PORTFOLIO SELECTION AND RETURN PERFORMANCE:
An Application of the Black-Litterman Method in the Istanbul Stock Exchange

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF APPLIED MATHEMATICS
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

MEHMET BURAK BOZDEMİR

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
THE DEPARTMENT OF FINANCIAL MATHEMATICS

SEPTEMBER 2011

Approval of the thesis:

PORTFOLIO SELECTION AND RETURN PERFORMANCE:
An Application of the Black-Litterman Method in the Istanbul Stock Exchange

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ABSTRACT

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September 2011, 58 pages

In this study, Black-Litterman method is examined, and an emprical study is conducted for Turkish Stock Market, using Black-Litterman method with quantitative views. As the quantitative model, AR(1) model is selected to generate the investor views. Expected returns implied by the Capital Asset Pricing Model (CAPM) is blended with the expected returns forecasted by AR(1) model. The performance of the resulting Black-Litterman portfolio is compared with the performance of the market portfolio. It is found that the Black-Litterman portfolio, with views coming from AR(1) model, does not perform better than the market portfolio. However, the difference between the two strategies is not found to be statistically significant.

Keywords: Portfolio Optimization, Portfolio Selection, Black-Litterman, CAPM, AR(1)

ÖZ

PORTFÖY SEÇİMİ VE GETİRİ PERFORMANSI:
Black-Litterman Metodu'nun Istanbul Menkul Kıymetler Borsası Üzerinde Uygulanması

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Eylül 2011, 58 Sayfa

Bu çalışmada Black-Litterman metodu incelenmiş, ve nicel görüşler ile Black-Litterman modeli kullanılarak Türkiye Hisse Senedi Piyasası üzerinde ampirik bir uygulama yapılmıştır. Yatırımcı görüşlerini oluşturmak için nicel bir model olan AR(1) modeli seçilmiştir. CAPM'den elde edilen beklenen getiriler AR(1) modeli tarafından tahmin edilen beklenen getiriler ile harmanlanmıştır. Elde edilen Black-Litterman portföyünün performansı pazar portföyünün performansı ile karşılaştırılmıştır. AR(1) modelinden gelen görüşler ile oluşturulmuş Black-Litterman portföyünün Pazar portföyünden daha iyi sonuçlar vermediği görülmüştür. Buna rağmen, iki strategy arasındaki farkın istatistiksel olarak anlamlı olmadığı anlaşılmıştır.

Anahtar Kelimeler: Portföy Optimizasyonu, Portföy Seçimi, Black-Litterman, CAPM, AR(1)

To my family

ACKNOWLEDGMENTS

First of all, I would like to express my gratitude to my advisor Assist. Prof. Dr. Seza Danışoğlu, for encourging and guiding me throughout this study.

I feel indebted to express my sincere thanks to my friend and collegue Mete Tepe, for his valuable ideas and kind suggestions.

Also, I am gratefull to my lovely and compassionate mother Gülsevim, and my precious sister Merve, for their endless love and support.

I want to thank Şehsuvar Selmanoğlu, for mentoring me throughout my path.

And I want to thank my dear friend İlker Kurşunlugil, for always standing near me in good times and bad times.

Last but not least, I want to thank all of the members of the Institute of Applied Mathematics, who contributed to the synergy around.

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CHAPTER 1

INTRODUCTION

One of the major concerns in financial literature is to find the best investment portfolio. Assuming that the average investor is risk averse, the best allocation should be the one that provides the highest return while having the minimum risk. To reach this optimal portfolio, different approaches may be applied. The most popular approach is the utulity maximization.

Unfortunately, the utulity maximization process is highly sensitive to the input data set. Small changes in the estimated returns or volatilities can result in drastic changes in the final allocation. The optimal portfolios generally result in large short positions or zero weights in many of the assets.

Moreover, the investors may have two different estimations in hand and want to impose both of them in the optimization process. For example, the investor may have expected return estimations coming from a quantitative model, but she has different subjective views for some of the included assets. To feed the optimizer, either the quantitative estimations or the subjective views should be chosen.

The motivation behind the study of Fisher Black and Robert Litterman (1990) [1] is to remedy these problems. Their method simply uses the CAPM equilibrium as the initial reference point and blends this prior information with the subjective analyst views in accordance with the confidence level of the investor about these views.

In our study, rather than using subjective analyst views, AR(1) model is used to generate the second information set. These AR(1) estimations are blended with the expected returns implied by CAPM, to reach the posterior risk-return estimation set. After that the posterior estimations are used to find the optimal portfolio weights. Finally, the performance of this portfolio strategy is compared with that of the CAPM strategy.

In the next chapter, firstly, the literature about the portfolio theory before Black-Litterman is briefly reviewed. After that, literature about the Black-Litterman method is investigated. Data and methodology is explained in the third chapter. In the fourth chapter, results are discussed. Conclusion is given in the fifth chapter.

CHAPTER 2

LITERATURE REVIEW

Literature review chapter is examined in two main parts. First part is the short review of portfolio theory before Black-Litterman method. In the second part, the literature about Black-Litterman method will be examined.

2.1 Background on Portfolio Theory Before Black-Litterman

The portfolio theory mainly stands on four main concepts: expected return, risk, risk-aversion and diversification. Using these four elements the portfolio construction decision is analysed from simplest basic to more complicated concepts. Firstly, diversification concept is ellaborated. Secondly, the construction of the efficient frontier and the capital market line is explained, to get a basic insight into portfolio decision. Thirdly, Markowitz's full covariance model is discussed. Fourth, the Treynor-Black Model, which is similar to full covariance model except its inputs come from simple linear regression, is examined. Fifth, CAPM, which deals with finding the optimal risky portfolio from an equilibrium perspective, is discussed. Lastly, multifactor models like Fama-French are investigated.

2.1.1 Diversification

In portfolio selection, a risk averse investor prefers higher risk and lower return. Moreover, the general observation on single assets is that the increase in the expected return comes with the expense of the increasing risk, in equilibrium condition. However, for portfolios with more than one asset, we observe that the increase in the expected return does not necessarily increases the risk. The explanation lies in the diversification effect, and any portfolio decision would be flawed without understanding this concept. Therefore, before constructing the efficient frontier, diversification effect is discussed.

To see the benefit of diversification clearly, a portfolio consisting of two assets can be examined. Suppose the risky assets A and B has expected returns and deviations of $\mathrm{E}(r_A)$, $\mathrm{E}(r_B)$ and σ_A, σ_B respectively. Further suppose the correlation between A and B is ρ_{AB}.

If A and B are combined with the corresponding weights w_A and w_B, the expected return of the portfolio is

$$\mathrm{E}(r_P) = w_A \mathrm{E}(r_A) + w_B \mathrm{E}(r_B).$$

On the other hand, the variance of the portfolio is

$$\sigma_P^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2 w_A^2 w_B^2 \sigma_A^2 \sigma_B^2 \rho_{AB}.$$

If the assets perfectly correlated, the correlation coefficient coefficient will be equal to 1, and the deviation of the portfolio will be the weighted average of the deviations of the components:

$$\sigma_P = w_A \sigma_A + w_B \sigma_B.$$

However, if the correlation coefficient is smaller than 1, then the deviation of the portfolio will be smaller than the weighted average. This observation leads us to the statement that if the assets are not perfectly correlated, the portfolio's deviation is less than the weighted average of the deviations of the assets, while the expected return of the portfolio is equal to the weighted average of the expected returns.

Practically, perfect correlation means that the assets are effectively the same. Therefore, it can be stated that if different assets are combined in a portfolio, there will be a gain coming from the decrease in the overall risk, and this is the benefit of diversification.

Armed with the insight of diversification, next section deals with the construction of the efficient frontier and the capital market line.

2.1.2 Efficient Frontier and Capital Market Line

Although, the capital market line can also be found in a simpler way by Markowitz's model numerically, reaching the same result by firstly constructing the efficient frontier provides a deeper insight into portfolio management. In this section, construction of the efficient frontier and determination of the optimum risky portfolio among the efficient portfolios is discussed first. After that, capital market line is constructed, utulizing the existence of the riskless asset.

Efficient frontier is the set of dominating portfolios among all of the possible portfolios in the asset universe. First step is the determination of the minimum variance frontier. If all the possible portfolios is represented in the mean-standart deviation plane, the minimum variance frontier will be the left border of the entire area. Since it is the left border, for every expected return level, the minimum variance frontier includes the portfolio with the minimum variance with the corresponding expected return level. The minimum variance frontier is shown in the figure below.

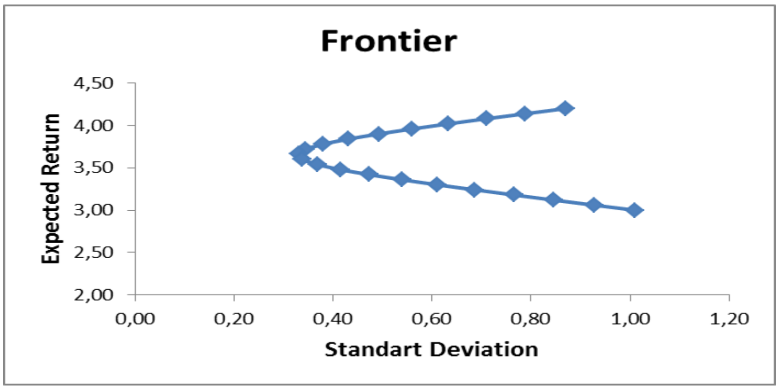

Figure 1

Although, the minimum variance frontier consists of the portfolios with the minimum variance, not all of them are efficient. To reach the efficient frontier, the minimum variance portfolio, which is the leftmost point of the curve, must be determined first. The lower part of the frontier is not efficient because there are portfolios with higher expected return for any standart deviation level. After excluding this unefficient part, the remainin upper side is the efficient frontier.

The portfolios in the efficient frontier dominates all of the possible portfolio space. For any standart deviation level, the point with the highest expected return always lies on this frontier. Likewise, for every expected return level, the less risky portfolio lies on this upperleft border.

After constructing the efficient frontier, it remains the question of which one of the portfolios should be chosen on the efficient frontier, because different risk aversion levels result in different indifference curves. That is to say, different investors may select different portfolios from the efficient frontier. This selection is made by comparing the capital allocation lines, which can be constructed by utulizing the riskless asset.

If an efficient portfolio is selected and combined with the riskless asset in different weights, the resulting portfolios can be represented as a line in the mean-standart deviation plane. This line is called as capital allocation line of the corresponding risky portfolio.

After drawing the capital allocation lines for all efficient portfolios, the line with the steepest slope will be the one tangent to the efficient frontier. This line is called the capital market line. It has the steepest slope, so that it dominates all of the other capital allocation lines. Therefore, the risky portfolio that is on the tangency point is the optimum risky portfolio in the market.

It is important to see that capital market line dominates all of the other capital alocation lines independent from the risk aversion level. Since this line has the steepest slope, the sharpe ratio of the points on this line is higher than any possible portfolio for every risk level. In other words, for every standart deviation value, the portfolio with the highest expected return belongs to the capital market line.

Another issue is that the tangency point changes depending on the risk-free rate. Therefore, for different risk-free rates, different optimal risky portfolios are obtained.

The next step is selecting the proper point on the capital market line for different risk aversion levels. In other words, the proper weighting between the risky portfolio and the riskless asset should be determined. The optimal allocation for an investor can be found where the indifference curves touch the tangency capital allocation line.

As previously mentioned, the optimal risky portfolio and the capital allocation line can simply be found by maximazing the sharpe ratio. Likewise, the optimal allocation between the risky portfolio and the riskless asset can be determined by utulity maximization. These optimization methods are used to reach the same results by Markowitz’s full covariance model, which is the topic of the next section.

2.1.3 Markowitz’s Full Covariance Model

Full Covariance model is developed by H. M. Markowitz (1952) [4]. It determines the optimal risky portfolio and the optimal allocation between the risky portfolio and the riskless asset simpler than the previously discussed graphical method. The inputs of the model are vector of expected returns and the covariance matrix of the assets and the risk aversion coefficient of the investor. Using these inputs, sharpe ratio is maximazed to find the optimal risky portfolio. After that, the optimal allocation is determined by utulity maximization. Sharpe ratio is a measure that captures the risk-return trade off of risky asset. It is formulated as

$$SR = \frac{\mathrm{E}(r) - r_f}{\sigma} .$$

The numerator is the excess return, and the denominator is the standart deviation of the portfolio. This expression can be read as the excess return per unit of risk. In other words, the return gained for bearing one unit of risk.

Since maximizing the sharpe ratio is same with maximizing the slope of the capital allocation line for the corresponding risky asset, the resulting weight vector belongs to the portfolio whose capital allocation line is tangent to the efficient frontier. Naturally, this portfolio is the optimal risky portfolio.

The maximization problem can be formulated as

$$\underset{w}{Max}\ \mathrm{SR} = \frac{w \cdot \mathrm{E}(r) - r_f}{\sigma\left[w \cdot r\right]} ,$$

where $\mathrm{E}(r)$ is the vector of expected returns, and w is the vector of weights, whose lengths are equal to the number of assets in the market.

Having found the optimal risky portfolio, determining the capital market line is straightforward. The next step is to choose the proper point on this line for a selected risk aversion level. This selection can be done by utulity maximization.

A frequently used utulity function by financial theorist and by the CFA institute is

$$U = \mathrm{E}(r) - \frac{1}{2}\delta\sigma^2 ,$$

where the letter δ stands for the risk aversion coefficient. Utulity score of a portfolio takes not only the risk-return characteristic, but also the risk aversion of the investor into account. Every unit of excess return is rewarded while every unit of volatility is penalized by the negative sign depending of the degree of risk aversion of the investor.

The utulity maximization problem can be represented as

$$\underset{w}{Max} \ \mathrm{U{=}E(r}_P) - \frac{1}{2}\delta\sigma_P^2$$
$$= r_f + w\left(E(r_{Risky}) - r_f\right) - \frac{1}{2}\delta w^2 \sigma_{Risky}^2$$

where w is the weight of the risky portfolio and A is the risk-aversion coefficient of the investor. If the derivative of this expression is set to zero, the optimal weight of the risky portfolio can be found as

$$w = \frac{\mathrm{E}(r_{risky}) - r_f}{\delta\sigma_{Risky}^2}.$$

To determine the optimal risky portfolio and the optimal allocation by Markowitz Model, the expected returns and the covariance matrix of the assets are needed. Same procedure is applied by obtaining the input list from simple linear regression in Treynor-Black Model, which is discussed in the next section.

