COMMUTATIVE AND NON-COMMUTATIVE INTEGRABLE EQUATIONS: LAX PAIRS, RECURSION OPERATORS

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ABSTRACT

COMMUTATIVE AND NON-COMMUTATIVE INTEGRABLE EQUATIONS: LAX PAIRS, RECURSION OPERATORS

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In this thesis, we investigate the integrability properties of some evolutionary type nonlinear equations in (1+1)-dimensions both with commutative and non-commutative variables. We construct the recursion operators, based on the Lax representation, for such equations. Finally, we question the notion of integrability for a certain one-component non-commutative equation. [We stress that calculations in this thesis are not original.]

Keywords: Integrability, Commutative, Non-commutative, Recursion operator, Lax pair

ÖZ

KOMUTATİF OLAN VE KOMUTATİF OLMAYAN İNTEGRALLENEBİLİNİR DENKLEMLER: LAX ÇİFTLERİ, SİMETRİ ADIM OPERATÖRLERİ

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Bu tez çalışmasında bazı evrimsel tipdeki çizgisel olmayan denklemlerin integrallenebilirlik özelliklerini (1+1) boyutta, komutatif ve komutatif olmayan değişkenlerle inceledik. Bu denklemler için Lax temsilini esas alarak, simetri adım operatörlerini kurduk. Son olarak, integrallenebilirlik kavramını bir bileşenli belli bir komutatif olmayan denklem için sorguladık. [Bu tezdeki hesaplar orijinal değildir.]

Anahtar Kelimeler: İntegrallenebilirlik, Komutatif olan, Komutatif olmayan, Simetri adım operatörü, Lax çifti

To my family

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CHAPTER 1

INTRODUCTION

A soliton is a solitary wave which preserves its shape and velocity upon nonlinear interaction with other solitary waves. The first observation of solitary waves was made by John Scott Russell, a naval engineer, in 1834 [1]. He observed that waves in one direction on the surface of a shallow channel keep their shape and velocity for a long time. In 1895, two Dutch scientists Korteweg and de Vries derived the equation for the propagation of waves in one direction on the surface of a shallow channel [2]; the dimensionless form of Korteweg-de Vries(KdV),

$$u_t = u_{3x} + uu_x,$$

where u represent the (small) elevation of the surface of the water above the normal depth, $u_x \equiv \frac{\partial u}{\partial x}$ and $u_{3x} \equiv \frac{\partial^3 u}{\partial x^3}$. In 1965, Zabusky and Kruskal were studying the Fermi-Pasta-Ulam problem [3] to investigate numerical solution of the KdV equation in [4]. They constructed the numerical solution of a train of solitary waves interacting elastically with periodic initial conditions. The solution they found is named solitons by Zabusky and Kruskal. After this introductory work on the KdV equation, *Inverse Spectral Transformation*(IST) method, a new method of solving a class of nonlinear partial differential equations was discovered for the exact solutions of the initial value problem for the KdV equation by Gardner, Greene, Kruskal and Miura [5]. The Lax pair theory was developed by Peter Lax [6] in 1968 as a way of generalizing the early work [5]. Later, the IST method was formulated in an algebraic form by Lax [6] who made it possible to obtain solvable physical models such as the modified KdV(mKdV), the nonlinear Schrödinger, Boussinesq, Kadomtsev-Petviashvili, sine-Gordon and many other equations. Since solitons theory arise in nonlinear theories, they have important applications in all areas of physics: fluid mechanics, nonlinear optics, plasma physics, classical and quantum field theories, solid state physics, astrophysics, biophysics, etc.

1.1 Integrability

The origin of the theory of integrable commutative nonlinear equations is based on the outstanding properties of the KdV equation: It possesses

a) N-soliton solutions

b) infinitely many symmetries and conserved quantities

c) Hamiltonian and bi-Hamiltonian structure

d) recursion operator

e) Lax pair representation

f) many other properties (Painleve property, Prolongation structure, Bäcklund transformation ...)

We note that many of the interrelations among these properties have not been rigorously established. Moreover, not all of these properties are shared by the other known nonlinear evolution equations. For example, Burgers equation is integrable but does not possess a Hamiltonian structure and Harry-Dym equation is integrable but does not possess the Painleve property. Based on each (or two) of these properties of KdV, one can define the concept of integrability and solvability. The most common definition of integrability is based on the existence of soliton solutions, i.e. the equations that can be solved by inverse scattering transform. We know that an equation is solvable by IST if it possesses a Lax pair, i.e. two linear operators L and A satisfy

$$L_t = [A, L] \equiv AL - LA.$$

Example 1.1.1 *The Korteweg-de Vries (KdV) equation*

$$u_t = u_{3x} + uu_x,$$

has the Lax pair (L, A)

$$L = 4D_x^2 + \frac{2}{3}u, \quad A = 4D_x^3 + uD_x + \frac{1}{2}u_x,$$

we can find the isospectal Lax equation

$$L_t + [L, A] = \frac{2}{3}(u_t - u_{3x} - uu_x) = 0.$$

However in the literature, there is no systematic way of finding whether a given evolution equation possesses a Lax representation and how one can construct the operators L, and A. In general, L, and A are determined by inspection. Therefore, the definition of the integrability in the sense of the existence of a Lax pair is too strong. Similarly one can define the term integrability with respect to the other properties of KdV. Furthermore, according to the terminology of Calogero [7] an equation is called *S-integrable* if it is solvable by IST and *C-integrable* if it can be linearized by a substitution (Cole-Hopf type).

Example 1.1.2 *The Burgers equation*

$$u_t = u_{2x} + 2uu_x \tag{1.1}$$

is an example of C-integrable equation which is reduced to the linear differential equation by nonlinear transformation, namely by Cole-Hopf transformation [8]. This transformation maps Burgers equation onto the linear heat equation. First let's introduce $u = \psi_x$*, inserting it in (1.1) and then integrating, we obtain*

$$\psi_t = \psi_{2x} + \psi_x^2,$$

where integration constant is zero, then introduce $\psi = \log \phi$ *to obtain*

$$\phi_t = \phi_{2x}.$$

In addition, according to the terminology of Fokas [9], "an equation is integrable if and only if it possesses infinitely many time-independent non-Lie point symmetries". At this point, we

can say that there is no consensus on the definition of integrability and solvability.
The definition of integrability, we use in this thesis is the existence of infinitely many symmetries (generalized symmetries) generated by a recursion operator.
Certain developments in string theory motivated the study of integrable nonlinear equations on non-commutative space-time [10, 11]. In connection with these developments many non-commutative integrable equations are obtained from the commutative integrable equations by replacing the ordinary product of dependent variables by the non-commutative Moyal $\star$-product. Moreover, there are many possible approaches to obtain the non-commutative integrable equations from commutative ones. In this thesis, we use the Olver-Sokolov [12] approach in which dependent variables take values in any non-commutative associative algebra (e.g. an algebra of matrices of functions).

The main concern of this thesis is to investigate the notion of integrability for a class of commutative and non-commutative nonlinear differential equations in (1+1)-dimensions in the context of recursion operators, produced by Lax representation. In chapter 2, we review the basic concepts about the symmetry and integrability of commutative nonlinear differential equations. The notion of the recursion operators is mentioned to obtain the generalized symmetries of such equations. Moreover, we deal with the correction of weak recursion operator which is an operator that does not give correct symmetries for a given integrable nonlinear differential equation. Then in the final section of chapter 2, we present the way of constructing recursion operators of integrable nonlinear differential equations from their Lax representations. In addition, this thesis is concerned with various integrable systems in a non-commutative setting. In chapter 3, we develop the non-commutative integrable nonlinear differential equations using the commutative integrable nonlinear equations where dependent variables take values in any non-commutative associative algebra and discuss their integrability. Non-commutativity is a well-known notion in quantum physics. Heisenberg uncertainty relation is the mostly encountered example of non-commutativity in nature where any pair of conjugate variables, such as position and momentum do not commute with each other. Non-commutative space and physics on such spaces has also been studied for over a decade now. Motivated also by results in string theory, formulation of and various aspects of non-commutative quantum field theories have been under investigation [10].

CHAPTER 2

PRELIMINARIES

2.1 Basic Definitions

In this chapter, we give some basic definitions about the symmetry and integrability of commutative nonlinear differential equations in a more general form for completeness.
Following the notation of Olver [13], a general system of n-th order nonlinear differential equations in p independent and q dependent variables,

$$\Delta_\alpha(x, u^{(n)}) = 0\,, \quad \alpha = 1, 2, \ldots, N, \tag{2.1}$$

where p-independent variables $x = (x_1, x_2, \ldots, x_p)$ are local coordinates on the Euclidean space X and q-dependent variables, $u = (u^1, u^2, \ldots, u^q)$ are coordinates on Euclidean space. $u^{(n)}$ denotes the derivatives of the u's with respect to p independent variables up to the order n. The system of N differential equations (2.1) can be abbreviated as $\Delta = 0$, and can be viewed as a smooth map from the prolongations of the total Euclidean space (jet space) $X \times U_{(n)}$ to some N dimensional Euclidean space:

$$\Delta : X \times U^{(n)} \to R^N.$$

A specific form of the general system of nonlinear differential equations (2.1) is said to be of evolution type if

$$\Delta = u_t - F[u] = 0, \tag{2.2}$$

where $t, x = (x_1, \ldots, x_{p-1})$ are p-independent and $u = (u_1, u_2, \ldots, u_q)$ are q-dependent variables and $F[u] = (F_1[u], F_2[u], \ldots, F_q[u])$ depend upon t, x, u and the x-derivatives of u only. In order to study the Lie symmetries and generalized symmetries of a given system of differential equations, we introduce the vector fields and their prolongations.

2.1.1 Vector Fields

A vector field, defined on $X \times U$, has a formal expression of the form:

$$v = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha},$$

where ξ^i and ϕ_α depend on only x and u. If we generalize that ξ^i and ϕ_α which are smooth differential functions depend on also derivative of u, then we define a formal expression of the generalized vector field:

$$v = \sum_{i=1}^{p} \xi^i[u]\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \phi_\alpha[u]\frac{\partial}{\partial u^\alpha}. \tag{2.3}$$

The prolongation of generalized vector field can be defined as

$$prv = \sum_{i=1}^{p} \xi^i\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q}\sum_{J} \phi_\alpha^J\frac{\partial}{\partial u_J^\alpha},$$

where

$$\phi_\alpha^J = D_J(\phi_\alpha - \sum_{i=1}^{p}\xi^i u_i^\alpha) + \sum_{i=1}^{p}\xi^i u_{J,i}^\alpha,$$

$J = (j_1, \ldots, j_p)$ are multi-indices and D_J is total derivative operator

$$D_J = D_{x_1}^{j_1} D_{x_2}^{j_2} \ldots D_{x_p}^{j_p}.$$

Also the abbreviations $u_J^\alpha = \frac{\partial^{j_1+\ldots+j_p} u^\alpha}{\partial^{j_1} x_1 \ldots \partial^{j_p} x_p}$ and $u_{J,i}^\alpha = \frac{\partial u_J^\alpha}{\partial x_i}$ are partial derivatives.

2.1.2 Generalized Infinitesimal Symmetry

We are now ready to give definition of an infinitesimal symmetry of a given system of differential equations.

Definition 2.1.1 *A generalized vector field v (2.3) is a generalized symmetry of a system of differential equations (2.1) if and only if*

$$prv[\Delta_\alpha[u]] = 0, \qquad \alpha = 1, 2, \ldots, N,$$

on every smooth solution $u = f(x)$

A similar definitions can be given for the Lie symmetry of a system of differential equations Any generalized vector field (2.3), has an evolutionary representative vector field V_Q of the form

$$V_Q = \sum_{\alpha=1}^{q} Q_\alpha[u] \frac{\partial}{\partial u^\alpha},$$

where the differential function $Q = (Q_1, Q_2, \ldots, Q_q)$ is the characteristic of the vector field with:

$$Q_\alpha = \phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha \ , \quad \alpha = 1, \ldots, q,$$

where $u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}$.
We noticed that these two vector fields v and V_Q generate the same generalized symmetry Q. In other words

$$prv(\Delta_\alpha[u]) = prV_Q(\Delta_\alpha[u]).$$

An alternative method for the construction of generalized symmetries of a given nonlinear differential equations is related to the linearization of nonlinear of differential equations.

2.1.3 Fréchet Derivative

Definition 2.1.2 *Let A be the space of differential functions depending on dependent variables, independent variables and derivatives of dependent variables. Let $P[u]$ be a r-tuple of differential functions and $Q[u]$ be a q-tuple of differential functions. Then the Fréchet derivative of P is the linear differential operator $F_P[u] = F_* : A^q \to A^r$ so that*

$$F_P[u](Q) = \frac{d}{d\varepsilon} P[u + \varepsilon Q[u]]|_{\varepsilon=0}, \tag{2.4}$$
$$= \sum_J \frac{\partial P}{\partial u_J} D_J Q, \tag{2.5}$$

where the summation is taken over all multi-indices J.

It is noticeable that prolongation of evolutionary vector field V_Q is equal to the Fréchet derivative of it.

$$prV_Q(P) = F_P[u](Q) = F_*(Q),$$

where $\Delta_v = P[u]$.
Using above observation, we can define the symmetry of a evolutionary differential equation $u_t = F[u]$ as follows.
If a differential function $\sigma[u]$ satisfies the symmetry condition

$$\sigma_t = F_* \sigma. \tag{2.6}$$

Then, it is called a symmetry of evolutionary differential equation $u_t = F[u]$.

