CONFORMAL SYMMETRY IN FIELD THEORY

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submitted by **ULAŞ HUYAL** in partial fulfillment of the requirements for the degree of **Master of Science in Physics Department, Middle East Technical University** by,

Prof. Dr. Canan Özgen
Dean, Graduate School of **Natural and Applied Sciences** ________________

Prof. Dr. Sinan Bilikmen
Head of Department, **Physics** ________________

Assoc. Prof. Dr. Bayram Tekin
Supervisor, **Physics Department** ________________

Examining Committee Members:

Prof. Dr. Ayşe Kalkanlı Karasu
Physics Department, METU ________________

Assoc. Prof. Dr. Bayram Tekin
Physics Department, METU ________________

Assoc. Prof. Dr. Seçkin Kürkçüoğlu
Physics Department, METU ________________

Prof. Dr. Ali Murat Güler
Physics Department, METU ________________

Assoc. Prof. Dr. Korkut Okan Ozansoy
Department of Engineering Physics, Ankara University ________________

Date: ________________

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Name, Last Name: ULAŞ HUYAL

Signature :

ABSTRACT

CONFORMAL SYMMETRY IN FIELD THEORY

Huyal, Ulaş
M.S., Department of Physics
Supervisor : Assoc. Prof. Dr. Bayram Tekin

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In this thesis, conformal transformations in d and two dimensions and the results of conformal symmetry in classical and quantum field theories are reviewed. After investigating the conformal group and its algebra, various aspects of conformal invariance in field theories, like conserved charges, correlation functions and the Ward identities are discussed. The central charge and the Virasoro algebra are briefly touched upon.

Keywords: conformal group, conformal field theories, conserved charges, correlation functions, Ward identities

ÖZ

ALAN KURAMINDA KONFORMAL SİMETRİ

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Bu tezde d ve iki boyutlarda konformal dönüşümler ve konformal simetrinin klasik ve kuantum alan kuramlarındaki sonuçları gözden geçirildi. Konformal grup ile cebiri incelendikten sonra, alan kuramlarında konformal değişmezliğin, korunumlu yükler, korelasyon fonksiyonları ve Ward özdeşlikleri gibi, bazı yönleri anlatılmıştır. Kısaca ana yük ve Virasoro cebirine değinilmiştir.

Anahtar Kelimeler: konformal grup, konformal alan kuramları, korunumlu yükler, korelasyon fonksiyonları, Ward özdeşlikleri

To my family and Fatma

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CHAPTER 1

INTRODUCTION

Today, conformal field theory has become an important framework to understand certain aspects of string theories, statistical mechanics, condensed matter physics and even lead to developments in mathematics. Symmetries have played an enormously important role in the development of modern physics in the 20$^{\text{th}}$ century. Especially the introduction of the notion of four dimensional spacetimes to investigate the Lorentz transformations helped to investigate and generalize the symmetry concept in physical theories. Inversion of coordinates was already utilized in electrostatics by Lord Kelvin and generalized in three dimensions by Liouville in the 19$^{\text{th}}$ century (see [1] and references therein). It was in 1909 when H. Bateman discussed some properties of conformal transformations in four dimensional Euclidean space and showed that the conformal group can be used to transform solutions of the classical wave equations of light into other solutions and applied to geometrical optics [1]. One year later Bateman tried to investigate all transformations that leave Maxwell's equations invariant and showed for the Maxwell equations of the free electron cloud that its symmetry group is isomorphic to the conformal group in four dimensional Euclidean space [2]. In that same year E. Cunningham also investigated the invariance properties of Maxwell's equations and one of the results proved was that Maxwell's equations in linear media are conformally invariant at points in space with vanishing current density [3]. Conformal symmetry in quantum field theory was utilized in the 1960's and 1970's, but it was in 1984 when Belavin, Polyakov and Zamolodchikov made an important discovery by investigating massless, interacting field theories in the critical dimension $d = 2$ and found that there corresponds an infinite-dimensional conformal group as the symmetry group to these theories. This symmetry property is exploited to examine exactly solvable conformal field theories [4]. The consideration of conformal symmetry in quantum field theories initially was based on arguments of scale invariance.

But one has to note that conformal symmetry itself is a generalization of scale invariance. We will see that conformal transformations locally represent scale transformations that are functions of positions [9]. But it remains an important question what the explicit relation between scale invariance and conformal invariance is. An answer to this kind of relation in two dimensions was given partly by Zamolodchikov [5] and was investigated further by Polchinski [6] and it shows that under unitarity together with a discrete spectrum of operator dimensions scale invariance does lead to conformal invariance in two dimensions [6]. For higher dimensions it is not clear whether scale invariance implies conformal invariance [6, 7]. An example of a non-unitary theory in two dimensions where scale invariance does not imply conformal invariance is given in reference [8].

In this work, first the concept of conformal transformations and the associated conformal group in $d > 2$ dimensional spacetimes are discussed. After a discussion of Noether's theorem and conserved charges, conformal symmetries which are present in classical and quantum field theories are examined. Subsequently, conserved charges are investigated, in particular, correlation functions and the Ward identities are considered. Finally conformal symmetry in $d = 2$ dimensional spacetimes is discussed.

CHAPTER 2

NOTATION AND CONVENTIONS

In this thesis the notation of Di Francesco, et al. [9] is extensively used. The dimension of our spacetime is denoted by d and the flat metric is taken as,

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, \cdots, -1), \tag{2.1}$$

for Minkowksi spacetimes and as,

$$\eta_{\mu\nu} = \mathrm{diag}(1, \cdots, 1), \tag{2.2}$$

for Euclidean spacetimes. The Einstein summation convention is employed, where one upper index together with the corresponding lower index indicates summation. Greek indices run through all spacetime indices, unless otherwise specified; whereas the Latin index i denotes spatial coordinates. Infinitesimal parameters ω_a are introduced, where the index a is taken as a label for the parameters and summation over the whole set of labels is understood when repeated indices occur (occasionally other Latin indices such as g may be used instead of a).

For the field theories that are considered, the fields are denoted generally as a whole using the symbol Φ, when seen necessary this collective symbol is replaced by the corresponding individual fields. The first order variation of the fields with respect to the infinitesimal parameters ω_a at a fixed spacetime point, $\boldsymbol{x}$, is denoted by,

$$\delta_\omega \Phi(\boldsymbol{x}) \equiv \Phi'(\boldsymbol{x}) - \Phi(\boldsymbol{x}). \tag{2.3}$$

This notation is also extended to general functions of the fields. Using this the following definition is made for the generator of an infinitesimal symmetry transformation,

$$-i\omega_a G_a \Phi(\boldsymbol{x}) \equiv \Phi'(\boldsymbol{x}) - \Phi(\boldsymbol{x}). \tag{2.4}$$

It is also assumed throughout this thesis that under infinitesimal transformations $\boldsymbol{x} \to \boldsymbol{x}'$, the transformed fields, $\Phi'(\boldsymbol{x}')$, can be written as a function of the original fields, $\Phi(\boldsymbol{x})$, i.e.

$$\Phi'(\boldsymbol{x}') = \mathcal{F}(\Phi(\boldsymbol{x})). \tag{2.5}$$

In this thesis, the definition of general correlation functions using the Feynman path integral formalism is given as,

$$\langle \Phi(\boldsymbol{x}_1) \cdots \Phi(\boldsymbol{x}_n) \rangle = \frac{1}{Z} \int \mathcal{D}\Phi \; \Phi(\boldsymbol{x}_1) \cdots \Phi(\boldsymbol{x}_n) \exp\left(-S[\Phi]\right), \tag{2.6}$$

where $S[\Phi]$ denotes the Euclidean action of the field theory and Z, the so-called vacuum functional, is given by,

$$Z = \int \mathcal{D}\Phi \; \exp\left(-S[\Phi]\right). \tag{2.7}$$

A scale transformation for $\boldsymbol{x}$ and a field, $\Phi(\boldsymbol{x})$, is defined as follows,

$$\begin{aligned} \boldsymbol{x}' &= \lambda \boldsymbol{x} \\ \Phi'(\lambda \boldsymbol{x}) &= \lambda^{-\Delta} \Phi(\boldsymbol{x}), \end{aligned} \tag{2.8}$$

where λ is the scale factor and Δ is called the scaling dimension of Φ.

For conformal symmetry in two dimensions holomorphic and antiholomorphic coordinates are used. Thus for $(z^0, z^1) \in \mathbb{R}^2$, the following definitions are made,

$$\begin{aligned} z &\equiv z^0 + iz^1 & \bar{z} &\equiv z^0 - iz^1 \\ \partial_z &\equiv \frac{1}{2}\left(\partial_0 - i\partial_1\right) & \partial_{\bar{z}} &\equiv \frac{1}{2}\left(\partial_0 + i\partial_1\right) \end{aligned} \tag{2.9}$$

where we also used, $\partial_0 \equiv \dfrac{\partial}{\partial z^0}$ and $\partial_1 \equiv \dfrac{\partial}{\partial z^1}$. z is called the holomorphic coordinate, whereas $\bar{z}$ is the antiholomorphic one.

CHAPTER 3

GLOBAL CONFORMAL INVARIANCE

3.1 The Conformal Group

Suppose we have a well-defined metric tensor in a d-dimensional spacetime with components, $g_{\mu\nu}(\boldsymbol{x})$. We have a conformal transformation, if under an invertible coordinate transformation $\boldsymbol{x} \to \boldsymbol{x}'$, our transformed metric components, $g'_{\mu\nu}(\boldsymbol{x}')$, can be written in the form,

$$g'_{\mu\nu}(\boldsymbol{x}') = \Lambda(\boldsymbol{x}) g_{\mu\nu}(\boldsymbol{x}), \tag{3.1}$$

where $\Lambda(\boldsymbol{x})$ is a well behaved function (i.e. non-vanishing and sufficiently smooth). These transformations form the so called conformal group, and we can see that for $\Lambda(\boldsymbol{x}) = 1$ we clearly get the symmetries of the Poincaré group, which is therefore a subgroup of the aforementioned group. The conformal transformations, as the name suggests, preserve the angle between two vectors defined at a spacetime point [9, 10].

Following the basic steps and notation of Di Francesco, et al. [9], let us consider an infinitesimal change in the coordinates, $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(\boldsymbol{x})$, and see how the metric changes accordingly. We note that $g_{\mu\nu}(\boldsymbol{x}) = \eta_{\mu\nu}$ is assumed from now on, where $\eta_{\mu\nu}$ is the flat Minkowski or Euclidean metric. Our transformed metric is given by,

$$g'_{\mu\nu}(\boldsymbol{x}') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(\boldsymbol{x}). \tag{3.2}$$

Keeping in mind that we have, $x^\mu = x'^\mu - \epsilon^\mu(\boldsymbol{x})$, and working to first order in ϵ, we have,

$$\frac{\partial x^\alpha}{\partial x'^\mu} = \delta^\alpha{}_\mu - \frac{\partial x^\beta}{\partial x'^\mu} \partial_\beta \epsilon^\alpha(\boldsymbol{x}), \tag{3.3}$$

or arranging the terms,

$$\frac{\partial x^\beta}{\partial x'^\mu} \left(\delta^\alpha{}_\beta + \partial_\beta \epsilon^\alpha \right) = \delta^\alpha{}_\mu, \tag{3.4}$$

and multiplying each side by $(\delta^{\nu}{}_{\alpha} - \partial_{\alpha}\epsilon^{\nu})$, we obtain,

$$\frac{\partial x^{\nu}}{\partial x'^{\mu}} = \delta^{\nu}{}_{\mu} - \partial_{\mu}\epsilon^{\nu}. \tag{3.5}$$

After inserting (3.5) into (3.2),

$$\begin{aligned} g'_{\mu\nu}(\boldsymbol{x}') &= \left(\delta^{\alpha}{}_{\mu} - \partial_{\mu}\epsilon^{\alpha}\right)\left(\delta^{\beta}{}_{\nu} - \partial_{\nu}\epsilon^{\beta}\right) g_{\alpha\beta}(\boldsymbol{x}) \\ &= g_{\mu\nu}(\boldsymbol{x}) - \left(\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}\right). \end{aligned} \tag{3.6}$$

According to (3.1), equation (3.6) and therefore $\left(\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}\right)$ must be proportional to $g_{\mu\nu}$ for conformal invariance to hold. Thus, we can write,

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = f(\boldsymbol{x})g_{\mu\nu}. \tag{3.7}$$

Taking the trace of (3.7) by multiplication with $g^{\mu\nu}$, we see that,

$$g^{\mu\nu}\partial_{\mu}\epsilon_{\nu} + g^{\mu\nu}\partial_{\nu}\epsilon_{\mu} = 2\partial_{\mu}\epsilon^{\mu} = d \cdot f(\boldsymbol{x}), \tag{3.8}$$

or simply,

$$f(\boldsymbol{x}) = \frac{2}{d}\partial_{\mu}\epsilon^{\mu}, \tag{3.9}$$

where we have used $g^{\mu\nu}g_{\mu\nu} = d$, which is valid for both Euclidean and Minkowskian space-times (or for any flat metric in d dimensions with signature (p, q) [10]). Now we want to find constraints on $f(\boldsymbol{x})$, or equivalently on ϵ, that are enforced upon us by conformal symmetry. For convenience we will write from now on $g_{\mu\nu}$ explicitly as $\eta_{\mu\nu}$. Take (3.7) and act on it with ∂_{ρ} to get, $\partial_{\rho}\partial_{\mu}\epsilon_{\nu} + \partial_{\rho}\partial_{\nu}\epsilon_{\mu} = \eta_{\mu\nu}\partial_{\rho}f$. Similarly we have, $\partial_{\mu}\partial_{\nu}\epsilon_{\rho} + \partial_{\mu}\partial_{\rho}\epsilon_{\nu} = \eta_{\nu\rho}\partial_{\mu}f$ and $\partial_{\nu}\partial_{\mu}\epsilon_{\rho} + \partial_{\nu}\partial_{\rho}\epsilon_{\mu} = \eta_{\mu\rho}\partial_{\nu}f$. Adding these two and subtracting the first equation, we arrive at the expression,

$$2\partial_{\mu}\partial_{\nu}\epsilon_{\rho} = \eta_{\nu\rho}\partial_{\mu}f + \eta_{\mu\rho}\partial_{\nu}f - \eta_{\mu\nu}\partial_{\rho}f. \tag{3.10}$$

