

A TWO-SIDED CUSUM FOR FIRST-ORDER INTEGER-VALUED
AUTOREGRESSIVE PROCESSES OF POISSON COUNTS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
INDUSTRIAL ENGINEERING

JUNE 2011

Approval of the thesis:

**A TWO-SIDED CUSUM FOR FIRST-ORDER INTEGER-VALUED
AUTOREGRESSIVE PROCESSES OF POISSON COUNTS**

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ABSTRACT

A TWO-SIDED CUSUM FOR FIRST-ORDER INTEGER-VALUED AUTOREGRESSIVE PROCESSES OF POISSON COUNTS

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June 2011, 103 pages

Count data are often encountered in manufacturing and service industries due to ease of data collection. These counts can be useful in process monitoring to detect shifts of a process from an in-control state to various out-of-control states. It is usually assumed that the observations are independent and identically distributed. However, in practice, observations may be autocorrelated and this may adversely affect the performance of the control charts developed under the assumption of independence. In this thesis, the cumulative sum (CUSUM) control chart for monitoring autocorrelated processes of counts is investigated. To describe the autocorrelation structure of counts, a Poisson integer-valued autoregressive moving average model of order 1, Poisson INAR(1), is employed. Changes in the process mean in both positive and negative directions are taken into account while designing the CUSUM chart. A trivariate Markov Chain approach is utilized for evaluating the performance of the chart.

Keywords: Autocorrelation, CUSUM Control Chart, Markov Chain, Poisson INAR(1) Model, Statistical Process Control.

ÖZ

BİRİNCİ DERECEDEDEN TAMSAYI DEĞERLİ OTOREGRESİF POISSON SAYIMLAR İÇİN ÇİFT TARAFLI CUSUM KONTROL GRAFİĞİ

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Haziran 2011, 103 sayfa

Sayım verilerine üretim ve servis sektörlerinde sıkça rastlanmaktadır. Bu sayımlar bir sürecin kontrol içi durumdan çeşitli kontrol dışı durumlara geçişlerini tespit etmek için süreç izlemede yararlı olabilirler. Genelde gözlemlerin bağımsız ve eş dağılımlı olduğu varsayılır. Ancak, pratikte gözlemler özilintili olabilir ve bu durum bağımsızlık varsayımı altında oluşturulan kontrol grafiklerinin performansını olumsuz etkileyebilir. Bu tezde, özilintili süreç sayımlarının izlenmesi için kümülatif toplam (CUSUM) kontrol grafiği araştırılmıştır. Sayımların özilinti yapısını tanımlamak için birinci derece Poisson tamsayı değerli otoregresif hareketli ortalama modeli, Poisson INAR(1), kullanılmıştır. CUSUM grafiği tasarlanırken süreç ortalama değerindeki pozitif ve negatif yönlerdeki değişimler dikkate alınmıştır. Grafiğin performansını değerlendirmek için üç-değişkenli bir Markov Zinciri yaklaşımından yararlanılmıştır.

Anahtar Kelimeler: Özilinti, CUSUM Kontrol Grafiği, Markov Zinciri, Poisson INAR(1) Modeli, İstatistiksel Süreç Kontrolü.

To My Family

ACKNOWLEDGEMENTS

I would like to express my thanks and gratitude to my co-supervisor Assoc. Prof. Dr. Murat Caner Testik for leading me in this research, for his contributions and for preparing me to an academic life throughout my studies.

I would also like to thank my supervisor Assist. Prof. Dr. Z. Pelin Bayındır for her support and guiding comments and for providing criticism to my studies.

I would like to extend my thanks to Dr. Christian Weiß, for his valuable contributions in the derivation of the two-sided CUSUM chart and his invaluable help throughout this research.

The technical assistance of my colleague Volkan Sönmez is gratefully acknowledged.

This research is supported by the National Scholarship Programme of TÜBİTAK (The Scientific and Technological Reserach Council of Turkey).

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CHAPTER 1

INTRODUCTION TO THE THESIS

Quality is one of the most important factors for differentiating between competitive products and services. Hence, a major objective of an organization should be continuous improvement of the quality of its products since improved quality leads to business success and growth. As quality is inversely proportional to variability, quality improvement requires reducing variability in processes.

Statistical methods play a fundamental role in quality improvement for variability reduction. The concept of statistical process control was first proposed by Walter A. Shewhart, who developed the first control charts in 1924 for this purpose. Shewhart's concept of statistical control is based on two sources of variability; common causes represent the natural variability of the process and special causes represent the variation that is not present in the process under normal operating conditions. As stated by Shewhart, keeping a process in statistical control is crucial for managing a process efficiently.

The use of statistical methods for quality improvement has increased since 1980's due to increased use of personal computers. Today, *statistical process control (SPC)* is one of the major tools used for monitoring processes. This involves using statistical techniques in order to measure the variation of processes and its aim is to keep processes under control by reducing variation.

One of the major tools of SPC is the control chart. A control chart is often used to detect presence of special causes of variation in a process as well as analyzing process capability and improving processes continuously.

The scope of quality has changed over the last decades. Although the methods were applied only in the manufacturing environment in the beginning, today quality improvement methods are also being applied to non-manufacturing sectors such as banking, marketing, sales, and healthcare.

1.1 Problem Definition

Count data are often encountered in manufacturing and service industries due to ease of data collection. One particular area where these counts can be useful is in process monitoring to detect shifts of a process from an in-control state to various out-of-control states. Hence, quality losses can be reduced and prevented through corrective actions to put the process back in a normal state.

Until recently, the Shewhart c -chart has been used for monitoring count data under the assumption of independent Poisson random variables. Shewhart-type charts use only the information in the last sample and ignores information given by the entire data sequence (Montgomery, 2005). Thus, the Shewhart c -charts are known to have poor performance in detecting small process shifts in the mean. The Poisson exponentially weighted moving average (EWMA)(Testik *et al.*, 2006) and Poisson cumulative sum (CUSUM) charts (Testik, 2007)are shown to increase effectiveness in detecting small shifts in the mean by accumulating information from the current and the previous samples.

Most of the research on count processes assume that the count data are independent and identically distributed. However, in practice, observations may be autocorrelated and this may adversely affect the performance of the control charts developed under the assumption of independence. Nevertheless, the issue of controlling autocorrelated count data has not been discussed much in the literature until recently.

As some examples, arrival of customers in a queue, bed utilization in a hospital, and the number of web page requests arriving at a server can be modeled by an autocorrelated count process. In a study on network monitoring conducted by Weiß and Testik (2009), it is shown that the count process is highly autocorrelated. The authors analyze counts of accesses to a web server within periods of 2-min length, where each count represents the number of different users registered within each period. The Poisson integer-valued autoregressive (INAR(1)) model is used for modeling the autocorrelated process of counts. In this study, a one-sided Poisson INAR(1) CUSUM chart is proposed in order to detect a positive shift from the in-control mean. A bivariate Markov Chain approach is then used for calculating the average run length (ARL) performance of the CUSUM chart.

This thesis extends the one-sided CUSUM chart proposed by Weiß and Testik (2009) to a two-sided chart. That is, both positive and negative shifts from the in-control mean are taken into account while monitoring the process. In this case, a trivariate Markov Chain is proposed for ARL computations. Numerical results from a simulation study are used to validate the Markov Chain approach. Design and performance of the proposed chart are discussed.

1.2 Organization of the Thesis

This thesis is organized as follows: In Chapter 2, a literature review concerning the research is presented. The methodology of the research given in Chapter 3 provides an overview of the one-sided Poisson INAR(1) CUSUM chart and the extension to the two-sided Poisson INAR(1) CUSUM chart. Next, in Chapter 4, Markov Chain results are presented and the performance of the chart is evaluated for various selected designs. Finally, some concluding remarks and future research plans are provided in Chapter 5.

CHAPTER 2

LITERATURE REVIEW

2.1 Basics of SPC

Statistical Process Control (SPC) is a collection of statistical and graphical tools used for process improvement. Although it was initially developed for use in manufacturing environments, nowadays it is also being widely used in nonmanufacturing environments. The tools of SPC include flow charts, Pareto charts, cause-and-effect diagrams and control charts. Note that all these tools complement each other in problem solving.

The most common tool of SPC is the control charts. A control chart consists of upper and lower control limits and a center line. Some statistics from the process data are plotted on the chart over time and an alarm is triggered if the plotted point on the chart exceeds any of the control limits. This is a signal showing that the process may have shifted to an out-of-control state. Following a signal, an investigation is performed and necessary corrective actions are taken in order to eliminate the causes of the variability and bring the process back in-control. Note that there are various types of control charts for both continuous (variables) and discrete (attributes) type data.

2.1.1 Control Charts for Variables

When a quality characteristic can be measured numerically, it is called a variable and can be monitored through one of several control charts for variables.

2.1.1.1 Shewhart-Type Control Charts

Shewhart-type control charts are very easy to implement and are used in SPC extensively to determine whether a process is in statistical control or not. Samples are taken from a process at different times and measures of a quality characteristic (a variable) are recorded over time. A statistic obtained from the sample, e.g. the mean, range or sample standard deviation, is plotted versus time order on the chart.

A Shewhart-type control chart generally has three components: a center line (CL), an upper control limit (UCL) and a lower control limit (LCL). The center line corresponds to the mean of the plotted statistic. Since the real process parameters are not usually known, these are generally estimated from a sample of observations. If $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_m$ are the averages of the subsamples collected, then the process mean, μ , can best be estimated by:

$$\bar{\bar{X}} = \frac{\bar{X}_1 + \bar{X}_2 + \dots + \bar{X}_m}{m}$$

where the grand average $\bar{\bar{X}}$ is used as the center line of the control chart. Upper and lower control limits are usually drawn at 3 standard deviations from the center line; where

$$UCL = \bar{\bar{X}} + 3\sigma_{\bar{X}}$$

$$LCL = \bar{\bar{X}} - 3\sigma_{\bar{X}}$$

The process is deemed to be in control as long as the points plot within the control limits. If a point falls outside of any of the limits, this is taken as an indication that the process mean has shifted from its in control value μ_0 to an out-of-control mean value μ_1 , hence presence of an assignable cause of variation should be investigated and eliminated.

It is usually assumed that the distribution of the quality characteristic is Normal. However, even if the underlying distribution is non-Normal, the Shewhart controlchart is approximately valid due to the central limit theorem (Montgomery, 2005).

Shewhart charts are effective in detecting large shifts, say 2σ or greater. Note that the statistic plotted on the chart takes only the most recent sample into account, ignoring all other past information and this reduces its sensitivity to smaller shifts in the mean. Thus, they are not very efficient for small shifts. On the other hand, EWMA and CUSUM charts are alternatives for detecting smaller shifts more effectively.

2.1.1.2 EWMA

When small shifts in the process mean are of interest, the exponentially weighted moving average (EWMA) control chart (also referred to as geometric moving average control chart) is a good alternative to the Shewhart chart. Since its introduction by Roberts (1959), many extensions of EWMA charts have been developed.

The EWMA statistic is an exponentially weighted average of the observations, which is displayed on the chart over time. The recursion for calculation of the EWMA statistic, Z_i , is defined by:

$$Z_i = \lambda X_i + (1 - \lambda)Z_{i-1} \quad (2.1)$$

where λ is the smoothing constant such that $\lambda \in (0, 1]$. A large value of λ gives more weight to the recent data and less weight to older data; whereas a small value of λ gives more weight to older data (see Montgomery, 2005). The smoothing constant is chosen depending on the required sensitivity of the chart for detecting changes in the process. In particular, smaller values of λ make the chart more sensitive to smaller shifts in the mean, whereas larger values of λ increase

the sensitivity to larger shifts. The special case $\lambda = 1$ corresponds to the Shewhart type charts.

The starting value of the EWMA statistic is usually the process target:

$$Z_0 = \mu_0$$

where μ_0 is the target value of the process.

Let the observations, X_i , be independent random variables with constant variance σ^2 . Then, the variance of Z_i is calculated by:

$$\sigma_{Z_i}^2 = \sigma^2 \left(\frac{\lambda}{2-\lambda} \right) [1 - (1-\lambda)^{2i}] \quad (2.2)$$

Hence, the center line and control limits are computed as follows, in a similar manner to the Shewhart type control charts but with the variance of the EWMA statistic:

$$\text{UCL} = \mu_0 + L\sigma \sqrt{\frac{\lambda}{2-\lambda} [1 - (1-\lambda)^{2i}]}$$

$$\text{CL} = \mu_0$$

$$\text{LCL} = \mu_0 - L\sigma \sqrt{\frac{\lambda}{2-\lambda} [1 - (1-\lambda)^{2i}]}$$

where L is the width of the control limits in terms of standard deviation units.

As i gets larger, the upper and lower control limits converge to their asymptotic values,

$$UCL = \mu_0 + L\sigma \sqrt{\frac{\lambda}{2 - \lambda}}$$

$$LCL = \mu_0 - L\sigma \sqrt{\frac{\lambda}{2 - \lambda}}$$

respectively, in the steady state and one may directly prefer to use steady-state control limits instead of the exact control limits.

2.1.1.3 CUSUM Control Chart

Cumulative sum (CUSUM) control charts are widely used for process control since their first introduction by Page (1954). Initially, CUSUM charts were designed for monitoring continuous variables; however, many extensions have been proposed to include non-Normal as well as discrete distribution models. As in the case of EWMA charts, CUSUM charts are also more effective in detecting small shifts than conventional Shewhart charts because they make use of information from all samples.

A CUSUM chart operates by cumulating the difference between observations, X_i , and a reference value k . The CUSUM statistic, C_i^+ , for detecting an increase in the process parameter is:

$$C_i^+ = \max[0, X_i - k + C_{i-1}^+] \quad (2.3)$$

and the CUSUM statistic, C_i^- , for detecting a decrease is:

$$C_i^- = \max[0, k - X_i + C_{i-1}^-] \quad (2.4)$$

where k is the reference value.

Both C_i^+ and C_i^- are reset to zero if they become negative. The two statistics, C_i^+ and C_i^- are plotted on the chart over time and an out of control signal is triggered if C_i^+ or C_i^- exceeds the decision interval, h . The signal is considered as an indication of a shift in the process mean, and an analysis follows to investigate existence of an assignable cause. If any special cause is found, corrective actions are taken to bring the process back in control again. Then, the CUSUM is restarted to monitor the process.

Page (1954) asserts that a CUSUM control scheme is a sequence of Wald sequential probability ratio tests (SPRT's) (Wald, 1947). Each SPRT tests the null hypothesis that the mean is μ_0 and the alternate hypothesis that the mean is $\mu_1 \neq \mu_0$.

There are two types of representations of CUSUMs, the tabular CUSUM and the V-mask. The V-mask CUSUM, although yields the same conclusions, has been widely used before the computer implementations; however, the tabular CUSUM is more convenient to use and it is the preferred method in this thesis.

A CUSUM chart usually sets the initial values $C_0^+ = 0$ and $C_0^- = 0$. However, when the Fast Initial Response (FIR) feature is used, one can start with a positive value such as $C_0 = h/2$. The FIR method was proposed by Lucas and Crosier (1982) in order to improve the sensitivity of the chart to process upsets at process start-up. If the process starts initially in control, the FIR feature has little effect. However, if it is initially out of control, the chart gives an out of control signal more quickly.

2.1.1.4 Performance Evaluation of the Control Charts

Performance of a control chart is usually evaluated by considering its run length distribution. The run length (RL) is the number of points plotted on the control chart until an out-of-control signal is triggered. The most common measure to

assess the control chart's performance is the average run length (ARL), which is the expected value of the RL. The ARL should be large when the process is in control, whereas it should be small when the process shifts out of control. The ARL distribution is usually geometric due to the independence assumption.

Pignatiello *et al.* (1995) defines an ARL-unbiased chart such that the out-of-control ARL is always smaller than the in-control ARL. In fact, a control chart may take longer to trigger an out-of-control alarm under a process shift than under a stable process. In this thesis, the concept of ARL unbiasedness is also considered for the proposed CUSUM chart.

Several techniques may be used to calculate the ARL performance of a control chart, e.g. simulation, integral equations and Markov Chain. An efficient method to calculate ARLs is developed by Brook and Evans (1972). Their method is based on considering transitions from an in-control state to an out-of-control state by using a Markov Chain approach. The method considered for ARL calculations in this thesis is based on the Markov Chain approach, which will be discussed in more detail in Section 2.1.4 and Chapter 3.

2.1.2 Control Charts for Attributes

When a quality characteristic cannot be represented as a measurement in a continuous scale, it is usually convenient to make a classification of the quality characteristic. The terms conforming/nonconforming or defective/nondefective can be used in this context in order to identify the classification of the item. Usually, the classification is measured by counting the numbers of conforming/nonconforming products in a sample. Then, the quality characteristic is called "an attribute". Therefore, count processes can be monitored by control charts for attributes. Attributes charts are especially useful in nonmanufacturing industries. Some details of the common attributes control charts are provided in

the following. Interested readers are also referred to Montgomery (2005) for further details.

2.1.2.1 The p-Control Chart

If a product does not conform to the customer requirements, it is called nonconforming. The “p-chart” displays the fraction of the nonconforming items in a sample. If the probability of nonconformance is p , and a random sample of n units of the product is selected, then the number of products that are nonconforming can be formulated by the binomial distribution as follows:

$$P\{D = x\} = \binom{n}{x} p^x (1 - p)^{n-x} \quad x = 0, 1, \dots, n$$

where D is the number of nonconforming products, and n is the size of the sample.

The center line and control limits of the pchart are as follows:

$$UCL = p + 3 \sqrt{\frac{p(1-p)}{n}}$$

$$\text{Center line} = p$$

$$LCL = p - 3 \sqrt{\frac{p(1-p)}{n}}$$

If a point plots outside of the control limits, it may be concluded that the fraction of the nonconforming units has shifted.

2.1.2.2 The np-Control Chart

The np-control chart monitors the number of nonconforming items in a sample.

The limits and center line of the np chart are as follows:

$$UCL = np + 3\sqrt{np(1-p)}$$

$$\text{Center line} = np$$

$$LCL = np - 3\sqrt{np(1-p)}$$

2.1.2.3 The c-Control Chart

Sometimes it is preferred to work with the number of defects rather than the fraction of nonconforming items because an item may have several defects on it and it could still be classified as conforming as long as the defects are minor. The c-control chart is based on the total number of defects in a unit. It is generally assumed that the occurrence of defects are modeled by Poisson distribution with parameter $c > 0$. Both mean and variance of the Poisson distribution are equal to c . Thus, the c-control chart is constructed as follows:

$$UCL = c + 3\sqrt{c}$$

$$\text{Center line} = c$$

$$LCL = c - 3\sqrt{c}$$

2.1.2.4 The u-Control Chart

The u control chart monitors the average number of defects per inspection unit.

Let D be the total number of defects in n inspection units. The average number of defects can then be computed as:

$$u = \frac{D}{n}$$

Consequently, the u-control chart can be constructed as follows:

$$UCL = u + 3\sqrt{\frac{\bar{u}}{n}}$$

$$\text{Center line} = u$$

$$LCL = u - 3\sqrt{\frac{\bar{u}}{n}}$$

2.1.2.5 The Poisson EWMA Control Chart

Occurrence of nonconformities in a process is usually modeled by the Poisson distribution. In addition to the c-control chart, various control charts have been proposed in literature to monitor Poisson models. Extension of the EWMA chart for i.i.d. count data is similar to the development of the variables EWMA chart. Borror *et al.* (1998) propose an EWMA control chart for a Poisson model and compute average run length (ARL) performance of the chart. Testik *et al.* (2006) further study the Poisson EWMA control chart for the case in which the process parameters are unknown. By using estimates from an in-control sample, effects of estimating the mean on the performance of the chart are investigated. The Poisson EWMA chart is proved to detect small shifts more quickly than the Shewhart charts.

The EWMA statistic for the Poisson case is obtained again as in Equation(2.1). Similar to the EWMA chart for variables, control limits of the Poisson EWMA chart are calculated by using the variance of the EWMA statistic, but this time using the Poisson distribution. The Poisson EWMA control chart triggers a signal when the EWMA statistic Z_i exceeds either of the control limits. If the lower control limit is less than zero, it should be set to zero;

because Z_i values are always nonnegative due to the Poisson distribution of the quality characteristic.

The Poisson EWMA control chart is very useful for monitoring count data following the Poisson distribution and it has been shown that these charts are more effective in contrast to Shewhart charts for detecting smaller shifts in the mean (Testik *et al.*, 2006).

2.1.2.6 The Poisson CUSUM Control Chart

CUSUM control charts for variables are generally used for process control. However, these schemes can also be designed to monitor processes of counts. A CUSUM procedure to control processes with Poisson marginals is first proposed by Brook and Evans (1972). Lucas (1985) describes counted data CUSUM's and provides tables of ARL values with several process shifts. Counted data CUSUM's have a similar design procedure to CUSUM's for variables (Lucas, 1976). However, by monitoring the counts of defects, the goal is to detect increases and/or decreases in the count level. It is usually assumed that the number of counts per interval follows a Poisson distribution when the number of counts in a specified interval is recorded.

Just like CUSUM charts for variables, Poisson CUSUM charts can also include the Fast Initial Response (FIR) feature. FIR provides more quick detection of an out of control situation at start-up. Lucas (1985) also demonstrates the benefits of the FIR feature for the Poisson CUSUM.

Similar to CUSUM charts for variables, counted data CUSUM's also calculate the difference of a count value X_i and the reference value k and plot the cumulative sum of these differences. The CUSUM statistics for detecting an increase and a decrease can be calculated as in Equations (2.3) and (2.4), respectively.

The CUSUM chart can be designed as one-sided or two-sided. One-sided CUSUM chart tries to detect an increase or decrease in the count level; whereas two-sided CUSUM monitors both an increase and a decrease. Two-sided schemes are obtained by implementing two one-sided schemes simultaneously. For a Poisson CUSUM, the parameter k is chosen to be between the acceptable count rate (μ_a) and the count rate that is needed to be detected quickly (μ_d). Lucas (1985) asserts that the k value should be close to

$$k = (\mu_d - \mu_a) / \ln(\mu_d) - \ln(\mu_a) \quad (2.5)$$

Note that k is usually rounded to the nearest integer, because Poisson CUSUM requires integer numbers. Then, the parameter h is determined by considering the desired in-control ARL performance. The h value should give a large ARL when the process is in-control and a small ARL when there is a shift in the process. Lucas (1985) presents tables of ARL values for several cases of interest.

Since the process mean (also the variance in this case) may be unknown but estimated, Testik (2007) studies the effects of estimated process mean on the conditional and marginal performance of the Poisson CUSUM when in-control process parameters are estimated. The effects of estimation have been shown to be significant under some circumstances.

2.1.2.7 Time-Between-Events CUSUM

When the number of counts per interval follows a Poisson distribution, the times between counts follow an exponential distribution. Lucas (1985) also proposes a time-between-events (TBE) CUSUM. TBE CUSUM is very convenient if it is possible to record the time between counts.

The design procedure for TBE CUSUM is similar to the Poisson CUSUM. The mean time between counts is the reciprocal of the number of counts per sampling

interval. Thus, the reference value, k , for the TBE CUSUM is the reciprocal of Equation (2.5), i.e.

$$k = (\ln(\mu_d) - \ln(\mu_a)) / (\mu_d - \mu_a).$$

For a TBE CUSUM, the h value is chosen from an ARL table. Lucas (1985) presents tables of ARL values with and without the FIR feature. In this case, the ARL is the average number of counts before an out-of-control signal occurs.

2.1.3 Time Series Models in Control Charts

2.1.3.1 ARIMA Models

Autoregressive integrated moving average models (ARIMA) are the most general class of models for forecasting a time series. In an ARIMA(p, d, q) model, p, d and q are non-negative integers that refer to the order of the autoregressive, integrated, and moving average parts of the model, respectively. The model can represent a wide range of time series data (See Montgomery *et al.*, 2008).

ARIMA models decompose the time series data into three components for modeling the serial correlation in the data. These three components are:

1. An autoregressive (AR) process having a memory of past events. The AR(p) model uses the p lags of the time series in the equation.
2. An integrated (I) process making the data stationary by differencing, thus making it easier to forecast. I(d) means differencing the data d times.
3. A moving average (MA) of the forecast errors. The MA(q) model uses the q lags of the forecast errors to improve the forecast.

ARIMA models therefore have three model parameters, one for the $AR(p)$ process, one for the $I(d)$ process, and one for the $MA(q)$ process, all interacting among each other.

2.1.3.2 INAR Models

Time series of counts are encountered in many cases. Several different approaches to model processes of counts have been proposed in the literature. Count processes have an integer structure. However, the ARMA recursion includes a scalar multiplication; therefore, it cannot be applied to the integer-valued case. In order to adapt the ARMA model to the integer-valued case, thinning operations may be used. Weiß (2008) reviews different thinning operations and summarizes essential properties of the specific thinning operations.

Binomial thinning, introduced by Steutel and Van Harn (1979), is one of the most frequently used thinning operations successfully applied to define integer-valued ARMA (INARMA) models. Binomial thinning can be applied to a discrete random variable N . Let $\alpha \in [0,1]$ be a parameter. Then, the binomial thinning operation is,

$$\alpha \circ N = \sum_{i=1}^N Y_i$$

where Y_i 's are i.i.d. Bernoulli random variables that are independent of N such that $P(Y_i = 1) = \alpha$. The basic idea of binomial thinning is that the count represents the size of a population. The population at time t has two components: the survivors of the population from time $t - 1$, and the newcomers. The thinning operation models which members of the population will survive. The members die independently of each other, and the probability of dying in a specified time interval is $1 - \alpha$.

The first INARMA model, the INAR(1) model, is proposed by McKenzie (1985). An INAR(1) model is a simple model for a sequence of integer-valued random variables with lag-one dependence and is the integer counterpart of the continuous-valued AR(1) process having similar correlation structure and distributional properties. The INAR(1) process can be defined by the recursion:

$$N_t = \alpha N_{t-1} + \varepsilon_t$$

where N_t is the number of counts at time t and ε_t is a sequence of independent integer-valued random variables. A thinning operation is performed at each time t .

INAR(1) model can be used for count processes in which an element of the process at time t is either the survival of an element of the process at time $t - 1$ or the outcome of an innovation process.

An INAR(1) process is stationary and it is a homogenous Markov Chain by definition because the observation at time t depends only on the observation at time $t - 1$. Properties of a stationary INAR(1) process are summarized below:

$$\begin{aligned} \mu_N &= \mu_\varepsilon / (1 - \alpha) \\ \sigma_N^2 &= (\alpha \mu_\varepsilon + \sigma_\varepsilon^2) / (1 - \alpha^2) \\ \rho_N(k) &= \rho^k \\ P(N_t = k | N_{t-1} = l) &= \sum_{j=0}^{\min(k,l)} \binom{l}{j} \alpha^j (1 - \alpha)^{l-j} \cdot P(\varepsilon_t = k - j) \quad (2.6) \\ E[N_t | N_{t-1}] &= \alpha \cdot N_{t-1} + \mu_\varepsilon \end{aligned}$$

The INAR(1) model is best suited for Poisson marginals. If ε_t 's are distributed according to the Poisson distribution with $\mu_\varepsilon = \lambda$, then N_t 's are also distributed according to Poisson distribution with $\mu_N = \lambda / (1 - \alpha)$. In case of Poisson marginals, the transition probability in Equation (2.6) becomes:

$$P(N_t = k | N_{t-1} = l) = \sum_{j=0}^{\min(k,l)} \binom{l}{j} \alpha^j (1 - \alpha)^{1-j} \cdot e^{-\lambda(1-\alpha) \frac{(\lambda(1-\alpha))^{k-j}}{(k-j)!}}$$

Interested readers are referred to Weiß and Testik (2009) for more details. Al-Osh and Alzaid (1987) also discuss the correlation structure and the distributional properties of the model and propose methods for estimating the parameters of the model. Weiß (2007) suggests approaches to control INAR(1) processes and compare their run length performance in a simulation study.

