FIXED POINT SCHEME OF THE HILBERT SCHEME UNDER A 1-DIMENSIONAL ADDITIVE ALGEBRAIC GROUP ACTION

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ENGİN ÖZKAN

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submitted by **ENGİN ÖZKAN** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Mathematics Department, Middle East Technical University** by,

Prof. Dr. Canan Özgen
Dean, Graduate School of **Natural and Applied Sciences** ________________

Prof. Dr. Zafer Nurlu
Head of Department, **Mathematics** ________________

Prof. Dr.Ersan Akyıldız
Supervisor, **Mathematics Department, METU** ________________

Assoc. Prof. Dr. Özgür Kişisel
Co-supervisor, **Mathematics Department, METU NCC** ________________

Examining Committee Members:

Prof. Dr. Hurşit Önsiper
Mathematics, METU ________________

Prof. Dr. Ersan Akyıldız
Mathematics, METU ________________

Prof. Dr.Ali Sinan Sertöz
Mathematics, Bilkent Üniv. ________________

Prof. Dr. Yıldıray Ozan
Mathematics, METU ________________

Assoc Prof. Yusuf Civan
Mathematics, Süleyman Demirel Üniv. ________________

Date: ________________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: ENGİN ÖZKAN

Signature :

ABSTRACT

FIXED POINT SCHEME OF THE HILBERT SCHEME UNDER A 1-DIMENSIONAL ADDITIVE ALGEBRAIC GROUP ACTION

Özkan, Engin
Ph.D, Department of Mathematics
Supervisor : Prof. Dr.Ersan Akyıldız
Co-Supervisor : Assoc. Prof. Dr. Özgür Kişisel

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In general we know that the fixed point locus of a 1-dimensional additive linear algebraic group,G_a, action over a complete nonsingular variety is connected. In thesis, we explicitly identify a subset of the G_a-fixed locus of the punctual Hilbert scheme of the d points,$Hilb^d(\mathbb{P}^2, 0)$,in $\mathbb{P}^2$. In particular we give an other proof of the fact that $Hilb^d(\mathbb{P}^2, 0)$ is connected.

Keywords: Hilbert Scheme, Fixed Point Scheme, G_a- action

ÖZ

HILBERT ŞEMASI'NIN BELİRLİ BİR 1-BOYUTLU TOPLAMSAL CEBİRSEL GRUP ETKİSİ ALTINDAKİ SABİT NOKTA ŞEMASI

Özkan, Engin
Doktora, Matematik Bölümü
Tez Yöneticisi : Prof. Dr. Ersan Akyıldız
Ortak Tez Yöneticisi : Doç. Dr. Özgür Kişisel

Mart 2011, 45 sayfa

Genel olarak, 1-boyutlu toplamsal lineer cebirsel grubun ,G_a, tam ve düzgün olan bir varyete üzerindeki sabit nokta lokusunun bağlantılı olduğu bilinmektedir. Tezimizde, projektif uzayda verilen d noktayı parametrize eden panktual Hilbert şemasının, $Hilb^d(\mathbb{P}^2, 0)$, G_a-etkisi altında sabit kalan lokusun bir altkümesini belirtiyoruz. Bu altküme yardımıyla $Hilb^d(\mathbb{P}^2, 0)$'nin bağlantılı olma özelliğinin farklı bir ispatını veriyoruz.

Anahtar Kelimeler: Hilbert Şeması, Sabit Nokta Lokusu , G_a-grup etkisi

To my beatiful girl who understood.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

In the introduction we will give some basic information about subschemes of $\mathbb{P}^2$, and their Hilbert polynomials, and afterwards we will give the functorial definition of the Hilbert scheme. See Hartshorne, R [1] for more details.

Let $R = \mathbb{C}[x_0, ..., x_n]$ be the polynomial ring over the complex field $\mathbb{C}$ and $\mathbf{m} =< x_0, ..., x_n >$ the irrelevant ideal of R. A homogenous ideal I of R not containing $\mathbf{m}$ defines a closed subscheme of $\mathbb{P}^n$ via the surjection $R \to R/I$. Conversely, for any subscheme $\mathbf{X} \subset \mathbb{P}^n$, the corresponding ideal sheaf $\mathcal{I}_\mathbf{X}$ is the kernel of the map $\mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_\mathbf{X}$. The direct sum $I = \bigoplus_{k\geq 0} H^0(\mathbb{P}^n, \mathcal{I}_\mathbf{X}(k))$ is a homogenous ideal of R.

We should remark that the correspondence explained above is not a bijection. This is because of the irrelevant ideal $\mathbf{m}$. In fact this correspondence gives a bijection between the subschemes of $\mathbb{P}^n$ and the saturated homogenous ideals of R. An ideal J of R is saturated if $(J : \mathbf{m}^\infty) = J$ where $(J : \mathbf{m}^\infty) = \{f \in R : f\mathbf{m}^i \subseteq J$ for some $i > 0$ $\}$.

The Hilbert polynomial of a homogenous ideal of R, (or of a subscheme of $\mathbb{P}^n$) is an invariant of the ideal (subscheme). The Hilbert polynomial is determined by the **Hilbert function** of the ideal. This is the function $\mathrm{H_{R/I}} : \mathbb{N} \to \mathbb{N}$ given by

$$\mathrm{H_{R/I}}(t) = dim_{\mathbb{C}}(R/I)_t,$$

where $(R/I)_t$ is the t-th graded piece of R/I which is a vector space over $\mathbb{C}$. The key fact is that the function $\mathrm{H_{R/I}}$ agrees with a polynomial $\mathrm{P}_{R/I}$ for large t, i.e $\mathrm{H_{R/I}}(t) = \mathrm{P}_{R/I}(t)$ for $t \gg 0$ (See Eisenbud,D [2], pg 42-43). The polynomial $\mathrm{P}_{R/I}$ is called the **Hilbert polynomial** of R/I. If X is the subscheme of $\mathbb{P}^n$ corresponding to I, then $P_{R/I}(t) = \chi(\mathcal{O}_X(t))$.

1.2 Functorial Definition and Existence of the Hilbert Scheme

Before passing to the definition of the Hilbert scheme, we will give some basic definitions about categories.

Definition 1.2.1 *A* ***category*** $\mathcal{C}$ *consists of:*

• *A collection* $Ob(\mathcal{C})$ *of objects,*

• *Sets* $Mor(X, Y)$ *of morphisms for each pair* X, Y *of objects, and also an identity morphism* 1_X *for each* $X \in \mathcal{C}$

• *A composition operation* $\circ$ *of morphisms such that for each* $X, Y, Z, W \in Ob(\mathcal{C})$ *and* $h \in Mor(X, Y), g \in Mor(Y, Z), f \in Mor(Z, W)$*, we have* $f \circ (g \circ h) = (f \circ g) \circ h$ *and* $h \circ 1_X = h$*,* $1_Y \circ h = h$*.*

Example 1.2.2 *Let* $Ob(\mathfrak{Sets})$ *consist of all sets and for any pair of sets* A, B *,* $Mor(A, B)$ *be the set of functions from A to B. Then* $\mathfrak{Sets}$ *is a category called the* ***category of sets****.*

Example 1.2.3 *Let* $Ob(\mathfrak{Sch})$ *consist of all schemes and for any pair of schemes* X, Y*, let* $Mor(X, Y)$ *be the set of morphisms of schemes from X to* Y*. Then* $\mathfrak{Sch}$ *is a category called the* ***category of schemes****.*

Definition 1.2.4 *Let* $\mathcal{C}$ *be a category. The* ***opposite category*** $\mathcal{C}^\circ$ *is a category such that its objects are the same as the objects of* $\mathcal{C}$ *(* $Ob(\mathcal{C}) = Ob(\mathcal{C}^\circ)$*) and its morphisms are given by* $Mor_{\mathcal{C}^\circ}(X, Y) = Mor_{\mathcal{C}}(Y, X)$*.*

Definition 1.2.5 *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be two categories. A covariant (contravariant)functor F from* $\mathcal{C}$ *to* $\mathcal{D}$ *consists in*

• *a map* $F : Ob(\mathcal{C}) \to Ob(\mathcal{D})$*.*

• *for every* $X, Y \in Ob(\mathcal{C})$*, a map*

$$F : Mor(X, Y) \to Mor(F(X), F(Y))$$

$$(F : Mor(X, Y) \to Mor(F(Y), F(X)))$$

such that

(i) $F(1_X) = 1_{F(X)}$*,* $\forall X \in Ob(\mathcal{C})$

(ii) The following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \big\downarrow F & & \big\downarrow F \\ F(X) & \xrightarrow{F(f)} & F(Y) \end{array} \qquad (\textbf{\textit{covariant}})$$

$$\left(\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \big\downarrow F & & \big\downarrow F \\ F(X) & \xleftarrow{F(f)} & F(Y) \end{array} \qquad \textbf{\textit{contravariant}}\right).$$

Definition 1.2.6 *Let X be a scheme. Let us define the functor h_X from the opposite category of the category of schemes $\mathfrak{Sch}$ to the category of sets $\mathfrak{Sets}$ by*

$$h_X(Y) = Mor(Y, X)$$

and for any morphism of schemes $f : Y \longrightarrow Z$, $h_X(f) : Mor(Y, X) \longrightarrow Mor(Z, X)$ is the induced function (see the diagrams below). The functor h_X is called the ***functor of points*** *of the scheme X.*

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ \big\downarrow h_X & & \big\downarrow h_X \\ Mor(Y,X) & \xrightarrow{h_X(f)} & Mor(Z,X) \end{array} \qquad (in\ \mathfrak{Sch}^\circ)$$

$$\begin{array}{ccc} Z & \xrightarrow{f} & Y \\ \big\downarrow h_X & & \big\downarrow h_X \\ Mor(Z,X) & \xleftarrow{h_X(f)} & Mor(Y,X) \end{array} \qquad (in\ \mathfrak{Sch})$$

where $h_X(f)(g) = g \circ f$ for $g \in Mor(Y, X)$.

Example 1.2.7 *Say $X = \mathrm{Spec}R$ is an affine variety and $Y = \mathrm{Spec}(k)$ where k is a field. Then $h_X(Y) = Mor(Speck(k), Spec(R))$ which is in 1-1 correspondence with morphisms $R \to k$. This in turn gives us the k-valued points of the scheme X.*

The relative version is also defined as follows: Let S be any scheme. If X is an S-scheme, the functor of points on the category of S-schemes is defined as:

$$h_{X/S}(Y) = (\mathrm{S-morphisms}\ f : Y \to X)$$

where S − morphisms are the morphisms such that the following diagram commutes:

$$\begin{array}{ccc} Y & \longrightarrow & X \\ & \searrow & \downarrow \\ & & S \end{array}$$

Example 1.2.8 *Let $S = \operatorname{Spec}(B)$ for any ring B, and $X = \operatorname{Spec}(A)$ where $A = B[x_1, ..., x_n]/(f_1, ..., f_r)$. If R is a B-algebra, then an R-valued point of X/S is a B-algebra homomorphism; this corresponds exactly to a solution of the equations $\{f_i = 0\}_{i=1,...,r}$ in R^n.*

Definition 1.2.9 *Let $\mathcal{C}$, $\mathcal{D}$ be two categories, and F, G two functors of the same type from $\mathcal{C}$ to $\mathcal{D}$ (assume both are covariant for the sake of exposition). A* ***natural transformation*** *α from F to G consists of a collection of morphisms $\alpha_X : F(X) \to G(X)$ such that for any $f \in Mor(X, Y)$,*

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & G(X) \\ \big\downarrow{\scriptstyle F(f)} & & \big\downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\alpha_Y} & G(Y) \end{array}$$

commutes.

Definition 1.2.10 *Let F, G be two functors of the same type from the category $\mathcal{C}$ to the category $\mathcal{D}$. We say that F and G are* ***isomorphic functors*** *if there exists natural transformations $\alpha : F \to G$ and $\gamma : G \to F$ such that their compositions are identity transformations ($\alpha\gamma = 1_{G(X)}$ and $\gamma\alpha = 1_{F(X)}$ $\forall\ X \in Ob(\mathcal{C})$). (We write $F \cong G$)*

Definition 1.2.11 *A covariant functor $F : \mathfrak{Sch}^\circ \to \mathfrak{Sets}$ is called* ***representable*** *if $F \cong h_X$ for some scheme X.*

Definition 1.2.12 *Let M be a module over an arbitrary ring $\boldsymbol{R}$. We say that M is* ***flat*** *over $\boldsymbol{R}$ if for every $\boldsymbol{R}$- module monomorphism $f : A \to B$ the induced map $1 \times f : M \otimes_{\boldsymbol{R}} A \to M \otimes_{\boldsymbol{R}} B$ given by $m \otimes a \to m \otimes f(a)$ is again a monomorphism.*

Definition 1.2.13 *Let X and Y be any schemes. We say that a morphism of scheme* $f : X \to Y$ *is* ***flat morphism*** *if for every subvariety* $U \subseteq X$ *with* $\overline{f(U)} = V \subseteq Y$, $O_{U,X}$ *is a flat* $O_{V,Y}$*- module via the pull back* f^* *of* f.

Example 1.2.14 *Let* $X = \text{Spec}\mathbb{C}[x,t]/(t-x)$ *and* $Y = \text{Spec}\mathbb{C}[t]$ *then the morphism* $f : X \to Y$ *induced by the homomorphism* $\mathbb{C}[t] \to \mathbb{C}[x,t]/(t-x)$ *taking t to t is a flat morphism.*

Example 1.2.15 *Say* $X = \text{Spec}\mathbb{C}[x,t]/(xt-t)$ *and* $Y = \text{Spec}\mathbb{C}[t]$ *then the morphism* $f : X \to Y$ *induced by* $\mathbb{C}[t] \to \mathbb{C}[x,t]/(xt-t)$ *taking t to t is not a flat morphism. Because, the inverse image of the point p corresponding to the prime ideal* (t) *has dimension* 1 *but for the points associated to primes* $(t-c)$ *where* $c \neq 0$ *the inverse image has dimension* 0.

Definition 1.2.16 *Suppose that the ground field is an arbitrary algebraically closed field k. Let X be a scheme over S. An algebraic family of closed subschemes of X/S, parameterized by an S-scheme T, is a closed subscheme Z of the fiber product* $X \times_S T$. *The algebraic family is flat if the projection morphism* $\pi : Z \to T$ *is a flat morphism. A fiber of the family is the pullback* $(1 \times t)^*(Z)$ *of Z defined by the commutative diagram:*

$$\begin{array}{ccc} (1 \times t)^*(Z) \subseteq X \times_S \text{Spec}(k) & \longrightarrow & \text{Spec}(k) \\ \big\downarrow{\scriptstyle (1\times t)} & & \big\downarrow{\scriptstyle t} \\ Z \subseteq X \times_S T & \longrightarrow & T \end{array}$$

where $t : \text{Spec}(k) \to T$ *is a (k-valued) point of T.*

Definition 1.2.17 *Let* $\mathbf{Hilb}_{X/S}(T)$ *be the set of flat algebraic families of closed subschemes Z of* $X \times_S T$ *parameterized by the S-scheme T. So we can visualize* $\mathbf{Hilb}_{X/S}$ *as a map:*

$$\begin{array}{rccl} \mathbf{Hilb}_{X/S} : & Ob(\mathfrak{Sch}) & \longrightarrow & Ob(\mathfrak{Sets}) \\ & T & \longmapsto & \mathbf{Hilb}_{X/S}(T) \end{array}$$

If $f : T' \to T$ *is any morphism of S-schemes, the diagram below*

$$\begin{array}{ccc} Z \times_T T' & \longrightarrow & Z \\ \big\downarrow & & \big\downarrow \\ T' & \xrightarrow{f} & T \end{array}$$

says that $Z \times_T T' \subset X \times_T T'$ is flat over T'.