Contracting (3.10) with $\eta_{\mu\nu}$, we get

$$2\eta^{\mu\nu}\partial_{\mu}\partial_{\nu}\epsilon_{\rho} = \eta^{\mu\nu}\eta_{\nu\rho}\partial_{\mu}f + \eta^{\mu\nu}\eta_{\mu\rho}\partial_{\nu}f - \eta^{\mu\nu}\eta_{\mu\nu}\partial_{\rho}f, \tag{3.11}$$

which leads to,

$$2\partial^{2}\epsilon_{\rho} = (2-d)\,\partial_{\rho}f. \tag{3.12}$$

We can now exploit the fact that order of differentiation on $f(\boldsymbol{x})$ does not matter, such that by acting ∂_{μ} on (3.12), we get $\partial_{\mu}\partial^{2}\epsilon_{\rho} = \partial_{\rho}\partial^{2}\epsilon_{\mu}$ or equivalently, $\partial^{2}\left(\partial_{\mu}\epsilon_{\rho} + \partial_{\rho}\epsilon_{\mu}\right) = (2-d)\,\partial_{\mu}\partial_{\rho}f$

Thus, after applying ∂^2 to (3.7) and combining it with the previous expression, one is left with,

$$(2-d)\,\partial_\mu \partial_\nu f = \eta_{\mu\nu} \partial^2 f, \tag{3.13}$$

which with the help of $\eta^{\mu\nu}$, finally gives us the constraint,

$$(d-1)\,\partial^2 f = 0. \tag{3.14}$$

We have several cases at hand, that have to be studied. First of all, the $d = 1$ case gives us trivially, $0 = 0$, which indicates there is no constraint on $f(\boldsymbol{x})$ in one dimension, other than having to provide us with an invertible coordinate transformation $\boldsymbol{x} \to \boldsymbol{x}'$. The $d = 2$ case will be postponed for now, we will discuss it in the next chapter. For the $d \geq 3$ case, (3.13) and (3.14) lead us to,

$$\partial^2 f = 0, \tag{3.15}$$

and more generally,

$$\partial_\mu \partial_\nu f = 0. \tag{3.16}$$

From equations (3.15) we see that our real scalar function, $f(\boldsymbol{x})$, should, in its most general form, depend linearly to x^μ, that is, with real constants denoted by A and B_μ, we have,

$$f(\boldsymbol{x}) = A + B_\mu x^\mu. \tag{3.17}$$

Using expression (3.17) in equation (3.10), we immediately get, $2\partial_\mu \partial_\nu \epsilon_\rho = B_\mu \eta_{\nu\rho} + B_\nu \eta_{\mu\rho} - B_\rho \eta_{\mu\nu}$, which is independent of x^μ, such that the most general expression for ϵ_μ, with a_μ, $b_{\mu\nu}$ and $c_{\mu\nu\rho}$ being real constants, is,

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho, \tag{3.18}$$

where in the last term we see clearly that $c_{\mu\nu\rho}$ must be symmetric in its last two indices, i.e. $c_{\mu\nu\rho} = c_{\mu\rho\nu}$. Now we can find conditions on the constants a_μ, $b_{\mu\nu}$, $c_{\mu\nu\rho}$ by inserting (3.18) into (3.7) and (3.9), then collecting terms of same order [11], or simply, as the x^μ are arbitrary, inserting each terms of different order independently into the relevant equations [9].

The constant term:

Plugging a_μ into (3.9) gives, $f(\boldsymbol{x}) = 0$. Thus, (3.7) and (3.10) are trivially satisfied for all real a_μ. We interpret a_μ as the infinitesimal translation in spacetime [9], or equivalently, as the inhomogeneous part of the Poincaré group.

The linear term:

(3.9) and (3.7) together give us,

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}\eta_{\mu\nu}\partial_\rho \epsilon^\rho. \tag{3.19}$$

Let us note that equations (3.19) are the *conformal Killing equations* in d-dimensional flat spacetimes. The expression for general spacetimes would be, $\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu = \dfrac{2}{d} g_{\mu\nu} \nabla \cdot \boldsymbol{\epsilon}$, where ∇_μ is the covariant derivative and $\nabla \cdot \boldsymbol{\epsilon}$ denotes the divergence of $\boldsymbol{\epsilon}$. Solutions of (3.19) are accordingly called conformal Killing fields [12].

Plugging the linear term into (3.19) we have,

$$\begin{aligned}
& \partial_\mu \left(b_{\nu\rho} x^\rho\right) + \partial_\nu \left(b_{\mu\rho} x^\rho\right) = \tfrac{2}{d}\eta_{\mu\nu}\partial_\lambda \left(b^\lambda{}_\rho x^\rho\right) \\
\Rightarrow \quad & b_{\nu\rho}\delta^\rho{}_\mu + b_{\mu\rho}\delta^\rho{}_\nu = \tfrac{2}{d} b^\lambda{}_\rho \delta^\rho{}_\lambda \eta_{\mu\nu} \\
\Rightarrow \quad & b_{\nu\mu} + b_{\mu\nu} = \tfrac{2}{d} b^\lambda{}_\lambda \eta_{\mu\nu}.
\end{aligned} \tag{3.20}$$

Now breaking $b_{\mu\nu}$ up into symmetric and antisymmetric parts, we see that in the previous equation only the symmetric part survives. We also recognize that the symmetric part must be proportional to $\eta_{\mu\nu}$ [11]. Thus, we are making the Ansatz,

$$b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu}, \tag{3.21}$$

where $m_{\mu\nu}$ is antisymmetric, i.e. $m_{\mu\nu} = -m_{\nu\mu}$, and α is a real constant. In fact, α directly stands for the trace part of $b_{\mu\nu}$, since $b_{\mu\nu} + b_{\nu\mu} = 2\alpha\eta_{\mu\nu}$ and that gives, $\alpha = b^\mu{}_\mu/d$. Here, α represents an infinitesimal scale transformation, whereas $m_{\mu\nu}$, depending on whether we have Euclidean or Lorentzian spacetimes, is identified with (pseudo-)orthogonal transformations [9].

The quadratic term:

When the last part of (3.18) is substituted into (3.10) we obtain,

$$2\partial_\mu\partial_\nu \left(c_{\rho\lambda\sigma} x^\lambda x^\sigma\right) = \frac{2}{d}\left[\eta_{\mu\rho}\partial_\nu\partial_\lambda \left(c^\lambda{}_{\sigma\kappa} x^\sigma x^\kappa\right) + \eta_{\nu\rho}\partial_\mu\partial_\lambda \left(c^\lambda{}_{\sigma\kappa} x^\sigma x^\kappa\right) - \eta_{\mu\nu}\left(c^\lambda{}_{\sigma\kappa} x^\sigma x^\kappa\right)\right], \tag{3.22}$$

and bearing in mind that $c_{\mu\nu\rho}$ is symmetric in its last two indices,

$$\begin{aligned}
\Rightarrow \quad 4c_{\rho\mu\nu} = \frac{2}{d}\Big[& \eta_{\mu\rho}\left(c^\lambda{}_{\sigma\kappa}\delta^\sigma{}_\lambda\delta^\kappa{}_\nu + c^\lambda{}_{\sigma\kappa}\delta^\sigma{}_\nu\delta^\kappa{}_\lambda\right) + \eta_{\nu\rho}\left(c^\lambda{}_{\sigma\kappa}\delta^\sigma{}_\lambda\delta^\kappa{}_\mu + c^\lambda{}_{\sigma\kappa}\delta^\sigma{}_\mu\delta^\kappa{}_\lambda\right) \\
& - \eta_{\mu\nu}\left(c^\lambda{}_{\sigma\kappa}\delta^\sigma{}_\lambda\delta^\kappa{}_\rho + c^\lambda{}_{\sigma\kappa}\delta^\sigma{}_\rho\delta^\kappa{}_\lambda\right)\Big] \\
\Rightarrow \quad c_{\rho\mu\nu} = \frac{1}{d} & \left(\eta_{\mu\rho} c^\lambda{}_{\lambda\nu} + \eta_{\nu\rho} c^\lambda{}_{\lambda\mu} - \eta_{\mu\nu} c^\lambda{}_{\lambda\rho}\right).
\end{aligned} \tag{3.23}$$

Defining $b_\mu \equiv \frac{1}{d} c^\lambda{}_{\lambda\mu}$, we finally get,

$$c_{\rho\mu\nu} = \eta_{\mu\rho} b_\nu + \eta_{\nu\rho} b_\mu - \eta_{\mu\nu} b_\rho. \tag{3.24}$$

Plugging the quadratic part of (3.18) into $x'^\mu = x^\mu + \epsilon^\mu$, we have,

$$\begin{aligned} x'^\mu &= x^\mu + \eta^{\mu\lambda} \left(\eta_{\rho\lambda} b_\nu + \eta_{\nu\lambda} b_\rho - \eta_{\rho\nu} b_\lambda \right) \\ &= x^\mu + \left(x^\mu b_\nu x^\nu + x^\mu b_\rho x^\rho - \boldsymbol{x}^2 b_\mu \right), \end{aligned} \tag{3.25}$$

or equivalently,

$$x'^\mu = x^\mu + 2 \left(\boldsymbol{b} \cdot \boldsymbol{x} \right) x^\mu - b^\mu \boldsymbol{x}^2. \tag{3.26}$$

This infinitesimal transformation is called the *special conformal transformation* [9].

Now the finite transformations of two of these infinitesimal transformations are well known members of the Poincaré group, namely the finite translations, $x'^\mu = x^\mu + a^\mu$, and the Lorentz and/or orthogonal transformations, $x'^\mu = M^\mu{}_\nu x^\nu$. The first term on the right-hand side of (3.21) describes a scaling of our coordinate, its finite version is clearly of the general form, $x'^\mu = \alpha x^\mu$ (note that this α is different from the previous one). To obtain these finite forms we have to take the exponential of the generators of the infinitesimal transformations. Though, to avoid complicated expressions for the exponentials it is better to investigate for example finite translations, scale transformations and rotations around a fixed axis seperately. In this way they form abelian subgroups of the conformal group, which leads to simple expressions of their exponentials [13]. Thus, the infinitesimal scale transformation, $x'^\mu = x^\mu + \alpha x^\mu$, becomes, replacing α with α'/N (where α' is finite), applying the infinitesimal transformation N times, and taking the limit $N \to \infty$,

$$\begin{aligned} x'^\mu &= \lim_{N\to\infty} \left(1 + \frac{\alpha'}{N} \right)^N x^\mu \\ &= \mathrm{e}^{\alpha'} x^\mu, \end{aligned} \tag{3.27}$$

or defining, $\alpha = \mathrm{e}^{\alpha'}$,

$$x'^\mu = \alpha x^\mu. \tag{3.28}$$

For the translations we do not even need exponentiation (actually for abelian groups addition plays the role of multiplication), applying a^μ/N, for $N \to \infty$, N times, should give us,

$$\begin{aligned} x'^\mu &= \lim_{N\to\infty} \left(x^\mu + N \cdot \frac{a^\mu}{N} \right) \\ &= x^\mu + a^\mu. \end{aligned} \tag{3.29}$$

Calculating along the same lines we are able to get the finite form for the second term on the right-hand side of (3.21). The finite form of the special conformal transformation is not so obvious at the first sight. But let us write down the finite form of it and demonstrate its validity by showing that it is a conformal transformation and evaluating its infinitesimal version [9]. The finite special conformal transformation is given by,

$$x'^{\mu} = \frac{x^{\mu} - b^{\mu}\boldsymbol{x}^2}{1 - 2\boldsymbol{b}\cdot\boldsymbol{x} + b^2\boldsymbol{x}^2}. \tag{3.30}$$

Taking b^{μ} infinitesimal, and evaluating our expressions up to $O(b^2)$,

$$\begin{aligned} x'^{\mu} &\approx \frac{x^{\mu} - b^{\mu}\boldsymbol{x}^2}{1 - 2b_{\nu}x^{\nu}} \\ &= \left(x^{\mu} - b^{\mu}\boldsymbol{x}^2\right)\left(1 + 2b_{\nu}x^{\nu} + O(b^2)\right) \\ &\approx x^{\mu} + 2b_{\nu}x^{\nu}x^{\mu} - b^{\mu}\boldsymbol{x}^2, \end{aligned} \tag{3.31}$$

which is exactly the infinitesimal special conformal transformation (SCT) we found before. To show that the finite SCT is also a conformal transformation, let us first take $\Lambda(\boldsymbol{x})$ in (3.1) as, $\Lambda(\boldsymbol{x}) = \left(1 - 2\boldsymbol{b}\cdot\boldsymbol{x} + b^2\boldsymbol{x}^2\right)^2$ [9], and see that it indeed satisfies our tensor transformation law (3.2) under the coordinate change, $x^{\mu} \to x'^{\mu} = \dfrac{x^{\mu} - b^{\mu}\boldsymbol{x}^2}{1 - 2\boldsymbol{b}\cdot\boldsymbol{x} + b^2\boldsymbol{x}^2}$. By writing the inverse coordinate transformation, we have,

$$g_{\mu\nu}(\boldsymbol{x}) = \frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\nu}}g'_{\alpha\beta}(\boldsymbol{x}'). \tag{3.32}$$

For the moment we want to leave $g_{\mu\nu}(\boldsymbol{x})$ as it is and see whether the above equation gives us the flat metric, $\eta_{\mu\nu}$, after inserting $g'_{\alpha\beta}(\boldsymbol{x}') = \Lambda(\boldsymbol{x})\eta_{\alpha\beta}$ and the corresponding expressions for $\dfrac{\partial x'^{\alpha}}{\partial x^{\mu}}$ and $\dfrac{\partial x'^{\beta}}{\partial x^{\nu}}$. Taking the derivative of (3.30) and denoting, for notational convenience, $I \equiv 1 - 2\boldsymbol{b}\cdot\boldsymbol{x} + b^2\boldsymbol{x}^2$, we obtain,

$$\begin{aligned} \frac{\partial x'^{\alpha}}{\partial x^{\mu}} &= \frac{1}{I^2}\Big[\left(\delta^{\alpha}{}_{\mu} - b^{\alpha}\eta_{\beta\gamma}\left(\delta^{\beta}{}_{\mu}x^{\gamma} + x^{\beta}\delta^{\gamma}{}_{\mu}\right)\right)I \\ &\qquad - \left(x^{\alpha} - b^{\alpha}\boldsymbol{x}^2\right)\left(-2b^{\beta}\eta_{\beta\gamma}\delta^{\gamma}{}_{\mu} + b^2\eta_{\beta\gamma}\left(\delta^{\beta}{}_{\mu}x^{\gamma} + x^{\beta}\delta^{\gamma}{}_{\mu}\right)\right)\Big] \\ &= \frac{1}{I^2}\left[I\left(\delta^{\alpha}{}_{\mu} - 2b^{\alpha}x_{\mu}\right) - \left(x^{\alpha} - b^{\alpha}\boldsymbol{x}^2\right)\left(-2b_{\mu} + 2b^2x_{\mu}\right)\right]. \end{aligned} \tag{3.33}$$

Let us contract the above expression with $\eta_{\alpha\beta}$, to get,

$$\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\eta_{\alpha\beta} = \frac{1}{I^2}\left[I\left(\eta_{\mu\beta} - 2b_{\beta}x_{\mu}\right) - 2\left(x_{\beta} - b_{\beta}\boldsymbol{x}^2\right)\left(b^2x_{\mu} - b_{\mu}\right)\right]. \tag{3.34}$$