Example 2.1.3 *The vector field $X = x\partial_x + 2t\partial_t$ is a symmetry of the Potential Burgers' equation*

$$u_t = u_{2x} + u_x^2.$$

We should use the condition in the equation (2.6) to check this vector field is symmetry. First of all, this vector field is generalized vector field, let's transform it into evolutionary vector field:

$$X = \sigma[u]\frac{\partial}{\partial u}\,, \quad \sigma = \phi[u] - \xi[u]u_x - \tau[u]u_t,$$

$$\begin{aligned} X = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} &= \sigma[u]\frac{\partial}{\partial u}, \\ &= (\phi - \xi u_x - \tau u_t)\frac{\partial}{\partial u}, \\ &= \phi\frac{\partial}{\partial u} - \xi\frac{\partial}{\partial x} - \tau\frac{\partial}{\partial t}, \end{aligned}$$

so $\phi = 0\,,\ \xi = -x\,,\ \tau = -2t$

$$\sigma = -xu_x - 2tu_t.$$

The right-hand side of (2.6) with Potential Burgers' equation can be written as

$$
\begin{aligned}
F_*[u]\sigma &= \frac{d}{d\epsilon}F[u+\epsilon\sigma]|_{\epsilon=0},\\
&= \sigma_{2x} + 2u_x\sigma_x,\\
&= (-xu_x - 2tu_t)_{2x} + 2u_x(-xu_x - 2tu_t)_x,\\
&= -2u_{2x} - 2u_x^2 - 2xu_xu_{2x} - 2tu_{2xt} - xu_{3x} - 4tu_xu_{xt},\\
&= -2(u_{2x} + u_x^2) - x(u_{2x} + u_x^2)_x - 2t(u_{2x} + u_x^2)_t,\\
F_*\sigma &= -2u_t - xu_{xt} - 2tu_{tt}.
\end{aligned}
$$

The left-hand side of (2.6) is;

$$
\begin{aligned}
\sigma_t &= (-xu_x - 2tu_t)_t,\\
\sigma_t &= -xu_{xt} - 2u_t - 2tu_{tt} = F_*\sigma.
\end{aligned}
$$

Since right side and left side of (2.6) is equal, the vector field X and its representation σ are the symmetries of the Potential Burgers' equation.

2.2 Recursion Operator

The method, given in (2.1.1), for the construction of the generalized symmetries of a given system of differential equations, is systematic but fails to characterize an infinite hierarchy of symmetries. To construct an infinite hierarchy of symmetries (if exists) for a given system, we introduce the notion of a recursion operator which connects symmetries. Therefore it guarantees the existence of infinite hierarchy of generalized symmetries.

The general form of the recursion operator first appeared in the context of generalized symmetries [13]. In general, recursion operators have local and non-local terms (pseudo-differential operator). Also, the recursion operator sometimes appear as the ratio of two differential operators which led to the construction of bi-Hamiltonian. In this section, we are interested in describing the notion of recursion operator for a class of nonlinear differential equations and hierarchies of their symmetries as well. Construction of recursion operator are investigated in the subsection (2.2.1).

Definition 2.2.1 *Let Δ be a system of differential equations. A recursion for Δ is a linear operator $R : A^q \to A^q$ in the space of q-tuples of differential functions with the property that whenever σ_i is an evolutionary symmetry of Δ, so is σ_{i+1} with*

$$\sigma_{i+1} = R\sigma_i. \tag{2.7}$$

As the definition makes clear, If we know the initial symmetry of a system of differential equation, we can construct a new symmetry of this system by applying the recursion operator to the initial symmetry. Also, we can form the infinite hierarchy of symmetries by applying the recursion operator on the last symmetry again and again endlessly.
There is a criteria for an operator to be a recursion operator:

Theorem 2.2.2 *Suppose $\Delta = u_t - F[u] = 0$ is a system of differential equations. A linear operator $R : A^q \to A^q$ is a standard recursion operator of the system if*

$$R_t = [F_*, R], \tag{2.8}$$

on the solutions of $\Delta = 0$.

Proof. According to (2.6)

$$\sigma_t = F_*\sigma,$$
$$(D_t - F_*)\sigma = 0.$$

If R satisfies (2.8), we get

$$(R_t - [F_*, R])\sigma = 0,$$
$$R_t\sigma - F_*R\sigma + RF_*\sigma = 0,$$
$$D_t(R\sigma) - F_*(R\sigma) = 0,$$
$$(D_t - F_*)(R\sigma) = 0,$$

where $R\sigma$ is also symmetry. Therefore, R is a recursion operator [13].

Example 2.2.3 *It is possible to find the several symmetries of Korteweg-de Vries(KdV) equation $u_t = u_{3x} + uu_x$ with the recursion operator $R = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_x D_x^{-1}$ where D_x is the total derivative operator with respect to x, and D_x^{-1} is the inverse of D_x. First of all, criteria of recursion operator (2.8) gives that*

$$\begin{aligned} R_t &= \frac{2}{3}u_t + \frac{1}{3}u_{tx}D_x^{-1}, \\ &= \frac{2}{3}(u_{3x} + uu_x) + \frac{1}{3}(u_{4x} + uu_{2x} + u_x^2)D_x^{-1}, \end{aligned}$$

is equal to

$$[F_*, R] = (D_x^3 + uD_x + u_x)(D_x^2 + \frac{2}{3}u + \frac{1}{3}u_x D_x^{-1}) - (D_x^2 + \frac{2}{3}u + \frac{1}{3}u_x D_x^{-1})(D_x^3 + uD + u_x).$$

Therefore, starting with the x-translation symmetry $-\partial_x$ which has characteristic $\sigma_0 = u_x$, we obtain

$$\begin{aligned} \sigma_1 &= R\sigma_0, \\ &= (D_x^2 + \frac{2}{3}u + \frac{1}{3}u_x D_x^{-1})u_x, \\ &= u_{3x} + \frac{2}{3}uu_x + \frac{1}{3}u_x u, \\ &= u_{3x} + uu_x, \\ \sigma_2 &= R\sigma_1, \\ &= (D_x^2 + \frac{2}{3}u + \frac{1}{3}u_x D_x^{-1})(u_{3x} + uu_x), \\ &= u_{5x} + \frac{5}{3}uu_{3x} + \frac{10}{3}u_x u_{2x} + \frac{5}{6}u^2 u_x, \end{aligned}$$

and

$$\begin{aligned} \sigma_3 &= R\sigma_2, \\ &= (D_x^2 + \frac{2}{3}u + \frac{1}{3}u_x D_x^{-1})(u_{5x} + \frac{5}{3}uu_{3x} + \frac{10}{3}u_x u_{2x} + \frac{5}{6}u^2 u_x, \\ &= u_{7x} + \frac{35}{3}u_{2x}u_{3x} + \frac{7}{3}uu_{5x} + 7u_x u_{4x} + \frac{35}{18}u^2 u_{3x} + \frac{70}{9}uu_x u_{2x} + \frac{35}{18}u_x^3 + \frac{35}{54}u_x u^3. \end{aligned}$$

The KdV equation $u_t = u_{3x} + uu_x$ possesses infinitely many symmetries produced by a recursion operator $R = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_x D_x^{-1}$.

Example 2.2.4 *The symmetries of Burgers equation $u_t = u_{2x} + 2uu_x$ can be found by recursion operator $R = D_x + u + u_x D_x^{-1}$. However, firstly we should check that this recursion operator is correct by using condition (2.8). The left hand sides of (2.8) gives us*

$$R_t = u_t + u_{xt} D_x^{-1} = u_{2x} + 2uu_x + (u_{3x} + 2u_x^2 + 2uu_{2x}) D_x^{-1},$$

and using Fréchet derivative (2.4), we obtain the right hand sides as follows

$$\begin{aligned}[F_*, R] &= (D_x^2 + 2uD_x + 2u_x)(D_x + u + u_x D_x^{-1}) - (D_x + u + u_x D_x^{-1}(D_x^2 + 2uD_x + 2u_x), \\ &= u_{2x} + 2u_x u + (u_{3x} + 2u_x^2 + 2uu_{2x}) D_x^{-1}.\end{aligned}$$

Both sides of (2.4) equal to each other , hence $R = D + u + u_x D^{-1}$ is correct recursion operator for Burgers equation. First basic symmetry is the x-translation symmetry $-\partial_x$ which has characteristic $\sigma_0 = u_x$, we can find the other symmetries by using (2.7). The first few are:

$$\begin{aligned}\sigma_1 &= R\sigma_0, \\ &= (D_x + u + u_x D_x^{-1}) u_x, \\ &= u_{2x} + 2uu_x, \\ \sigma_2 &= R\sigma_1, \\ &= (D_x + u + u_x D_x^{-1})(u_{2x} + 2uu_x), \\ &= u_{3x} + 3u_x^2 + 3uu_x + 3u^2 u_x, \\ \sigma_3 &= R\sigma_2, \\ &= (D_x + u + u_x D_x^{-1})(u_{3x} + 3u_x^2 + 3uu_x + 3u^2 u_x), \\ &= u_{4x} + 10u_x u_{2x} + 4uu_{3x} + 12uu_x^2 + 6u^2 u_{2x} + 4u^3 u_x.\end{aligned}$$

In the following section we will investigate the hierarchy of symmetries of integrable equations generated by time-dependent recursion operators (i.e. the coefficients of recursion operators depend on time explicitly).

2.2.1 Weak Recursion Operator

The time-dependent recursion operators, under the rule $D_x^{-1}D_x = 1$, do not generate the hierarchy of symmetries of integrable evolution equations. This type of recursion operators is called weak recursion operators in the work of Sanders and Wang [14]. They introduced a method to construct the time-dependent hierarchy of symmetries from a corrected recursion operator obtained from the weak one. Later, Gürses, Karasu and Turhan [15] showed that time-dependent recursion operators need modification due to the violation of associativity. For illustration, let's consider the cylindrical KdV(cKdV) equation in the form

$$\upsilon_\tau = \upsilon_{\xi\xi\xi} + \upsilon\upsilon_\xi - \frac{\upsilon}{2\tau}.$$

The point transformation

$$t = -2\tau^{-\frac{1}{2}}, \quad x = \xi\tau^{-\frac{1}{2}}, \quad u = \tau\upsilon + \frac{1}{2}\xi \tag{2.9}$$

transforms the cKdV equation to the KdV equation

$$u_t = u_{3x} + uu_x.$$

In example 2.2.3, some of the symmetries of KdV equation have been already calculated. Now, using the invertible point transformations (2.9) with a relation $\delta u = \tau\delta\upsilon$, we can derive the symmetries of cKdV from those of KdV.

$$
\begin{aligned}
\rho_1 &= \tau^{\frac{1}{2}}\upsilon_\xi + \frac{1}{2}\tau^{-\frac{1}{2}},\\
\rho_2 &= \tau^{\frac{3}{2}}\upsilon_\tau + \tau^{\frac{1}{2}}(\upsilon + \frac{\xi}{2}\upsilon_\xi) + \tau^{-\frac{1}{2}}\frac{\xi}{4},\\
\rho_3 &= \tau^{\frac{5}{2}}(\upsilon_{5\xi} + \frac{5}{3}\upsilon\upsilon_{3\xi} + \frac{10}{3}\upsilon_\xi\upsilon_{2\xi} + \frac{5}{6}\upsilon^2\upsilon_\xi)\\
&\quad + \tau^{\frac{3}{2}}(\frac{5}{6}\xi\upsilon_{3\xi} + \frac{5}{3}\upsilon_{2\xi} + \frac{5}{6}\xi\upsilon\upsilon_\xi + \frac{5}{12}\upsilon^2)\\
&\quad + \tau^{\frac{1}{2}}(\frac{5}{12}\xi\upsilon + \frac{5}{24}\xi^2\upsilon_\xi) + \tau^{-\frac{1}{2}}\frac{5}{48}\xi^2,\\
\rho_4 &= \tau^{\frac{7}{2}}(\upsilon_{7\xi} + \frac{7}{3}\upsilon\upsilon_{5\xi} + 7\upsilon_\xi\upsilon_{4\xi} + \frac{35}{18}\upsilon^2\upsilon_{3\xi} + \frac{35}{3}\upsilon_{2\xi}\upsilon_{3\xi} + \frac{70}{9}\upsilon\upsilon_\xi\upsilon_{2\xi}\\
&\quad + \frac{35}{18}\upsilon_\xi^3 + \frac{35}{54}\upsilon^3\upsilon_\xi) + \tau^{\frac{5}{2}}(\frac{7}{6}\xi\upsilon_{5\xi} + \frac{7}{2}\upsilon_{4\xi} + \frac{35}{18}\xi\upsilon\upsilon_{3\xi} + \frac{35}{9}\xi\upsilon_\xi\upsilon_{2\xi}\\
&\quad + \frac{35}{9}\upsilon\upsilon_{2\xi} + \frac{35}{12}\upsilon_\xi^2 + \frac{35}{36}\xi\upsilon^2\upsilon_\xi + \frac{35}{108}\upsilon^3) + \tau^{\frac{3}{2}}(\frac{35}{72}\xi^2\upsilon_{3\xi} + \frac{35}{18}\xi\upsilon_{2\xi}\\
&\quad + \frac{35}{72}\xi^2\upsilon\upsilon_\xi + \frac{35}{24}\upsilon_\xi + \frac{35}{72}\xi\upsilon^2) + \tau^{\frac{1}{2}}(\frac{35}{432}\xi^3\upsilon_\xi + \frac{35}{144}\xi^2\upsilon + \frac{35}{144})\\
&\quad + \tau^{-\frac{1}{2}}\frac{35}{864}\xi^3,
\end{aligned}
$$

where all ρ are the symmetries of the system. We have used the transformation (2.9) to get these symmetries; however, let's check whether they satisfy the symmetry condition (2.6).