Define

$$\begin{array}{rll} \mathbf{Hilb}_{X/S}(f): & \mathbf{Hilb}_{X/S}(T) & \longrightarrow \mathbf{Hilb}_{X/S}(T') \\ & Z & \longmapsto \mathbf{Hilb}_{X/S}(f)(Z) = Z \times_T T' \end{array}$$

This consideration of $\mathbf{Hilb}_{X/S}$ and the map makes $\mathbf{Hilb}_{X/S}$ a contravariant functor on the category of S-schemes. Now let us check that $\mathbf{Hilb}_{X/S}$ is a functor:

(i) $\mathbf{Hilb}_{X/S}(id_T) = id_{\mathbf{Hilb}_{X/S}(T)}$.

Since $Z \times_T T$ is isomorphic to Z, we have that $\mathbf{Hilb}_{X/S}(id_T)$ is equal to $id_{\mathbf{Hilb}_{X/S}(T)}$.

(ii) Is the composition operation preserved under $\mathbf{Hilb}_{X/S}$?

For any given morphisms, $T'' \xrightarrow{g} T' \xrightarrow{f} T$

$$\begin{array}{ccccc} Z \times_T T' \times_{T'} T'' & \longrightarrow & Z \times_T T' & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ T'' & \xrightarrow{g} & T' & \xrightarrow{f} & T \end{array}$$

This diagram says that $Z \times_T T''$ is flat over T'' and since $Z \times_T T' \times_{T'} T'' \simeq Z \times_T T''$, so the composition operator is preserved. The main question at this point is: Is $\mathbf{Hilb}_{X/S}$ is representable?

*If $\mathbf{Hilb}_{X/S} \cong h_Y$ for some S-scheme Y, then Y is called the **Hilbert scheme** of X/S.*

Definition 1.2.18 *Let C be a category and F be a functor from C to the category of sets. A functor G from C to the category of sets is called a **subfunctor** of F if;*

(i) For all objects $X \in C$, $G(X) \subseteq F(X)$ (as sets)

(ii) For any morphism $f : X \to Y$, $G(f)$ is the restriction of $F(f) : F(X) \to F(Y)$.

Let again $Z \subseteq X \times_S T$ be a flat family over T and let $p : Z \to T$ be the projection. The Hilbert polynomial of Z at t is defined by $P_t(Z)(m) = \chi(O_{Z_t}(m))$ here $Z_t = p^{-1}(t)$. For every polynomial $P(x) \in \mathbb{C}[x]$ let us define $\mathbf{Hilb}^P_{X/S}$ to be the following subfunctor of $\mathbf{Hilb}_{X/S}$:

$$\mathbf{Hilb}^P_{X/S}(T) = \left\{ Z \subseteq X_T = X \times_S T \,\middle|\, Z \textit{ is flat over } T \textit{ and } P_t(Z) = P \textit{ for all } t \in T \right\}$$

Theorem 1.2.19 *(Grothendieck [3]) Assume that X is projective over S. Then $\mathbf{Hilb}^P_{X/S}(T)$ is representable by a scheme $Hilb^d(\mathbb{P}^2)$ and $Hilb^d(\mathbb{P}^2)$ is projective over S.*

Proof 1.2.20 *See Grothendieck ,A [3]* ■

1.3 Hilbert Schemes of Points On Surfaces

In this subsection we will introduce the Hilbert scheme of points in the plane and we'll show that this is a smooth and irreducible variety.

Let $\mathbb{A}^2$= Spec $\mathbb{C}[x, y]$ be the affine plane over the complex field $\mathbb{C}$. As a set the Hilbert scheme of d-points in the plane, $Hilb^d(\mathbb{A}^2)$, is the set of ideals $I \subseteq \mathbb{C}[x, y]$ such that the dimension of $\mathbb{C}[x, y]/I$ as a vector space over $\mathbb{C}$ is d. This scheme can also be viewed in another way: $Hilb^d(\mathbb{A}^2)$ is the scheme parameterizing subschemes $X \subseteq \mathbb{A}^2$ such that X=Spec($\mathbb{C}[x, y]/I$) is zero dimensional of length d. Moreover if we put the condition $\sqrt{I} = (x, y)$ in algebraic definition then Supp(X) = $(0, 0)$ and the resulting scheme is called the **Punctual Hilbert scheme**. Denote it by $Hilb^d(\mathbb{A}^2, 0)$.

Before giving more details about $Hilb^d(\mathbb{A}^2)$, we should remark that this Hilbert scheme has special properties. In fact Fogarty, J [4] showed that it is smooth and irreducible but for general n, $Hilb^d(\mathbb{A}^n)$ is neither smooth nor irreducible. For example Iarrobino, A [5] showed that the Hilbert scheme parameterizing 0-dimensional subschemes of length d of a nonsingular projective variety of dimension bigger than 2 is reducible.

First of all, we will see the variety structure on $Hilb^d(\mathbb{A}^2)$ explicitly by identifying it with a subvariety of a Grassmannian.

For any partition λ of d, define the set $\mathcal{B}_\lambda = \{x^k y^s : (k, s) \in \lambda\}$ and the ideal I_λ generated by monomials not belonging to $\mathcal{B}_\lambda$. Let k index the rows of λ, and s the columns.

Example 1.3.1 *For $d = 14$, consider the partition $\lambda : 4 + 3 + 3 + 2 + 2$, $\mathcal{B}_{(4,3,3,2,2)}$ consists of the monomials*

x^4	x^4y		
x^3	x^3y		
x^2	x^2y	x^2y^2	
x	xy	xy^2	
1	y	y^2	y^3

The ideal I_λ is generated by x^5, x^3y^2, xy^3, y^4 and its diagram is of the following form:

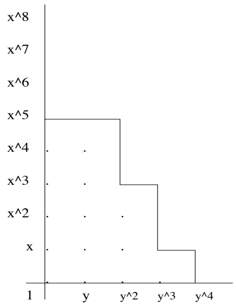


Figure 1.1: diagram of $I =< x^5, x^3y^2, xy^3, y^4 >$

For each nonnegative integer n, let V_n be the subspace of $\mathbb{C}[x, y]$ spanned by the $\binom{n+2}{2}$-monomials of degree at most n.

Example 1.3.2 *For $n = 3$, V_n is spanned by the* $1, x, y, xy, x^2, y^2, x^2y, xy^2, x^3, y^3$.

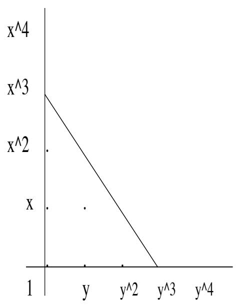


Figure 1.2: Figure of V_n

Lemma 1.3.3 *For any ideal* I *of length d, the image of V_n in $\mathbb{C}[x, y]/I$ spans $\mathbb{C}[x, y]/I$ as a vector space whenever $n \geq d$.*

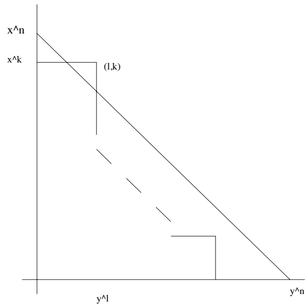


Figure 1.3: Such figures not possible when $n \geq d$

Proof 1.3.4 *Suppose not .Then the figure 1.3 of I is over the figure of V_n.The graph of V_n crosses the x-axis and y-axis respectively at points $(n, 0)$ and $(0, n)$. If the figure of I has at least one point, say (k,l), above the line $y = n - x$ then $k + l > n$. But the length of I is d, so we have $d \geq k + l > n$ when $n \geq d$.So we get a contradiction.* ∎

1.3.1 Smoothness and Connectedness

In this subsection, the references mainly consulted are [6] and Haiman, M [7] Now we define the following set for each partition λ

$$U_\lambda = \{I \in Hilb^d(\mathbb{A}^2) : \mathcal{B}_\lambda \; spans \; \mathbb{C}[x, y]/I\}$$

Since $dim_{\mathbb{C}}\mathbb{C}[x, y]/I = d$, $\mathcal{B}_\lambda$ is a basis. This means that the monomials outside I_λ constitutes a vector space basis for $\mathbb{C}[x, y]/I$.

Since $\mathcal{B}_\lambda$ is a basis for $\mathbb{C}[x, y]/I$, if we mod out any given monomial $x^r y^s$ with respect to I, the class $x^r y^s + I$ can be written uniquely as a linear combination of the elements of $\mathcal{B}_\lambda$. This means that the given monomial can be written as follows:

$$x^r y^s \equiv \sum_{(h,k)\in\lambda} c^{rs}_{hk} x^h y^k \text{mod}(I) \qquad (2.1)$$

Here, $(h, k) \in \lambda$ means that $x^h y^k \in \mathcal{B}_\lambda$.

Lemma 1.3.5 *The sets U_λ are open affine subvarieties which cover $Hilb^d(\mathbb{A}^2)$. The affine* *coordinate ring O_{U_λ} is generated by the functions c^{rs}_{hk}, for $(h,k) \in \lambda$ and all (r,s) such that $\{x^r y^s\}$ is a generating set of monomials for I_λ.*

Proof 1.3.6 *Let us first show that $\{U_\lambda\}_{|\lambda|=d}$ set theoretically covers $Hilb^d(\mathbb{A}^2)$. Let $\mathcal{B}$ be the set of monomials not in $I' =< in(I) >$. Note that every divisor of a monomial in $\mathcal{B}$ is also in $\mathcal{B}$. Therefore $\mathcal{B} = \mathcal{B}_\lambda$ for some partition λ of d. Therefore $I \in U_\lambda$. By this fact the sets U_λ cover $Hilb^d(\mathbb{A}^2)$.*
Now let us show that $\{U_\lambda\}$ is a subvariety of a Grassmanian. The intersection $I \cap V_n$ is a vector subspace in V_n of codimension d. Because of the lemma 1.3.3 above I is generated by $I \cap V_n$ when $n \geq d$. Thus $Hilb^d(\mathbb{A}^2)$ is contained as a set in the Grassmanian, $Gr^d(V_n)$, of codimension d subspaces of V_n.

The set of codimension d subspaces $W \subset V_n$ for which the monomials outside I_λ span V_n/W constitutes a standard open affine subvariety of $Gr^d(V_n)$. Here in(I) is the initial ideal of I with respect to grlex *order. This means that W has a unique $\mathbb{C}$-basis consisting in elements of the form*

$$x^r y^s - \sum_{(h,k)\in\lambda} c^{rs}_{hk} x^h y^k \qquad (*)$$

The affine chart of the Grassmanian is the affine space such that its coordinate ring is the polynomial ring in the coefficients c^{rs}_{hk}. Here $x^s y^r$ are monomials in $I \cap V_n$ so $\#\{(r,s)\} = \binom{n+2}{2} - d$. Since $\#\{(h,k)\} = d$ we retrieve $dim(Gr^d(V_n)) = d.(\binom{n+2}{2} - d)$.
If in addition $W = I \cap V_n$ then this situation gives rise to some relations in the polynomial ring $\mathbb{C}[c^{rs}_{hk}]$ and these relations generate a radical ideal. Let us see how we can get these relations. The ideal imposes the multiplication relation. This means multiplication with x or y preserves the ideal. Explicitly, if $x^{r+1}y^s \in V_n$, then multiplying (*) with x we get $x^{r+1}y^s - \sum c^{rs}_{hk} x^{h+1} y^s$ inside $V_n \cap I$. Now some terms $x^{h+1}y^s$ lie in I_λ, so using (*) again we have to expand such terms.

$$x^{r+1}y^s - \Big(\sum_{h+1,k\in\lambda} c^{rs}_{hk} x^{h+1} y^k + \sum_{h+1,k\notin\lambda} c^{rs}_{hk} \sum_{p,q\in\lambda} c^{h+1,k}_{pq} x^p y^q\Big) \in I \qquad (**)$$

If we equate the coefficients of $x^h y^k$ in (*) and (2.2) we get the relations in $\mathbb{C}[c^{rs}_{hk}]$. So U_λ is an algebraic subset of an open cell in the Grassmannian. ■

Example 1.3.7 *(See Miller.E, Strumfels,B [6] pg 359) Let $d = 4$ and λ partition $2+2$. Every* *ideal I in U_{2+2} is generated by the four polynomials:*

$$x^2 - c^{20}_{11}xy - c^{20}_{01}y - c^{20}_{10}x - c^{20}_{00} \tag{1.1}$$

$$x^2y - c^{21}_{11}xy - c^{21}_{01}y - c^{21}_{10}x - c^{21}_{00} \tag{1.2}$$

$$y^2 - c^{20}_{11}xy - c^{20}_{10}x - c^{20}_{01}y - c^{20}_{00} \tag{1.3}$$

$$xy^2 - c^{02}_{11}xy - c^{02}_{10}x - c^{02}_{01}y - c^{02}_{00} \tag{1.4}$$

Let us call $a = c^{20}_{11}, e = c^{20}_{01}, p = c^{20}_{10}, t = c^{20}_{00}, b = c^{21}_{11}, f = c^{21}_{01}, q = c^{21}_{10}, u = c^{21}_{00}, c = c^{02}_{11}, g = c^{02}_{01}, r = c^{02}_{10}, v = c^{02}_{00}, d = c^{11}_{11}, h = c^{11}_{10}, s = c^{11}_{01}, w = c^{11}_{00}$. *Before continuing the example we will show that these four polynomials are enough to generate I.*

Lemma 1.3.8 *The generators numbered with* (1.1), (1.2), (1.3), (1.4) *are enough to generate I.*

Proof 1.3.9 *Let us show first we do not need generators of the form:*

$$x^2y^2 - c^{22}_{11}xy - c^{22}_{01}y - c^{22}_{10}x - c^{22}_{00} \tag{1.5}$$

Suppose we have a such generator. Then by multiplying (1.2) *by y we get*

$$x^2y^2 - bxy^2 - fy^2 - qxy - uy \tag{1.6}$$

now rewriting xy^2 and y^2 in terms of $\{1, x, y, xy\}$ *in* (1.6) *we get the following expression:*

$$x^2y^2 - b(dxy + hx + sy + w) - f(cxy + gx + ry + v) - qxy - uy \tag{1.7}$$

If we use (1.5) *and* (1.7) *we obtain relation:*

$$xy(bd + fc + q - c^{22}_{11}) + x(bh + gf - c^{22}_{10}) + y(bs + fr + u - c^{22}_{01}) + (bw + fv - c^{22}_{00}) = 0$$

The coefficents in last equality are nonzero so we have a relation between $1, x, y, xy$*. This says us that the length of I is less than or equal* 3 *but this is a contradiction. In general to prove that we do not need generators of the form* $x^my^n - c^{mn}_{11}xy - c^{mn}_{10}x - c^{mn}_{01}y - c^{mn}_{00}$*, we use the same method inductively.* ■

Remark 1.3.10 *For a general d, if we have such a diagram for a partition λ of d, we omit the corners of the diagram labeled by (×) and we take all monomials on the boundary of I_λ labelled by (o) to get all generators of I. See the following figure*

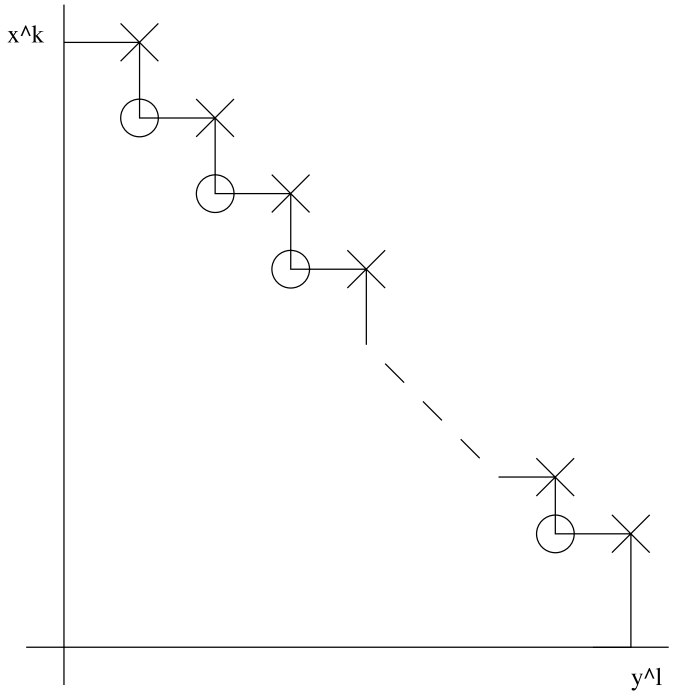


Figure 1.4: General case

The quotient ring $\mathbb{C}[x, y]/I$ *has a* $\mathbb{C}$*-basis* $(1, x, y, xy)$ *if and only if*

$$p = b - ad - ec, q = ah + eg, r = d - ag - bc, s = cf + eg$$

$$t = f - ed - acf + bce, u = aw + adeg - aceh - beg + eh$$

$$v = h - bg - ach + adg, w = cu - bceg - acfd + deg + fg$$

The parameters p, q, r, s, t, u, v *can be written in terms of* $\{a, b, c, d, e, f, g, h\}$*. We have also one further relation*

$$w(1 - ac) = P(a, b, c, d, e, f, g, h)$$

where $P(a, b, c, d, e, f, g, h)$ *denotes a polynomial with variable set* $\{a, b, c, d, e, f, g, h\}$*. Therefore the affine chart* U_{2+2} *is a smooth hypersurface in* $\mathbb{C}^9$*.*

Before saying that $Hilb^d(\mathbb{A}^2)$ is connected and smooth, we will give two basic definitions, tools and lemmas that will be used in the proofs. Fix a vector $v \in \mathbb{N}^n$. For any polynomial $f = \sum_{u \in \mathbb{N}^n} c_u x^u$, we set the initial term of f to be $in_v f = \sum c_u x^u$ where the sum is taken over $u \in \mathbb{N}^n$ such that the scalar product $v.u$ is maximal for all u with coefficient nonzero i.e $c_u \neq 0$

Example 1.3.11 *For* $n = 5$, $v = (1, 1, 0, 0, 1)$, $f = x_0x_1x_2 + x_1x_2x_3x_4 - x_0x_4 + x_0x_1x_4$ *then* $in_w f$ *is equal to* $x_0x_1x_4$.