2.1.4 Markov Chain Approach for ARL Performance Evaluation

Several approaches have been proposed in the literature to evaluate the performance of control charts. The Markov Chain (MC) approach proposed by Brook and Evans (1972) is an effective method and can be used for performance evaluation of control charts including the CUSUM and EWMA charts and their various extensions. Weiß (2011) discusses the MC approach for computing ARLs for different types of control charts. It is concluded that MC approach is the method of choice for serially dependent attributes data processes since it allows to compute the ARLs exactly.

In the case of a counted data CUSUM chart where the counts are independent of each other, let X_t be the observed count, k be the reference value and h be the decision interval, having positive integer values. Then the cumulative sum statistic C_t can only take one of the integer values $0, 1, \dots, h$. If $C_t = i$, the CUSUM is in state E_i at time t . Here, it is assumed the process is initially in state E_0 and E_h is the absorbing state. When C_t value reaches the decision interval, the Markov process enters the absorbing state E_h .

The process is a Markov Chain and the transition probability matrix, \mathbf{P} , is formed by using the given values of k and h . The transition probabilities are calculated by the probability distribution of X_t as follows:

$$p_{i0} = P(X \leq k - i)$$

$$p_{ij} = P(X = k + j - i) \text{ such that } j = 1, \dots, h$$

$$p_{ih} = P(X \geq k + h - i)$$

where p_{ij} is the probability of a transition from state E_i to state E_j .

\mathbf{P} is an $(h + 1) \times (h + 1)$ matrix and the last row and column of the \mathbf{P} matrix represent the absorbing state. A new matrix, \mathbf{Q} , is formed by removing the last row and column of the \mathbf{P} matrix. \mathbf{Q} is an $h \times h$ matrix and is composed of the transition probabilities among the in-control states. Then, the h -dimensional ARL vector is obtained as,

$$(\mathbf{I} - \mathbf{Q})^{-1} \mathbf{1}$$

where \mathbf{I} is the $h \times h$ identity matrix and the vector $\mathbf{1}$ has each of its h elements equal to unity. Each element of this vector represents the time of the first transition of the process to the absorbing state starting from the corresponding in-control state. In other words, the vector calculates the average of the first passage times to the absorbing state, thus calculating the ARL performance.

A computer code is presented in White and Keats (1996) for implementing the MC approach to calculate ARL values and higher-order moments for a Poisson CUSUM control scheme under the assumption of independent counts.

CHAPTER 3

METHODOLOGY

Processes of count data can be monitored by using various control schemes. Such count data can often be modeled by Poisson distribution. If the process is independent, control schemes such as c-chart, u-chart, EWMA chart and CUSUM chart can be used. However, count processes usually exhibit autocorrelation. The autocorrelation structure of the processes with Poisson marginals can often be modeled by integer valued ARMA (INARMA) models. In this thesis, the autocorrelated count processes are assumed to follow a stationary Poisson INAR(1) model. We consider CUSUM control chart for monitoring the counts of a Poisson INAR(1) process. The outline of this chapter is as follows: First, the one-sided CUSUM chart for monitoring a Poisson INAR(1) model proposed by Weiß and Testik (2009) is presented in Section 3.1. Next, this one-sided chart is extended to a two-sided CUSUM chart for detecting both increases and decreases in the count rate of the process in Section 3.2. Then, an illustrated example is given and a simulation study for the validation of the ARL calculation proposed in this thesis is provided.

3.1 One-Sided Poisson INAR(1) CUSUM Charts

3.1.1 A One-sided INAR(1) CUSUM for Detecting Increases

Weiß and Testik (2009) proposed a one-sided Poisson INAR(1) CUSUM chart for monitoring a process in order to detect positive shifts from the in-control mean. In this section, the procedure of their work is presented. A one-sided Poisson CUSUM chart for detecting increases monitors the statistic:

$$C_t^+ = \max[0, N_t - k^+ + C_{t-1}^+] \quad (3.1)$$

where N_t is the observed count at time t and k^+ is the reference value. Note that $k^+ \in \mathbb{N}$ and $\lambda_0 \leq k^+$. The statistic C_t^+ is calculated according to Equation (3.1) after observing each count N_t . Then, these statistics C_t^+ are plotted on the control chart with control region $\mathcal{C}(h^+) = \{0, \dots, h^+ - 1\}$, $h^+ \in \mathbb{N}$. The process is in-control as long as $C_t^+ \in \mathcal{C}(h^+)$.

The probabilities $p_k = P(N_t = k)$ of a Poisson distribution, $\text{Pois}(\lambda)$, and the transition probabilities $p_{k|l} = P(N_t = k | N_{t-1} = l)$ are needed to compute the ARLs of the Poisson CUSUM chart. The probabilities can be computed recursively as follows:

$$\begin{aligned} p_0 &= e^{-\lambda}, \\ p_{k+1} &= \frac{\lambda}{k+1} \cdot p_k, \\ p_{0|l} &= (1 - \alpha)^l \cdot e^{-\lambda(1-\alpha)}, \\ k \cdot p_{k|l} &= \lambda(1 - \alpha) \cdot p_{k-1|l} + \alpha \cdot l \cdot p_{k-1|l-1}, \quad k \geq 1, \quad l \geq 0 \end{aligned} \quad (3.2)$$

Suppose $(N_t)_{\mathbb{N}}$ is a stationary Poisson INAR(1) process. Then (N_t, C_t^+) is a bivariate Markov Chain with range \mathbb{N}_0^2 and with the following transition probabilities:

$$\begin{aligned} p(a, b|c, d) &= P(N_t = a, C_t^+ = b | N_{t-1} = c, C_{t-1}^+ = d) \\ &= P(C_t^+ = b | N_t = a, N_{t-1} = c, C_{t-1}^+ = d) \\ &\quad \cdot P(N_t = a | N_{t-1} = c, C_{t-1}^+ = d) \\ &= P(C_t^+ = b | N_t = a, C_{t-1}^+ = d) \cdot P(N_t = a | N_{t-1} = c) \\ &= \delta_{b, \max(0, a - k^+ + d)} \cdot p_{a|c} \end{aligned} \quad (3.3)$$

where $\delta_{x,y}$ denotes the Kronecker delta such that:

$$\delta_{x,y} = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}$$

Then, the probability of the first transition is calculated as follows:

$$\begin{aligned} p_1(a, b|c) &= P(N_1 = a, C_1^+ = b | C_0^+ = c) \\ &= P(C_1^+ = b | N_1 = a, C_0^+ = c) \cdot P(N_1 = a | C_0^+ = c) \\ &= \delta_{b, \max(0, a - k^+ + c)} \cdot p_a \end{aligned} \quad (3.4)$$

The speed and efficiency of the calculations depend on the number of states of the Markov Chain. If there are too many states, then the transition probability matrix will be very large and it will take a long time to calculate the ARL. Thus, the dimension of the matrix used for the Markov Chain approach affects the speed of the ARL computations. In fact, not all in-control states are reachable, so working only with the reachable in-control states will save time and space. As a result, specifying the set of reachable in-control values of (N_t, C_t^+) reduces the dimension of the transition matrix, thus speeding up the calculations.

Let $(N_t, C_t^+) = (n, i)$. Then $i = \max[0, n - k^+ + C_{t-1}^+]$. The statistic C_t^+ exceeds h^+ iff $N_t - k^+ + C_{t-1}^+ \geq h^+$. Therefore, a value of $N_t \geq k^+ + h^+$ will lead to an out-of-control signal. Then, the set of in-control pairs (n, i) can be reached if and only if

$$n \leq i + k^+ \text{ and } n \geq i + k^+ - h^+ + 1$$

As a result, the set of in-control pairs (n, i) is given by:

$$\mathcal{J}(h^+, k^+) = \{(n, i) \mid \max[0, i + k^+ - h^+ + 1] \leq n \leq i + k^+\} \quad (3.5)$$

3.1.2 ARL Computation of the Upper-sided INAR(1) CUSUM Chart

Since (N_t, C_t^+) is a discrete time discrete space Markov Chain, the ARLs of the CUSUM chart can be computed exactly by the Markov Chain approach of Brook and Evans (1972). The procedure of Weiß and Testik(2009) is presented in this section.

Let $(N_t, C_t^+)_{\mathbb{N}}$ be a stationary Poisson INAR(1) process with $C_0^+ \in \mathbb{N}_0$ and $\mathcal{C} = \mathcal{C}(h^+)$ be the control region of the process. Let $\mathcal{J}(h^+, k^+)$ be the set of in-control pairs (n, i) . Then,

$$p_{m,a}(r) = P(C_{r+1}^+ \notin \mathcal{C}, C_r^+, \dots, C_2^+ \in \mathcal{C} \mid C_1^+ = a, N_1 = m)$$

where $p_{m,a}(r), r \geq 1$ is the probability that the $(r+1)^{st}$ statistic C_t^+ leads to the first out-of-control signal, given that $(N_1, C_1^+) = (m, a) \in \mathcal{J}(h^+, k^+)$.

Let $\mu_{m,a}$ be the expected number of in-control signals before the first-out-of-control signal. Then, $\mu_{m,a}$ can be calculated as:

$$\mu_{m,a} = \sum_{r=1}^{\infty} r \cdot p_{m,a}(r).$$

If $\boldsymbol{\mu} = (\dots, \mu_{n,i}, \dots)^T$ and $\mathbf{Q}^T = (p(n, i \mid m, a))_{(n,i),(m,a) \in \mathcal{J}(h^+, k^+)}$, then summing over the possible in-control states (n, i) , one obtains:

$$\begin{aligned} p_{m,a}(r) &= P(C_{r+1}^+ \notin \mathcal{C}, C_r^+, \dots, C_2^+ \in \mathcal{C} \mid C_1^+ = a, N_1 = m) \\ &= \sum_{(n,i) \in \mathcal{J}(h^+, k^+)} p_{n,i}(r-1) \cdot p(n, i \mid m, a) \end{aligned}$$

Therefore,

$$\begin{aligned}
\mu_{m,a} &= \sum_{r=1}^{\infty} r \cdot p_{m,a}(r) \\
&= \sum_{r=1}^{\infty} p_{m,a}(r) + \sum_{r=2}^{\infty} (r-1) \cdot \sum_{(n,i) \in \mathcal{J}(h^+, k^+)} p_{n,i}(r-1) \cdot p(n, i | m, a) \\
&= 1 + \sum_{(n,i) \in \mathcal{J}(h^+, k^+)} p(n, i | m, a) \cdot \sum_{s=1}^{\infty} s \cdot p_{n,i}(s) \\
&= 1 + \sum_{(n,i) \in \mathcal{J}(h^+, k^+)} p(n, i | m, a) \cdot \mu_{n,i}
\end{aligned}$$

which can be written in matrix form as:

$$\boldsymbol{\mu} = \mathbf{1} + \mathbf{Q} \cdot \boldsymbol{\mu}$$

As a result, one can obtain:

$$\begin{aligned}
(\mathbf{I} - \mathbf{Q}) \cdot \boldsymbol{\mu} &= \mathbf{1} \\
\boldsymbol{\mu} &= \mathbf{1} \cdot (\mathbf{I} - \mathbf{Q})^{-1}
\end{aligned}$$

The ARL depends on the starting value $C_0^+ = c_0$ and is defined as:

$$ARL(c_0) = 1 + \sum_{(m,a) \in \mathcal{J}(h^+, k^+)} \mu_{m,a} \cdot p_1(m, a | c_0)$$

The dimension of \mathbf{Q} and $\boldsymbol{\mu}$ is determined by the size of the set of in-control pairs.

3.1.3 A One-sided INAR(1) CUSUM for Detecting Decreases

This section presents the one-sided CUSUM for detecting decreases proposed in a working report of Weiß and Testik (2010). The approach is similar to the case of the one-sided upper CUSUM.

A one-sided CUSUM chart for detecting a decrease monitors the statistics:

$$C_t^- = \max[0, k^- - N_t + C_{t-1}^-] \quad (3.6)$$

where N_t is the observed count at time t and k^- is the reference value. Note that $k^- \in \mathbb{N}$ and $k^- \leq \lambda_0$.

The statistic C_t^- is calculated according as in Equation (3.6) after observing each count N_t . Then these statistics C_t^- are plotted on the control chart with control region $\mathcal{C}(h^-) = \{0, \dots, h^- - 1\}$, $h^- \in \mathbb{N}$. The process is in-control as long as $C_t^- \in \mathcal{C}(h^-)$.

Let $(N_t)_{\mathbb{N}}$ be a stationary Poisson INAR(1) process. Then (N_t, C_t^-) is a bivariate Markov Chain with range \mathbb{N}_0^2 and with the following transition probabilities:

$$\begin{aligned} p(a, b|c, d) &= P(N_t = a, C_t^- = b \mid N_{t-1} = c, C_{t-1}^- = d) \\ &= P(C_t^- = b \mid N_t = a, N_{t-1} = c, C_{t-1}^- = d) \\ &\quad \cdot P(N_t = a \mid N_{t-1} = c, C_{t-1}^- = d) \\ &= P(C_t^- = b \mid N_t = a, C_{t-1}^- = d) \cdot P(N_t = a \mid N_{t-1} = c) \\ &= \delta_{b, \max(0, k^- - a + d)} \cdot p_{a|c} \end{aligned} \quad (3.7)$$

The probabilities $p_k = P(N_t = k)$ and $p_{k|l} = P(N_t = k \mid N_{t-1} = l)$ are computed as in Equation (3.2).

The probability of the first transition is calculated as follows:

$$\begin{aligned} p_1(a, b|c) &= P(N_1 = a, C_1^- = b \mid C_0^- = c) \\ &= P(C_1^- = b \mid N_1 = a, C_0^- = c) \cdot P(N_1 = a \mid C_0^- = c) \\ &= \delta_{b, \max(0, k^- - a + c)} \cdot p_a \end{aligned} \quad (3.8)$$

The set of reachable in-control values of (N_t, C_t^-) should be specified in order to speed up the calculations. Let $(N_t, C_t^-) = (n, i)$. Then $i = \max[0, k^- - n + C_{t-1}^-]$. Similar to the upper-sided CUSUM, not all pairs from $\mathbb{N}_0 \times \mathcal{C}(h^-)$ can be reached by (N_t, C_t^-) from an in-control state. A finite set can be obtained that contains the reachable in-control states. This set is given by:

$$\mathcal{J}(h^-, k^-) = \{(n, i) \mid \max[0, k^- - i] \leq n, \text{ and } n < h^- + k^- - i \text{ iff } i > 0\}$$

3.1.4 ARL Computation of Lower-sided INAR(1) CUSUM Chart

Since (N_t, C_t^-) is a discrete Markov chain, the ARLs of the lower-sided INAR(1) CUSUM chart can again be computed exactly by the Markov Chain approach of Brook and Evans (1972). Weiß and Testik (2010) present the procedure in their working report as follows.

Suppose that $(N_t, C_t^-)_{\mathbb{N}}$ is a stationary Poisson INAR(1) process with $C_0^- \in \mathbb{N}_0$ and $\mathcal{C} = \mathcal{C}(h^-)$ is the control region of the process. Let $\mathcal{J}(h^-, k^-)$ be the set of in-control pairs (n, i) . Then,

$$p_{m,a}(r) = P(C_{r+1}^- \notin \mathcal{C}, C_r^-, \dots, C_2^- \in \mathcal{C} \mid C_1^- = a, N_1 = m)$$

where $p_{m,a}(r)$, with $r \geq 1$ is the probability that the $(r + 1)^{st}$ statistic C_t^- leads to the first out-of-control signal, given that $(N_1, C_1^-) = (m, a) \in \mathcal{J}(h^-, k^-)$.

Let $\mu_{m,a}$ denote the expected number of in-control signals before the first-out-of-control signal. Then, $\mu_{m,a}$ can be calculated as,

$$\mu_{m,a} = \sum_{r=1}^{\infty} r \cdot p_{m,a}(r).$$

By summing over the possible in-control states (n, i) , one obtains:

$$\begin{aligned} p_{m,a}(r) &= P(C_{r+1}^- \notin \mathcal{C}, C_r^-, \dots, C_2^- \in \mathcal{C} \mid C_1^- = a, N_1 = m) \\ &= \sum_{(n,i) \in \mathcal{J}(h^-, k^-)} p_{n,i}(r-1) \cdot p(n, i \mid m, a) \end{aligned}$$

Hence,

$$\begin{aligned} \mu_{m,a} &= \sum_{r=1}^{\infty} r \cdot p_{m,a}(r) \\ &= \sum_{r=1}^{\infty} p_{m,a}(r) + \sum_{r=2}^{\infty} (r-1) \cdot \sum_{(n,i) \in \mathcal{J}(h^-, k^-)} p_{n,i}(r-1) \cdot p(n, i \mid m, a) \\ &= 1 + \sum_{(n,i) \in \mathcal{J}(h^-, k^-)} p(n, i \mid m, a) \cdot \sum_{s=1}^{\infty} s \cdot p_{n,i}(s) \\ &= 1 + \sum_{(n,i) \in \mathcal{J}(h^-, k^-)} p(n, i \mid m, a) \cdot \mu_{n,i} \end{aligned}$$

The statistic C_t^- exceeds h^- iff $k^- - N_t + C_{t-1}^- \geq h^-$. Therefore, an upper bound for N_t cannot be found, i.e. there are infinitely many states to consider. As a result, the exact ARL cannot be computed. Weiß and Testik (2010) offer an upper bound, n_{max} , that can limit the number of in-control states. n_{max} must be chosen sufficiently large. An appropriate value may be 8 standard deviations away from the mean count level; that is,

$$n_{max} = \lceil \lambda + 8 \cdot \sqrt{\lambda} \rceil.$$

By using an approximate upper bound, an approximate ARL can be computed. The states of the set below are considered while calculating the ARLs,

$$\mathcal{J}^*(h^-, k^-, n_{max}) = \{(n, i) \mid \max[0, k^- - i] \leq n \leq n_{max}, \text{ and } n < h^- + k^- - i \text{ iff } i > 0\}$$

If $\boldsymbol{\mu} = (\dots, \mu_{n,i}, \dots)^T$ and $\boldsymbol{Q}^T = (p(n, i | m, a))_{(n,i),(m,a) \in \mathcal{J}^*(h^-, k^-, n_{max})}$, then the system of linear equations can be approximated by the following system:

$$\begin{aligned} (\boldsymbol{I} - \boldsymbol{Q}) \cdot \boldsymbol{\mu} &= \mathbf{1} \\ \boldsymbol{\mu} &= \mathbf{1} \cdot (\boldsymbol{I} - \boldsymbol{Q})^{-1} \end{aligned}$$

Note that the approximate ARL depends on the initial value $C_0^- = c_0$ and it is obtained as

$$ARL(c_0) = 1 + \sum_{(m,a) \in \mathcal{J}^*(h^-, k^-, n_{max})} \mu_{m,a} \cdot p_1(m, a | c_0)$$

3.2 Two-sided Poisson INAR(1) CUSUM Charts

The procedure of the two-sided INAR(1) CUSUM is the extension of the work by Weiß and Testik (2009) and is the main research topic of this thesis. The procedures of the one-sided upper and lower INAR(1) CUSUMs are combined and necessary adjustments are made in order to design a two-sided Poisson INAR(1) CUSUM chart.

A two-sided Poisson CUSUM chart monitors the following two statistics, the first being the upper and the second being the lower CUSUM statistics, respectively,

$$\begin{aligned} C_t^+ &= \max[0, N_t - k^+ + C_{t-1}^+] \\ C_t^- &= \max[0, k^- - N_t + C_{t-1}^-] \end{aligned}$$

where N_t is the observed count at time t and k^+ and k^- are the reference values for the upper and lower CUSUMs, respectively. Note that $k^-, k^+ \in \mathbb{N}$ and $k^- \leq \lambda_0 \leq k^+$.

In operating the chart, the statistics C_t^+ and C_t^- are calculated after observing each count N_t . Then these statistics (C_t^+, C_t^-) are plotted on the control chart with control region $\mathcal{C}(h^+, h^-) = \{0, \dots, h^+ - 1\} \times \{0, \dots, h^- - 1\}$, $h^+, h^- \in \mathbb{N}$. The process is in-control as long as $(C_t^+, C_t^-) \in \mathcal{C}(h^+, h^-)$.

If $(N_t)_{\mathbb{N}}$ is a stationary Poisson INAR(1) process, then (N_t, C_t^+, C_t^-) is a trivariate Markov Chain with the following transition probabilities:

$$\begin{aligned}
p(a, b, c | d, e, f) &= P(N_t = a, C_t^+ = b, C_t^- = c \mid N_{t-1} = d, C_{t-1}^+ = e, C_{t-1}^- = f) \\
&= P(C_t^+ = b \mid N_t = a, N_{t-1} = d, C_{t-1}^+ = e, C_t^- = c, C_{t-1}^- = f) \\
&\quad \cdot P(C_t^- = c \mid N_t = a, N_{t-1} = d, C_{t-1}^+ = e, C_{t-1}^- = f) \\
&\quad \cdot P(N_t = a \mid N_{t-1} = d, C_{t-1}^- = e) \\
&= P(C_t^+ = b \mid N_t = a, C_{t-1}^- = e) \\
&\quad \cdot P(C_t^- = c \mid N_t = a, C_{t-1}^- = e) \cdot P(N_t = a \mid N_{t-1} = d) \\
&= \delta_{b, \max(0, a - k^+ + e)} \cdot \delta_{c, \max(0, k^- - a + f)} \cdot p_{a|d}
\end{aligned} \tag{3.9}$$

Then, the probability of the first transition is calculated as follows:

$$\begin{aligned}
p_1(a, b, c | d, e) &= P(N_1 = a, C_1^+ = b, C_1^- = c \mid C_0^+ = d, C_0^- = e) \\
&= P(C_1^+ = b \mid N_1 = a, C_0^+ = d, C_1^- = c, C_0^- = e) \\
&\quad \cdot P(C_1^- = c \mid N_1 = a, C_0^+ = d, C_0^- = e) \\
&\quad \cdot P(N_1 = a \mid C_0^+ = d, C_0^- = e) \\
&= \delta_{b, \max(0, a - k^+ + d)} \cdot \delta_{c, \max(0, k^- - a + e)} \cdot p_a
\end{aligned} \tag{3.10}$$

The probabilities $p_k = P(N_t = k)$ and $p_{k|l} = P(N_t = k \mid N_{t-1} = l)$ are needed to compute the ARLs of the two-sided Poisson CUSUM chart. These probabilities can be computed as in Equation (3.2).

The two-sided CUSUM chart is the combination of upper one-sided and lower one-sided CUSUM charts. Thus, the restrictions concerning the reachable in-control states also apply to the two-sided CUSUM chart. It is not necessary to

consider all states, but only those of the finite set formed by combining the sets of the two one-sided CUSUM charts, i.e.

$$\mathcal{J}(h^+, k^+, h^-, k^-) = \{(n, i, j) \in \mathbb{N}_0 \times \mathcal{C}(h^+, h^-) \mid \max[0, i + k^+ - h^+ + 1] \leq n \leq i + k^+, \max[0, k^- - j] \leq n, \text{ and } n < h^- + k^- - j \text{ iff } j > 0\}$$

3.2.1 An Illustrated Example

In order to illustrate a two-sided INAR(1) CUSUM chart, a simulated example of a selected design is presented in the following. The parameters are selected such that:

- $\lambda_0 = 5, \alpha_0 = 0.25$
- $h^+ = 21, h^- = 16$
- $k^+ = 6, k^- = 4$
- $C_0^+ = 0, C_0^- = 0$

First, 25 random INAR(1) process counts are generated when the process is assumed to be in-control. Then, the mean level of counts is shifted such that the mean is $\lambda = 7.5$ for the following observations generated. Figure 3.1 represents the graphical display for the two-sided INAR(1) CUSUM. The chart is constructed by plotting C_i^+ and C_i^- for each observation. Each point represents the values of C_i^+ and C_i^- . The decision intervals are also plotted on the chart. Notice that the lower CUSUM is the negative of the C_i^- values so that the two-sided chart can be visualized more easily. The lower decision interval value is also taken to be the negative of h^- .

According to Figure 3.1, the CUSUM statistics plot inside of the decision interval for the first 25 observations. After the 26th observation, there is a positive shift in the mean count level and the upward drift of the C_i^+ statistics can be seen. Point 28

exceeds the upper decision interval value, indicating that the process shifted to an out-of-control state.

A similar illustration is also used under a negative shift. After 25 in-control INAR(1) process counts are generated, the mean level of counts is shifted to $\lambda = 2.5$. Figure 3.2 presents the two-sided INAR(1) CUSUM for this case. The negative trend in C_i^- values after the 26th point can be seen on the control chart. Point 32 exceeds the lower decision interval value, thus giving an out of control signal.

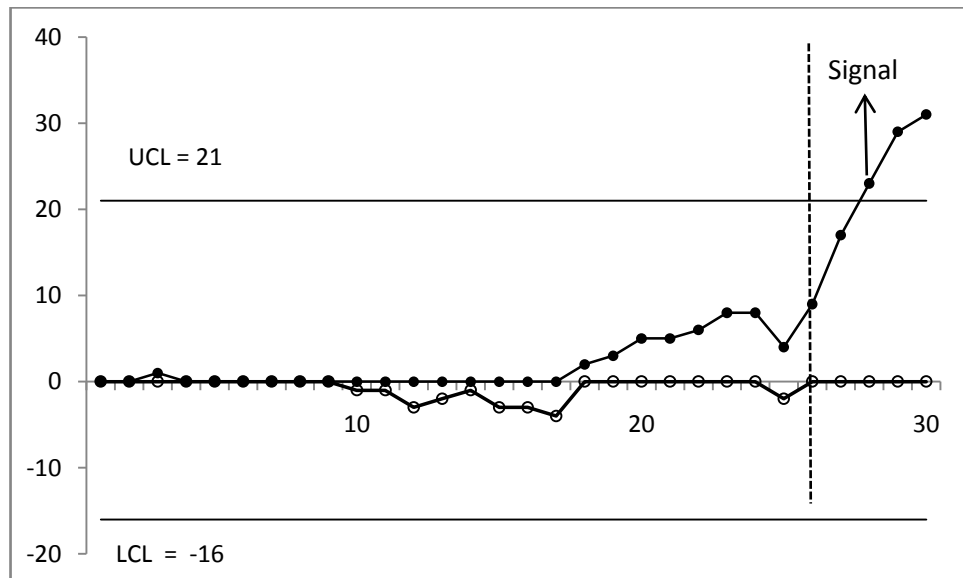


Figure 3.1 The Two-Sided INAR(1) CUSUM Chart Detecting an Increase in the Count Level

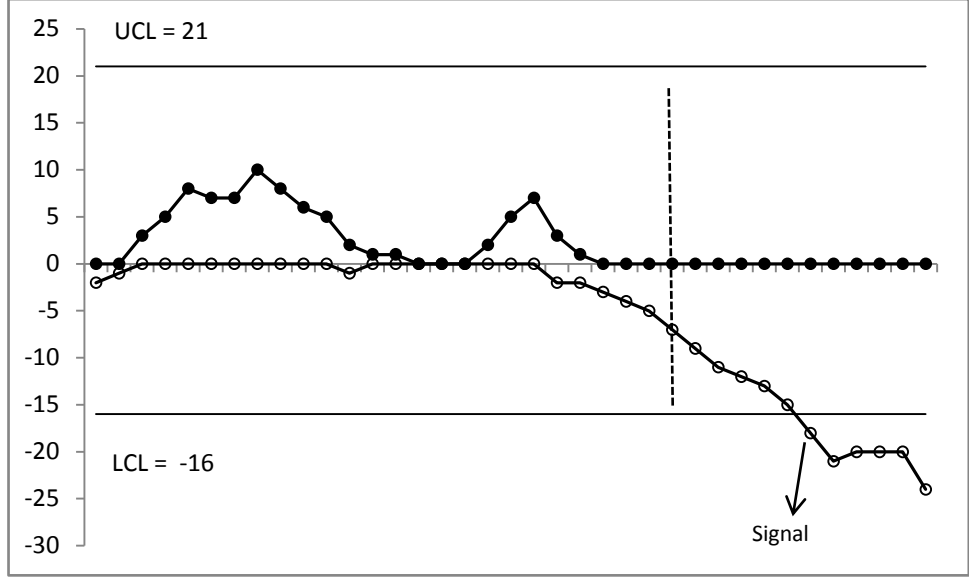


Figure 3.2 The Two-Sided INAR(1) CUSUM Chart Detecting a Decrease in the Count Level

3.2.2 ARL Computation of the Two-sided INAR(1) CUSUM Chart

Since (N_t, C_t^+, C_t^-) is a discrete Markov Chain, the ARLs of the CUSUM chart can again be computed exactly by the Markov Chain approach of Brook and Evans (1972).