2.1.4 Treynor-Black Single Index Model

Single index model was first suggested by William Sharpe (1963) [5], and developed by Jack Treynor and Fisher Black (1973) [6]. The main difference between the full covariance model and the single index model is the way to obtain the input variables of the models. Full covariance model uses the expected returns vector and the full covariance matrix of the assets involved. On the other hand, single index model obtains the expected returns and covariances from the below simple linear regression:

$$R_i = \alpha_i + \beta_i R_M + e_i,$$

where R_i is the excess return of the i^{th} asset; and R_M is the excess return of the index portfolio. Although, there is no spesific limitation on the selection of the index portfolio, a diversified market index like S&P 500 is conventionally used as the index portfolio.

The first regression coefficient, α_i, can be interpretted as the abnormal return of the i^{th} asset. The second coefficient, β_i, is the sensitivity of the i^{th} asset to the movements in the market index. The last coefficient, e_i, is used to estimate the firm spesific risk, $\sigma^2\left(e_i\right)$, of the i^{th} asset.

After determining the regression coefficients of the assets in the market, the coefficients of a portfolio constructed by these assets can be found from the following formulas:

$$\alpha_P = \sum_{i=1}^{n} w_i \alpha_i \, ,$$

$$\beta_P = \sum_{i=1}^{n} w_i \beta_i \, ,$$

$$\sigma^2\left(e_P\right) = \sum_{i=1}^{n} w_i^2 \sigma^2\left(e_i\right).$$

Here, the calculation of the $\sigma^2(e_P)$ relies on the assumption that firm spesific risks are independent from each other.

Using the calculated portfolio coefficients, portfolio expected return and portfolio variance can be computed as

$$\mathrm{E}\left(R_p\right) = \alpha_P + \beta_P \mathrm{E}\left(R_M\right),$$

$$\sigma^2\left(R_P\right) = \beta_P^2 \sigma^2\left(R_M\right) + \sigma^2\left(e_P\right).$$

Being able to calculate the risk and the return of any portfolio, the same optimization procedures, sharpe ratio optimization and utulity optimization, can be applied to find the optimal risky portfolio and the optimal allocation between the risky portfolio and the riskless asset.

In fact, Treynor-Black model provides a simplicity: the weight vector of the optimal risky portfolio can be found explicitly, instead of solving the two maximization problems. To do this, the optimal risky portfolio can be divided into two parts: the pasive and active portfolios.

The passive portfolio is the index portfolio, which would be selected as the optimal risky portfolio in the absence of any security analysis. Its beta is equal to 1, and its alpha and firm spesific risk are equal to zero by definition.

The active portfolio is composed to provide the maximum information ratio, which can be represented as the abnormal return per unit of firm spesific risk. Then, the sharpe ratio of the this portfolio can be analysed as

$$SR_P = SR_M + \left[\frac{\alpha_A}{\sigma\left(e_A\right)}\right],$$

where SR_P is the sharpe ratio of the overall portfolio; SR_M is the sharpe ratio of the index portfolio; α_A is the abnormal return of the active portfolio; and $\sigma(e_A)$ is the firm spesific risk of the active portfolio.

Therefore, the maximization of the sharpe ratio of the overall portfolio may be reduced to the maximization of the information ratio of the active portfolio. According to Treynor and Black, the optimal weight of an asset in the active portfolio is proportional to its information ratio. That means, if the abnormal return of an asset is zero, its active weight will be equal to zero. Further, if no abnormal returns is detected in the market, then the overall portfolio will be the index portfolio.

This result is consistent with CAPM, which is elaborated in the next section. In an efficient market which is in equilibrium, no abnormal returns is expected and the market portfolio, which is the market capitalization weighted portfolio in CAPM, is the optimal risky portfolio.

The explicit solution for the optimal weights in Treynor-Black model is

$$w_i^* = w_A^* \frac{\dfrac{\alpha_i}{\sigma^2(e_i)}}{\sum_{i=1}^{n} \dfrac{\alpha_i}{\sigma^2(e_i)}},$$

where w_A^* is the relative weight of the active portfolio in the overall risky portfolio. The weights of the active and passive portfolios can be calculated as

$$w_A^* = \frac{\dfrac{\alpha_A / \sigma^2(e_A)}{\mathrm{E}(R_M) / \sigma^2(R_M)}}{1 + (1-\beta_A)\left\{\dfrac{\alpha_A / \sigma^2(e_A)}{\mathrm{E}(R_M) / \sigma^2(R_M)}\right\}},$$

$$w_M^* = 1 - w_A^*.$$

Moreover, this method is applicable to not only the whole asset universe, but also a small subset of it. For example, if the investment company has limited resources to conduct security analysis, the number of assets in the active portfolio can be reduced to a subset of the asset universe. In other words, secutiry analysis can be conducted to a small subset, and active portfolio can be composed following the same method described above. This flexibility of Treynor-Black makes it more suitable to active portfolio management, compared to full covariance model.

Another simplicity of Treynor-Black model is that the number of inputs is less than that of the Markowitz model. To conduct Treynor-Black model n estimates of expected returns and n estimates of error terms are enough. On the other hand, Markowitz model needs additional $(n^2 - n)/2$ estimates of covariances. This additional number of inputs become an issue as the number of assets increases.

To summarize, Treynor-Black model requires less inputs and is more flexible compared to full covariance model. Further, the alpha estimates obtained by Treynor-Black makes it favorable in active portfolio management.

2.1.5 Capital Asset Pricing Model (CAPM)

Capital Asset Pricing Model was developed by the studies of William Sharpe (1964) [7], John Linter (1965) [8], and Jan Mossin (1966) [9]. It investigates the optimal risky portfolio and the risk-return characteristics of assets in an equilibrium perspective. Although CAPM has similarities to single index model, its results provided important extensions to portfolio theory.

Before explaining the results, the basic assumptions of the model should be explained. One of the assumptions is that all investors are rational mean-variance optimizers with homogeneous expectations and holding periods. Another assumption is that all of the investors are price takers, which means no investor has the power to manipulate the market. A third assumption is the unexistence of taxes and transaction costs, and the existance of unique riskfree rate. Lastly, the investment universe does not contain non-traded assets like human capital.

The main result of the model is that the optimal risky portfolio is the market capitalization weighted portfolio; and all investors hold this market portfolio in equilibrium condition.

Three steps can be followed to arrive this result [10]. First step is that since all investors are rational mean-variance optimizers with homogeneous expectations, all of them have to hold the same risky portfolio. Now, the question of whether this unique portfolio should include all of the investment universe remains. As the second step, suppose that any one of the assets is not included in the portfolio. In this case, since the demand on this asset is zero, the price of the asset would decrease rapidly, making the asset attractive to include in the portfolio. Eventually, this stock would be included in the portfolio and its price would increase to a degree consistent with its risk level. Therefore, relying on the assumptions of the model, all investors have to hold the same portfolio and this portfolio includes all the assets in the asset universe. As the third step, it can be stated that the only way for these two statements to be true is that the optimal risky portfolio is the market capitalization weighted portfolio.

Second important result of CAPM is about the average risk aversion in the market. Model states the below equation:

$$\delta = \frac{\mathrm{E}\left(R_{MktCap}\right)}{\sigma^2\left(R_{MktCap}\right)},$$

where δ is the average risk aversion in the market and R_M is the excess return of the market capitalization weighted portfolio.

This result can be achieved by modifying the last equation of section 2.1.3. Firstly, it should be mentioned that since for every short position in the risk-free asset has its corresponding long position, the net capitalization of riskless asset is zero. Therefore, according to CAPM, the weight of the riskless asset in the overall portfolio should be zero. Now, replace the risk aversion coefficient A by the average risk aversion δ, and substitute the return of the optimal risky porfolio with that of the market capitalization weighted portfolio. After these substitutions, the weight of market portfolio, w, in the overall portfolio should be equal to one. Eventually, this equation turns out to be

$$1 = w = \frac{\mathrm{E}(R_{MktCap})}{\delta \cdot \sigma^2\left(R_{MktCap}\right)},$$

which leads to the average risk aversion equality above.

Third result of CAPM is that the excess return of an asset is proportional to its contribution to the total risk of the market portfolio. Mathematically, the excess return of an asset can be represented as

$$\mathrm{E}\left(R_i\right) = \beta_i \cdot \mathrm{E}\left(R_{MktCap}\right),$$

Where β is the sensitivity of the asset to the movements in the return of the market portfolio.

To arrive this result, we have to utulize the fact that the reward-to-risk ratio of all the investments should be the same. If any portfolio offers a higher ratio, its price increases, decreasing the ratio to the market level, and vice versa. This pricing mechanism balances and equates the reward-to-risk ratio in the whole market. Thus the following equation holds:

$$\frac{\mathrm{E}\left(R_i\right)}{Cov\left(R_i, R_{MktCap}\right)} = \frac{\mathrm{E}\left(R_{MktCap}\right)}{\sigma^2\left(R_{MktCap}\right)},$$

where R_i is the return of any asset in the market. Modifying this equality leads to the third equality of CAPM:

$$\begin{aligned}\mathrm{E}\left(R_i\right) &= \frac{Cov\left(R_i, R_{MktCap}\right)}{\sigma^2\left(R_{MktCap}\right)}\mathrm{E}\left(R_{MktCap}\right)\\ &= \beta_i \cdot \mathrm{E}\left(R_{MktCap}\right).\end{aligned}$$

Furthermore, utulizing the first equation of CAPM, $\mathrm{E}(R_{MktCap})$ can be replaced with $\delta \cdot \sigma^2(R_{MktCap})$:

$$\begin{aligned}\mathrm{E}\left(R_i\right) &= \frac{Cov\left(R_i, R_{MktCap}\right)}{\sigma^2\left(R_{MktCap}\right)} \cdot \delta \cdot \sigma^2\left(R_{MktCap}\right)\\ &= \delta \cdot Cov\left(R_i, R_{MktCap}\right).\end{aligned}$$

This equation is used frequently used to calculate excess return of any asset implied by CAPM.

The weak point of CAPM is the strong assumptions that it stands on. For example, to reach the real market portfolio non-tradable assets should be included. Moreover, all the investors does not have homogeneous expectations since different investors may use different methods to estimate the expected returns. Nevertheless, CAPM is an elegant equilibrium model, and a reasonable candidate to be a starting point. If it can be conbined with active methods like Treynor-Black or Black-Littreman, it may become more functional in an investment decision process. It may provide a solid reference path while the oscilations from this path can be captured by the active methods. In fact, that is probably why CAPM is chosen as the prior model in the original paper of Black and Litterman.

2.1.6 Multifactor Models

Multifactor models are extended versions of single factor models. The motivation of the multifactor models is that one single factor may not capture the systematic risk completely. To remedy that problem, the number of factors may be increased, composing a multifactor model. A multifactor model of n factors can be represented as

$$r_i = \mathrm{E}(r_i) + \beta_{i1} \cdot F_1 + \beta_{i2} \cdot F_2 + ... + \beta_{in} \cdot F_n + e_i ,$$

where r_i is the return of the i^{th} security; F_k is the unexpected departure of the k^{th} factor; β_{ik} is the sensitivity of the i^{th} security to the k^{th} factor; and e_i is the idiosyncratic risk of the i^{th} security. In this setup, since the independent variables are the unexpected departures of the macrofactors, the expected value of these variables are zero. That is to say, these are the error terms of the macrofactors. Likewise, the expectation of the unsystematic error term, e_i , is zero.

Although this model states the relation between a single security and the macrofactors, it is not practical in use for several reasons. First of all this equation is not suitable for estimation purposes, since the expectation of the unexpected deviations of the macro factors are zero. Therefore, if we take expectation of both sides, all of the terms including the macrofactors and the idiosyncratic risk cancel out. The remaining equation is that expectation of the security return is equal to itself.

A more convenient model uses the risk premiums of the macrofactors as the independent variable. This model is represented as

$$r_i - r_f = \beta_{i1} \cdot RP_1 + \beta_{i2} \cdot RP_2 + ... + \beta_{in} \cdot RP_n + e_i ,$$

where RP_i is the risk premium of the i^{th} macrofactor. The beta coefficients are the equal to the ones in the first model. This can simply be shown in one factor case by calculating the covariance of a single security with the market. Using the first model the covariance is calculated as

$$\begin{aligned} Cov(r_i, r_M) &= Cov\left(\mathrm{E}(r_i) + \beta_{first} F + e_i ; F\right) \\ &= Cov\left(\mathrm{E}(r_i); F\right) + Cov\left(\beta_{first} F; F\right) + Cov\left(e_i; F\right) \\ &= 0 + \beta_{first} Cov\left(F; F\right) + 0 \\ &= \beta_{first} \sigma^2(r_M). \end{aligned}$$

Same covariance can also be calculated by using the second model as

$$\begin{aligned} Cov(r_i, r_M) &= Cov(R_i, R_M) \\ &= Cov(\beta_{\text{sec}ond} R_M + e_i; R_M) \\ &= Cov(\beta_{\text{sec}ond} R_M; R_M) + Cov(e_i; R_M) \\ &= \beta_{\text{sec}ond} Cov(R_M; R_M) + 0 \\ &= \beta_{\text{sec}ond} \sigma^2 (R_M). \end{aligned}$$

Since variance of total returns is equal to variance of excess returns, beta of the first model, β_{first}, must be equal to the beta of the second model, $\beta_{\text{sec}ond}$.

One important issue about the factor models is the diversification issue, because the Arbitrage Pricing Theory of Stephen Ross (1976) [10] stands on the assumptions that security returns can be modeled by factor models, and the unsystematic risk can be eliminated by constructing well diversified portfolios.