Writing the corresponding expression for $\dfrac{\partial x'^{\beta}}{\partial x^{\nu}}$ and combining this with the previous equation, we are able to write,

$$\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\nu}}\eta_{\alpha\beta} = \frac{1}{I^4}\Big[I^2\left(\eta_{\mu\nu} - 2b_\nu x_\mu - 2b_\mu x_\nu + 4b^2 x_\mu x_\nu\right) + 4\boldsymbol{x}^2 I\left(b^2 x_\mu - b_\mu\right)\left(b^2 x_\nu - b_\nu\right)$$
$$- 2I\left(\left(x_\mu - b_\mu \boldsymbol{x}^2 - 2\left(\boldsymbol{x}\cdot\boldsymbol{b}\right)x_\mu + 2b^2\boldsymbol{x}^2\right)\left(b^2 x_\nu - b_\nu\right)\right)$$
$$- 2I\left(\left(x_\nu - b_\nu \boldsymbol{x}^2 - 2\left(\boldsymbol{x}\cdot\boldsymbol{b}\right)x_\nu + 2b^2\boldsymbol{x}^2\right)\left(b^2 x_\mu - b_\mu\right)\right)\Big], \tag{3.35}$$

simplifying this further, we have,

$$\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\nu}}\eta_{\alpha\beta} = \frac{1}{I^3}\Big[I\left(\eta_{\mu\nu} - 2b_\nu x_\mu - 2b_\mu x_\nu + 4b^2 x_\mu x_\nu\right) + 4\boldsymbol{x}^2\left(b^2 x_\mu - b_\mu\right)\left(b^2 x_\nu - b_\nu\right)$$
$$- 2\left(x_\mu I + \left(x_\mu b^2 - b_\mu\right)\boldsymbol{x}^2\right)\left(b^2 x_\nu - b_\nu\right)$$
$$- 2\left(x_\nu I + \left(x_\nu b^2 - b_\nu\right)\boldsymbol{x}^2\right)\left(b^2 x_\mu - b_\mu\right)\Big]$$
$$= \frac{1}{I^2}\left(\eta_{\mu\nu} - 2b_\nu x_\mu - 2b_\mu x_\nu + 4b^2 x_\mu x_\nu - 2b^2 x_\mu x_\nu + 2x_\mu b_\nu - 2b^2 x_\nu x_\mu + 2x_\nu b_\mu\right), \tag{3.36}$$

all terms cancel, except $\eta_{\mu\nu}$, therefore we get,

$$\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\nu}}\eta_{\alpha\beta} = \frac{\eta_{\mu\nu}}{I^2}. \tag{3.37}$$

If we insert (3.37) and $g'_{\alpha\beta}(\boldsymbol{x}') = \Lambda(\boldsymbol{x})\eta_{\alpha\beta} = I^2\eta_{\alpha\beta}$ into (3.32), we get, $g_{\mu\nu}(\boldsymbol{x}) = \eta_{\mu\nu}$, as expected. So, (3.30) really is a conformal transformation, whose infinitesimal form is given by (3.26).

Now let us find the generators of the conformal group using the infinitesimal transformations we derived earlier. We will represent the generators with the standard coordinate basis, $\{\partial_\mu\}$. Until now we considered symmetries of the metric tensor. But we also have to consider symmetries of field theories.

3.2 Continuous Symmetry Transformations

Following closely [9], let us take a collection of fields, for which we use, shorthandedly, the notation Φ. Also, let us consider a Lagrangian (or Lagrangian density) that depends on the collection of fields, Φ, and their first derivatives, $\partial_\mu\Phi$. Namely, we have a Lagrangian, $\mathcal{L}(\Phi, \partial_\mu\Phi)$. Considering a transformation that has the following general form,

$$\boldsymbol{x} \to \boldsymbol{x}' \qquad\qquad \Phi(\boldsymbol{x}) \to \Phi'(\boldsymbol{x}') \tag{3.38}$$

we are able to discover how fields behave under certain coordinate transformations, which directly links us to the physics of our system. With the help of the Lagrangian, we can determine which symmetries our system possesses and furthermore, in the case of continuous symmetries, as we will see, also obtain conserved "currents" and correspondingly conserved "charges". We assume that $\Phi'(\boldsymbol{x}')$ can be written as a functional of $\Phi(\boldsymbol{x})$, i.e. we have,

$$\Phi'(\boldsymbol{x}') = \mathcal{F}(\Phi(\boldsymbol{x})). \tag{3.39}$$

If we take the infinitesimal form of our transformations, we end up working with terms up to and not including $O(\omega^2)$,

$$x'^\mu = x^\mu + \delta x^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a} \tag{3.40}$$

$$\Phi'(\boldsymbol{x}') = \Phi(\boldsymbol{x}) + \delta\mathcal{F}(\boldsymbol{x}) = \Phi(\boldsymbol{x}) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(\boldsymbol{x}), \tag{3.41}$$

where the ω_a are infinitesimal parameters. Note that the index a here is a generic index, which depends on the transformation we are looking at. Let us define the generator, G_a,

$$-i\omega_a G_a \Phi(\boldsymbol{x}) \equiv \Phi'(\boldsymbol{x}) - \Phi(\boldsymbol{x}) \quad \Big(\equiv \delta_\omega \Phi(\boldsymbol{x})\Big) \tag{3.42}$$

We took the difference at the same spacetime point, $\boldsymbol{x}$, this means that we consider the generator of this continuous transformation to be an indicator of the change in our field, Φ. If we take, $x^\mu = x'^\mu - \omega_a \frac{\delta x^\mu}{\delta \omega_a}$ and expand $\Phi(\boldsymbol{x})$ and $\frac{\delta \mathcal{F}}{\delta \omega_a}(\boldsymbol{x})$ around $\boldsymbol{x}'$, we get the approximations,

$$\Phi(\boldsymbol{x}) = \Phi(\boldsymbol{x}') + (x^\mu - x'^\mu)\,\partial_\mu \Phi(\boldsymbol{x})\Big|_{\boldsymbol{x}=\boldsymbol{x}'} + \cdots = \Phi(\boldsymbol{x}') - \omega_a \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \Phi(\boldsymbol{x}') + O(\omega^2), \tag{3.43}$$

$$\frac{\delta \mathcal{F}}{\delta \omega_a}(\boldsymbol{x}) = \frac{\delta \mathcal{F}}{\delta \omega_a}(\boldsymbol{x}') + O(\omega). \tag{3.44}$$

Using these expressions in (3.41) and neglecting higher order terms, we get,

$$\Phi'(\boldsymbol{x}') = \Phi(\boldsymbol{x}') - \omega_a \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \Phi(\boldsymbol{x}') + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(\boldsymbol{x}'), \tag{3.45}$$

thus,

$$iG_a \Phi = \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \Phi - \frac{\delta \mathcal{F}}{\delta \omega_a}. \tag{3.46}$$

Now we suppose that the fields are not changed by our conformal transformation, for the moment, which means we take them as scalar fields, $\mathcal{F}(\Phi(\boldsymbol{x})) = \Phi(\boldsymbol{x})$ [9, 14].

With this assumption it is now easy to establish the generators of the conformal group using the infinitesimal transformations found previously. First, let us consider the infinitesimal translation, for which the variation in coordinates takes the form, $\frac{\delta x^\mu}{\delta \omega^\nu} = \delta^\mu{}_\nu$. With $\mathcal{F}(\Phi(\boldsymbol{x})) = \Phi(\boldsymbol{x})$, we also clearly have, $\frac{\delta \mathcal{F}}{\delta \omega^\nu} = 0$, giving us,

$$iG_\nu \Phi = \delta^\mu{}_\nu \partial_\mu \Phi \qquad \Rightarrow \qquad G_\nu = -i\partial_\nu \quad (\equiv P_\nu)\,, \tag{3.47}$$

where the last step follows from the arbitrariness of Φ.

If we consider the infinitesimal scale transformation, $x^\mu \to x'^\mu = x^\mu + \alpha x^\mu$, we have, keeping in mind that ω in this case should be a scalar (thus representing the scaling parameter),

$$x'^\mu = x^\mu + \alpha x^\mu = x^\mu + \omega x^\mu \tag{3.48}$$

$$\delta x^\mu = \omega x^\mu. \tag{3.49}$$

We get then, using $\frac{\delta x^\mu}{\delta \omega} = x^\mu$,

$$iG\Phi = x^\mu \partial_\mu \Phi, \tag{3.50}$$

again using the arbitrariness of Φ, we get,

$$G = -ix^\mu \partial_\mu \quad (\equiv D)\,. \tag{3.51}$$

Taking the infinitesimal Lorentz transformation in (3.21), i.e. $x^\mu \to x'^\mu = x^\mu + m^\mu{}_\nu x^\nu$, we see that,

$$\begin{aligned} x'^\mu &= x^\mu + \omega^\mu{}_\nu x^\nu \\ &= x^\mu + \omega_{\rho\nu} \eta^{\rho\mu} x^\nu, \end{aligned} \tag{3.52}$$

but we have to note that our $\omega_{\rho\nu}$ is antisymmetric, thus it is better to take the antisymmetric part of $\eta^{\rho\mu} x^\nu$ in the indices, ρ and ν, which is clearly given by, $\left(\eta^{\rho\mu} x^\nu - \eta^{\nu\mu} x^\rho\right)/2$. This way it is more transparent to see the number of independent parameters, which is given by $(d^2 - d)/2$. Thus we write,

$$x'^\mu = x^\mu + \omega_{\rho\nu} \frac{1}{2}\left(\eta^{\rho\mu} x^\nu - \eta^{\nu\mu} x^\rho\right), \tag{3.53}$$

which leaves us with,

$$\frac{\delta x^\mu}{\delta \omega_{\rho\nu}} = \frac{1}{2}\left(\eta^{\rho\mu} x^\nu - \eta^{\nu\mu} x^\rho\right). \tag{3.54}$$

Using this expression in our definition of the generator and adding an additional $1/2$ in the definition of $G^{\rho\nu}$ to avoid overcounting the parameters of the Lorentz symmetry [9], we have,

$$
\begin{aligned}
i\frac{1}{2}G^{\rho\nu}\Phi &= \frac{1}{2}\left(\eta^{\rho\mu}x^{\nu} - \eta^{\nu\mu}x^{\rho}\right)\partial_{\mu}\Phi, \\
G^{\rho\nu} &= i\left(x^{\rho}\partial^{\nu} - x^{\nu}\partial^{\rho}\right) \quad (\equiv L^{\rho\nu}). \qquad (3.55)
\end{aligned}
$$

And finally, let us consider the infinitesimal SCT, $x^{\mu} \to x'^{\mu} = x^{\mu} + 2\,(\boldsymbol{b}\cdot\boldsymbol{x})\,x^{\mu} - b^{\mu}\boldsymbol{x}^2$. For these transformations we obtain,

$$
\begin{aligned}
x'^{\mu} &= x^{\mu} + 2\,(\boldsymbol{b}\cdot\boldsymbol{x})\,x^{\mu} - b^{\mu}\boldsymbol{x}^2 \\
&= x^{\mu} + 2x^{\mu}x_{\nu}b^{\nu} - \boldsymbol{x}^2\delta^{\mu}{}_{\nu}b^{\nu} \\
&= x^{\mu} + \left(2x^{\mu}x_{\nu} - \boldsymbol{x}^2\delta^{\mu}{}_{\nu}\right)\omega^{\nu}, \qquad (3.56)
\end{aligned}
$$

where we used $\omega^{\nu} \equiv b^{\nu}$ in the last line. That means that we have $\frac{\delta x^{\mu}}{\delta\omega^{\nu}} = 2x^{\mu}x_{\nu} - \boldsymbol{x}^2\delta^{\mu}{}_{\nu}$. This therefore gives us,

$$
\begin{aligned}
iG_{\nu}\Phi &= \left(2x^{\mu}x_{\nu} - \boldsymbol{x}^2\delta^{\mu}{}_{\nu}\right)\partial_{\mu}\Phi \\
G_{\nu} &= -i\left(2x_{\nu}x^{\mu}\partial_{\mu} - \boldsymbol{x}^2\partial_{\nu}\right) \quad (\equiv K_{\nu}). \qquad (3.57)
\end{aligned}
$$

Therefore we have shown that the generators of the conformal group are given by;

$$P_{\mu} = -i\partial_{\mu} \quad \text{(translation)} \qquad (3.58)$$

$$D = -ix^{\mu}\partial_{\mu} \quad \text{(dilation)} \qquad (3.59)$$

$$L_{\mu\nu} = i\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right) \quad \text{(rotation)} \qquad (3.60)$$

$$K_{\mu} = -i\left(2x_{\mu}x^{\nu}\partial_{\nu} - \boldsymbol{x}^2\partial_{\mu}\right) \quad \text{(SCT)} \qquad (3.61)$$