Definition 1.3.12 *The* ***initial ideal*** *of I is the ideal* $in_v(I) := (in_v f : f \in I)$.

Generally, $in_v I$ does not have to be generated by the initial terms of the minimal generating set of I. Given a set $\mathcal{G} = \{f_1, ..., f_s\}$ of polynomials in the ideal I, we say that $\mathcal{G}$ is a **Gröbner basis** for I if $in_v I =< in_v f_1, ..., in_v f_s >$. For any ideal I and for any polynomial $f = \sum_u c_u x^u \in I$ define the $f' = t^d \sum_u c_u x^u t^{-v.u}$, where $d = max_{c_u \neq 0} v.u$. By this definition at least one term has no t term. Set $I_t = (f' : f \in I)$.

Theorem 1.3.13 *For any ideal* $I \subseteq \mathbb{C}[x, y]$, *the* $\mathbb{C}[t]$-*algebra* $\mathbb{C}[x, y][t]/I_t$ *is flat* $\mathbb{C}[t]$-*module, we also have;*

$$\mathbb{C}[x, y][t]/I_t \otimes_{\mathbb{C}[t]} \mathbb{C}[t, t^{-1}] \cong \mathbb{C}[x, y]/I[t, t^{-1}]$$

and

$$\mathbb{C}[x, y][t]/I_t \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/(t) \cong \mathbb{C}[x, y]/in_v(I)$$

Therefore $\mathbb{C}[x, y][t]/I_t$ *is a flat family over* $\mathbb{C}[t]$ *such that its fiber over 0 is* $\mathbb{C}[x, y]/in_v(I)$ *and over any point* $(t - a)$ *of* $\mathrm{Spec}(\mathbb{C}[t])$ *for* $0 \neq a \in \mathbb{C}$ *is* $\mathbb{C}[x, y]/I$

Proof 1.3.14 *See Eisenbud, D [2] pg: 343-344* ■

Remark 1.3.15 *The theorem says that we have a degeneration from an ideal to its initial ideal. Such a degeneration is called a* ***Gröbner degeneration****. If I is homogenous, we replace* $Spec(R[t]/I_t)$ *by* $Proj(R[t]/I_t)$ *and in that case t has degree 0.*

Lemma 1.3.16 *Every ideal* $I \in Hilb^d(\mathbb{A}^2)$ *can be connected to a monomial ideal by a rational curve.*

Proof 1.3.17 *Without loss of generality fix a monomial order "<", i.e it is a total order on the monomials in* $\mathbb{C}[x, y]$ *compatible with multiplication. With this monomial order define the ideal* $J = in_<(I)$. *Since the Gröbner degeneration is a flat family* I_t *over* $\mathbb{A}^1$ *we stay in* $Hilb^d(\mathbb{A}^2)$. *For* $t = 0$, *it gives I and for* $t = 1$ *we get the monomial ideal J.* ■

Lemma 1.3.18 *For every partition λ of d , the point $I_\lambda \in Hilb^d(\mathbb{A}^2)$ lies in the closure of the locus of all radical ideals in the Hilbert scheme $Hilb^d(\mathbb{A}^2)$.*

Proof 1.3.19 *First of all let us choose any monomial order "<". Now consider the exponents (h, k) of monomials $x^h y^k$ which is outside of I_λ. The collection of these exponents is a subset $\mathcal{E}$ of $\mathbb{C}^2$ and the order of the set $\mathcal{E}$ is d. Call the radical ideal of these points the* ***distraction*** *of I_λ and denote it by I'_λ. Suppose that the ideal I_λ is of the form $< x^{a_1}y^{b_1}, ..., x^{a_n}y^{b_n} >$. Consider the polynomials*

$$f_i = x(x-1)(x-2)...(x-a_i+1)y(y-1)...(y-b_i+1)$$

Since these polynomials vanish at $\mathcal{E}$ =, so $< f_1, ..., f_m > \subseteq I'_\lambda$. The leading terms of the f_i's are the generators of the ideal I_λ which has length d. So the length of the ideal $< f_1, ..., f_m >$ is less than or equal to d. Therefore

$$< f_1, ..., f_m >= I'_\lambda$$

Moreover, for the given monomial order $<$, I_λ is the initial monomial ideal of I'_λ. The ideal $(I'_\lambda)_t$ constructed from Gröbner degeneration is radical for each $t \neq 0$. So the proof is completed.
■

Example 1.3.20 *The distraction of the ideal $I_{2+1+1} =< x^4, x^2y, xy^2, y^3 >$ is the ideal*

$$I'_{2+1+1} =< x(x-1)(x-2)(x-3), x(x-1)y, xy(y-1), y(y-1)(y-2) >$$

Theorem 1.3.21 *The Hilbert scheme $Hilb^d(\mathbb{A}^2)$ is connected.*

Proof 1.3.22 *For any two points I and J in $Hilb^d(\mathbb{A}^2)$ we can find a path as follows. First go from the ideal I to the initial monomial ideal I_λ and then to its distraction I'_λ. Do the same for J. After these process we have two radical ideals I'_λ and J'_μ of the d points in $\mathbb{A}^2$. By passing from one point configuration to the another we can connect these two radical ideals. So we are done.* ■

Let us consider the $\mathbb{C}^*$ torus action on $Hilb^d(\mathbb{A}^2)$. We have such an action since the action of $\mathbb{C}^*$ on $\mathbb{A}^2$ induces an action on $S = \mathbb{C}[x, y]$ and this action on S induces a $\mathbb{C}^*$-action on ideals $I \in S$ with $dim_k(S/I) = d$. So we have a $\mathbb{C}^*$- action on $Hilb^d(\mathbb{A}^2)$. The following will give all fixed points of $\mathbb{C}^*$.

Lemma 1.3.23 *The fixed points of the $\mathbb{C}^*$-action on $Hilb^d(\mathbb{A}^2)$ are monomial ideals of length d.*

Proof 1.3.24 *Since the torus action scales each monomial in $\mathbb{C}[x, y]$, monomial ideals are fixed under this action. To show the other inclusion, consider $w \in \mathbb{N}^2$, and let us consider the one parameter subgroup $\phi_w : \mathbb{C}^* \to \mathbb{C}^{*2}$ given by the formula $\phi_w(t) = (t^{w_1}, t^{w_2})$, so the action of $\mathbb{C}^*$ on R is $\phi_w(t)(x, y) = (t^{-w_1}x, t^{-w_2}y)$. Let's consider $lim_{t\to 0}\phi_w(t)I$. This limit is equal to in_wI for any given ideal in $Hilb^d(\mathbb{A}^2)$. If we choose w sufficiently generic then the ideal in_wI is a monomial ideal. So if $I \in Hilb^d(\mathbb{A}^2)$ is not a monomial ideal then $lim_{t\to 0}\phi_w(t)I \neq I$,this means that I is not a fixed point of the $\mathbb{C}^*$-action. Therefore the fixed point set of $Hilb^d(\mathbb{A}^2)$ under the torus action just consists of monomial ideals.* ■

Remark 1.3.25 *If $Hilb^d(\mathbb{A}^2)$ is smooth at every monomial ideal then $Hilb^d(\mathbb{A}^2)$ is smooth. Since the singular locus of $Hilb^d(\mathbb{A}^2)$ is fixed under the torus action (torus action is a kind of automorphism so it must preserve geometric properties of points) and it is closed, so if it not empty then the singular locus must contain a $\mathbb{C}^*$-fixed point and so a monomial ideal.*

By the remark above it is enough to check smoothness of $Hilb^d(\mathbb{A}^2)$ only at monomial ideals. To prove smoothness we will compute the dimension of the cotangent space at a monomial ideal of the form I_λ and we will put a bound on the dimension of $Hilb^d(\mathbb{A}^2)$ via the Hilbert-Chow morphism. For each element $I \in Hilb^d(\mathbb{P}^2)$, we can define a 0-cycle $\sum_i m_i x_i$ where x_i are the points of $\mathbb{A}^2$ in the support of the subscheme determined by the ideal I, and m_i are the multiplicities equal to the length of the local ring $O_{R,x_i} = (\mathbb{C}[x, y]/I)_{x_i}$ so $\sum m_i = d$. The symmetric group S_d acts on $(\mathbb{A}^2)^d$ by permuting the coordinates and we can parameterize the 0-cycle of $\mathbb{A}^2$ by $(\mathbb{A}^2)^d/S_d$. Let us define the morphism:

$$Hilb^d(\mathbb{A}^2) \to (\mathbb{A}^2)^d/S_d$$

sending $I \in Hilb^d(\mathbb{A}^2)$ to its 0-cycle. This morphism is called the **Hilbert-Chow** morphism. This morphism is surjective.

Lemma 1.3.26 *$Hilb^d(\mathbb{A}^2)$ is at least $2d$ dimensional.*

Proof 1.3.27 *$(\mathbb{A}^2)^d/S_d$ is $2d$-dimensional and the Hilbert-Chow morphism is surjective, so $dim(Hilb^d(\mathbb{A}^2) \geq 2d$* ■

Now for a monomial ideal I in $Hilb^d(\mathbb{A}^2)$ we will give a combinatorial description of cotangent space and show that its dimension is at most $2d$.

Lemma 1.3.28 *For any partition λ of d, $dim_{\mathbb{C}}(m_{I_\lambda}/m_{I_\lambda}^2) \leq 2d$. Here m_{I_λ} is the maximal ideal corresponding to the monomial ideal $I_\lambda \in Hilb^d(\mathbb{A}^2)$.*

Proof 1.3.29 *We know from lemma 1.3.4 that U_λ is an affine cover for $Hilb^d(\mathbb{A}^2)$ and its coordinate ring is generated by c_{hk}^{rs} for all $(h,k) \in \lambda$ and all (r,s). Here $(h,k) \in \lambda$ means that $x^h y^k$ is not in the ideal I_λ.*

We associate an arrow for each c_{hk}^{rs} starting from the box (h,k) and ending at (r,s).

We set c_{hk}^{rs}=0 when $h < 0$ or $k < 0$ or $(r,s) \in \lambda$ and set $c_{hk}^{rs} = 1$ when $(r,s) = (h,k)$. The maximal ideal $\mathfrak{m}_{I_\lambda}$ associated to the point $I_\lambda \in Hilb^d(\mathbb{A}^2)$ contains the c_{hk}^{rs} for which $(h,k) \in \lambda$ and $(r,s) \notin \lambda$ so we can omit these coordinate functions from $\mathfrak{m}_{I_\lambda}$.

Now let us make the following analysis:

If we multiply $x^r y^s \equiv \sum_{(h,k)\in\lambda} c_{hk}^{rs} x^h y^k (mod I_\lambda)$ by x we get $x^{r+1}y^s - \sum c_{hk}^{rs} x^{h+1} y^s$. Now some terms $x^{h+1}y^s$ are not in I_λ so expanding these terms again with respect to (2.1) and comparing the coefficients we get that

$$c_{hk}^{r+1,s} = \sum_{(l,m)\in\lambda} c_{lm}^{rs} c_{hk}^{l+1,m} \qquad (2.2).$$

Apply the similar process for y in order to get the relation

$$c_{hk}^{r,s+1} = \sum_{(l,m)\in\lambda} c_{lm}^{rs} c_{hk}^{l,m+1}. \qquad (2.3)$$

Since the terms $c_{lm}^{rs} c_{hk}^{l+1,m}$ are in $\mathfrak{m}_\lambda^2$ for $(l+1,m) \notin \lambda$ and for $(l+1,m) \in \lambda$ but $(l+1,m) \neq (h,k)$ in the quotient $\mathfrak{m}_\lambda/\mathfrak{m}_\lambda^2$ such terms reduce to zero. Taking quotient in (2.2) *the remaining term is $c_{h-1,k}^{rs}$. This means that:*

$$c_{hk}^{r+1,s} \equiv c_{h-1,k}^{rs} mod(\mathfrak{m}_\lambda/\mathfrak{m}_\lambda^2). \qquad (2.4)$$

Similarly the process of multiplying by y gives the equality

$$c_{hk}^{r,s+1} \equiv c_{h,k-1}^{rs} mod(\mathfrak{m}_\lambda/\mathfrak{m}_\lambda^2). \qquad (2.5)$$

This analysis says that if we move the tail and head of an arrow one box corresponding to the function c_{hk}^{rs} horizontally or vertically, this movement does not effect the residue class of the

arrow (or c^{rs}_{hk}) as long as the head of the arrow is outside and the tail is inside the staircase diagram of I_λ.
We should remark that this analysis also contains the case where $h < 0$ or $k < 0$ i.e the arrow crosses the x or y-axis. For example $c^{45}_{13} \equiv c^{35}_{03} \equiv c^{36}_{04} \equiv c^{37}_{05} \equiv c^{38}_{06} \equiv 0$ from the equalities (2.4) and (2.5). So we have a restriction when we move the arrow. First of all if the arrow crosses an axis then it is zero and such an arrow does not effect the quotient $\mathfrak{m}_\lambda/\mathfrak{m}^2_\lambda$. Count the number of arrows such that for each (h, k) the tail of the north-west pointing arrow lies inside the square (h, k) and the head of the arrow lies just above square of column 1 outside λ and the south-east pointing arrows such that its tail lies inside column k and its head lies inside row 1 outside λ. The total number of such north-west and south-east pointing arrows is equal to $2d$. Therefore the cotangent space has at most dimension $2d$. ■

In the light of lemmas 1.3.26 and 1.3.28 the combinatorial tangent space has dimension $2d$. So $Hilb^d(\mathbb{A}^2)$ is smooth.

1.3.2 Homology Groups of $Hilb^d(\mathbb{P}^2)$ and the B-B Decomposition

Ellingsrud and Strømme in [8] computed the Betti numbers of $Hilb^d(\mathbb{P}^2)$ by using the Bialynicki-Birula decomposition [9], [10].We would like to explain this computation below.