To see how the unsystematic risk can be diversified away, consider the variance of the idiosyncratic error term of a well diversified portfolio modeled by single factor model:

$$\sigma^2(e_P) = \sigma^2\left(\sum_{i=1}^{n} w_i \cdot e_i\right) = \sum_{i=1}^{n} w_i^2 \cdot \sigma^2(e_i).$$

As the number of assets in the portfolio increases, the weights of the assets decreases. Since weight of an asset is a rational number between 0 and 1, the square of it will approach to zero when n gets large, resulting the variance of the unsystematic risk to be zero. Moreover, the expectation of the unsystematic risk is zero by definition. A random variable with zero expectation and zero variance is practically zero. In other words, as n gets large, the unsystematic risk disappears. Therefore, the factor model representation of a well diversified portfolio does not include the error term. In light of this fact, two useful equations for a well diversified portfolio are

$$r_p = \mathrm{E}(r_P) + \beta_P \cdot F$$

and

$$\mathrm{E}(r_P) = r_f + \beta_P \cdot RP_M.$$

These equations can be extended to more than one factors. As no unsystematic error exists in these equations, risk arbitrage arguments can be applied to well diversified portfolios using these factor models. For example, two different portfolios with same beta coefficients should provide the same expected returns, otherwise the demand on the underpriced portfolio forces the prices to get balanced. However, this decision depends on the validity of the model and the accuracy of the estimations. Therefore, it is crucial to remember that risk arbitrage is not totally riskless.

Riskless arbitrage can be applied to the cases where the security under concern can perfectly be mimiced. For example, derivative securities can be priced depending on strict arbitrage rules. However, primitive securities can not be perfectly mimiced, so strict arbitrage rules does not apply to them. The reason is that the arbitrage profit depends on the realisation of the securities under concern. Only risk arbitrage may be applied on the well diversified portfolios composed by them.

Another important issue about multifactor models is the determination of the explanatory factors in the model. There is not a single set of macrofactors that everyone agree upon. One of the most famous ones is the model proposed by Eugene F. Fama and Kenneth R. French (1996) [11]. It is a tree- factor model which can be represented as

$$r_i = \alpha_i + \beta_{i,M} \cdot RP_{M,t} + \beta_{i,SmB} \cdot RP_{SmB} + \beta_{i,HmL} \cdot RP_{HmL} + e_i$$

RP_{SmB} is the factor called Small Minus Big. It is the premium of the portfolio of small stocks over the portfolio of big stocks. RP_{HmL} is the factor called High Minus Low which represents the premium of the portolio of stocks with high book-to-market ratios over the portfolio of stocks with low book-to-market value. Lastly, RP_M is the risk premium of the broad market index.

The idea behind RP_{SmB} is that small companies are more sensitive to the changes in the macroeconomy. Therefore the premium on them reflects the macroeconomic conditions and expectations effectively. Likewise, the premium of firms with high book-to-market ratios reflects the direction of the macroeconomy, since they are under financial distress and relatively more sensitive to macroeconomic conditions. Together with the market risk premium, these three factors are expected to capture the effects of the unknown fundamental macrofactors in the economy.

Fama and French tested the validity of their model in different markets for different time periods, and showed that the model has significant predictive power on security returns.
Up to this point the portfolio theory since Markowitz model is mentioned. Next section is about the Black-Litterman method, which is a tool to combine different models explained untill here.

2.2 Literature about Black-Litterman

Black-Litterman method is a tool to combine different output sets, rather than a model trying to explain the evolution of prices or the state of equilibrium in the market. Since there is not a model that perfectly explains and predicts the security prices, the model choice is a decision that every investor has to face. CAPM, Fama-French, time series analysis, fundamental analysis, or subjective qualitative analysis can be used to make the investment decision. The strengh of Black-Litterman method is that it enables us to choose and blend the outputs of different methods. Moreover, the confidence of the investor about the outputs of the selected models can be imposed on the results by this method. In the original paper, Black and Litterman chose CAPM as the prior model and blended the outputs of CAPM with subjective analyst views to reach the posterior outputs.

In this part, firstly the original paper will be investigated. After that other important papers about Black-Litterman method will be reviewed.

2.2.1 The Original Paper

The study of Fisher Black and Robert Litterman was first appeared in the publication of Goldman, Sachs & Company in 1990. The authors extended and published their studies in 1991[2] and 1992 [3].

Their motivation was the inflexibility of the quantitative portfolio management tools in use. First of all, since mean-variance optimization in very sensitive to expected returns, and expected returns are very difficult to estimate, the resulting portfolios are unbalanced in most of the cases. Secondly, quantitative methods use the past information to predict the future. On the other hand, it may be the case that an investor wants to impose her view depending on a present news. Unfortunately, quantitative methods generally do not allow investors to impose subjective views on the results.

Black and Litterman based their study to remedy these problems. This model sets CAPM equilibrium as the reference point and blends the subjective views of the investor to reach the final optimal portfolio composition. The departures detected by the user are captured meanwhile the weight composition of the final portfolio gravitates to the initially assigned balanced portfolio.

The tool that is utulized to make the blending is Theil's mixed estimation method (1971) [12, 13]. Although there are brief discussions about the estimation method and the variables, the full setup is not given in a clear and detailed manner. Intermediate steps and the derivations are also absent in the paper. Therefore, to keep simple, the mixed estimation procedure will be summarized in a different notation than the original paper.

To begin with, suppose there are two information sets about the future returns in the market: the equilibrium expected returns, Π, with the risk element of $\tau\Sigma$; and the investor views, Q, with the risk element of Ω.

Further suppose the investor believes that both of these expected returns are driven by a common factor, $\mathrm{E}(R)$. The mathematical representation of the models can be represented as

$$\Pi = I \cdot \mathrm{E}(R) + u$$
$$Q = P \cdot \mathrm{E}(R) + v$$

where I is the identity matrix; P is the matrix that assigns the location of the corresponding views; u is the error term which has a mean of zero and variance of $\tau\Sigma$; v is the error term which has a mean of zero and variance of Ω.

Alternatively, the distributions implied by the above equations are

$$\Pi \sim N\left(I \cdot \mathrm{E}(R); \tau\Sigma\right)$$
$$Q \sim N\left(P \cdot \mathrm{E}(R); \Omega\right).$$

The desired forecast is the estimation of the common factor, $\mathrm{E}(R)$, around which Π and Q are centered. To do this, two equations are consolidated and Theil's mixed estimation method is used:

$$\begin{vmatrix} \Pi \\ Q \end{vmatrix} = \begin{vmatrix} I \\ P \end{vmatrix} \cdot \mathrm{E}(R) + \begin{vmatrix} u \\ v \end{vmatrix}$$

By least squares method, the common factor, $\mathrm{E}(R)$, can be found to be normally distributed with mean

$$\mu_{BL} = \left[\left(\tau\Sigma\right)^{-1} + P'\Omega^{-1}P\right]^{-1}\left[\left(\tau\Sigma\right)^{-1}\Pi + P'\Omega^{-1}Q\right],$$

and variance

$$\Sigma_{BL} = \left[\left(\tau\Sigma \right)^{-1} + P'\Omega^{-1}P \right]^{-1}.$$

After some matrix operations, the same formulae can also be represented as

$$\mu_{BL} = \Pi + \left(\tau\Sigma \right) P' \left(P \left(\tau\Sigma \right) P' + \Omega \right)^{-1} \left(Q - P\Pi \right)$$

$$\Sigma_{BL} = \tau\Sigma - \tau^2 \Sigma P' \left(\tau P \Sigma P' + \Omega \right)^{-1} P\Sigma.$$

The derivations can be found in the study of Meucci (2009), which will be investigated later in this section.

After finding the mean and variance of $\mathrm{E}(R)$, the mean-variance optimization is used to reach the optimal portfolio.

In the paper, a three asset example is illustrated to show the effect of a relative view on the final weight composition. They show that even the view is not related to the third asset, the corresponding weight is effected by the view, depending on the covariance structure among the assets. That is to say, as the number of assets or the number of views are increased, the interaction between the views and the final weight composition may become intraceable.

The intraceablity does not mean that the increase in the number of views or in the number of assets results in unbalanced portfolios. The final weight vector still gravitates to the prior equilibrium portfolio if the confidence level is propoerly set. To see the overall deportment of the model, an example of seven countries is given, and the behaviour of the posterior portfolio under different confidence levels are discussed. The results are reasonable as they expected.

The issue of balancing the posterior portfolio by decreasing the confidence of the views is also included in the paper. They discussed that if the resulting portfolio is unbalanced, i.e. there are extreme departures from the equilibrium weights, the confidence level should be decreased instead of forcing extra virtual constraints on weights.

2.2.2 Following Literature

In this part the following literature after the original paper will be surveyed. Most of the literature is about the derivation, explanation, and illustration of the model. Although there are contributions about selection of the confidence levels and the views, the studies on the performance of method is few. The main reason is that, because of the mixed nature of the model, the performance test of Black-Litterman gives a joint result, i.e. the performance of the method cannot be extracted from the precision of the estimation and the quality of views. In the case that quantitative methods are utulized to attain the views, the a joint result is meaningfull. On the other hand, if subjective analyst views are used, the success of the method will highly depend on the performance of the human factor.

Firstly, the main studies that demystify and extend the model will be elaborated. After that the variations that attain views and confidences using different methods will be reviewed. In these variations, the ones that use quantitative methods to attain views have similarities with our study.

One of the papers that shed light on the method is the study of Satchell and Scowcroft (2000) [14]. The study used bayesian procedure and reached the same formula. Before mentioning the procedure in Satchell and Scowcroft, the Bayes' formula is stated below.

$$pdf\left(A|B\right)=\frac{pdf\left(B|A\right)\cdot pdf\left(A\right)}{pdf\left(B\right)}$$

In application, it is convenient to represent the formula in the following form for application purposes.

$$pdf\left(\theta|data\right)=\frac{pdf\left(data|\theta\right)\cdot pdf\left(\theta\right)}{pdf\left(data\right)}$$

where θ is the variable to be estimated, and $data$ is the observed sample derived by θ. Roughly, the logic is that a prior belief is assigned by $pdf\left(\theta\right)$ first, and then this belief is updated by the probability of observing the sampled data set given that the distribution of θ is represented by $pdf\left(\theta\right)$.

To use Bayes' formula, Satchell and Scowcroft has made two initial assumptions:

$$P\cdot \mathrm{E}(R)\sim N\left(Q,\Omega\right)$$

$$\Pi|P\cdot \mathrm{E}(R)\sim N\left(P\cdot \mathrm{E}(R),\tau\Sigma\right).$$

First assumption represents the beliefs and states that the expected returns are distributed normally around the assigned beliefs. The statement of the second assumption is that, given the expected returns, the CAPM equilibrium returns are distributed normally around the given returns. In terms of the generic Bayes' formula, $P \cdot \mathrm{E}(R)$ and $\Pi \big| P \cdot \mathrm{E}(R)$ can be thought of as the prior belief and the updating expression respectively.

The Bayes' formula using these assumptions is represented as

$$pdf\left(\mathrm{E}(R)\big|\Pi\right) = \frac{pdf\left(\Pi\big|\mathrm{E}(R)\right) \cdot pdf\left(\mathrm{E}(R)\right)}{pdf\left(\Pi\right)}.$$

Following the bayesion procedure $pdf(\Pi)$ behaves like a constant in terms of the variable $\mathrm{E}(R)$, and cancelled out. The remaining expression is the kernell of normal distribution. In the end, it is found that

$$\mathrm{E}(R)\big|\Pi \sim N\left(\mu_{BL}, \Sigma_{BL}\right),$$

where

$$\mu_{BL} = \left[\left(\tau\Sigma\right)^{-1} + P \cdot \Omega^{-1} \cdot P\right]^{-1} \left[\left(\tau\Sigma\right)^{-1} \Pi + P \cdot \Omega^{-1} \cdot Q\right],$$

$$\Sigma_{BL} = \left[\left(\tau\Sigma\right)^{-1} + P \cdot \Omega^{-1} \cdot P'\right]^{-1}.$$

Satchell and Scowcroft also made an extension to the model and reciprocal of τ is defined as a gamma distributed random variable. This second model setup has four initial assumptions:

$$P \cdot \mathrm{E}(R) \sim N\left(Q, \tau\Omega\right)$$

$$\Pi \big| P \cdot \mathrm{E}(R), \tau \sim N\left(P \cdot \mathrm{E}(R), \tau\Sigma\right)$$

$$\omega \sim Gamma\left(\frac{m}{2}, \frac{\lambda}{2}\right)$$

$$\omega \amalg \Pi,$$

where ω is defined as the reciprocal of τ.

Following the same bayesian arguments, the conditional distribution of the expected returns are found to be multivariate student-t :

$$\mathrm{E}(R)\big|\Pi \sim Multi-T\left(\left(H^{-1} \cdot C\right), \left(\frac{m \cdot H}{\lambda + A - C' H^{-1} C}\right)\right),$$

where

$$A = Q' \cdot \Omega^{-1} \cdot Q + \Pi' \cdot \left(\tau\Sigma\right)^{-1} \Pi$$

$$C = \left(\tau\Sigma\right)^{-1} \Pi + P' \cdot \Omega^{-1} \cdot Q$$

$$H = \left(\tau\Sigma\right)^{-1} + P' \cdot \Omega^{-1} \cdot P .$$

Another study that provides deeper insight into the Black-Litterman model, is the study of He and Litterman (2002) [15]. Using the results in Black-Litterman, they stated the distribution of the returns instead of expected returns. After that, they represented the solution of the optimal posterior weights in a way that is easier to interpret.

