic maps ϕ_α from E to $U_\alpha \times F$ and that these satisfy:

$$\forall U_\alpha \exists \phi_\alpha : \Pi^{-1}(U_\alpha) \rightarrow U_\alpha \times F , \quad (\text{C.1})$$

which maps a union of fibers $\Pi^{-1}(U_\alpha)$ which is a subspace of E into the locally trivialized product space $U_\alpha \times F$. Since ϕ_α is a homeomorphism, it has a continuous inverse, and we can use it to go from $U_\alpha \times F$ to E in such a way that:

$$\Pi\phi_\alpha^{-1}(x, f) = x , \quad (\text{C.2})$$

where $x \in U_\alpha \subset X$ and $f \in F$ is an arbitrary point on a fibre or a fibre point. The homeomorphic map ϕ_α is hence called a **local trivialization**.

These objects can be used to construct what is called a “fibre bundle”. Although one can show that to define a fibre bundle uniquely, the sufficient and necessary objects are only a subset of those above.

Now we can discuss the structure group G . To find the constituents of this group, we consider the following problem: Suppose that a point $x \in X$ remains in the intersection of the two open neighborhoods U_α and U_β , thus $U_\alpha \cap U_\beta \neq \emptyset$. Due to our formalism, we know that corresponding to U_α and U_β are unique local trivializations, ϕ_α and ϕ_β respectively. We can map the point x to E via Π^{-1} , but now we have two local trivializations to go to $(U_\alpha \cap U_\beta) \times F$. They both will map the point x to a different fibre point f in the product space. We can use the map $\phi_\alpha \cdot \phi_\beta^{-1}$ (which is a homeomorphism itself)

$$\phi_\alpha \cdot \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times F \rightarrow (U_\alpha \cap U_\beta) \times F , \quad (\text{C.3})$$

to move between the respective fibre points in the locally trivialized space. Hence we naturally have an object that shows us how to go from one fibre point to another, and due to this reason that object is called a **transition function**. So label the fibre point that ϕ_α maps $\Pi^{-1}(x)$ as f , ϕ_β maps as f' . The transition functions show us how to glue these points together for the overlap region of two neighborhoods;

$$f = g_{\alpha\beta} f' , \quad (\text{C.4})$$

$$g_{\alpha\beta} \equiv \phi_\alpha \cdot \phi_\beta^{-1} . \quad (\text{C.5})$$

Now we have the tool to define our structure group G : the structure group G is defined as the set of all transition functions,

$$G = \{g_{\alpha\beta}\} . \quad (\text{C.6})$$

We can show that this set really forms a group by showing that it satisfies the group axioms, for $g_{\alpha\beta} \in G$:

- There exists an identity element : $g_{\alpha\alpha} = \phi_\alpha \cdot \phi_\alpha^{-1} = 1$
- There exist inverses: $(g_{\alpha\beta})^{-1} = (\phi_\alpha \phi_\beta^{-1})^{-1} = \phi_\beta \phi_\alpha^{-1} = g_{\beta\alpha}$, $g_{\beta\alpha} \in G$ and one can easily find that for the group operation (\cdot) this really is the inverse of $g_{\alpha\beta}$

$$g_{\alpha\beta} \cdot g_{\beta\alpha} = \phi_\alpha \phi_\beta^{-1} \phi_\beta \phi_\alpha^{-1} = 1$$

- Associativity: $(\phi_\alpha \phi_\beta) \phi_\sigma = \phi_\alpha (\phi_\beta \phi_\sigma)$

Another concept that we often make use of is the concept of a section.

Section: A section s of E is a continuous map from the total to the base space, $s : X \rightarrow E$ which has the property that:

$$\Pi s(x) = x \quad (\text{C.7})$$

for all $x \in X$. That the definition is made for all $x \in X$ makes it global; if the section is defined only for some open neighborhood in X , then it is called a local section. Note that not all bundles allow for a section, but one can always find local sections for any bundle.

Now that we have the basic construction at hand, we can give brief definitions of some types of fibre bundles that we make use of. They usually are named after the properties of their fibres.

We said that usually the global topology of a bundle is a complicated one, and hence we make use of local trivializations in order to overcome the difficulties that naturally follow from such a non-trivial construct. Because even if the global structure of a bundle is complicated, locally there are no difficulties. This means that from a global point of view, our bundle can have a very difficult structure to comprehend made up of twists and etc., but when we map it to a

product space $U_\alpha \times F$ using a *local* bundle coordinate (U_α, ϕ_α) , the picture obtained then is easily manageable. We emphasize again that generally this consideration works for a local consideration only. And this brings us to our first type of fibre bundle:

Trivial Bundle : A trivial bundle is a bundle for which the total space E is exactly equal to $X \times F$, meaning that there is no twisting of the fibres and hence the global topology is a trivial one. This also means that all the transition functions are identically 1.

Vector Bundle: A vector bundle is a bundle whose standard fibre F (and therefore all fibres F_x) are vector spaces. Depending whether it is a real or a complex vector space, the structure group is the general linear group $GL(N, \mathbb{C})$ or $GL(N, \mathbb{R})$ respectively.