Let $(N_t, C_t^+, C_t^-)_{\mathbb{N}}$ be a stationary Poisson INAR(1) process with $C_0^+, C_0^- \in \mathbb{N}_0$ and $\mathcal{C} = \mathcal{C}(h^+, h^-)$ be the control region of the process. Let $\mathcal{J}(h^+, k^+, h^-, k^-)$ be the set of in-control points. Then,

$$p_{m,a,b}(r) = P((C_{r+1}^+, C_{r+1}^-) \notin \mathcal{C}, (C_r^+, C_r^-), \dots, (C_2^+, C_2^-) \in \mathcal{C} \mid (C_1^+, C_1^-) = (a, b), N_1 = m)$$

where $p_{m,a,b}(r)$ with $r \geq 1$ is the probability that the $(r+1)^{st}$ pair (C_t^+, C_t^-) leads to the first out-of-control signal, given that $(N_1, C_1^+, C_1^-) = (m, a, b)$.

Let $\mu_{m,a}$ be the expected number of in-control signals before the first-out-of-control signal. Then, $\mu_{m,a}$ can be calculated as

$$\mu_{m,a,b} = \sum_{r=1}^{\infty} r \cdot p_{m,a,b}(r)$$

If $\boldsymbol{\mu} = (\dots, \mu_{n,i,j}, \dots)^T$ and $\mathbf{Q}^T = (p(n, i, j | m, a, b))_{(n,i,j),(m,a,b) \in \mathcal{J}(h^+, k^+, h^-, k^-)}$,

then summing over the possible in-control states (n, i, j) , one obtains:

$$\begin{aligned} p_{m,a,b}(r) &= P((C_{r+1}^+, C_{r+1}^-) \notin \mathcal{C}, (C_r^+, C_r^-), \dots, (C_2^+, C_2^-) \in \mathcal{C} | (C_1^+, C_1^-) = (a, b), N_1 = m) \\ &= \sum_{(n,i,j) \in \mathcal{J}(h^+, k^+, h^-, k^-)} p_{n,i,j}(r-1) \cdot p(n, i, j | m, a, b) \end{aligned}$$

Hence,

$$\begin{aligned} \mu_{m,a,b} &= \sum_{r=1}^{\infty} r \cdot p_{m,a,b}(r) = \sum_{r=1}^{\infty} p_{m,a,b}(r) \\ &\quad + \sum_{r=2}^{\infty} (r-1) \cdot \sum_{(n,i,j) \in \mathcal{J}(h^+, k^+, h^-, k^-)} p_{n,i,j}(r-1) \cdot p(n, i, j | m, a, b) \\ &= 1 + \sum_{(n,i,j) \in \mathcal{J}(h^+, k^+, h^-, k^-)} p(n, i, j | m, a, b) \cdot \sum_{s=1}^{\infty} s \cdot p_{n,i,j}(s) \\ &= 1 + \sum_{(n,i,j) \in \mathcal{J}(h^+, k^+, h^-, k^-)} p(n, i, j | m, a, b) \cdot \mu_{n,i,j} \end{aligned}$$

As a result, one obtains:

$$\boldsymbol{\mu} = \mathbf{1} \cdot (\mathbf{I} - \mathbf{Q})^{-1}$$

Note that the ARL depends on the initial values $(C_0^+, C_0^-) = (i_0, j_0)$ and it can be computed as;

$$ARL(i_0, j_0) = 1 + \sum_{(m,a,b) \in \mathcal{J}(h^+, k^+, h^-, k^-)} \mu_{m,a,b} \cdot p_1(m, a, b | i_0, j_0)$$

3.2.3 Validation of the Markov Chain Model through Simulation

The ARL computations of the two-sided INAR(1) CUSUM chart is coded in MATLAB. However, in order to verify this code and the developed equations, a simulation study is also conducted. In this simulation study, the counts of the INAR(1) Poisson process are generated in MATLAB. The code continues to generate random numbers and operate the chart until either of the cumulative sum values C_t^+ or C_t^- exceeds the decision interval values and then calculates the run length (RL). The code conducts 100,000 replications and calculates the average run length (ARL) performance by taking the average of the run lengths generated. Standard errors of the ARLs are also calculated in order to measure the accuracy of the ARLs.

After the simulation results are obtained for the selected designs, a comparison is made between the simulated ARLs and exact ARLs obtained by the Markov Chain approach proposed. The errors in percentages are calculated.

Table 3.1 gives some results from the simulation study. The exact ARL values and simulated ARL values are given and the errors in percentages are shown in the last column. According to the results, the error rates are so small that the Markov Chain approach used for calculation of ARL performance is validated. Complete results of this study are also provided in Appendix A.

Table 3.1 Some Results From Simulation Study

λ	α	h^+	h^-	k^+	k^-	ARL_0	ARL_{sim}	Error(%)
2.50	0.25	6.00	4.00	3.00	1.00	40.57	40.49	0.21
2.50	0.25	6.00	4.00	3.00	2.00	15.07	15.11	0.24
2.50	0.25	6.00	4.00	4.00	1.00	209.50	208.68	0.39
2.50	0.25	6.00	4.00	4.00	2.00	22.08	22.03	0.21
2.50	0.25	6.00	4.00	5.00	1.00	858.92	853.83	0.59
2.50	0.25	6.00	4.00	5.00	2.00	24.14	24.13	0.08
2.50	0.25	6.00	5.00	3.00	1.00	41.34	41.21	0.31

CHAPTER 4

PERFORMANCE EVALUATION

This chapter presents a computational analysis of the performance of the proposed two-sided INAR(1) CUSUM chart. In Section 4.1, the design procedure for the parameters of the chart is given. Then, in-control ARL_0 values are calculated and analyzed in Section 4.2. Next, an approximation to calculate ARLs from the literature is presented in Section 4.3. In Section 4.4, one-sided ARL_0 performances of the two-sided CUSUM chart are evaluated. In Section 4.5, out-of-control ARL values are calculated with positive and negative shifts from the in-control mean.

4.1 Selection of the Parameters for Designing the Two-Sided Poisson INAR(1) CUSUM

This section provides the design procedure for the Poisson INAR(1) CUSUM chart for monitoring a shift in the count rate in both positive and negative directions. The design of the chart requires values for the parameters; the reference values, k^+ and k^- , the decision interval values, h^+ and h^- , and the start values, C_0^+ and C_0^- .

In order to set the parameters of the chart, the parameters of the process should be determined first. Remember from Section 2.1.3.2 that the Poisson INAR(1) process model can be defined as:

$$N_t = \alpha N_{t-1} + \varepsilon_t$$

where α is the probability of survival from the previous time interval. The model depends on two parameters: λ , the count rate, and α , the dependence parameter. The design parameters $h^+, h^-, k^+, k^-, C_0^+$ and C_0^- are then selected according to the in-control values of the process parameters. In selecting the design parameters, one needs to consider the performance and hence, the Markov Chain approach in Section 3.2 or the results provided in the following can be utilized.

In this thesis, several integer combinations of h^+, h^-, k^+ and k^- values are considered such that the in-control ARL_0 is between 450 and 550, as suggested in Weiß and Testik (2009). ARL values are calculated for the considered in-control mean values $\lambda_0 = 2.5$ and 5. The in-control dependence values are considered to be $\alpha_0 = 0.25, 0.50$ and 0.75.

The parameters k^+ and k^- are the reference values of the two-sided INAR(1) CUSUM for the upper and lower parts, respectively. For a Poisson CUSUM under independent observations, the reference values are determined by the acceptable process mean and the mean level of counts that is to be detected quickly. The closer the reference values are to the mean, the faster the CUSUM chart can detect the small shifts. Thus k^+ and k^- values are chosen to be close to the in-control mean value λ_0 . The selected reference values for two different lambda values are given in Table 4.1.

Table 4.1 Selected Reference Values

	k^-	k^+
$\lambda_0 = 2.5$	1, 2	3, 4, 5
$\lambda_0 = 5$	3, 4	6, 7

The parameters h^+ and h^- are determined with respect to the desired in-control ARL . The value of these parameters should give a large ARL when the count rate

is at its in-control value. They should also be chosen to give a small ARL value when the process is running at a shifted count level. After reference values are selected, the decision interval values h^+ and h^- are chosen among those which result in an ARL value between 450 and 550.

The initial values are $C_0^+ = 0$ and $C_0^- = 0$ without the FIR feature. The values with the FIR feature are selected such that $C_0^+ = h^+/2$ and $C_0^- = h^-/2$ as recommended in the literature.

Note that although many other design combinations are considered in the thesis (see the results in Appendix B), only the preferred cases are provided here, in order to improve readability.

4.2 Results of In-Control ARL_0 Calculations

Recall that the control chart performance is usually evaluated by examining the distribution of the run length (RL). The expected value of the RL, namely ARL is often considered as the primary performance measure to evaluate the control charts. When designing a control chart, the parameters of the chart should be selected such that the rate of false alarms is at an acceptable level given that the process is in control. The average run length of an in-control process is denoted by ARL_0 in the following.

After parameter selection is completed, the Markov Chain approach of Brook and Evans (1972) is implemented in MATLAB. The MATLAB code is run under six different INAR(1) process combinations with several design parameters. Considered INAR(1) processes are provided in Table 4.2.

Table 4.2 Selected Process Parameters

Considered Processes	Combinations for ARL Calculation
Process 1	$\lambda_0 = 2.5$ and $\alpha_0 = 0.25$
Process 2	$\lambda_0 = 2.5$ and $\alpha_0 = 0.5$
Process 3	$\lambda_0 = 2.5$ and $\alpha_0 = 0.75$
Process 4	$\lambda_0 = 5$ and $\alpha_0 = 0.25$
Process 5	$\lambda_0 = 5$ and $\alpha_0 = 0.5$
Process 6	$\lambda_0 = 5$ and $\alpha_0 = 0.75$

For each of the six processes, the MATLAB code is run with a wide range of h^+ and h^- values. In order to achieve desired ARL_0 performance, ranges for the decision interval values are specified for the in-control parameters of the process. The settings are given in Table 4.3.

Table 4.3 Considered Decision Interval Parameters

Settings	Ranges of the decision interval parameters
Process 1	$h^+ \in [6,25], h^- \in [3,25]$
Process 2	$h^+ \in [6,30], h^- \in [3,30]$
Process 3	$h^+ \in [6,34], h^- \in [3,45]$
Process 4	$h^+ \in [8,30], h^- \in [5,30]$
Process 5	$h^+ \in [8,35], h^- \in [5,35]$
Process 6	$h^+ \in [8,42], h^- \in [5,55]$

The MATLAB code developed works very fast with the small value of the count rate $\lambda_0 = 2.5$ and the small survival rate $\alpha_0 = 0.25$. It takes about 10 seconds of CPU time to solve ARL_0 for each set of parameters. In Table 4.4, some sample results for Process 1, where $\lambda_0 = 2.5$ and $\alpha_0 = 0.25$ are provided. Complete results are provided in Appendix B.

Table 4.4 Sample ARL_0 Results for Process 1

λ_0	α_0	h^+	h^-	k^+	k^-	ARL_0
2.50	0.25	10.00	3.00	3.00	1.00	92.83
2.50	0.25	10.00	3.00	3.00	2.00	13.30
2.50	0.25	10.00	3.00	4.00	1.00	306.39
2.50	0.25	10.00	3.00	4.00	2.00	14.92
2.50	0.25	10.00	3.00	5.00	1.00	367.34
2.50	0.25	10.00	3.00	5.00	2.00	15.06
2.50	0.25	10.00	4.00	3.00	1.00	116.10
2.50	0.25	10.00	4.00	3.00	2.00	20.28
2.50	0.25	10.00	4.00	4.00	1.00	882.44
2.50	0.25	10.00	4.00	4.00	2.00	24.17
2.50	0.25	10.00	4.00	5.00	1.00	1683.19

After the ARL_0 values are calculated, the values that are between 450 and 550 are filtered so that desirable in-control values could be obtained. These designs are considered to be candidates since the out-of-control ARL performances should be considered together with the in-control ARLs in selecting better designs.

Similarly, sample results for Processes 2-6 are provided in Tables 4.5-4.9, respectively.

Table 4.5 Sample ARL_0 Results for Process 2

λ_0	α_0	h^+	h^-	k^+	k^-	ARL_0
2.50	0.50	15.00	10.00	3.00	1.00	160.55
2.50	0.50	15.00	10.00	3.00	2.00	60.74
2.50	0.50	15.00	10.00	4.00	1.00	1644.11
2.50	0.50	15.00	10.00	4.00	2.00	94.20
2.50	0.50	15.00	10.00	5.00	1.00	17334.71
2.50	0.50	15.00	10.00	5.00	2.00	99.61
2.50	0.50	15.00	11.00	3.00	1.00	160.69
2.50	0.50	15.00	11.00	3.00	2.00	69.10
2.50	0.50	15.00	11.00	4.00	1.00	1658.95
2.50	0.50	15.00	11.00	4.00	2.00	115.37
2.50	0.50	15.00	11.00	5.00	1.00	19134.19
2.50	0.50	15.00	11.00	5.00	2.00	123.52

Table 4.6 Sample ARL_0 Results for Process3

λ_0	α_0	h^+	h^-	k^+	k^-	ARL_0
2.50	0.75	20.00	15.00	3.00	1.00	141.68
2.50	0.75	20.00	15.00	3.00	2.00	59.12
2.50	0.75	20.00	15.00	4.00	1.00	710.98
2.50	0.75	20.00	15.00	4.00	2.00	93.61
2.50	0.75	20.00	15.00	5.00	1.00	3699.70
2.50	0.75	20.00	15.00	5.00	2.00	106.25
2.50	0.75	20.00	16.00	3.00	1.00	142.19
2.50	0.75	20.00	16.00	3.00	2.00	63.03
2.50	0.75	20.00	16.00	4.00	1.00	723.16
2.50	0.75	20.00	16.00	4.00	2.00	103.13

Table 4.7 Sample ARL_0 Results for Process 4

λ_0	α_0	h^+	h^-	k^+	k^-	ARL_0
5.00	0.25	15.00	16.00	6.00	3.00	292.08
5.00	0.25	15.00	16.00	6.00	4.00	228.22
5.00	0.25	15.00	16.00	7.00	3.00	2883.66
5.00	0.25	15.00	16.00	7.00	4.00	769.34
5.00	0.25	15.00	17.00	6.00	3.00	292.13
5.00	0.25	15.00	17.00	6.00	4.00	241.81
5.00	0.25	15.00	17.00	7.00	3.00	2888.63
5.00	0.25	15.00	17.00	7.00	4.00	947.98
5.00	0.25	15.00	18.00	6.00	3.00	292.16
5.00	0.25	15.00	18.00	6.00	4.00	252.97
5.00	0.25	15.00	18.00	7.00	3.00	2890.99
5.00	0.25	15.00	18.00	7.00	4.00	1144.93
5.00	0.25	15.00	19.00	6.00	3.00	292.17
5.00	0.25	15.00	19.00	6.00	4.00	261.94
5.00	0.25	15.00	19.00	7.00	3.00	2892.12
5.00	0.25	15.00	19.00	7.00	4.00	1353.67

Table 4.8 Sample ARL_0 Results for Process 5

λ_0	α_0	h^+	h^-	k^+	k^-	ARL_0
5.00	0.50	20.00	21.00	6.00	3.00	243.80
5.00	0.50	20.00	21.00	6.00	4.00	173.80
5.00	0.50	20.00	21.00	7.00	3.00	1500.77
5.00	0.50	20.00	21.00	7.00	4.00	435.94
5.00	0.50	20.00	22.00	6.00	3.00	243.97
5.00	0.50	20.00	22.00	6.00	4.00	182.15
5.00	0.50	20.00	22.00	7.00	3.00	1507.24
5.00	0.50	20.00	22.00	7.00	4.00	491.64
5.00	0.50	20.00	23.00	6.00	3.00	244.08
5.00	0.50	20.00	23.00	6.00	4.00	189.78
5.00	0.50	20.00	23.00	7.00	3.00	1511.46
5.00	0.50	20.00	23.00	7.00	4.00	550.46
5.00	0.50	20.00	24.00	6.00	3.00	244.16
5.00	0.50	20.00	24.00	6.00	4.00	196.68
5.00	0.50	20.00	24.00	7.00	3.00	1514.21
5.00	0.50	20.00	24.00	7.00	4.00	611.80

The MATLAB code slows down as the count rate λ_0 and survival rate α_0 get larger. It takes about 50 minutes of CPU time to solve ARL_0 for each set of parameters for Process 6 with parameters $\lambda_0 = 5$ and $\alpha_0 = 0.75$.

Table 4.9 Sample ARL_0 Results for Process 6

λ_0	α_0	h^+	h^-	k^+	k^-	ARL_0
5.00	0.75	37.00	33.00	6.00	3.00	346.66
5.00	0.75	37.00	33.00	6.00	4.00	196.99
5.00	0.75	37.00	33.00	7.00	3.00	1566.08
5.00	0.75	37.00	33.00	7.00	4.00	360.58
5.00	0.75	37.00	34.00	6.00	3.00	347.76
5.00	0.75	37.00	34.00	6.00	4.00	203.94
5.00	0.75	37.00	34.00	7.00	3.00	1588.13
5.00	0.75	37.00	34.00	7.00	4.00	383.87
5.00	0.75	37.00	35.00	6.00	3.00	348.68
5.00	0.75	37.00	35.00	6.00	4.00	210.75
5.00	0.75	37.00	35.00	7.00	3.00	1606.84

The ARL_0 values are more sensitive to reference values than decision interval values. The sensitivity also differs with respect to the dependence parameter, α . Figure 4.1 gives an analysis of ARL_0 values with respect to changing parameters. All parameters except for the upper reference value, k^+ , are the same. The design is chosen such that $\lambda = 2.5$, $h^+ = 9$, $h^- = 15$ and $k^- = 1$. The horizontal axis gives the differing k^+ values and the vertical axis shows ARL_0 values. For each different α value, the behavior of ARL_0 values with respect to the k^+ values is investigated.

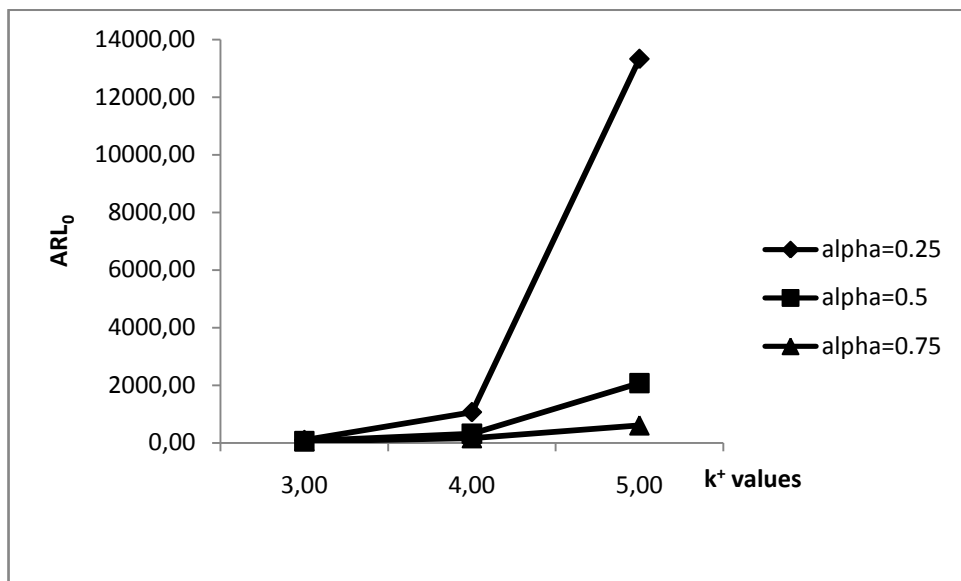


Figure 4.1 ARL_0 performance for different values of the upper reference value

According to Figure 4.1, the ARL_0 value changes the most when α value gets smaller. It is not very sensitive to changes in the upper reference level, especially when $\alpha = 0.75$. It can be concluded that ARL_0 performance is more sensitive to changes in the reference value as the dependence of the process to previous time interval decreases.

4.3 An Approximation for the Average Run Length of a Two-Sided INAR(1) CUSUM

As stated in the previous sections, the performance of a two-sided chart can be evaluated directly through the Markov Chain approach. The time to compute the ARL values takes longer as the parameters get larger. The dimension of the transition matrix of the Markov Chain may become extremely large and it takes so long that it is not efficient to calculate the exact ARL performances. In these cases, an approximation from the literature, as discussed below, can be used. The approximation is based on the fact that two-sided schemes can also be evaluated by combining two one-sided schemes. Note that the calculation of the run length of a one-sided scheme takes shorter time, making it easier to calculate the two-sided ARL performances from the ones of the one-sided charts.

Siegmund (1985) provides an approximation to the ARL calculation using two one-sided schemes. The approximation he proposes is as follows:

$$\frac{1}{ARL} = \frac{1}{ARL_h} + \frac{1}{ARL_l} \quad (4.1)$$

where ARL_h and ARL_l are the average run lengths for the upper and lower sides of the chart, respectively.

Lucas (1985) also proposes a formula that approximates the ARL of a two-sided scheme with head-start values C_0^+ and C_0^- as follows:

$$ARL(C_0^+, C_0^-) = \frac{ARL_h(C_0^+)ARL_l(0) + ARl_h(0)ARL_l(C_0^-) - ARl_h(0)ARL_l(0)}{ARL_h(0) + ARL_l(0)} \quad (4.2)$$

where the subscripts h and l refer to upper and lower sides, respectively. Here, the head-start values are indicated in parentheses.

Notice that when no head-start is used, Equation (4.2) reduces to Siegmund's approximation given in Equation (4.1).

In order to shorten the time for ARL computations and evaluate the errors of the approximation, Siegmund's approximation in Equation (4.1) is used. First, ARLs for the lower and upper CUSUMs are calculated separately by the one-sided ARL calculation implemented in MATLAB code provided by Weiß and Testik (2009). Then, Siegmund's approximation is used as a design approach. Since the calculation is the harmonic mean of two numbers, it can be easily concluded that if the one-sided charts are designed such that the upper and lower ARLs are around 1000, the two-sided ARL turns out to be around 500. This procedure has some potential advantages. Using the approximation is faster than using the two-sided code right from the beginning, because the one-sided codes are faster. Since the one-sided design has fewer parameters, the search space is narrower, again easing the calculations. Furthermore, if the upper and lower-sided charts are designed as being equally robust, unbiased ARL performance might be reached. Some example ARL_0 results using Siegmund's approximation are given in Table 4.10. Complete results can be found in Appendix C.

According to Table 4.10, the approximate ARLs and exact ARLs are very close. To see how much they differ, the errors in percentages are provided in the last column.

The error rates of the approximation are very small, ranging from 0.01% to 0.80%. According to the results, the exact ARL is always less than the approximate ARL, though this is a very small difference. In addition, the deviation from the exact ARL increases slightly with increasing α for a given λ value.

Basseville and Nikiforov(1993, pp 177-178) suggest that the ARL function of the two-sided CUSUM algorithm can be computed from the ARL functions of the two one-sided CUSUM algorithms as follows,

$$\frac{1}{L^T} \geq \frac{1}{L^l} + \frac{1}{L^u} \quad (4.3)$$

where L^T is the ARL function of the two-sided CUSUM, L^l is the ARL function of the lower one-sided CUSUM and L^u is the ARL function of the upper one-sided CUSUM. According to Equation (4.3), the exact ARL is always smaller than or equal to the approximate ARL. Our results confirm that this inequality holds.

Table 4.10 A Comparison Between Approximate and Exact In-Control ARLs

λ	α	h^+	h^-	k^+	k^-	Upper ARL	Lower ARL	Approximate ARL	Exact ARL	Error (%)
2.50	0.25	6.00	14.00	5.00	2.00	1646.26	806.79	541.44	540.96	0.09
2.50	0.25	7.00	13.00	5.00	2.00	3348.70	594.31	504.73	504.45	0.06
2.50	0.25	8.00	4.00	4.00	1.00	651.59	1798.62	478.31	477.89	0.09
2.50	0.25	8.00	13.00	5.00	2.00	6714.74	594.31	545.98	545.82	0.03
2.50	0.25	8.00	16.00	4.00	2.00	651.59	1474.09	451.86	451.45	0.09
2.50	0.25	8.00	17.00	4.00	2.00	651.59	1986.36	490.64	490.29	0.07
2.50	0.25	8.00	18.00	4.00	2.00	651.59	2672.69	523.87	523.57	0.06
2.50	0.25	9.00	14.00	4.00	2.00	1065.85	806.79	459.20	458.73	0.10
2.50	0.25	9.00	15.00	4.00	2.00	1065.85	1091.86	539.35	538.87	0.09
2.50	0.25	10.00	14.00	4.00	2.00	1734.46	806.79	550.65	550.24	0.08
2.50	0.25	11.00	13.00	4.00	2.00	2812.05	594.31	490.62	490.34	0.06
2.50	0.25	12.00	13.00	4.00	2.00	4546.89	594.31	525.61	525.41	0.04
2.50	0.25	13.00	13.00	4.00	2.00	7337.65	594.31	549.78	549.65	0.02
2.50	0.25	16.00	5.00	3.00	1.00	501.38	8618.93	473.82	473.72	0.02

4.4 Evaluation of the One-Sided ARL_0 Performances of the Two-Sided INAR(1) CUSUMS

Using the exact results for the two-sided INAR(1) CUSUM where the ARL_0 values are between 450 and 550, the upper and lower ARLs are calculated individually for the designs. The designs that have one-sided ARLs around 1000 are selected. The interval of one-sided ARLs for selection is (900,1100), i.e. the two-sided designs that give upper and lower ARLs between 900 and 1100 are selected for further investigation. The aim is to make both sides of the two-sided CUSUM chart equally robust to false signals.

After specifying the designs which have close upper and lower ARLs, the focus will be on reaching unbiased ARLs in case of a shift in the process parameters, as will be discussed next. Table 4.11 presents 32 designs that have close upper and lower ARLs; that is, the designs have similar upper and lower performances.