To model the returns, He and Litterman added one more assumption that

$$R \sim N\left(\mathrm{E}(R), \Sigma\right),$$

where R is the security returns. This assumption can be formulated as

$$R \overset{d}{=} \mathrm{E}(R) + z$$

where $z \sim N\left(0, \Sigma\right)$. This relation impies that

$$R \middle| Q \overset{d}{=} \left(\mathrm{E}(R) \middle| Q\right) + z .$$

Using the previous result, since

$$\mathrm{E}\left(R\right) \middle| Q \sim N\left(\mu_{BL}, \Sigma_{BL}\right),$$

the conditional distribution of the returns can be written as

$$R \middle| Q \sim N\left(\mu_{BL}, \Sigma_{BL} + \Sigma\right).$$

We know that covariance of posterior expected returns can also be represented as

$$\Sigma_{BL} = \tau\Sigma - \tau^2 \Sigma P' \left(\tau P \Sigma P' + \Omega\right)^{-1} P \Sigma .$$

In terms of this alternative representation, covariance matrix for the returns simply becomes

$$\Sigma_{BL}^{(R)} = \left(1 + \tau\right) \Sigma - \tau^2 \Sigma P' \left(\tau P \Sigma P' + \Omega\right)^{-1} P \Sigma .$$

Another contribution of He and Litterman is the intuitive representation of the posterior weights. To reach this formula, they started with the utulity maximization problem of

$$\underset{w}{Max}\ w'\mu_{BL} - \frac{\delta}{2} w'\Sigma_{BL}^{(R)} w.$$

The first order conditions provide us the solution that

$$w_{BL} = \frac{1}{\delta}\left(\Sigma_{BL}^{(R)}\right)^{-1} \mu_{BL}.$$

After the implementation of the mean and variance, the equation becomes

$$w_{BL}^{(R)} = \frac{1}{\delta}\left(\Sigma_{BL}^{(R)}\right)^{-1}\left[\left(\tau\Sigma\right)^{-1} + P'\Omega^{-1}P\right]^{-1}\left[\left(\tau\Sigma\right)^{-1}\Pi + P'\Omega^{-1}Q\right].$$

With some algebra the above equation can be simplified to

$$w_{BL}^{(R)} = \frac{1}{1+\tau}\left(w_{MktCap} + P'\Lambda\right),$$

where

$$\Lambda = \left(\frac{\tau}{\delta}\Omega^{-1}Q\right) - \left(A^{-1}P\cdot\Sigma\cdot w_{MktCap}\frac{1}{1+\tau}\right) - \left(A^{-1}\frac{\tau}{1+\tau}P\cdot\Sigma\cdot P'\Omega^{-1}Q\frac{1}{\delta}\right)$$

$$A = \frac{\Omega}{\tau} + \frac{P'\cdot\Sigma\cdot P}{1+\tau}.$$

This representation provides a different perspective into posterior weights. The expression implies that, the portfolio optimization process starts with market capitalization weights. After that Λ is calculated, taking Ω, Q, Σ, and Π into account. The view portfolio is weighted with that value of lamda afterwards. Finally, the resulting posterior portfolio is the summation of the initial market capitalization weighted portfolio plus the the view portfolio which is weighted by lamda value. It can be directly seen that, increase in the lamda value results an increase in the relative effect of the view portfolio.

The interpretation of Λ is also intuitive. In the first term, if the views are bullish, i.e. Q is large, then the value of Λ increases. Likewise, if the investor has a high confidence on views, i.e. Ω is low, the value of Λ increases again. In the second term, the covariance between the market capitalization weighted portfolio and the view portfolio, $\left(P\cdot\Sigma\cdot w_{MktCap}\right)$, is penalized because of the negative sign. The idea behind this penalization is that, if view portfolio carries similar information with the initial market portfolio, there exists redundant information in the view portfolio. Therefore the effect of the view portfolio decreases. In the third expression, the covariance between the individual views, $\left(P \cdot \Sigma \cdot P'\right)$, is penalized with the negative sign, because of the same reason.

Another study that derived the Black-Litterman formula in bayesian techniques is the study of Attilio Meucci (2009) [16]. Meucci has started with different assumptions and used bayesian arguments to reach the original formula. As a second contribution, Meucci changed the initial assumptions and modified the method to the extend to which the final formula directly gives the estimation of realized returns instead of expected returns.
There are two initial assumptions of the first model setup:

$$\mathrm{E}(R) \sim N\left(\Pi, \tau\Sigma\right)$$

$$Q \middle| P \cdot \mathrm{E}(R) \sim N\left(P \cdot \mathrm{E}(R), \Omega\right).$$

After setting the assumptions, the Bayes' formula is used to find the conditional distribution of expected returns given the view returns:

$$pdf\left(\mathrm{E}(R) \middle| Q\right) = \frac{pdf\left(Q \middle| \mathrm{E}(R)\right) \cdot pdf\left(\mathrm{E}(R)\right)}{pdf\left(Q\right)}.$$

In this setup, CAPM equilibrium returns constitutes the prior belief, while conditional distribution coming from the view returns are used as the updating information. Solving the above equation using bayesian arguments, it is found that

$$\mathrm{E}(R) \middle| Q \sim N\left(\mu_{BL}, \Sigma_{BL}\right),$$

where μ_{BL} and Σ_{BL} are the mean and the variance in the original paper.

Meucci also provided a second version for Black-Litterman formula. In this model setup, the initial assumptions are build on the future realized returns rather than expected returns. These two assumptions can be formulated as

$$R \sim N\left(\Pi, \Sigma\right)$$

$$Q \middle| P \cdot R \sim N\left(P \cdot R, \Omega\right).$$

Following the same bayesian procedures, the conditional distribution of future realized returns are found to be

$$R \middle| Q \sim N\left(\mu_{BL}^{m}, \Sigma_{BL}^{m}\right),$$

where

$$\mu_{BL}^{m} = \left[\Sigma^{-1} + P \cdot \Omega^{-1} \cdot P\right]^{-1} \left[\Sigma^{-1}\Pi + P \cdot \Omega^{-1} \cdot Q\right],$$

$$\Sigma_{BL}^{m} = \left[\Sigma^{-1} + P \cdot \Omega^{-1} \cdot P'\right]^{-1}.$$

Realize that, although the extension of He and Litterman also models the returns instead of expected returns, the resulting distribution is different than the one proposed by Meucci.

Another important paper about the Black-Litterman method is the one written by Thomas M. Idzorek (2004) [17]. In his study, Idzorek gives detailed explanations and clear examples about the model. In additon he proposes a method to assign confidence on subjective views. He explains his methodology in six steps.

In the first step, the posterior returns under 100% confidence on views are calculated:

$$\mu_{BL}^{100\%} = \Pi + (\tau\Sigma) P' \left(P (\tau\Sigma) P'\right)^{-1} (Q - P\Pi).$$

In the second step the posterior optimal weights under full confidence is found using the expected returns found in the previous step:

$$w_{BL}^{100\%} = (\delta\Sigma)^{-1} \mu_{BL}^{100\%}.$$

In the third step, the difference between the market capitalization weights and the posterior weights under full confidence is calculated:

$$D_{100\%} = w_{BL}^{100\%} - w_{MktCap}.$$

This differential can be thought of as size of the maximum effect that can be impose on the posterior weights by the views.

In the fourth step, user decides how much of this effect she wants to impose on the posterior portfolio. If the desired size is represented by C, the size of the imposition can be calculated as

$$Tilt = D_{100\%} C.$$

In the fifth step, these excess weights are added to the initial market capitalization weights, and the posterior weights representing the degree of investor’s confidence is found:

$$w_{BL}^{Tilt} = w_{MktCap} + Tilt.$$

In the final step, the confidence matrix implied by the posterior weights which reflects the degree of confidence of the investor is calculated by solving the below minimization problem:

$$\underset{\Omega}{Min}\ \left(w_{BL}^{Tilt} - w_{BL}\right)$$

where

$$w_{BL} = \frac{1}{\delta}\left(\Sigma_{BL}\right)^{-1}\left[\left(\tau\Sigma\right)^{-1} + P'\Omega^{-1}P\right]^{-1}\left[\left(\tau\Sigma\right)^{-1}\Pi + P'\Omega^{-1}Q\right].$$

Idzorek's methods enables the investor to assign her confidence on views as a proportion of the maximum attainable tilt in the weights.

Some other papers focus on attaining the views via quantitative models. The study of Jones, Lim and Zangari (2007) [18] employed the factor model of Carhart (1997) [19], which is an extension of the three-factor model developed by Fama and French. The authors also give a numerical illustration for the method they proposed. The study of Beach and Orlov (2006) [20] is another example that attains the views by quantitative methods. They obtain the views via E-GARCH model. Their portfolio surpasses the market equilibrium weighted portfolio and the portfolio composed according to Markowitz mean-variance technique.

In the study of Palomba (2006) [21], Multivariate GARCH model is used to forecast security returns. These forecasted returns are used as the equilibrium returns, and this returns are combined with personal views. This paper differs from the most of the literature as it uses a time varying model to obtain equilibrium returns.

Da Silva, Lee, and Pornrojnangkool (2009) [22] discussed the utulity optimization problem used for obtaining the optimal weights in active portfolio management. They stated that the utulity maximization problem used in Black-Litterman method uses the unconstrained Sharpe ratio optimization:

$$\underset{w_a}{Max}\ \left(w_a + w_p\right)'\Pi - \delta\left(w_a + w_p\right)'\Sigma\left(w_a + w_p\right),$$

where w_a and w_p are the weights of the active and passive portfolios respectively. They propose that, for active portfolio management, information ratio must be optimized instead of Sharpe ratio, to obtain both the equilibrium and posterior weights. The optimization problem that maximizes the information ratio is

$$\underset{w_a}{Max}\ \left(w_a + w_p\right)' \Pi - \delta w_a{}' \cdot \Sigma \cdot w_a$$
$$s.to \quad \mathrm{w}_a \bullet \vec{1} = 0.$$

Martellini and Zeimann (2007) [23] applied Black-Litterman method to hedge funds, and proposed to use four-factor CAPM to obtain the equilibrium returns. Moreover, 95% VaR is chosen as the risk factor in the optimization processes. They generated the views from conditional factor analysis.

Bewan and Winkelmann (1998) [24] used two types of views: macro views and micro views. Macro views are generated by a calibration process with constraints on information ratio. Micro views are assigned subjectively with three levels: High, medium, and low. They estimated the covariance matrix by the method mentioned in the study of Litterman and Winkelmann (1998) [25]. To attain the posterior weights optimization arguments are utulized by assigning constraints on tracking-error and on market exposure.

Giacometti, Bertocchi, Rachev, and Fabozzi (2007) [26] applied Black-Litterman method under different distributions: Normal distribution, Student-t distribution, and α-stable distribution. Applying three different distributions they obtained the equilibrium returns as

$$\Pi = \delta \bullet \Sigma \bullet w,$$
$$\Pi = \frac{\delta}{2}\left(CVaR_{\lambda} \frac{\Sigma \bullet w}{\sqrt{w' \cdot \Sigma \cdot w}} - \mathrm{E}(R) \right),$$
$$\Pi = \frac{\delta}{2}\left(VaR_{\lambda} \frac{\Sigma \bullet w}{\sqrt{w' \cdot \Sigma \cdot w}} - \mathrm{E}(R) \right),$$

where the risk parameters of the second and third expressions are $\lambda\%$ conditional value at risk , $CVaR_{\lambda}$, and $\lambda\%$ value at risk, VaR_{λ}, respectively.

The paper of Jay Walters (2009) [27] includes detailed explanations, illustrations, and full derivations of Black-Litterman method. For most of the derivations of the expressions appeared in other papers, and for a detailed literature survey about Black-Litterman, Walter's paper can be refered.

CHAPTER 3

DATA AND METHODOLOGY

3.1 Data

Our data consists of three main data sets: daily values of turkish industrial price indexes, compounded returns of 3-month government securities, and market capitalizations of turkish industrial indexes. The first two are collected for 124 months and the third is collected for 64 months. The asset universe is composed to include 23 mutually exclusive industrial indexes, and 3 month goverment securities. Since the included industrial indexes cover nearly all the sectors from real estate to technology, it is a good proxy that is able to mimic the asset universe available in Turkish market.

The daily values of turkish industrial indexes are obtained from the official web site of Istanbul Stock Exchange Market: www.imkb.gov.tr. The data covers the period between 31/07/2000 and 30/11/2010. 23 industrial indexes are selected: banking, information technologies, electric, leasing, food, real estate investment trusts, services, holdings, telecominication, paper, chemistry, finance, metal(main), metal(stuff), defense, insurance, stone & soil, technology, textile, commerce, tourism, transportation, and mutual funds. Sports industry index is excluded because it does not exist before 2004. The chosen indexes are mutually exclusive and are covering a very high majority of the stocks in İstanbul Stock Exchange Market.

Instead of dealing with single stocks, we have chosen to deal with industrial indexes and treated them as assets in order to decrease the computation procedure and so the estimation errors. Moreover, even if the views were not quantitative, it would not be so realistic to come up with sound subjective views for hundreds of single stocks every updating period, which is one month in our study. Therefore, if the views are quantitative, as it is in our study, dealing with industrial indexes decreases the estimation errors; if views are qualitative then it is more effective to make subjective monthly forecasts for 23 indexes than it is for hundreds of single stocks.

The compounded returns of 3-month government securities are aso obtained from the web site of Istanbul Stock Exchange Market and covers the same period between 31/07/2000 and 30/11/2010. Due to the non-availability of 90-day government bills every month, the ones which have the nearest maturity to 90 days are chosen. Therefore, the maturities of the sellected securities varies from 68 days to 187 days.

Because we are dealing with returns of stocks which theoretically have infinite maturity, it may be more intuitive to choose longer maturities like 10 years. On the other hand, using 1 month risk-free rates would be more consistent with our monthly updating periods. However, the conjoncture of the turkish economy leaded us to choose a maturity in between. After the 1994 and 2001 economic crises, and in the high inflationary enviroment, turkish governments had diffucilties while issuing bonds. There were times when governments couldn't find demand for desired maturities. Short term interest rates were highly volatile and long term bonds such as 5 to 10 years were inexistent because of the high risk perception in the market. In the light of these facts, we have chosen the maturity as 3 months instead of 10 years or 1 month.

For the market capitalizations, we collected the data from the web site of Istanbul Stock Exchange Market for the time period between 31/08/2005 and 30/11/2010. These market capitalizations is used to calculate the market capitalization weights.

3.2 Methodology

In majority of portfolio optimization processes, a utulity function is set and subjected to maximization procedure. The optimization may end up maximizing the sharpe ratio, information ratio, or some other quantity depending on the objective utulity function and on the constraints. Our objective function is the below utulity function,

$$U = w'\mathrm{E}(R) - \frac{1}{2}\delta w'\Sigma w$$

where

w : $nx1$ weight vector;

$E(R)$: $nx1$ vector of expected excess returns;

δ : risk-aversion coefficient;

Σ : nxn covariance matrix.

In Black-Litterman method, the aim is to find and blend two risk-return information sets with a desired confidence level, and feed the above utulity function with the resulting risk-return information to find the optimum weights. Our first information set, namely the prior, contains the excess returns implied by CAPM and the historical covariance matrix of these excess returns.

The second information set, namely the views, consists of AR(1) forecast for the asset returns, and the historical covariance matrix, which is same as the one in prior.