Let us at last write out the algebra of the conformal group, using the generators we found. The commutation relations can be found by applying the generators on an arbitrary test function, $f(\boldsymbol{x})$, as follows,

$$
\begin{aligned}
[D, P_{\mu}]\,f(\boldsymbol{x}) &= -x^{\nu}\partial_{\nu}\partial_{\mu}f + \partial_{\mu}\left(x^{\nu}\partial_{\nu}f\right) \\
&= -x^{\nu}\partial_{\nu}\partial_{\mu}f + \delta^{\nu}{}_{\mu}\partial_{\nu}f + x^{\nu}\partial_{\mu}\partial_{\nu}f \\
&= \partial_{\mu}f = iP_{\mu}f(\boldsymbol{x}), \qquad (3.62)
\end{aligned}
$$

$$
\begin{aligned}
[D, K_\mu]\, f(\boldsymbol{x}) &= -x^\nu \partial_\nu \left(2x_\mu x^\rho \partial_\rho f - \boldsymbol{x}^2 \partial_\mu f\right) + \left(2x_\mu x^\rho \partial_\rho - \boldsymbol{x}^2 \partial_\mu\right)(x^\nu \partial_\nu f) \\
&= -2x^\nu \eta_{\mu\rho} \delta^\sigma{}_\nu x^\rho \partial_\rho f - 2x^\nu x_\mu \delta^\rho{}_\nu \partial_\rho f - 2x_\mu x^\nu x^\rho \partial_\nu \partial_\rho f \\
&\quad + 2x^\nu x_\nu \partial_\mu f + x^\nu \boldsymbol{x}^2 \partial_\nu \partial_\mu f + 2x_\mu x^\rho \delta^\nu{}_\rho \partial_\nu f + 2x_\mu x^\rho x^\nu \partial_\rho \partial_\nu f \\
&\quad - \boldsymbol{x}^2 \delta^\nu{}_\mu \partial_\nu f - \boldsymbol{x}^2 x^\nu \partial_\mu \partial_\nu f \\
&= -2x_\mu x^\rho \partial_\rho f + \boldsymbol{x}^2 \partial_\mu f = -iK_\mu f(\boldsymbol{x}),
\end{aligned} \tag{3.63}
$$

$$
\begin{aligned}
[K_\mu, P_\nu]\, f(\boldsymbol{x}) &= -\left(2x_\mu x^\rho \partial_\rho - \boldsymbol{x}^2 \partial_\mu\right) \partial_\nu f + \partial_\nu \left(2x_\mu x^\rho \partial_\rho - \boldsymbol{x}^2 \partial_\mu\right) f \\
&= -2x_\mu x^\rho \partial_\rho \partial_\nu f + \boldsymbol{x}^2 \partial_\mu \partial_\nu f + 2\eta_{\mu\nu} x^\rho \partial_\rho f + 2x_\mu \partial_\nu f \\
&\quad + 2x_\mu x^\rho \partial_\nu \partial_\rho f - 2x_\nu \partial_\mu f - \boldsymbol{x}^2 \partial_\nu \partial_\mu f \\
&= 2\eta_{\mu\nu} x^\rho \partial_\rho f + 2x_\mu \partial_\nu f - 2x_\nu \partial_\mu f = 2i\left(\eta_{\mu\nu} D - L_{\mu\nu}\right) f(\boldsymbol{x}),
\end{aligned} \tag{3.64}
$$

$$
\begin{aligned}
[K_\rho, L_{\mu\nu}]\, f(\boldsymbol{x}) &= \left(2x_\rho x^\sigma \partial_\sigma - \boldsymbol{x}^2 \partial_\rho\right)\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) f - \left(x_\mu \partial_\nu - x_\nu \partial_\mu\right)\left(2x_\rho x^\sigma \partial_\sigma - \boldsymbol{x}^2 \partial_\rho\right) f \\
&= 2x_\rho x_\mu \partial_\nu f + 2x_\rho x_\mu x^\sigma \partial_\sigma \partial_\nu f - 2x_\rho x_\nu \partial_\mu f - 2x_\rho x_\nu x^\sigma \partial_\sigma \partial_\mu f \\
&\quad - \boldsymbol{x}^2 \eta_{\mu\rho} \partial_\nu f - \boldsymbol{x}^2 x_\mu \partial_\rho \partial_\nu f + \boldsymbol{x}^2 \eta_{\nu\rho} \partial_\mu f + \boldsymbol{x}^2 x_\nu \partial_\rho \partial_\mu f \\
&\quad - 2x_\mu \eta_{\rho\nu} x^\sigma \partial_\sigma f - 2x_\mu x_\rho \partial_\nu f - 2x_\mu x_\rho x^\sigma \partial_\nu \partial_\sigma f + \boldsymbol{x}^2 x_\mu \partial_\nu \partial_\rho f \\
&\quad + 2x_\nu \eta_{\rho\mu} x^\sigma \partial_\sigma f + 2x_\nu x_\rho \partial_\mu f + 2x_\nu x_\rho x^\sigma \partial_\mu \partial_\sigma f - \boldsymbol{x}^2 x_\nu \partial_\mu \partial_\rho f \\
&= \eta_{\rho\mu}\left(2x_\nu x^\sigma \partial_\sigma - \boldsymbol{x}^2 \partial_\nu\right) f - \eta_{\rho\nu}\left(2x_\mu x^\sigma \partial_\sigma - \boldsymbol{x}^2 \partial_\mu\right) f \\
&= i\left(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu\right) f(\boldsymbol{x}),
\end{aligned} \tag{3.65}
$$

$$
\begin{aligned}
[P_\rho, L_{\mu\nu}]\, f(\boldsymbol{x}) &= \partial_\rho (x_\mu \partial_\nu - x_\nu \partial_\mu) f - (x_\mu \partial_\nu - x_\nu \partial_\mu) \partial_\rho f \\
&= \eta_{\mu\lambda} \delta^\lambda{}_\rho \partial_\nu f + x_\mu \partial_\rho \partial_\nu f - \eta_{\nu\lambda} \delta^\lambda{}_\rho \partial_\mu f + x_\nu \partial_\rho \partial_\mu f - x_\mu \partial_\nu \partial_\rho f + x_\nu \partial_\mu \partial_\rho f \\
&= \left(\eta_{\mu\rho} \partial_\nu - \eta_{\nu\rho} \partial_\mu\right) f = i\left(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu\right) f(\boldsymbol{x}),
\end{aligned} \tag{3.66}
$$

Finally let us evaluate the commutation relation for rotations among themselves. We note that the previous commutator $[P_\rho, L_{\mu\nu}]$ together with the following commutator $[L_{\mu\nu}, L_{\rho\sigma}]$ constitutes a well-known subalgebra of the conformal algebra, the Poincaré algebra. We can directly see that the generators, P_μ and $L_{\mu\nu}$ form a closed algebra.

$$\begin{aligned}
[L_{\mu\nu}, L_{\rho\sigma}]\, f(\boldsymbol{x}) &= -(x_\mu\partial_\nu - x_\nu\partial_\mu)(x_\rho\partial_\sigma - x_\sigma\partial_\rho)f + (x_\rho\partial_\sigma - x_\sigma\partial_\rho)(x_\mu\partial_\nu - x_\nu\partial_\mu)f \\
&= -x_\mu\eta_{\rho\lambda}\delta^\lambda{}_\nu\partial_\sigma f - x_\mu x_\rho\partial_\nu\partial_\sigma f + x_\mu\eta_{\sigma\lambda}\delta^\lambda{}_\nu\partial_\rho f + x_\mu x_\sigma\partial_\nu\partial_\rho f \\
&\quad + x_\nu\eta_{\rho\lambda}\delta^\lambda{}_\mu\partial_\sigma f + x_\nu x_\rho\partial_\mu\partial_\sigma f - x_\nu\eta_{\sigma\lambda}\delta^\lambda{}_\mu\partial_\rho f - x_\nu x_\sigma\partial_\mu\partial_\rho f \\
&\quad + x_\rho\eta_{\mu\lambda}\delta^\lambda{}_\sigma\partial_\nu f + x_\rho x_\mu\partial_\sigma\partial_\nu f - x_\rho\eta_{\nu\lambda}\delta^\lambda{}_\sigma\partial_\mu f - x_\rho x_\nu\partial_\sigma\partial_\mu f \\
&\quad - x_\sigma\eta_{\mu\lambda}\delta^\lambda{}_\rho\partial_\nu f - x_\sigma x_\mu\partial_\rho\partial_\nu f + x_\sigma\eta_{\nu\lambda}\delta^\lambda{}_\rho\partial_\mu f + x_\sigma x_\nu\partial_\rho\partial_\mu f \\
&= \eta_{\nu\rho}(x_\sigma\partial_\mu - x_\mu\partial_\sigma)f + \eta_{\nu\sigma}(x_\mu\partial_\rho - x_\rho\partial_\mu)f + \eta_{\mu\rho}(x_\nu\partial_\sigma - x_\sigma\partial_\nu)f \\
&\quad + \eta_{\mu\sigma}(x_\rho\partial_\nu - x_\nu\partial_\rho)f \\
&= i\left(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}\right) f(\boldsymbol{x}). \qquad (3.67)
\end{aligned}$$

The remaining commutators for these generators vanish. This is obvious for $[D, D]$ and $[P_\mu, P_\nu]$, thus let us investigate the possibilities that are left, namely, $[K_\mu, K_\nu]$ and $[D, L_{\mu\nu}]$.

$$\begin{aligned}
[K_\mu, K_\nu]\, f(\boldsymbol{x}) &= -(2x_\mu x^\rho\partial_\rho - \boldsymbol{x}^2\partial_\mu)(2x_\nu x^\sigma\partial_\sigma - \boldsymbol{x}^2\partial_\nu)f \\
&\quad + (2x_\nu x^\sigma\partial_\sigma - \boldsymbol{x}^2\partial_\nu)(2x_\mu x^\rho\partial_\rho - \boldsymbol{x}^2\partial_\mu)f \\
&= -4x_\mu x^\rho\eta_{\rho\nu}x^\sigma\partial_\sigma f - 4x_\mu x^\rho x_\nu\partial_\rho(x^\sigma\partial_\sigma f) + 4x_\mu x^\rho x_\rho\partial_\nu f \\
&\quad + 2\boldsymbol{x}^2\eta_{\mu\nu}x^\sigma\partial_\sigma f + 2\boldsymbol{x}^2 x_\nu\delta^\sigma{}_\mu\partial_\sigma f + 2\boldsymbol{x}^2 x_\nu x^\sigma\partial_\mu\partial_\sigma f - 2\boldsymbol{x}^2 x_\mu\partial_\nu f \\
&\quad + 4x_\nu x^\sigma\eta_{\sigma\mu}x^\rho\partial_\rho f + 4x_\nu x^\sigma x_\mu\partial_\sigma(x^\rho\partial_\rho f) - 4x_\nu x^\sigma x_\sigma\partial_\mu f \\
&\quad - 2\boldsymbol{x}^2\eta_{\nu\mu}x^\rho\partial_\rho f - 2\boldsymbol{x}^2 x_\mu\delta^\rho{}_\nu\partial_\rho f - 2\boldsymbol{x}^2 x_\mu x^\rho\partial_\nu\partial_\rho f + 2\boldsymbol{x}^2 x_\nu\partial_\mu f \\
&\quad + 2x_\mu x^\rho\boldsymbol{x}^2\partial_\rho\partial_\nu f - 2x_\nu x^\sigma\boldsymbol{x}^2\partial_\sigma\partial_\mu f - \boldsymbol{x}^4\partial_\mu\partial_\nu f + \boldsymbol{x}^4\partial_\nu\partial_\mu f \\
&= 0 \qquad (3.68)
\end{aligned}$$

and

$$\begin{aligned}
[D, L_{\mu\nu}]\, f(\boldsymbol{x}) &= x^\rho\partial_\rho(x_\mu\partial_\nu - x_\nu\partial_\mu)f - (x_\mu\partial_\nu - x_\nu\partial_\mu)x^\rho\partial_\rho f \\
&= x^\rho\eta_{\mu\lambda}\delta^\lambda{}_\rho\partial_\nu f + x^\rho x_\mu\partial_\rho\partial_\nu f - x^\rho\eta_{\nu\lambda}\delta^\lambda{}_\rho\partial_\mu f - x^\rho x_\nu\partial_\rho\partial_\mu f \\
&\quad - x_\mu\delta^\rho{}_\nu\partial_\rho f - x_\mu x^\rho\partial_\nu\partial_\rho f + x_\nu\delta^\rho{}_\mu\partial_\rho f + x_\nu x^\rho\partial_\mu\partial_\rho f \\
&= x_\mu\partial_\nu f + x_\mu x^\rho\partial_\rho\partial_\nu f - x_\nu\partial_\mu f - x_\nu x^\rho\partial_\rho\partial_\mu f \\
&\quad - x_\mu\partial_\nu f - x_\mu x^\rho\partial_\nu\partial_\rho f + x_\nu\partial_\mu f + x_\nu x^\rho\partial_\mu\partial_\rho f \\
&= 0. \qquad (3.69)
\end{aligned}$$

Collecting the results we found, the conformal algebra is given by,

$$\begin{aligned}
[D, P_\mu] &= iP_\mu, \\
[D, K_\mu] &= -iK_\mu, \\
[K_\mu, P_\nu] &= 2i\left(\eta_{\mu\nu}D - L_{\mu\nu}\right), \\
[K_\rho, L_{\mu\nu}] &= i\left(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu\right), \\
[P_\rho, L_{\mu\nu}] &= i\left(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu\right), \\
[L_{\mu\nu}, L_{\rho\sigma}] &= i\left(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}\right),
\end{aligned} \tag{3.70}$$

where the commutators which are not written down explicitly, vanish.

3.3 Noether's theorem

Let us return now to our Lagrangian, $\mathcal{L}(\Phi, \partial_\mu\Phi)$, and its corresponding action integral,

$$S = \int d^d x \, \mathcal{L}(\Phi, \partial_\mu\Phi). \tag{3.71}$$

Under the transformations (3.38), we have to carefully determine how the action changes. That means the integration measure and the derivative of Φ have also to be adjusted with respect to the coordinate transformation. Di Francesco, et al. [9] first plug in, $\Phi(\boldsymbol{x}) \to \Phi'(\boldsymbol{x})$, and then consider the change, $\boldsymbol{x} \to \boldsymbol{x}'$. That gives us,

$$S' = \int d^d x \, \mathcal{L}(\Phi'(\boldsymbol{x}), \partial_\mu\Phi'(\boldsymbol{x})). \tag{3.72}$$

If $\boldsymbol{x}$ is taken to $\boldsymbol{x}'$ we have,

$$S' = \int d^d x' \, \mathcal{L}(\Phi'(\boldsymbol{x}'), \partial'_\mu\Phi'(\boldsymbol{x}')), \tag{3.73}$$

where we defined $\partial'_\mu = \dfrac{\partial}{\partial x'^\mu}$. Now using $\Phi'(\boldsymbol{x}') = \mathcal{F}(\Phi(\boldsymbol{x}))$ we obtain,

$$S' = \int d^d x' \, \mathcal{L}(\mathcal{F}(\Phi(\boldsymbol{x})), \partial'_\mu\mathcal{F}(\Phi(\boldsymbol{x}))). \tag{3.74}$$

Finally by considering the Jacobian of the coordinate transformation we can write,

$$S' = \int d^d x \left|\frac{\partial \boldsymbol{x}'}{\partial \boldsymbol{x}}\right| \mathcal{L}(\mathcal{F}(\Phi(\boldsymbol{x})), (\partial x^\nu/\partial x'^\mu)\partial_\nu\mathcal{F}(\Phi(\boldsymbol{x}))). \tag{3.75}$$

Now we have a global symmetry of our action if under the coordinate transformation of the previous section,

$$x'^{\mu} = x^{\mu} + \omega_a \frac{\delta x^{\mu}}{\delta \omega_a} \tag{3.76}$$

$$\Phi'(\boldsymbol{x}') = \Phi(\boldsymbol{x}) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(\boldsymbol{x}), \tag{3.77}$$

the parameters ω_a are independent of the coordinates x^{μ} and the action does not change with such a rigid transformation [9, 14]. We will consider actions that are invariant under global symmetry transformations. But for the moment we take the parameters ω_a as slowly varying functions of $\boldsymbol{x}$, despite the definition of global parameters, more precisely we have, $|\omega_a| \ll 1$ and $l\,|\partial_{\mu}\omega_a| \ll |\omega_a|$, where l is considered as a characteristic scale of variation of $\Phi(\boldsymbol{x})$ [14]. To calculate the variation of the action, $\delta S = S' - S$, let us first express the relevant quantities of (3.72) explicitly in terms of our transformation parameters. We clearly have,

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \delta^{\mu}{}_{\nu} + \partial_{\nu}\left(\omega_a \frac{\delta x^{\mu}}{\delta \omega_a}\right). \tag{3.78}$$