Table 4.11 Designs Having Close Upper and Lower ARLs

λ_0	α_0	h^+	h^-	k^+	k^-	Upper ARL	Lower ARL	Exact ARL
2.50	0.25	9.00	15.00	4.00	2.00	1065.85	1091.86	538.87
2.50	0.25	19.00	15.00	3.00	2.00	960.75	1091.86	510.68
2.50	0.50	7.00	5.00	5.00	1.00	910.28	967.19	467.49
2.50	0.50	7.00	22.00	5.00	2.00	910.28	988.08	472.48
2.50	0.50	13.00	5.00	4.00	1.00	975.25	967.19	484.28
2.50	0.50	13.00	22.00	4.00	2.00	975.25	988.08	489.62
2.50	0.50	29.00	5.00	3.00	1.00	986.37	967.19	487.22
2.50	0.50	29.00	22.00	3.00	2.00	986.37	988.08	492.59
2.50	0.75	11.00	39.00	5.00	2.00	959.51	972.70	479.55
2.50	0.75	11.00	40.00	5.00	2.00	959.51	1052.95	498.56
2.50	0.75	22.00	39.00	4.00	2.00	955.06	972.70	478.69
2.50	0.75	22.00	40.00	4.00	2.00	955.06	1052.95	497.61
2.50	0.75	23.00	39.00	4.00	2.00	1077.70	972.70	508.05
2.50	0.75	23.00	40.00	4.00	2.00	1077.70	1052.95	529.38
5.00	0.25	12.00	7.00	7.00	3.00	975.75	1073.67	510.63
5.00	0.25	12.00	16.00	7.00	4.00	975.75	1048.74	504.97
5.00	0.25	21.00	7.00	6.00	3.00	1051.62	1073.67	530.78

Table 4.11 Designs Having Close Upper and Lower ARLs(*continued*)

5.00	0.25	21.00	16.00	6.00	4.00	1051.62	1048.74	524.66
5.00	0.50	18.00	10.00	7.00	3.00	1016.41	1031.50	510.60
5.00	0.50	18.00	24.00	7.00	4.00	1016.41	1027.61	509.75
5.00	0.50	31.00	10.00	6.00	3.00	929.26	1031.50	487.64
5.00	0.50	31.00	24.00	6.00	4.00	929.26	1027.61	486.87
5.00	0.50	32.00	10.00	6.00	3.00	1044.53	1031.50	517.76
5.00	0.50	32.00	24.00	6.00	4.00	1044.53	1027.61	516.89
5.00	0.75	31.00	17.00	7.00	3.00	1001.08	922.60	476.55
5.00	0.75	31.00	42.00	7.00	4.00	1001.08	932.41	479.45
5.00	0.75	31.00	43.00	7.00	4.00	1001.08	1006.37	498.54
5.00	0.75	31.00	44.00	7.00	4.00	1001.08	1085.87	517.56
5.00	0.75	32.00	17.00	7.00	3.00	1094.98	922.60	497.16
5.00	0.75	32.00	42.00	7.00	4.00	1094.98	932.41	500.29
5.00	0.75	32.00	43.00	7.00	4.00	1094.98	1006.37	521.09
5.00	0.75	32.00	44.00	7.00	4.00	1094.98	1085.87	541.88

4.5 Calculation of Out-of-Control ARL (ARL_1) Values

After the in-control ARL_0 performances are calculated and the designs with the specified performances are selected, the next step is the calculation of out-of-control ARLs (ARL_1). The shifts in the count rate are considered in both positive and negative directions. Let δ be the shift in the process mean such that $\lambda = \lambda_0 + \delta$. The shifts δ are considered to be multiples of λ_0 . Considered δ values for the positive shifts are; $0.05 * \lambda_0, 0.1 * \lambda_0, 0.2 * \lambda_0, 0.5 * \lambda_0, 1 * \lambda_0$ and $2 * \lambda_0$. For decreases, considered negative shifts for δ are; $-0.05 * \lambda_0, -0.1 * \lambda_0, -0.2 * \lambda_0, -0.5 * \lambda_0$ and $-0.75 * \lambda_0$. The positive and negative shifts are evaluated separately for both $\lambda_0 = 2.5$ and $\lambda_0 = 5$ and the corresponding out-of-control ARL performances are obtained. The primary aim is to determine better designs in detecting shifts and to have unbiased ARLs so that the out-of-control ARLs are smaller than the in-control ARLs.

4.5.1 Positive Shifts

First, the positive shifts are implemented in the MATLAB code. Among the 32 designs provided in Table 4.11, there are 14 designs for $\lambda_0 = 2.5$ and there are 18 designs for $\lambda_0 = 5$. The shifts are applied separately for each set of process parameters. Table 4.12 represents the count levels after the considered process shifts for $\lambda_0 = 2.5$:

Table 4.12 Shifted Count Levels for $\lambda_0 = 2.5$

Magnitude of shifts	Shifted count rates
$\lambda = \lambda_0 + 0.05 * \lambda_0$	$\lambda=2.625$
$\lambda = \lambda_0 + 0.1 * \lambda_0$	$\lambda=2.75$
$\lambda = \lambda_0 + 0.2 * \lambda_0$	$\lambda=3$
$\lambda = \lambda_0 + 0.5 * \lambda_0$	$\lambda=3.75$
$\lambda = \lambda_0 + 1 * \lambda_0$	$\lambda=5$
$\lambda = \lambda_0 + 2 * \lambda_0$	$\lambda=7.5$

ARL₁ performance results for the selected charts when $\lambda_0 = 2.5$ are provided in Table 4.13. In the table, highlighted cells indicate bias since those ARL values are greater than the corresponding in-control ARL values (see column $\lambda = 2.5$ for the ARL₀ performances).

Table 4.13 indicates that the two-sided Poisson INAR(1) CUSUM chart is effective in detecting a positive shift δ , especially a shift such as $\delta = 0.5 \cdot \lambda_0$ in the process mean. The out-of-control ARL performance of the chart increases when the reference values k^+ and k^- are closer to the mean. Since $\lambda_0 = 2.5$, the chart gives better performance when $k^+ = 3$ and $k^- = 2$.

Table 4.13 Out-of- Control ARLs for $\lambda_0 = 2.5$ (positive shifts)

ARLs of Poisson INAR(1) CUSUM Chart with $\lambda_0=2.5$											
α_0	h^+	h^-	k^+	k^-	$\lambda=2.5$	$\lambda=2.625$	$\lambda=2.75$	$\lambda=3$	$\lambda=3.75$	$\lambda=5$	$\lambda=7.5$
0.25	9.00	15.00	4.00	2.00	538.87	523.78	401.61	197.76	37.21	9.38	3.35
0.25	19.00	15.00	3.00	2.00	510.68	396.37	246.26	100.54	25.10	10.43	4.90
0.50	7.00	5.00	5.00	1.00	467.49	437.87	374.37	239.39	64.15	15.78	4.11
0.50	7.00	22.00	5.00	2.00	472.48	480.31	412.02	250.14	64.24	15.78	4.11
0.50	13.00	5.00	4.00	1.00	484.28	442.05	363.52	214.22	51.43	14.04	4.81
0.50	13.00	22.00	4.00	2.00	489.62	485.30	398.89	222.79	51.50	14.04	4.81
0.50	29.00	5.00	3.00	1.00	487.22	391.43	279.66	138.78	38.79	16.05	7.36
0.50	29.00	22.00	3.00	2.00	492.59	424.95	300.16	142.35	38.82	16.05	7.36
0.75	11.00	39.00	5.00	2.00	479.55	489.30	443.96	304.38	94.96	26.68	6.87
0.75	11.00	40.00	5.00	2.00	498.56	503.07	451.45	305.75	94.96	26.68	6.87
0.75	22.00	39.00	4.00	2.00	478.69	477.14	420.60	274.28	81.40	24.88	8.26
0.75	22.00	40.00	4.00	2.00	497.61	490.21	427.31	275.39	81.40	24.88	8.26
0.75	23.00	39.00	4.00	2.00	508.05	515.72	458.94	298.72	86.17	25.86	8.56
0.75	23.00	40.00	4.00	2.00	529.38	530.99	466.93	300.03	86.17	25.86	8.56

Table 4.14 Out-of- Control ARLs for $\lambda_0 = 2.5$ with FIR feature (positive shifts)

ARLs of Poisson INAR(1) CUSUM Chart with $\lambda_0=2.5$ (FIR)													
α_0	h^+	h^-	k^+	k^-	C_0^+	C_0^-	$\lambda=2.5$	$\lambda=2.625$	$\lambda=2.75$	$\lambda=3$	$\lambda=3.75$	$\lambda=5$	$\lambda=7.5$
0.25	9.00	15.00	4.00	2.00	5.00	8.00	479.03	476.18	368.09	178.85	28.99	5.77	1.89
0.25	19.00	15.00	3.00	2.00	10.00	8.00	431.29	331.34	199.05	73.02	14.40	5.60	2.68
0.50	7.00	5.00	5.00	1.00	4.00	3.00	435.61	410.24	351.67	224.42	57.25	12.12	2.52
0.50	7.00	22.00	5.00	2.00	4.00	11.00	425.45	443.96	386.79	236.84	57.72	12.13	2.52
0.50	13.00	5.00	4.00	1.00	7.00	3.00	445.02	406.34	332.92	192.61	41.12	9.11	2.72
0.50	13.00	22.00	4.00	2.00	7.00	11.00	434.58	439.99	365.07	202.40	41.48	9.12	2.72
0.50	29.00	5.00	3.00	1.00	15.00	3.00	423.39	330.15	226.54	101.87	22.98	8.78	3.98
0.50	29.00	22.00	3.00	2.00	15.00	11.00	412.42	352.84	242.75	105.69	23.21	8.79	3.98
0.75	11.00	39.00	5.00	2.00	6.00	20.00	418.17	439.48	407.22	285.09	86.73	21.60	4.39
0.75	11.00	40.00	5.00	2.00	6.00	20.00	439.04	455.57	416.85	287.51	86.78	21.60	4.39
0.75	22.00	39.00	4.00	2.00	11.00	20.00	410.53	419.63	375.82	247.15	68.16	17.53	4.93
0.75	22.00	40.00	4.00	2.00	11.00	20.00	431.04	434.76	384.43	249.18	68.20	17.53	4.93
0.75	23.00	39.00	4.00	2.00	12.00	20.00	434.90	452.42	408.73	267.74	71.12	17.71	4.94
0.75	23.00	40.00	4.00	2.00	12.00	20.00	457.71	469.75	418.70	270.04	71.17	17.71	4.94

When evaluated for each value of α_0 separately, the chart shows different behavior for small shifts and large shifts. For small to moderate shifts, reference values are dominant and they affect the performance of the chart more than the decision interval values. For example, when $\lambda_0 = 2.5$ and $\alpha_0 = 0.25$, the design

h^+	h^-	k^+	k^-
19.00	15.00	3.00	2.00

has good performance. Similarly, for $\lambda_0 = 2.5$ and $\alpha_0 = 0.5$, the design

h^+	h^-	k^+	k^-
29.00	22.00	3.00	2.00

also shows good performance. Notice that these designs both have reference values close to $\lambda_0 = 2.5$. However, for larger shifts, the decision interval values begin to dominate the ARL performances. The tighter the limits are, the faster the chart detects the larger shifts.

Table 4.14 gives the ARL_1 performances with the same designs but with FIR feature. The headstart values used are $C_0^+ = h^+/2$ and $C_0^- = h^-/2$. According to the results in Table 4.14, the FIR feature improves the performance of the chart for process shifts at the start-up by reducing the ARL_1 values. However, it increases the number of biased designs. This may be due to the fact that the chart already starts with a shift when FIR is implemented.

For the cases where $\lambda_0 = 5$, values for the shifted count rates are provided in Table 4.15 and the ARL performances are provided in Table 4.16. Note again that the column corresponding to $\lambda = 5$ is the in-control ARL performance.

Table 4.15 Shifted Count Levels for $\lambda_0 = 5$

Magnitude of shifts	Shifted count rates
$\lambda = \lambda_0 + 0.05 * \lambda_0$	$\lambda=5.25$
$\lambda = \lambda_0 + 0.1 * \lambda_0$	$\lambda=5.5$
$\lambda = \lambda_0 + 0.2 * \lambda_0$	$\lambda=6$
$\lambda = \lambda_0 + 0.5 * \lambda_0$	$\lambda=7.5$
$\lambda = \lambda_0 + 1 * \lambda_0$	$\lambda=10$
$\lambda = \lambda_0 + 2 * \lambda_0$	$\lambda=15$

Table 4.16 indicates that the two-sided Poisson INAR(1) CUSUM chart is effective in detecting a positive shift δ , especially a shift such as $\delta = 0.5 \cdot \lambda_0$ in the process mean. The out-of-control ARL performance of the chart increases when the reference values k^+ and k^- are closer to the mean. Since $\lambda_0 = 5$, the chart gives better performance when $k^+ = 6$ and $k^- = 4$. When the reference values and the dependence parameter, α , are the same, tighter decision intervals, h^+ and h^- , give better results in terms of ARL_1 performance. For small to moderate shifts, reference values affect the performance of the chart more than the decision interval values. For example, when $\lambda_0 = 5$ and $\alpha_0 = 0.25$, the design

h^+	h^-	k^+	k^-
21.00	16.00	6.00	4.00

has a good performance in terms of ARL_1 . Similarly, for $\lambda_0 = 5$ and $\alpha_0 = 0.5$, the design

h^+	h^-	k^+	k^-
31.00	10.00	6.00	3.00

again has good ARL performance. For larger shifts, the smaller the limits, the faster the chart detects the shifts irrespective of the reference values.

Table 4.16 Out-of- Control ARLs for $\lambda_0 = 5$ (positive shifts)

ARLs of Poisson INAR(1) CUSUM Chart with $\lambda_0=5$											
α_0	h^+	h^-	k^+	k^-	$\lambda=5$	$\lambda=5.25$	$\lambda=5.5$	$\lambda=6$	$\lambda=7.5$	$\lambda=10$	$\lambda=15$
0.25	12.00	7.00	7.00	3.00	510.63	405.71	263.69	103.43	16.32	4.92	2.11
0.25	12.00	16.00	7.00	4.00	504.97	429.82	273.70	104.11	16.32	4.92	2.11
0.25	21.00	7.00	6.00	3.00	530.78	357.57	195.53	67.13	14.76	6.06	2.92
0.25	21.00	16.00	6.00	4.00	524.66	376.17	200.99	67.42	14.76	6.06	2.92
0.50	18.00	10.00	7.00	3.00	510.60	432.62	304.89	135.67	24.91	7.43	2.97
0.50	18.00	24.00	7.00	4.00	509.75	458.44	318.22	136.97	24.91	7.43	2.97
0.50	31.00	10.00	6.00	3.00	487.64	359.86	224.25	91.69	22.15	8.92	4.14
0.50	31.00	24.00	6.00	4.00	486.87	377.59	231.41	92.29	22.15	8.92	4.14
0.50	32.00	10.00	6.00	3.00	517.76	387.73	240.32	96.21	22.82	9.18	4.25
0.50	32.00	24.00	6.00	4.00	516.89	408.36	248.56	96.88	22.82	9.18	4.25
0.75	31.00	17.00	7.00	3.00	476.55	432.69	341.23	184.35	43.21	13.20	4.90
0.75	31.00	42.00	7.00	4.00	479.45	456.23	358.64	187.72	43.23	13.20	4.90
0.75	31.00	43.00	7.00	4.00	498.54	467.00	362.55	188.04	43.23	13.20	4.90
0.75	31.00	44.00	7.00	4.00	517.56	477.26	366.12	188.32	43.23	13.20	4.90
0.75	32.00	17.00	7.00	3.00	497.16	457.48	362.49	194.63	44.62	13.54	5.03
0.75	32.00	42.00	7.00	4.00	500.29	483.81	382.15	198.38	44.64	13.54	5.03
0.75	32.00	43.00	7.00	4.00	521.09	495.92	386.58	198.73	44.64	13.54	5.03
0.75	32.00	44.00	7.00	4.00	541.88	507.48	390.63	199.05	44.64	13.54	5.03

Table 4.17 Out-of- Control ARLs for $\lambda_0 = 5$ with FIR feature (positive shifts)

ARLs of Poisson INAR(1) CUSUM Chart with $\lambda_0=5$ (FIR)													
α_0	h^+	h^-	k^+	k^-	C_0^+	C_0^-	$\lambda=5$	$\lambda=5.25$	$\lambda=5.5$	$\lambda=6$	$\lambda=7.5$	$\lambda=10$	$\lambda=15$
0.25	12.00	7.00	7.00	3.00	6.00	4.00	478.73	379.88	244.55	91.97	11.75	3.02	1.31
0.25	12.00	16.00	7.00	4.00	6.00	8.00	467.93	402.44	255.11	93.17	11.77	3.02	1.31
0.25	21.00	7.00	6.00	3.00	11.00	4.00	482.14	315.44	164.19	49.22	8.44	3.34	1.65
0.25	21.00	16.00	6.00	4.00	11.00	8.00	470.86	331.82	169.69	49.81	8.46	3.34	1.65
0.50	18.00	10.00	7.00	3.00	9.00	5.00	484.06	408.40	284.61	121.53	18.34	4.56	1.75
0.50	18.00	24.00	7.00	4.00	9.00	12.00	466.80	425.24	294.74	122.66	18.35	4.56	1.75
0.50	31.00	10.00	6.00	3.00	16.00	5.00	445.08	318.74	189.93	69.17	13.13	4.93	2.33
0.50	31.00	24.00	6.00	4.00	16.00	12.00	428.67	328.24	194.32	69.60	13.14	4.93	2.33
0.50	32.00	10.00	6.00	3.00	16.00	5.00	476.07	346.80	206.01	73.70	13.81	5.19	2.45
0.50	32.00	24.00	6.00	4.00	16.00	12.00	458.59	358.55	211.26	74.18	13.82	5.19	2.45
0.75	31.00	17.00	7.00	3.00	16.00	9.00	442.00	399.96	312.35	162.83	32.19	8.01	2.75
0.75	31.00	42.00	7.00	4.00	16.00	21.00	425.59	410.83	323.63	165.36	32.23	8.01	2.75
0.75	31.00	43.00	7.00	4.00	16.00	22.00	441.41	419.93	326.91	165.61	32.23	8.01	2.75
0.75	31.00	44.00	7.00	4.00	16.00	22.00	461.34	431.40	331.42	166.18	32.23	8.01	2.75
0.75	32.00	17.00	7.00	3.00	16.00	9.00	462.72	424.72	333.60	173.17	33.65	8.38	2.89
0.75	32.00	42.00	7.00	4.00	16.00	21.00	445.71	437.64	346.73	176.04	33.69	8.38	2.89
0.75	32.00	43.00	7.00	4.00	16.00	22.00	463.06	447.96	350.49	176.32	33.69	8.38	2.89
0.75	32.00	44.00	7.00	4.00	16.00	22.00	484.77	460.79	355.54	176.94	33.70	8.38	2.89

Table 4.17 gives the ARL_1 performances with the same designs but with FIR feature. The headstart values used are $C_0^+ = h^+/2$ and $C_0^- = h^-/2$. The results indicate that the FIR feature improves the performance of the chart for mean shifts at the start-up. Especially for small shifts, the ARL_1 values are reduced.

4.5.2 Negative Shifts

The negative shifts considered when $\lambda_0 = 2.5$ are provided in Table , where shifted count levels are provided in the second column.

Table 4.18 Shifted Count Levels for $\lambda_0 = 2.5$

Magnitude of shifts	Shifted count rates
$\lambda = \lambda_0 - 0.05 * \lambda_0$	$\lambda=2.375$
$\lambda = \lambda_0 - 0.1 * \lambda_0$	$\lambda=2.2$
$\lambda = \lambda_0 - 0.2 * \lambda_0$	$\lambda=2$
$\lambda = \lambda_0 - 0.5 * \lambda_0$	$\lambda=1.25$
$\lambda = \lambda_0 - 0.75 * \lambda_0$	$\lambda=0.625$

ARL performance results when $\lambda_0 = 2.5$ are provided in Table 4.19. Here, again the highlighted cell indicates bias since the ARL value is greater than the corresponding in-control ARL value, which is the corresponding entry in column $\lambda = 2.5$.

Table 4.19 indicates that the two-sided Poisson INAR(1) CUSUM chart is effective in detecting a negative shift δ , especially a shift such as $\delta = -0.5 \cdot \lambda_0$, in the process mean.

Table 4.19 Out-of- Control ARLs for $\lambda_0 = 2.5$ (negative shifts)

ARLs of Poisson INAR(1) CUSUM Chart with $\lambda_0=2.5$										
α_0	h^+	h^-	k^+	k^-	$\lambda=2.5$	$\lambda=2.375$	$\lambda=2.25$	$\lambda=2$	$\lambda=1.25$	$\lambda=0.625$
0.25	9.00	15.00	4.00	2.00	538.87	401.03	245.56	92.20	19.92	11.24
0.25	19.00	15.00	3.00	2.00	510.68	417.94	254.27	92.85	19.92	11.24
0.50	7.00	5.00	5.00	1.00	467.49	444.26	377.43	220.72	36.58	11.66
0.50	7.00	22.00	5.00	2.00	472.48	378.03	263.17	120.45	29.41	16.50
0.50	13.00	5.00	4.00	1.00	484.28	463.37	390.79	223.68	36.58	11.66
0.50	13.00	22.00	4.00	2.00	489.62	391.82	269.64	121.34	29.41	16.50
0.50	29.00	5.00	3.00	1.00	487.22	496.12	417.77	228.69	36.58	11.66
0.50	29.00	22.00	3.00	2.00	492.59	415.02	282.29	122.83	29.41	16.50
0.75	11.00	39.00	5.00	2.00	479.55	413.26	324.08	182.60	52.47	29.36
0.75	11.00	40.00	5.00	2.00	498.56	432.43	339.20	189.58	53.80	30.08
0.75	22.00	39.00	4.00	2.00	478.69	417.54	328.17	183.90	52.48	29.36
0.75	22.00	40.00	4.00	2.00	497.61	437.09	343.67	190.97	53.81	30.08
0.75	23.00	39.00	4.00	2.00	508.05	433.94	335.43	184.98	52.49	29.36
0.75	23.00	40.00	4.00	2.00	529.38	455.07	351.62	192.14	53.82	30.08

According to Table 4.19, the out-of-control ARL performance of the CUSUM chart for $\lambda_0 = 2.5$ is satisfactory when the lower reference value k^- is at its closest value to the mean, i.e. $k^- = 2$.

For the negative shifts applied when $\lambda_0 = 5$, Table 4.20 gives shifted count levels and Table 4.21 provides the obtained ARL performances.

Table 4.20 Shifted Count Levels for $\lambda_0 = 5$

Magnitude of shifts	Shifted count rates
$\lambda = \lambda_0 - 0.05 * \lambda_0$	$\lambda=4.75$
$\lambda = \lambda_0 - 0.1 * \lambda_0$	$\lambda=4.5$
$\lambda = \lambda_0 - 0.2 * \lambda_0$	$\lambda=4$
$\lambda = \lambda_0 - 0.5 * \lambda_0$	$\lambda=2.5$
$\lambda = \lambda_0 - 0.75 * \lambda_0$	$\lambda=1.25$

Table 4.21 indicates that the two-sided Poisson INAR(1) CUSUM chart is effective in detecting a negative shift δ , especially a shift of magnitude $\delta = -0.5 \cdot \lambda_0$, in the process mean. The chart seems to show better performance when the decision interval values, h^+ and h^- , are not much different from each other.

Table 4.21 Out-of- Control ARLs for $\lambda_0 = 5$ (negative shifts)

ARLs of Poisson INAR(1) CUSUM Chart with $\lambda_0=5$										
α_0	h^+	h^-	k^+	k^-	$\lambda=5$	$\lambda=4.75$	$\lambda=4.5$	$\lambda=4$	$\lambda=2.5$	$\lambda=1.25$
0.25	12.00	7.00	7.00	3.00	510.63	455.67	308.65	112.38	11.31	4.47
0.25	12.00	16.00	7.00	4.00	504.97	357.46	190.65	58.70	11.18	6.27
0.25	21.00	7.00	6.00	3.00	530.78	490.42	320.43	112.79	11.31	4.47
0.25	21.00	16.00	6.00	4.00	524.66	378.53	195.10	58.81	11.18	6.27
0.50	18.00	10.00	7.00	3.00	510.60	454.23	322.60	132.30	16.39	6.38
0.50	18.00	24.00	7.00	4.00	509.75	382.00	227.49	82.17	16.86	9.28
0.50	31.00	10.00	6.00	3.00	487.64	464.75	330.37	132.86	16.39	6.38
0.50	31.00	24.00	6.00	4.00	486.87	389.44	231.36	82.39	16.86	9.28
0.50	32.00	10.00	6.00	3.00	517.76	478.83	333.67	132.95	16.39	6.38
0.50	32.00	24.00	6.00	4.00	516.89	399.29	232.98	82.43	16.86	9.28
0.75	31.00	17.00	7.00	3.00	476.55	434.49	338.52	169.88	28.32	10.96
0.75	31.00	42.00	7.00	4.00	479.45	394.56	276.55	127.09	29.86	16.13
0.75	31.00	43.00	7.00	4.00	498.54	413.80	289.53	131.44	30.52	16.49
0.75	31.00	44.00	7.00	4.00	517.56	433.52	302.91	135.86	31.19	16.86
0.75	32.00	17.00	7.00	3.00	497.16	445.62	342.69	170.23	28.32	10.96
0.75	32.00	42.00	7.00	4.00	500.29	403.73	279.35	127.28	29.86	16.13
0.75	32.00	43.00	7.00	4.00	521.09	423.87	292.59	131.65	30.52	16.49
0.75	32.00	44.00	7.00	4.00	541.88	444.58	306.25	136.08	31.19	16.86

According to Table 4.21, notice that the out-of-control ARL performance of the CUSUM chart for $\lambda_0 = 5$ is satisfactory when the lower reference value k^- is at its closest value to the mean, i.e. $k^- = 4$.

For a better interpretation, the results of the negative and positive shifts are combined and provided in Tables 4.22 and 4.23 for the cases $\lambda_0 = 2.5$ and $\lambda_0 = 5$, respectively.

In the tables, first four columns are the CUSUM chart parameters and the rest are the ARL values for the corresponding count rates. The column with $\lambda = 2.5$ ($\lambda = 5$) gives in-control ARL_0 values. The highlighted cells are the biased ARL values. Recall that the out-of-control ARLs should be smaller than the in-control ARLs for an unbiased performance.

As can be seen from Table 4.23, there are no biased designs when $\lambda = 5$. For such a large mean, the Poisson distribution is nearly symmetrical and it can be concluded that this property carries over to the distribution of the CUSUM statistics. In addition, it is observed that the unbiasedness can be achieved if both individual (one-sided) ARLs nearly coincide.