While blending the two information sets, the relative confidence between prior information and views may be assigned in two ways: by view covariance matrix; and by tilda value. In our study, historical covariance matrix is used to proxy the risk element both for prior and for views. Therefore, the relative confidence is reflected via the tilda value. Since we are using a naive quantitative model, AR(1), to compose our view returns, we assign our confidence on view returns very low by seting tilda very close to zero, increasing the effect of prior, and then gradually increased it to relatively higher levels, imposing more weight on views . As we increase the value of tilda, we expect the blended (posterior) returns to diverge from the prior returns. As it converges to zero, we expect the posterior returns to converge to the prior returns.

After assigning the confidence and blending the information sets, the optimal weights are found by utulity maximization. However, the main motivation is to see whether the Black-Litteman strategy with AR(1) views beats the prior strategy, namely the CAPM.

To make meaningfull comparisons, a significiant sample size is needed. By using a rolling window, a sample size of 64 is reached. The size of the rolling window is set to be 60, and it is rolled forward 64 times to reach 64 couples of weights: the prior strategy weights and the posterior strategy weights. Using these weights, out-of-sample montly returns of these strategies are found and four comparisons are made between the returns of these two strategies.

The first comparison is made on 64 months compound returns. Secondly, two strategies are compared in mean-variance basis. After that the utulities of these strategies are calculated under the assumption that the mean of the risk-aversion coefficients for the last 64 months represent the risk-aversion of the turkish market. Lastly, student-t test test is conducted to test whether there is statistically significant difference between the mean returns of the strategies.

To sum up, the first step is to obtain the CAPM excess returns and historical covariance matrix as the prior information set. Secondly, the AR(1) model is used to obtain the quantitative views; together with these views the historical covariance matrix will be the view information set. Thirdly, these informations are blended via the Black-Litterman method with the desired confidence level. Fourth step is to use these posterior information in the optimization process in order to find the optimum weights. This procedure is executed 64 times by the rolling window method, and we end up with 64 posterior weight vectors. As the fifth step, these vectors are used to calculate the monthly returns for 64 months, and finally these returns are compared with the ones obtained by following the prior (CAPM) strategy. The fifth and the sixth steps are repeated 5 times for different tilda values.

3.2.1 The Prior

To obtain the excess returns implied by CAPM, the following maximization problem needs to be solved,

$$\max_w \ U = w'\mathrm{E}(R) - \frac{1}{2}\delta w'\Sigma w$$

We know that this maximization problem has an analytical solution :

$$w = (\delta\Sigma)^{-1}\mathrm{E}(R) .$$

If we know the optimum weights and want to find the excess returns implied by these weights we transform it the following reverse optimization formula,

$$\mathrm{E}(R) = \delta\Sigma w .$$

The optimum weights implied by CAPM are nothing but the market capitalization weights. We have the market capitalizations for the 23 indexes, so the market capitalization weights can be calculated by the simple formula:

$$w_{MktCap}(i) = \frac{MktCap(i)}{\sum_{i=1}^{23} MktCap(i)} .$$

To obtain the covariance matrix, risk-free rates and then excess returns of the industry indexes should be calculated. By the following formula, the annualized 3-month compounded risk-free rates are transformed to 1-month risk-free rates,

$$r_f = (1 + r_{f,annual})^{\frac{1}{12}} - 1 .$$

By using the 1-month risk-free rates, we simply obtain the covariance matrix. For the risk aversion coefficient the following formula is used,

$$\delta = \frac{\mathrm{E}(r_m) - r_f}{\sigma^2(r_m)} .$$

The return of the market portfolio, r_m, is obtained by multiplying the index returns with the corresponding market capitalization weights. After taking the mean and the variance inside the rolling window containing 60 samples, the risk aversion coefficient for the corresponding period is calculated.

After obtaining the risk-aversion coefficients, the excess returns implied by CAPM can be found by utulizing the reverse optimization formula mentioned above.

Of course the above procedure is done 64 times iteratively, in a rolling window manner. After calculating implied CAPM returns and historical covariance matrixes for 64 time points, the second information set, the views, should be generated.

3.2.2 The Views

Although the basic motivation of Black-Litterman method is to combine CAPM model with subjective analyst views, it enables us to combine any couple of information sets about the risks and returns in the asset universe. In the original paper, a quantitative model was combined with qualitative views. Alternatively, if a quantitative model is used to generate views and blended with CAPM, then the posterior information will be a product of two quantitative models. If both the prior information and the views are obtained from subjective analyses, then the blending is between two qualitative information sets.

Even more, although there is not an example in the literature, theoretically it is possible to blend more than two information sets by Black-Litterman method. For example, if an institution had 3 analysts, the views of the first two would be blended and the posterior would be combined with the view of the third analyst. Of course, assigning confidences should be done in a different way, and this issue may be the subject of another study.

At the beginning of our study, we intented to use CAPM as the prior and subjective analyst reports as the views. However, finding analyst reports are not only expensive to attain, but also hard to find for a non-institutional researcher. Black and Litterman (1991) were researchers of Goldman Sachs and used the analyst views of the same company. For Turkish stock market, it is even harder to find that kind of subjective analyts report covering the whole market.

Another limitation of using subjective views in an emprical study is that the results would be highly affected by the quality of analyst reports used. Even if Black-Litterman strategy provides signaficantly different returns than the benchmark, it is not possible to say that the difference is due to the Black-Litterman method or to the quality of analyst reports. Therefore, in this study a quantitative model is used to generate views instead of subjective analyst reports.

As a quantitative model, GARCH may be the most suitable time series model, as it enables both return forecasting and risk forecasting. However, in turkish stock market we couldn't find a GARCH model which is consistantly statistically significant for all periods and for all industry indexes. In some cases ARMA(1,1)-GARCH(1,1) model is not significant, but ARMA(1,1)-EGARCH(1,1) model is significant. For some cases even ARMA(1,1) is not statistically significant. Therefore, we minimized the model until it becomes statistically significant for all industries and all periods. The resulting model is AR(1).

An autoregressive model of order p, denoted as AR(p), has the following structure,

$$y_t = \phi_0 + \sum_{i=1}^{p} \phi_i y_{t-i} + e_t$$

where

$$e_t \overset{iid}{\sim} \mathrm{N}(0, \sigma^2) \,.$$

In our case the model has order 1 and the model becomes

$$y_t = \phi_0 + \phi_1 y_{t-1} + e_t \,.$$

After estimating the coefficients for 23 indexes for 60 periods, these coefficient estimates are used to make forecasting by the following equation

$$y_{t+1} = \tilde{\phi}_0 + \tilde{\phi}_1 y_t \,.$$

The estimation and forecasting procedure is done in E-views. After obtaining the forecasts, these forecasted prices are transformed to returns. The transformation is done with the following equation

$$r_{t+1} = \frac{\tilde{y}_{t+1} - y_t}{y_t} \,.$$

Consequently, the return element of the view information set is obtained. The other element is the risk. Since AR(1) model is not designed to forecast volatility, historical covariance matrix is used as the risk element of the view information set.

For AR(1) model, although error variances could be used to construct a covariance matrix, the non-diagonal elements would be zero, as the model doesn't take into account any covariance structure between diffrent assets, and the error terms are assumed to be independent. Moreover, homoskedasticity is assumed in the model. The estimation is done inside a rolling window with the assumption that homoskedasticity exist. If we tried to estimate heteroskedastic variance as the window is rolled, then the methodology would not be consistent.

Beside the unforecastibility of the covariance matrix, AR(1) model has one more drawback that it doesn't carry any economic interpretation compared to a regression model whose independent variables are macro factors, or to a GARCH type model whose mean equation includes economic variables.

Since AR(1) model is an inferior model, we set the relative confidence of the views very low. The first tool to adjust the relative confidence is the covariance matrix of the view information set. The second way is to utulize tilda which is a constant variable in the Black-Litterman formula. Since we used the same covariance matrix for both prior and the views, we used a very small tilda value to impose a low relative confidence to view information set, and then gradually increased it to higher levels to see the overall picture.

Having obtained the prior information and views, the next step is to blend these information by Black-Litterman method to reach the posterior information set which is subjected to the utulity maximization problem to reach the optimal weights.

3.2.3 Blending by Black-Litterman

In Black-Litterman method the the blending is done in a bayesian manner and it states that

$$\mathrm{E}_{BL}(R)\Big|\Pi \sim \mathrm{N}\left\{\left[\left(\tau\Sigma\right)^{-1}+\mathrm{P}'\Omega^{-1}\mathrm{P}\right]^{-1}\left[\left(\tau\Sigma\right)^{-1}\Pi+\mathrm{P}'\Omega^{-1}Q\right]\ ;\ \left[\left(\tau\Sigma\right)^{-1}+\mathrm{P}'\Omega^{-1}\mathrm{P}\right]^{-1}\right\},$$

where

R : vector of excess returns $(nx1)$,

Π : vector of prior excess returns $(nx1)$,

Q : vector of view excess returns $(nx1)$,

Σ : covariance matrix implied by the prior (nxn),

Ω : covariance matrix implied by the views (nxn),

P : views assigner matrix (nxn),

τ : real valued constant variable tilda, which justifies the relative confidence.

Namely, Black-Litterman states that, given the prior returns, the posterior (blended) expected excess returns are distributed as multivariate normal with the given mean and variance.

In our study, specificly, these variables are defined as follows, and the process is illustrated in the following chart.

R : vector of blended excess returns $(23x1)$,

Π : excess returns implied by CAPM $(23x1)$,

Q : excess returns obtained by AR(1) model $(23x1)$,

Σ : historical covariance matrix $(23x23)$,

Ω : historical covariance matrix $(23x23)$,

P : identity matrix $(23x23)$,

τ : real valued variable which is very close to zero.

Roughly, Black-Litterman posterior returns are nothing but a weighted average of the prior returns and the views, where the corresponding weights are the inverses of the variances.

To see that, let us simplify the situation. Assume there are only two returns, Π and Q; and the corresponding weights are w_Π and w_Q. The weighted average can be formulated as

$$\mathrm{E}_{BL}(R) = \frac{w_\Pi \Pi + w_Q Q}{w_\Pi + w_Q}.$$

Since the weights are the inverses of the variances of Π and Q, the equation becomes

$$\mathrm{E}_{BL}(R) = \frac{(\frac{1}{\mathrm{var}_\Pi})\Pi + (\frac{1}{\mathrm{var}_Q})Q}{\frac{1}{\mathrm{var}_\Pi} + \frac{1}{\mathrm{var}_Q}}$$

$$= \left(\frac{1}{\mathrm{var}_\Pi} + \frac{1}{\mathrm{var}_Q}\right)^{-1} \left(\frac{1}{\mathrm{var}_\Pi}\Pi + \frac{1}{\mathrm{var}_Q}Q\right)$$

We also know that $\mathrm{var}_\Pi = \tau\Sigma$ and $\mathrm{var}_Q = \Omega$. After inserting these values, we end up with

$$E_{BL}(R)=\left[\left(\tau\Sigma\right)^{-1}+\Omega^{-1}\right]^{-1}\left[\left(\tau\Sigma\right)^{-1}\Pi+\Omega^{-1}Q\right].$$

The view assigner matrix, P, is the identity matrix and it is equal to 1 in our simplified one-dimentional example. Therefore, the above equation can also be written as

$$E_{BL}(R)=\left[\left(\tau\Sigma\right)^{-1}+P'\Omega^{-1}P\right]^{-1}\left[\left(\tau\Sigma\right)^{-1}\Pi+P'\Omega^{-1}Q\right],$$

which is the exact same formula for the mean of the posterior Black-Litterman excess returns. Thus, we may roughly say that the posterior returns are the weighted average of the prior returns and the view returns where the weights are the inverses of the corresponding variances.

This result is intuitive. When the variance of prior increases, the corresponding uncertainity increases. Therefore the weight of the prior should decrase, penalizing this increasing uncertainity. On the other hand, if the variance of the prior decreases, then the weight of it increases to exploit the increased level of certainity. The same logic applies to the views.

In our study we used the same covariance matrix both for the CAPM and the AR(1) views. However, we imposed our subjective degree of relative uncertainity by utulizing the tilda value. As our views are coming from an inferior time series model, AR(1), significantly low tilda values are assigned to decrease our relative confidence on the views. With a low tilda value the variance of CAPM, $\left(\tau\Sigma\right)$, decreases while the variance of AR(1) side, $\Omega=\Sigma$, stays the same. Therefore decreasing the tilda value increases the weight of the CAPM returns.

Consequently, the weighted average of the CAPM returns and AR(1) returns is taken by Black-Litterman formula, where $\left(\tau\Sigma\right)^{-1}$ is the weight of the CAPM and Σ^{-1} is the weight of AR(1) views. Together with the historical covarince matrix, the resulting posterior returns is subjected to utulity maximization problem to obtain the optimal weights implied by the posterior information set.