Line Bundle: A line bundle is a vector bundle whose fibre is a 1-dimensional vector space.

What follows is the critical definition of a principal bundle:

Principal Bundle: One forms a principal bundle by taking the standard fibre F and the structure group G as identical to each other. This means that they are diffeomorphic to each other as differentiable manifolds. When considering the principal bundle, E will be denoted as P . The fact that the group G has both the structure of a group and is diffeomorphic to a differentiable manifold turns it into a Lie Group. This Lie group brings a right-action $R : G \times P \rightarrow P$ (and a left-action, but it is not used within our consideration) on P , that is denoted as

$$R(g, p_1) \equiv R_g(p_1) , \quad (C.8)$$

and is defined as:

$$p_1 \xrightarrow{g} p_2 = R_g(p_1) \equiv \phi_\alpha^{-1}(x)[\phi_\alpha(x)[p_1] \cdot g] , \quad (C.9)$$

where $x = \Pi(p_1)$, $x \in X$ and (\cdot) is the group operation in G . Here the p are elements of the fibre where g is an element of the Lie group, hence one does not automatically have the mathematical operation between them. So how we made the above definition of the right action is as follows: In order to relate p_1 to p_2 via the right action we have to move to familiar territory, the locally trivialized space. First we map $p_1 \in F_x$ to $F = G$ using the local trivialization ϕ_α . What we have obtained this way, $\phi_\alpha(x)[p_1]$ is an element of the group G , hence we can use the group operation on it now. After the group multiplication, we need to map the resulting, $(\phi_\alpha(x)[p_1] \cdot g)$ back to P , and for that we use ϕ_α^{-1} . So we may conclude here that the right action of G on P lets one move points along individual fibres, since we can relate all the points on the fibre this way.

We use the notations $P(M, G)$ and (P, M, Π, G) for a principal bundle interchangeably, where P is the total space, M the base space and G is the structure group.

As useful as this might be, we still need an object that enables one to move between fibres. That object is rightfully called a connection, to which we shall give a brief definition shortly. First, we define the concept of associated bundle:

Associated Bundle: One can associate a principal fibre bundle to a vector bundle at hand. To see how to do so, we first pick a vector bundle $E: (E, X, \pi, G)$, since it is a vector bundle G is $GL(N, \mathbb{R})$ or $GL(N, \mathbb{C})$ (note that they are indeed Lie groups). To this vector bundle, we can associate a principal bundle $P': (P', X, \pi', G)$ by replacing the fibres of E with the copies of the structure group G . The transition functions of P' shall be chosen as the transition functions of E , that is to say that the transition functions of the vector bundle defines the way to glue the fibres of the associated principal bundle. The fact that the structure group is the same for both bundles justifies this choice. The principal fibre bundle P' is called the associated principal bundle to E .

Now we can start our definition of the connection on a principal bundle. Above we discussed how to move along individual fibres. What the connection does is to define the way to move between different fibres, so that we can talk about the concept of parallel transport. Due to the immense nature of the topic, we can only give a brief introduction here. For a more detailed discussion (and one that includes definitions of the items that we use here without defining) we refer to [10][25][26][27].

We start formulating the connection by first separating the tangent space of a fibre point u into two spaces, the vertical and the horizontal subspace. We will see that the horizontal subspace together with a one-form will let us define the connection. Now we see how to construct these subspaces. The discussion here will mainly follow [25].

Choose the fibre point u to be an element of the principal bundle $P(M, G)$, $u \in P(M, G)$ and that u resides on the fibre G_p with $\Pi(u) = p$. Now we define the vertical subspace $V_u P$. Since it is a subspace of the total tangent space $T_u P$ at u , it needs to be tangent to the fibre G_p . To make use of the tangent space concept efficiently, we try to define a curve in P through u . Making use of the right action of the Lie group G , our structure group, we can obtain such a curve. We need to pick an element A from \mathcal{G} (the Lie algebra of G), $A \in \mathcal{G}$. The right action

on u is:

$$R_{\exp(tA)}u = u \exp(tA) . \quad (\text{C.10})$$

This defines a curve through u in P . We also know that this curve lies on the fibre G_p , since it is the fibre obtained through inverse bijection Π^{-1} (i.e. $\Pi(u) = \Pi(u \exp(tA)) = p$), and the right action is used for moving points along individual fibres. By using a smooth function which maps fibre points to real numbers $f : P \rightarrow \mathbb{R}$, we define the vector \tilde{A}

$$\tilde{A}f(u) = \frac{d}{dt}f(u \exp(tA))|_{t=0} . \quad (\text{C.11})$$

This vector is tangent to the fibre at the point u , and is also an element of the vertical subspace V_uP of T_uP which is defined as [26]

$$V_uP \equiv \{\tilde{A} \in T_uP \mid \Pi_*\tilde{A} = 0\} , \quad (\text{C.12})$$

where Π_* is the pushforward map induced by the bijection Π . This says that elements of V_uP are mapped to the zero vector on T_pM , the tangent space at point p on the base space M . One can find all vectors like \tilde{A} for each point u , these vectors form the vector field called "the fundamental vector field" generated by A and we denote it as X_A . Now that we have the definition of the vertical subspace V_uP , we define the horizontal subspace H_uP as its complement in the total tangent space T_uP .