Table 4.22 ARL_1 Results of the Control Chart when $\lambda_0 = 2.5$

ARLs of Poisson INAR(1) CUSUM Chart with $\lambda_0=2.5$																
α_0	h^+	h^-	k^+	k^-	$\lambda=0.625$	$\lambda=1.25$	$\lambda=2$	$\lambda=2.25$	$\lambda=2.375$	$\lambda=2.5$	$\lambda=2.625$	$\lambda=2.75$	$\lambda=3$	$\lambda=3.75$	$\lambda=5$	$\lambda=7.5$
0.25	9	15	4	2	11.24	19.92	92.20	245.56	401.03	538.87	523.78	401.61	197.76	37.21	9.38	3.35
0.25	19	15	3	2	11.24	19.92	92.85	254.27	417.94	510.68	396.37	246.26	100.54	25.10	10.43	4.90
0.50	7	5	5	1	11.66	36.58	220.72	377.43	444.26	467.49	437.87	374.37	239.39	64.15	15.78	4.11
0.50	7	22	5	2	16.50	29.41	120.45	263.17	378.03	472.48	480.31	412.02	250.14	64.24	15.78	4.11
0.50	13	5	4	1	11.66	36.58	223.68	390.79	463.37	484.28	442.05	363.52	214.22	51.43	14.04	4.81
0.50	13	22	4	2	16.50	29.41	121.34	269.64	391.82	489.62	485.30	398.89	222.79	51.50	14.04	4.81
0.50	29	5	3	1	11.66	36.58	228.69	417.77	496.12	487.22	391.43	279.66	138.78	38.79	16.05	7.36
0.50	29	22	3	2	16.50	29.41	122.83	282.29	415.02	492.59	424.95	300.16	142.35	38.82	16.05	7.36
0.75	11	39	5	2	29.36	52.47	182.60	324.08	413.26	479.55	489.30	443.96	304.38	94.96	26.68	6.87
0.75	11	40	5	2	30.08	53.80	189.58	339.20	432.43	498.56	503.07	451.45	305.75	94.96	26.68	6.87
0.75	22	39	4	2	29.36	52.48	183.90	328.17	417.54	478.69	477.14	420.60	274.28	81.40	24.88	8.26
0.75	22	40	4	2	30.08	53.81	190.97	343.67	437.09	497.61	490.21	427.31	275.39	81.40	24.88	8.26
0.75	23	39	4	2	29.36	52.49	184.98	335.43	433.94	508.05	515.72	458.94	298.72	86.17	25.86	8.56
0.75	23	40	4	2	30.08	53.82	192.14	351.62	455.07	529.38	530.99	466.93	300.03	86.17	25.86	8.56

Table 4.23 ARL_1 Results of the Control Chart when $\lambda_0 = 5$

ARLs of Poisson INAR(1) CUSUM Chart with $\lambda_0=5$																
α_0	h^+	h^-	k^+	k^-	$\lambda=1.25$	$\lambda=2.5$	$\lambda=4$	$\lambda=4.5$	$\lambda=4.75$	$\lambda=5$	$\lambda=5.25$	$\lambda=5.5$	$\lambda=6$	$\lambda=7.5$	$\lambda=10$	$\lambda=15$
0.25	12	7	7	3	4.47	11.31	112.38	308.65	455.67	510.63	405.71	263.69	103.43	16.32	4.92	2.11
0.25	12	16	7	4	6.27	11.18	58.70	190.65	357.46	504.97	429.82	273.70	104.11	16.32	4.92	2.11
0.25	21	7	6	3	4.47	11.31	112.79	320.43	490.42	530.78	357.57	195.53	67.13	14.76	6.06	2.92
0.25	21	16	6	4	6.27	11.18	58.81	195.10	378.53	524.66	376.17	200.99	67.42	14.76	6.06	2.92
0.50	18	10	7	3	6.38	16.39	132.30	322.60	454.23	510.60	432.62	304.89	135.67	24.91	7.43	2.97
0.50	18	24	7	4	9.28	16.86	82.17	227.49	382.00	509.75	458.44	318.22	136.97	24.91	7.43	2.97
0.50	31	10	6	3	6.38	16.39	132.86	330.37	464.75	487.64	359.86	224.25	91.69	22.15	8.92	4.14
0.50	31	24	6	4	9.28	16.86	82.39	231.36	389.44	486.87	377.59	231.41	92.29	22.15	8.92	4.14
0.50	32	10	6	3	6.38	16.39	132.95	333.67	478.83	517.76	387.73	240.32	96.21	22.82	9.18	4.25
0.50	32	24	6	4	9.28	16.86	82.43	232.98	399.29	516.89	408.36	248.56	96.88	22.82	9.18	4.25
0.75	31	17	7	3	10.96	28.32	169.88	338.52	434.49	476.55	432.69	341.23	184.35	43.21	13.20	4.90
0.75	31	42	7	4	16.13	29.86	127.09	276.55	394.56	479.45	456.23	358.64	187.72	43.23	13.20	4.90
0.75	31	43	7	4	16.49	30.52	131.44	289.53	413.80	498.54	467.00	362.55	188.04	43.23	13.20	4.90
0.75	31	44	7	4	16.86	31.19	135.86	302.91	433.52	517.56	477.26	366.12	188.32	43.23	13.20	4.90
0.75	32	17	7	3	10.96	28.32	170.23	342.69	445.62	497.16	457.48	362.49	194.63	44.62	13.54	5.03
0.75	32	42	7	4	16.13	29.86	127.28	279.35	403.73	500.29	483.81	382.15	198.38	44.64	13.54	5.03
0.75	32	43	7	4	16.49	30.52	131.65	292.59	423.87	521.09	495.92	386.58	198.73	44.64	13.54	5.03
0.75	32	44	7	4	16.86	31.19	136.08	306.25	444.58	541.88	507.48	390.63	199.05	44.64	13.54	5.03

According to the results, bias occurs only when $\lambda_0 = 2.5$. However, this is only a very moderate bias, visible only for the very small shifts; that is, for the cases $\delta = \lambda_0 \pm 0.05 * \lambda_0$. For such a small λ , the Poisson distribution is skewed and it can be concluded again that the skewness carries over to the CUSUM statistics.

Since one would prefer designs with unbiased ARL performances, the biased designs should be further analyzed. In order to reach an unbiased ARL performance, the skewness of the Poisson distribution must be taken into account. In this study, it is observed that if the lower part of the CUSUM chart is chosen to be slightly more robust (in terms of less false alarms), then it may be possible to obtain unbiased ARLs.

As a result, more experiments on the biased designs are conducted by making the lower side more robust. To analyze the biased cases with respect to one-sided performances, the ratio of the lower ARL to the upper ARL is considered, i.e.

$$\frac{ARL_l}{ARL_u}$$

where ARL_l and ARL_u are the one-sided lower and upper ARLs, respectively. This ratio is close to “1” for the original designs, because the upper and lower sides are designed as being “almost equal” in terms of ARL_0 performances. If the ratio above is varied, one can reach unbiased designs.

Let us analyze the first biased design, where $\lambda_0 = 2.5$, i.e. the design

α_0	h^+	h^-	k^+	k^-
0.50	7.00	22.00	5.00	2.00

The ratio of the one-sided lower and upper ARLs can be changed by varying the parameter h^- . An analysis of the design mentioned above is provided in Table 4.24, where the h^- parameter is varied in the range 19 to 30.

According to the results in Table 4.24, when the lower part of the CUSUM is made more robust, the bias disappears. When the h^- value is increased, the lower ARL increases and the ratio ARL_l/ARL_u increases. However, if h^- value is decreased, the process works the opposite way. With a lower ARL the ratio decreases, which results in an increase in the bias.

Now let us analyze the other biased designs. Consider again the case of $\lambda_0 = 2.5$ but with the design

α_0	h^+	h^-	k^+	k^-
0.75	11.00	39.00	5.00	2.00
0.75	11.00	40.00	5.00	2.00

Table 4.25 provides the ARL results when h^- is varied in the range 39 to 45. Likewise, notice that the bias disappears as the ratio gets larger.

Table 4.24 Analysis for a Biased Design

ARLs of Poisson INAR(1) CUSUM Chart with $\lambda_0=2.5$														
α_0	h^+	h^-	k^+	k^-	$\lambda=2.5$	$\lambda=2.625$	$\lambda=2.75$	$\lambda=3$	$\lambda=3.75$	$\lambda=5$	$\lambda=7.5$	Upper ARL	Lower ARL	Ratio ARLI/ARLu
0.50	7.00	19.00	5.00	2.00	353.32	393.24	369.37	244.65	64.24	15.78	4.11	910.28	580.81	0.64
0.50	7.00	20.00	5.00	2.00	392.66	424.66	386.17	247.07	64.24	15.78	4.11	910.28	694.54	0.76
0.50	7.00	21.00	5.00	2.00	432.57	453.78	400.30	248.85	64.24	15.78	4.11	910.28	829.05	0.91
0.50	7.00	22.00	5.00	2.00	472.48	480.31	412.02	250.14	64.24	15.78	4.11	910.28	988.08	1.09
0.50	7.00	23.00	5.00	2.00	511.82	504.07	421.62	251.08	64.24	15.78	4.11	910.28	1176.02	1.29
0.50	7.00	24.00	5.00	2.00	550.06	525.05	429.41	251.76	64.24	15.78	4.11	910.28	1398.08	1.54
0.50	7.00	25.00	5.00	2.00	586.74	543.33	435.68	252.26	64.25	15.78	4.11	910.28	1660.36	1.82
0.50	7.00	26.00	5.00	2.00	621.47	559.09	440.70	252.61	64.25	15.78	4.11	910.28	1970.10	2.16
0.50	7.00	27.00	5.00	2.00	653.96	572.55	444.69	252.87	64.25	15.78	4.11	910.28	2335.83	2.57
0.50	7.00	28.00	5.00	2.00	684.01	583.94	447.86	253.06	64.25	15.78	4.11	910.28	2767.58	3.04
0.50	7.00	29.00	5.00	2.00	711.51	593.53	450.36	253.19	64.25	15.78	4.11	910.28	3277.23	3.60
0.50	7.00	30.00	5.00	2.00	736.45	601.54	452.34	253.29	64.25	15.78	4.11	910.28	3878.75	4.26

Table 4.25 Analysis for a Biased Design

ARLs of Poisson INAR(1) CUSUM Chart with $\lambda_0=2.5$														
α_0	h^+	h^-	k^+	k^-	$\lambda=2.5$	$\lambda=2.625$	$\lambda=2.75$	$\lambda=3$	$\lambda=3.75$	$\lambda=5$	$\lambda=7.5$	Upper ARL	Lower ARL	Ratio ARLl/ARLu
0.75	11.00	39.00	5.00	2.00	479.55	489.30	443.96	304.38	94.96	26.68	6.87	959.51	972.70	1.01
0.75	11.00	40.00	5.00	2.00	498.56	503.07	451.45	305.75	94.96	26.68	6.87	959.51	1052.95	1.10
0.75	11.00	41.00	5.00	2.00	517.40	516.25	458.37	306.94	94.96	26.68	6.87	959.5	1139.29	1.19
0.75	11.00	42.00	5.00	2.00	536.02	528.82	464.74	307.98	94.97	26.68	6.87	959.5	1232.16	1.28
0.75	11.00	43.00	5.00	2.00	554.36	540.78	470.60	308.88	94.97	26.68	6.87	959.5	1332.04	1.39
0.75	11.00	44.00	5.00	2.00	572.40	552.13	475.97	309.67	94.97	26.68	6.87	959.5	1439.45	1.50
0.75	11.00	45.00	5.00	2.00	590.08	562.86	480.88	310.36	94.97	26.68	6.87	959.5	1554.95	1.62

Note that up to this point, the designs were biased under positive shifts. Now, let us analyze the design that has bias under negative shifts. Consider the following design again for $\lambda_0 = 2.5$ but with a bias when there is a negative shift.

α_0	h^+	h^-	k^+	k^-
0.50	29.00	5.00	3.00	1.00

It can be observed from the ARL results in Table 4.26 (where the h^+ values are varied in the range 27 to 35) that the opposite of the conclusions for the positive shifts applies here. When h^+ is increased, the upper ARL is increased and thus the ratio ARL_l/ARL_u is decreased. Notice that the bias disappears as the ratio gets smaller and increases as it becomes larger.

Table 4.26 Analysis for a Biased Design

ARLs of Poisson INAR(1) CUSUM Chart with $\lambda_0=2.5$													
α_0	h^+	h^-	k^+	k^-	$\lambda=2.5$	$\lambda=2.375$	$\lambda=2.25$	$\lambda=2$	$\lambda=1.25$	$\lambda=0.625$	Upper ARL	Lower ARL	Ratio ARLl/ARLu
0.50	27.00	5.00	3.00	1.00	428.91	456.33	400.75	227.30	36.58	11.66	774.29	967.19	1.25
0.50	28.00	5.00	3.00	1.00	458.07	476.88	409.81	228.07	36.58	11.66	874.30	967.19	1.11
0.50	29.00	5.00	3.00	1.00	487.22	496.12	417.77	228.69	36.58	11.66	986.37	967.19	0.98
0.50	30.00	5.00	3.00	1.00	516.15	514.00	424.70	229.19	36.58	11.66	1111.93	967.19	0.87
0.50	31.00	5.00	3.00	1.00	544.67	530.51	430.72	229.58	36.58	11.66	1252.57	967.19	0.77
0.50	32.00	5.00	3.00	1.00	572.62	545.65	435.93	229.90	36.58	11.66	1410.08	967.19	0.69
0.50	33.00	5.00	3.00	1.00	599.84	559.45	440.41	230.15	36.58	11.66	1586.45	967.19	0.61
0.50	34.00	5.00	3.00	1.00	626.17	571.96	444.27	230.36	36.58	11.66	1783.91	967.19	0.54
0.50	35.00	5.00	3.00	1.00	651.51	583.25	447.57	230.52	36.58	11.66	2004.97	967.19	0.48

In conclusion, by varying the parameters of the chart, the symmetry of the upper and lower sides are changed and unbiased designs are obtained for $\lambda_0 = 2.5$.

Figure 4.2 is an illustration with 3 design settings and various shifts in the process mean when $\lambda_0 = 5$ and $\alpha_0 = 0.25$. The design settings are shown as $(h^+, h^-, k^+, k^-, C_0^+, C_0^-)$. The horizontal axis represents the shifted λ values and the vertical axis represents the corresponding ARL performances.

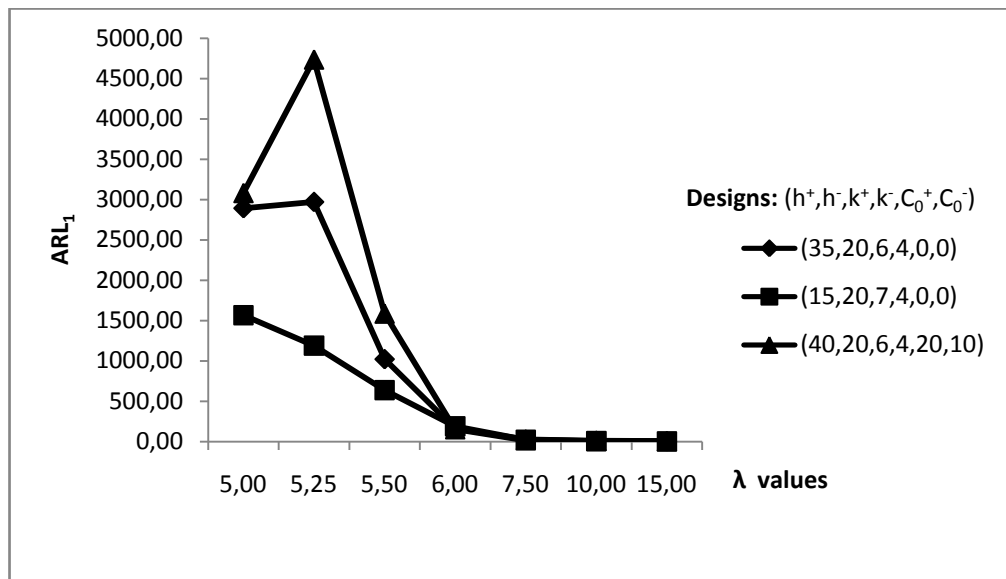


Figure 4.2 ARL performance of the CUSUM chart for different values of process parameters

According to Figure 4.2, the ARL performances differ only for the small shifts. The design with the FIR feature gives a biased performance. For large shifts, the performance of the chart does not differ since the design parameters have close values. Thus, by changing the parameters of the chart, the sensitivity of the chart to small shifts can be adjusted.

CHAPTER 5

CONCLUSIONS

In this thesis, the one-sided INAR(1) CUSUM chart proposed by Weiß and Testik(2009) is extended to a two-sided INAR(1) CUSUM chart. The two-sided chart is developed by combining the properties of the upper and lower one-sided CUSUM charts. The performance of the proposed control chart is evaluated by calculating the ARL performances considering both positive and negative shifts from the in-control mean of the process. The Markov Chain method of Brook and Evans(1972) is used to calculate these ARL performances.

In designing a two-sided Poisson INAR(1) CUSUM chart, parameters h^+ , h^- , k^+ and k^- should be determined. Parameter selection for the CUSUM chart is an important aspect for the performance. The design of the chart depends on the two parameters of the process, which are the mean level of counts λ and the dependence parameter α . The reference values k^+ and k^- are selected according to the required sensitivity of the chart. The value of the control limits, h^+ and h^- , are decided with respect to the reference values and the desired ARL_0 performances. In the literature, an ARL_0 value between 450 and 550 is suggested for practical applications.

In this study, two different mean levels for the counts, $\lambda = 5$ and $\lambda = 2.5$, have been studied. For $\lambda = 5$, the Poisson distribution is nearly symmetrical. Thus, the distribution of the CUSUM statistic can be considered as being symmetrical. Designing the upper and lower-sided charts as being equally robust (in terms of equal ARL_0 performances for both the upper and lower sides) provides unbiased ARL results.

For $\lambda = 2.5$, the performance of the CUSUM chart is affected by the skewness of the Poisson distribution. Some biased designs have been observed with very small shifts, that is, for the cases $\delta = \lambda_0 \pm 0.05 * \lambda_0$. Consequently, by varying the values for the decision intervals, the ARL performance of one of the individuals charts is adjusted so that unbiased designs can be reached for $\lambda = 2.5$ as well.

After analyzing the ARL performances for selected designs and excluding the biased designs, several unbiased designs for the two-sided INAR(1) CUSUM chart are obtained. Table 5.1 provides recommended designs after the analyses. The biased designs are replaced by the unbiased designs by adjusting the decision interval values.

In general, the CUSUM charts with reference values close to the in-control mean λ_0 show better performance in detecting small to moderate shifts. According to Weiß and Testik (2009), the best value for the upper reference value k^+ is $\lceil \lambda_0 + 1 \rceil$. The results in this thesis confirm that for $\lambda = 2.5$, the best k^+ value is 3, whereas for $\lambda = 5$, it is 6. In the same way, the best value for the lower reference value k^- is $\lfloor \lambda_0 - 1 \rfloor$. For $\lambda = 2.5$, the best k^- value is 2, whereas for $\lambda = 5$, it is 4.

Application of the Fast Initial Response (FIR) feature might improve the detection performance for shifts at process start-ups; however, this would also slightly degrade the ARL_0 performance. In this study, it is observed that the FIR feature gives better detection performance with especially smaller shifts, with a slight effect on the false alarm performance.

An approximation from the literature is also presented for the cases when it is difficult to calculate the ARL performance for the two-sided chart with the given parameters. The approximation gives good results with very small error rates, thus it can be used to speed up the calculation of ARL performances.

Finally, a simulation study is conducted for validation of the Markov Chain approach. The simulation results show that the Markov Chain approach is a convenient tool for calculating the ARL performance of the two-sided INAR(1) CUSUM chart.

Table 5.1 Recommended Designs for a Two-Sided INAR(1) CUSUM Chart

λ_0	α_0	h^+	h^-	k^+	k^-
2.5	0.25	9.00	15.00	4.00	2.00
2.5	0.25	19.00	15.00	3.00	2.00
2.5	0.50	7.00	5.00	5.00	1.00
2.5	0.50	7.00	23.00	5.00	2.00
2.5	0.50	13.00	5.00	4.00	1.00
2.5	0.50	13.00	22.00	4.00	2.00
2.5	0.50	29.00	5.00	3.00	1.00
2.5	0.50	29.00	22.00	3.00	2.00
2.5	0.50	30.00	5.00	3.00	1.00
2.5	0.50	31.00	5.00	3.00	1.00
2.5	0.50	32.00	5.00	3.00	1.00
2.5	0.75	11.00	41.00	5.00	2.00
2.5	0.75	11.00	42.00	5.00	2.00
2.5	0.75	11.00	43.00	5.00	2.00
2.5	0.75	22.00	39.00	4.00	2.00
2.5	0.75	22.00	40.00	4.00	2.00
2.5	0.75	23.00	41.00	4.00	2.00
5.0	0.25	12.00	7.00	7.00	3.00
5.0	0.25	12.00	16.00	7.00	4.00
5.0	0.25	21.00	7.00	6.00	3.00
5.0	0.25	21.00	16.00	6.00	4.00
5.0	0.50	18.00	10.00	7.00	3.00
5.0	0.50	18.00	24.00	7.00	4.00
5.0	0.50	31.00	10.00	6.00	3.00
5.0	0.50	31.00	24.00	6.00	4.00
5.0	0.50	32.00	10.00	6.00	3.00
5.0	0.50	32.00	24.00	6.00	4.00
5.0	0.75	31.00	17.00	7.00	3.00
5.0	0.75	31.00	42.00	7.00	4.00
5.0	0.75	31.00	43.00	7.00	4.00
5.0	0.75	31.00	44.00	7.00	4.00
5.0	0.75	32.00	17.00	7.00	3.00
5.0	0.75	32.00	42.00	7.00	4.00
5.0	0.75	32.00	43.00	7.00	4.00
5.0	0.75	32.00	44.00	7.00	4.00

5.1 Future Research Directions

In this thesis, a two-sided CUSUM chart is proposed for the counts modeled by Poisson INAR(1) process. In a future research, application areas in real-life situations, such as in healthcare applications, will be investigated. The proposed chart may be applied in many real-life problems and its benefits should be investigated.

As another future research, extension of the proposed CUSUM chart to other types of integer valued time series data may be studied. There are different discrete time series models and it may be possible to design effective two-sided CUSUM charts for discrete time series models other than the Poisson INAR(1) model.

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APPENDIX A

SIMULATION STUDY

Table A.1 Results of Simulation Study

λ	α	h^+	h^-	k^+	k^-	ARL_0	ARL_{sim}	Error(%)
2.50	0.25	6.00	4.00	3.00	1.00	40.57	40.49	0.21
2.50	0.25	6.00	4.00	3.00	2.00	15.07	15.11	0.24
2.50	0.25	6.00	4.00	4.00	1.00	209.50	208.68	0.39
2.50	0.25	6.00	4.00	4.00	2.00	22.08	22.03	0.21
2.50	0.25	6.00	4.00	5.00	1.00	858.92	853.83	0.59
2.50	0.25	6.00	4.00	5.00	2.00	24.14	24.13	0.08
2.50	0.25	6.00	5.00	3.00	1.00	41.34	41.21	0.31
2.50	0.25	6.00	5.00	3.00	2.00	19.52	19.59	0.38
2.50	0.25	6.00	5.00	4.00	1.00	230.99	228.41	1.12
2.50	0.25	6.00	5.00	4.00	2.00	32.66	32.54	0.36
2.50	0.25	6.00	5.00	5.00	1.00	1381.92	1355.21	1.93
2.50	0.25	6.00	5.00	5.00	2.00	37.26	37.30	0.11
2.50	0.25	6.00	6.00	3.00	1.00	41.50	41.69	0.46
2.50	0.25	6.00	6.00	3.00	2.00	23.65	23.65	0.01
2.50	0.25	6.00	6.00	4.00	1.00	236.04	234.82	0.51
2.50	0.25	6.00	6.00	4.00	2.00	45.63	45.44	0.42
2.50	0.25	6.00	6.00	5.00	1.00	1582.96	1551.24	2.00
2.50	0.25	6.00	6.00	5.00	2.00	54.96	55.19	0.42
2.50	0.25	6.00	7.00	3.00	1.00	41.54	41.68	0.34
2.50	0.25	6.00	7.00	3.00	2.00	27.33	27.33	0.01
2.50	0.25	6.00	7.00	4.00	1.00	237.12	235.72	0.59
2.50	0.25	6.00	7.00	4.00	2.00	61.05	60.79	0.43
2.50	0.25	6.00	7.00	5.00	1.00	1632.61	1608.31	1.49
2.50	0.25	6.00	7.00	5.00	2.00	78.77	78.53	0.30
2.50	0.25	6.00	8.00	3.00	1.00	41.54	41.67	0.30
2.50	0.25	6.00	8.00	3.00	2.00	30.44	30.40	0.14
2.50	0.25	6.00	8.00	4.00	1.00	237.35	235.83	0.64
2.50	0.25	6.00	8.00	4.00	2.00	78.46	78.67	0.27
2.50	0.25	6.00	8.00	5.00	1.00	1643.39	1608.46	2.13
2.50	0.25	6.00	8.00	5.00	2.00	110.08	109.74	0.31

Table A.1 Results of Simulation Study(*continued*)

2.50	0.25	6.00	9.00	4.00	2.00	97.26	97.23	0.03
2.50	0.25	6.00	9.00	5.00	1.00	1645.66	1617.15	1.73
2.50	0.25	6.00	9.00	5.00	2.00	150.66	149.88	0.52
2.50	0.25	6.00	10.00	3.00	1.00	41.55	41.72	0.42
2.50	0.25	6.00	10.00	3.00	2.00	35.01	35.10	0.24
2.50	0.25	6.00	10.00	4.00	1.00	237.40	234.91	1.05
2.50	0.25	6.00	10.00	4.00	2.00	116.58	116.40	0.16
2.50	0.25	6.00	10.00	5.00	1.00	1646.13	1610.28	2.18
2.50	0.25	6.00	10.00	5.00	2.00	202.25	201.12	0.56
2.50	0.25	6.00	11.00	3.00	1.00	41.55	41.56	0.03
2.50	0.25	6.00	11.00	3.00	2.00	36.59	36.49	0.29
2.50	0.25	6.00	11.00	4.00	1.00	237.41	235.26	0.90
2.50	0.25	6.00	11.00	4.00	2.00	135.53	135.44	0.06
2.50	0.25	6.00	11.00	5.00	1.00	1646.23	1617.17	1.77
2.50	0.25	6.00	11.00	5.00	2.00	266.46	264.95	0.57
2.50	0.25	6.00	12.00	3.00	1.00	41.55	41.64	0.23
2.50	0.25	6.00	12.00	3.00	2.00	37.81	37.66	0.40
2.50	0.25	6.00	12.00	4.00	1.00	237.41	236.51	0.38
2.50	0.25	6.00	12.00	4.00	2.00	153.29	152.41	0.57
2.50	0.25	6.00	12.00	5.00	1.00	1646.25	1612.69	2.04
2.50	0.25	6.00	12.00	5.00	2.00	344.38	340.70	1.07
2.50	0.25	6.00	13.00	3.00	1.00	41.55	41.61	0.16
2.50	0.25	6.00	13.00	3.00	2.00	38.74	38.45	0.75
2.50	0.25	6.00	13.00	4.00	1.00	237.41	236.82	0.25
2.50	0.25	6.00	13.00	4.00	2.00	169.26	168.29	0.57
2.50	0.25	6.00	13.00	5.00	1.00	1646.26	1614.84	1.91
2.50	0.25	6.00	13.00	5.00	2.00	436.24	433.17	0.71
2.50	0.25	6.00	14.00	3.00	1.00	41.55	41.58	0.08
2.50	0.25	6.00	14.00	3.00	2.00	39.44	39.47	0.07
2.50	0.25	6.00	14.00	4.00	1.00	237.41	236.62	0.33
2.50	0.25	6.00	14.00	4.00	2.00	183.10	181.75	0.74
2.50	0.25	6.00	14.00	5.00	1.00	1646.26	1618.10	1.71
2.50	0.25	6.00	14.00	5.00	2.00	540.96	536.62	0.80
2.50	0.25	6.00	15.00	3.00	1.00	41.55	41.44	0.25
2.50	0.25	6.00	15.00	3.00	2.00	39.97	39.97	0.02
2.50	0.25	6.00	15.00	4.00	1.00	237.41	236.94	0.20
2.50	0.25	6.00	15.00	4.00	2.00	194.73	193.78	0.49
2.50	0.25	6.00	15.00	5.00	1.00	1646.26	1616.01	1.84
2.50	0.25	6.00	15.00	5.00	2.00	655.94	649.06	1.05
2.50	0.25	6.00	16.00	3.00	1.00	41.55	41.36	0.45

Table A.1 Results of Simulation Study (*continued*)