Definition 1.3.30 *Let X be projective scheme over $\mathbb{C}$. A cell decomposition of X is a filtration*

$$X = Y_n \supset Y_{n-1} \supset Y_{n-2} \supset ... \supset Y_0 \supset Y_{-1} = \emptyset$$

*such that each $Y_i - Y_{i-1}$ is a disjoint union of affine schemes $A^{m_{ij}}$ for all $i = 0, ..., n$. We call these affine schemes the **cells** of the decomposition.*

Theorem 1.3.31 *(Fulton,W [11]) Let X be a scheme over $\mathbb{C}$ with a cell decomposition. Then for 1≤ i ≤ dimX we have*
(1)The cycle map $cl : A_*(X) \to H_*(X)$ is an isomorphism
(2)$H_{2i+1}(X) = 0$
(3)$H_{2i}(X)$ is free abelian group generated by the homology classes of the i-dimensional cells.

Ellingsrud and Strømme constructed the cell decomposition of the $Hilb^d(\mathbb{P}^2)$ by using G_m-actions where G_m is the 1-dimensional multiplicative algebraic group . In general if X is a smooth projective variety over $\mathbb{C}$ with an action of G_m and $x \in X$ is a fixed point of this action, then there exists an induced action of G_m on $T_{X,x}$. Let $T^+_{X,x}$ denotes the positive weight space of G_m. The following is proved by Bialynicki-Birula:

Theorem 1.3.32 *(B-B decomposition [9], [10])Let X be a smooth projective scheme over $\mathbb{C}$ with a G_m-action. Assume that the fixed point set of G_m is a finite set $\{x_1, ..., x_r\}$, and let $X_i = \{x \in X \mid lim_{t\to 0} t.x = x_i\}$. Then*
(i) X has a cellular decomposition with cells X_i
(ii) $T_{X_i,x_i} = T^+_{X,x_i} \bigoplus T^-_{X,x_i}$

The main theorem in [8] is the following.

Theorem 1.3.33 *(Ellingsrud,Strømme [8]) i) Let X denote the one of the schemes $Hilb^d(\mathbb{P}^2)$, $Hilb^d(\mathbb{A}^2)$ or $Hilb^d(\mathbb{A}^2, 0)$. Then the cycle map $cl : A_*(X) \to H_*(X)$ is an isomorphism, and in particular the odd homology vanishes.*

ii)

$$b_{2k}(Hilb^d(\mathbb{P}^2)) = \sum_{d_0+d_1+d_2=d} \sum_{p+r=k-d_1} P(p, d_0 - p)P(d_1)P(2d_2 - r, r - d_2)$$

and

$$\chi(Hilb^d(\mathbb{P}^2)) = \sum_{d_0+d_1+d_2=d} P(d_0)P(d_1)P(d_2)$$

iii)

$$b_{2k}(Hilb^d(\mathbb{A}^2)) = P(2d - k, k - d) \quad and \quad \chi(Hilb^d(\mathbb{A}^2)) = P(d)$$

(iv)

$$b_{2k}(Hilb^d(\mathbb{A}^2, 0)) = P(k, d - k) \quad and \quad \chi(Hilb^d(\mathbb{A}^2, 0)) = P(d)$$

Here $P(m, n)$ for $m \geq n$ denotes the number of partitions of m into n-parts and $P(m)$ is the number of the partitions of m. We assume that $P(m, n) = 0$ if m or n is negative.

Proof 1.3.34 *(sketch of proof)We will shortly give the sketch proof in [8]. Let us first show that the part* (i) *of theorem* 2.28. *Let* x_0, x_1, x_2 *homogenous coordinates for* $\mathbb{P}^2$ *and* $G \subset SL(3, \mathbb{C})$ *be the maximal torus consisting of all diagonal matrices. Let* ξ_0, ξ_1, ξ_2 *are characters of G such that for all* $g \in G$ *we have* $g = diag(\xi_0(g), \xi_1(g), \xi_2(g))$*.Then the action of G on* $\mathbb{P}^2$ *is given by,*

$$g.x_i = \xi_i(g)x_i$$

Clearly the fixed points of this action are $p_1 = (1 : 0 : 0), p_2 = (0 : 1 : 0), p_3 = (0 : 0 : 1)$
Let $\phi : G_m \to G$ *be a one-parameter subgroup of G such that this one parameter subgroup has the same fixed point set as G.*
Let L be the line $x_0 = 0$*, consider the sequence,* $\emptyset = Y_0 \subset Y_1 = (p_3) \subset Y_2 = L \subset Y_3 = \mathbb{P}^2$. *Then* $F_i = Y_i - Y_{i-1} \simeq \mathbb{A}^i$ *for* $i = 1, 2, 3$ *and by definition* (2.25) *these* $F_i's$ *define a cell decomposition of* $\mathbb{P}^2$. *Let us consider the one-parameter subgroups* $\phi : G_m \to G$ *of the form* $\phi(t) = diag(t^{m_0}, t^{m_1}, t^{m_2})$ *where* $m_0 < m_1 < m_2$ *and* $m_0 + m_1 + m_2 = 0$. *We take such one-parameter subgroup to get the same fixed point set with the maximal torus. Because if we consider the limits we get the following conclusions when* $m_0 < m_1 < m_2$*:*

$$\begin{aligned} X_0 &= \{[x_0 : x_1 : x_2] \in \mathbb{P}^2 : lim_{t\to 0} t \cdot [x_0 : x_1 : x_2] = [0 : 0 : 1]\} \\ &= \{[x_0 : x_1 : x_2] \in \mathbb{P}^2 : lim_{t\to 0}[t^{m_0}x_0 : t^{m_1}x_1 : t^{m_2}x_2] = [0 : 0 : 1]\} \\ &= \{[x_0 : x_1 : x_2] \in \mathbb{P}^2 : lim_{t\to 0}[t^{m_0-m_2}x_0/x_2 : t^{m_1-m_2}x_1/x_2 : 1] = [0 : 0 : 1]\} \end{aligned}$$

The only $[x_0 : x_1 : x_2]$ *satisfying* $lim_{t\to 0}[x_0 : x_1 : x_2] = [0 : 0 : 1]$ *is* $[0 : 0 : 1]$. *This means that* $X_0 = [0 : 0 : 1]$

$$\begin{aligned} X_1 &= \{[x_0 : x_1 : x_2] \in \mathbb{P}^2 : lim_{t\to 0} t \cdot [x_0 : x_1 : x_2] = [0 : 1 : 0]\} \\ &= \{[x_0 : x_1 : x_2] \in \mathbb{P}^2 : lim_{t\to 0}[t^{m_0}x_0 : t^{m_1}x_1 : t^{m_2}x_2] = [0 : 1 : 0]\} \\ &= \{[x_0 : x_1 : x_2] \in \mathbb{P}^2 : lim_{t\to 0}[t^{m_0-m_1}x_0/x_1 : 1 : t^{m_2-m_1}x_2/x_1] = [0 : 1 : 0]\} \end{aligned}$$

This limit is equal to $[0 : 1 : 0]$ *if and only if* $X_1 = L - [0 : 0 : 1]$ *where* $L = \{x_0 = 0\}$.

$$\begin{aligned} X_2 &= \{[x_0 : x_1 : x_2] \in \mathbb{P}^2 : lim_{t\to 0} t \cdot [x_0 : x_1 : x_2] = [1 : 0 : 0]\} \\ &= \{[x_0 : x_1 : x_2] \in \mathbb{P}^2 : lim_{t\to 0}[t^{m_0}x_0 : t^{m_1}x_1 : t^{m_2}x_2] = [1 : 0 : 0]\} \\ &= \{[x_0 : x_1 : x_2] \in \mathbb{P}^2 : lim_{t\to 0}[1 : t^{m_1-m_0}x_1/x_0 : t^{m_2-m_0}x_2/x_0] = [1 : 0 : 0]\} \end{aligned}$$

As a conclusion the above limit is equal to $[1 : 0 : 0]$ *if and only if* $X_2 = \mathbb{P}^2 - L$.
By theorem 2.27 *these* $X_i = F_i$ *are cells of* $\mathbb{P}^2$. *The action of G induces an action on* $Hilb^d(\mathbb{P}^2)$ *because G acts on the ideals in* $\mathbb{C}[x_0, x_1, x_2]$. *If a point of* $Z \in Hilb^d(\mathbb{P}^2)$ *is fixed under G then the corresponding homogenous ideal* I_Z *is generated by monomial ideals and the number of*

such ideals is finite so the number of fixed points set of G on $Hilb^d(\mathbb{P}^2)$ is finite. By using Theorem 2.27, part (i) of Theorem 2.28 is proved since we know that $Hilb^d(\mathbb{P}^2)$ is smooth and projective.

For any finite length subscheme Z_i supported at p_i, we have $supp(lim_{t\to 0}\phi(t).Z_i) = lim_{t\to 0}(\phi(t).supp(Z_i)) = p_i$. Therefore for any subscheme $Z \subset Hilb^d(\mathbb{P}^2)$ fixed under the G-action, the support of Z is contained in the fixed point set $\{p_1, p_2, p_3\}$. So we can write any such Z of the form $Z = Z_1 \cup Z_2 \cup Z_3$ where support of each closed subscheme Z_i is contained in X_i for $i = 1, 2, 3$.

Let us put $d_i(Z) = lenght(O_{Z_i})$. For each triple (d_1, d_2, d_3) satisfying $d_1 + d_2 + d_3 = d$, we define $W(d_1, d_2, d_3)$ to be the subset of $Hilb^d(\mathbb{P}^2)$ corresponding to the finite length subschemes Z with $d_i = d_{Z_i}$, for $i = 1, 2, 3$. So we can write $Hilb^d(\mathbb{P}^2)$ as a union of $W's$ i.e $Hilb^d(\mathbb{P}^2) = \cup_{d_1+d_2+d_3=d} W(d_1, d_2, d_3)$.

Since we have an isomorphism

$$W(d_0, d_1, d_2) = W(d_0, 0, 0) \times W(0, d_1, 0) \times W(0, 0, d_2)$$

so we have

Lemma 1.3.35

$$b_{2k}(Hilb^d(\mathbb{P}^2) = \sum_{d_0+d_1+d_2=d} \sum_{p+q+r=k} b_{2p}(W(d_0, 0, 0))b_{2q}(W(0, d_1, 0))b_{2r}(W(0, 0, d_2)).$$

The last equality reduces the problem of computing $b_{2k}(Hilb^d(\mathbb{P}^2)$ to the problem of computing the Betti numbers of $W(d_1, 0, 0), W(0, d_2, 0), W(0, 0, d_3)$.

Each W is a union of cells from the cell decomposition of $Hilb^d(\mathbb{P}^2)$. For example the cells contained in $W(d_1, 0, 0)$ are the subschemes corresponding to the fixed points supported at p_0.Since every G-invariant subscheme concentrated at a fixed point is also contained in G-invariant affine plane, we will deal with the ideals in $R = \mathbb{C}[x, y]$ of length d and invariant under the action of the two dimensional torus T. The action of T, is given on R as $t.x = \chi(t)x$ and $t.y = \eta(t)y$ for two linearly independent characters χ and η of T.

Let I be such a T-invariant ideal, so it is generated by the monomials in x and y. We put

$$a_j = min\{k \mid x^j y^k \in I\}$$

because of the invariance, such a number exists for all integers $j \geq 0$. Let r be the least integer making $a_r > 0$.Then $(a_0, a_1, ..., a_r)$ is a partition of d and $y^{a_0}, xy^{a_1}, x^2y^{a_2}, ..., x^{r+1}$ are

generators for the T-invariant ideal I. So we get a bijection between partitions of d and the T-invariant ideals I of length d in R.
*If $\mathbb{T}$ is the tangent space of $Hilb^d(\mathbb{A}^2)$ at the point I, we have an identification $\mathbb{T} \simeq Hom_R(I, R/I)$, see [3]. Ellingsrud and Strømme calculated the representation of T on $\mathbb{T}$. Before giving this calculation we will give some conventions for the notation following **Ellingsrud-Strømme**.*
Let us denote by $R[\alpha,\beta]$ the R-module R with the action of T is given by:

$$t.x^m y^n = \chi(t)^{m-\alpha}\eta(t)^{n-\beta}x^m y^n$$

for any pair of integers (α,β).
In the representation ring of T let us write $R[\alpha,\beta] = \sum_{p\geq-\alpha,q\geq-\beta}\chi^p\eta^q$.

Lemma 1.3.36 *There is a T-equivariant resolution*

$$0 \to \bigoplus_{i=1}^{r} R[-i,-a_{i-1}] \xrightarrow{M} \bigoplus_{i=0}^{r} R[-i,-a_i] \to I \to 0$$

Furthermore if $e_i = a_{i-1} - a_i$ for $1 \leq i \leq r$ then $\mathbf{M} = \begin{bmatrix} x & 0 & 0 & \dots & 0 \\ y^{e_1} & x & 0 & \dots & 0 \\ 0 & y^{e_2} & x & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & \dots & \dots & \dots & x \\ 0 & \dots & \dots & \dots & y^{e_r} \end{bmatrix}$.

Ellingsrud-Strømme gave the representation of T on the tangent space $\mathbb{T}$ as:

Lemma 1.3.37 *In the representation ring of T we have the identity*

$$Hom_R(I, R/I) = \sum_{1\leq i\leq j\leq r}\sum_{s=a_j}^{a_{j-1}-1}(\chi^{i-j-1}\eta^{a_{i-1}-s-1} + \chi^{j-i}\eta^{s-a_i}).$$

Proof 1.3.38 *The following T-equivariant short exact sequence*

$$0 \to I \to R \to R/I \to 0$$

induce the sequence

$$0 \to Hom_R(I, R/I) \xrightarrow{g} Ext^1_R(I, I) \xrightarrow{f} Ext^1_R(I, R) \xrightarrow{h} Ext^1_R(I, R/I) \to 0$$

The map h is an isomorphism since

$$Ext^1_R(I,R) \cong Ext^2_R(R/I,R) \cong Ext^2_R(R/I,R/I) \cong Ext^1_R(I,R/I).$$

Now let us consider the following T *equivariant complex to compute* $Ext^1_R(I,I)$

$$\check{E}_0 \otimes E_1 \xrightarrow{A} (\check{E}_0 \otimes E_0) \oplus (\check{E}_1 \otimes E_1) \xrightarrow{B} \check{E}_1 \otimes E_0$$

where $E_0 = \oplus_{i=0}^r R[-i,-a_i]$, $E_1 = \oplus_{i=1}^r R[-i,-a_{i-1}]$*. The maps A and B are given by respectively* $A = (id_{\check{E}_0} \otimes M, \check{M} \otimes id_{E_1})$*,* $B = (\check{M} \otimes id_{E_0}, -id_{\check{E}_1} \otimes M)$*. The cokernel of B is* $Ext^1_R(I,I)$*, the middle homology is* $Hom_R(I,I) = R$*, and A is injective. So in the representation ring we have*
$Ext^1_R(I,I) = R + \sum_{1\le i\le r, 0\le j\le r} R[i-j, a_{i-1}-a_i] - \sum_{1\le i,j\le r} R[i-j, a_{i-1}-a_{j-1}]$
$- \sum_{0\le i,j\le r} R[i-j, a_i - a_j] + \sum_{1\le i\le r, 0\le j\le r} R[j-i, a_j - a_{i-1}]$
For $1 \le i \le j \le r$ *define*

$$K_{ij} = R[j-i+1, a_{j-1}-a_{i-1}] - R[j-i, a_{j-1}-a_{i-1}] - R[j-i+1, a_j - a_{i-1}] + R[j-i, a_j - a_{i-1}]$$

and

$$L_{ij} = R[i-j, a_{i-1}-a_j] - R[i-j, a_{i-1}-a_{j-1}] - R[i-1-j, a_{i-1}-a_j] + R[i-1-j, a_{i-1}-a_{j-1}]$$

By this formulation we get $Ext^1_R(I,I) = \sum_{1\le i,j\le r} K_{ij} + L_{ij}$ *and*

$$K_{ij} = \sum_{p\ge i-j-1, q\ge a_{i-1}-a_{j-1}} \chi^p\eta^q - \sum_{p\ge i-j, q\ge a_{i-1}-a_{j-1}} \chi^p\eta^q - \sum_{p\ge i-j-1, q\ge a_{i-1}-a_j} \chi^p\eta^q + \sum_{p\ge i-j, q\ge a_{i-1}-a_j} \chi^p\eta^q$$