To calculate the approximate form (up to first order) of the Jacobian of (3.78), we use the relation [15],

$$\det(\exp M) = \exp(\mathrm{Tr}\, M), \tag{3.79}$$

where M is any $n \times n$ complex matrix. Substituting $M = \ln(1 + E)$, where 1 is the identity matrix and E a small matrix (i.e. a matrix with $|E_{ij}| \ll 1$) into (3.79) and working to first order in E, we have,

$$\begin{aligned} \det(1 + E) &= \exp\left[\mathrm{Tr}\left(\ln(1 + E)\right)\right] \\ &= \exp\left(\mathrm{Tr}\, E + O(E^2)\right) \\ &= 1 + \mathrm{Tr}\, E + O(E^2) \approx 1 + \mathrm{Tr}\, E. \end{aligned} \tag{3.80}$$

Thus, using (3.80) we can write as a first order approximation for the Jacobian in (3.72),

$$\left|\frac{\partial \boldsymbol{x}'}{\partial \boldsymbol{x}}\right| = 1 + \delta^{\nu}{}_{\mu}\, \partial_{\nu}\left(\omega_a \frac{\delta x^{\mu}}{\delta \omega_a}\right) = 1 + \partial_{\mu}\left(\omega_a \frac{\delta x^{\mu}}{\delta \omega_a}\right). \tag{3.81}$$

Furthermore, from (3.78) we can also write the inverse matrix as,

$$\frac{\partial x^{\nu}}{\partial x'^{\mu}} = \delta^{\nu}{}_{\mu} - \partial_{\mu}\left(\omega_a \frac{\delta x^{\nu}}{\delta \omega_a}\right), \tag{3.82}$$

because we have to first order,

$$\begin{aligned}\frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\rho} &= \left(\delta^\mu{}_\nu + \partial_\nu\left(\omega_a\frac{\delta x^\mu}{\delta\omega_a}\right)\right)\left(\delta^\nu{}_\rho - \partial_\rho\left(\omega_a\frac{\delta x^\nu}{\delta\omega_a}\right)\right)\\ &= \delta^\mu{}_\rho - \partial_\rho\left(\omega_a\frac{\delta x^\mu}{\delta\omega_a}\right) + \partial_\rho\left(\omega_a\frac{\delta x^\mu}{\delta\omega_a}\right)\\ &= \delta^\mu{}_\rho. \end{aligned} \tag{3.83}$$

Plugging then equations (3.81) and (3.82) into (3.72), we have,

$$\begin{aligned} S' &= \int d^dx\,\left(1+\partial_\rho\left(\omega_a\frac{\delta x^\rho}{\delta\omega_a}\right)\right)\\ &\quad\times\mathcal{L}\left(\Phi+\omega_a\frac{\delta\mathcal{F}}{\delta\omega_a},\left[\delta^\nu{}_\mu-\partial_\mu\left(\omega_a\frac{\delta x^\nu}{\delta\omega_a}\right)\right]\left[\partial_\nu\Phi+\partial_\nu\left(\omega_a\frac{\delta\mathcal{F}}{\delta\omega_a}\right)\right]\right)\\ &= \int d^dx\,\left(1+\partial_\rho\left(\omega_a\frac{\delta x^\rho}{\delta\omega_a}\right)\right)\\ &\quad\times\mathcal{L}\left(\Phi+\omega_a\frac{\delta\mathcal{F}}{\delta\omega_a},\partial_\mu\Phi+\partial_\mu\left(\omega_a\frac{\delta\mathcal{F}}{\delta\omega_a}\right)-\partial_\mu\left(\omega_a\frac{\delta x^\nu}{\delta\omega_a}\right)\partial_\nu\Phi+O(\omega^2)\right)\\ &= \int d^dx\,\left(1+\partial_\rho\left(\omega_a\frac{\delta x^\rho}{\delta\omega_a}\right)\right)\\ &\quad\times\left[\mathcal{L}(\Phi,\partial_\mu\Phi)+\omega_a\frac{\delta\mathcal{F}}{\delta\omega_a}\frac{\partial\mathcal{L}}{\partial\Phi}+\left(\partial_\mu\left(\omega_a\frac{\delta\mathcal{F}}{\delta\omega_a}\right)-\partial_\mu\left(\omega_a\frac{\delta x^\nu}{\delta\omega_a}\right)\partial_\nu\Phi\right)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}+O(\omega^2)\right]\\ &= \int d^dx\,\left[\mathcal{L}(\Phi,\partial_\mu\Phi)+\omega_a\frac{\delta\mathcal{F}}{\delta\omega_a}\frac{\partial\mathcal{L}}{\partial\Phi}+\left(\omega_a\partial_\mu\left(\frac{\delta\mathcal{F}}{\delta\omega_a}\right)-\omega_a\partial_\mu\left(\frac{\delta x^\nu}{\delta\omega_a}\right)\partial_\nu\Phi\right)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\right.\\ &\quad\left.+\left(\partial_\mu\omega_a\frac{\delta\mathcal{F}}{\delta\omega_a}-\partial_\mu\omega_a\frac{\delta x^\nu}{\delta\omega_a}\partial_\nu\Phi\right)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}+\omega_a\partial_\mu\left(\frac{\delta x^\mu}{\delta\omega_a}\right)\mathcal{L}\partial_\mu\omega_a\frac{\delta x^\mu}{\delta\omega_a}\mathcal{L}+O(\omega^2)\right]. \end{aligned} \tag{3.84}$$

If we arrange the terms with common factors, ω_a and $\partial_\mu\omega_a$, and take the difference, $\delta S = S' - S$, we get, borrowing the notation from Maggiore [14],

$$\delta S = \int d^dx\,\left(\omega_a K_a(\Phi) - j^\mu_a\partial_\mu\omega_a + O(\omega^2)\right), \tag{3.85}$$

where we have defined $K_a(\Phi)$ as the overall factor of ω_a after arrangement, whose explicit form is immaterial for us, as we will see shortly, and for j^μ_a, we have defined,

$$j^\mu_a = \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_\nu\Phi - \delta^\mu{}_\nu\,\mathcal{L}\right]\frac{\delta x^\nu}{\delta\omega_a} - \frac{\delta\mathcal{F}}{\delta\omega_a}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}. \tag{3.86}$$

j^μ_a is called the canonical current of the continuous symmetry transformation we considered [9]. Now let us consider the factor $K_a(\Phi)$. We took ω_a as slowly varying parameters, but for our global symmetry transformation, the ω_a are actually independent from the coordinates, as mentioned earlier. Therefore, if the global symmetry condition holds, our variation of the action should vanish (i.e. $\delta S = 0$) with the parameters ω_a being constants. As all $\partial_\mu \omega_a$ now vanish, the only coefficient left in the variation of the action is $K_a(\Phi)$, which means we have,

$$0 = \int d^d x\ \omega_a K_a(\Phi), \tag{3.87}$$

for arbitrary Φ and arbitrary ω_a. Thus, we must conclude that, regardless of the expression we calculated before, $K_a(\Phi)$ identically vanishes for all Φ. Even if we consider now slowly varying ω_a again, we can say that, because $K_a(\Phi)$ is already independent of ω_a, it still vanishes [14]. We finally have then,

$$\delta S = -\int d^d x\ j^\mu_a \partial_\mu \omega_a. \tag{3.88}$$

Integrating (3.88) by parts, using Stokes' theorem and keeping in mind that our fields behave well enough at infinity (which means they vanish fast enough) such that the current also vanishes at infinity fast enough (for example if we have Euclidean spacetime with $r \equiv \|\boldsymbol{x}\|$, then j^μ_a must vanish faster than $O(1/r^{d-1})$ as $r \to \infty$),

$$\begin{aligned} \delta S &= -\int_\Omega d^d x\ \partial_\mu \left(j^\mu_a \omega_a \right) + \int d^d x\ \omega_a \partial_\mu j^\mu_a \\ &= -\int_{\partial\Omega} d\Sigma\ n_\mu \left(j^\mu_a \omega_a \right) + \int d^d x\ \omega_a \partial_\mu j^\mu_a \\ &= \int d^d x\ \omega_a \partial_\mu j^\mu_a, \end{aligned} \tag{3.89}$$

here Ω is the spacetime (which we actually take Euclidean for simplicity) we integrate on, $\partial\Omega$ is its boundary at infinity, $d\Sigma$ the hypersurface element and n_μ is the unit normal to the hypersurface we integrate on. Here in using Stokes' theorem in its divergence form we assumed that our hypersurface $\partial\Omega$ is not null or, equivalently, that the induced metric on $\partial\Omega$ is nondegenerate [12].

Up to this point we did not impose any constraint on the fields, Φ, they were arbitrary (besides behaving well enough at infinity). But if we require the fields to satisfy the classical equations of motion, then we can say that the variation of our action, δS, should vanish, no matter how we choose our parameters, ω_a [9]. Hence, we have shown that for classical fields we can write,

$$\partial_\mu j^\mu_a = 0, \tag{3.90}$$

which indicates nothing else than the conservation of the current(s), j^μ_a. This result is called Noether's theorem, after the mathematician Emmy Noether. We can also find the charge corresponding to the conserved current, which with a sufficiently fast vanishing current is constant in time. Consequently, using the definition for the charge corresponding to j_a^μ, we obtain,

$$Q_a = \int d^{d-1}x\ j_a^0, \tag{3.91}$$

where j_a^0 is taken as the time component of our current and $d^{d-1}x$ is the spatial integration measure, the following expression for the time derivative of the charge,

$$\dot{Q}_a = \int_V d^{d-1}x\ \partial_0 j_a^0, \tag{3.92}$$

where we denoted space by V. Recalling that the divergence vanishes, i.e. $\partial_\mu j_a^\mu = 0$ and labeling the spatial indices by latin indices, i, one can write,

$$\begin{aligned} \dot{Q}_a &= -\int_V d^{d-1}x\ \partial_i j_a^i \\ &= -\int_{\partial V} dS\, n_i' j_a^i, \end{aligned} \tag{3.93}$$

where we again used Stokes' theorem, with ∂V, the boundary at spatial infinity, dS, the hypersurface element and, n_i', the unit normal to the spatial hypersurface element. If we assume good enough behavior of the current at spatial infinity, as before, we clearly have, $\dot{Q}_a = 0$. We can actually make a gauge transformation that does not change the conservation status of the current. Using the symmetry property of partial differentiation, we write,

$$j_a'^\mu = j_a^\mu + \partial_\nu B_a^{\nu\mu}, \tag{3.94}$$

where we take $B_a^{\nu\mu} = -B_a^{\mu\nu}$. We clearly have, $\partial_\mu j_a'^\mu = \partial_\mu j_a^\mu + \partial_\mu \partial_\nu B_a^{\nu\mu} = 0$, which is also a conserved current.

3.4 Transformation of the Correlation Functions

Now let us consider the effects of a continuous symmetry transformation, such as (3.38), on a quantum field theory. The previous discussion on conserved currents, as we have noted, does mainly apply to classical field theories. But we will see that continuous symmetry transformations give constraints for the correlation functions [9]. We take again the action of our theory to be invariant under the coordinate transformation we consider. In writing the correlation functions for our fields we will consider the Euclidean path integral. For the correlation function we have therefore,

$$\langle \Phi(\boldsymbol{x}_1) \cdots \Phi(\boldsymbol{x}_n) \rangle = \frac{1}{Z} \int \mathcal{D}\Phi\ \Phi(\boldsymbol{x}_1) \cdots \Phi(\boldsymbol{x}_n) \exp\left(-S[\Phi]\right), \tag{3.95}$$

where we have, $Z = \int \mathcal{D}\Phi \ \exp\left(-S\left[\Phi\right]\right)$, which is called the vacuum functional. We can now write,

$$\langle \Phi(\boldsymbol{x}_1') \cdots \Phi(\boldsymbol{x}_n') \rangle = \frac{1}{Z} \int \mathcal{D}\Phi \ \Phi(\boldsymbol{x}_1') \cdots \Phi(\boldsymbol{x}_n') \exp\left(-S\left[\Phi\right]\right), \tag{3.96}$$

where we took only the coordinate transformation and did not change our fields, Φ, for the moment. Now as path integration is performed over all possible values of the fields, Φ, we can relabel the fields, while leaving the path integral invariant,

$$\langle \Phi(\boldsymbol{x}_1') \cdots \Phi(\boldsymbol{x}_n') \rangle = \frac{1}{Z} \int \mathcal{D}\Phi' \ \Phi'(\boldsymbol{x}_1') \cdots \Phi'(\boldsymbol{x}_n') \exp\left(-S\left[\Phi'\right]\right). \tag{3.97}$$

Furthermore we note that the path integral measure is invariant, i.e. $\mathcal{D}\Phi' = \mathcal{D}\Phi$ [9], and we also assumed that our action is invariant under the transformation, i.e. $S\left[\Phi'\right] = S\left[\Phi\right]$. We therefore have, using also $\Phi'(\boldsymbol{x}') = \mathcal{F}(\Phi(\boldsymbol{x}))$, the following,

$$\begin{aligned} \langle \Phi(\boldsymbol{x}_1') \cdots \Phi(\boldsymbol{x}_n') \rangle &= \frac{1}{Z} \int \mathcal{D}\Phi \ \mathcal{F}(\Phi(\boldsymbol{x}_1)) \cdots \mathcal{F}(\Phi(\boldsymbol{x}_n)) \exp\left(-S\left[\Phi\right]\right) \\ &= \langle \mathcal{F}(\Phi(\boldsymbol{x}_1)) \cdots \mathcal{F}(\Phi(\boldsymbol{x}_n)) \rangle. \end{aligned} \tag{3.98}$$

Now we should as in Noether's theorem try to use the invariance of the action to arrive at further properties using (3.72), but according to Di Francesco, et al. this is not so straightforward as in the classical case. As specific examples for the relation (3.98), let us consider translational invariance, Lorentz invariance and scale invariance in our action. For translations we have, $x'^{\mu} = x^{\mu} + a^{\mu}$ and also we take, $\mathcal{F}(\Phi'(\boldsymbol{x}')) = \Phi(\boldsymbol{x})$, such that we have,

$$\langle \Phi(\boldsymbol{x}_1 + \boldsymbol{a}) \cdots \Phi(\boldsymbol{x}_n + \boldsymbol{a}) \rangle = \langle \Phi(\boldsymbol{x}_1) \cdots \Phi(\boldsymbol{x}_n) \rangle, \tag{3.99}$$

which means that the correlation function is also translationally invariant. For Lorentz invariance of the action let us consider scalar fields, i.e. $\mathcal{F}(\Phi'(\boldsymbol{x}')) = \Phi(\boldsymbol{x})$, then we obtain,

$$\langle \Phi(\Lambda\boldsymbol{x}_1) \cdots \Phi(\Lambda\boldsymbol{x}_n) \rangle = \langle \Phi(\boldsymbol{x}_1) \cdots \Phi(\boldsymbol{x}_n) \rangle, \tag{3.100}$$

where Λ is not to be confused with $\Lambda(\boldsymbol{x})$, the local scale factor of conformal transformations. Finally, if we investigate scale invariant actions for fields, ϕ_i with scaling dimensions Δ_i (for definition see chapter 2), we have [9],

$$\langle \phi(\lambda\boldsymbol{x}_1) \cdots \phi(\lambda\boldsymbol{x}_n) \rangle = \lambda^{-\Delta_1} \cdots \lambda^{-\Delta_n} \langle \phi(\boldsymbol{x}_1) \cdots \phi(\boldsymbol{x}_n) \rangle. \tag{3.101}$$

Now we are ready to describe the Ward identities.