2.50	0.25	6.00	16.00	3.00	2.00	40.37	40.08	0.72
2.50	0.25	6.00	16.00	4.00	1.00	237.41	235.27	0.90
2.50	0.25	6.00	16.00	4.00	2.00	204.25	203.58	0.33
2.50	0.25	6.00	16.00	5.00	1.00	1646.26	1614.14	1.95
2.50	0.25	6.00	16.00	5.00	2.00	777.17	767.44	1.25
2.50	0.25	6.00	17.00	3.00	1.00	41.55	41.34	0.50
2.50	0.25	6.00	17.00	3.00	2.00	40.67	40.73	0.15
2.50	0.25	6.00	17.00	4.00	1.00	237.41	235.02	1.00
2.50	0.25	6.00	17.00	4.00	2.00	211.89	209.61	1.07
2.50	0.25	6.00	17.00	5.00	1.00	1646.26	1609.58	2.23
2.50	0.25	6.00	17.00	5.00	2.00	899.66	890.08	1.06
2.50	0.25	6.00	18.00	3.00	1.00	41.55	41.46	0.21
2.50	0.25	6.00	18.00	3.00	2.00	40.89	41.00	0.26
2.50	0.25	6.00	18.00	4.00	1.00	237.41	234.21	1.35
2.50	0.25	6.00	18.00	4.00	2.00	217.90	216.27	0.75
2.50	0.25	6.00	18.00	5.00	1.00	1646.26	1618.61	1.68
2.50	0.25	6.00	18.00	5.00	2.00	1018.24	1007.93	1.01
2.50	0.25	6.00	19.00	3.00	1.00	41.55	41.41	0.32
2.50	0.25	6.00	19.00	3.00	2.00	41.06	41.24	0.45
2.50	0.25	6.00	19.00	4.00	1.00	237.41	236.37	0.44
2.50	0.25	6.00	19.00	4.00	2.00	222.58	222.03	0.25
2.50	0.25	6.00	19.00	5.00	1.00	1646.26	1618.28	1.70
2.50	0.25	16.00	4.00	3.00	1.00	391.78	391.76	0.01
2.50	0.25	16.00	4.00	3.00	2.00	23.34	23.33	0.05
2.50	0.25	16.00	4.00	4.00	1.00	1698.70	1693.69	0.29
2.50	0.25	16.00	4.00	4.00	2.00	24.52	24.45	0.28
2.50	0.25	16.00	4.00	5.00	1.00	1796.30	1793.49	0.16
2.50	0.25	16.00	4.00	5.00	2.00	24.54	24.51	0.12
2.50	0.25	16.00	5.00	3.00	1.00	473.72	473.61	0.02
2.50	0.25	16.00	5.00	3.00	2.00	35.38	35.45	0.20
2.50	0.25	16.00	5.00	4.00	1.00	6725.16	6646.77	1.17
2.50	0.25	16.00	5.00	4.00	2.00	38.12	38.19	0.19
2.50	0.25	16.00	5.00	5.00	1.00	8566.13	8487.41	0.92
2.50	0.25	16.00	5.00	5.00	2.00	38.17	38.04	0.33
2.50	0.25	16.00	6.00	3.00	1.00	495.33	489.91	1.10
2.50	0.25	16.00	6.00	3.00	2.00	50.99	51.08	0.18
2.50	0.50	6.00	5.00	3.00	1.00	32.77	32.85	0.25
2.50	0.50	6.00	5.00	3.00	2.00	14.47	14.46	0.10
2.50	0.50	6.00	5.00	4.00	1.00	112.22	112.49	0.24
2.50	0.50	6.00	5.00	4.00	2.00	22.45	22.37	0.36

Table A.1 Results of Simulation Study (*continued*)

2.50	0.50	6.00	5.00	5.00	1.00	365.30	364.25	0.29
2.50	0.50	6.00	5.00	5.00	2.00	26.75	26.69	0.22
2.50	0.50	6.00	6.00	3.00	1.00	33.56	33.62	0.19
2.50	0.50	6.00	6.00	3.00	2.00	16.91	16.91	0.03
2.50	0.50	6.00	6.00	4.00	1.00	121.24	121.66	0.34
2.50	0.50	6.00	6.00	4.00	2.00	28.27	28.14	0.48
2.50	0.50	6.00	6.00	5.00	1.00	477.35	472.88	0.94
2.50	0.50	6.00	6.00	5.00	2.00	35.13	35.07	0.17
2.50	0.50	6.00	7.00	3.00	1.00	33.88	33.81	0.18
2.50	0.50	6.00	7.00	3.00	2.00	19.18	19.19	0.08
2.50	0.50	6.00	7.00	4.00	1.00	125.09	125.56	0.37
2.50	0.50	6.00	7.00	4.00	2.00	34.58	34.63	0.16
2.50	0.50	6.00	7.00	5.00	1.00	541.04	537.57	0.64
2.50	0.50	6.00	7.00	5.00	2.00	45.05	45.20	0.34
2.50	0.50	6.00	8.00	3.00	1.00	34.00	33.84	0.46
2.50	0.50	6.00	8.00	3.00	2.00	21.23	21.13	0.47
2.50	0.50	6.00	8.00	4.00	1.00	126.64	127.34	0.55
2.50	0.50	6.00	8.00	4.00	2.00	41.21	41.02	0.46
2.50	0.50	6.00	8.00	5.00	1.00	570.36	564.88	0.96
2.50	0.50	6.00	8.00	5.00	2.00	56.58	56.46	0.21
2.50	0.50	6.00	9.00	3.00	1.00	34.05	33.97	0.23
2.50	0.50	6.00	9.00	3.00	2.00	23.06	23.24	0.77
2.50	0.50	6.00	9.00	4.00	1.00	127.25	127.42	0.13
2.50	0.50	6.00	9.00	4.00	2.00	48.05	48.30	0.52
2.50	0.50	6.00	9.00	5.00	1.00	582.55	579.28	0.56
2.50	0.50	6.00	9.00	5.00	2.00	69.86	69.49	0.52
2.50	0.50	6.00	10.00	3.00	1.00	34.07	34.03	0.10
2.50	0.50	6.00	10.00	3.00	2.00	24.67	24.69	0.08
2.50	0.50	6.00	10.00	4.00	1.00	127.49	127.88	0.31
2.50	0.50	6.00	10.00	4.00	2.00	54.96	55.06	0.17
2.50	0.50	6.00	10.00	5.00	1.00	587.40	583.59	0.65
2.50	0.50	6.00	10.00	5.00	2.00	84.96	84.75	0.24
2.50	0.50	6.00	11.00	3.00	1.00	34.07	34.11	0.12
2.50	0.50	6.00	11.00	3.00	2.00	26.07	26.04	0.10
2.50	0.50	6.00	11.00	4.00	1.00	127.58	127.62	0.03
2.50	0.50	6.00	11.00	4.00	2.00	61.81	62.29	0.78
2.50	0.50	6.00	11.00	5.00	1.00	589.30	580.52	1.49
2.50	0.50	6.00	11.00	5.00	2.00	101.93	101.46	0.46
2.50	0.50	6.00	12.00	3.00	1.00	34.08	34.16	0.25
2.50	0.50	6.00	12.00	3.00	2.00	27.28	27.25	0.12

Table A.1 Results of Simulation Study (*continued*)

2.50	0.50	6.00	12.00	4.00	1.00	127.62	128.69	0.84
2.50	0.50	6.00	12.00	4.00	2.00	68.47	68.42	0.07
2.50	0.50	6.00	12.00	5.00	1.00	590.03	585.99	0.69
2.50	0.50	6.00	12.00	5.00	2.00	120.76	120.46	0.25
2.50	0.50	6.00	13.00	3.00	1.00	34.08	33.93	0.43
2.50	0.50	6.00	13.00	3.00	2.00	28.32	28.38	0.21
2.50	0.50	6.00	13.00	4.00	1.00	127.63	127.83	0.15
2.50	0.50	6.00	13.00	4.00	2.00	74.84	74.68	0.21
2.50	0.50	6.00	13.00	5.00	1.00	590.32	586.56	0.64
2.50	0.50	6.00	13.00	5.00	2.00	141.41	140.74	0.47
2.50	0.50	6.00	14.00	3.00	1.00	34.08	34.27	0.56
2.50	0.50	6.00	14.00	3.00	2.00	29.20	29.17	0.10
2.50	0.50	6.00	14.00	4.00	1.00	127.64	128.31	0.52
2.50	0.50	6.00	14.00	4.00	2.00	80.83	81.24	0.51
2.50	0.50	6.00	14.00	5.00	1.00	590.43	584.92	0.93
2.50	0.50	6.00	14.00	5.00	2.00	163.72	162.57	0.71
2.50	0.50	6.00	15.00	3.00	1.00	34.08	34.29	0.62
2.50	0.50	6.00	15.00	3.00	2.00	29.96	29.98	0.07
2.50	0.50	6.00	15.00	4.00	1.00	127.64	127.80	0.13
2.50	0.50	6.00	15.00	4.00	2.00	86.39	86.77	0.44
2.50	0.50	6.00	15.00	5.00	1.00	590.47	585.21	0.89
2.50	0.50	6.00	15.00	5.00	2.00	187.52	186.53	0.53
2.50	0.50	6.00	16.00	3.00	1.00	34.08	34.01	0.20
2.50	0.50	6.00	16.00	3.00	2.00	30.60	30.62	0.06
2.50	0.50	6.00	16.00	4.00	1.00	127.64	128.19	0.43
2.50	0.50	6.00	16.00	4.00	2.00	91.49	90.86	0.70
2.50	0.50	6.00	16.00	5.00	1.00	590.49	587.92	0.43
2.50	0.50	6.00	16.00	5.00	2.00	212.53	212.44	0.04
2.50	0.50	6.00	17.00	3.00	1.00	34.08	33.98	0.28
2.50	0.50	6.00	17.00	3.00	2.00	31.14	31.15	0.01
2.50	0.50	6.00	17.00	4.00	1.00	127.64	127.99	0.27
2.50	0.50	6.00	17.00	4.00	2.00	96.12	95.89	0.24
2.50	0.50	6.00	17.00	5.00	1.00	590.50	590.00	0.08
2.50	0.50	6.00	17.00	5.00	2.00	238.45	237.31	0.48
2.50	0.50	6.00	18.00	3.00	1.00	34.08	34.27	0.57
2.50	0.50	6.00	18.00	3.00	2.00	31.60	31.59	0.04
2.50	0.50	6.00	18.00	4.00	1.00	127.64	128.38	0.58
2.50	0.50	6.00	18.00	4.00	2.00	100.27	100.53	0.27
2.50	0.50	6.00	18.00	5.00	1.00	590.50	587.43	0.52
2.50	0.50	6.00	18.00	5.00	2.00	264.90	263.91	0.37

Table A.1 Results of Simulation Study(*continued*)

2.50	0.50	6.00	19.00	3.00	1.00	34.08	34.04	0.11
2.50	0.50	6.00	19.00	3.00	2.00	31.99	32.04	0.14
2.50	0.50	6.00	19.00	4.00	1.00	127.64	127.26	0.30
2.50	0.50	6.00	19.00	4.00	2.00	103.96	104.56	0.58
2.50	0.50	6.00	19.00	5.00	1.00	590.50	585.07	0.92
2.50	0.50	6.00	19.00	5.00	2.00	291.50	290.97	0.18
2.50	0.50	6.00	20.00	3.00	1.00	34.08	34.23	0.44
2.50	0.50	6.00	20.00	3.00	2.00	32.32	32.31	0.02
2.50	0.50	6.00	20.00	4.00	1.00	127.64	127.57	0.06
2.50	0.50	6.00	20.00	4.00	2.00	107.22	107.00	0.21
2.50	0.50	6.00	20.00	5.00	1.00	590.50	589.67	0.14
2.50	0.50	6.00	20.00	5.00	2.00	317.85	316.00	0.58
2.50	0.50	6.00	21.00	3.00	1.00	34.08	34.10	0.08
2.50	0.50	6.00	21.00	3.00	2.00	32.60	32.59	0.02
2.50	0.50	6.00	21.00	4.00	1.00	127.64	127.99	0.28

APPENDIX B

IN-CONTROL ARL_0 PERFORMANCE

Table B.1 Results of In-Control ARL_0 Calculations

λ	α	h^+	h^-	k^+	k^-	ARL_0
2.50	0.25	6.00	3.00	3.00	1.00	37.22
2.50	0.25	6.00	3.00	3.00	2.00	10.76
2.50	0.25	6.00	3.00	4.00	1.00	144.50
2.50	0.25	6.00	3.00	4.00	2.00	14.06
2.50	0.25	6.00	3.00	5.00	1.00	303.45
2.50	0.25	6.00	3.00	5.00	2.00	14.91
2.50	0.25	6.00	4.00	3.00	1.00	40.57
2.50	0.25	6.00	4.00	3.00	2.00	15.07
2.50	0.25	6.00	4.00	4.00	1.00	209.50
2.50	0.25	6.00	4.00	4.00	2.00	22.08
2.50	0.25	6.00	4.00	5.00	1.00	858.92
2.50	0.25	6.00	4.00	5.00	2.00	24.14
2.50	0.25	6.00	5.00	3.00	1.00	41.34
2.50	0.25	6.00	5.00	3.00	2.00	19.52
2.50	0.25	6.00	5.00	4.00	1.00	230.99
2.50	0.25	6.00	5.00	4.00	2.00	32.66
2.50	0.25	6.00	5.00	5.00	1.00	1381.92
2.50	0.25	6.00	5.00	5.00	2.00	37.26
2.50	0.25	6.00	6.00	3.00	1.00	41.50
2.50	0.25	6.00	6.00	3.00	2.00	23.65
2.50	0.25	6.00	6.00	4.00	1.00	236.04
2.50	0.25	6.00	6.00	4.00	2.00	45.63
2.50	0.25	6.00	6.00	5.00	1.00	1582.96
2.50	0.25	6.00	6.00	5.00	2.00	54.96
2.50	0.25	6.00	7.00	3.00	1.00	41.54
2.50	0.25	6.00	7.00	3.00	2.00	27.33
2.50	0.25	6.00	7.00	4.00	1.00	237.12
2.50	0.25	6.00	7.00	4.00	2.00	61.05
2.50	0.25	6.00	7.00	5.00	1.00	1632.61

Table B.1 Results of In-Control ARL_0 Calculations (*continued*)

2.50	0.25	6.00	8.00	3.00	2.00	30.44
2.50	0.25	6.00	8.00	4.00	1.00	237.35
2.50	0.25	6.00	8.00	4.00	2.00	78.46
2.50	0.25	6.00	8.00	5.00	1.00	1643.39
2.50	0.25	6.00	8.00	5.00	2.00	110.08
2.50	0.25	6.00	9.00	3.00	1.00	41.55
2.50	0.25	6.00	9.00	3.00	2.00	32.99
2.50	0.25	6.00	9.00	4.00	1.00	237.39
2.50	0.25	6.00	9.00	4.00	2.00	97.26
2.50	0.25	6.00	9.00	5.00	1.00	1645.66
2.50	0.25	6.00	9.00	5.00	2.00	150.66
2.50	0.25	6.00	10.00	3.00	1.00	41.55
2.50	0.25	6.00	10.00	3.00	2.00	35.01
2.50	0.25	6.00	10.00	4.00	1.00	237.40
2.50	0.25	6.00	10.00	4.00	2.00	116.58
2.50	0.25	6.00	10.00	5.00	1.00	1646.13
2.50	0.25	6.00	10.00	5.00	2.00	202.25
2.50	0.25	6.00	11.00	3.00	1.00	41.55
2.50	0.25	6.00	11.00	3.00	2.00	36.59
2.50	0.25	6.00	11.00	4.00	1.00	237.41
2.50	0.25	6.00	11.00	4.00	2.00	135.53
2.50	0.25	6.00	11.00	5.00	1.00	1646.23
2.50	0.25	6.00	11.00	5.00	2.00	266.46
2.50	0.25	6.00	12.00	3.00	1.00	41.55
2.50	0.25	6.00	12.00	3.00	2.00	37.81
2.50	0.25	6.00	12.00	4.00	1.00	237.41
2.50	0.25	6.00	12.00	4.00	2.00	153.29
2.50	0.25	6.00	12.00	5.00	1.00	1646.25
2.50	0.25	6.00	12.00	5.00	2.00	344.38
2.50	0.25	6.00	13.00	3.00	1.00	41.55
2.50	0.25	6.00	13.00	3.00	2.00	38.74
2.50	0.25	6.00	13.00	4.00	1.00	237.41
2.50	0.25	6.00	13.00	4.00	2.00	169.26
2.50	0.25	6.00	13.00	5.00	1.00	1646.26
2.50	0.25	6.00	13.00	5.00	2.00	436.24
2.50	0.25	6.00	14.00	3.00	1.00	41.55
2.50	0.25	6.00	14.00	3.00	2.00	39.44
2.50	0.25	6.00	14.00	4.00	1.00	237.41
2.50	0.25	6.00	14.00	4.00	2.00	183.10
2.50	0.25	6.00	14.00	5.00	1.00	1646.26

Table B.1 Results of In-Control ARL_0 Calculations (*continued*)

2.50	0.25	6.00	14.00	5.00	2.00	540.96
2.50	0.25	6.00	15.00	3.00	1.00	41.55
2.50	0.25	6.00	15.00	3.00	2.00	39.97
2.50	0.25	6.00	15.00	4.00	1.00	237.41
2.50	0.25	6.00	15.00	4.00	2.00	194.73
2.50	0.25	6.00	15.00	5.00	1.00	1646.26
2.50	0.25	6.00	15.00	5.00	2.00	655.94
2.50	0.25	6.00	16.00	3.00	1.00	41.55
2.50	0.25	6.00	16.00	3.00	2.00	40.37
2.50	0.25	6.00	16.00	4.00	1.00	237.41
2.50	0.25	6.00	16.00	4.00	2.00	204.25
2.50	0.25	6.00	16.00	5.00	1.00	1646.26
2.50	0.25	6.00	16.00	5.00	2.00	777.17
2.50	0.25	6.00	17.00	3.00	1.00	41.55
2.50	0.25	6.00	17.00	3.00	2.00	40.67
2.50	0.25	6.00	17.00	4.00	1.00	237.41
2.50	0.25	6.00	17.00	4.00	2.00	211.89
2.50	0.25	6.00	17.00	5.00	1.00	1646.26
2.50	0.25	6.00	17.00	5.00	2.00	899.66
2.50	0.25	6.00	18.00	3.00	1.00	41.55
2.50	0.25	6.00	18.00	3.00	2.00	40.89
2.50	0.25	6.00	18.00	4.00	1.00	237.41
2.50	0.25	6.00	18.00	4.00	2.00	217.90
2.50	0.25	6.00	18.00	5.00	1.00	1646.26
2.50	0.25	6.00	18.00	5.00	2.00	1018.24
2.50	0.25	6.00	19.00	3.00	1.00	41.55
2.50	0.25	6.00	19.00	3.00	2.00	41.06
2.50	0.25	6.00	19.00	4.00	1.00	237.41
2.50	0.25	6.00	19.00	4.00	2.00	222.58
2.50	0.25	6.00	19.00	5.00	1.00	1646.26
2.50	0.25	6.00	19.00	5.00	2.00	1128.41
2.50	0.25	6.00	20.00	3.00	1.00	41.55
2.50	0.25	6.00	20.00	3.00	2.00	41.18
2.50	0.25	6.00	20.00	4.00	1.00	237.41
2.50	0.25	6.00	20.00	4.00	2.00	226.19
2.50	0.25	6.00	20.00	5.00	1.00	1646.26
2.50	0.25	6.00	20.00	5.00	2.00	1226.93
2.50	0.25	6.00	21.00	3.00	1.00	41.55
2.50	0.25	6.00	21.00	3.00	2.00	41.27
2.50	0.25	6.00	21.00	4.00	1.00	237.41

Table B.1 Results of In-Control ARL₀ Calculations (*continued*)

2.50	0.25	6.00	21.00	4.00	2.00	228.95
2.50	0.25	6.00	21.00	5.00	1.00	1646.26
2.50	0.25	6.00	21.00	5.00	2.00	1312.06
2.50	0.25	6.00	22.00	3.00	1.00	41.55
2.50	0.25	6.00	22.00	3.00	2.00	41.34
2.50	0.25	6.00	22.00	4.00	1.00	237.41
2.50	0.25	6.00	22.00	4.00	2.00	231.04
2.50	0.25	6.00	22.00	5.00	1.00	1646.26
2.50	0.25	6.00	22.00	5.00	2.00	1383.48
2.50	0.25	6.00	23.00	3.00	1.00	41.55
2.50	0.25	6.00	23.00	3.00	2.00	41.39
2.50	0.25	6.00	23.00	4.00	1.00	237.41
2.50	0.25	6.00	23.00	4.00	2.00	232.63
2.50	0.25	6.00	23.00	5.00	1.00	1646.26
2.50	0.25	6.00	23.00	5.00	2.00	1441.94
2.50	0.25	6.00	24.00	3.00	1.00	41.55
2.50	0.25	6.00	24.00	3.00	2.00	41.43
2.50	0.25	6.00	24.00	4.00	1.00	237.41
2.50	0.25	6.00	24.00	4.00	2.00	233.82
2.50	0.25	6.00	24.00	5.00	1.00	1646.26
2.50	0.25	6.00	24.00	5.00	2.00	1488.83
2.50	0.25	6.00	25.00	3.00	1.00	41.55
2.50	0.25	6.00	25.00	3.00	2.00	41.46
2.50	0.25	6.00	25.00	4.00	1.00	237.41
2.50	0.25	6.00	25.00	4.00	2.00	234.72
2.50	0.25	6.00	25.00	5.00	1.00	1646.26
2.50	0.25	6.00	25.00	5.00	2.00	1525.83
2.50	0.25	7.00	3.00	3.00	1.00	48.38
2.50	0.25	7.00	3.00	3.00	2.00	11.61
2.50	0.25	7.00	3.00	4.00	1.00	191.30
2.50	0.25	7.00	3.00	4.00	2.00	14.45
2.50	0.25	7.00	3.00	5.00	1.00	335.06
2.50	0.25	7.00	3.00	5.00	2.00	14.99
2.50	0.25	7.00	4.00	3.00	1.00	54.13
2.50	0.25	7.00	4.00	3.00	2.00	16.72
2.50	0.25	7.00	4.00	4.00	1.00	323.85
2.50	0.25	7.00	4.00	4.00	2.00	23.00
2.50	0.25	7.00	4.00	5.00	1.00	1169.57
2.50	0.25	7.00	4.00	5.00	2.00	24.34
2.50	0.25	7.00	5.00	3.00	1.00	55.50

Table B.1 Results of In-Control ARL_0 Calculations (*continued*)

2.50	0.25	7.00	5.00	3.00	2.00	22.31
2.50	0.25	7.00	5.00	4.00	1.00	378.00
2.50	0.25	7.00	5.00	4.00	2.00	34.66
2.50	0.25	7.00	5.00	5.00	1.00	2411.20
2.50	0.25	7.00	5.00	5.00	2.00	37.71
2.50	0.25	7.00	6.00	3.00	1.00	55.79
2.50	0.25	7.00	6.00	3.00	2.00	27.82
2.50	0.25	7.00	6.00	4.00	1.00	391.66
2.50	0.25	7.00	6.00	4.00	2.00	49.56
2.50	0.25	7.00	6.00	5.00	1.00	3096.97
2.50	0.25	7.00	6.00	5.00	2.00	55.95
2.50	0.25	7.00	7.00	3.00	1.00	55.85
2.50	0.25	7.00	7.00	3.00	2.00	33.00
2.50	0.25	7.00	7.00	4.00	1.00	394.64
2.50	0.25	7.00	7.00	4.00	2.00	68.22
2.50	0.25	7.00	7.00	5.00	1.00	3292.72
2.50	0.25	7.00	7.00	5.00	2.00	80.78
2.50	0.25	7.00	8.00	3.00	1.00	55.87
2.50	0.25	7.00	8.00	3.00	2.00	37.60
2.50	0.25	7.00	8.00	4.00	1.00	395.27
2.50	0.25	7.00	8.00	4.00	2.00	90.62
2.50	0.25	7.00	8.00	5.00	1.00	3336.83
2.50	0.25	7.00	8.00	5.00	2.00	114.02
2.50	0.25	7.00	9.00	3.00	1.00	55.87
2.50	0.25	7.00	9.00	3.00	2.00	41.52
2.50	0.25	7.00	9.00	4.00	1.00	395.40
2.50	0.25	7.00	9.00	4.00	2.00	116.56
2.50	0.25	7.00	9.00	5.00	1.00	3346.21
2.50	0.25	7.00	9.00	5.00	2.00	158.12
2.50	0.25	7.00	10.00	3.00	1.00	55.87
2.50	0.25	7.00	10.00	3.00	2.00	44.74
2.50	0.25	7.00	10.00	4.00	1.00	395.43
2.50	0.25	7.00	10.00	4.00	2.00	145.33
2.50	0.25	7.00	10.00	5.00	1.00	3348.17
2.50	0.25	7.00	10.00	5.00	2.00	215.86
2.50	0.25	7.00	11.00	3.00	1.00	55.87
2.50	0.25	7.00	11.00	3.00	2.00	47.33
2.50	0.25	7.00	11.00	4.00	1.00	395.43
2.50	0.25	7.00	11.00	4.00	2.00	175.90
2.50	0.25	7.00	11.00	5.00	1.00	3348.59

Table B.1 Results of In-Control ARL_0 Calculations (*continued*)

2.50	0.25	7.00	11.00	5.00	2.00	290.53
2.50	0.25	7.00	12.00	3.00	1.00	55.87
2.50	0.25	7.00	12.00	3.00	2.00	49.37
2.50	0.25	7.00	12.00	4.00	1.00	395.44
2.50	0.25	7.00	12.00	4.00	2.00	206.91
2.50	0.25	7.00	12.00	5.00	1.00	3348.67
2.50	0.25	7.00	12.00	5.00	2.00	385.59
2.50	0.25	7.00	13.00	3.00	1.00	55.87
2.50	0.25	7.00	13.00	3.00	2.00	50.95
2.50	0.25	7.00	13.00	4.00	1.00	395.44
2.50	0.25	7.00	13.00	4.00	2.00	236.99
2.50	0.25	7.00	13.00	5.00	1.00	3348.69
2.50	0.25	7.00	13.00	5.00	2.00	504.45
2.50	0.25	7.00	14.00	3.00	1.00	55.87
2.50	0.25	7.00	14.00	3.00	2.00	52.16
2.50	0.25	7.00	14.00	4.00	1.00	395.44
2.50	0.25	7.00	14.00	4.00	2.00	264.95