$K_{ij} = \sum_{q\ge a_{i-1}-a_{j-1}} \chi^{i-j-1}\eta^q - \sum_{q\ge a_{i-1}-a_j} \chi^{i-j-1}\eta^q$
$K_{ij} = \sum_{s=a_j}^{a_{j-1}-1} \chi^{i-j-1}\eta^{a_{i-1}-s-1}$
If we compute L_{ij} *similarly we get that*

$$L_{ij} = \sum_{p\ge j-i, q\ge a_j-a_{i-1}} \chi^p\eta^q - \sum_{p\ge j-i, q\ge a_{j-1}-a_{i-1}} \chi^p\eta^q - \sum_{p\ge j-i+1, q\ge a_j-a_{i-1}} \chi^p\eta^q + \sum_{p\ge j-i+1, q\ge a_{j-1}-a_{i-1}} \chi^p\eta^q$$

$L_{ij} = \sum_{s=a_j}^{a_{j-1}-1} \chi^{j-i}\eta^{s-a_{i-1}}$
Using the formulation $Ext^1_R(I,I) = \sum_{1\le i,j\le r} K_{ij} + L_{ij}$ *we get the conclusion*

$$Ext^1_R(I,I) \cong Hom_R(I,R/I) = \sum_{1\le i\le j\le r} \sum_{s=a_j}^{a_{j-1}-1} (\chi^{i-j-1}\eta^{a_{i-1}-s-1} + \chi^{j-i}\eta^{s-a_i})$$

■

Proposition 1.3.39 *$b_{2k}(W(0,0,d)) = P(2d-k, k-d)$.*

Proof 1.3.40 *All subschemes of $\mathbb{P}^2$ corresponding to the points in $W(0,0,d)$ are contained in $Spec\mathbb{C}[x_0/x_2, x_1/x_2]$. We may take $G = T$ and $\lambda = \xi_0\xi_2^{-1}$ and $\mu = \xi_1\xi_2^{-1}$.*
Now choosing a one-parameter subgroup $\phi : \mathbb{G}_m \to G$ given by $\phi(t) = diag(t^{w_0,w_1,w_2})$ where $w_0 < w_1 < w_2$ and $w_0 + w_1 + w_2 = 0$. Then for any character $\chi^\alpha\eta^\beta$ of G we have $\chi^\alpha\eta^\beta \circ \phi(t) = t^{\alpha(w_0-w_2)+\beta(w_1-w_2)}$.
Let U be a cell from the cell decomposition of $Hilb^d(\mathbb{P}^2)$ defined by ϕ and contained in $W(0,0,d)$. The cell U corresponds to a fixed point of G in $Hilb^d(\mathbb{P}^2)$ and contained in $Spec\mathbb{C}[x_0/x_2, x_1/x_2]$, so corresponds to an G-invariant ideal I in $\mathbb{C}[x,y]$ where $x = x_0/x_2$ and $y = x_1/x_2$. By B-B decomposition theorem $dimU = dim\mathbb{T}^+$. Again we have a G-invariant identification $\mathbb{T} = Hom_R(I, R/I)$ where $R = \mathbb{C}[x,y]$, [see {3]. Since $w_o < w_1 < w_2$, it is possible to take the quotient $w_o - w_2/w_1 - w_2$ so large if we need. Then any one dimensional representation $\chi^\alpha\eta^\beta$ has positive weight to the ϕ if and only if $\alpha < 0$ or $\alpha = 0 and \beta < 0$. From lemma 1.27 it follows that

$$\mathbb{T}^+ = \sum_{1\leq i\leq j\leq r}\sum_{s=a_j}^{a_{j-1}-1}\chi^{i-j-1}\eta^{a_{i-1}-s-1} + \sum_{j=1}^{r}\sum_{s=a_j}^{a_{j-1}-1}\eta^{s-a_j-1}$$

. The number of summand in the first sum is $\sum_{i=1}^r\sum_{j=1}^r(a_{j-1}-a_j) = \sum_{i=1}^r a_{i-1} = d$ and in the second sum $\sum_{j=1}^r(a_{j-1}-a_j) = b_0$. So the $dimU = dim\mathbb{T}^+ = d + b_0$. Since there is a one-to-one correspondence between invariant ideals of length d and partitions $a_0 \geq a_1 \geq \ldots \geq a_r = 0$ of d, and $b_{2k}(W(0,0,d))$ is equal to the number of k-dimensional cells $b_{2k}(W(0,0,d))$ is the number of partitions of $2d-k$ in parts bounded by $k-d$.
Using same method in the case $W(0,0,d)$ it can be proved the other two cases i.e $b_{2k}(W(d,0,0)) = P(k, d-k)$ and $b_{2k}(W(0,d,0)) = P(d)$. ∎

Remark 1.3.41 *Since $W(0,0,d) = Hilb^d(\mathbb{A}^2)$,the (iii) part of Theorem 1.24 is proved.*

If these computations are put in Lemma 1.26 we get the (ii) part of Theorem 1.24. ∎

CHAPTER 2

A G_a-ACTION ON $Hilb^d(\mathbb{P}^2)$ AND ITS FIXED LOCUS

2.1 Multiplicative and Additive Group Actions on the Hilbert Scheme

Let G_m and G_a denote the 1-dimensional connected multiplicative and additive algebraic groups respectively. The group $GL(3, \mathbb{C})$ acts on $\mathbb{C}^3$ by matrix multiplication. Take linear representations of G_m and G_a on $\mathbb{C}^3$,then view the images as subgroups of $GL(3, \mathbb{C})$. So these representations induce actions on $\mathbb{P}^2$. Let $T \subseteq GL(3, \mathbb{C})$ be the maximal torus consisting of the diagonal matrices and $\lambda_0, \lambda_1, \lambda_2$ denote the characters of T such that any element $t \in T$ is given by;

$$t = diag(\lambda_0(t), \lambda_1(t), \lambda_2(t))$$

Let x_0, x_1, x_2 be the homogenous coordinates of $\mathbb{P}^2$. The torus T acts on $\mathbb{P}^2$ by;

$$\lambda : T \times \mathbb{P}^2 \to \mathbb{P}^2$$

$$(t, [x_0 : x_1 : x_2]) \to [\lambda_0(t)x_0 : \lambda_1(t)x_1 : \lambda_2(t)x_2]$$

This action of T on $\mathbb{P}^2$ induces an action on $Hilb^d(\mathbb{P}^2)$ because T acts on the homogenous ideals in $\mathbb{C}[x_0, x_1, x_2]$.

Now we will give some definitions and facts about (G_m, G_a)- varieties. These definitions and facts based on references [12], [13], [14].

Definition 2.1.1 *Let X be a nonsingular n-dimensional complex projective variety having G_m and G_a-actions*

$$\lambda : G_m \times X \to X, ((t, x) \to \lambda(t) \cdot x)$$

$$\theta : G_a \times X \to X, ((u, x) \to \theta(u) \cdot x)$$

such that

(i) The G_a-action has a unique fixed point,say s_0

(ii) There exists a positive integer $p \geq 1$ such that

$\lambda(t) \cdot \theta(u) \cdot \lambda(t^{-1}) = \theta(t^p \cdot u)$ for all $t \in G_m$ and $u \in G_a$.

We call such a variety X a (G_m, G_a)-variety.

In [12], it is shown that if X is a (G_m, G_a)-variety then the fixed points X^{G_m} of the G_m-action is a finite set of points and $s_0 \in X^{G_m}$.
Let $X^{G_m} = (s_0, s_1, ..., s_r)$.

Theorem 2.1.2 (Bialynicki – Birula decomposition) *Let X be a smooth projective variety over an algebraically closed field k with an action of G_m. Suppose that the fixed point set of the G_m-action is the finite set $(s_0, s_1, ..., s_r)$. Let*

$$X_i^- := \{x \in X \mid \lim_{t \to \infty} t.x = s_i\} \quad for \quad i = 1, \cdots, r$$

Then X has a cell decomposition such that its cells are the X_i^-.

Proof 2.1.3 *See [9], [10].* ■

The G_m-action λ induces an action $d\lambda$ of G_m, via derivation, on the tangent space $T_{s_i}(X)$ of X at the fixed points s_i for all $i = 0, ..., r$. By [5], all the weights of the induced action on $T_{s_i}X$ are nonzero. Let $T_{s_i}(X)^-$ denote the negative weight space of the induced action on $T_{s_i}(X)$.

Theorem 2.1.4 *(i) $T_{s_0}(X) = T_{s_0}(X)^-$*

(ii)Each minus cell X_i^- is G_m-equivariantly isomorphic to $T_{s_i}(X)^-$

(iii)X_0^- is Zariski open in X.

Proof 2.1.5 *See [9], [10].* ■

Now let V be the holomorphic vector field associated to θ i.e $V = \frac{d\theta}{du} |_{u=0}$, and let Z be the zero scheme of V. In [14], it is shown that the fixed point scheme X^{G_a} is equal to Z as a scheme and the support of Z is (s_0). From the identity $\lambda(t).\theta(u).\lambda(t^{-1}) = \theta(t^p.u)$ we can say that the fixed point scheme X^{G_a} is a G_m-invariant closed subscheme of X. So Z is a G_m-invariant subscheme of X_0^-. After this point let $X_0^- = U$. The G_m-action λ on the affine space U induces a G_m- action on the coordinate ring $A(U)$ of U as below;

$$(\lambda(t) \cdot f)(x) = f(\lambda(t^{-1}) \cdot x)$$

This G_m-action induces a graded algebra structure on $A(U) = \bigoplus_{l=0}^{\infty} A(U)_l$, where

$$A(U)_l = (f \in A(U) : \lambda(t) \cdot f = t^l \cdot f \quad \forall t \in G_m)$$

is the l-weight space. Then the coordinate ring $A(Z)$ of Z has a graded algebra structure, since Z is a G_m-invariant subscheme of U. The ideal $I(Z)$ is homogenous ideal in $A(U)$.
By theorem 1.1.7, we can identify $A(U)$ with $Sym(T_{s_0}(X)^\vee)$ as follows: If $e_1, e_2, ..., e_n$ is an eigenbasis of $T_{s_0}(X)$ with weights $a_1, a_2, ..., a_n$, respectively, and $x_1, x_2, ..., x_n$ is the dual basis, then $Sym(T_{s_0}(X)^\vee) = \mathbb{C}[x_1, ..., x_n]$ such that each x_i is homogenous of degree $-a_i$. Since all weights of the G_m-action on T_{s_0} are negative, each x_i has positive degree.

Viewing V as a derivation of $S = \mathbb{C}[x_1, ..., x_n]$, define $\phi_i = V(x_i)$. So $V = \Sigma \phi_i e_i$

Lemma 2.1.6 *$\phi_1, \phi_2, ..., \phi_n$ is a homogenous regular sequence in S with $deg(\phi_i) = deg(x_i) + p$.*

Proof 2.1.7 *See, [13]* ■

The following lemma is a general rule for the factorization of the Poincare polynomial of a graded algebra satisfying certain conditions.

Lemma 2.1.8 *Let S be the polynomial ring $\mathbb{C}[x_1, ..., x_n]$ graded by the degree of $x_i = b_i \geq 1 \quad for \quad i = 1, \ldots, n$. If $\phi_1, \phi_2, ..., \phi_n$ is a homogenous regular sequence in S, then the Poincare polynomial $P(S/(\phi_1, \phi_2, ..., \phi_n), t)$ of the graded algebra $S/(\phi_1, \phi_2, ..., \phi_n)$ has the following factorization:*

$$P(S/(\phi_1, \phi_2, ..., \phi_n), t) = \prod_{i=1}^{n} \frac{1 - t^{deg(\phi_i)}}{1 - t^{deg(x_i)}}.$$

The basic result for a (G_m, G_a)-variety X is:

Theorem 2.1.9 *There exists an algebra isomorphism $\varphi : A(Z) \to H^*(X, \mathbb{C})$ which carries $A(Z)_{pi}$ onto $H^{2i}(X, \mathbb{C})$. In particular $A(Z)_l$ is trivial unless $l = ip$ for some $i, 0 \leq i \leq n$.*

Proof 2.1.10 *See [12] and [13].* ■

An easy corollary for a (G_m, G_a)-variety X is the following: The Poincare polynomial $P(X, t^{p/2})$ of X has the following factorization:

$$P(X, t^{p/2}) = \prod_{i=1}^{n} \frac{1 - t^{p-a_i}}{1 - t^{-a_i}}$$

2.2 G_a-Fixed Point Locus

2.3 Questions about the G_a-fixed locus of $Hilb^d(\mathbb{P}^2)$

The G_a and G_m-actions on $\mathbb{P}^2$ naturally induce compatible G_a and G_m-actions on $Hilb^d(\mathbb{P}^2)$. However, $Hilb^d(\mathbb{P}^2)$ is not a (G_m, G_a)-variety in the manner described, since in general there is more than one G_a-fixed point. The most general fact known for the fixed point scheme X^{G_a} in general is that X^{G_a} is connected. Nevertheless it is interesting to answer the following questions in order to uncover the structure of $Hilb^d(\mathbb{P}^2)$,

i) What is the dimension of fixed locus of the G_a-action on a given smooth scheme X?

ii) If the fixed locus 1-dimensional then is it rational?

iii) What can be said about the G_a-fixed locus in question(irreducible, rational, uniruled etc.)?

In this thesis we try to get a path to questions formulated above.

2.3.1 An Example

In this section we'll try to see the fixed points, the weights of the G_m-action and the factorization pattern of Poincare polynomial by assuming that the G_a-action has unique fixed point on $Hilb^2(\mathbb{P}^2)$, the Hilbert scheme of 2-points in projective space over complex field.We conjecture that there is indeed a unique fixed point.

Let us fix the $\mathbb{G}_a$ and $\mathbb{G}_m$ -action on $\mathbb{P}^2$ as:

$$\theta : \mathbb{G}_a \times \mathbb{P}^2 \longrightarrow \mathbb{P}^2$$

$$\left(\begin{bmatrix} 1 & a & a^2/2 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}, [x_0 : x_1 : x_2] \right) \rightarrow [x_0 + ax_1 + \frac{a^2}{2}x_2 : x_1 + ax_2 : x_2]$$

$$\lambda : \mathbb{G}_m \times \mathbb{P}^2 \longrightarrow \mathbb{P}^2$$

$$(t, [x_0 : x_1 : x_2]) \to [x_0 : tx_1 : t^2 x_2]$$

where $a \in \mathbb{G}_a$, $t \in \mathbb{G}_m$ and $[x_0 : x_1 : x_2]$ are the homogenous coordinates of $\mathbb{P}^2$

$[1 : 0 : 0], [0 : 0 : 1], [0 : 1 : 0]$ are G_m fixed points and $[1 : 0 : 0]$ is the unique G_a-fixed point.

Let us define U as the open set $(x_0 \neq 0)$ and say $x = \frac{x_1}{x_0}$, $y = \frac{x_2}{x_0}$ are the affine coordinates of U. Then the induced G_m and G_a-actions are of the following form:

$$\theta : \mathbb{G}_a \times \mathbb{A}^2 \longrightarrow \mathbb{A}^2$$

$$(a, (x, y)) \longrightarrow \left(\frac{x + ay}{1 + ax + \frac{a^2}{2}y}, \frac{y}{1 + ax + \frac{a^2}{2}y}\right)$$

$$\lambda : \mathbb{G}_m \times \mathbb{A}^2 \longrightarrow \mathbb{A}^2$$

$$(t, (x, y)) \longrightarrow (tx, t^2 y)$$

Using these actions on $\mathbb{A}^2$ we will try to compute the fixed locus of $Hilb^2(\mathbb{A}^2)$. It is equivalent to finding the G_a-invariant ideals I in $R = \mathbb{C}[[x, y]]_{(1+ax+\frac{a^2}{2}y)}$ such that $dim_{\mathbb{C}} \frac{R}{I} = 2$ by definition of the Hilbert scheme. We want to look at the G_a and G_m-fixed locus. Since these ideals are G_m-invariant they must be monomial ideals. Also we look at the ideals supported at $(0, 0)$ since $(0, 0)$ is the only G_a-fixed point.