3.5 Ward Identities

Let us consider again the change of the field in terms of the generator of the symmetry transformation,

$$\Phi'(\boldsymbol{x}) = \Phi(\boldsymbol{x}) - i\omega_a G_a \Phi(\boldsymbol{x}). \tag{3.102}$$

We take ω_a as functions of the coordinates, $\boldsymbol{x}$, and consider the above change in the fields in (3.95). Although we are now not considering classical fields we still can use (3.89), but we see that the variation of the action cannot vanish in general and our action is not invariant under the transformation. Following again closely the notation of [9], we denote collectively the product, $\Phi(\boldsymbol{x}_1)\cdots\Phi(\boldsymbol{x}_n)$, by X and its variation by $\delta_\omega X$. We note again that we assume that the path integral measure does not change under the transformations of the fields ($\mathcal{D}\Phi' = \mathcal{D}\Phi$). With the chosen notation we have,

$$\begin{aligned}\langle X\rangle &= \langle \Phi(\boldsymbol{x}_1)\cdots\Phi(\boldsymbol{x}_n)\rangle \\ &= \frac{1}{Z}\int \mathcal{D}\Phi\ \Phi(\boldsymbol{x}_1)\cdots\Phi(\boldsymbol{x}_n)\exp\left(-S\left[\Phi\right]\right).\end{aligned} \tag{3.103}$$

Now we relabel the field Φ to Φ' and note that,

$$\begin{aligned}X + \delta X &= \Phi(\boldsymbol{x}_1)\cdots\Phi(\boldsymbol{x}_n) + \delta(\Phi(\boldsymbol{x}_1)\cdots\Phi(\boldsymbol{x}_n)) \\ &= \Phi'(\boldsymbol{x}_1)\cdots\Phi'(\boldsymbol{x}_n).\end{aligned} \tag{3.104}$$

We also take advantage of expression (3.89) to describe the variation in our action,

$$\delta S = \int d^d x\ \omega_a(\boldsymbol{x})\partial_\mu j_a^\mu, \tag{3.105}$$

such that we have the following,

$$\begin{aligned}\langle X\rangle &= \frac{1}{Z}\int \mathcal{D}\Phi'\ \Phi'(\boldsymbol{x}_1)\cdots\Phi'(\boldsymbol{x}_n)\exp\left(-S\left[\Phi'\right]\right) \\ &= \frac{1}{Z}\int \mathcal{D}\Phi\ \ (X+\delta X)\exp\left(-S\left[\Phi\right] - \int d^d x\ \omega_a(\boldsymbol{x})\partial_\mu j_a^\mu\right),\end{aligned} \tag{3.106}$$

where we also had to consider, $S\left[\Phi'\right] = S\left[\Phi\right] + \delta S\left[\Phi\right]$. If the above expression is expanded to first order in ω_a we get,

$$\langle X\rangle = \frac{1}{Z}\int \mathcal{D}\Phi\ \ (X+\delta X)\exp\left(-S\left[\Phi\right]\right)\left(1 - \int d^d x\ \omega_a(\boldsymbol{x})\partial_\mu j_a^\mu + O(\omega^2)\right), \tag{3.107}$$

and keeping in mind that δX is, of course, linear in the term $\omega_a(\boldsymbol{x})$, we obtain,

$$\begin{aligned}\langle X\rangle = \frac{1}{Z}\int &\mathcal{D}\Phi(X\exp\left(-S\left[\Phi\right]\right) + \delta X\exp\left(-S\left[\Phi\right]\right) \\ &- X\int d^d x\ \omega_a(\boldsymbol{x})\partial_\mu j_a^\mu \exp\left(-S\left[\Phi\right]\right) + O(\omega^2)),\end{aligned} \tag{3.108}$$

or writing only the first order terms, we clearly see that,

$$\langle X\rangle = \langle X\rangle + \langle \delta X\rangle - \int d^d x\ \omega_a(\boldsymbol{x})\langle X\partial_\mu j_a^\mu\rangle, \tag{3.109}$$

or equivalently we have,

$$\langle \delta X\rangle = \int d^d x\ \omega_a(\boldsymbol{x})\langle X\partial_\mu j_a^\mu\rangle. \tag{3.110}$$

Finally let us write out δX in terms of first order variations of the fields.

$$\begin{aligned}\Phi'(\boldsymbol{x}_1)\cdots\Phi'(\boldsymbol{x}_n) &= (\Phi(\boldsymbol{x}_1)+\delta_\omega\Phi(\boldsymbol{x}_1))\cdots(\Phi(\boldsymbol{x}_n)+\delta_\omega\Phi(\boldsymbol{x}_n))\\ &= \Phi(\boldsymbol{x}_1)\cdots\Phi(\boldsymbol{x}_n)+\delta_\omega\Phi(\boldsymbol{x}_1)\Phi(\boldsymbol{x}_2)\cdots\Phi(\boldsymbol{x}_n)\\ &\quad +\cdots+\Phi(\boldsymbol{x}_1)\cdots\delta_\omega\Phi(\boldsymbol{x}_k)\cdots\Phi(\boldsymbol{x}_n)+O(\omega^2)\\ &= \Phi(\boldsymbol{x}_1)\cdots\Phi(\boldsymbol{x}_n) - i\sum_{k=1}^{n}\Phi(\boldsymbol{x}_1)\cdots G_a\Phi(\boldsymbol{x}_k)\cdots\Phi(\boldsymbol{x}_n)\omega_a(\boldsymbol{x}_k),\end{aligned} \tag{3.111}$$

where we used the definition of the generator,

$$-i\omega_a(\boldsymbol{x}_k)G_a\Phi(\boldsymbol{x}_k) \equiv \delta_\omega\Phi(\boldsymbol{x}_k) \equiv \Phi'(\boldsymbol{x}_k)-\Phi(\boldsymbol{x}_k). \tag{3.112}$$

Thus, δX is given by,

$$\delta X = -i\sum_{k=1}^{n}\Phi(\boldsymbol{x}_1)\cdots G_a\Phi(\boldsymbol{x}_k)\cdots\Phi(\boldsymbol{x}_n)\omega_a(\boldsymbol{x}_k). \tag{3.113}$$

We can also express the above result in terms of Dirac delta functions as,

$$\delta X = -i\int d^d x\ \omega_a(\boldsymbol{x})\sum_{k=1}^{n}\Phi(\boldsymbol{x}_1)\cdots G_a\Phi(\boldsymbol{x}_k)\cdots\Phi(\boldsymbol{x}_n)\delta(\boldsymbol{x}-\boldsymbol{x}_k). \tag{3.114}$$

This relation and equation (3.110) are valid for any arbitrary infinitesimal functions, $\omega_a(\boldsymbol{x})$. Therefore, comparing these two expressions we can see that we finally have,

$$\frac{\partial}{\partial x^\mu}\langle j_a^\mu(\boldsymbol{x})\Phi(\boldsymbol{x}_1)\cdots\Phi(\boldsymbol{x}_n)\rangle = -i\sum_{k=1}^{n}\delta(\boldsymbol{x}-\boldsymbol{x}_k)\langle\Phi(\boldsymbol{x}_1)\cdots G_a\Phi(\boldsymbol{x}_k)\cdots\Phi(\boldsymbol{x}_n)\rangle. \tag{3.115}$$

This relation is known as the Ward identity for the current $j_a^\mu(\boldsymbol{x})$.

3.6 Conformal Invariance in Classical Field Theory

In the previous parts we made the assumption that the fields were unaffected by the conformal transformation and therefore we derived solely the conformal algebra for the spacetime symmetry of the physical systems being considered. Now we will also take into account transformations of our fields first in classical systems then in the next section in quantum systems.

We stress that conformal symmetry in classical field theories does not necessarily imply that conformal symmetry also holds in the corresponding quantum field theories [9].

Sticking to [9], we consider an infinitesimal conformal transformation, ω_g, such that,

$$\Phi'(\boldsymbol{x}') = (1 - i\omega_g T_g)\Phi(\boldsymbol{x}), \tag{3.116}$$

here T_g is a generator (a matrix representation) that complements the spacetime conformal symmetry we described by generators we found before (in equation (3.46) we see that in the general case the term, $\dfrac{\delta\mathcal{F}}{\omega_g}$ will not vanish). To determine the constraints we have on the generator, T_g, we consider a method that is used for the Poincaré algebra. Citing from [9], we take the Lorentz subgroup of the Poincaré group which leaves the point, $x = 0$ invariant. After that we define the spin operator, $S_{\mu\nu}$, as the matrix representation which satisfies,

$$L_{\mu\nu}\Phi(0) = S_{\mu\nu}\Phi(0). \tag{3.117}$$

Now using, $[P_\rho, L_{\mu\nu}] = i\left(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu\right)$, and the Baker-Campbell-Hausdorff formula, given by,

$$\mathrm{e}^{-A}B\mathrm{e}^{A} = B + [B, A] + \frac{1}{2!}[[B, A], A] + \frac{1}{3!}[[[B, A], A], A] + \cdots, \tag{3.118}$$

we see that,

$$\mathrm{e}^{ix^\rho P_\rho}L_{\mu\nu}\mathrm{e}^{-ix^\rho P_\rho} = S_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu. \tag{3.119}$$

Using these, we then can write the most general form of the action of the generators on our fields,

$$P_\mu\Phi(\boldsymbol{x}) = -i\partial_\mu\Phi(\boldsymbol{x}) \tag{3.120}$$

$$L_{\mu\nu}\Phi(\boldsymbol{x}) = i(x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi(\boldsymbol{x}) + S_{\mu\nu}\Phi(\boldsymbol{x}) \tag{3.121}$$

Analogously we apply the same method to the conformal algebra. We can see that the point $x = 0$ is held fixed when we consider rotations (or generally Lorentz transformation in Minkowski spacetimes), dilations and, of course, the special conformal transformations. Taking as in the previous example matrix representations corresponding to the full generators of the conformal algebra, as $S_{\mu\nu}$, $\tilde{\Delta}$ and κ_μ associated with $L_{\mu\nu}$, D and K_μ respectively, and acting on $\Phi(0)$, we have the following algebra,

$$
\begin{aligned}
[\tilde{\Delta}, S_{\mu\nu}] &= 0 \\
[\tilde{\Delta}, \kappa_\mu] &= -i\kappa_\mu \\
[\kappa_\mu, \kappa_\nu] &= 0 \\
[\kappa_\rho, S_{\mu\nu}] &= i\left(\eta_{\rho\mu}\kappa_\nu - \eta_{\rho\nu}\kappa_\mu\right) \\
[S_{\mu\nu}, S_{\rho\sigma}] &= i\left(\eta_{\nu\rho}S_{\mu\sigma} + \eta_{\mu\sigma}S_{\nu\rho} - \eta_{\mu\rho}S_{\nu\sigma} - \eta_{\nu\sigma}S_{\mu\rho}\right)
\end{aligned}
\tag{3.122}
$$

Again using (3.118), and the relations (3.70), we get,

$$
\mathrm{e}^{ix^\rho P_\rho} D \mathrm{e}^{-ix^\rho P_\rho} = D + x^\nu P_\nu \tag{3.123}
$$

$$
\mathrm{e}^{ix^\rho P_\rho} K_\mu \mathrm{e}^{-ix^\rho P_\rho} = K_\mu + 2x_\mu D - 2x^\nu L_{\mu\nu} + 2x_\mu(x^\nu P_\nu) - \boldsymbol{x}^2 P_\mu, \tag{3.124}
$$

and using this we obtain,

$$
D\Phi(\boldsymbol{x}) = (-ix^\nu\partial_\nu + \tilde{\Delta})\Phi(\boldsymbol{x}) \tag{3.125}
$$

$$
K_\mu\Phi(\boldsymbol{x}) = \left[\kappa_\mu + 2x_\mu\tilde{\Delta} - x^\nu S_{\mu\nu} - 2ix_\mu x^\nu\partial_\nu + i\boldsymbol{x}^2\partial_\mu\right]\Phi(\boldsymbol{x}) \tag{3.126}
$$

If we assume that the $S_{\mu\nu}$ are finite dimensional irreducible representations, we can use $[\tilde{\Delta}, S_{\mu\nu}] = 0$ and Schur's lemma [16] to conclude that, because $\tilde{\Delta}$ commutes with $S_{\mu\nu}$ for all spacetime points, it must be a multiple of the identity matrix acting on the fields $\Phi(\boldsymbol{x})$. With $\tilde{\Delta}$ proportional to the identity, (3.122) forces all the κ_μ to vanish, i.e.

$$
[\tilde{\Delta}, \kappa_\mu] = -i\kappa_\mu \quad \Rightarrow \quad 0 = -i\kappa_\mu. \tag{3.127}
$$

Thus we take $\tilde{\Delta}$ as a number equal to $-i\Delta$, where Δ is the so-called scaling dimension. To see what Δ exactly is, take a scale transformation of the coordinates, $\boldsymbol{x} \to \boldsymbol{x}' = \lambda\boldsymbol{x}$, then the scale transformation of our field, $\Phi(\boldsymbol{x})$, is defined as, $\Phi(\boldsymbol{x}) \to \Phi'(\lambda\boldsymbol{x}') = \lambda^{-\Delta}\Phi(\boldsymbol{x})$ [9].

For a spinless field $\phi(\boldsymbol{x})$, for which $S_{\mu\nu} = 0$, we have the following transformation,

$$
\phi(\boldsymbol{x}) \to \phi'(\boldsymbol{x}') = \left|\frac{\partial\boldsymbol{x}'}{\partial\boldsymbol{x}}\right|^{-\Delta/d}\phi(\boldsymbol{x}), \tag{3.128}
$$

where the Jacobian, $\left|\frac{\partial\boldsymbol{x}'}{\partial\boldsymbol{x}}\right|$, is related to $\Lambda(\boldsymbol{x})$ of (3.1) by,

$$
\left|\frac{\partial\boldsymbol{x}'}{\partial\boldsymbol{x}}\right| = \Lambda(\boldsymbol{x})^{-d/2}. \tag{3.129}
$$

If a field transforms in this way it is called a quasi-primary field.