$$\rho_{1_\tau} = F_*\rho_1. \tag{2.10}$$

The Fréchet derivative of cKdV system is

$$
\begin{aligned}
F_*[\upsilon]\rho &= \frac{d}{d\epsilon}F[\upsilon + \epsilon\rho]|_{\epsilon=0},\\
&= \rho_{3\xi} + \upsilon\rho_\xi + \rho\upsilon_\xi - \frac{\rho}{2\tau}.
\end{aligned}
$$

The right hand sides of (2.10) becomes

$$
\begin{aligned}
F_*\rho &= \tau^{\frac{1}{2}}\upsilon_{4\xi} + \tau^{\frac{1}{2}}\upsilon\upsilon_{2\xi} + (\tau^{\frac{1}{2}}\upsilon_\xi + \frac{1}{2}\tau^{-\frac{1}{2}})\upsilon_\xi - \frac{1}{2\tau}(\tau^{\frac{1}{2}}\upsilon_\xi + \frac{1}{2}\tau^{-\frac{1}{2}}),\\
&= \tau^{\frac{1}{2}}\upsilon_{4\xi} + \tau^{\frac{1}{2}}\upsilon\upsilon_{2\xi} + \tau^{\frac{1}{2}}\upsilon^2 - \frac{1}{4}\tau^{-\frac{3}{2}},
\end{aligned}
$$

and by using $\upsilon_\tau = \upsilon_{3\xi} + \upsilon\upsilon_\xi - \frac{\upsilon}{2\tau}$, the left hand sides of (2.10) can be obtained as follows

$$\rho_\tau = \tau^{\frac{1}{2}}\upsilon_{4\xi} + \tau^{\frac{1}{2}}\upsilon\upsilon_{2\xi} + \tau^{\frac{1}{2}}\upsilon^2 - \frac{1}{4}\tau^{-\frac{3}{2}}.$$

The right hand sides and left hand sides are equal to each other. Therefore, ρ_1 is the correct symmetries of the cKdV system. For the second symmetry $\rho_2 = \tau^{\frac{3}{2}}\upsilon_\tau + \tau^{\frac{1}{2}}(\upsilon + \frac{1}{2}\xi\upsilon_\xi) + \frac{1}{4}\tau^{-\frac{1}{2}}\xi$,

$$\begin{aligned}
\rho_\tau &= F_*\rho, \\
&= \rho_{3\xi} + \rho_\xi\upsilon + \upsilon_\xi\rho - \frac{\rho}{2\tau}, \\
&= (\tau^{\frac{3}{2}}\upsilon_{\tau\xi\xi\xi} + \tau^{\frac{1}{2}}(\frac{5}{2}\upsilon_{3\xi} + \frac{1}{2}\xi\upsilon_{4\xi}) + [\tau^{\frac{3}{2}}\upsilon_{\tau\xi} + \tau^{\frac{1}{2}}(\frac{3}{2}\upsilon_\xi + \frac{1}{2}\xi\upsilon_{2\xi}) + \frac{1}{4}\tau^{-\frac{1}{2}})]\upsilon \\
&+ \upsilon_\xi[\tau^{\frac{3}{2}}\upsilon_\tau + \tau^{\frac{1}{2}}(\upsilon + \frac{1}{2}\xi\upsilon_\xi) + \frac{1}{4}\tau^{-\frac{1}{2}}\xi] - \frac{1}{2\tau}[\tau^{\frac{3}{2}}\upsilon_\tau + \tau^{\frac{1}{2}}(\upsilon + \frac{1}{2}\xi\upsilon_\xi) + \frac{1}{4}\tau^{-\frac{1}{2}}\xi], \\
&= \tau^{\frac{3}{2}}(\upsilon_{\tau\xi\xi\xi} + \upsilon_{\tau\xi}\upsilon + \upsilon_\tau\upsilon_\xi) + \tau^{\frac{1}{2}}(2\upsilon_{3\xi} + 2\upsilon_\xi\upsilon + \frac{1}{2}\xi\upsilon_{4\xi} + \frac{1}{2}\xi\upsilon\upsilon_{2\xi} + \frac{1}{2}\xi\upsilon_\xi^2) - \frac{1}{8}\tau^{-\frac{3}{2}}\xi.
\end{aligned}$$

If we take the derivative of ρ_2 with respect to τ, we get

$$\begin{aligned}
\rho_\tau &= \frac{3}{2}\tau^{\frac{1}{2}}\upsilon_\tau + \tau^{\frac{3}{2}}\upsilon_{\tau\tau} + \frac{1}{2}\tau^{-\frac{1}{2}}(\upsilon + \frac{1}{2}\xi\upsilon_\xi) + \tau^{\frac{1}{2}}(\upsilon_\tau + \frac{1}{2}\upsilon_{\xi\tau}) - \frac{1}{8}\tau^{-\frac{3}{2}}\xi, \\
&= \tau^{\frac{3}{2}}(\upsilon_{\tau\xi\xi\xi} + \upsilon_{\tau\xi}\upsilon + \upsilon_\tau\upsilon_\xi) + \tau^{\frac{1}{2}}(2\upsilon_{3\xi} + 2\upsilon_\xi\upsilon + \frac{1}{2}\xi\upsilon_{4\xi} + \frac{1}{2}\xi\upsilon\upsilon_{2\xi} + \frac{1}{2}\xi\upsilon_\xi^2) - \frac{1}{8}\tau^{-\frac{3}{2}}\xi.
\end{aligned}$$

So, ρ_2 is also one of the symmetries of cKdV system. Similarly, for the symmetries ρ_3 and ρ_4, it is possible to show that they satisfy symmetry condition. Hence, ρ_3 and ρ_4 are also symmetries of this system.

By using the transformation (2.9), we can also transform the recursion operator of KdV

$$\mathcal{R}_{KdV} = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_x D_x^{-1},$$

to cKdV with $D_x = \tau^{\frac{1}{2}}D_\xi$, $D_x^2 = \tau D_\xi^2$ and $D_x^{-1} = \tau^{-\frac{1}{2}}D_\xi^{-1}$.

$$\mathcal{R}_{cKdV} = \tau D_\xi^2 + \frac{2}{3}(\tau\upsilon + \frac{1}{2}\xi) + \frac{1}{3}\tau^{\frac{1}{2}}(\tau\upsilon_\xi + \frac{1}{2})\tau^{-\frac{1}{2}}D_\xi^{-1}, \tag{2.11}$$

$$\mathcal{R}_{cKdV} = \tau D_\xi^2 + \frac{2}{3}\tau\upsilon + \frac{1}{3}\xi + \frac{1}{6}(1 + 2\tau\upsilon_\xi)D_\xi^{-1}. \tag{2.12}$$

Although this recursion operator satisfies the condition (2.8), it is not the correct recursion operator. The third symmetry σ_3 which are obtained by the recursion operator (2.12) does not satisfy the symmetry condition (2.6). As a result of this problem, the correction terms should be added to a weak recursion operator so as to give the correct symmetries, or symmetries which do not satisfy condition (2.6) should be corrected by adding some terms. In this section, we will correct some weak recursion operators and symmetries respectively.

2.2.1.1 Construction of Corrected Recursion Operator

In general a recursion operator may have more complicated nonlocal terms; however, let's consider a recursion operator in the form $\mathcal{R}_w = R_1 + aD_x^{-1}$ where R_1 is the local part of the recursion operator and a is a function of jet coordinates, x and t. Now, let $\mathcal{R} = \mathcal{R}_w + \frac{a}{g}H$ where H is an operator and g is required function so that $\frac{a}{g}$ is a symmetry. If we use our assumption in the eigenvalue equation $\mathcal{R}\sigma = \lambda\sigma$ where $\sigma \in \mathcal{A}$ (the space of symmetries of an evolution equation) are the symmetries of a system, then we get

$$\mathcal{R}_w\sigma + \frac{a}{g}H\sigma = \lambda\sigma.$$

Taking the derivative of the eigenvalue equation with respect to time, we obtain

$$\mathcal{R}_{wt}(\sigma) + \mathcal{R}_w\sigma_t + (\frac{a}{g})_t(H\sigma) + \frac{a}{g}(H\sigma)_t = \lambda\sigma_t.$$

Using (2.6) and (2.8) and paying attention to the order of parenthesis because of nonlocal terms (D_x^{-1}) in recursion operator, the above equation becomes

$$a(H(\sigma))_t + g[As(\mathcal{R}_w, F_*, \sigma) - As(F_*, \mathcal{R}_w, \sigma)] = 0, \tag{2.13}$$

where $As(P, Q, \sigma) = P(Q(\sigma)) - (PQ)(\sigma)$ for any operators P, Q and any σ. The associators $As(P, Q, \sigma)$ is vanished for local cases. For the correction of recursion operator, only a time-dependent constant should be added, therefore, the operator H contains a projection operator Π such that $\Pi\sigma = \lim_{x,q,q_x,\ldots\to 0}\sigma$ = a time-dependent function. As a result of this, (2.13) becomes

$$(H\sigma_0)_t + g[As(D_x^{-1}, F_{*0}, \sigma_0) - As(F_{*0}, D_x^{-1}, \sigma_0)] = 0, \quad (2.14)$$

where $F_{*0} = \lim_{q,q_x,\ldots\to 0} F_*$ and σ_0 is the part of the symmetries depends only x and t. We can add correct terms $\frac{a}{g}H$ to the recursion operator by calculating H in (2.14) [15].

Example 2.2.5 *There are two well known recursion operators of Burgers equation in the form $u_t = u_{2x} + uu_x$ such that*

$$\mathcal{R}_1 = D_x + \frac{1}{2}u + \frac{1}{2}u_x D_x^{-1}, \quad \mathcal{R}_2 = tD_x + \frac{1}{2}tu + \frac{1}{2}x + \frac{1}{2}(1 + tu_x)D_x^{-1},$$

for the first one, there is no problem in the calculation of symmetries as we have done in the example 2.2.4. On the other hand, second recursion operator does not investigate the correct symmetries of the Burgers equation. To find the correction of $\mathcal{R}_2$, we first choose g such that $\frac{a}{g}$ is a symmetry of the system with $a = \frac{1}{2}(1 + tu_x)$. For $g = 1$, the symmetry condition (2.6) should be true. Now, let $\sigma_0 = a_1(t) + a_2(t)x + a_3(t)x^2 + \ldots$ then by using (2.14), we obtain

$$(H\sigma_0)_t + D_x^{-1}(F_{*0}(\sigma_0)) - (D_x^{-1}F_{*0})(\sigma_0) - F_{*0}(D_x^{-1}\sigma_0) + (F_{*0}D_x^{-1})\sigma_0 = 0,$$

with $F_{*0} = lim_{u,u_x,\ldots\to 0}F_*[u] = lim_{u,u_x,\ldots\to 0}(u_x + uD_x + D_x^2) = D_x^2$, equation (2.14) becomes

$$(H\sigma_0)_t = a_2,$$

which means

$$H = D_t^{-1}\Pi D_x.$$

Therefore, the corrected second recursion operator of Burgers equation is

$$\mathcal{R}_2 = tD_x + \frac{1}{2}tu + \frac{1}{2}x + \frac{1}{2}(1 + tu_x)D_x^{-1} + \frac{1}{2}(1 + tu_x)D_x^{-1}\Pi D_x.$$

Example 2.2.6 *We have obtained the recursion operator of Cylindirical Korteweg de Vries (cKdV) equation $\upsilon_\tau = \upsilon_{3\xi} + \upsilon\upsilon_\xi - \frac{1}{2\tau}\upsilon$ by using the transformation (2.9) as follows*

$$\mathcal{R}_w = \tau D_\xi^2 + \frac{2}{3}\tau\upsilon + \frac{1}{3}\xi + \frac{1}{6}(1 + 2\tau\upsilon_\xi)D_\xi^{-1},$$

where $a = \frac{1}{6}(1 + 2\tau\upsilon_\xi)$ and $F_{*0} = D_\xi^3 - \frac{1}{2\tau}$. Because of $\frac{a}{g}$ should be a symmetry, g must take the value $g = \sqrt{\tau}$. Using similar assumption for σ_0 in the above example, equation (2.14) can be written

$$\begin{aligned}
&(H\sigma_0)_\tau + \sqrt{\tau}[As(D_\xi, D_\xi^3 - \frac{1}{2\tau}, \sigma_0) - As(D_\xi^3 - \frac{1}{2\tau}, D_\xi^{-1}, \sigma_0)] = 0,\\
&(H\sigma)_\tau = 2\tau a_3,
\end{aligned}$$

this means that

$$H = D_\tau^{-1}\sqrt{\tau}\Pi D_\xi^2.$$

Hence the recursion operator of cKdV equation with correction term is

$$\mathcal{R} = \tau D_\xi^2 + \frac{2}{3}\tau\upsilon + \frac{1}{3}\xi + \frac{1}{6}(1 + 2\tau\upsilon_\xi)D_\xi^{-1} + \frac{1}{6\sqrt{\tau}}D_\tau^{-1}\sqrt{\tau}\Pi D_\xi^2.$$

2.2.1.2 Construction of Corrected Symmetries

We have added some terms in nonlocal part of a weak recursion operator to get correct one, another way is keeping recursion operators as they are and introduce correction on the symmetries. Firstly, let's introduce the action of D_x^{-1} such that we take $D_x^{-1}G_x = G$ where $G \in \mathcal{A}_1$ and $D_x^{-1}H_x = H + h(t)$ where $H \in \mathcal{A}_0$ and h is a function of t.