APPENDIX C

APPROXIMATION RESULTS

Table C.1 Results of Siegmund's Approximation

λ	α	h^+	h^-	k^+	k^-	Upper ARL	Lower ARL	Approx. ARL	Exact ARL	Error (%)
2.50	0.25	6.00	14.00	5.00	2.00	1646.26	806.79	541.44	540.96	0.09
2.50	0.25	7.00	13.00	5.00	2.00	3348.70	594.31	504.73	504.45	0.06
2.50	0.25	8.00	4.00	4.00	1.00	651.59	1798.62	478.31	477.89	0.09
2.50	0.25	8.00	13.00	5.00	2.00	6714.74	594.31	545.98	545.82	0.03
2.50	0.25	8.00	16.00	4.00	2.00	651.59	1474.09	451.86	451.45	0.09
2.50	0.25	8.00	17.00	4.00	2.00	651.59	1986.36	490.64	490.29	0.07
2.50	0.25	8.00	18.00	4.00	2.00	651.59	2672.69	523.87	523.57	0.06
2.50	0.25	9.00	14.00	4.00	2.00	1065.85	806.79	459.20	458.73	0.10
2.50	0.25	9.00	15.00	4.00	2.00	1065.85	1091.86	539.35	538.87	0.09
2.50	0.25	10.00	14.00	4.00	2.00	1734.46	806.79	550.65	550.24	0.08
2.50	0.25	11.00	13.00	4.00	2.00	2812.05	594.31	490.62	490.34	0.06
2.50	0.25	12.00	13.00	4.00	2.00	4546.89	594.31	525.61	525.41	0.04
2.50	0.25	13.00	13.00	4.00	2.00	7337.65	594.31	549.78	549.65	0.02
2.50	0.25	16.00	5.00	3.00	1.00	501.38	8618.93	473.82	473.72	0.02
2.50	0.25	16.00	6.00	3.00	1.00	501.38	41226.93	495.36	495.33	0.00
2.50	0.25	16.00	20.00	3.00	2.00	501.38	4823.09	454.17	454.04	0.03
2.50	0.25	16.00	21.00	3.00	2.00	501.38	6471.53	465.33	465.23	0.02
2.50	0.25	16.00	22.00	3.00	2.00	501.38	8678.56	474.00	473.92	0.02
2.50	0.25	16.00	23.00	3.00	2.00	501.38	11633.21	480.66	480.60	0.01
2.50	0.25	16.00	24.00	3.00	2.00	501.38	15588.51	485.76	485.71	0.01
2.50	0.25	16.00	25.00	3.00	2.00	501.38	20883.09	489.62	489.59	0.01
2.50	0.25	17.00	4.00	3.00	1.00	624.12	1798.62	463.34	463.00	0.07
2.50	0.25	17.00	17.00	3.00	2.00	624.12	1986.36	474.91	474.63	0.06
2.50	0.25	17.00	18.00	3.00	2.00	624.12	2672.69	505.97	505.74	0.05
2.50	0.25	17.00	19.00	3.00	2.00	624.12	3591.99	531.73	531.54	0.04
2.50	0.25	18.00	4.00	3.00	1.00	775.12	1798.62	541.68	541.31	0.07
2.50	0.25	18.00	15.00	3.00	2.00	775.12	1091.86	453.31	452.94	0.08
2.50	0.25	18.00	16.00	3.00	2.00	775.12	1474.09	508.00	507.65	0.07

Table C.1 Results of Siegmund's Approximation (*continued*)

2.50	0.25	19.00	15.00	3.00	2.00	960.75	1091.86	511.06	510.68	0.07
2.50	0.25	20.00	14.00	3.00	2.00	1188.89	806.79	480.63	480.26	0.08
2.50	0.25	21.00	14.00	3.00	2.00	1469.14	806.79	520.79	520.44	0.07
2.50	0.25	23.00	13.00	3.00	2.00	2235.90	594.31	469.51	469.25	0.05
2.50	0.25	24.00	13.00	3.00	2.00	2754.65	594.31	488.84	488.62	0.05
2.50	0.25	25.00	13.00	3.00	2.00	3391.34	594.31	505.69	505.49	0.04
2.50	0.50	6.00	6.00	5.00	1.00	590.50	2515.60	478.24	477.35	0.19
2.50	0.50	6.00	7.00	5.00	1.00	590.50	6522.64	541.48	541.04	0.08
2.50	0.50	6.00	26.00	5.00	2.00	590.50	1970.10	454.32	453.40	0.20
2.50	0.50	6.00	27.00	5.00	2.00	590.50	2335.83	471.34	470.51	0.18
2.50	0.50	6.00	28.00	5.00	2.00	590.50	2767.58	486.66	485.92	0.15
2.50	0.50	6.00	29.00	5.00	2.00	590.50	3277.23	500.35	499.68	0.13
2.50	0.50	6.00	30.00	5.00	2.00	590.50	3878.75	512.48	511.89	0.11
2.50	0.50	7.00	5.00	5.00	1.00	910.28	967.19	468.94	467.49	0.31
2.50	0.50	7.00	22.00	5.00	2.00	910.28	988.08	473.79	472.48	0.28
2.50	0.50	7.00	23.00	5.00	2.00	910.28	1176.02	513.11	511.82	0.25
2.50	0.50	8.00	20.00	5.00	2.00	1381.58	694.54	462.19	461.01	0.26
2.50	0.50	8.00	21.00	5.00	2.00	1381.58	829.05	518.13	516.89	0.24
2.50	0.50	9.00	19.00	5.00	2.00	2072.25	580.81	453.66	452.75	0.20
2.50	0.50	9.00	20.00	5.00	2.00	2072.25	694.54	520.19	519.19	0.19
2.50	0.50	10.00	19.00	5.00	2.00	3079.75	580.81	488.66	487.94	0.15
2.50	0.50	11.00	6.00	4.00	1.00	563.20	2515.60	460.17	459.38	0.17
2.50	0.50	11.00	7.00	4.00	1.00	563.20	6522.64	518.43	518.05	0.07
2.50	0.50	11.00	8.00	4.00	1.00	563.20	16890.25	545.02	544.86	0.03
2.50	0.50	11.00	19.00	5.00	2.00	4543.77	580.81	514.98	514.44	0.10
2.50	0.50	11.00	27.00	4.00	2.00	563.20	2335.83	453.78	453.04	0.16
2.50	0.50	11.00	28.00	4.00	2.00	563.20	2767.58	467.97	467.30	0.14
2.50	0.50	11.00	29.00	4.00	2.00	563.20	3277.23	480.61	480.01	0.12
2.50	0.50	11.00	30.00	4.00	2.00	563.20	3878.75	491.79	491.26	0.11
2.50	0.50	12.00	19.00	5.00	2.00	6664.05	580.81	534.25	533.85	0.07
2.50	0.50	12.00	23.00	4.00	2.00	742.49	1176.02	455.14	454.00	0.25
2.50	0.50	12.00	24.00	4.00	2.00	742.49	1398.08	484.95	483.86	0.22
2.50	0.50	12.00	25.00	4.00	2.00	742.49	1660.36	513.06	512.04	0.20
2.50	0.50	12.00	26.00	4.00	2.00	742.49	1970.10	539.26	538.31	0.18
2.50	0.50	13.00	5.00	4.00	1.00	975.25	967.19	485.60	484.28	0.27
2.50	0.50	13.00	18.00	5.00	2.00	9725.79	484.70	461.69	461.45	0.05
2.50	0.50	13.00	19.00	5.00	2.00	9725.79	580.81	548.08	547.79	0.05

Table C.1 Results of Siegmund's Approximation (*continued*)

2.50	0.50	13.00	22.00	4.00	2.00	975.25	988.08	490.81	489.62	0.24
2.50	0.50	13.00	23.00	4.00	2.00	975.25	1176.02	533.13	531.95	0.22
2.50	0.50	14.00	5.00	4.00	1.00	1277.15	967.19	550.38	549.08	0.24
2.50	0.50	14.00	18.00	5.00	2.00	14135.70	484.70	468.63	468.46	0.04
2.50	0.50	14.00	21.00	4.00	2.00	1277.15	829.05	502.72	501.57	0.23
2.50	0.50	15.00	18.00	5.00	2.00	20473.13	484.70	473.49	473.37	0.03
2.50	0.50	15.00	20.00	4.00	2.00	1668.44	694.54	490.40	489.40	0.20
2.50	0.50	16.00	18.00	5.00	2.00	29562.52	484.70	476.88	476.79	0.02
2.50	0.50	16.00	19.00	4.00	2.00	2175.23	580.81	458.41	457.61	0.17
2.50	0.50	16.00	20.00	4.00	2.00	2175.23	694.54	526.45	525.57	0.17
2.50	0.50	17.00	18.00	5.00	2.00	42575.97	484.70	479.24	479.18	0.01
2.50	0.50	17.00	19.00	4.00	2.00	2831.25	580.81	481.94	481.26	0.14
2.50	0.50	18.00	19.00	4.00	2.00	3680.02	580.81	501.64	501.07	0.11
2.50	0.50	19.00	19.00	4.00	2.00	4777.67	580.81	517.86	517.39	0.09
2.50	0.50	20.00	19.00	4.00	2.00	6196.64	580.81	531.04	530.66	0.07
2.50	0.50	21.00	18.00	4.00	2.00	8030.37	484.70	457.11	456.85	0.06
2.50	0.50	21.00	19.00	4.00	2.00	8030.37	580.81	541.64	541.33	0.06
2.50	0.50	22.00	18.00	4.00	2.00	10399.37	484.70	463.11	462.91	0.04
2.50	0.50	22.00	19.00	4.00	2.00	10399.37	580.81	550.09	549.85	0.04
2.50	0.50	23.00	8.00	3.00	1.00	471.26	16890.25	458.47	458.35	0.03
2.50	0.50	23.00	9.00	3.00	1.00	471.26	43712.01	466.24	466.19	0.01
2.50	0.50	23.00	18.00	4.00	2.00	13459.09	484.70	467.85	467.69	0.03
2.50	0.50	24.00	7.00	3.00	1.00	534.52	6522.64	494.03	493.71	0.07
2.50	0.50	24.00	8.00	3.00	1.00	534.52	16890.25	518.12	517.99	0.03
2.50	0.50	24.00	9.00	3.00	1.00	534.52	43712.01	528.06	528.01	0.01
2.50	0.50	24.00	18.00	4.00	2.00	17409.98	484.70	471.57	471.44	0.03
2.50	0.50	24.00	29.00	3.00	2.00	534.52	3277.23	459.56	459.07	0.11
2.50	0.50	24.00	30.00	3.00	2.00	534.52	3878.75	469.78	469.34	0.09
2.50	0.50	25.00	6.00	3.00	1.00	605.48	2515.60	488.02	487.30	0.15
2.50	0.50	25.00	18.00	4.00	2.00	22510.55	484.70	474.48	474.38	0.02
2.50	0.50	25.00	26.00	3.00	2.00	605.48	1970.10	463.14	462.41	0.16
2.50	0.50	25.00	27.00	3.00	2.00	605.48	2335.83	480.84	480.17	0.14
2.50	0.50	25.00	28.00	3.00	2.00	605.48	2767.58	496.80	496.19	0.12
2.50	0.50	25.00	29.00	3.00	2.00	605.48	3277.23	511.06	510.52	0.11
2.50	0.50	25.00	30.00	3.00	2.00	605.48	3878.75	523.73	523.25	0.09
2.50	0.50	26.00	6.00	3.00	1.00	685.07	2515.60	538.44	537.66	0.14
2.50	0.50	26.00	18.00	4.00	2.00	29094.13	484.70	476.76	476.68	0.02

Table C.1 Results of Siegmund's Approximation (*continued*)

2.50	0.50	26.00	24.00	3.00	2.00	685.07	1398.08	459.77	458.87	0.20
2.50	0.50	26.00	25.00	3.00	2.00	685.07	1660.36	484.97	484.12	0.17
2.50	0.50	26.00	26.00	3.00	2.00	685.07	1970.10	508.31	507.53	0.15
2.50	0.50	26.00	27.00	3.00	2.00	685.07	2335.83	529.71	528.99	0.14
2.50	0.50	26.00	28.00	3.00	2.00	685.07	2767.58	549.14	548.49	0.12
2.50	0.50	27.00	18.00	4.00	2.00	37590.52	484.70	478.53	478.47	0.01
2.50	0.50	27.00	23.00	3.00	2.00	774.29	1176.02	466.89	465.91	0.21
2.50	0.50	27.00	24.00	3.00	2.00	774.29	1398.08	498.31	497.38	0.19
2.50	0.50	27.00	25.00	3.00	2.00	774.29	1660.36	528.04	527.16	0.17
2.50	0.50	28.00	5.00	3.00	1.00	874.30	967.19	459.20	458.07	0.25
2.50	0.50	28.00	22.00	3.00	2.00	874.30	988.08	463.86	462.84	0.22
2.50	0.50	28.00	23.00	3.00	2.00	874.30	1176.02	501.48	500.48	0.20
2.50	0.50	28.00	24.00	3.00	2.00	874.30	1398.08	537.91	536.95	0.18
2.50	0.50	29.00	5.00	3.00	1.00	986.37	967.19	488.34	487.22	0.23
2.50	0.50	29.00	22.00	3.00	2.00	986.37	988.08	493.61	492.59	0.21
2.50	0.50	29.00	23.00	3.00	2.00	986.37	1176.02	536.44	535.43	0.19
2.50	0.50	30.00	5.00	3.00	1.00	1111.93	967.19	517.26	516.15	0.22
2.50	0.50	30.00	21.00	3.00	2.00	1111.93	829.05	474.94	473.94	0.21
2.50	0.50	30.00	22.00	3.00	2.00	1111.93	988.08	523.18	522.16	0.20
2.50	0.75	8.00	12.00	5.00	1.00	512.68	4149.99	456.31	454.84	0.32
2.50	0.75	8.00	13.00	5.00	1.00	512.68	6354.22	474.41	473.37	0.22
2.50	0.75	8.00	14.00	5.00	1.00	512.68	9723.80	487.01	486.29	0.15
2.50	0.75	8.00	15.00	5.00	1.00	512.68	14874.58	495.60	495.12	0.10
2.50	0.75	8.00	16.00	5.00	1.00	512.68	22747.87	501.38	501.06	0.06
2.50	0.75	9.00	10.00	5.00	1.00	636.75	1765.16	467.95	465.01	0.63
2.50	0.75	9.00	11.00	5.00	1.00	636.75	2708.17	515.54	513.21	0.45
2.50	0.75	9.00	12.00	5.00	1.00	636.75	4149.99	552.05	550.31	0.32
2.50	0.75	10.00	9.00	5.00	1.00	784.30	1148.48	466.04	462.40	0.79
2.50	0.75	10.00	10.00	5.00	1.00	784.30	1765.16	543.02	539.81	0.60
2.50	0.75	10.00	41.00	5.00	2.00	784.30	1139.29	464.52	461.17	0.73
2.50	0.75	10.00	42.00	5.00	2.00	784.30	1232.16	479.25	475.95	0.69
2.50	0.75	10.00	43.00	5.00	2.00	784.30	1332.04	493.64	490.41	0.66
2.50	0.75	10.00	44.00	5.00	2.00	784.30	1439.45	507.68	504.52	0.63
2.50	0.75	10.00	45.00	5.00	2.00	784.30	1554.95	521.34	518.26	0.60
2.50	0.75	11.00	9.00	5.00	1.00	959.51	1148.48	522.76	519.01	0.72
2.50	0.75	11.00	38.00	5.00	2.00	959.51	898.10	463.89	460.42	0.75
2.50	0.75	11.00	39.00	5.00	2.00	959.51	972.70	483.03	479.55	0.73

Table C.1 Results of Siegmund's Approximation (*continued*)

2.50	0.75	11.00	40.00	5.00	2.00	959.51	1052.95	502.03	498.56	0.70
2.50	0.75	11.00	41.00	5.00	2.00	959.51	1139.29	520.85	517.40	0.67
2.50	0.75	11.00	42.00	5.00	2.00	959.51	1232.16	539.44	536.02	0.64
2.50	0.75	12.00	8.00	5.00	1.00	1167.26	745.32	454.87	451.26	0.80
2.50	0.75	12.00	36.00	5.00	2.00	1167.26	764.36	461.90	458.56	0.73
2.50	0.75	12.00	37.00	5.00	2.00	1167.26	828.77	484.66	481.27	0.70
2.50	0.75	12.00	38.00	5.00	2.00	1167.26	898.10	507.57	504.14	0.68
2.50	0.75	12.00	39.00	5.00	2.00	1167.26	972.70	530.57	527.11	0.66
2.50	0.75	12.00	40.00	5.00	2.00	1167.26	1052.95	553.58	550.10	0.63
2.50	0.75	13.00	8.00	5.00	1.00	1413.25	745.32	487.97	484.53	0.71
2.50	0.75	14.00	8.00	5.00	1.00	1704.11	745.32	518.53	515.30	0.63
2.50	0.75	14.00	34.00	5.00	2.00	1704.11	648.92	469.96	467.16	0.60
2.50	0.75	14.00	35.00	5.00	2.00	1704.11	704.51	498.45	495.54	0.59
2.50	0.75	14.00	36.00	5.00	2.00	1704.11	764.36	527.68	524.67	0.57
2.50	0.75	15.00	8.00	5.00	1.00	2047.61	745.32	546.42	543.43	0.55
2.50	0.75	15.00	33.00	5.00	2.00	2047.61	597.30	462.41	459.95	0.54
2.50	0.75	15.00	34.00	5.00	2.00	2047.61	648.92	492.76	490.19	0.53
2.50	0.75	15.00	35.00	5.00	2.00	2047.61	704.51	524.17	521.48	0.51
2.50	0.75	16.00	33.00	5.00	2.00	2452.81	597.30	480.33	478.11	0.47
2.50	0.75	16.00	34.00	5.00	2.00	2452.81	648.92	513.16	510.82	0.46
2.50	0.75	16.00	35.00	5.00	2.00	2452.81	704.51	547.31	544.86	0.45
2.50	0.75	17.00	12.00	4.00	1.00	511.57	4149.99	455.43	454.06	0.30
2.50	0.75	17.00	13.00	4.00	1.00	511.57	6354.22	473.45	472.48	0.20
2.50	0.75	17.00	14.00	4.00	1.00	511.57	9723.80	486.00	485.33	0.14
2.50	0.75	17.00	15.00	4.00	1.00	511.57	14874.58	494.56	494.11	0.09
2.50	0.75	17.00	16.00	4.00	1.00	511.57	22747.87	500.32	500.01	0.06
2.50	0.75	17.00	32.00	5.00	2.00	2930.24	549.37	462.63	460.75	0.41
2.50	0.75	17.00	33.00	5.00	2.00	2930.24	597.30	496.16	494.17	0.40
2.50	0.75	17.00	34.00	5.00	2.00	2930.24	648.92	531.27	529.17	0.40
2.50	0.75	18.00	11.00	4.00	1.00	581.49	2708.17	478.70	476.66	0.43
2.50	0.75	18.00	12.00	4.00	1.00	581.49	4149.99	510.02	508.51	0.30
2.50	0.75	18.00	13.00	4.00	1.00	581.49	6354.22	532.73	531.65	0.20
2.50	0.75	18.00	14.00	4.00	1.00	581.49	9723.80	548.67	547.93	0.14
2.50	0.75	18.00	32.00	5.00	2.00	3492.24	549.37	474.69	473.03	0.35
2.50	0.75	18.00	33.00	5.00	2.00	3492.24	597.30	510.06	508.29	0.35
2.50	0.75	18.00	34.00	5.00	2.00	3492.24	648.92	547.24	545.36	0.34
2.50	0.75	19.00	10.00	4.00	1.00	659.76	1765.16	480.26	477.48	0.58

Table C.1 Results of Siegmund's Approximation (*continued*)

2.50	0.75	19.00	11.00	4.00	1.00	659.76	2708.17	530.52	528.31	0.42
2.50	0.75	19.00	32.00	5.00	2.00	4153.13	549.37	485.19	483.72	0.30
2.50	0.75	19.00	33.00	5.00	2.00	4153.13	597.30	522.20	520.64	0.30
2.50	0.75	19.00	45.00	4.00	2.00	659.76	1554.95	463.22	460.54	0.58
2.50	0.75	20.00	10.00	4.00	1.00	747.37	1765.16	525.06	522.12	0.56
2.50	0.75	20.00	31.00	5.00	2.00	4929.64	504.87	457.97	456.77	0.26
2.50	0.75	20.00	32.00	5.00	2.00	4929.64	549.37	494.28	493.00	0.26
2.50	0.75	20.00	33.00	5.00	2.00	4929.64	597.30	532.75	531.38	0.26
2.50	0.75	20.00	42.00	4.00	2.00	747.37	1232.16	465.20	462.19	0.65
2.50	0.75	20.00	43.00	4.00	2.00	747.37	1332.04	478.76	475.80	0.62
2.50	0.75	20.00	44.00	4.00	2.00	747.37	1439.45	491.95	489.07	0.59
2.50	0.75	20.00	45.00	4.00	2.00	747.37	1554.95	504.76	501.95	0.56
2.50	0.75	21.00	9.00	4.00	1.00	845.40	1148.48	486.95	483.52	0.71
2.50	0.75	21.00	31.00	5.00	2.00	5841.24	504.87	464.71	463.66	0.23
2.50	0.75	21.00	32.00	5.00	2.00	5841.24	549.37	502.14	501.02	0.22
2.50	0.75	21.00	33.00	5.00	2.00	5841.24	597.30	541.89	540.69	0.22
2.50	0.75	21.00	40.00	4.00	2.00	845.40	1052.95	468.92	465.75	0.68
2.50	0.75	21.00	41.00	4.00	2.00	845.40	1139.29	485.29	482.16	0.65
2.50	0.75	21.00	42.00	4.00	2.00	845.40	1232.16	501.39	498.29	0.62
2.50	0.75	21.00	43.00	4.00	2.00	845.40	1332.04	517.17	514.12	0.59
2.50	0.75	21.00	44.00	4.00	2.00	845.40	1439.45	532.60	529.61	0.56
2.50	0.75	21.00	45.00	4.00	2.00	845.40	1554.95	547.65	544.72	0.54
2.50	0.75	22.00	9.00	4.00	1.00	955.06	1148.48	521.44	517.95	0.67
2.50	0.75	22.00	31.00	5.00	2.00	6910.61	504.87	470.50	469.59	0.19
2.50	0.75	22.00	32.00	5.00	2.00	6910.61	549.37	508.91	507.94	0.19
2.50	0.75	22.00	33.00	5.00	2.00	6910.61	597.30	549.78	548.74	0.19
2.50	0.75	22.00	38.00	4.00	2.00	955.06	898.10	462.85	459.64	0.70
2.50	0.75	22.00	39.00	4.00	2.00	955.06	972.70	481.90	478.69	0.67
2.50	0.75	22.00	40.00	4.00	2.00	955.06	1052.95	500.81	497.61	0.64
2.50	0.75	22.00	41.00	4.00	2.00	955.06	1139.29	519.54	516.35	0.62
2.50	0.75	22.00	42.00	4.00	2.00	955.06	1232.16	538.03	534.87	0.59
2.50	0.75	23.00	7.00	5.00	1.00	8164.12	481.85	454.99	454.17	0.18
2.50	0.75	23.00	31.00	5.00	2.00	8164.12	504.87	475.47	474.68	0.17
2.50	0.75	23.00	32.00	5.00	2.00	8164.12	549.37	514.73	513.89	0.16
2.50	0.75	23.00	37.00	4.00	2.00	1077.70	828.77	468.49	465.33	0.68
2.50	0.75	23.00	38.00	4.00	2.00	1077.70	898.10	489.87	486.68	0.65
2.50	0.75	23.00	39.00	4.00	2.00	1077.70	972.70	511.26	508.05	0.63

Table C.1 Results of Siegmund's Approximation (*continued*)

2.50	0.75	23.00	40.00	4.00	2.00	1077.70	1052.95	532.59	529.38	0.61
2.50	0.75	23.00	41.00	4.00	2.00	1077.70	1139.29	553.82	550.61	0.58
2.50	0.75	24.00	7.00	5.00	1.00	9632.50	481.85	458.89	458.18	0.15
2.50	0.75	24.00	8.00	4.00	1.00	1214.83	745.32	461.92	458.60	0.72
2.50	0.75	24.00	31.00	5.00	2.00	9632.50	504.87	479.73	479.05	0.14
2.50	0.75	24.00	32.00	5.00	2.00	9632.50	549.37	519.73	519.00	0.14
2.50	0.75	24.00	36.00	4.00	2.00	1214.83	764.36	469.16	466.11	0.65
2.50	0.75	24.00	37.00	4.00	2.00	1214.83	828.77	492.67	489.57	0.63
2.50	0.75	24.00	38.00	4.00	2.00	1214.83	898.10	516.36	513.22	0.61
2.50	0.75	24.00	39.00	4.00	2.00	1214.83	972.70	540.18	537.00	0.59
2.50	0.75	25.00	7.00	5.00	1.00	11351.47	481.85	462.23	461.62	0.13
2.50	0.75	25.00	8.00	4.00	1.00	1368.12	745.32	482.48	479.26	0.67
2.50	0.75	25.00	31.00	5.00	2.00	11351.47	504.87	483.37	482.79	0.12
2.50	0.75	25.00	32.00	5.00	2.00	11351.47	549.37	524.01	523.38	0.12
2.50	0.75	25.00	35.00	4.00	2.00	1368.12	704.51	465.04	462.15	0.63
2.50	0.75	25.00	36.00	4.00	2.00	1368.12	764.36	490.38	487.42	0.61
2.50	0.75	25.00	37.00	4.00	2.00	1368.12	828.77	516.12	513.09	0.59
2.50	0.75	25.00	38.00	4.00	2.00	1368.12	898.10	542.18	539.10	0.57
2.50	0.75	26.00	7.00	5.00	1.00	13362.60	481.85	465.08	464.55	0.11
2.50	0.75	26.00	8.00	4.00	1.00	1539.45	745.32	502.19	499.08	0.62
2.50	0.75	26.00	31.00	5.00	2.00	13362.60	504.87	486.49	485.99	0.10
2.50	0.75	26.00	32.00	5.00	2.00	13362.60	549.37	527.67	527.13	0.10
2.50	0.75	26.00	34.00	4.00	2.00	1539.45	648.92	456.50	453.80	0.59
2.50	0.75	26.00	35.00	4.00	2.00	1539.45	704.51	483.32	480.54	0.58
2.50	0.75	26.00	36.00	4.00	2.00	1539.45	764.36	510.76	507.90	0.56
2.50	0.75	26.00	37.00	4.00	2.00	1539.45	828.77	538.74	535.80	0.55
2.50	0.75	27.00	7.00	5.00	1.00	15714.19	481.85	467.51	467.06	0.10
2.50	0.75	27.00	8.00	4.00	1.00	1730.89	745.32	520.98	518.01	0.57
2.50	0.75	27.00	31.00	5.00	2.00	15714.19	504.87	489.16	488.72	0.09
2.50	0.75	27.00	32.00	5.00	2.00	15714.19	549.37	530.81	530.34	0.09
2.50	0.75	27.00	34.00	4.00	2.00	1730.89	648.92	471.98	469.41	0.55
2.50	0.75	27.00	35.00	4.00	2.00	1730.89	704.51	500.71	498.06	0.53
2.50	0.75	27.00	36.00	4.00	2.00	1730.89	764.36	530.22	527.47	0.52
2.50	0.75	28.00	7.00	5.00	1.00	18462.43	481.85	469.59	469.20	0.08
2.50	0.75	28.00	8.00	4.00	1.00	1944.77	745.32	538.82	535.99	0.53
2.50	0.75	28.00	30.00	5.00	2.00	18462.43	463.58	452.22	451.88	0.08
2.50	0.75	28.00	31.00	5.00	2.00	18462.43	504.87	491.43	491.06	0.08

Table C.1 Results of Siegmund's Approximation (*continued*)