Let us consider the monomial ideal $I = (x^2, y)$.

Lemma 2.3.1 $dim_{\mathbb{C}} \frac{R}{(x^2,y)} = 2$.

Proof 2.3.2 $(1, x)$ *is a* $\mathbb{C}$*-basis for* $\frac{R}{(x^2,y)}$*. Since* $y, x^2 \in I$ *then* $y, x^2 \equiv 0$ *in* $\frac{R}{(x^2,y)}$*. The only monomials which are not in I are 1 and x. So we can write every element of* $\frac{R}{(x^2,y)}$ *as* $f \equiv \mathbb{C}.1 + \mathbb{C}.x$*. And also 1 and x are* $\mathbb{C}$*-linearly independent.* ■

Lemma 2.3.3 $I = (x^2, y)$ *is* G_a*-invariant.*

Proof 2.3.4 *Under the* G_a*-action* $I = (x^2, y)$ *is sent to the ideal* $J = \left(\frac{(x+ay)^2}{(1+ax+\frac{a^2}{2}y)^2}, \frac{y}{1+ax+\frac{a^2}{2}y}\right)$*. Since* $1 + ax + \frac{a^2}{2}y$ *is invertible in R ,* $J = ((x + ay)^2, y)$ *so* $y \in J$*. Since* $y \in J$ *we can cancel*

out all y terms in $(x + ay)^2$ *so* $x^2 \in I$*. This shows that* $I \subset J$*.*
For the other direction; Since $(x+ay)^2 = \underbrace{x^2}_{\in I} + \underbrace{2ax}_{\in R} \underbrace{y}_{\in I} + \underbrace{y}_{\in R} . \underbrace{y}_{\in I}$*, so* $(x+ay)^2 \in I$ *and also* $y \in I$*,* $J \subset I$*.* ■

We assumed that the G_a-action has a unique fixed point. So $I = (x^2, y)$ is the unique fixed point. We have seen before that $X = Hilb^2(\mathbb{A}^2)$ is smooth and of dimension 4. So in order to apply corollary 2.11, we have to find the G_m-weights of $T_I X$. So we look at all possible first order deformations at the point $I = (x^2, y)$.

Lemma 2.3.5 *The first order deformations at the point* $I = (x^2, y)$ *are as follows;*

$$I^1_\epsilon = (x^2, y - \epsilon)$$

$$I^2_\epsilon = (x^2 - \epsilon, y)$$

$$I^3_\epsilon = (x^2 - \epsilon x, y)$$

$$I^4_\epsilon = (x^2, y + \epsilon x)$$

where $\epsilon^2 = 0$*.*

Proof 2.3.6 *First of all we have to show that these elements are in the Hilbert scheme i.e their lengths are equal to 2. Claim: Length of* I^i_ϵ *is equal to 2 i.e* $dim_{\mathbb{C}} \frac{R[\epsilon]/(\epsilon^2)}{I^i_\epsilon}$ *for* $i = 1, ..., 4$ *and* I^i_ϵ *for all* $i = 1, 2, 3, 4$ *are linearly independent. Let us say* $S = \frac{R[\epsilon]/(\epsilon^2)}{I^i_\epsilon}$

For $i = 1$:
1 and x are not in $(x^2, y - \epsilon)$*.* $\underbrace{(y-\epsilon)}_{\in I^1_\epsilon}\underbrace{(y+\epsilon)}_{\in I^1_\epsilon} = y^2 - \epsilon^2 \equiv y^2$ *in so* $y^2 \in I^1_\epsilon$*. Also,* $\underbrace{(y-\epsilon)}_{\in I^1_\epsilon}\underbrace{(x-\epsilon)}_{\in I^1_\epsilon} = xy - \epsilon x - \epsilon y + \epsilon^2 \equiv -\epsilon x - \epsilon \underbrace{1\epsilon}_{y=1\epsilon} \equiv -\epsilon x$ *so xy is equivalent to x in* S*.* *Therefore every element in* I^1_ϵ *can be written as a* $\mathbb{C}$*-linear combination of 1 and x. i.e* $(1, x)$ *is a basis for* I^1_ϵ*. So the length of* I^1_ϵ *is equal to 2.*

For the case $i = 2$:

Since $y \in I^2_\epsilon$*, xy and* y^2 *are in* I^2_ϵ*.* $x^2 = 1.\epsilon$ *so 1 and* x^2 *are equivalent in* S*. The only remaining elements which are not in I are* $(1, x)$*. Since 1 and x are linearly independent, the length of* I^2_ϵ

is equal to 2.

Similarly, we look at the case $i = 3$:

Since $y \in I$ *we can cancel out all powers of* y *in* S. *Also* $x^2 = \epsilon x$, *so all powers of* x *in* S *can be represented by* x. *So length of* I_ϵ^3 *is equal to 2.*

Finally the case $i = 4$:

Since $y = -\epsilon x$, y *and* x *are equivalent in* S. *So* $\{1, x\}$ *is a basis for* I_ϵ^4.

Now let us show that I_ϵ^i *for all* $i = 1, 2, 3, 4$ *is linearly independent. This is equivalent to show that these 4 deformations are independent in the tangent space of Hilbert scheme.*
Suppose that there exists nonzero $a, b, c, d \in \mathbb{C}$ *such that*

$$\underbrace{(x^2 - a\epsilon x + b\epsilon, y - c\epsilon x + d\epsilon)}_{J} = \underbrace{(x^2, y)}_{I}$$

i.e suppose that they are dependent. where this equivalence is in $\mathbb{C}[[x, y]][\epsilon]/\epsilon^2$.
Since $x^2 \in J$, *then we get that* $a\epsilon x + b\epsilon \in J$. *Let us take into a* $b\epsilon$ *paranthesis. Then,* $a\epsilon x + b\epsilon = b\epsilon(\frac{a}{b}x + 1)$. *Since* $\frac{a}{b}x + 1$ *is invertible in* $\mathbb{C}[[x, y]][\epsilon]/\epsilon^2$ *we get* $b\epsilon \in J = I$. *But* $b\epsilon$ *is in* I *if and only if* $b = 0$.
Similarly, since $y \in J$ *we get that* $-c\epsilon x + d\epsilon = d\epsilon\underbrace{(-\frac{c}{d}x + 1)}_{invertible} \in J$. *So* $d\epsilon$ *is in* $J = I$. *This holds if and only if* $d = 0$.
Since $b = d = 0$, J *is, now, of the form* $J = (x^2 - a\epsilon x, y - c\epsilon x)$. *But since* x^2 *and* y *is in* $I = J$, *so* $a\epsilon x$ *and* $c\epsilon x$ *is in* $J = I$. *In the ideal* I, *the smallest degree of* x *is 2. So* $a\epsilon x$ *and* $c\epsilon x$ *is in* I *if and only if* $a = c = 0$. *So we showed these four deformations are independent. Recall that the tangent space to an ideal in the Hilbert scheme can be identified with the first order deformations of* I, *see [9] section VI. Since* $X = Hilb^2(\mathbb{A}^2)$ *is smooth of dimension 4 there exists no other deformation.* ■

Now let us compute the weights of the G_m-action for each I_ϵ^i.

$$t.I_\epsilon^1 = (t^2x^2, t^2y - \epsilon) = (x^2, y - \epsilon t^{-2}x) = I^1_{\epsilon t^{-2}}$$

$$t.I_\epsilon^2 = (t^2x^2 - \epsilon, t^2y) = (x^2 - t^{-2}\epsilon, y) = I^2_{\epsilon t^{-2}}$$

$$t.I_\epsilon^3 = (t^2x^2 - \epsilon tx, t^2y) = (x^2, \epsilon t^{-1}x, y) = I^3_{\epsilon t^{-1}}$$

$$t.I_\epsilon^4 = (t^2x^2, t^2y + \epsilon tx) = (x^2, y + \epsilon t^{-1}x) = I^4_{\epsilon t^{-1}}$$

So I_ϵ^1 and I_ϵ^2 have weight -2, I_ϵ^3 and I_ϵ^4 have weight -1.

Therefore, the Poincaré polynomial has the following factorization,

$$P(Hilb^2(\mathbb{A}^2), t^{1/2}) = \frac{1-t^2}{1-t}.\frac{1-t^2}{1-t}.\frac{1-t^3}{1-t^2}.\frac{1-t^3}{1-t^2} = t^4 + 2t^3 + 3t^2 + 2t + 1$$

These coefficients agree with the numbers given in table 1 in [8]. This motivates our conjecture that there is a unique G_a-fixed point.

2.4 G_a -invariant monomial ideals

In this section we will answer the following question: "Which monomial ideals in $Hilb^d(\mathbb{P}^2)$ are G_a invariant"?

Lemma 2.4.1 *$I = (x^d, y)$ is G_a-invariant.*

Proof 2.4.2 *Under the G_a-action this ideal goes to $J = (\frac{(x+ay)^d}{(1+ax+\frac{a^2}{2}y)^d}, \frac{y}{1+ax+\frac{a^2}{2}y})$. Now let us check these two ideals are equal in R. We can cancel out denominators $1 + ax + \frac{a^2}{2}y$ and $(1+ax+\frac{a^2}{2}y)^d$ since $1+ax+\frac{a^2}{2}y$ is invertible in R. So it is enough to consider $J = ((x+ay)^d, y)$. It is obvious that $J \subset I$. So we need to show $I \subset J$.*

Look at the power

$$(x + ay)^d = x^d + d.x^{d-1}y + \frac{d.(d-1)}{2}x^{d-2}y^2 + ... + \frac{d.(d-1)}{2}xy^{d-1} + y^d$$

$$= x^d + y\underbrace{(d.x^{d-1} + \frac{d.(d-1)}{2}x^{d-2}y + ... + \frac{d.(d-1)}{2}xy^{d-2} + y^{d-1})}_{f(x,y)}$$

So

$$x^d = x^d + y\underbrace{(d.x^{d-1} + \frac{d.(d-1)}{2}x^{d-2}y + ... + \frac{d.(d-1)}{2}xy^{d-2} + y^{d-1})}_{f(x,y)} - y.f(x.y)$$ *Since* $y \in J$ *then* $x^d \in J$, *therefore we are done.* ■

Lemma 2.4.3 *Ideals of the form* $I = (x^{m_0}, x^{m_1}y, x^{m_2}y^2, ..., x^{m_{k-1}}y^{s-2}, x^{m_k}y^{s-1}, y^s)$ *are* G_a*-invariant where the powers of y are increasing by 1 and the sum of the powers of the x's is equal to d(= length) such that* $m_0 > m_1 > ... > m_{k-1} > m_k > 0$ *are positive integers.*

Proof 2.4.4 *Under the* G_a*-action the ideal I goes to:*

$$J = (\frac{(x+ay)^{m_0}}{(1+ax+\frac{a^2}{2}y)^{m_0}}, \frac{(x+ay)^{m_1}}{(1+ax+\frac{a^2}{2}y)^{m_1}}\frac{y}{(1+ax+\frac{a^2}{2})}, ..., \frac{y^s}{(1+ax+\frac{a^2}{2}y)^s})$$

By the same reason as in proof 5.2 we can forget all denominators, so;

$$J = ((x+ay)^{m_0}, (x+ay)^{m_1}y, (x+ay)^{m_2}y^2, ..., (x+ay)^{m_{k-1}}y^{s-2}, (x+ay)^{m_k}y^{s-1}, y^s)$$

Let us show that $J \subset I$.

$$(x+ay)^{m_0} = \underbrace{x^{m_0}}_{\in I} + \binom{m_0}{1}ax^{m_0-1}y + \binom{m_0}{2}a^2x^{m_0-2}y^2 + ... + a^{m_0}y^{m_0}$$

$$= \underbrace{x^{m_0}}_{\in I} + \binom{m_0}{1}a \underbrace{x^u}_{u+m_1=m_0-1} \underbrace{x^{m_1}y}_{\in I} + \binom{m_0}{2}a^2 \underbrace{x^v}_{v+m_2=m_0-2} \underbrace{x^{m_2}y^2}_{\in I} + ... + a^{m_0} \underbrace{y^s}_{since m_0 \geq s} \underbrace{y^{m_0-s}}_{\in I}$$

so $(x+ay)^{m_0} \in I$.

$$(x+ay)^{m_1}y = \underbrace{x^{m_1}y}_{\in I} + m_1a \underbrace{x^u}_{m_1-1\geq m_2} \underbrace{x^{m_2}y^2}_{\in I} + \binom{m_1}{2}a^2 \underbrace{x^v}_{m_1-2\geq m_3} \underbrace{x^{m_3}y^3}_{\in I} + ... + a^{m_1}y^{m_1-s} \underbrace{y^s}_{\in I}$$ *. i.e* $(x+ay)^{m_1}y \in I$.

$$(x+ay)^{m_k}y^{s-1} = x^{m_k}y^{s-1} + m_kax^{m_k-1}y^s + \binom{m_k}{2}a^2x^{m_k-2}y^{s+1} + ... + a^{m_k}y^{m_k+s-1} = \underbrace{x^{m_k}y^{s-1}}_{\in I} + \underbrace{y^s}_{\in I}(f(x,y))$$

for some polynomial $f(x,y)$.

Thus every generator of J is an element of I. So $J \subset I$.

In the other direction look at the term $(x+ay)^{m_k}y^{s-1}$.

$$\begin{aligned}(x+ay)^{m_k}y^{s-1} &= (x^{m_k} + m_kx^{m_k-1}y + \binom{m_k}{2}a^2x^{m_k-2}y^2 + ... + a^{m_k-1}m_kxy^{m_k-1} + a^{m_k}y^{m_k})y^{s-1} \\ &= x^{m_k}y^{s-1} + y^s\underbrace{(x^{m_k-1} + \binom{m_k}{2}a^2x^{m_k-2}y + ... + a^{m_k-1}m_kxy^{m_k-2} + y^{m_k-1})}_{g(x,y)}\end{aligned}$$

Since $y^s \in J$, $x^{m_k}y^{s-1} = (x+ay)^{m_k}y^{s-1} - y^s.g(x,y) \in J$. $(*)$

Similarly;

$$\begin{aligned}(x+ay)^{m_{k-1}}y^{s-2} &= (x^{m_{k-1}} + m_{k-1}x^{m_{k-1}-1}y + ... + a^{m_{k-1}-1}m_{k-1}xy^{m_{k-1}-1} + a^{m_{k-1}}y^{m_{k-1}})y^{s-2} \\ &= x^{m_{k-1}}y^{s-2} + x^{m_{k-1}-1}y^{s-1} + y^s(h(x,y))\end{aligned}.$$

Since $m_{k-1}-1 \geq m_k$, *write* $x^{m_{k-1}-1}y^{s-1}$ *as* $x^{m_k-1}y^{s-1} = x^{m_{k-1}-m_k-1}.\underbrace{x^{m_k}y^{s-1}}_{\in J(*)}$

So

$$J \ni x^{m_{k-1}}y^{s-2} = (x+ay)^{m_{k-1}}y^{s-2} - \underbrace{y^s}_{\in J}\underbrace{h(x,y)}_{\in \mathbb{C}[x,y]} - x^{m_{k-1}-m_k-1}.\underbrace{x^{m_k}y^{s-1}}_{\in J}$$

Applying the same process inductively for each monomial generator of J we get a monomial generator of I. Thus $J \subset I$, *and consequently,* $I = J$. ■

Lemma 2.4.5 *Suppose that the ideal is of the form*
$I = (x^{m_0}, x^{m_1}y^{s_1}, x^{m_2}y^{s_2}, ..., x^{m_s}y^{s_k}, x^{m_{s+1}}y^{s_{k+1}}, ..., x^{m_{l-1}}y^{s_{l-1}}, y^{s_l})$
where $m_0 > m_1 > ... > m_{s+1} > ... > m_l > 0$ *and* $1 < 2 < ... < s_k < s_{k+1} < ... < s_n$.
If there exists at least one consequtive pair $(s_k, s_{k+1}), s_k \geq 1$ *in the power of y such that* $s_{k+1} - s_k \geq 2$ *then the ideal I is not* G_a*-invariant.*

Proof 2.4.6 *Under the* G_a*-action I is sent to:*

$$J = (\frac{(x+ay)^{m_0}}{(1+ax+\frac{a^2}{2}y)^{m_0}}, ..., \frac{(x+ay)^{m_s}y^{s_k}}{(1+ax+\frac{a^2}{2}y)^{m_s+s_k}}, \frac{(x+ay)^{m_{s+1}}y^{s_{k+1}}}{(1+ax+\frac{a^2}{2}y)^{m_{s+1}+s_{k+1}}}, ..., \frac{y^{s_n}}{(1+ax+\frac{a^2}{2}y)^{s_n}}).$$

Forgetting denominators;

$$J = ((x+ay)^{m_0}, ..., (x+ay)^{m_s}y^{s_k}, (x+ay)^{m_{s+1}}y^{s_{k+1}}, ..., y^{s_n}).$$

Suppose that $I = J$ *and the ideal I contains consequtive terms,* $x^{m_s}y^k$ *and* $x^{m_{s+1}}y^u$ *such that* $u \geq k+2$.