Definition 2.2.7 *Let $\mathcal{R}_w$ be a recursion operator of the form*

$$\mathcal{R}_w = R_1 + R_0,$$

where $R_0 = \mathcal{R}_w \mid_{q,q_x,\cdots\to 0}$, and let σ_n be symmetries of the system, generated by the $\mathcal{R}_w$, of the form

$$\sigma_n = \sigma_n^1 + \sigma_n^0,$$

where $\sigma_0 = \sigma_n \mid_{q,q_x,\cdots\to 0}$.

The following proposition is necessary at this point.

Proposition 2.2.1 *Let the function F vanish in the limit when the jet space coordinates go to zero, i.e $\lim_{q,q_x,\cdots} F = 0$. Then the operator $R_0 = \lim_{q,q_x,\cdots\to 0} \mathcal{R}_w$ satisfies $\sigma_{n+1}^0 = R_0\sigma_n^0$ and $R_{0_t} = [F_{*0}, R_0]$ where F_{*0} is the Fréchet derivative when q and all the derivatives of q also go to zero.*

Now, using this proposition we can find the missing terms in symmetries and the difference between the weak symmetries (the ones obtained by $\mathcal{R}_w$) and the corrected symmetries comes from σ_0 part of the symmetries. The general corrected symmetry σ should be in the form

$$\sigma = \bar{\sigma} + \frac{a}{g}h, \tag{2.15}$$

where $\bar{\sigma}$ is obtained by the weak recursion operator, when we find the correction term $h(t)$ for σ_0. The corresponding corrected recursion operator takes the form

$$\mathcal{R} = \mathcal{R}_w + \frac{a}{g}H, \tag{2.16}$$

and $h = H\sigma$ [15].

Example 2.2.8 *The Burgers equation*

$$u_t = u_{2x} + uu_x$$

possesses a recursion operator of the form

$$\mathcal{R}_w = tD_x + \frac{1}{2}tu + \frac{1}{2}x + \frac{1}{2}(1 + tu_x)D_x^{-1},$$

so

$$R_0 = tD_x + \frac{1}{2}x + \frac{1}{2}D_x^{-1}.$$

Let $\sigma_n^0 = a_1(t) + a_2(t)x + a_3(t)x^2 + \cdots$ *. Using (2.6) we obtain* $\sigma_{n_t}^0 = \sigma_{n_{2x}}^0$ *where* $F_* = u_x + uD_x + D_x^2$ *and* $F_{*0} = D_x^2$*.*

$$a_{1_t} + a_{2_t}x + a_{3_t}x^2 + \cdots = 2a_3 + 6a_4x + 12a_5x^2 + \cdots ,$$

As a result of this, the undetermined coefficients related to each other as follows

$$a_{1_t} = 2a_3, \quad a_{2_t} = 6a_4, \quad a_{3_t} = 12a_5, \tag{2.17}$$

by using 2.2.1, we can obtain

$$\begin{aligned}
\sigma_{n+1}^0 &= (tD_x + \frac{1}{2}x + \frac{1}{2}D_x^{-1})\sigma_n^0, \\
&= t(a_2 + 2a_3x + 3a_4x^2 + \cdots) + \frac{1}{2}x(a_1 + a_2x + a_3x^2 + \cdots) + \frac{1}{2}(a_1x + \frac{1}{2}a_2x^2 \\
&+ \frac{1}{3}a_3x^3 + \cdots + h(t)), \\
\sigma_{n+1}^0 &= (ta_2 + \frac{h}{2}) + (2a_3t + a_1) + (3a_4t + \frac{3}{4}a_2)x^2 + \cdots ,
\end{aligned}$$

again using (2.6) for σ_{n+1}^0 *and the following system of equations for* a_i *and h can be found by equating the coefficients at power of x to zero*

$$
\begin{aligned}
(ta_2 + \frac{1}{2}h)_t &= 2(3a_4 t + \frac{3}{4}a_2),\\
(2ta_3 + a_1)_t &= 6(4a_5 t + \frac{2}{3}a_3),\\
\vdots &= \vdots
\end{aligned}
$$

with (2.17) the first equation gives $h_t = a_2$ and all the others are satisfies identically. Therefore, h is

$$h = D_t^{-1}(\Pi D_x \sigma_n^0).$$

The equation (2.15) can be rewritten for σ_{n+1}^0 such as

$$\sigma_{n+1}^0 = \bar{\sigma_{n+1}}^0 + \frac{1}{2}D_t^{-1}(\Pi D_x \sigma_n^0).$$

It can be generalized for σ_n by adding the constant of integration $h(t)$ in general symmetry equation (2.15)

$$\sigma_{n+1} = \bar{\sigma_{n+1}} + \frac{1}{2}(1 + tu_x)D_t^{-1}(\Pi D_x \sigma_n^0),$$

where $\bar{\sigma_{n+1}}$ is the symmetry obtained by standard application of the operator D_x^{-1}. The correction of symmetries allows us to define corrected recursion operator by (2.16)

$$\mathcal{R} = \mathcal{R}_w + \frac{1}{2}(1 + tu_x)D_t^{-1}\Pi D_x.$$

.

Example 2.2.9 *The cylindrical Korteweg-de Vries equation (cKdV)*

$$u_t = u_{3x} + uu_x - \frac{1}{2t}u$$

possesses a recursion operator of the form

$$\mathcal{R}_w = tD_x^2 + \frac{2}{3}tu + \frac{1}{3}x + \frac{1}{6}(1 + 2tu_x)D_x^{-1},$$

where $R_0 = tD_x^2 + \frac{1}{3}x + \frac{1}{6}D_x^{-1}$. *Let's use similar ansatz* σ_0 *in (2.2.8)*

$$\sigma_n^0 = a_1(t) + a_2(t)x + a_3(t)x^2 + \cdots .$$

.

The symmetry condition (2.6) gives us the relation:

$$\sigma_{n_t}^0 = \sigma_{n_{3x}}^0 - \frac{1}{2t}\sigma_n^0,$$

because of $F_* = u_x - \frac{1}{2t} + uD_x + D_x^3$ *and* $F_{*_0} = \lim_{q,q_x,\cdots\to 0} F_* = D_x^3 - \frac{1}{2t}$. *Inserting our ansatz for* σ_0 *in this equation we get*

$$a_{1_t} = 6a_4 - \frac{1}{2t}a_1, \quad a_{2_t} = 24a_5 - \frac{1}{2t}a_2, \quad a_{3_t} = 60a_6 - \frac{1}{2t}a_3, \cdots .$$

Then

$$\begin{aligned}\sigma_{n+1}^0 &= R_0\sigma_n^0, \\ &= (2a_3t + \frac{1}{6}h) + (6a_4t + \frac{1}{2}a_1)x + \cdots .\end{aligned}$$

Using (2.6) for σ_{n+1}, *we obtain*

$$\begin{aligned}\sigma_{n+1\,t}^0 &= F_{*_0}\sigma_{n+1}^0, \\ \sigma_{n+1\,t}^0 &= (D_x^3 - \frac{1}{2t})\sigma_{n+1}^0,\end{aligned}$$

$$(2a_3t + \frac{1}{6}h)_t + (6a_4t + \frac{1}{2}a_1)_t x + (12a_5t + \frac{5}{12}a_2)_t x^2 + \cdots = 6(20a_6t + \frac{7}{18}a_3) - a_3 - \frac{h}{12t} + \cdots ,$$

and $h_t + \frac{1}{2t}h = 2a_3$. *It gives* $h = \frac{1}{\sqrt{t}}D_t^{-1}(\sqrt{t}\Pi D_x^2\sigma_n^0)$. *Hence the symmetry equation for cKdV equation and corresponding recursion operator are respectively*

$$\sigma_{n+1} = \bar{\sigma}_{n+1} + \frac{1}{6}(2tu_x + 1)\frac{1}{\sqrt{t}}D_t^{-1}(\sqrt{t}\Pi D_x^2\sigma_n^0),$$

and

$$\mathcal{R} = \mathcal{R}_w + \frac{1}{6}(1 + 2tu_x)\frac{1}{\sqrt{t}}D_t^{-1}\sqrt{t}\Pi D_x^2.$$

2.3 Lax pairs, Recursion operator

A Lax pair contains two linear operators L and A such that

$$L\psi = \lambda\psi, \tag{2.18}$$

$$\psi_t = A\psi. \tag{2.19}$$

Here (2.18) represents the spectral equation for L and (2.19) represents the time evolution of the eigenfunctions ψ.

Differentiating (2.18) with respect to t gives

$$L_t\psi + L\psi_t = \lambda_t\psi + \lambda\psi_t.$$

Using (2.19), we obtain

$$L_t + (LA - AL)\psi = \lambda_t\psi.$$

Hence, if $\lambda_t = 0$, then

$$L_t = [A, L] \tag{2.20}$$

is called Lax equation and contains commutative nonlinear evolution equation for suitable L and A . Another form of the isospectral problem is

$$\phi_x = U(u, \lambda)\phi, \quad \lambda_t = 0,$$
$$\phi_t = V(u, \lambda)\phi,$$

and then necessary condition for Lax pair is

$$U_t - V_x + [U, V] = 0.$$

This equation is called zero-curvature equation [16].

The basic problem in the Lax representation of an integrable evolution equation is to find all the operators A for a given L. Lax found such a family of operators, say A_n, for KdV equation. Later, Gel'fand-Dikii [17] gave a construction of all the operators for KdV-type equations, based on the fractional powers of differential operator L. In this construction the linear operator L has the form

$$L = D_x^m + u_{m-2}D_x^{m-2} + \ldots + u_0,$$

and all the operators A_n, n is a positive integer, has the form

$$A_n = (L^{\frac{n}{m}})_+. \tag{2.21}$$

.

Here $L^{\frac{n}{m}}$ has the pseudo differential series form

$$L^{\frac{n}{m}} = \sum u_i D_x^i,$$

where $(L^{\frac{n}{m}})_+$ is the formal series containing the differential operator of degree greater than or equal to zero. Therefore, the set of system

$$L_{t_n} = [A_n, L], \tag{2.22}$$

with fixed m and all n is called the n^{th} KdV-type hierarchy equation. Now, we can rewrite the fractional power of L as follows

$$L^{\frac{n}{m}} = (L^{\frac{n}{m}})_+ + (L^{\frac{n}{m}})_-,$$

then, we get

$$L_{t_n} = [A_n, L] = [L^{\frac{n}{m}}, L] - [(L^{\frac{n}{m}})_-, L].$$

Since, by construction, $[L^{\frac{n}{m}}, L] = 0$, we have

$$L_{t_n} = [A_n, L] = -[(L^{\frac{n}{m}})_-, L]. \tag{2.23}$$

The left-hand side of (2.23) is a differential operator of order $n-2$ but right-hand side contains the commutator of two operator of orders $Ord((L^{\frac{n}{m}})_-) + Ord(L)$. Its order is equal to or less than $-1+n-1 = n-2$. Therefore, both sides of (2.22) are differential operators of order$\leq n-2$, for each positive integer n [18].