CHAPTER 4

CONFORMAL INVARIANCE IN TWO DIMENSIONS

In the previous chapter we restricted ourselves to conformal invariance in $d > 2$ dimensions. There we found constraints that conformal transformations should satisfy and evaluated the corresponding generators for the conformal group. For the global conformal symmetry we used everywhere well-defined functions, but now in two dimensions we will also consider local transformations which may not be well-defined on the whole plane [9].

4.1 The Conformal Group in Two Dimensions

If we take the plane with coordinates, (z^0, z^1), and consider the transformation of the inverse metric tensor under the coordinate change, $z^\mu \to w^\mu(z)$,

$$g'^{\mu\nu}(w) = \frac{\partial w^\mu}{\partial z^\alpha} \frac{\partial w^\nu}{\partial z^\beta} g^{\alpha\beta}(z). \tag{4.1}$$

Together with the definition of conformal transformation (3.1) and the Euclidean flat metric, $g_{\mu\nu}(z) = \delta_{\mu\nu}$, we have,

$$g'_{\mu\nu}(w) = \Lambda(z)\, \delta_{\mu\nu}. \tag{4.2}$$

Thus we get,

$$\left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2 = \Lambda(z), \tag{4.3}$$

$$\left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2 = \Lambda(z), \tag{4.4}$$

or correspondingly,

$$\left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2 = \left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2. \tag{4.5}$$

In addition we have,

$$\frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1} = 0. \tag{4.6}$$

First multiply both sides of (4.5) by $\frac{\partial w^1}{\partial z^0}\frac{\partial w^0}{\partial z^1}$ to obtain,

$$\left(\frac{\partial w^0}{\partial z^1}\right)^3 \frac{\partial w^1}{\partial z^0} + \left(\frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0}\right)\frac{\partial w^0}{\partial z^0}\frac{\partial w^0}{\partial z^1} = \left(\frac{\partial w^1}{\partial z^0}\right)^3 \frac{\partial w^0}{\partial z^1} + \left(\frac{\partial w^1}{\partial z^1}\frac{\partial w^0}{\partial z^1}\right)\frac{\partial w^1}{\partial z^0}\frac{\partial w^1}{\partial z^1}. \tag{4.7}$$

Using then equation (4.6) to write, $\frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0} = -\frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1}$, and substitute it in the above equation, we have,

$$\left(\frac{\partial w^0}{\partial z^1}\right)^2 \left(-\frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^1} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^0}\right) = \left(\frac{\partial w^1}{\partial z^0}\right)^2 \left(\frac{\partial w^1}{\partial z^0}\frac{\partial w^0}{\partial z^1} - \frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^1}\right), \tag{4.8}$$

or

$$\left[\left(\frac{\partial w^0}{\partial z^1}\right)^2 - \left(\frac{\partial w^1}{\partial z^0}\right)^2\right]\left(\frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^0} - \frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^1}\right) = 0. \tag{4.9}$$

To satisfy this equation we have three possibilities: either $\frac{\partial w^1}{\partial z^0} = -\frac{\partial w^0}{\partial z^1}$ and/or $\frac{\partial w^1}{\partial z^0} = \frac{\partial w^0}{\partial z^1}$ and/or $\frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^1} = \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^0}$. We note at once that the first two possibilities are mutually exclusive unless both expressions are equal to zero. Thus let us consider four cases.

Case 1: $\frac{\partial w^1}{\partial z^0} = -\frac{\partial w^0}{\partial z^1} \neq 0$

With the given condition and (4.6), we get,

$$\frac{\partial w^1}{\partial z^0}\left(\frac{\partial w^0}{\partial z^0} - \frac{\partial w^1}{\partial z^1}\right) = 0. \tag{4.10}$$

As $\frac{\partial w^1}{\partial z^0}$ is assumed nonzero, we immediately obtain,

$$\frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1}. \tag{4.11}$$

Furthermore we can say that, because of these conditions, we have,

$$\left(\frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^0} - \frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^1}\right) = -\left(\frac{\partial w^0}{\partial z^0}\right)^2 - \left(\frac{\partial w^0}{\partial z^1}\right)^2. \tag{4.12}$$

This expression cannot be equal to zero, otherwise we see that with the help of the constraint (4.5) all derivatives $\frac{\partial w^\mu}{\partial z^\nu}$ must vanish identically which contradicts the condition we assumed. This exhausts all possible subcases for this particular case.

Case 2: $\dfrac{\partial w^1}{\partial z^0} = \dfrac{\partial w^0}{\partial z^1} \neq 0$

Using (4.6) and this condition, we have,

$$\frac{\partial w^1}{\partial z^0}\left(\frac{\partial w^0}{\partial z^0} + \frac{\partial w^1}{\partial z^1}\right) = 0. \tag{4.13}$$

Again, because $\dfrac{\partial w^1}{\partial z^0}$ is nonzero, we get,

$$\frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1}. \tag{4.14}$$

These conditions, as in the previous case, give rise to,

$$\left(\frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^0} - \frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^1}\right) = \left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2. \tag{4.15}$$

We argue once more that this expression should not be equal to zero or we would get trivial results for the derivatives, $\dfrac{\partial w^\mu}{\partial z^\nu}$, which, in turn, would conflict with our starting assumption.

Case 3: $\dfrac{\partial w^1}{\partial z^0} = \dfrac{\partial w^0}{\partial z^1} = 0$

Now we can consider the case we excluded before. Using (4.5) we have,

$$\left(\frac{\partial w^0}{\partial z^0}\right)^2 = \left(\frac{\partial w^1}{\partial z^1}\right)^2. \tag{4.16}$$

This equation leaves us with two choices, namely, either,

$$\frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1} \tag{4.17}$$

or

$$\frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1}. \tag{4.18}$$

This time, the trivial solution is allowed. But note that in such a case the determinant of $\dfrac{\partial w^\mu}{\partial z^\nu}$ vanishes at the point of the trivial solution which means our coordinate transformation will no longer be invertible around that point. We see that this trivial solution agrees with the expressions derived in the first two cases, if we allow the derivatives to vanish. In other words, we have a consistent set of equations up to this point.

Case 4: $\dfrac{\partial w^0}{\partial z^1}\dfrac{\partial w^1}{\partial z^0} = \dfrac{\partial w^0}{\partial z^0}\dfrac{\partial w^1}{\partial z^1}$

For this case we use the equation, $\frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^0} - \frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^1} = 0$, which is our constraint, and equation (4.6). First multiply our constraint on both sides by $\frac{\partial w^0}{\partial z^1}$, and then (4.6) on both sides by $\frac{\partial w^0}{\partial z^0}$, such that we have,

$$\left(\frac{\partial w^0}{\partial z^1}\right)^2 \frac{\partial w^1}{\partial z^0} - \frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^1}\frac{\partial w^0}{\partial z^1} = 0 \tag{4.19}$$

$$\left(\frac{\partial w^0}{\partial z^0}\right)^2 \frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^0}\frac{\partial w^1}{\partial z^1} = 0. \tag{4.20}$$

Adding these two expressions one gets,

$$\frac{\partial w^1}{\partial z^0}\left[\left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2\right] = 0. \tag{4.21}$$

Now using again the same equations, but this time multiplying the constraint on both sides by $\frac{\partial w^1}{\partial z^0}$ and equation (4.6) by $\frac{\partial w^1}{\partial z^1}$, we see that,

$$\frac{\partial w^0}{\partial z^1}\left(\frac{\partial w^1}{\partial z^0}\right)^2 - \frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^1}\frac{\partial w^1}{\partial z^0} = 0 \tag{4.22}$$

$$\frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^1}\frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1}\left(\frac{\partial w^1}{\partial z^1}\right)^2 = 0. \tag{4.23}$$

Thus if we add the above equations,

$$\frac{\partial w^0}{\partial z^1}\left[\left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2\right] = 0. \tag{4.24}$$

The first subcase is when either, $\left[\left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2\right]$ or $\left[\left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2\right]$ is equal to zero. Because of (4.5) we directly see that if one of them vanishes, the other one vanishes too. Thus this subcase corresponds again to the trivial solution.

Another subcase will be when either $\frac{\partial w^1}{\partial z^0}$ or $\frac{\partial w^0}{\partial z^1}$ vanishes. Then because of our constraint we will have,

$$\frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^1} = 0. \tag{4.25}$$

That means that at least one of these factors must vanish, too. Now if $\frac{\partial w^1}{\partial z^0}$ and $\frac{\partial w^1}{\partial z^1}$ are zero, using (4.5) eventually makes the other factors also zero. If $\frac{\partial w^1}{\partial z^0}$ and $\frac{\partial w^0}{\partial z^0}$ vanish, we see that (4.6) leads us to the conclusion that,

$$\frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1} = 0, \tag{4.26}$$

which means again that at least one of the factors must be equal to zero. Thus (4.5) enables us to solve for the last unknown factor, which is obviously equal to zero. Finally if $\frac{\partial w^0}{\partial z^1}$ and $\frac{\partial w^1}{\partial z^1}$ vanish we recover the same situation as the previous one. And if $\frac{\partial w^0}{\partial z^1}$ and $\frac{\partial w^0}{\partial z^0}$ are zero, we see with the help of (4.5) that the remaining factors vanish, too. Thus for this case all solutions turn out to be trivial. The last two cases either trivially satisfy the equations in *Case 1* and *Case 2* or are special cases of them, thus no inconsistency occurs when we consider that the equations derived in the first two cases provide us with the most general solution space (including also the trivial solutions).

We covered all cases and we can see that by the equations of *Case 1* we recover the Cauchy-Riemann equations [17, 18] for holomorphic functions,

$$\frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1} \qquad \text{and} \qquad \frac{\partial w^1}{\partial z^0} = -\frac{\partial w^0}{\partial z^1} \tag{4.27}$$

and by the equations of *Case 2* the Cauchy-Riemann equations for antiholomorphic functions,

$$\frac{\partial w^1}{\partial z^0} = \frac{\partial w^0}{\partial z^1} \qquad \text{and} \qquad \frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1}. \tag{4.28}$$

Now using this resemblance to the Cauchy-Riemann conditions, we can define the complex variables, z and $\bar{z}$, in the following way [9],

$$\begin{aligned} z &\equiv z^0 + iz^1 & z^0 &= \frac{1}{2}(z + \bar{z}) \\ \bar{z} &\equiv z^0 - iz^1 & z^1 &= \frac{1}{2i}(z - \bar{z}) \\ \partial_z &\equiv \frac{1}{2}(\partial_0 - i\partial_1) & \partial_0 &= \partial_z + \partial_{\bar{z}} \\ \partial_{\bar{z}} &\equiv \frac{1}{2}(\partial_0 + i\partial_1) & \partial_1 &= i(\partial_z - \partial_{\bar{z}}) \end{aligned} \tag{4.29}$$

Let us state that we call a function holomorphic if it is complex differentiable on its domain [18]. An antiholomorphic function, analogously, is a function that is differentiable in the antiholomorphic coordinate (i.e. $\bar{z}$, the complex conjugate of z). Instead of ∂_z and $\partial_{\bar{z}}$ we will also sometimes use ∂ and $\bar{\partial}$, when there is no risk of confusion. Our metric tensor in terms of holomorphic and antiholomorphic coordinates can be found by writing it explicitly. Using $dz^0 = (dz + d\bar{z})/2$ and $dz^1 = (dz - d\bar{z})/2i$, we can convert the real metric into its holomorphic form. Using the notation, z^μ, where the index μ denotes either the holomorphic coordinate or the antiholomorphic one, i.e. $z^\mu \in \{z, \bar{z}\}$, we get,

$$
\begin{aligned}
g_{\mu\nu}(z,\bar{z})\, dz^\mu \otimes dz^\nu &= dz^0 \otimes dz^0 + dz^1 \otimes dz^1 \\
g_{\mu\nu}(z,\bar{z})\, dz^\mu \otimes dz^\nu &= \frac{1}{4}\,(dz \otimes dz + dz \otimes d\bar{z} + d\bar{z} \otimes dz + d\bar{z} \otimes d\bar{z}) \\
&\quad - \frac{1}{4}\,(dz \otimes dz - dz \otimes d\bar{z} - d\bar{z} \otimes dz + d\bar{z} \otimes d\bar{z})
\end{aligned}
\tag{4.30}
$$

which means we have,

$$
g_{\mu\nu}(z,\bar{z})\, dz^\mu \otimes dz^\nu = \frac{1}{2}\,(dz \otimes d\bar{z} + d\bar{z} \otimes dz)\,. \tag{4.31}
$$

Thus the metric and the inverse metric components become,

$$
g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad\qquad g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \tag{4.32}
$$

Trying to rewrite the Cauchy-Riemann equations in holomorphic form, we have, using (4.27) with $w(z,\bar{z}) = w^0 + iw^1$, $\partial_0 = \partial_z + \partial_{\bar{z}}$ and $\partial_1 = i\,(\partial_z - \partial_{\bar{z}})$, the following,

$$
\frac{\partial w^0}{\partial z^0} - \frac{\partial w^1}{\partial z^1} = 0 \quad \Rightarrow \quad \partial_z w^0 + \partial_{\bar{z}} w^0 - i\partial_z w^1 + i\partial_{\bar{z}} w^1 = 0, \tag{4.33}
$$

$$
\frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1} = 0 \quad \Rightarrow \quad \partial_z w^1 + \partial_{\bar{z}} w^1 + i\partial_z w^0 - i\partial_{\bar{z}} w^0 = 0, \tag{4.34}
$$

which enables us to write,

$$
\partial_z \bar{w}(z,\bar{z}) + \partial_{\bar{z}} w(z,\bar{z}) = 0 \tag{4.35}
$$

and

$$
i\partial_z \bar{w}(z,\bar{z}) - i\partial_{\bar{z}} w(z,\bar{z}) = 0, \tag{4.36}
$$

or equivalently,

$$
\partial_{\bar{z}} w(z,\bar{z}) = 0 \tag{4.37}
$$

and

$$
\partial_z \bar{w}(z,\bar{z}) = 0. \tag{4.38}
$$

We observe that, because the derivative of $w(z,\bar{z})$ with respect to $\bar{z}$ is zero, w only depends on the holomorphic coordinate, z. Similarly $\bar{w}$ depends only on the antiholomorphic coordinate, $\bar{z}$. This result is remarkable, since our requirement of conformal invariance led us to the general result that any holomorphic function, $w(z)$, is an admissible coordinate transformation for conformal invariance.

The theory of complex variables tells us that any holomorphic function is infinitely continuously differentiable and can be expressed as the infinite sum of a convergent Taylor series. Within its radius of convergence the series has very desirable properties, such as uniform and absolute convergence, which enables us to work easily with the Taylor series of a function [17]. Now the fact that any holomorphic function can be expanded in its Taylor series does imply that we, in fact, need an infinite number of parameters, i.e. the coefficients of the series being evaluated at a certain point, to determine a holomorphic function. This will cause the algebra of the two dimensional conformal group to have an infinite number of generators. Furthermore, this property gives a rich structure to our conformal field theories in two dimensions and we can extract much information from them [9].