2.50	0.75	28.00	32.00	5.00	2.00	18462.43	549.37	533.49	533.09	0.08
2.50	0.75	28.00	33.00	4.00	2.00	1944.77	597.30	456.96	454.63	0.51
2.50	0.75	28.00	34.00	4.00	2.00	1944.77	648.92	486.57	484.14	0.50
2.50	0.75	28.00	35.00	4.00	2.00	1944.77	704.51	517.16	514.64	0.49
2.50	0.75	28.00	36.00	4.00	2.00	1944.77	764.36	548.70	546.08	0.48
2.50	0.75	29.00	7.00	5.00	1.00	21672.60	481.85	471.37	471.03	0.07
2.50	0.75	29.00	30.00	5.00	2.00	21672.60	463.58	453.87	453.57	0.07
2.50	0.75	29.00	32.00	5.00	2.00	21672.60	549.37	535.79	535.44	0.06
2.50	0.75	29.00	33.00	4.00	2.00	2183.67	597.30	469.01	466.83	0.47
2.50	0.75	29.00	34.00	4.00	2.00	2183.67	648.92	500.26	497.98	0.46
2.50	0.75	29.00	35.00	4.00	2.00	2183.67	704.51	532.66	530.27	0.45
2.50	0.75	30.00	32.00	5.00	2.00	25420.55	549.37	537.75	537.45	0.06
2.50	0.75	30.00	33.00	4.00	2.00	2450.49	597.30	480.24	478.20	0.43
2.50	0.75	30.00	34.00	4.00	2.00	2450.49	648.92	513.06	510.91	0.42
2.50	0.75	30.00	35.00	4.00	2.00	2450.49	704.51	547.19	544.95	0.41
2.50	0.75	31.00	32.00	4.00	2.00	2748.43	549.37	457.85	456.05	0.40
2.50	0.75	31.00	32.00	5.00	2.00	29794.41	549.37	539.42	539.17	0.05
2.50	0.75	31.00	33.00	4.00	2.00	2748.43	597.30	490.67	488.76	0.39
2.50	0.75	31.00	34.00	4.00	2.00	2748.43	648.92	524.97	522.97	0.38
2.50	0.75	32.00	32.00	4.00	2.00	3081.08	549.37	466.24	464.57	0.36
2.50	0.75	32.00	32.00	5.00	2.00	34896.52	549.37	540.85	540.63	0.04
2.50	0.75	32.00	33.00	4.00	2.00	3081.08	597.30	500.31	498.54	0.35
2.50	0.75	32.00	34.00	4.00	2.00	3081.08	648.92	536.03	534.16	0.35
2.50	0.75	34.00	32.00	4.00	2.00	3866.91	549.37	481.03	479.61	0.30
2.50	0.75	34.00	32.00	5.00	2.00	47780.05	549.37	543.12	542.96	0.03
2.50	0.75	34.00	33.00	4.00	2.00	3866.91	597.30	517.38	515.88	0.29
5.00	0.25	10.00	11.00	7.00	3.00	467.17	21243.30	457.12	457.07	0.01
5.00	0.25	10.00	12.00	7.00	3.00	467.17	44762.36	462.35	462.32	0.00
5.00	0.25	10.00	25.00	7.00	4.00	467.17	14760.03	452.84	452.78	0.01
5.00	0.25	10.00	26.00	7.00	4.00	467.17	19765.81	456.38	456.34	0.01
5.00	0.25	11.00	8.00	7.00	3.00	676.28	2265.10	520.79	520.40	0.08
5.00	0.25	11.00	17.00	7.00	4.00	676.28	1410.94	457.16	456.73	0.09
5.00	0.25	11.00	18.00	7.00	4.00	676.28	1896.06	498.48	498.10	0.08
5.00	0.25	11.00	19.00	7.00	4.00	676.28	2545.72	534.33	534.01	0.06
5.00	0.25	12.00	7.00	7.00	3.00	975.75	1073.67	511.19	510.63	0.11
5.00	0.25	12.00	16.00	7.00	4.00	975.75	1048.74	505.47	504.97	0.10
5.00	0.25	13.00	15.00	7.00	4.00	1404.28	778.39	500.80	500.34	0.09

Table C.1 Results of Siegmund's Approximation (*continued*)

5.00	0.25	15.00	14.00	7.00	4.00	2893.13	576.65	480.81	480.54	0.06
5.00	0.25	16.00	6.00	7.00	3.00	4144.93	506.29	451.18	450.96	0.05
5.00	0.25	16.00	14.00	7.00	4.00	4144.93	576.65	506.22	506.01	0.04
5.00	0.25	17.00	6.00	7.00	3.00	5933.23	506.29	466.48	466.32	0.03
5.00	0.25	17.00	14.00	7.00	4.00	5933.23	576.65	525.57	525.41	0.03
5.00	0.25	18.00	6.00	7.00	3.00	8487.41	506.29	477.79	477.67	0.02
5.00	0.25	18.00	9.00	6.00	3.00	558.28	4783.22	499.93	499.75	0.04
5.00	0.25	18.00	10.00	6.00	3.00	558.28	10082.19	528.98	528.89	0.02
5.00	0.25	18.00	11.00	6.00	3.00	558.28	21243.30	543.98	543.93	0.01
5.00	0.25	18.00	14.00	7.00	4.00	8487.41	576.65	539.96	539.84	0.02
5.00	0.25	18.00	19.00	6.00	4.00	558.28	2545.72	457.87	457.61	0.06
5.00	0.25	18.00	20.00	6.00	4.00	558.28	3415.58	479.85	479.64	0.04
5.00	0.25	18.00	21.00	6.00	4.00	558.28	4580.19	497.62	497.46	0.03
5.00	0.25	18.00	22.00	6.00	4.00	558.28	6139.30	511.74	511.61	0.03
5.00	0.25	18.00	23.00	6.00	4.00	558.28	8226.43	522.80	522.70	0.02
5.00	0.25	18.00	24.00	6.00	4.00	558.28	11020.28	531.36	531.28	0.01
5.00	0.25	18.00	25.00	6.00	4.00	558.28	14760.03	537.93	537.87	0.01
5.00	0.25	18.00	26.00	6.00	4.00	558.28	19765.81	542.94	542.90	0.01
5.00	0.25	18.00	27.00	6.00	4.00	558.28	26466.08	546.74	546.71	0.01
5.00	0.25	18.00	28.00	6.00	4.00	558.28	35434.30	549.62	549.59	0.00
5.00	0.25	19.00	6.00	7.00	3.00	12134.85	506.29	486.01	485.93	0.02
5.00	0.25	19.00	8.00	6.00	3.00	690.33	2265.10	529.08	528.73	0.07
5.00	0.25	19.00	17.00	6.00	4.00	690.33	1410.94	463.53	463.16	0.08
5.00	0.25	19.00	18.00	6.00	4.00	690.33	1896.06	506.07	505.74	0.07
5.00	0.25	19.00	19.00	6.00	4.00	690.33	2545.72	543.06	542.78	0.05
5.00	0.25	20.00	6.00	7.00	3.00	17342.73	506.29	491.93	491.87	0.01
5.00	0.25	20.00	7.00	6.00	3.00	852.50	1073.67	475.19	474.72	0.10
5.00	0.25	20.00	16.00	6.00	4.00	852.50	1048.74	470.25	469.82	0.09
5.00	0.25	20.00	17.00	6.00	4.00	852.50	1410.94	531.42	531.01	0.08
5.00	0.25	21.00	6.00	7.00	3.00	24777.74	506.29	496.15	496.11	0.01
5.00	0.25	21.00	7.00	6.00	3.00	1051.62	1073.67	531.27	530.78	0.09
5.00	0.25	21.00	16.00	6.00	4.00	1051.62	1048.74	525.09	524.66	0.08
5.00	0.25	22.00	6.00	7.00	3.00	35391.26	506.29	499.15	499.12	0.01
5.00	0.25	22.00	15.00	6.00	4.00	1296.07	778.39	486.32	485.92	0.08
5.00	0.25	23.00	6.00	7.00	3.00	50540.89	506.29	501.27	501.25	0.00
5.00	0.25	23.00	15.00	6.00	4.00	1596.09	778.39	523.22	522.84	0.07
5.00	0.25	25.00	14.00	6.00	4.00	2416.11	576.65	465.54	465.27	0.06

Table C.1 Results of Siegmund's Approximation (*continued*)

5.00	0.25	26.00	14.00	6.00	4.00	2970.49	576.65	482.90	482.67	0.05
5.00	0.25	27.00	14.00	6.00	4.00	3650.65	576.65	497.99	497.78	0.04
5.00	0.25	28.00	6.00	6.00	3.00	4485.09	506.29	454.93	454.76	0.04
5.00	0.25	28.00	14.00	6.00	4.00	4485.09	576.65	510.95	510.78	0.03
5.00	0.25	29.00	6.00	6.00	3.00	5508.75	506.29	463.67	463.53	0.03
5.00	0.25	29.00	14.00	6.00	4.00	5508.75	576.65	522.00	521.86	0.03
5.00	0.25	30.00	6.00	6.00	3.00	6764.50	506.29	471.03	470.91	0.03
5.00	0.25	30.00	14.00	6.00	4.00	6764.50	576.65	531.35	531.23	0.02
5.00	0.50	15.00	13.00	7.00	3.00	550.10	3822.72	480.90	480.31	0.12
5.00	0.50	15.00	14.00	7.00	3.00	550.10	5906.86	503.24	502.82	0.08
5.00	0.50	15.00	15.00	7.00	3.00	550.10	9123.97	518.82	518.53	0.06
5.00	0.50	15.00	16.00	7.00	3.00	550.10	14089.45	529.43	529.24	0.04
5.00	0.50	15.00	17.00	7.00	3.00	550.10	21753.65	536.54	536.41	0.02
5.00	0.50	15.00	18.00	7.00	3.00	550.10	33583.01	541.24	541.15	0.02
5.00	0.50	15.00	19.00	7.00	3.00	550.10	51840.59	544.33	544.27	0.01
5.00	0.50	15.00	30.00	7.00	4.00	550.10	2830.30	460.58	459.91	0.15
5.00	0.50	15.00	31.00	7.00	4.00	550.10	3345.86	472.43	471.83	0.13
5.00	0.50	15.00	32.00	7.00	4.00	550.10	3954.18	482.92	482.39	0.11
5.00	0.50	15.00	33.00	7.00	4.00	550.10	4671.91	492.15	491.69	0.09
5.00	0.50	15.00	34.00	7.00	4.00	550.10	5518.70	500.24	499.83	0.08
5.00	0.50	15.00	35.00	7.00	4.00	550.10	6517.74	507.29	506.93	0.07
5.00	0.50	16.00	11.00	7.00	3.00	676.24	1597.92	475.16	474.03	0.24
5.00	0.50	16.00	12.00	7.00	3.00	676.24	2472.62	531.02	530.11	0.17
5.00	0.50	16.00	26.00	7.00	4.00	676.24	1443.57	460.52	459.44	0.23
5.00	0.50	16.00	27.00	7.00	4.00	676.24	1709.40	484.55	483.55	0.21
5.00	0.50	16.00	28.00	7.00	4.00	676.24	2023.13	506.83	505.90	0.18
5.00	0.50	16.00	29.00	7.00	4.00	676.24	2393.38	527.27	526.42	0.16
5.00	0.50	16.00	30.00	7.00	4.00	676.24	2830.30	545.83	545.06	0.14
5.00	0.50	17.00	10.00	7.00	3.00	829.73	1031.50	459.84	458.51	0.29
5.00	0.50	17.00	11.00	7.00	3.00	829.73	1597.92	546.14	544.93	0.22
5.00	0.50	17.00	24.00	7.00	4.00	829.73	1027.61	459.06	457.84	0.27
5.00	0.50	17.00	25.00	7.00	4.00	829.73	1218.37	493.59	492.40	0.24
5.00	0.50	17.00	26.00	7.00	4.00	829.73	1443.57	526.89	525.74	0.22
5.00	0.50	18.00	10.00	7.00	3.00	1016.41	1031.50	511.95	510.60	0.26
5.00	0.50	18.00	23.00	7.00	4.00	1016.41	866.03	467.61	466.38	0.26
5.00	0.50	18.00	24.00	7.00	4.00	1016.41	1027.61	510.99	509.75	0.24
5.00	0.50	19.00	22.00	7.00	4.00	1243.39	729.21	459.64	458.49	0.25

Table C.1 Results of Siegmund's Approximation (*continued*)

5.00	0.50	19.00	23.00	7.00	4.00	1243.39	866.03	510.48	509.28	0.24
5.00	0.50	20.00	9.00	7.00	3.00	1519.30	664.69	462.39	461.25	0.25
5.00	0.50	20.00	22.00	7.00	4.00	1519.30	729.21	492.72	491.64	0.22
5.00	0.50	20.00	23.00	7.00	4.00	1519.30	866.03	551.61	550.46	0.21
5.00	0.50	21.00	9.00	7.00	3.00	1854.57	664.69	489.32	488.27	0.22
5.00	0.50	21.00	21.00	7.00	4.00	1854.57	613.37	460.93	460.00	0.20
5.00	0.50	21.00	22.00	7.00	4.00	1854.57	729.21	523.41	522.40	0.19
5.00	0.50	22.00	9.00	7.00	3.00	2261.88	664.69	513.72	512.77	0.19
5.00	0.50	22.00	21.00	7.00	4.00	2261.88	613.37	482.52	481.69	0.17
5.00	0.50	22.00	22.00	7.00	4.00	2261.88	729.21	551.43	550.52	0.17
5.00	0.50	23.00	9.00	7.00	3.00	2756.63	664.69	535.56	534.71	0.16
5.00	0.50	23.00	21.00	7.00	4.00	2756.63	613.37	501.73	500.99	0.15
5.00	0.50	24.00	21.00	7.00	4.00	3357.44	613.37	518.62	517.97	0.13
5.00	0.50	25.00	20.00	6.00	3.00	454.78	80019.35	452.21	452.18	0.01
5.00	0.50	25.00	20.00	7.00	4.00	4086.95	515.32	457.62	457.12	0.11
5.00	0.50	25.00	21.00	7.00	4.00	4086.95	613.37	533.33	532.76	0.11
5.00	0.50	26.00	13.00	6.00	3.00	513.26	3822.72	452.51	451.99	0.11
5.00	0.50	26.00	14.00	6.00	3.00	513.26	5906.86	472.23	471.87	0.08
5.00	0.50	26.00	15.00	6.00	3.00	513.26	9123.97	485.93	485.68	0.05
5.00	0.50	26.00	16.00	6.00	3.00	513.26	14089.45	495.22	495.06	0.03
5.00	0.50	26.00	17.00	6.00	3.00	513.26	21753.65	501.43	501.32	0.02
5.00	0.50	26.00	18.00	6.00	3.00	513.26	33583.01	505.54	505.46	0.01
5.00	0.50	26.00	19.00	6.00	3.00	513.26	51840.59	508.23	508.18	0.01
5.00	0.50	26.00	20.00	7.00	4.00	4972.57	515.32	466.93	466.51	0.09
5.00	0.50	26.00	21.00	7.00	4.00	4972.57	613.37	546.02	545.53	0.09
5.00	0.50	26.00	32.00	6.00	4.00	513.26	3954.18	454.29	453.84	0.10
5.00	0.50	26.00	33.00	6.00	4.00	513.26	4671.91	462.46	462.06	0.09
5.00	0.50	26.00	34.00	6.00	4.00	513.26	5518.70	469.59	469.24	0.07
5.00	0.50	26.00	35.00	6.00	4.00	513.26	6517.74	475.79	475.49	0.06
5.00	0.50	27.00	12.00	6.00	3.00	578.77	2472.62	468.99	468.24	0.16
5.00	0.50	27.00	13.00	6.00	3.00	578.77	3822.72	502.67	502.11	0.11
5.00	0.50	27.00	14.00	6.00	3.00	578.77	5906.86	527.12	526.73	0.08
5.00	0.50	27.00	15.00	6.00	3.00	578.77	9123.97	544.25	543.98	0.05
5.00	0.50	27.00	20.00	7.00	4.00	6047.59	515.32	474.86	474.50	0.08
5.00	0.50	27.00	29.00	6.00	4.00	578.77	2393.38	466.07	465.37	0.15
5.00	0.50	27.00	30.00	6.00	4.00	578.77	2830.30	480.51	479.89	0.13
5.00	0.50	27.00	31.00	6.00	4.00	578.77	3345.86	493.42	492.86	0.11

Table C.1 Results of Siegmund's Approximation (*continued*)

5.00	0.50	27.00	32.00	6.00	4.00	578.77	3954.18	504.88	504.38	0.10
5.00	0.50	27.00	33.00	6.00	4.00	578.77	4671.91	514.98	514.54	0.08
5.00	0.50	27.00	34.00	6.00	4.00	578.77	5518.70	523.84	523.45	0.07
5.00	0.50	27.00	35.00	6.00	4.00	578.77	6517.74	531.57	531.24	0.06
5.00	0.50	28.00	11.00	6.00	3.00	652.14	1597.92	463.13	462.12	0.22
5.00	0.50	28.00	12.00	6.00	3.00	652.14	2472.62	516.04	515.23	0.16
5.00	0.50	28.00	20.00	7.00	4.00	7352.33	515.32	481.57	481.26	0.06
5.00	0.50	28.00	27.00	6.00	4.00	652.14	1709.40	472.05	471.16	0.19
5.00	0.50	28.00	28.00	6.00	4.00	652.14	2023.13	493.17	492.35	0.17
5.00	0.50	28.00	29.00	6.00	4.00	652.14	2393.38	512.50	511.75	0.15
5.00	0.50	28.00	30.00	6.00	4.00	652.14	2830.30	530.02	529.34	0.13
5.00	0.50	28.00	31.00	6.00	4.00	652.14	3345.86	545.77	545.16	0.11
5.00	0.50	29.00	11.00	6.00	3.00	734.30	1597.92	503.11	502.05	0.21
5.00	0.50	29.00	20.00	7.00	4.00	8935.72	515.32	487.22	486.96	0.05
5.00	0.50	29.00	25.00	6.00	4.00	734.30	1218.37	458.17	457.12	0.23
5.00	0.50	29.00	26.00	6.00	4.00	734.30	1443.57	486.72	485.73	0.20
5.00	0.50	29.00	27.00	6.00	4.00	734.30	1709.40	513.65	512.72	0.18
5.00	0.50	29.00	28.00	6.00	4.00	734.30	2023.13	538.76	537.89	0.16
5.00	0.50	30.00	10.00	6.00	3.00	826.29	1031.50	458.78	457.57	0.26
5.00	0.50	30.00	11.00	6.00	3.00	826.29	1597.92	544.65	543.55	0.20
5.00	0.50	30.00	20.00	7.00	4.00	10857.07	515.32	491.97	491.75	0.04
5.00	0.50	30.00	24.00	6.00	4.00	826.29	1027.61	458.01	456.91	0.24
5.00	0.50	30.00	25.00	6.00	4.00	826.29	1218.37	492.37	491.30	0.22
5.00	0.50	30.00	26.00	6.00	4.00	826.29	1443.57	525.50	524.47	0.20
5.00	0.50	31.00	10.00	6.00	3.00	929.26	1031.50	488.86	487.64	0.25
5.00	0.50	31.00	20.00	7.00	4.00	13188.34	515.32	495.94	495.76	0.04
5.00	0.50	31.00	24.00	6.00	4.00	929.26	1027.61	487.98	486.87	0.23
5.00	0.50	31.00	25.00	6.00	4.00	929.26	1218.37	527.18	526.09	0.21
5.00	0.50	32.00	10.00	6.00	3.00	1044.53	1031.50	518.99	517.76	0.24
5.00	0.50	32.00	20.00	7.00	4.00	16016.73	515.32	499.26	499.10	0.03
5.00	0.50	32.00	23.00	6.00	4.00	1044.53	866.03	473.47	472.37	0.23
5.00	0.50	32.00	24.00	6.00	4.00	1044.53	1027.61	518.00	516.89	0.22
5.00	0.50	33.00	10.00	6.00	3.00	1173.55	1031.50	548.97	547.76	0.22
5.00	0.50	33.00	20.00	7.00	4.00	19448.01	515.32	502.02	501.89	0.02
5.00	0.50	33.00	23.00	6.00	4.00	1173.55	866.03	498.30	497.22	0.22
5.00	0.50	33.00	24.00	6.00	4.00	1173.55	1027.61	547.87	546.76	0.20
5.00	0.50	34.00	20.00	7.00	4.00	23610.44	515.32	504.31	504.21	0.02

Table C.1 Results of Siegmund's Approximation (*continued*)

5.00	0.50	34.00	22.00	6.00	4.00	1317.93	729.21	469.46	468.44	0.22
5.00	0.50	34.00	23.00	6.00	4.00	1317.93	866.03	522.62	521.55	0.20
5.00	0.50	35.00	9.00	6.00	3.00	1479.51	664.69	458.64	457.59	0.23
5.00	0.50	35.00	20.00	7.00	4.00	28659.51	515.32	506.22	506.13	0.02
5.00	0.75	23.00	29.00	7.00	3.00	477.11	9197.01	453.58	452.91	0.15
5.00	0.75	23.00	30.00	7.00	3.00	477.11	11123.36	457.49	456.93	0.12
5.00	0.75	23.00	31.00	7.00	3.00	477.11	13452.00	460.77	460.30	0.10
5.00	0.75	23.00	32.00	7.00	3.00	477.11	16266.95	463.52	463.12	0.09
5.00	0.75	23.00	33.00	7.00	3.00	477.11	19669.71	465.81	465.48	0.07
5.00	0.75	23.00	34.00	7.00	3.00	477.11	23783.03	467.73	467.45	0.06
5.00	0.75	23.00	35.00	7.00	3.00	477.11	28755.25	469.33	469.10	0.05
5.00	0.75	24.00	24.00	7.00	3.00	524.94	3546.76	457.26	455.66	0.35
5.00	0.75	24.00	25.00	7.00	3.00	524.94	4292.87	467.75	466.36	0.30
5.00	0.75	24.00	26.00	7.00	3.00	524.94	5194.86	476.77	475.57	0.25
5.00	0.75	24.00	27.00	7.00	3.00	524.94	6285.28	484.48	483.46	0.21
5.00	0.75	24.00	28.00	7.00	3.00	524.94	7603.48	491.04	490.18	0.18
5.00	0.75	24.00	29.00	7.00	3.00	524.94	9197.01	496.60	495.87	0.15
5.00	0.75	24.00	30.00	7.00	3.00	524.94	11123.36	501.29	500.67	0.12
5.00	0.75	24.00	31.00	7.00	3.00	524.94	13452.00	505.23	504.71	0.10
5.00	0.75	24.00	32.00	7.00	3.00	524.94	16266.95	508.53	508.10	0.09
5.00	0.75	24.00	33.00	7.00	3.00	524.94	19669.71	511.30	510.93	0.07
5.00	0.75	24.00	34.00	7.00	3.00	524.94	23783.03	513.61	513.30	0.06
5.00	0.75	31.00	17.00	7.00	3.00	1001.08	922.60	480.12	476.55	0.75
5.00	0.75	31.00	18.00	7.00	3.00	1001.08	1120.11	528.63	525.06	0.68
5.00	0.75	31.00	41.00	7.00	4.00	1001.08	863.60	463.64	460.34	0.72
5.00	0.75	31.00	42.00	7.00	4.00	1001.08	932.41	482.76	479.45	0.69
5.00	0.75	31.00	43.00	7.00	4.00	1001.08	1006.37	501.86	498.54	0.67
5.00	0.75	31.00	44.00	7.00	4.00	1001.08	1085.87	520.88	517.56	0.64
5.00	0.75	31.00	45.00	7.00	4.00	1001.08	1171.32	539.76	536.47	0.61
5.00	0.75	32.00	17.00	7.00	3.00	1094.98	922.60	500.71	497.16	0.72
5.00	0.75	32.00	40.00	7.00	4.00	1094.98	799.61	462.14	458.89	0.71
5.00	0.75	32.00	41.00	7.00	4.00	1094.98	863.60	482.81	479.54	0.68
5.00	0.75	32.00	42.00	7.00	4.00	1094.98	932.41	503.59	500.29	0.66
5.00	0.75	32.00	43.00	7.00	4.00	1094.98	1006.37	524.40	521.09	0.64
5.00	0.75	32.00	44.00	7.00	4.00	1094.98	1085.87	545.20	541.88	0.61
5.00	0.75	33.00	16.00	7.00	3.00	1197.10	759.27	464.60	461.19	0.74
5.00	0.75	33.00	17.00	7.00	3.00	1197.10	922.60	521.04	517.51	0.68

Table C.1 Results of Siegmund's Approximation (*continued*)

5.00	0.75	33.00	39.00	7.00	4.00	1197.10	740.09	457.34	454.20	0.69
5.00	0.75	33.00	40.00	7.00	4.00	1197.10	799.61	479.39	476.20	0.67
5.00	0.75	33.00	41.00	7.00	4.00	1197.10	863.60	501.68	498.45	0.65
5.00	0.75	33.00	42.00	7.00	4.00	1197.10	932.41	524.15	520.88	0.63
5.00	0.75	34.00	16.00	7.00	3.00	1308.14	759.27	480.42	477.09	0.70
5.00	0.75	34.00	17.00	7.00	3.00	1308.14	922.60	541.02	537.55	0.65
5.00	0.75	34.00	39.00	7.00	4.00	1308.14	740.09	472.67	469.60	0.65
5.00	0.75	34.00	40.00	7.00	4.00	1308.14	799.61	496.26	493.13	0.64
5.00	0.75	34.00	41.00	7.00	4.00	1308.14	863.60	520.19	517.00	0.62
5.00	0.75	34.00	42.00	7.00	4.00	1308.14	932.41	544.38	541.15	0.60
5.00	0.75	35.00	16.00	7.00	3.00	1428.86	759.27	495.81	492.55	0.66
5.00	0.75	35.00	38.00	7.00	4.00	1428.86	684.73	462.90	459.98	0.63
5.00	0.75	35.00	39.00	7.00	4.00	1428.86	740.09	487.55	484.56	0.62
5.00	0.75	35.00	40.00	7.00	4.00	1428.86	799.61	512.70	509.63	0.60
5.00	0.75	35.00	41.00	7.00	4.00	1428.86	863.60	538.27	535.15	0.58
5.00	0.75	36.00	16.00	7.00	3.00	1560.11	759.27	510.72	507.55	0.62
5.00	0.75	37.00	15.00	7.00	3.00	1702.80	624.22	456.77	453.95	0.62
5.00	0.75	37.00	16.00	7.00	3.00	1702.80	759.27	525.12	522.06	0.59
5.00	0.75	37.00	37.00	7.00	4.00	1702.80	633.25	461.59	458.95	0.57
5.00	0.75	37.00	38.00	7.00	4.00	1702.80	684.73	488.35	485.62	0.56
5.00	0.75	37.00	39.00	7.00	4.00	1702.80	740.09	515.87	513.06	0.55
5.00	0.75	37.00	40.00	7.00	4.00	1702.80	799.61	544.10	541.21	0.54
5.00	0.75	38.00	15.00	7.00	3.00	1857.89	624.22	467.24	464.53	0.58
5.00	0.75	38.00	16.00	7.00	3.00	1857.89	759.27	539.00	536.04	0.55
5.00	0.75	38.00	37.00	7.00	4.00	1857.89	633.25	472.28	469.75	0.54
5.00	0.75	38.00	38.00	7.00	4.00	1857.89	684.73	500.33	497.71	0.53
5.00	0.75	39.00	15.00	7.00	3.00	2026.47	624.22	477.22	474.63	0.55
5.00	0.75	39.00	16.00	7.00	3.00	2026.47	759.27	552.33	549.47	0.52
5.00	0.75	39.00	36.00	7.00	4.00	2026.47	585.38	454.18	451.86	0.51
5.00	0.75	39.00	37.00	7.00	4.00	2026.47	633.25	482.48	480.06	0.50
5.00	0.75	40.00	15.00	7.00	3.00	2209.68	624.22	486.72	484.25	0.51
5.00	0.75	40.00	36.00	7.00	4.00	2209.68	585.38	462.78	460.57	0.48
5.00	0.75	41.00	15.00	7.00	3.00	2408.80	624.22	495.75	493.39	0.48
5.00	0.75	42.00	15.00	7.00	3.00	2625.18	624.22	504.31	502.07	0.45
5.00	0.75	42.00	31.00	6.00	3.00	469.07	13452.00	453.27	452.84	0.10
5.00	0.75	42.00	32.00	6.00	3.00	469.07	16266.95	455.93	455.56	0.08