Example 2.3.1 *The Lax operator of KdV equation $u_t = \frac{1}{4}(u_{3x} + 6uu_x)$ is $L = D_x^2 + u$, to construct the differential operator A_n, let's start with trying to find square root of Lax operator L. The square root of this operator can be written as an infinite series in inverse powers of D_x;*

$$L^{\frac{1}{2}} = D_x + a_0(u) + \sum_{n=1}^{\infty} a_n(u) D_x^{-n}.$$

To determine $a_0(u)$ and the each of the $a_n(u)$'s, we should square this formal series and require it to be equal to L. Firstly, let's expand $L^{\frac{1}{2}}$ up to the D_x^{-1} terms, we get

$$L^{\frac{1}{2}} = D_x + a_0(u) + a_1(u) D_x^{-1},$$
$$(L^{\frac{1}{2}})_+ = D_x + a_0(u),$$

then,

$$\begin{aligned}L &= L^{\frac{1}{2}}L^{\frac{1}{2}},\\ D_x^2 + u &= (D_x + a_0(u) + a_1(u)D_x^{-1})(D_x + a_0(u) + a_1(u)D_x^{-1}),\end{aligned}$$

equating coefficients of order D_x, we can find $a_0(u) = 0$ and $a_1(u) = \frac{u}{2}$, then the differential operator A_1 should be constructed as

$$A_1 = (L^{\frac{1}{2}})_+ = D_x.$$

If we expand $L^{\frac{1}{2}}$ up to the D_x^{-2}, then our undetermined coefficients can be found as follows:

$$(L(t))^{\frac{1}{2}} = D_x + a_0(u) + a_1(u)D_x^{-1} + a_2(u)D_x^{-2},$$

we know that

$$\begin{aligned}L(t) &= (L(t))^{\frac{1}{2}}(L(t))^{\frac{1}{2}},\\ D_x^2 + u &= (D_x + a_0(u) + a_1(u)D_x^{-1} + a_2(u)D_x^{-2})(D_x + a_0(u) + a_1(u)D_x^{-1} + a_2(u)D_x^{-2}),\\ D_x^2 + u &= D_x^2 + (a_0)_x + a_0D_x + (a_1)_xD_x^{-1} + a_1 + (a_2)_xD_x^{-2} + a_2D_x^{-1} + a_0D_x + (a_0)^2\\ &\quad + a_0a_1D_x^{-1} + a_0a_2D_x^{-2} + a_1 + a_1(D_x^{-1}a_0) + a_1(D_x^{-1}a_1)D_x^{-1} + a_1(D_x^{-1}a_2)D_x^{-2}\\ &\quad + a_2D_x^{-1} + a_2(D_x^{-2}a_0) + a_2(D_x^{-2}a_1)D_x^{-1} + a_2(D_x^{-2}a_2)D_x^{-2},\end{aligned}$$

and by equating the terms of the different power of D_x, we obtain the undetermined coefficient as follows: For the order of D_x

$$\begin{aligned}2a_0D_x &= 0,\\ a_0 &= 0.\end{aligned}$$

For the order of D_x^0

$$a_0^2 + 2(a_1)_x = u,$$
$$a_1 = \frac{u}{2}.$$

For the order of D_x^{-1}

$$(a_1)_x + 2a_2 + a_0 a_1 = 0,$$
$$a_2 = -\frac{u_x}{4}.$$

So we get,

$$(L(t))^{\frac{1}{2}} = D_x + \frac{1}{2}uD_x^{-1} - \frac{1}{4}u_x D_x^{-2}.$$

This expansion gives us again the differential operator A_1, on the other hand, the differential operator A_3 can be constructed by expansion of $\frac{3}{2}$ powers of L as follows

$$\begin{aligned}
(L(t))^{\frac{3}{2}} &= (D_x^2 + u)(L(t))^{\frac{1}{2}}, \\
&= D_x^3 + \frac{1}{2}(u_{2x}D_x^{-1} + 2u_x + uD_x) - \frac{1}{4}(u_{3x}D_x^{-2} + 2u_{2x}D_x^{-1} + u_x) + uD_x + \frac{1}{2}u^2 D_x^{-1} \\
&- \frac{1}{4}uu_x D_x^{-2}, \\
&= D_x^3 + \frac{3}{2}uD_x + \frac{3}{4}u_x + \frac{1}{2}u^2 D_x^{-1} - \frac{1}{4}(u_{3x} + uu_x)D_x^{-2}.
\end{aligned}$$

The positive powers of differential operator of this equation gives us the Lax operator A_3. i.e

$$A_3 = (L(t))_+^{\frac{3}{2}} = D_x^3 + \frac{3}{2}uD_x + \frac{3}{4}u_x = D_x^3 + \frac{3}{4}(D_x u + uD_x).$$

In the previous example 2.3.1, we exemplify the constructing A operator from L operator by Gel'fand-Dikii formalism. However, Gel'fand-Dikii formalism is based on an ansatz for the operator L. In general, there are three types of formalism for operator L [16] in the forms

$$\begin{aligned}
k = 0: &\quad L = c_m D_x^m + c_{m-1} D_x^{m-1} + u_{m-2} D_x^{m-2} + \cdots + u_0, \\
k = 1: &\quad L = c_m D_x^m + u_{m-1} D_x^{m-1} + \cdots + u_0 + D_x^{-1} u_{-1}, \\
k = 2: &\quad L = u_m D_x + u_{m-1} D_x^{m-1} + \cdots + u_0 + D_x^{-1} u_{-1} + D_x^{-2} u_{-2}.
\end{aligned}$$

with the three admissible Lax hierarchies

$$L_{t_n} = [(L^{\frac{n}{m}})_{\geq k}, L] = -[L^{\frac{n}{m}}_{<k}, L], \tag{2.24}$$

$k = 0$ is the case for Gel'fand-Dikii formalism and the cases $k = 1, 2$ were introduced by Kupershmidt [19]. For illustration, the three choice of k provides us evaluate the hierarchies of Lax equations for three nonlinear differential equations; $k = 0$ for the KdV equation, $k = 1$ for the modified KdV equation and $k = 2$ for the Harry-Dym equation. Constructing a recursion operator for a given integrable nonlinear differential equation is a nontrivial task. Recently, Gürses, Karasu and Sokolov gave a construction for the recursion operator for such equations which is based on Lax representation [20].

Proposition 2.3.1 *For any n*

$$A_{n+m} = LA_n + R_n,$$

where R_n is a differential operator of order $\leq m - 1$, called remainder.

Proof. Since $L^{\frac{n+m}{m}} = LL^{\frac{n}{m}}$, then using (2.21) we write

$$A_{n+m} = (LL^{\frac{n}{m}})_+ = (L(L^{\frac{n}{m}})_+)_+ + (L(L^{\frac{n}{m}})_-)_+.$$

It gives

$$L_{t_{n+m}} = [A_{n+m}, L] = LL_{t_n} + [(L(L^{\frac{n}{m}})_-)_+, L].$$

For any n, we can rewrite the Lax operator A_{m+n} like this

$$A_{m+n} = L(L^{\frac{n}{m}})_+ + (L(L^{\frac{n}{m}})_-)_+,$$

by using (2.21)

$$A_{n+m} = LA_n + (L(L^{\frac{n}{m}})_-)_+.$$

If we substitute $R_n = (L(L^{\frac{n}{m}})_-)_+$ with the $Ord(R_n) \leq Ord(L) + Ord(L^{\frac{m}{n}}) = m - 1$, the proof is finished.

The result of this proposition leads to

$$L_{t_{n+m}} = [A_{n+m}, L] = [LA_n + R_n, L] = L[A_n, L] + [R_n, L] = LL_{t_n} + [R_n, L],$$

$$L_{t_{n+m}} = LL_{t_n} + [R_n, L], \tag{2.25}$$

and it is called *the recursion relation*.

Remark 2.3.1 *It follows from the formula*

$$A_{n+m} = (L^{\frac{n}{m}}L)_+ = (L^{\frac{n}{m}})_+L + ((L^{\frac{n}{m}})_-L)_+,$$

that

$$A_{n+m} = A_nL + \overline{R_n}, \tag{2.26}$$

and

$$L_{t_{n+m}} = L_{t_n} L + [L, \overline{R_n}],$$

where $\overline{R_n}$ is a differential operator of order$\leq m-1$.

By equating the coefficient of different powers of $D_x^i,\ i = 2m-2, \ldots, m-2$ in (2.25), we can easily determine R_n in terms of the coefficient of operator L_{t_n}. The necessary condition of the resulting formula is the linearity in the coefficient of L_{t_n}. The remaining coefficients of $D_x^i,\ \ i = m-2, \ldots, 0$ in (2.25) give us the relation

$$\begin{pmatrix} u_0 \\ \vdots \\ u_{m-2} \end{pmatrix}_{t_{n+m}} = \mathcal{R} \begin{pmatrix} u_0 \\ \vdots \\ u_{m-2} \end{pmatrix}_{t_n},$$

where $\mathcal{R}$ is a recursion operator. Also, we can use proposition (2.26) instead of (2.25), the corresponding recursion operators coincide [20].

Example 2.3.2 *The KdV equation*

$$u_t = \frac{1}{4}(u_{3x} + 6uu_x),$$

has a Lax representation with

$$L = D^2 + u, \quad A = (L^{\frac{3}{2}})_+.$$

Since $L_{t_n} = u_{t_n} \equiv u_n$ and $L_{t_{n+2}} = u_{t_{n+2}} \equiv u_{n+2}$, (2.25) becomes

$$u_{n+2} = (D^2 + u)u_n + [R_n, L], \tag{2.27}$$

with $R_n = a_n D + b_n$ (note that $R_n \leq 1$).

To determine the R_n, let's find unknowns in (2.27)

$$
\begin{aligned}
[R_n, L] &= R_nL - LR_n, \\
&= (a_nD + b_n)(D^2 + u) - (D^2 + u)(a_nD + b_n), \\
&= a_nu_x - a_{nxx}D - 2a_{nx}D^2 - b_{nxx} - 2b_{nx}D,
\end{aligned}
$$

then, (2.27) becomes

$$u_{n+2} = u_{nxx} + 2u_{nx}D + u_nD^2 + uu_n + a_nu_x - a_{nxx}D - 2a_{nx}D^2 - b_{nxx} - 2b_{nx}D.$$

Now, if we equate to zero the coefficients of $D_x^2,\ D_x,$ and D_x^0 in above equation, we get

$$a_n = \frac{1}{2}D^{-1}(u_n), \quad b_n = \frac{3}{4}u_n,$$

and

$$u_{n+2} = (\frac{1}{4}D^2 + u + \frac{1}{2}u_xD^{-1})u_n,$$

that gives the standard recursion operator for the KdV equation:

$$\mathcal{R} = \frac{1}{4}D^2 + u + \frac{1}{2}u_xD^{-1}.$$

Example 2.3.3 *Recursion operators of the Burgers equation $u_t = u_{2x} + 2uu_x$ with the Lax operator $L = D_x + u$ can be found by*

$$L_{t_{n+1}} = LL_{t_n} + [R_n, L], \tag{2.28}$$

where $R_n = a_nD_x + b_n$, $L_{t_{n+1}} = u_{t_{n+1}} = u_{n+1}$, and $L_{t_n} = u_{t_n} = u_n$, to define R_n we should find $[R_n, L]$,

$$[R_n, L] = R_nL - LR_n = (a_nD_x + b_n)(D_x + u) - (D_x + u)(a_nD_x + b_n),$$
$$= a_nu_x - a_{n_x}D_x - b_{n_x},$$

then, (2.28) becomes

$$u_{n+1} = u_{n_x} + u_nD_x + uu_n + a_nu_x - a_{n_x}D_x - b_{n_x}.$$

There is no order of D_x in the left hand sides of this equation, therefore we can easily define a_n such that $a_n = D_x^{-1}(u_n)$, and for the choice $b_n = 0$, we get one of the recursion operators of general Burgers equation.

$$u_{n+1} = u_{n_x} + uu_n + D_x^{-1}(u_n)u_x,$$
$$u_{n+1} = (D_x + u + u_xD_x^{-1})u_n,$$
$$\mathcal{R} = D_x + u + u_xD_x^{-1}.$$

The other recursion operator for the Burgers equation by inspection is given by $R = tD_x + tu + \frac{x}{2} + (tu_x + \frac{1}{2})D_x^{-1}$. Later, we have seen that this time-dependent recursion operator does not connect the symmetry correctly.

2.3.1 Symmetric and skew-symmetric reductions of differential Lax operator

There are two conditions, $L^* = L$ or $L^* = -L$ for the standard reductions of the Gel'fand-Dikii systems. Here $*$ denotes the adjoint operation defined as follows.

Definition 2.3.4 *Let L be a differential operator, $L = \sum a_iD_x^i$, then its adjoint L^* is given by*

$$L^* = \sum(-D_x)^i \cdot a_i.$$

It is noticeable that the order of L; m must be an even integer when $L^* = L$, and odd integer when $L^* = -L$. The compatibility of (2.21) provides that $(A_n)^* = -A_n$, so all possible A_n are defined by (2.21), where n takes odd integer values.

If $L^* = L$, the recursion operator can be found from (2.25) and (2.26) because $n + m$ is an odd integer in the formula $A_{n+m} = (LL^{\frac{n}{m}})_+ = (L^{\frac{n+m}{m}})_+$. Therefore, in this case proposition (2.3.1) is still valid and this formula gives the correct A_n-operator . On the other hand, if $L^* = -L$, $m + n$ is even integer because both m and n are odd. Hence, $(L^{\frac{n+m}{m}})_+$ is not suitable to be an A_n-operator. As a result of this, we have to work on $A_{n+2m} = (L^{\frac{n+2m}{m}})_+ = (L^2L^{\frac{n}{m}})_+$ to find the recursion operator.

Proposition 2.3.2 *If $L^* = -L$, then*

$$A_{n+2m} = L^2A_n + R_n, \tag{2.29}$$

where $ord(R_n) < 2ord(L)$. It follow from (2.29) that

$$L_{t_{n+2m}} = L^2L_{t_n} + [R_n, L].$$

Remark 2.3.2 *Instead of (2.29), we can use the ansatz*

$$A_{n+2m} = LA_nL + \overline{R_n}, \tag{2.30}$$

or

$$A_{n+2m} = A_nL^2 + \overline{\overline{R_n}}. \tag{2.31}$$

then, recursion operator obtained by the utility of (2.29), (2.30) and (2.31) all coincide.

2.3.2 Matrix L-operator of the first order

In previous section, we have worked on the scalar L-operator in the form $L = D_x^m + u_{m-2}D_x^{m-2} + \cdot + u_0$ to define recursion operator. In this section, we consider L is a matrix operator of the form

$$L = D_x + \lambda a + q(x,t), \tag{2.32}$$

where q and a belong to a Lie algebra, λ is the spectral parameter.

Proposition 2.3.3 *Let L be a matrix operator of the form (2.32) then corresponding recursion relation is*

$$L_{t_{n+1}} = \lambda L_{t_n} + [R_n, L], \tag{2.33}$$

where R_n is a matrix operator.

Using proposition 2.3.3, one can easily find the corresponding recursion operator [20].