Under the G_a*-action these terms are sent to* $(x+ay)^{m_s}y^k \in J$ *and* $(x+ay)^{m_{s+1}}y^u \in J$, *respectively.*

Now let us expand the term, $(x+ay)^{m_s}y^k \in J$

$$\underbrace{(x+ay)^{m_s}y^k} = \underbrace{x^{m_s}y^k + am_s x^{m_s-1}y^{k+1} + \binom{m_s}{2}x^{m_s-2}y^{k+2} + ... + a^{m_s}y^{m_s+k}}_{\in J}.$$

Since we have assumption that $I = J$ *so,*
$\underbrace{x^{m_s}y^k + am_s x^{m_s-1}y^{k+1} + \binom{m_s}{2}x^{m_s-2}y^{k+2} + ... + a^{m_s}y^{m_s+k}}_{\in I}$. *Therefore we can write this term as:*

$x^{m_s}y^k + am_s x^{m_s-1}y^{k+1} + \binom{m_s}{2}x^{m_s-2}y^{k+2} + ... + a^{m_s}y^{m_s+k} = x^{m_0}f_0(x,y) + x^{m_1}yf_1(x,y) + ... + y^s f_s(x,y)$, then $am_s x^{m_s-1}y^{k+1} = x^{m_0}f_0(x,y) + x^{m_1}yf_1(x,y) + ... + y^s f_s(x,y) - x^{m_s}y^k - \binom{m_s}{2}x^{m_s-2}y^{k+2} - ... - a^{m_s}y^{m_s+k}$.

If we consider this equality modula the ideal $P = (x^{m_s}, y^{k+2})$ and forgetting the coefficients, we have that

$$\underbrace{x^{m_s-1}y^{k+1}}_{\neq 0 (mod P)} = \underbrace{x^{m_0}f_0(x,y) + ... + y^s f_s(x,y) - x^{m_s}y^k - x^{m_s-2}y^{k+2} - ... - y^{m_s+k}}_{=0(mod P)},$$

which is a contradiction. Therefore the ideals I in the lemma are not G_a-invariant. ■

2.4.1 Patterns of Degeneration

In lemma 5.2 and 5.3 we determined all the G_a-invariant monomial ideals. Since the G_a-fixed point scheme is connected these monomial ideals must be connected at least via some curves in the Hilbert scheme. Here we will demonstrate explicitly rational curves that connect monomial ideal locus.

Example 2.4.7 *Let us fix the length as d. If $I = (x^d, y)$ and $J = (x^{d-1}, xy, y^2)$ we can connect I and J by the degeneration $J_t = ((x^{d-1} + ty, xy, y^2)$, and J_t is G_a-invariant for all $t \in \mathbb{C}$.*

Proof 2.4.8 *First of all $dim_{\mathbb{C}} \frac{R}{J_t} = d$.*
Now let us show that J_t is G_a-invariant. Under the G_a-action, J_t is sent to

$$J_t^{'} = (\frac{(x+ay)^{d-1}}{(1+ax+\frac{a^2}{2}y)^{d-1}} + t\frac{y}{1+ax+\frac{a^2}{2}y}, \frac{x}{1+ax+\frac{a^2}{2}}\frac{y}{1+ax+\frac{a^2}{2}y}, \frac{y^2}{(1+ax+\frac{a^2}{2}y)^2})$$

Since we work in $\mathbb{C}[[x,y]]_{1+ax+\frac{a^2}{2}y}$ we cancel out the denominators. After some simplification, we get

$$J_t^{'} = ((x+ay)^{d-1} + ty(1+ax+\frac{a^2}{2}y)^{d-2}, xy, y^2).$$

Look at the power expansion of $(x+ay)^{d-1} + ty(1+ax+\frac{a^2}{2}y)^{d-2}$:
$(x+ay)^{d-1} + ty(1+ax+\frac{a^2}{2}y)^{d-2} = x^{d-1} + xyf(x,y) + a^{d-1}y^{d-1} + ty + xy.g(x,y) + ty^{d-1}$ for certain polynomials $f(x,y)$, $g(x,y)$.
Since $xy \in J_t^{'}$ and $d-1 \geq 2$,

$$x^{d-1} + ty = (x+ay)^{d-1} + ty(1+ax+\frac{a^2}{2}y)^{d-2} - \underbrace{xy}_{\in J_t^{'}}(f(x,y)+g(x,y)) - \underbrace{y^2}_{\in J_t^{'}}(a^{d-1}y^{d-3} + ty^{d-3}).$$

i.e $x^{d-1} + ty \in J_t^{'}$.*This shows that* $J_t \subset J_t^{'}$.

Conversely, we will show that $J_t^{'} \subset J_t$.

Since xy *and* y^2 *are in both* $J_t^{'}$*and* J_t, *it is enough to check that* $(x+ay)^{d-1} + ty(1+ax+\frac{a^2}{2}y)^{d-2}$ *is in* J_t *or not. Let us call this term as* $z(x,y,t)$*,then*

$$\begin{aligned} z(x,y,t) &= x^{d-1} + xy(f(x,y)) + a^{d-1}y^{d-1} + ty[(1+ax)^{d-2} + ... + (\tfrac{a^2}{2})^{d-2}y^{d-2}] \\ &= x^{d-1} + xyf(x,y) + a^{d-1}y^{d-1} + ty[1 + xg(x) + \tfrac{a^2}{2}y + xyh(x,y) + ... + (\tfrac{a^2}{2})^{d-2}y^{d-2}] \\ &= x^{d-1} + xy(f(x,y)) + a^{d-1}y^{d-1} + ty + xy(tg(x)) + \tfrac{a^2}{2}y^2 + ... + t(\tfrac{a^2}{2}y^{d-1}) \\ &= \underbrace{x^{d-1} + ty}_{\in J_t} + \underbrace{xy}_{\in J_t}(f^{'}(x,y)) + \underbrace{y^2}_{\in J_t}(g^{'}(y)) \end{aligned}$$

where $f^{'}(x,y) = f(x,y) + tyh(x,y) + tg(x)$ *and for certain polynomials* $f^{'}(x,y), g^{'}(y)$. *Here we can take* y^2 *parentheses since* $d-1 \geq 2$.

So $J_t^{'} \subset J_t$. *Therefore,* $J_t^{'}$ *is* G_a*-invariant.i.e* $J_t^{'} = J_t$.

For $t = 0$ *we get* J.*When* $t \neq 0$, $J_t = (\frac{(x^{d-1}}{t} + y, xy, y^2)$. *If we take the limit* $t \to \infty$ *we see that* $y \in$ *the limiting ideal. Also* $x(x^{d-1} + ty) - y(xy) = x^d$, *so* x^d *is in* J_t *for all* t *hence in the limiting ideal. So the limit of* J_t *is equal to* J. *Namely we found a rational curve in* $Hilb^d(\mathbb{P}^2)$ *connecting* (x^d, y) *to* (x^{d-1}, xy, y^2). ■

Remark 2.4.9 *If* $I = (x^u, \underbrace{x^{m_1}y}_{1^{st}-position}, ..., \underbrace{x^{m_k}y^k}_{k^{th}-position}, ..., x^{m_{s-1}}y^{s-1}, y^s)$ *then there are two cases for the ideal* J *such that its first monomial is of the form* x^{u-1}. *The first case is that the other monomial generators of* J *is equal to monomial generators of* I *except one monomial generator since length of* I= *length of* J. *Say the difference is at position* $k, 1 \leq k \leq s$. *This means that* $x^{m_k}y^k \in I$ *is different from* $x^{m_k^{'}}y^k \in J$.*Since* I *and* J *have the same length we must have* $m_k^{'} - m_k = 1$.

Lemma 2.4.10 *If*

$$I = (x^u, x^{m_1}y, x^{m_2}y^2, ..., x^{m_k}y^k, ..., x^{m_{s-1}}y^{s-1}, y^s)$$

and

$$J = (x^{u-1}, x^{m_1}y, x^{m_2}y^2, ..., x^{m_k^{'}}y^k, ..., x^{m_{s-1}}y^{s-1}, y^s)$$

such that $1 \leq k \leq s$ *Then there a exists a rational curve in* $Hilb^d(\mathbb{P}^2)$ *connecting* I *and* J, *all of whose points are* G_a*-invariant. Explicitly, its points are:*

$$J_t = (x^{u-1} + tx^{m_k}y^k, x^{m_1}y, x^{m_2}y^2, ..., x^{m_k^{'}}y^k, ..., x^{m_{s-1}}y^{s-1}, y^s).$$

Proof 2.4.11 *Let us show that J_t is G_a-invariant. Forgetting and equalizing denominators, J_t is sent to:*

$$J_t^{'} = ((x+ay)^{u-1} + t(x+ay)^{m_k}y^k(1+ax+\frac{a^2}{2}y))^{u-m_k-k-1}, (x+ay)^{m_1}y, ..., (x+ay)^{m_k^{'}}y^k, ..., y^s)$$

Start from $(x+ay)^{m_{s-1}}y^{s-1} = x^{m_{s-1}}y^{s-1} + y^s(f(x,y))$. Subtracting $\underbrace{y^s}_{\in J_t^{'}}(f(x,y))$ from $(x+ay)^{m_{s-1}}y^{s-1}$ we see that $x^{m_{s-1}}y^{s-1}$ is in $J_t^{'}$. Similarly, $(x+ay)^{m_{s-2}}y^{s-2} = x^{m_{s-2}}y^{s-2} + x^{m_{s-2}-1}y^{s-1} + y^s(g(x,y))$. Since $m_{s-2} - 1 \geq m_{s-1}$ write this one as:

$$(x+ay)^{m_{s-2}}y^{s-2} = x^{m_{s-2}}y^{s-2} + x^{m_{s-2}-1-m_{s-1}}(x^{m_{s-1}}y^{s-1}) + y^s(g(x,y)).$$

So we get that $x^{m_{s-2}}y^{s-2}$ is in $J_t^{'}$. Continuing like that we see easily that $x^{m_1}y, x^{m_2}y^2, ..., x^{m_k^{'}}y^k, ..., x^{m_{s-1}}y^{s-1}, y^s$ are in $J_t^{'}$.

Let us expand the term:

$(x+ay)^{u-1} + t(x+ay)^{m_k}y^k(1+ax+\frac{a^2}{2}y))^{u-m_k-k-1}$. Let us call it as $z(x,y,t)$

$$\begin{aligned} z(x,y,t) &= x^{u-1} + ... + y^{u-1} + t(x+ay)^{m_k}y^k(1+x(f(x)+y(g(x,y)))) \\ &= x^{u-1} + ax^{u-2}y + ... + y^{u-1} + t(x^{m_k} + ... + y^{m_k})y^k(1+x(f(x)+y(g(x,y)))) \quad (*) \end{aligned}$$

Since $u-i \geq m_i$,$u-1 \geq s$ we can cancel the terms $x^{u-2}y, ..., y^{u-1}$: We can rewrite (*) as:

$x^{u-1} + tx^{m_k}y^k + t(ax^{m_k-1}y + ... + y^{m_k})(y^k + xy^kf(x) + y^{k+1}g(x,y))$

$x^{u-1} + tx^{m_k}y^k + \underbrace{tax^{m_k-1}y^{k+1} + tax^{m_k}y^{k+1}f(x) + ... + y^{m_k+k+1}g(x,y)}$

We can cancel out all the underbraced terms by the monomials $x^{m_k}y^k, x^{m_{k+1}}y^{k+1}, ..., y^s$ by degree reasons, $m_{k+i} \leq m_k$for all $i \geq 0$ and $m_k + k \geq s$.

So $x^{u-1} + tx^{m_k}y^k \in J_t^{'}$.Therefore $J_t \subset J_t^{'}$.

Now we will show that $J_t \supset J_t^{'}$

$$\begin{aligned} (x+ay)^{m_k}y^k &= x^{m_k}y^k + am_kx^{m_k-1}y^{k+1} + ... + a^{m_k}y^{m_k+k} \\ &= \underbrace{x^{m_k}y^k}_{\in J_t} + am_k \underbrace{x^u}_{m_{k+1}+u=mk-1} \underbrace{x^{m_{k+1}}y^{k+1}}_{\in J_t} + ... + a^{m_k} \underbrace{y^v}_{m_k+k=v+s} \underbrace{y^s}_{\in J_t}. \end{aligned}$$

So $(x+ay)^{m_k}y^k \in J_t$ for all $1 \leq k \leq s-1$

For the term , $(x+ay)^{u-1} + t(x+ay)^{m_k}y^k(1+ax+\frac{a^2}{2}y)^{\overbrace{u-m_k-k-1}^{p}}$, if we expand powers and put the necessary parenthesis it can be written of the the following form:

$x^{u-1} + ax^{u-2}y + a^2x^{u-3}y^2 + ... + a^{u-1}y^{u-1} + ty^k(x^{m_k} + x^{m_k-1}y + ... + y^{m_k})(1+f(x,y))$ for certain polynomial $f(x,y)$. It is equal to the term $\underbrace{x^{u-1} + tx^{m_k}y^k}_{\in J_t} + t \underbrace{x^{m_k-1}y^{k+1}}_{\in J_t, m_k-1 \geq m_{k+1}} (1+f(x,y)) + ... + t \underbrace{y^{m_k+k}}_{\in J_t, m_k+k \geq s} (1+f(x,y)) + \underbrace{ax^{u-2}y + a^2x^{u-3}y^2 + ... + a^{u-1}y^{u-1}}_{\in J_t}$.

So all generators of the $J_t^{'}$ *is also in* J_t. *Hence* $J_t^{'} \subset J_t$. *Therefore,* $J_t^{'}$ *is* G_a *invariant. Also length(I)= length(* $J_t^{'}$*). Let's show that in the limiting positions* $J_t^{'}$ *gives I and J. For* $t = 0$ *we have J. If* $t \neq 0$,

$$x(x^{u-1} + tx^{m_k}y^k) - t(x^{m_k^{'}}y^k) = x^u + tx^{\overbrace{m_k + 1}^{=m_k^{'}}}y^k - t(x^{m_k^{'}}y^k) = x^u.$$

So x^u *is in* J_t *for all t, hence also in the limiting ideal.*
When $t \to \infty$, $\frac{x^{u-1}}{t} + x^{m_k}y^k \to x^{m_k}y^k$, *therefore* $x^{m_k}y^k$ *is in the limiting ideal. We have had an assumption that all monomials in I and J are equal except one monomial. Therefore we can connect I and J by* J_t. ■

Remark 2.4.12 *If* $I = (x^u, x^{m_1}y, ..., x^{m_k}y^k, ..., x^{m_{s-1}}y^{s-1}, y^s)$ *then the second case for the ideal J such that it's first monomial is* x^{u-1} *is that;*

$$J = (x^{u-1}, x^{m_1}y, ..., x^{m_k}y^k, ..., x^{m_{s-1}}y^{s-1}, \underbrace{xy^s}_{newterm}, y^{s+1})$$

Lemma 2.4.13 *The ideals I and J in remark 6.6 are connected with the rational curve of* G_a*-invariant ideals*

$$J_t = (x^{u-1} + ty^s, x^{m_1}y, ..., x^{m_k}y^k, ..., x^{m_{s-1}}y^{s-1}, xy^s, y^{s+1}).$$

Proof 2.4.14 *First we need to show that* J_t *is* G_a*-invariant.*
Applying the action we get

$$J_t^{'} = ((x + ay)^{u-1} + ty^s(1 + ax + \frac{a^2}{2}y)^{u-s-1}, (x + ay)^{m_1}y, ..., (x + ay)y^s, y^{s+1})$$

$(x + ay)y^s = xy^s + ay^{s+1}$ *so subtracting a* $\underbrace{y^{s+1}}_{\in J_t^{'}}$ *we see that* $xy^s \in J_t^{'}$.