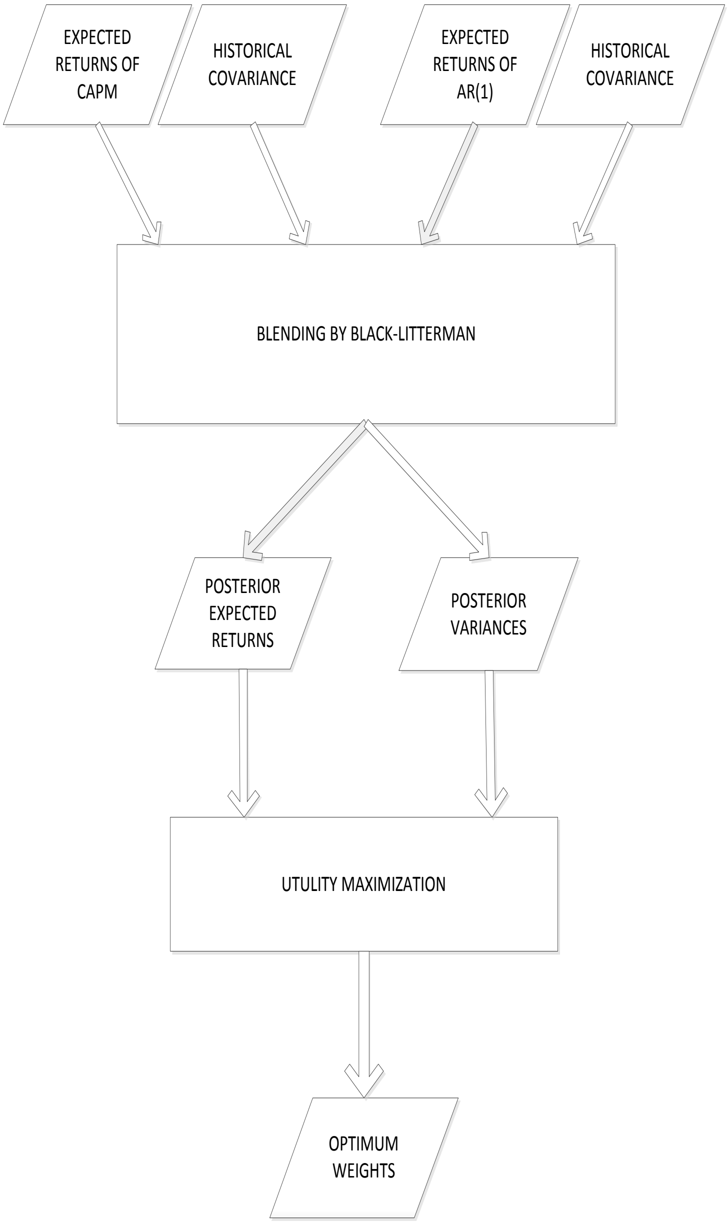

Figure 2

It is important to note that although Black-Litterman method enables us to reach both posterior returns and posterior covariance, we only employed the model to obtain the posterior returns. In other words, the Black-Litterman method is utulized only to blend the CAPM and AR(1) returns proportioned to our relative confidence level imposed by our subjective selection of tilda value. Therefore, instead of posterior covariance matrix, the historical covariance matrix is used in the utulity maximization problem. In fact, since the view assigner matrix, P, is identity matrix, the posterior covariance would be nothing but a mixture of the historical covariance matrix, Σ, with the scaled version of itshelf, $\left(\tau\Sigma\right)$.

3.2.4 Optimal Posterior Weights

To find the optimal posterior weights, the utulity maximization problem is fed by the posterior excess returns that we found. The problem is the same as the one that was solved to obtain the excess returns implied by CAPM, except that the returns are now posterior returns:

$$\max_w \; U = w'\mathrm{E}_{BL}(R) - \frac{1}{2}\delta w'\Sigma w$$

The analytical solution of this problem is know to be

$$w^{BL} = (\delta\Sigma)^{-1}\mathrm{E}_{BL}(R).$$

The resulting posterior weights should be summed up and subtracted from 1 to reach the weight for the risk-free asset. For example, if the summation is smaller than 1 then it means that the remainder should be invested in the risk-free asset. If the summation is larger than one, we have to borrow from the risk-free asset to finance the excess amount.

In fact, the utulity maximization problem implicitly has a constraint on weights, although it seems to be an unconstrained maximization. However, the weight constraint does not affect the solution of the optimal risky weights. The constraint comes into play after the optimal risky weights are found.

It can simply be shown that solving the unconstrained maximization problem for risky assets and equating the total weights to one by adding or subtracting the weight of the risk-free asset is the same as solving the constrained maximization problem including the risk-free asset.

The constrained maximization can be formulated as

$$\max_w \; U = (w_{new})\mathrm{E}(R_{new}) - \frac{1}{2}\delta(w_{new}')\Sigma_{new}(w_{new})$$
$$s.to \; \sum_{i=1}^{n+1}(w_{new})_i = 1.$$

Here, the sizes of $\mathrm{E}(R_{new}), \Sigma_{new}$ and w_{new} are increased to $((n+1)x1)$, $((n+1)x(n+1))$ and $(n+1)$ respectively, because of the inclusion of the risk-free asset. The above expression for 2 risky and 1 riskless assets, can be expressed as

$$\max_w \; U = \begin{vmatrix} w_f & w_1 & w_2 \end{vmatrix} \begin{vmatrix} r_f - r_f \\ r_1 - r_f \\ r_2 - r_f \end{vmatrix} - \left(\frac{\delta}{2}\right) \begin{vmatrix} w_f & w_1 & w_2 \end{vmatrix} \begin{vmatrix} \sigma_{ff} & \sigma_{f1} & \sigma_{f2} \\ \sigma_{f1} & \sigma_{11} & \sigma_{12} \\ \sigma_{f2} & \sigma_{12} & \sigma_{22} \end{vmatrix} \begin{vmatrix} w_f \\ w_1 \\ w_2 \end{vmatrix}$$

$$s.to \; \mathrm{w}_f = 1 - w_1 - w_2.$$

If we implement the weight constraint in the utulity equation, we can drop it from the expression. Also we know that the excess risk-free return and the covariances involving the risk-free return are zero. Then the maximization problem becomes

$$\max_w \; U = \begin{vmatrix} (1-w_1-w_2) & w_1 & w_2 \end{vmatrix} \begin{vmatrix} 0 \\ R_1 \\ R_2 \end{vmatrix} -$$

$$\left(\frac{\delta}{2}\right) \begin{vmatrix} (1-w_1-w_2) & w_1 & w_2 \end{vmatrix} \begin{vmatrix} 0 & 0 & 0 \\ 0 & \sigma_{11} & \sigma_{12} \\ 0 & \sigma_{12} & \sigma_{22} \end{vmatrix} \begin{vmatrix} (1-w_1-w_2) \\ w_1 \\ w_2 \end{vmatrix}.$$

If we make the matrix multiplications we reach

$$\max_w \; U = w_1 R_1 - w_2 R_2 - \frac{\delta}{2}\left(\sigma_{11} w_1{}^2 + 2\sigma_{12} w_1 w_2 + \sigma_{22} w_2{}^2\right).$$

In matrix notation, the above equation is

$$\max_w \; U = w'\mathrm{E}(R) - \frac{1}{2}\delta w'\Sigma w$$

where

$$w_f = 1 - w.$$

That means, the constrained maximization problem including the risk-free asset can be simpliflied to the unconstrained maximization problem. In other words, the same result can be achieved via solving the constrained maximization problem which includes the risk-free asset or first solving the unconstrained maximization problem, which doesn’t include risk-free asset, and then equating the summation of the resulting weights by using the weight of risk-free asset.

The second way, solving the unconstrained problem, is easier to solve. Moreover, if the utulity function is chosen as the one we used, there exists an analitycal solution. Therefore, in our study we preferred to use the second way.

There is one more issue to be discussed: if everything else is fixed, whether changing the expected return of a single asset changes the weight of the corresponding asset in the same direction or not. Our expectation was to find a direct relationship between the expected returns and corresponding weights. However, the relationship is more complex than we expected; it changes depending on the covariance structure between assets.

To understand this issue, a simple example with 2 assets can be elaborated. The weights are found by the below analytical solution of the unconstrained utulity maximization problem:

$$w^* = (\delta\Sigma)^{-1}\mathrm{E}(R),$$

or in matrix representation

$$w^* = \begin{vmatrix} w_1^* \\ w_2^* \end{vmatrix} = \frac{1}{\delta}\begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix}^{-1} \begin{vmatrix} \mathrm{E}(R_1) \\ \mathrm{E}(R_2) \end{vmatrix}.$$

The above equation can also be expressed more simply as 2 seperate equations.

$$w_1 = \frac{1}{\delta}\left[\sigma_{11}{}^{(-1)}\mathrm{E}(R_1) + \sigma_{12}{}^{(-1)}\mathrm{E}(R_2)\right]$$

$$w_2 = \frac{1}{\delta}\left[\sigma_{22}{}^{(-1)}\mathrm{E}(R_2) + \sigma_{12}{}^{(-1)}\mathrm{E}(R_1)\right]$$

where the expression, $\sigma_{ij}{}^{(-1)}$, stands for the ij^{th} term of the inverse of the covariance matrix. This means that if the diagonal entries of the inverse of the covariance matrix are negative and the other terms are positive, then increasing the expected return of the first asset decreases the weight of the first asset and increases the second, and vice versa.

By trial and error we found that if the variances of the assets are not equal and the covariance of these assets is between these variances, then the diagonal entries of the inverse of the covariance matrix are negative while the remaining terms are positive.
To illustrate this fact, let

$$\mathrm{cov}(R_1, R_2) = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} = \begin{vmatrix} 4 & 3 \\ 3 & 2 \end{vmatrix}$$

$$\mathrm{E}(R) = \begin{vmatrix} \mathrm{E}(R_1) \\ \mathrm{E}(R_2) \end{vmatrix} = \begin{vmatrix} 3.5 \\ 2.5 \end{vmatrix}$$

$$\delta = 1.$$

Then the reverse of the covariance matrix comes in the desired form:

$$\begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix}^{-1} = \begin{vmatrix} 4 & 3 \\ 3 & 2 \end{vmatrix}^{-1} = \begin{vmatrix} -2 & 3 \\ 3 & -4 \end{vmatrix}.$$

By the reverse optimization formula, we can obtain the optimal weights as

$$w^* = \begin{vmatrix} w_1^* \\ w_2^* \end{vmatrix} = \begin{vmatrix} 4 & 3 \\ 3 & 2 \end{vmatrix}^{-1} \begin{vmatrix} 3.5 \\ 2.5 \end{vmatrix} = \begin{vmatrix} 0.5 \\ 0.5 \end{vmatrix}.$$

Now, let us increase the expected return of the first asset and keep everything else constant. The new expected return vector is

$$\mathrm{E}(R_{new}) = \begin{vmatrix} \mathrm{E}(R_{1,new}) \\ \mathrm{E}(R_{2,new}) \end{vmatrix} = \begin{vmatrix} 3.55 \\ 2.5 \end{vmatrix}.$$

The resulting weight vector is calculated as

$$w_{new}{}^* = \begin{vmatrix} w_{1,new}^* \\ w_{2,new}^* \end{vmatrix} = \begin{vmatrix} 4 & 3 \\ 3 & 2 \end{vmatrix}^{-1} \begin{vmatrix} 3.55 \\ 2.5 \end{vmatrix} = \begin{vmatrix} 0.40 \\ 0.65 \end{vmatrix}.$$

As can be seen from the illustration, an increase in the expected return of a single asset decreased the weight of the corresponding asset. Moreover, as the summation is greater than one, we have to borrow from the risk-free rate to finance the additonal 5%.

Therefore, the relation is not direct.; it depends on the covariance structure. This illustration is based on an asset universe consisting of only 2 asset. If the size of the asset universe increases then the intercorrelations will be more and more sophisticated, making the relation hardly traceable.

After finding the optimum weights for the 2 strategies, we obtained and compared the returns of the strategies for a time interval of 64 months, between 31/08/2005 and 30/11/2010. This process is discussed in the next section.

3.2.5 Obtaining and Compairing the Strategy Returns

The strategy returns on a time point are obtained by multiplying the weight vectors by the realized return vectors. This procedure is repeated for 64 months to get a time series matrix of strategy returns.

To compare the performances of the strategies, four methods are preffered: compound return analysis, mean-variance analysis, utulity analysis and unpaired student-t test.

The first method to compare the two strategy returns is the 64-months compounded returns. The idea is that if both strategy returns were followed with monthly updating and without any endowment or any comsumption, which one of the strategies would provide a higher return.

To make this comparison, compounded returns are calculated by the following formula:

$$r_{compounded} = \prod_{i=61}^{124}(1+r_i)\,.$$

The second method is mean-variance analysis. The means and standart deviations of two strategies are represented in two dimensional space, and it is visually checked whether there exists a clear dominance between two strategies for different tilda values.

Thirdly, the utulities of two strategies are calculated and compared. In this calculation the risk-aversion coefficient is calculated by averaging the monthly risk-aversion coefficients in the out-of-sample period.

Lastly, an unpaired student-t test is conducted. Unpaired student-t test is used instead of paired student-t test, because our main concern is the compound returns of self-financing portfolios managed and updated by different strategies in any period, which is 64 months in our study. The compound return doesn't change by altering the order of the returns. To see that, suppose C is the initial capital, and R_a, R_b and R_c are the returns of the first, second and third months respectively. The compounded return of this three month period would be the same, if the order of the returns was changed; because

$$C(1+R_a)(1+R_b)(1+R_c) = C(1+R_c)(1+R_b)(1+R_a).$$

Therefore, our test should be insensitive to the ordering of the sample. However, the paired test is sensitive to the ordering. For example, if the order of a sample is changed and a paired test is conducted between the original and the generated sample, it is highly possible to reject the hypothesis that the means of the two samples are the same. On the other hand, unpaired test is not sensitive to the ordering. If unpaired test was used for the above case, the hypothesis would not be rejected. So, unpaired student-t test is preferred instead of the paired test.

To conduct two sample unpaired student-t test, firstly the t-statistic is calculated by the following formula:

$$t = \frac{\overline{x}_{CAPM} - \overline{x}_{BL}}{\sqrt{\frac{\sigma^2_{CAPM}}{n_{CAPM}} - \frac{\sigma^2_{BL}}{n_{BL}}}}$$

where the degrees of freedom is calculated as

$$DoF = \frac{\left(\frac{\sigma^2_{CAPM}}{n_{CAPM}} + \frac{\sigma^2_{BL}}{n_{BL}}\right)^2}{\frac{\left(\frac{\sigma^2_{CAPM}}{n_{CAPM}}\right)^2}{n_{CAPM} - 1} + \frac{\left(\frac{\sigma^2_{BL}}{n_{BL}}\right)^2}{n_{BL} - 1}}.$$

After finding the t-statistic and the degrees-of-freedom, p-value can be calculated by using the cumulative density function (cdf) of the student-t distribution. If the t-statistic is smaller than 0.5 then the value obtained from the cdf is the desired p-value, otherwise the value obtained from the cdf is subtracted from 1 to find the p-value.

The hypothesis is that the strategy returns are equal to each other. If the p-value is smaller than the subjective confidence level than the hypothesis is rejected, otherwise it is not rejected.

CHAPTER 4

RESULTS

4.1 Results

Following the steps described in the methodology chapter, firstly CAPM implied returns are calculated as the prior returns. Secondly, the AR(1) forecasts are obtained as the quantitative views. Thirdly, prior returns and quantitative views are blended by Black-Litterman method, to obtain the posterior returns. After that, these posterior returns are used to solve the utulity maximization problem in order to reach the optimal weights of the industry indexes. Lastly, Black-Litterman strategy was compared with the CAPM strategy by four different methods.