Example 2.3.5

$$\begin{aligned} u_t &= -\frac{1}{2}u_{2x} + u^2 v, \\ v_t &= \frac{1}{2}v_{2x} - v^2 u, \end{aligned} \tag{2.34}$$

is equivalent to the nonlinear Schrödinger equation, has a Lax operator

$$L = D_x + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda + \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}.$$

Let's use the proposition 2.3.3 to construct the recursion operator of the nonlinear Schrödinger equation with $R_n = \begin{pmatrix} a_n & b_n \\ c_n & -a_n \end{pmatrix}$, $L_{t_n} = \begin{pmatrix} 0 & u_n \\ v_n & 0 \end{pmatrix}$ and $L_{t_{n+1}} = \begin{pmatrix} 0 & u_{n+1} \\ v_{n+1} & 0 \end{pmatrix}$. We can find the undetermined coefficients by comparing the coefficients of the powers of $\lambda^i, i = 0, 1$ as follows

$$\begin{aligned} a_n &= \frac{1}{2}D_x^{-1}(u_n v + v_n u), \\ b_n &= \frac{1}{2}u_n, \quad c_n = -\frac{1}{2}v_n, \end{aligned}$$

hence, the recursion operator of the system (2.34) can be found like this

$$\mathcal{R} = \begin{pmatrix} uD_x^{-1} - \frac{1}{2}D_x & uD_x^{-1}u \\ -vD_x^{-1}v & -vD_x^{-1}u + \frac{1}{2}D_x \end{pmatrix}.$$

CHAPTER 3

NON-COMMUTATIVE INTEGRABLE EQUATIONS

So far, we have examined the integrability properties of some commutative nonlinear evolution equations in (1+1)-dimensions with dependent variables taking values in a commutative associative algebra. For the criterion of integrability, we have used the existence of recursion operators based on the Lax representations of such equations. In this chapter, we shall examine the same properties of non-commutative nonlinear evolution equations in (1+1)-dimensions with dependent variables taking values in a non-commutative algebra.

Flat non-commutative space is spanned by the coordinates x^0, x^1 which fulfill the $\star$-commutation relations

$$[x^i, x^j] = x^i \star x^j - x^j \star x^i = \mathrm{i}\theta^{ij},$$

where θ^{ij} are real constants, called the *NC parameters* [21]. To obtain the non-commutative multiplication, ordinary products of the coordinates and their functions in commutative space is replaced with the Moyal $\star$-product which is given as

$$\begin{aligned} f \star g(x) &:= exp(\frac{1}{2}\mathrm{i}\theta^{ij}\partial_i^{(x')}\partial_j^{(x'')})f(x')g(x'') \mid_{x'=x''=x}, \\ &= f(x)g(x) + \frac{1}{2}\mathrm{i}\theta^{ij}\partial_i f(x)\partial_j g(x) + 0(\theta^2). \end{aligned}$$

Hence, for non-commutative nonlinear evolution equations, products of dependent variables should be replaced with the Moyal $\star$-product. There are various methods to construct non-commutative integrable equations from a commutative one. To give some example, we can mention that Lax pair generating technique, non-commutative version of the usual Lax representation, bicomplex method, non-commutative zero curvature representation and the reduction of self-dual Yang-Mills equations.

In this chapter, we will use the Lax pair generating technique and non-commutative version of the usual Lax representation with non-commutative multiplication for all products. e.g $uu_x \neq u_x u$. To distinguish the non-commutative multiplication, we introduce the operators of left and right multiplication [12] such as

$$L_u(v) = uv, \qquad R_u(v) = vu,$$

where u, v are the element of a linear associative algebra, L is the left multiplication operator and R is a right multiplication operator. Moreover, we have

$$L_{\alpha\cdot\beta}(v) = L_\alpha(v) \cdot L_\beta(v), \quad R_{\alpha\cdot\beta}(v) = R_\alpha(v) \cdot R_\beta(v),$$
$$L_{\alpha+\beta}(v) = L_\alpha(v) + L_\beta(v), \quad R_{\alpha+\beta}(v) = R_\alpha(v) + R_\beta(v)$$

where α, β is any component of jet space.

Now, we are ready to exemplify the recursion operators for non-commutative nonlinear differential equations, namely, the non-commutative KdV equation and the non-commutative nonlinear Schrödinger equation [20].

Example 3.0.6 *In the example 2.3.2, we have found the recursion operator of KdV equation. Let's work out the non-commutative version of this equation given by*

$$u_t = \frac{1}{4}(u_{3x} + 3uu_x + 3u_x u),$$

with Lax pairs $L = D_x^2 + L_u, \quad A = (L^{\frac{3}{2}})_+$. *According to the technique which is mentioned in [20], the recursion relation (2.25) of the NC KdV equation is*

$$L_{t_{n+1}} = LL_{t_n} + [R_n, L]. \tag{3.1}$$

.

By inserting the remainder $R_n = a_n D_x + b_n$ *in (3.1), we obtain*

$$
\begin{aligned}
u_{n+1} &= (D_x^2 + L_u)u_n + [R_n, L],\\
u_{n+1} &= (D_x^2 + L_u)u_n + (a_n D_x + b_n)(D_x^2 + L_u) - (D_x^2 + L_u)(a_n D_x + b_n),\\
u_{n+1} &= u_{n_{2x}} + 2u_{n_x} D_x + uu_n + a_n u_x + b_n u - ub_n - b_{n_{2x}} + a_n u D_x - a_{n_{2x}} D_x - ua_n D_x\\
&\quad + u_n D_x^2 - 2a_{n_x} D_x^2 - 2b_{n_x} D_x.
\end{aligned}
$$

Let's equate the terms of different powers of D_x,
For the order of D_x^2,

$$
\begin{aligned}
u_n &= 2a_{n_x},\\
a_n &= \frac{1}{2} D_x^{-1} u_n.
\end{aligned}
$$

For the order of D_x,

$$
\begin{aligned}
2u_{n_x} + a_n u - a_{n_{2x}} - ua_n - 2b_{n_x} &= 0,\\
b_n &= \frac{3}{4}u_n + \frac{1}{4}D_x^{-1}(L_u - R_u)(D_x^{-1}u_n),
\end{aligned}
$$

then, equation (3.1) becomes

$$
\begin{aligned}
u_{n+1} &= \frac{1}{4}u_{n_{2x}} + \frac{1}{2}uu_n + \frac{1}{2}u_n u + \frac{1}{4}D_x^{-1}(u_n)u_x + \frac{1}{4}u_x D_x^{-1}(u_n) + \frac{1}{4}D_x^{-1}(L_u - R_u)(D_x^{-1}u_n)u\\
&\quad - u\frac{1}{4}D_x^{-1}(L_u - R_u)(D_x^{-1}u_n),\\
u_{n+1} &= (\frac{1}{4}D_x^2 + \frac{1}{2}(L_u + R_u) + \frac{1}{4}(R_{u_x} + L_{u_x})D_x^{-1} + \frac{1}{4}(L_u - R_u)D_x^{-1}(L_u - R_u)D_x^{-1})u_n.
\end{aligned}
$$

Hence, the recursion operator of non-commutative KdV equation is

$$
\mathcal{R} = \frac{1}{4}D_x^2 + \frac{1}{2}(L_u + R_u) + \frac{1}{4}(R_{u_x} + L_{u_x})D_x^{-1} + \frac{1}{4}(L_u - R_u)D_x^{-1}(L_u - R_u)D_x^{-1}.
$$

Now, we can construct the generalized symmetries of non-commutative KdV equation by using the recursion operator as follows: Starting with x-translation symmetry $\sigma_0 = u_x$ and using the property of recursion operator (2.7), we can get the first few symmetries

$$
\begin{aligned}
\sigma_1 &= \mathcal{R}\sigma_0,\\
\sigma_1 &= (\frac{1}{4}D_x^2 + \frac{1}{2}(L_u + R_u) + \frac{1}{4}(R_{u_x} + L_{u_x})D_x^{-1} + \frac{1}{4}(L_u - R_u)D_x^{-1}(L_u - R_u)D_x^{-1})u_x,\\
\sigma_1 &= \frac{1}{4}(u_{3x} + 3uu_x + 3u_xu),
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma_2 &= \mathcal{R}\sigma_1,\\
\sigma_2 &= (\frac{1}{4}D_x^2 + \frac{1}{2}(L_u + R_u) + \frac{1}{4}(R_{u_x} + L_{u_x})D_x^{-1}\\
&\quad + \frac{1}{4}(L_u - R_u)D_x^{-1}(L_u - R_u)D_x^{-1})\frac{1}{4}(u_{3x} + 3uu_x + 3u_xu),\\
\sigma_2 &= \frac{1}{16}u_{5x} + \frac{5}{8}u_xu_{2x} + \frac{5}{8}u_{2x}u_x + \frac{5}{16}uu_{3x} + \frac{5}{16}u_{3x}u + \frac{5}{8}u^2u_x + \frac{5}{8}u_xu^2 + \frac{5}{8}uu_xu.
\end{aligned}
$$

Example 3.0.7 *In the section 2.3.2, we have found the recursion operator of the nonlinear Schrödinger equation which has L operator in matrix form. Let's work on the non-commutative version of this equation given by*

$$
\begin{aligned}
u_t &= -\frac{1}{2}u_{2x} + uvu,\\
v_t &= \frac{1}{2}v_{2x} + vuv,
\end{aligned}
$$

and the Lax operator of this system is given by

$$
L = D_x + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda + \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}.
$$

To construct the recursion operator we should find the matrix operator in the proposition 2.3.3 with $R_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$*. Because of* $L_{t_n} = \begin{pmatrix} 0 & u_n \\ v_n & 0 \end{pmatrix}$ *and* $L_{t_{n+1}} = \begin{pmatrix} 0 & u_{n+1} \\ v_{n+1} & 0 \end{pmatrix}$*, we get*

$$
\begin{pmatrix} 0 & u_{n+1} \\ v_{n+1} & 0 \end{pmatrix} = \begin{pmatrix} b_nv - a_{n_x} - uc_n & \lambda u_n - 2b_n\lambda + a_nu - b_{n_x} - ud_n \\ \lambda v_n + 2c_n\lambda + d_nv - va_n - c_{n_x} & c_nu - vb_n - d_{n_x} \end{pmatrix},
$$

by comparing the coefficients of the powers of $\lambda^i, i = 0, 1,$ *we find that*

$$a_n = \frac{1}{2}D_x^{-1}(u_n v + u v_n),$$
$$b_n = \frac{1}{2}u_n,$$
$$c_n = -\frac{1}{2}v_n,$$
$$d_n = -\frac{1}{2}D_x^{-1}(v_n u + v u_n)$$

then, recursion operator of the non-commutative Schrödinger equation is constructed as

$$\mathcal{R} = \frac{1}{2}\begin{pmatrix} -D_x + R_u D_x^{-1} R_v + L_u D_x^{-1} L_v & R_u D_x^{-1} L_u + L_u D_x^{-1} R_u \\ -R_v D_x^{-1} L_v - L_v D_x^{-1} R_v & D_x - R_v D_x^{-1} R_u - L_v D_x^{-1} L_u \end{pmatrix}.$$

3.1 Non-commutative Burgers Equation and Non-commutative Mixed Burgers Equation

In this section, we present the non-commutative version of Burgers equation, then construct the recursion operators, symmetries of the non-commutative Burgers equation and discuss its integrability. Two non-commutative versions of Burgers equation are formed by the classification of non-commutative extension of the integrable nonlinear evolution in (1+1) dimensions according to symmetry based integrability in [12]. The forms of these types are

$$u_t = u_{2x} + 2u_x u, \quad u_t = u_{2x} + 2uu_x,$$

and are called left- and right-handed, respectively. Let's work on the right-handed NC Burgers $u_t = u_{2x} + 2uu_x$ to investigate the generalized symmetries of NC Burgers equation by constructing recursion operators because the recursion operators for left-handed one is obtained by only interchanging of left multiplication $L_\psi(\phi) = \psi\phi$ with the right multiplication $R_\psi(\phi) = \phi\psi$ in all the results for the right-handed one. Let's start the Lax representation and hierarchy of the right-handed NC Burgers equation.

The Lax representation for the right-handed NC Burgers hierarchy with

$$L = D_x + L_u,$$

is given by

$$L_{t_n} = [A_n, L]. \tag{3.2}$$

In section 2.3, we have mentioned that Gel'fand-Dikii method to generate the hierarchy of a given system with its Lax representation. For this method, a second operator of Lax pair, A, is defined by choosing the power $\frac{n}{m}$ of L and taking its series part which contains differential operators greater than or equal to zero. Hence, the hierarchy of Lax equation could be written by (2.23) and more general by (2.24). For $k = 0$, when we start with purely differential operators, integer powers of L does not generate the Lax hierarchy because $L^{\frac{n}{m}}_{\geq 0} = L^{\frac{n}{m}}$ and $L_{t_n} = [L^{\frac{n}{m}}, L] = 0$. There are two way to eliminate this problem. One of them is consider the fractional power of L for the second operator of Lax pair (like we have done in example (2.3.1) for the KdV equation) and the second one is choosing k different than zero [16].