At this point it is important to say that, although z and $\bar{z}$ are complex conjugates and (z^0, z^1) is a pair of real numbers, it is beneficient to consider z and $\bar{z}$ as independent coordinates. This can be done by extending the real coordinates, z^0 and z^1, to the complex plane, increasing the number of parameters of z and $\bar{z}$ which allows us to think of them as independent complex coordinates [9]. Now the transformations we defined can be considered as well-defined functions in the complex plane, $\mathbb{C}^2$. To recover physically acceptable results we restrict ourselves to the case where $\bar{z}$ is actually the complex conjugate of z, in other words we take the real subspace of the complex spacetime we defined [9, 11].

Furthermore we can point out that our choice of the Euclidean flat metric does not constitute a problem anymore, because of our complexification of the coordinates. That is, by redefinition of the timelike component (i.e. multiplying the timelike component by i and relabeling it as a new coordinate in Euclidean space) we perform a Wick rotation. This Wick rotation enables us to handle path integrals and propagators in quantum field theory more comfortably with respect to their convergence properties. But, as indicated, for all this to be feasible, analytic continuation into the complex plane must be performed which may not always be possible [11]. Note that by using the proper transformations at the first place, we are able to take advantage of the well established theory of complex variables (had we chosen the Minkowski metric for $g^{\mu\nu}(z)$ we could not extract the Cauchy-Riemann equations).

As we saw in the derivation of the equivalence of the conformal transformation to the Cauchy-

Riemann equations, we can have transformations that are not invertible. The Cauchy-Riemann conditions do not give us information about global properties, such that well-definedness of the transformations on the whole plane and invertibility is not a priori given to us. To have a conformal group we must be able to find inverse transformations of every conformal transformation and furthermore the transformations have to map the whole plane into itself. We have to seperate global conformal transformations which give us a genuine group, from local conformal transformations which are not everywhere well-defined [9].

The global conformal transformations may be easily expressed by the Möbius transformations [17],

$$f(z) = \frac{az+b}{cz+d}, \quad \text{with } ad - bc \neq 0. \tag{4.39}$$

We note that, a, b, c and d are complex numbers. As we have, $\frac{\lambda az + \lambda b}{\lambda cz + \lambda d} = \frac{az+b}{cz+d}$, for any nonzero complex number, λ, and we have the condition $ad - bc \neq 0$, we can set, $ad - bc = 1$, without loss of generality. The Möbius transformations form a group, as can be seen easily. Two arbitrary Möbius transformations, say, $f_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1}$ and $f_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$, when applied successively, give us,

$$\begin{aligned}
(f_2 \circ f_1)(z) &= \frac{a_2\left(\frac{a_1 z + b_1}{c_1 z + d_1}\right) + b_2}{c_2\left(\frac{a_1 z + b_1}{c_1 z + d_1}\right) + d_2} \\
&= \frac{a_2 a_1 z + a_2 b_1 + b_2 c_1 z + b_2 d_1}{c_1 z + d_1} \, \frac{c_1 z + d_1}{c_2 a_1 z + c_2 b_1 + c_1 d_2 z + d_1 d_2} \\
&= \frac{(a_1 a_2 + b_2 c_1) z + (a_2 b_1 + b_2 d_1)}{(a_1 c_2 + c_1 d_2) z + (b_1 c_2 + d_1 d_2)},
\end{aligned} \tag{4.40}$$

which is another Möbius transformation. We can also consider the transformations, f_1 and f_2, as matrices,

$$M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \tag{4.41}$$

$$M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \tag{4.42}$$

respectively and get,

$$M_2 M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{bmatrix} a_1 a_2 + c_1 b_2 & a_2 b_1 + b_2 d_1 \\ a_1 c_2 + c_1 d_2 & b_1 c_2 + d_1 d_2 \end{bmatrix}, \tag{4.43}$$

which clearly reproduces the composition rule (4.40).

In addition if we consider, $f^{-1}(z) = \dfrac{dz - b}{-cz + a}$, and use (4.40) for the composition of two transformations, we have,

$$
\begin{aligned}
(f \circ f^{-1})(z) &= \frac{(da - bc)\, z + (-ab + ba)}{(dc - cd)\, z + (-bc + ad)} \\
&= z,
\end{aligned} \tag{4.44}
$$

where we used $ad - bc = 1$. We also get,

$$
\begin{aligned}
(f^{-1} \circ f)(z) &= \frac{(ad - bc)\, z + (db - bd)}{(-ac + ca)\, z + (-bc + da)} \\
&= z,
\end{aligned} \tag{4.45}
$$

where we again used $ad - bc = 1$. Therefore, $f^{-1}(z) = \dfrac{dz - b}{-cz + a}$ is indeed the inverse transformation of $f(z) = \dfrac{az + b}{cz + d}$ (this inverse clearly works also for the matrix representation). We call this group the special conformal group [9]. The corresponding matrices we defined conserve the structure of the special conformal group and are also invertible ($\det M = ad - bc = 1$), thus there is a homomorphism between the group $SL(2, \mathbb{C})$ and the special conformal group. In fact, if we consider the quotient group $SL(2, \mathbb{C})/\mathbb{Z}_2$ we get an isomorphism with the special conformal group. Furthermore $SL(2, \mathbb{C})/\mathbb{Z}_2$ and $SO(3, 1)$ are known to be isomorphic, thus the special conformal group (or the global conformal group) is isomorphic to the four-dimensional Lorentz group [9, 10].

With the definition $\mathbb{C}_\infty \equiv \mathbb{C} \cup \{\infty\}$, which is called the extended complex plane [17] or the Riemann sphere [9] and $f(\infty) = a/c$ and $f(-d/c) = \infty$, we can extend the special conformal group to $\mathbb{C}_\infty$ [17].

4.2 Conformal Generators

Now let us consider the group of all admissable conformal transformations in two dimensions, i.e. all holomorphic transformations and find the generators of the algebra of the local conformal group. Consider any infinitesimal holomorphic transformation,

$$
z' = z + \epsilon(z). \tag{4.46}
$$

Because of the analyticity of this transformation for a certain region and also assuming it is analytical at $z = 0$ we can write $\epsilon(z)$ in terms of its Laurent series around $z = 0$,

$$
\epsilon(z) = \sum_{n=-\infty}^{\infty} c_n z^{n+1}, \tag{4.47}
$$

where we used the convention of [9]. Similarly for the antiholomorphic coordinate we take the transformation,

$$\bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z}), \tag{4.48}$$

where,

$$\bar{\epsilon}(\bar{z}) = \sum_{n=-\infty}^{\infty} \bar{c}_n \bar{z}^{n+1}. \tag{4.49}$$

Taking a spinless scalar field, $\phi(z, \bar{z})$, we know that we should have,

$$\phi'(z', \bar{z}') = \phi(z, \bar{z}). \tag{4.50}$$

Furthermore let us expand $\phi(z', \bar{z}')$ around $(z, \bar{z})$ up to first order in ϵ and $\bar{\epsilon}$, such that we get,

$$\phi(z', \bar{z}') = \phi(z, \bar{z}) + \epsilon(z)\partial_{z'}\phi(z, \bar{z}) + \bar{\epsilon}(\bar{z})\partial_{\bar{z}'}\phi(z, \bar{z}) + \cdots . \tag{4.51}$$

Approximating also $\epsilon(z)$, $\bar{\epsilon}(\bar{z})$ and $\phi(z, \bar{z})$ in terms of functions of z' and $\bar{z}'$, we see that we have,

$$\begin{aligned} \epsilon(z) &= \epsilon(z') + O(\epsilon^2) \\ \bar{\epsilon}(\bar{z}) &= \bar{\epsilon}(\bar{z}') + O(\bar{\epsilon}^2) \\ \phi(z, \bar{z}) &= \phi(z', \bar{z}') + O(\epsilon, \bar{\epsilon}) \end{aligned} \tag{4.52}$$

Using these approximate forms in the above expansion and writing only terms up to first order, we see that we obtain,

$$\phi(z', \bar{z}') = \phi(z, \bar{z}) + \epsilon(z')\partial_{z'}\phi(z', \bar{z}') + \bar{\epsilon}(\bar{z}')\partial_{\bar{z}'}\phi(z', \bar{z}'), \tag{4.53}$$

or

$$\phi(z, \bar{z}) = \phi(z', \bar{z}') - \epsilon(z')\partial'\phi(z', \bar{z}') - \bar{\epsilon}(\bar{z}')\bar{\partial}'\phi(z', \bar{z}'), \tag{4.54}$$

where we used $\partial' \equiv \partial_{z'}$ and $\bar{\partial}' \equiv \partial_{\bar{z}'}$. Plugging the above expression into (4.50) we clearly have,

$$\phi'(z', \bar{z}') = \phi(z', \bar{z}') - \epsilon(z')\partial'\phi(z', \bar{z}') - \bar{\epsilon}(\bar{z}')\bar{\partial}'\phi(z', \bar{z}'). \tag{4.55}$$

We take the variation of ϕ by considering the difference $\phi'(z', \bar{z}') - \phi(z', \bar{z}')$, and therefore get taking into account that the coordinates $(z', \bar{z'})$ are arbitrary,

$$\delta\phi = -\epsilon(z)\partial\phi(z, \bar{z}) - \bar{\epsilon}(\bar{z})\bar{\partial}\phi(z, \bar{z}). \tag{4.56}$$

The only thing left to do is to insert the Laurent expansions we wrote for ϵ and $\bar{\epsilon}$. Hence we obtain,

$$\delta\phi = -\sum_{n=-\infty}^{\infty} c_n z^{n+1}\partial_z\phi - \sum_{n=-\infty}^{\infty} \bar{c}_n \bar{z}^{n+1}\partial_{\bar{z}}\phi. \tag{4.57}$$

With the following definitions,

$$l_n \equiv -z^{n+1}\partial_z \qquad \bar{l}_n \equiv -\bar{z}^{n+1}\partial_{\bar{z}}, \tag{4.58}$$

we finally can write,

$$\delta\phi = \sum_{n=-\infty}^{\infty} \left(c_n l_n \phi(z,\bar{z}) + \bar{c}_n \bar{l}_n \phi(z,\bar{z})\right). \tag{4.59}$$

We note that using the previous definition of a generator and taking the l_n and $\bar{l}_n$ as independent operators which is the case if we consider z and $\bar{z}$ as independent variables, the l_n and $\bar{l}_n$ become the generators for the algebra of the local conformal group. Here, of course, the coefficients c_n and $\bar{c}_n$ are the infinitesimal parameters that were in the previous chapter denoted by ω_a. Thus we see that we need an infinite number of generators to describe this algebra.

Let us also determine the commutators of this algebra. We clearly have,

$$\begin{aligned}
[l_n, l_m]\phi &= l_n l_m \phi - l_m l_n \phi \\
&= z^{n+1}\partial_z(z^{m+1}\partial_z\phi) - z^{m+1}\partial_z(z^{n+1}\partial_z\phi) \\
&= z^{n+1}(m+1)z^m\partial_z\phi + z^{n+1}z^{m+1}\partial_z^2\phi - z^{m+1}(n+1)z^n\partial_z\phi - z^{m+1}z^{n+1}\partial_z^2\phi \\
&= z^{n+m+1}(m-n)\partial_z\phi \\
&= (n-m)l_{n+m}\phi
\end{aligned} \tag{4.60}$$

Similarly for $[\bar{l}_n, \bar{l}_m]$ we find,

$$\begin{aligned}
[\bar{l}_n, \bar{l}_m]\phi &= \bar{l}_n \bar{l}_m \phi - \bar{l}_m \bar{l}_n \phi \\
&= \bar{z}^{n+1}\partial_{\bar{z}}(\bar{z}^{m+1}\partial_{\bar{z}}\phi) - \bar{z}^{m+1}\partial_{\bar{z}}(\bar{z}^{n+1}\partial_{\bar{z}}\phi) \\
&= (n-m)\bar{l}_{n+m}\phi
\end{aligned} \tag{4.61}$$

Finally we have the last commutator $[l_n, \bar{l}_m]$,

$$\begin{aligned}
[l_n, \bar{l}_m]\phi &= l_n \bar{l}_m \phi - \bar{l}_m l_n \phi \\
&= z^{n+1}\partial_n(\bar{z}^{m+1}\partial_{\bar{z}}\phi) - \bar{z}^{m+1}\partial_{\bar{z}}(z^{n+1}\partial_z\phi) \\
&= z^{n+1}\bar{z}^{m+1}\partial_{z\bar{z}}\phi - \bar{z}^{m+1}z^{n+1}\partial_{\bar{z}z}\phi \\
&= 0,
\end{aligned} \tag{4.62}$$

where we used the independency of z and $\bar{z}$ and also the commutativity of the partial derivatives. Thus we have shown that the algebra of the local conformal group is completely determined by,

$$
\begin{aligned}
[l_n, l_m] &= (n-m)l_{n+m} \\
[\bar{l}_n, \bar{l}_m] &= (n-m)\bar{l}_{n+m} \\
[l_n, \bar{l}_m] &= 0
\end{aligned}
\tag{4.63}
$$

This algebra is called the Witt algebra (or sometimes the Virasoro algebra without central extension) [9]. By taking the generators l_{-1}, l_0 and l_1 together with $\bar{l}_{-1}$, $\bar{l}_0$ and $\bar{l}_1$ we get the subalgebra that generates the global conformal algebra we considered in the previous section [9, 10].

Let us again point out that we take $(z, \bar{z}) \in \mathbb{C}^2$ such that we consider these coordinates as independent ones. Together with the third subalgebra which shows that the generators l_n and $\bar{l}_n$ do not mix, this enables us to consider the first two subalgebras of the Witt algebra as two isomorphic subalgebras. To get to the physically relevant case where $(z_0, z_1) \in \mathbb{R}^2$ it suffices to take the linear combination of the generators such that we have,

$$l_n + \bar{l}_n \tag{4.64}$$

$$i(l_n - \bar{l}_n) \tag{4.65}$$

as our modified generators [9].

CHAPTER 5

CONCLUSION

In this thesis, we have reviewed conformal symmetry and its implications on classical and quantum field theories. We mainly used the book by Di Francesco, et al. [9] in discussing conformal invariance and conformal field theories. First, we defined the conformal transformation then dicussed the conformal group in $d > 2$ dimensions to some extent. We derived the generators of the conformal group in d dimensions and established its algebra. Then Noether's theorem was described leading to the discussion of conserved charges. This brought us to conformal invariance in classical field theories and the representation of the conformal group in d dimensions. Correlation functions and the Ward identities were investigated. Finally, conformal invariance in two dimensions was considered and the corresponding generators of the algebra of the local conformal group were established.

Conformal invariance and the consequences that arise in two dimensions have received much attendance in the physical community [9] and it continues to extend its implications on field theories. It is therefore of importance to consider further studies of conformal field theories.

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