A substep of obtaining the CAPM implied returns is calculating the risk aversion coefficients in the rolling window period. The risk aversion coefficients are smaller than we expected. The mean of the coefficients for 64 periods is 0.8474. The reason is that untill recent years turkish governments were covering their budget deficits by domestic debt. Together with the inflationary enviroment and increased risk perception after the two economic crises, the huge amount of government bond supply carried the interest rates of government bonds to virtually high levels. Because of this fact, the difference between returns in the stock exchange market and the risk-free rates had decreased, decreasing the risk aversion coefficient.

Although we used these risk-aversion coefficients without any modification, we do not think that these values represent the real risk aversion in the market. It may be that the market did not perceive the turkish government bond as risk-free, although it should be by definition. Using other non-government bonds of high rated companies as a proxy for risk-free rates might be a remedy. However, it is highly possible to reach the same result.

Another alternative explanation may be that in an emerging market which had gone through two economic crises, the sample of size 64 is too small to successfully represent the population, because the data includes shocks and jumps. However, the price data for the industrial indexes are avaliable only after 1997, limiting the sample size.

Despite of these limitations, we used these risk aversion coefficients, since it seems that a better alternative is not available.

Another issue is the behaviour of the posterior weights. At first we expected that the posterior weights would move to the same direction with the view return. For example, if the AR(1) forecasted return for an industry index was greater than the corresponding CAPM implied return, we would expect the posterior weight of this industry index to be greater than the market capitalization weight. However, we could not detect such a direct or indirect relationship between the view returns and the posterior weights. In table.1, the differences in the returns [(CAPM implied Returns) – (Black-Litterman Posterior Returns)] and the corresponding differences in the weights [(Market Capitalization Weights)-(Black-Litterman Posterior Weights)] are given. These values are obtained by setting tilda equal to 0.00001. It can easily be seen that there is not a relationship as we expected.

The scatter plot shown in figure.3 represents these points in 2 dimensional space. The unexistence of any patterns means that there is no direct or reverse relationship between the change in returns and the change in weights.

Although this observation seems unintuitive at first sight, the unexistence of a direct relationship does not really imply a mistake. After a deeper study, it can be seen that the relationship actually depends on the covariance structure of returns. In methodology chapter it is discussed and stated that for some covariance structures, there may even be a reverse relationship. The illustration in the previous chapter was done for a quiete small asset universe: 2 assets. Obviously, the more the size of the asset universe increases, the more complex relationships will be observed; and it will be harder to trace them. Therefore, this observation is not inconsistent with the methodology indeed.

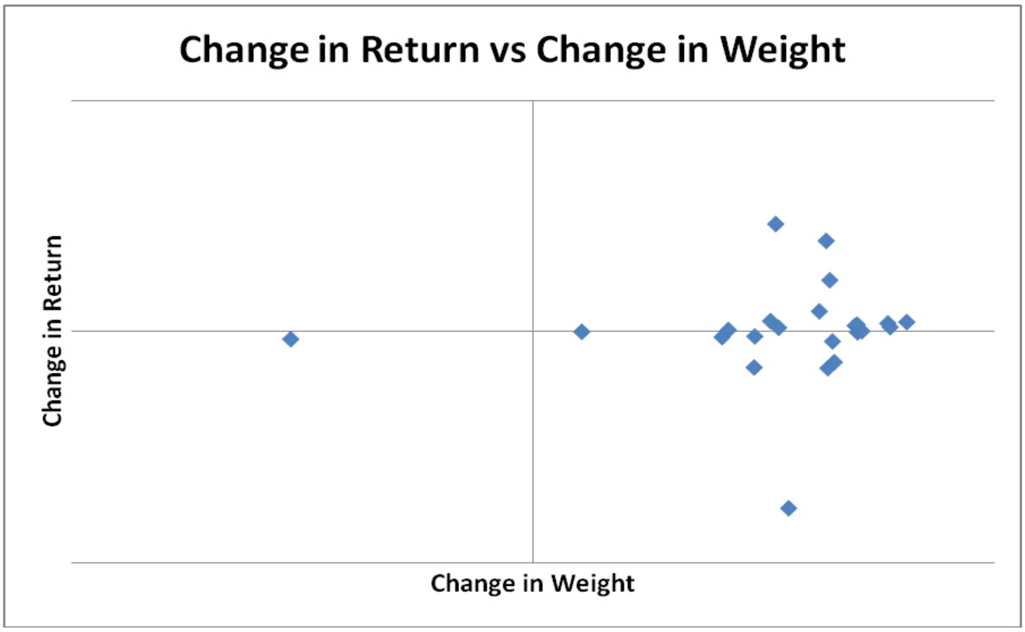

Figure 3

Table 1

Industry Index	Return Diff at	Weight Diff at 31/08/05
Banking	0,00000035	-0,00992976
Information Tech.	0,00000063	-0,00113781
Electric	-0,00000002	0,00025704
Leasing	0,00000032	-0,00010302
Food	0,00000009	-0,00005414
Real Estate Inv. Tr.	0,00000032	-0,00054587
Services	0,00000034	-0,00388227
Holdings	0,00000036	-0,00429856
Telecominication	0,00000032	0,00156479
Paper	0,00000012	-0,00000297
Chemistry	0,00000050	0,00013580
Finance	0,00000039	0,01580641
Metal(main)	0,00000014	-0,00003641
Metal(stuff)	-0,00000006	-0,00069475
Defense	0,00000032	-0,00055510
Insurance	0,00000036	-0,00073402
Stone & soil	0,00000046	0,00040207
Technology	0,00000063	0,00177611
Textile	0,00000017	0,00000076
Commerce	0,00000069	0,00144051
Tourism	0,00000073	0,00013964
Transportation	0,00000068	0,00026946
Mutual Funds	0,00000016	0,00009254

Before making the comparison between the strategies, time series graphs and histograms of the prior returns and the view returns are checked.

As can be see from the graphs, CAPM strategy returns are nearly smooth compared to highly volatile AR(1) strategy returns. AR(1) returns have more than 10 extreme values like -3000 % and 2000 %. Moreover, there are two large negative jumps which can be treated as outliers. In time series graph of AR(1), the maximum and minimum values are set to be 50 and -50 respectively, so two negative jump points are not seen in the graph. The reason for excluding these jumps in the graph is that when they are included, the scale gets too small, making the graph not understandable.

This highly volatile behaviour and the existence of extreme negative returns are the main reasons why we exclude pure AR(1) strategy in the comparisons. In AR(1) strategy, we have observed that for some months there are some negative returns smaller than -100%. The reason is that, since we used an unconstrained optimization, there are no limitations on short selling, and the summation of the weights of risky assets can be different than 1. Therefore, for some periods the total weight of risky assets are observed to be at extreme levels, like -2500 %. This means, we have to short sell risky assets with an amount which is 25 times our budget, and invest this cash in risk-free rate for one month. It is obvious that in case of a strong bullish movement in the stock market, returns smaller than -100% will be generated, resulting a negative wealth. In other words, not only the principal will be lost, but also the strategy will end up with debt. If this happens, the strategy will not be admissible anymore because a new portfolio can not be composed without additional endowment. In fact, in the first month, AR(1) strategy results in negative wealth with a return of -635 %, so the strategy should be terminated very early.

On the other hand, CAPM returns are ossilating around the close neighborhood of 0 %. The maximum return is about 27 % and the minimum return is nearly -23 %. Since there is not a return smaller than -100 %, the strategy is admissible.

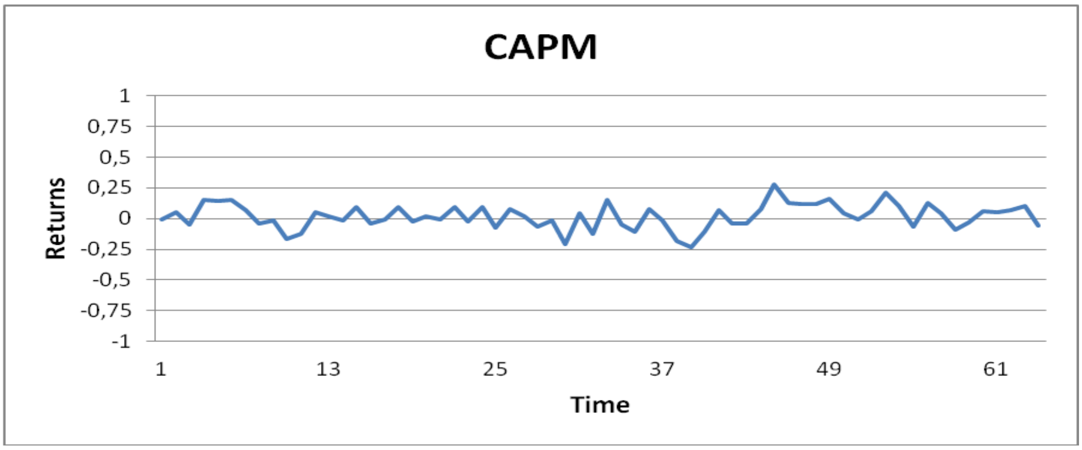

Figure 4

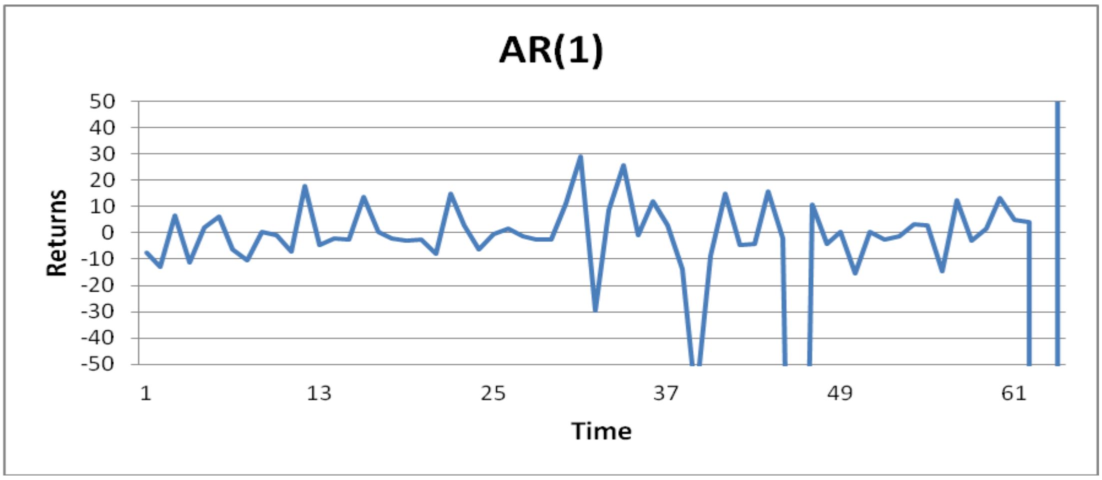

Figure 5

The below histograms show the frequency distributions of prior and view returns. Although, both of them are centered around zero, AR(1) returns are spreaded in a more distant range and have extremes values, while capm returns are distributed in a relatively smaller range: $\pm 25\%$.

To make the graph readable, four outliers are excluded from AR(1) returns . After excluding the outliers in view returns, AR(1) has a mean of 42.67 and standart deviation of 10.11, while the mean and standart deviation of CAPM are 0.018 and 0.10 respectively.

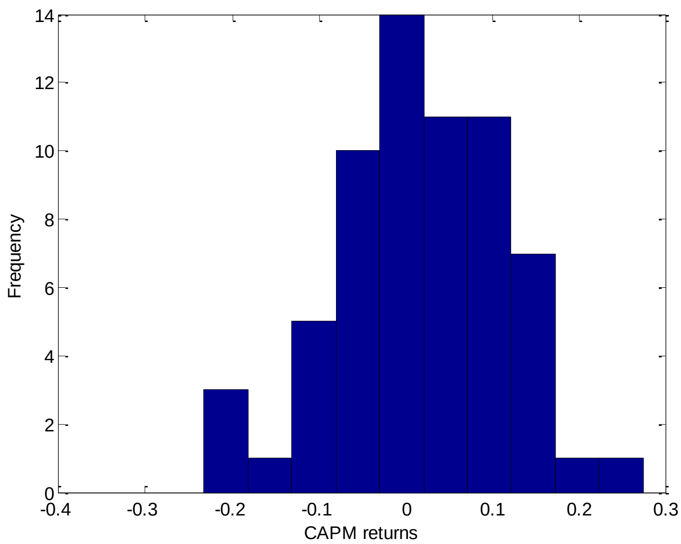

Figure 6

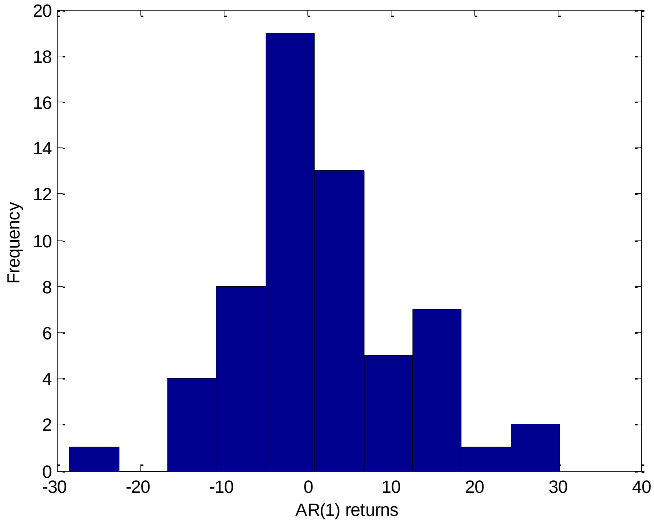

Figure 7

The first comparison will be made on 64 months compound returns. CAPM strategy generates a 64 month compound return of 132 %. The return of Black-Litterman strategy depends on tilda. As tilda increases, the weight of CAPM decreases and the compound return of Black-Litterman diverges from that of CAPM in a negative direction. For small tilda values it is nearly equal to CAPM return. As tilda increases the Black-Litterman strategy return decreases. When tilda is equal to its lowest level, $6.25\,10^{-6}$, tilda imposes the highest weight to AR(1), and Black-Litterman compounded return is equal to -24%. Although it is a negative return, it does not cause a negative wealth, because it is not smaller than -100%. The below graph shows the relation visually.