Let's evaluate the Lax hierarchy of NC Burgers by using second way. Starting Lax operator $L = D_x + L_u$, We can find the powers of L as follows

$$\begin{aligned}
L^2 &= LL = (D_x + L_u)(D_x + L_u) = D_x^2 + 2uD_x + u_x + u^2, \\
L^3 &= LL^2 = (D_x + L_u)(D_x^2 + 2uD_x + u_x + u^2), \\
&= D_x^3 + 3u_xD_x + 3uD_x^2 + 3u^2D_x + u_{2x} + 2uu_x + u_xu + u^3, \\
L^4 &= LL^3 = (D_x + L_u)(D_x^3 + 3u_xD_x + 3uD_x^2 + 3u^2D_x + u_{2x} + 2uu_x + u_xu + u^3), \\
&= D_x^4 + 5u_{2x}D_x + 6u_xD_x^2 + 4uD_x^3 + 4u_xuD_x + 8uu_xD_x + 6u^2D_x^2 + 4u^3D_x + 3u^2u_x \\
&\quad + 2uu_xu + 3uu_{2x} + 3u_x^2 + u_{2x}u + u_xu^2 + u_{3x} + u^4.
\end{aligned}$$

Instead of $k = 0$, let's choose $k = 1$ in (2.24) with $m = 1$, then we get

$$L_{t_n} = [L^n_{\geq 1}, L],$$

or

$$L_{t_n} = -[L^n_{<1}, L].$$

Both of these equation give the same results for Lax hierarchy, choosing the second equation with $L^2_{<1} = u_x + u^2$, $L^3_{<1} = u_{2x} + 2uu_x + u_x u + u^3$ and $L^4_{<1} = 3u^2 u_x + 2uu_x u + 3uu_{2x} + 3u_x^2 + u_{2x}u + u_x u^2 + u_{3x} + u^4$, we obtain the first few Lax hierarchy of NC Burgers equation as follows

$$\begin{aligned} u_{t_2} &= -[u_x + u^2, D_x + u], \\ u_{t_2} &= 2uu_x + u_{2x}, \end{aligned}$$

for $A_2 = L^2_{<1}$

and

$$\begin{aligned} u_{t_3} &= -[u_{2x} + 2uu_x + u_x u + u^3, D_x + u], \\ u_{t_3} &= u_{3x} + 3u_x^2 + 3uu_{2x} + 3u^2 u_x, \end{aligned}$$

for $A_3 = L^3_{<1}$,

and

$$\begin{aligned} u_{t_4} &= -[3u^2 u_x + 2uu_x u + 3uu_{2x} + 3u_x^2 + u_{2x}u + u_x u^2 + u_{3x} + u^4, D_x + u], \\ u_{t_4} &= 4u^3 u_x + 6u^2 u_{2x} + 8uu_x^2 + 4uu_{3x} + 4u_x uu_x + 6u_x u_{2x} + 4u_{2x}u_x + u_{4x}, \end{aligned}$$

for $A_4 = L^4_{<1}$.

Now, we investigate the recursion operators of NC Burgers equation using the methods introduced in [20]. In previous chapter, we have used it to construct the recursion operator of KdV and Burgers equations. Now, let's try to apply this method for NC Burgers equation.

According to the technique which is mentioned in [20], the *recursion relation* 2.25 of the NC Burgers equation is

$$L_{t_{n+1}} = LL_{t_n} + [R_n, L]. \tag{3.3}$$

Inserting the ansatz the remainder $R_n = a_n D_x + b_n$ in (3.3), the undetermined coefficients a_n, b_n could be found as follows

$$\begin{aligned}
u_{n+1} &= (D_x + L_u)u_n + [R_n, L], \\
u_{n+1} &= (D_x + L_u)u_n + (a_n D_x + b_n)(D_x + L_u) - (D_x + L_u)(a_n D_x + b_n), \\
u_{n+1} &= u_{n_x} + u_n D_x + u u_n + a_n u_x + b_n u - u b_n - b_{n_x} + a_n u D_x - a_{n_x} D_x - u a_n D_x.
\end{aligned}$$

By equating the terms at the order of D_x, we get

$$u_n = (D_x + L_u - R_u)a_n.$$

Using the notation $ad_L = [L, \cdot]$, a_n can be rewritten such as

$$a_n = ad_L^{-1} u_n,$$

then (3.3) becomes

$$u_{n+1} = (D_x + L_u)u_n + R_{u_x} ad_L^{-1} u_n - ad_L b_n. \tag{3.4}$$

The different choice of b_n gives the time independent or time dependent recursion operators of NC Burgers equation. Firstly, Let's choose $b_n = 0$, then the time independent recursion operator of NC Burgers equation is

$$\mathcal{R}_1 = D_x + L_u + R_{u_x} ad_L^{-1}.$$

To check that this recursion operator is a conventional recursion operator for the NC Burgers hierarchy, it should satisfy the condition (2.8) $\mathcal{R}_t = [\mathcal{F}_*, \mathcal{R}]$ with the Fréchet derivative of the right-handed NC Burgers equation $\mathcal{F}_* = D_x^2 + 2L_u D_x + 2R_{u_x}$. In spite of using this formulation, it is more convenient to transform its equivalent form [22] where recursion operator take in the form $\mathcal{R} = \mathcal{M}N^{-1}$. By inserting $\mathcal{R} = \mathcal{M}N^{-1}$ in (2.8), we get

$$\mathcal{M}_t - \mathcal{F}_*\mathcal{M} = \mathcal{M}\mathcal{N}^{-1}(\mathcal{N}_t - \mathcal{F}_*\mathcal{N}), \tag{3.5}$$

where $(\mathcal{N}^{-1})_t = -\mathcal{N}^{-1}\mathcal{N}_t\mathcal{N}^{-1}$. For the time independent recursion operator $\mathcal{R}_1 = D_x + L_u + R_{u_x}ad_L^{-1}$, $\mathcal{M}$ and $\mathcal{N}$ can be defined as follows

$$\mathcal{M} = (D_x + L_u)ad_L + R_{u_x}, \quad \mathcal{N} = ad_L.$$

It is more useful to multiply both sides of (3.5) by a function $f(x)$ so that the differential operator D_x reduces to partial derivative of functions $f(x)$ and $u, u_x, u_{2x}\cdots$. The right hand sides of (3.5) should be found as follows

$$\begin{aligned}
\mathcal{M}\mathcal{N}^{-1}(\mathcal{N}_t - \mathcal{F}_*\mathcal{N})f(x) &= \mathcal{M}\mathcal{N}^{-1}(-f_{2x} - uf_{2x} + f_{2x}u - 2u_xf_x - 2uf_{2x} - 2u^2f_x + 2uf_xu),\\
&= -\mathcal{M}\mathcal{N}^{-1}(\mathcal{N}(f_{2x} + 2uf_x)),\\
&= -\mathcal{M}((f_{2x} + 2uf_x)),\\
&= -[(D_x + L_u)ad_L + R_{u_x}](f_{2x} + 2uf_x),\\
\mathcal{M}\mathcal{N}^{-1}(\mathcal{N}_t - \mathcal{F}_*\mathcal{N})f(x) &= -[f_{4x} + 4uf_{3x} + 5(u_x + u^2)f_{2x} + (2u_{2x} + 4uu_x + 2u^3)f_x - f_{3x}u\\
&\quad - 3uf_{2x}u - 2u_xf_xu - 2u^2f_xu + 2u_xuf_x],
\end{aligned}$$

and the left hand sides of (3.5) is

$$
\begin{aligned}
(\mathcal{M}_t - \mathcal{F}_*\mathcal{M})f(x) &= [(D_x + L_u)ad_L]_t f(x) + R_{u_{xt}} f(x) - (D_x^2 + 2L_u D_x + 2R_{u_x})[((D_x + L_u)ad_L \\
&\quad + R_{u_x})f], \\
&= 2u_{2x}f_x + 4uu_xf_x + u_{2x}uf - u_{2x}fu + 2uu_xuf - 2uu_xfu + u_{3x}f - f_xu_{2x} \\
&\quad + 2u_x^2f + 3uu_{2x}f - 2f_xuu_x - ufu_{2x} + 2u^2u_xf - 2ufuu_x - (f_{4x} + u_{3x}f \\
&\quad + 4u_{2x}f_x + 5u_xf_{2x} + 4uf_{3x} - f_{3x}u - f_xu_{2x} + u_{2x}uf + 2u_x^2f + 2u_xuf_x \\
&\quad + 3uu_{2x}f - 3uf_{2x}u + 5u^2f_{2x} + 8uu_xf_x - u_{2x}fu - 2u_xf_xu - ufu_{2x} \\
&\quad + 2uu_xuf - 2uu_xfu + 2u^2u_xf - 2u^2f_xu - 2f_xuu_x - 2ufuu_x + 2u^3f_x), \\
(\mathcal{M}_t - \mathcal{F}_*\mathcal{M})f(x) &= -[f_{4x} + 4uf_{3x} + 5(u_x + u^2)f_{2x} + (2u_{2x} + 4uu_x + 2u^3)f_x - f_{3x}u \\
&\quad - 3uf_{2x}u + 2u_xuf_x - 2u_xf_xu - 2u^2f_xu].
\end{aligned}
$$

The right and left hand sides of (3.5) equal to each other, so $\mathcal{R}_1$ is one of the recursion operators of NC Burgers equation and generates an infinite hierarchy of symmetries by mapping a symmetry to another. Starting from the symmetries $\sigma_0 = u_x$ and using (2.7) , we get new symmetries:

$$
\begin{aligned}
\sigma_1 &= \mathcal{R}\sigma_0, \\
&= (D_x + L_u - R_{u_x}ad_L^{-1})\sigma_0, \\
&= u_{2x} + uu_x + R_{u_x}adL^{-1}u_x, \\
\sigma_1 &= u_{2x} + 2uu_x,
\end{aligned}
$$

where $ad_L^{-1}u_x = u$. Similarly, other symmetries σ_2 could be found

$$
\begin{aligned}
\sigma_2 &= \mathcal{R}\sigma_1, \\
&= (D_x + L_u - R_{u_x}ad_L^{-1})(u_{2x} + 2uu_x), \\
&= u_{3x} + 2u_x^2 + 3uu_{2x} + 2u^2u_x - R_{u_x}ad_L^{-1}(u_{2x} + 2uu_x), \\
\sigma_2 &= u_{3x} + 3u_x^2 + 3uu_{2x} + 3u^2u_x,
\end{aligned}
$$

where $ad_L^{-1}u_x = u_x + u^2$ and for σ_3

$$
\begin{aligned}
\sigma_3 &= \mathcal{R}\sigma_2, \\
&= (D_x + L_u - R_{u_x} ad_L^{-1})(u_{3x} + 3u_x^2 + 3uu_{2x} + 3u^2u_x), \\
&= u_{4x} + 6u_xu_{2x} + 4uu_{3x} + 3u_{2x}u_x + 3u_xuu_x + 6uu_x^2 + 6u^2u_{2x} + 3u^3u_x \\
&+ R_{u_x} ad_L^{-1}(u_{3x} + 3u_x^2 + 3uu_{2x} + 3u^2u_x), \\
\sigma_3 &= u_{4x} + 4uu_{3x} + 4u_xuu_x + 4u_{2x}u_x + 6u^2u_{2x} + 4u^3u_x + 6u_xu_{2x} + 8uu_x^2.
\end{aligned}
$$

Another recursion operator of the NC-Burgers equation which is explicitly time-dependent like the commutative Burgers equation is found by inspection such as

$$\mathcal{R}_2 = ad_L(tD_x + tL_u + \frac{x}{2})ad_L^{-1}.$$

To check that $\mathcal{R}_2$ is conventional recursion operator, it is more convenient to verify

$$\mathcal{M}_t - \mathcal{F}_*M = \mathcal{M}\mathcal{N}^{-1}(\mathcal{N}_t - F_*N), \tag{3.6}$$

in spite of $\mathcal{R}_t = [\mathcal{F}_*, \mathcal{R}]$ because this recursion operator also in the form $\mathcal{R} = \mathcal{M}\mathcal{N}^{-1}$. With $\mathcal{M} = (D_x + L_u - R_u)(tD_x + tL_u + \frac{x}{2})$ and $\mathcal{N} = (D_x + L_u - R_u)$, the right hand sides of (3.6) becomes

$$
\begin{aligned}
\mathcal{M}\mathcal{N}^{-1}(\mathcal{N}_t - \mathcal{F}_*\mathcal{N})f(x) &= \mathcal{M}(-f_{2x} - 2uf_x), \\
&= (D_x + L_u - R_u)(tD_x + tL_u + \frac{x}{2})(-f_{2x} - 2uf_x), \\
&= t(-f_{4x} - 5u_xf_{2x} - 4uf_{3x} - 2u_{2x}f_x - 2u_xuf_x - 4uu_xf_x - 5u^2f_{2x} \\
&- 2u^3f_x + f_{3x}u + 3uf_{2x}u + 2u_xf_xu + 2u^2f_xu) + x(-\frac{1}{2}f_{3x} - u_xf_x \\
&- \frac{3}{2}uf_{2x} - u^2f_x + \frac{1}{2}f_{2x} + uf_xu) - \frac{1}{2}f_{2x} - uf_x,
\end{aligned}
$$

and the left hand sides of (3.6) could be found as follows

$$
\begin{aligned}
(\mathcal{M}_t - \mathcal{F}_*\mathcal{M})f(x) &= (D_x + L_u - R_u)_t(tD_x + tL_u + \frac{x}{2})f(x) + (D_x + L_u - R_u)(tD_x + tL_u \\
&+ \frac{x}{2})_t f(x) - (D_x^2 + 2L_uD_x + 2R_{u_x})(D_x + L_u - R_u)(tD_x + tL_u + \frac{x}{2})f(x), \\
&= t(-f_{4x} - 5u_xf_{2x} - 4uf_{3x} - 2u_{2x}f_x - 2u_xuf_x - 4uu_xf_x - 5u^2f_{2x} \\
&- 2u^3f_x + f_{3x}u + 3uf_{2x}u + 2u_xf_xu + 2u^2f_xu) + x(-\frac{1}{2}f_{3x} - u_xf_x \\
&- \frac{3}{2}uf_{2x} - u^2f_x + \frac{1}{2}f_{2x} + uf_xu) - \frac{1}{2}f_{2x} - uf_x.
\end{aligned}
$$

Both sides of equation (3.6) equal to each other which is essential to be conventional recursion operator. However, this time-dependent recursion operator $\mathcal{R}_2$ is a weak recursion operator because it fails to generate higher order symmetries correctly. Let's start with the first symmetry $\sigma_0 = 1/2 + u_xt$ to get symmetries of NC Burgers and then check that they are correct symmetries or not by using (2.6). By action of the recursion operator to the symmetry, we obtain the first few symmetries of NC Burgers such as

$$
\begin{aligned}
\sigma_1 &= \mathcal{R}_2\sigma_0 = t^2(u_{2x} + 2uu_x) + t(u_xx + u) + \frac{x}{2}, \\
\sigma_2 &= \mathcal{R}_2\sigma_1 = t^3(u_{3x} + 3u_x^2 + 3u^2u_x + 3uu_{2x}) + t^2(\frac{3}{2}u_{2x}x + \frac{5}{2}u_x + \frac{3}{2}u^2 + 3uu_xx) \\
&+ t(\frac{3}{4}u_xx^2 + \frac{3}{2}ux + \frac{1}{2}) + \frac{3}{8}x^2.
\end{aligned}
$$

Although the second symmetry σ_1 satisfies the symmetry condition (2.6), the third symmetry σ_2 does not fulfill this condition.