$x^{m_s}y^{s-1} = (x + ay)^{ms}y^{s-1} - x^{m_{s-2}}\underbrace{(xy^s)}_{\in J_t^{'}} - f(x, y)\underbrace{y^{s+1}}_{\in J_t^{'}}$.

Therefore $x^{m_s}y^{s-1} \in J_t^{'}$.
Continuing this process we see that all monomial generators of $J_t^{'}$ *up to up to* $(x + ay)^{u-1} + ty^s(1 + ax + \frac{a^2}{2}y)^{u-s-1}$ *is in* $J_t^{'}$.
Now consider the term:$(x + ay)^{u-1} + ty^s(1 + ax + \frac{a^2}{2}y)^p$. *Let us call it as* $z(x, y, t)$ *then*

$$\begin{aligned} z(x, y, t) &= x^{u-1} + x^{u-2}y + ... + y^{u-1} + ty^s((1 + ax)^p + (1 + ax)^{p-1}y + ... + y^p) \\ &= x^{u-1} + x^{u-2}y + ... + y^{u-1} + ty^s(1 + ax)^p + y^{s+1}(f(x, y)) \\ &= x^{u-1} + xy^{u-2} + y^{u-1} + ty^s + xy^s(g(x)) + y^{s+1}(f(x, y)) \end{aligned}$$

We can cancel out the term $y^{s+1}(f(x,y))$ since $y^{s+1} \in J_t'$.
Since $u-2 \geq m_1, u-3 \geq m_2, ..., u-1 \geq s$, we can also cancel out $x^{u-2}y, ..., xy^{u-2}, y^{u-1}$ by elements of J_t', $x^{m_1}y, x^{m_2}y^2, ..., xy^s, y^{s+1}$ as a conclusion we get that $x^{u-1} + ty^s$ is in J_t'. So J_t is invariant under the G_a-action. The length of J_t' is equal to length of I. Since $x(x^{u-1} + ty^s) - txy^s = x^u$, so the term x^u is in the limiting ideal. For $t = 0$ we get the ideal J.
For $t \neq 0$, if take the limit of $\frac{x^{u-1}}{t} + y^s$ as $t \to \infty$ we get y^s. Other monomials $x^{m_1}y, x^{m_2}y^2, ..., x^{m_{s-1}}y^{s-1}$ are also in limiting ideal. Since y^s is in we can forget the monomial xy^s. Therefore the limiting ideal gives us the ideal I. ■

Example 2.4.15 *For length $d = 8$ we will show all invariant monomial ideals and their degenerations described above.*
$I_1 = (x^8, y)$
$I_2 = (x^7, xy, y^2)$
$I_3 = (x^6, x^2y, y^2)$
$I_4 = (x^5, x^3y, y^2)$
$I_5 = (x^5, x^2y, xy^2, y^3)$
$I_6 = (x^4, x^3y, xy^2, y^3)$

$$I_1 \overbrace{\longleftarrow}^{(x^7+ty,xy,y^2)} I_2 \overbrace{\longleftarrow}^{(x^6+txy,x^2y,y^2)} I_3 \overbrace{\longleftarrow}^{(x^5+tx^2y,x^3y,y^2)} I_4 \overbrace{\longleftarrow}^{(x^4+ty^2,x^3y,xy^2,y^3)} I_6$$

also we can connect I_5 and I_3 by the degeneration $(x^5+ty^2, x^2y, xy^2, y^3)$. So we get a connected graph for the fixed point locus of G_a.

The followings figures are Ferrer's diagrams of $I_i's$.

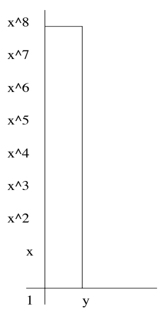


Figure 2.1: Figure of I_1

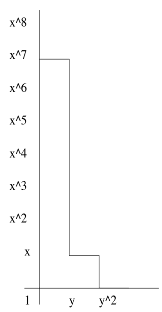


Figure 2.2: Figure of I_2

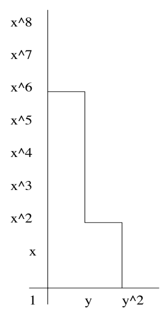


Figure 2.3: Figure of I_3

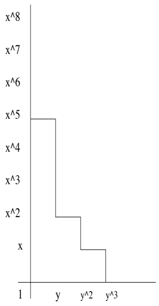


Figure 2.4: Figure of I_4

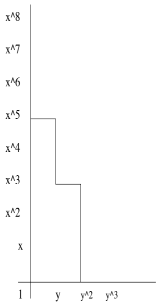


Figure 2.5: Figure of I_5

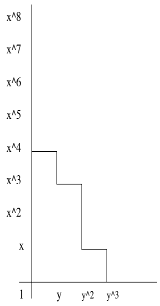
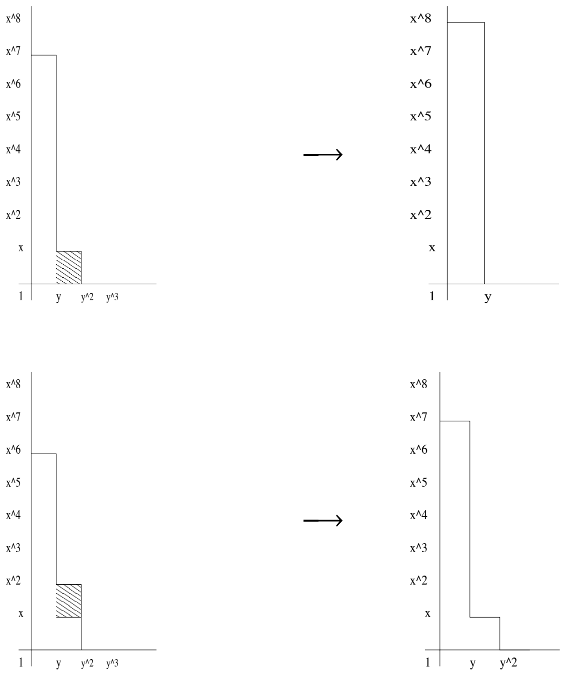
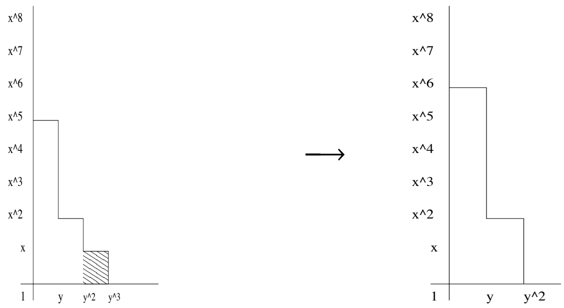


Figure 2.6: Figure of I_6

Now we will try to see how we can connect each figure to others by playing on figures: For example in order to connect I_2 to I_1, we take out the shaded region in the figure of I_2 and put it into first column of figure of I_2 to get the figure of I_1. This gives a quick visulation of degeneration. On figures we have;

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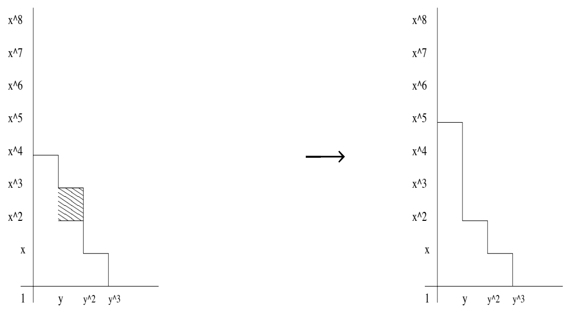
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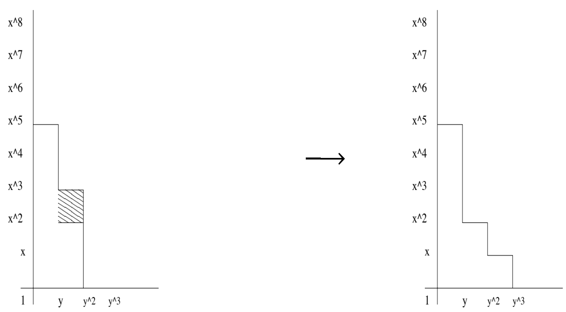
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2.4.2 Connectedness of $Hilb^d(\mathbb{P}^2, 0)$

Lemma 2.4.16 *Every ideal of the form*

$$I =< x^{s_0}, x^{s_1}y^{m_1}, ..., x^{s_k}y^{m_k}, x^{s_{k+1}}y^{m_{k+1}}, ..., y^{m_p} >$$

where $s_i > s_j$ for $i < j$ and $m_i < m_j$ for $i < j$,can be connected to

$$J =< x^{s_0}, x^{s_1}y^{m_1}, ..., x^{s_k+1}y^{m_k}, x^{s_k}y^{m_k+1}, x^{s_k-1}y^{m_{k+1}-1}, x^{s_{k+1}}y^{m_{k+1}}, ..., y^{m_p} >$$

by a family

$$I_t =< x^{s_0}, x^{s_1}y^{m_1}, ..., x^{s_k}y^{m_k} + tx^{s_k-1}y^{m_{k+1}-1}, x^{s_{k+1}}y^{m_{k+1}}, ..., y^{m_p} >$$

when $m_{k+1} - m_k \geq 2$.

Proof 2.4.17 *First of all the family I_t is a flat family. In the case that $t = 0$ we get the ideal I. Now we will show that when $t \to \infty$ we get the ideal J. If we multiply the term $x^{s_k}y^{m_k} + t.x^{s_k-1}y^{m_{k+1}-1}...(*)$ with y we get that $x^{s_k}y^{m_k+1} + tx^{s_k-1}y^{m_{k+1}}...(**)$. Since $s_k - 1 \geq s_{k+1}$ and $x^{s_{k+1}}y^{m_{k+1}} \in I_t$, the monomial $x^{s_k-1}y^{m_{k+1}}$ is also in I_t. If we subtract the $x^{s_k-1}y^{m_{k+1}}$ from (**) we get that $x^{s_k}y^{m_k+1} \in I_t$. Now let us multiply (*) with x then we have $x^{s_k+1}y^{m_k}+tx^{s_k}y^{m_{k+1}-1}...(***)$. By subtracting $\underbrace{tx^{s_k}y^{m_{k+1}-1}}_{\in I_t}$ from (***) we get that $x^{s_k+1}y^{m_k} \in I_t$. Here $x^{s_k}y^{m_{k+1}-1} \in I_t$, since $m_{k+1} - m_k \geq 2$.Finally if we take the limit of $x^{s_1}y^{m_1} + t.x^{s_l}y^{m_l-1}$ when $t \to \infty$ we see that $x^{s_k}y^{m_k} + t.x^{s_k-1}y^{m_{k+1}-1}$ is in I_∞. So in the case $t \to \infty$ we get that $I_\infty = J$.* ■

Remark 2.4.18 *The above lemma has the following meaning in terms of Ferrer's diagrams of I and J. We can get the diagram of J from the diagram of I by taking out the square which it's left bottom corner at point $(x^{s_k-1}, y^{m_{k+1}-1})$ in the diagram of I and put it into place such that new left bottom corner coordinate of taken square is (x^{s_k}, y^{m_k}). The final diagram is the diagram of J.*

Remark 2.4.19 *In the Ferrer's diagram of a monomial ideal*

$$I =< x^{s_0}, x^{s_1}y^{m_1}, ..., x^{s_k}y^{m_k}, ..., y^{m_p} >$$

we have line segments connecting monomial generators of I. We have horizontal and vertical line segments. Each inner corner point and the intersection points with the axes are monomial *generators of I. At the horizontal line segments we will define* ***steps***. *It is a part of a horizontal line segment such that the difference of powers of y's corresponding to its end points is one. We will give a different positive integer to the left end of each step if there is another step before it. This positive integer is the power of y corresponding to the left end of the step. So we have increasing positive integers from left to right.*

In the lemma above we have seen that we could connect two ideals if one of them can be gotten by the other one by moving one square in Ferrer's diagram. Also before we had determined which kinds of Ferrer's diagrams are invariant under the G_a-action.

Now suppose that we have a general Ferrer's diagram corresponding to an arbitrary ideal

$$I =< x^{s_0}, x^{s_1}y^{m_1}, ..., x^{s_k}y^{m_k}, ...x^{s_{p-1}}y^{m_{p-1}}, y^{m_p} >$$

and we have a number of finite positive integers $m_1 < m_2 < ... < m_{p-1}$ for each **step**. Let us start from the right-most horizontal line and move the square in this horizontal line such that its left bottom corner coordinate is $(x^{s_{p-1}}, y^{m_{p-1}})$. Sliding this square to some other place to the left of itself will kill the integer m_{p-1} and introduce a smaller integer. In this way we get a new ideal J and by the lemma above we can connect these two ideals I and J. If the right-most horizontal line has length bigger than 1 then continue this process. In each step we strictly decrease the sum of m_i's. We can continue this process until all horizontal lines in the Ferrer's diagram of I has length one. We had seen before that Ferrer's diagrams are invariant if all horizontal lines have length one.So together with lemma 2.4.16, we proved that

Theorem 2.4.20 *$Hilb^d(\mathbb{P}^2, 0)$ is connected.*

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VITA

PERSONAL INFORMATION

Surname, Name:ÖZKAN , Engin

Nationality: Turkish (T. C.)

Date and Place of Birth: 20 July 1980,Kırıkkale

Marital Status: Single

email: enozkan1234@gmail.com, enozkan@metu.edu.tr

EDUCATION

Degree	**Institution**	**Year of Graduation**
PhD	METU, Mathematics	2011
BS	Ankara University, Mathematics	2001
High School	Mamak Niğbolu Lisesi	1997

WORK EXPERIENCE

Year	**Place**	**Enrollment**
2002-2011	METU, Mathematics	Teaching Assistant

TEACHING EXPERIENCE

- Fundamentals of Mathematics, Metu.
- Calculus with Analytic Geometry, Metu.
- Introduction to Differential Equations , Metu.

- Abstract Algebra, Metu.
- Complex Calculus, Metu
- Linear Algebra , Metu.
- Differential Geometry, Metu.

DISSERTATION TOPIC

FIXED POINT SCHEME OF THE HILBERT SCHEME UNDER A 1-DIMENSIONAL ADDITIVE ALGEBRAIC GROUP ACTION,
Thesis Advisor: Assoc. Prof. Dr. Ali Ulaş Özgür Kişisel.

TALKS GIVEN

- (G_m, G_a)-actions on toric surfaces–Gael XIV (Geometrie Algebrique en Liberte), Bedlewo, Poland 2006.
- Compatible (G_m, G_a)-actions on toric surfaces –ODTU-Bilkent Algebraic Geometry Seminars, Bilkent, October 2006.
- Normality of Toric Orbit Closures in G/P–ODTU-Bilkent Algebraic Geometry Seminars, METU, December 2007.

FOREIGN LANGUAGE

- Advanced English.