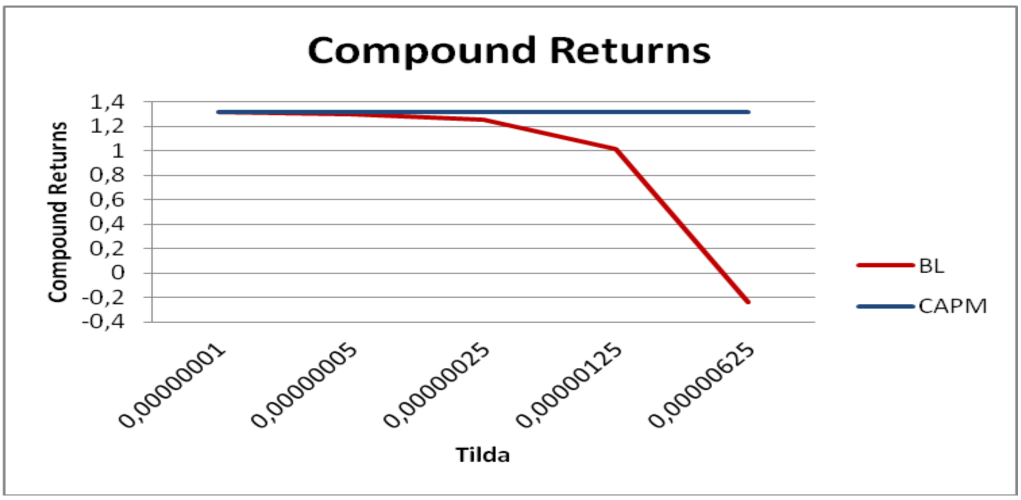

Figure 8

Second comparison is the mean-variance analysis. To see whether there is an obvious dominance, CAPM strategy returns and Black-Litterman strategy returns are represented on mean-standart deviation plane. There are five Black-Litterman points where BL1 represents the posterior return for the smallest tilda and BL5 represents the posterior return for the largest tilda.

In the first graph, all the six points are included. However, the graph is not readable, because of the distant location of BL5. In order to make the graph more readable, the scale is increased by excluding BL5.

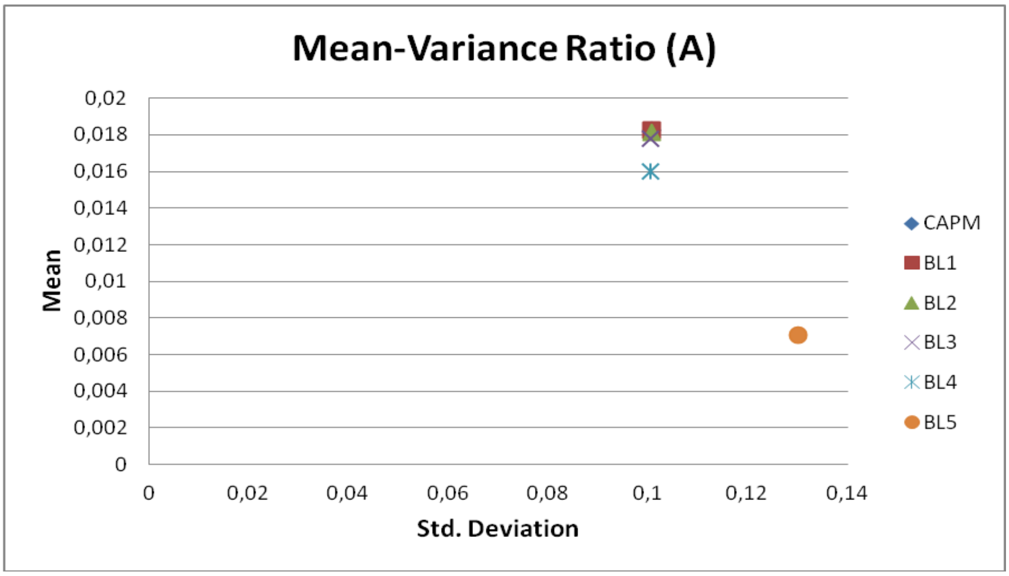

Figure 9

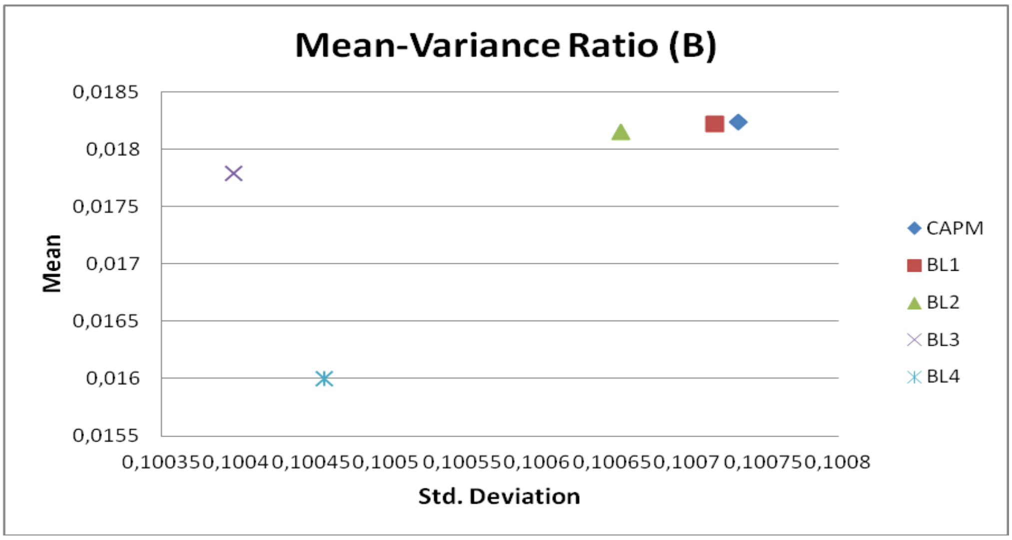

Figure 10

In the second graph, it can be seen that until BL3, the points move to the southwest direction as tilda increases. After that, the direction moves to southeast, making the last two Black-Litterman portfolios unefficient.

From this graph we may say that BL4 and BL5 are dominated by the other strategies. However, we cannot make a clear distinction between CAPM, BL1, BL2 and BL3. Although, the increase in tilda decreases the mean returns, it decreases the standart deviation in the meantime. Therefore, investors with different risk-aversion coefficients may choose different strategies among these.

Third method is utulity analysis. In this analysis, expected utulities are calculated for 6 different strategies. In mean-variance analysis, the distinction could not be made between the first four strategies. However, if we assign a risk-aversion coefficient for the market, the expected utulities can be calculated, and the selection becomes more clear.

To estimate the risk-aversion of the market, we used the mean of the monthly risk-aversion coefficients for the out-of-sample period. Using this estimated coefficient, the expected utulities are calculated. The below graph shows the results. The expected utulity decreases, while tilda value increases.

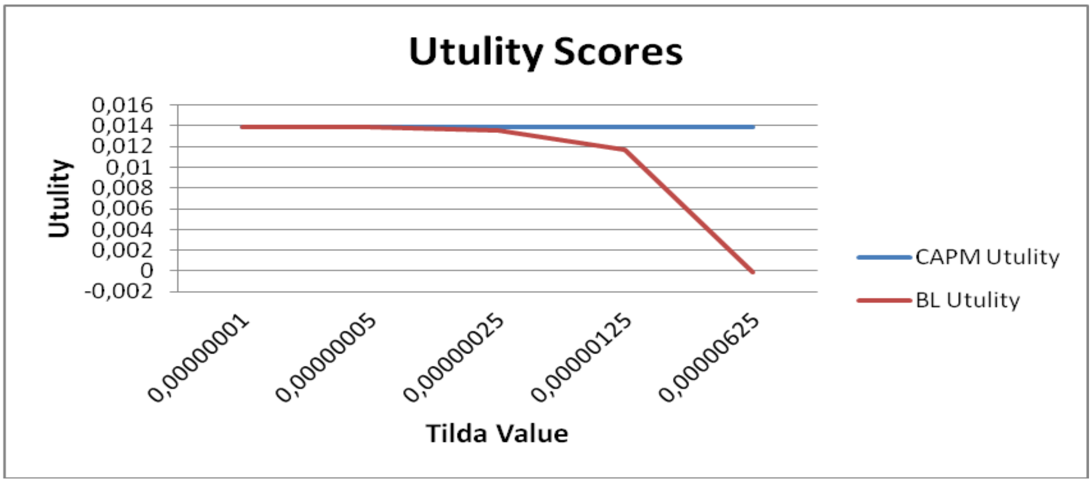

Figure 11

The strength of utulity analysis is that a precise decision is possible by comparing the expected utulities. However, it relies on the assumption that the mean risk-aversion coefficient is a good estimate for the market. The unexpectedly small risk-aversion coefficients in mind, we may doubt that our estimate may not be a good enough estimate to represent the market. The problem about the risk-aversion coefficients was discussed in methodology chapter.

The last comparison will be made on the mean strategy returns. The compound return analysis shows us the relationship between strategies for one sample of size 64. On the other hand, to test whether this result is due to chance, a statistical hypothesis test must be executed. The idea is that if two strategies were coming from the same population, and the mean returns of different samples of the same size coming from different time periods were calculated, what would be the probability to observe the sample that we work on.

Our null hypothesis is that both of the strategy returns of size 64 are coming from the same population, and the differences are due to chance. If the null hypothesis is rejected then it can be stated that the difference between strategy returns are statistically significant.

Unpaired student t-test is conducted to test the difference between CAPM and Black-Litterman. Same test is executed for different tilda values and the corresponding p-values are obtained. The below graph shows the relationship between tilda and p-value. It is easly seen that as tilda increases, the p-value decreases, which means the probability of observing such a difference decreases. In other words, the difference between CAPM strategy returns and Black-Litterman Strategy returns is getting more significant.

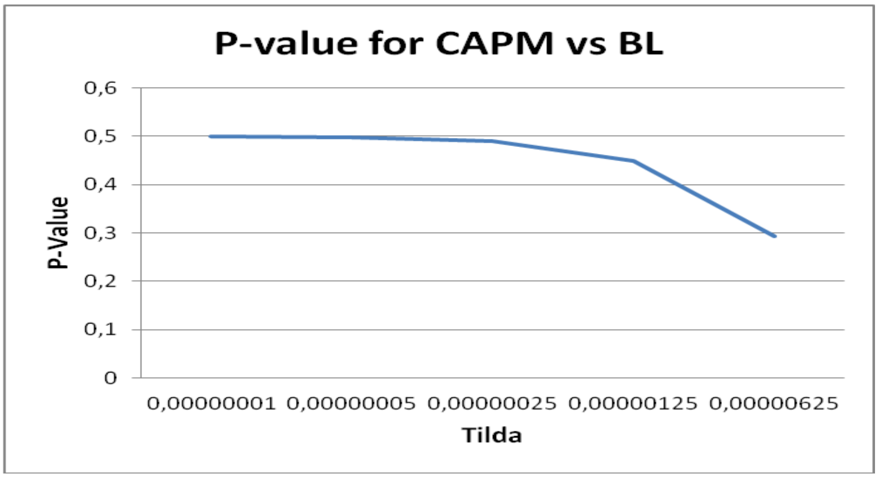

Figure 12

This test result is consistant with our expectations. Since tilda imposes the relative weight between the prior and the views, the increase in tilda gives less weight to the prior returns. Therefore, as tilda increases, the difference between the posterior and the prior increases. Nevertheless, the difference is not statistically significant even for the highest tilda value. The smallest p-value obtained to be 0.3, which is not small enough in any manner to reject the null hypothesis. Consenquently, Black-Litterman strategy returns diverges from CAPM strategy returns in negative direction, however the difference is not found to be statistically significant.

CHAPTER 5

CONCLUSION

In this thesis, Black-Litterman method is elaborated and an emprical study is conducted on the Turkish stock market. Unlike the original study of Black and Litterman, quantitative views are used instead of qualitative analyst views. Returns of the posterior portfolios are compared with that of the market portfolio for four different criteria. The results showed that, with AR(1) views, the Black-Litterman portfolio does not beat the market portfolio.

The first chapter of this study gives a brief introduction and explaines the motivation of using Black-Litterman methodology in portfolio management. The main idea of blending two different information sets is mentioned.

In the second chapter, the portfolio theory is reviewed, beginning from the studies of Markowitz . In the first section, the portfolio theory before Black-Litterman metdod is reviewed. In the second section, the literature about Black-Litterman method is investigated.

Third chapter explaines the steps proceeded while conducting our emprical study. Some obstacles are faced because of the underperformance of AR(1) forecasts, and the untraceable interactions arising from the size of the asset universe. Nevertheless, dealing with these obstacles provided a deeper insight in to the mechanisim and dynamics of the model. These inferences are shared by detailed discussions throughout the chapter.

In the fourth chapter, the results of the comparisons between the performances of the market portfolio and the posterior portfolio is presented. Black-Litterman portfolio performed worse than the market portfolio, in terms of the first three criteria: compound return, mean-variance ratio, and utulity score. As tilda gets large, the difference between the two strategies become more visible. On the other hand, the observed difference is not statistically evidenced according to the unpaired student-t test, the fourth criterion. Nevertheless, it is clear that the difference would enter the critical zone, should the increase in the tilda value proceed.

Consequently, Black-Litterman method does not provide a superior performance compared to the passive market portfolio, if AR(1) forecasts are selected to be the quantitative views. The AR(1) model can be blamed for the infrior performance of the Black-Litterman model, since AR(1) model only gives naive and unsophisticated forecasts about the future returns of the assets. This fact was speculated beforehand, however some obstacles constrained us to use this simple model. Firstly, qualitative views are not used because of the relative ellusiveness of sound qualitative analyst views, and the dependency of the final performance results to the human factor involved. Secondly, among so many quantitative models, only AR(1) model is found to be consistently statistically significant for all time periods and for all asset clases.

Further direciton may be to find and test another quantitative model which may better suit the turkish market. Also more sophisticated covariance estimation methods may improve the performance of the Black-Litterman method.

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