In section 2.2, we have found the correct recursion operator for commutative Burgers equation. For NC Burgers, we should do the similar calculation with $L = D_x + L_u$, $F_* = u_x + L_uD_x + D_x^2$ and $\frac{a}{g} = \frac{1}{2}(\frac{1}{2} + tL_{u_x})$ which is the symmetry of NC Burgers equation because it satisfies the symmetry condition (2.6). With $F_{*0} = D_x^2$ and ansatz $\sigma_0 = a_1(t) + a_2(t)x + a_3(t)x^2 + \dots$ equation (2.14) becomes

$$(H\sigma_0)_t + D_x^{-1}(D_x^2\sigma_0) - (D_x^{-1}D_x^2)\sigma_0 - D_x^2(D_x^{-1}\sigma_0) + (D_x^2D_x^{-1})\sigma_0 = 0.$$

Then, we could find H as follows

$$
\begin{aligned}
(H\sigma_0)_t &= a_2, \\
H &= D_t^{-1}\Pi ad_L.
\end{aligned}
$$

With inserting H in the equation (2.2.1.1), the correct recursion operator can be written as follows

$$\mathcal{R}_2 = ad_L(tD_x + tL_u + \frac{x}{2})ad_L^{-1} + \frac{1}{2}(\frac{1}{2} + tL_{u_x})D_t^{-1}\Pi ad_L.$$

Starting with the symmetry $\sigma_0 = \frac{1}{2} + tu_x$ and using the correct recursion operator R_2, the first few symmetries can be constructed:

$$\begin{aligned}\sigma_1 &= \mathcal{R}_2\sigma_0,\\ &= (ad_L(tD_x + tL_u + \frac{x}{2})ad_L^{-1} + \frac{1}{2}(\frac{1}{2} + tL_{u_x})D_t^{-1}\Pi ad_L)(\frac{1}{2} + tu_x),\\ &= t^2(u_{2x} + 2uu_x) + t(u_x x + u) + \frac{x}{2},\end{aligned}$$

where $ad_L^{-1}(\frac{1}{2} + tu_x) = tu + \frac{x}{2}$,

$$\begin{aligned}\sigma_2 &= \mathcal{R}_2\sigma_1,\\ &= (ad_L(tD_x + tL_u + \frac{x}{2})ad_L^{-1} + \frac{1}{2}(\frac{1}{2} + tL_{u_x})D_t^{-1}\Pi ad_L)(t^2(u_{2x} + 2uu_x) + t(u_x x + u) + \frac{x}{2}),\\ &= t^3(u_{3x} + 3uu_{2x} + 3u_x^2 + 3u^2u_x) + t^2(3u_x + \frac{3}{2}xu_{2x} + 3xuu_x + \frac{3}{2}u^2)\\ &+ t(\frac{3}{4} + \frac{3}{4}x^2u_x + \frac{3}{2}uu_x) + \frac{3}{8}x^2.\end{aligned}$$

These all symmetries satisfy the symmetry condition (2.6).

In literature, one of the methods of obtaining the non-commutative integrable equations from commutative ones is the *Lax-pair generating technique* [23, 24, 25]. Recently, Hamanaka and Toda [21] have found a non-commutative version of Burgers equation (mixed NC Burgers equation) by using this technique. This technique includes an ansatz for a corresponding A-operator for a given L-operator:

$$A = A' - D_i^n L^m. \tag{3.7}$$

The L-operator of the mixed NC Burgers equation is given by

$$L_{Burgers} = D_x + u,$$

so, for the case $n = 1$ the general ansatz (3.7) reduces

$$\begin{aligned} A_{Burgers} &= A' - D_x L_{Burgers}, \\ &= A' - (D_x^2 + u_x + uD_x). \end{aligned}$$

Now, one can construct the Lax equation (2.20). The right hand sides of (2.20) can be written as

$$\begin{aligned} [A, L] &= AL - LA, \\ &= (A' - D_x^2 - u_x - uD_x)(D_x + u) - (D_x + u)(A' - D_x^2 - u_x - uD_x), \\ &= A'D_x + A'u - u_x D_x - u_x u - D_x A' - uA', \\ &= [A', D_x] + [A', u] - u_x D_x - u_x u. \end{aligned}$$

The left hand sides of (2.20) is obviously equal to u_t. By equating both sides of the Lax equation (2.20), one can obtain the formula

$$L_t = u_t = [A', D_x + u] - u_x D_x - u_x u, \tag{3.8}$$

$$[A', D_x + u] = u_t + u_x D_x + u_x u, \tag{3.9}$$

which is useful to construct A'-operator. To find appropriate A'-operator, first of all, one should define the form of it. Since the right hand sides of (3.9) contains D_x term, the form of the A'-operator should take as following form

$$A' = aD_x + b, \tag{3.10}$$

where a and b are polynomials of u, u_x, u_t etc. This is the suitable form of A'-operator to eliminate D_x term in the equation (3.9). The undetermined constants a, b can be defined by inserting (3.10) in (3.9).

$$
\begin{aligned}
[A', D_x + u] &= u_t + u_x D_x + u_x u, \\
(aD_x + b)(D_x + u) - (D_x + u)(aD_x + b) &= u_t + u_x D_x + u_x u, \\
(-a_x + [a, u] - u_x)D_x + [b, u] - b_x + au_x - u_t - u_x u &= 0.
\end{aligned}
$$

Then, the Lax equation takes in the form $fD_x + g = 0$, and the conditions $f = 0$ and $g = 0$ gives us some part of a, b and the Burgers equations respectively.
The condition $f = 0$ is

$$
-a_x + [u, a] = u_x.
$$

The solution is $a = -u$. The second condition $g = 0$ becomes

$$
b_x + [u, b] + uu_x + u_x u + u_t = 0,
$$

by inserting a in it. If one chooses $b = -cu_x - du^2$, then one can get the NC version of the Burgers equation as follows

$$
u_t - cu_{2x} + (1 + c - d)u_x u + (1 - c - d)uu_x = 0, \tag{3.11}
$$

where c, d are constants, and Lax pairs of NC version of Burgers equation is

$$
L = D_x + u, \tag{3.12}
$$

$$
A = A' - (D_x^2 + u_x + uD_x) = -(D_x^2 + 2uD_x + (c + 1)u_x + du^2). \tag{3.13}
$$

In commutative limit $uu_x = u_x u$, the equation (3.11) reduces the Burgers equation;

$$u_t - cu_{2x} + 2(1-d)uu_x = 0$$

[21].

Although in commutative limit equation (3.11) reduces the Burgers equation, one should also consider the Cole-Hopf transformation for this equation. As mentioned before, in commutative case, the Burgers equation is linearized by the Cole-Hopf transformation

$$u = \frac{1}{a} D_x log\phi = \frac{1}{a}\frac{\phi_x}{\phi},$$

where a is constant, by taking this transformation for the Burgers equation (3.11), one can get

$$\phi_t = c\phi_{2x} - (c - \frac{d-1}{a})\frac{\phi_x^2}{\phi}$$

The only way that equation (3.11) reduces to the heat equation is choosing $ca = d - 1$, then (3.11) becomes

$$u_t - cu_{2x} - 2cauu_x = 0$$

In NC case, there are two possibilities to non-commutative version of Cole-Hopf transformation:

$$u = \frac{1}{a}\phi_x\phi^{-1}$$

or

$$u = \frac{1}{a}\phi^{-1}\phi_x.$$

For both cases, the derivative of u with respect to x and t should be taken carefully because derivative of ϕ^{-1} is different from the commutative case, for example, in the first cases, In the first case, the derivative of u with respect to x and t is given by

$$u_t = \frac{1}{a}\phi_{xt}\phi^{-1} - \frac{1}{a}\phi_x\phi^{-1}\phi_t\phi^{-1},$$
$$u_x = \frac{1}{a}\phi_{2x}\phi^{-1} - \frac{1}{a}\phi_x\phi^{-1}\phi_x\phi^{-1},$$

then, taking the transformation for (3.11), it is noticeable that when $c+d = 1, a = -1$, equation (3.11) reduces to the equation

$$(D_x - \phi_x\phi^{-1})(\phi_t - c\phi_{2x}) = 0,$$

so one can get the heat equation $\phi_t = c\phi_{2x}$ by using this transformation. Then, (3.11) becomes

$$u_t - cu_{2x} + 2cu_x u = 0.$$

Similarly, in second case, one can get same equation when $c - d = -1, a = 1$.

Although, by the convenient choice of c, d, equation (3.11) can be reduced the right- and left-handed NC Burgers equations $u_t = u_{2x} + 2uu_x$, $u_t = u_{2x} + 2u_x u$, respectively, other choice of c, d gives us a differential equation includes a parametric mixture of right- and left-handed NC Burgers equations (mixed NC Burgers). As a result of this whole calculation for non-commutative versions of (left- and right-handed) Burgers equation , we investigate their integrability both obtaining infinitely many symmetries by recursion operator and linearization by a non-commutative version of the Cole-Hopf transformation. For mixed NC Burgers equation, integrability of this equation is a controversial subject. Although the mixed NC Burgers equations admits Lax formulation (3.12), we can not construct the recursion operator by using the method introduced in section (2.3). This method only provides the recursion operator for left- and right-handed Burgers equation. Although, recently there is no recursion operator for mixed NC Burgers equation, Gürses, Karasu and Turhan [15] tried to find the possible integrable mixed version of NC Burgers equation for higher symmetry. They generated the general form of the candidate symmetry to check that whether it satisfies the symmetry condition (2.6) or not. As a result of this, they stated the following proposition.

Proposition 3.1.1 *The equation of form*

$$u_t = u_{2x} + auu_x + bu_x u$$

with $a, b \in \mathbb{R}$ and $ab \neq 0$, u is non-commutative, does not admit any higher symmetry from the class of equations

$$
\begin{aligned}
u_t = \nu(x,t) &+ \sum_{i=0}^{4} \alpha^i(x,t) u_{ix} + \sum_{i,j=0}^{4} \beta^{ij}(x,t) u_{ix} u_{jx} + \sum_{i,j,k=0}^{4} \gamma^{ijk}(x,t) u_{ix} u_{jx} u_{kx} \\
&+ \sum_{i,j,k,l=0}^{4} \delta^{ijkl}(x,t) u_{ix} u_{jx} u_{kx} u_{lx}.
\end{aligned}
$$

This proposition shows that there is no possible higher order symmetries for the mixed NC Burgers equation, so it is claimed that there is no a recursion operator for the mixed NC Burgers equation. We can say that the only integrable non-commutative versions of Burgers equation are the left- and right-handed ones. Although the mixed NC Burgers equation admits Lax representation (3.12), it is not an integrable nonlinear differential equation.

CHAPTER 4

CONCLUSION

In this thesis, we have discussed the integrability of some commutative and non-commutative nonlinear differential equations in (1+1)-dimensions in the context of recursion operator obtained from Lax representations. Moreover, we have shown that the existence of a recursion operator is a sufficient condition for integrability. It is well known that if an evolution equation possesses a time-independent recursion operator, then a hierarchy of infinitely many symmetries can be produced recursively. However, time-dependent recursion operators do not generate the hierarchies of infinitely many symmetries correctly. In this thesis, we have analyzed this case and found the corrected recursion operators for some integrable evolution equations.

We have obtained the non-commutative evolutionary type integrable equations from the commutative ones by using the non-commutative version of Lax representations. We have also constructed the recursion operators for such equations. Finally, we have questioned the integrability of mixed non-commutative Burgers equation obtained from the Lax representation. We have shown that there does not exist either a hierarchy of symmetry or a recursion operator for this equation.

Motivated by the power of the recursion operators in (1+1)-dimensional integrable equations, one can construct the recursion operators of (2+1)-dimensional commutative and non-commutative integrable equations from Lax representations. And also one can study their algebraic and geometrical structures.

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