Definition A3.1: A connection on a principal bundle assigns smoothly to each fibre point $u \in P(M, G)$ a unique subspace called the horizontal subspace H_uP such that:

1. $T_uP = H_uP \oplus V_uP$
2. One can smoothly decompose a smooth vector field A into $X^H \in H_uP$ and $X^V \in V_uP$ such that $X = X^H + X^V$
3. $R_{g*}H_uP = H_{ug}P \quad \forall u \in P \text{ and } g \in G$

The last condition above shows that we can use the push-forward map induced by the right action to relate the horizontal subspaces at different points on the same bundle; similar to the

way we use the right action itself to relate different points on the same bundle. If we know the horizontal subspace $H_u P$ at a fibre point u , we can generate all the other horizontal subspaces via the suitable right action. Thus if we parallel transport u , all the other fibre points are parallel transported too.

Definition of a Lie-algebra-valued one-form (it maps tangent vectors to the elements of a Lie algebra) lets us define the connection in a more useful manner [25][26]:

Definition A3.2: A connection one-form $w \in T^*P \otimes \mathcal{G}$ is a projection of $T_u P$ onto $V_u P$ satisfying

1. $w(X_A) = A, \quad A \in \mathcal{G}$
2. $R_g^* w = Ad_{g^{-1}} w$, where Ad is the adjoint map defined as $Ad_g : \mathcal{G} \rightarrow \mathcal{G}, A \mapsto gAg^{-1}$ and R_g^* is the pullback map induced by the right action.

With the above relation, we have for an arbitrary tangent vector $X \in T_u P$

$$R_g^* w_{ug}(X) = w_{ug}(R_{g*} X) = g^{-1} w_u(X) g . \quad (C.13)$$

Once we have the connection one-form $w : T_u P \rightarrow \mathcal{G}$, we can simply define the horizontal subspace $H_u P$ by its kernel:

$$H_u P = \{X \in T_u P | w(X) = 0\} . \quad (C.14)$$

Again we have $T_u P = H_u P \oplus V_u P$, and $H_u P$ defined this way again satisfies Condition (3) from Definition A3.1. Therefore we see that both definitions for a connection on a principal bundle are equivalent; we note that Definition A3.2 is more suitable for physics in that it regards the connection as a one-form rather than a collection of vector spaces. We can show that studying with a connection that has the form of a \mathcal{G} -valued one-form gives us an easy way to relate this consideration to gauge potentials. Now we show how this can be done [10].

Pick any arbitrary point $x \in M$; one can always find an open chart (U_α, ϕ_α) that includes p , and remember that there is always a local section $s : U_\alpha \rightarrow P$. Here, we state that it is also always possible to find a \mathcal{G} -valued one-form A_α^s on M such that [10]:

$$s^*(w) = A_\alpha^s, \quad (\text{C.15})$$

on U_α . This \mathcal{G} -valued one-form is called the local connection one-form. The notation is chosen based on the fact that this one-form depends on the local section s , and the chart (U_α, ϕ_α) . Once we choose a particular section s_α related to each chart (subscript α denoting the association with the chart), the dependence of the local connection one-form on the section disappears. This particular section is called the canonical local section, and is defined as

$$\phi_\alpha(s_\alpha(x)) \equiv e \in G, \quad \forall x \in U_\alpha, \quad (\text{C.16})$$

where e is the identity element of the structure group G . Hence the canonical section is realized when the local trivialization maps each $s_\alpha(x) \in P$ to e . What makes the canonical local sections useful is that one can reach any local section from the canonical one by a suitable application of the right action. So for any $x \in U_\alpha$, there exists $g(x) \in G$ such that:

$$s(x) = s_\alpha(x) \cdot g(x), \quad (\text{C.17})$$

where “ \cdot ” denotes group multiplication. This result generalizes to [10]

$$s_1(x) \rightarrow s_2(x) = s_1(x) \cdot g(x). \quad (\text{C.18})$$

C.1 HOLONOMY

Before finishing this section, we consider one last item briefly, namely the subject of holonomy. Again, for a rigorous discussion, we refer to [25] especially, and [10][26] also.

It can be found that the elements that relates coinciding points on a loop actually form a group called the Holonomy group [25]. Furthermore, one can find that when one considers a gauge potential $A_i = A_{i\mu} dx^\mu$ for a loop C in an open neighborhood U_i , one obtains for the holonomy

$$g_C = \mathcal{P} \exp \left(- \oint_C A_{i\mu} dx^\mu \right), \quad (\text{C.19})$$

where \mathcal{P} denotes the path-ordered exponential [25]. For considerations having Abelian groups as the structure group, the path-ordering vanishes.