KINEMATIC AND FORCE ANALYSES OF OVERCONSTRAINED MECHANISMS

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ABSTRACT

KINEMATIC AND FORCE ANALYSES OF OVERCONSTRAINED MECHANISMS

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This thesis comprises a study on the kinematic and force analyses of the overconstrained mechanisms. The scope of the overconstrained mechanisms is too wide and difficult to handle. Therefore, the study is restricted to the planar overconstrained mechanisms. Although the study involves only the planar overconstrained mechanisms, the investigated methods and approaches could be extended to the spatial overconstrained mechanisms as well.

In this thesis, kinematic analysis is performed in order to investigate how an overconstrained mechanism can be constructed. Four methods are used. These are the analytical method, the method of cognates, the method of combining identical modules and the method of extending an overconstrained mechanism with extra links.

This thesis also involves the force analysis of the overconstrained mechanisms. A method is introduced in order to eliminate the force indeterminacy encountered in the overconstrained mechanisms. The results are design based and directly associated with the assembly phase of the mechanism.

Keywords: Overconstrained mechanisms, kinematic analysis, force analysis, cognates, planar mechanisms.

ÖZ

FAZLA KISITLI MEKANİZMALARIN KİNEMATİK VE KUVVET ANALİZLERİ

Üstün, Deniz

Yüksek Lisans, Makine Mühendisliği Bölümü

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Bu tez, fazla kısıtlı mekanizmaların kinematik ve kuvvet analizleri üzerine bir çalışmadır. Fazla kısıtlı mekanizmalar geniş kapsamlı ve ele alması zor bir konudur. Bu yüzden, çalışma düzlemsel fazla kısıtlı mekanizmaları kapsayacak biçimde sınırlandırılmıştır. Çalışma sadece düzlemsel fazla kısıtlı mekanizmaları içerse de incelenen metotlar ve yaklaşımlar uzaysal fazla kısıtlı mekanizmalar için de kullanılabilir.

Bu tezde, kinematik analiz fazla kısıtlı mekanizmaların nasıl oluşturulabileceğini incelemek için yapılmıştır. Dört metot kullanılmıştır. Bunlar analitik metot, kökteş mekanizmalar metodu, aynı modüllerin birleştirilmesi metodu ve fazla kısıtlı bir mekanizmaya ekstra uzuvlar ekleme metodudur.

Bu tez aynı zamanda fazla kısıtlı mekanizmaların kuvvet analizini de içerir. Fazla kısıtlı mekanizmalarda karşılaşılan kuvvet belirsizliklerini ortadan kaldırmak için bir metot sunulmuştur. Sonuçlar tasarım temellidir ve direkt olarak mekanizmanın montaj fazı ile ilişkilendirilmiştir.

Anahtar Kelimeler: Fazla kısıtlı mekanizmalar, kinematik analiz, kuvvet analizi, kökdeş mekanizmalar, düzlemsel mekanizmalar.

To my family

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CHAPTER 1

INTRODUCTION

A linkage whose degree of freedom obtained by the general degree of freedom formula is less than its actual degree of freedom is called an overconstrained mechanism. Overconstrained mechanisms usually have zero or less degree of freedom (DOF) according to the general degree of freedom formula but do have full cycle mobility. The first overconstrained mechanism is proposed by Sarrus in 1853. Bennett (1914), Delassus (1922), Bricard (1927), Myard (1931), Goldberg (1943), Waldron (1967, 1968, 1969), Wohlhart (1987, 1991) are the researchers who had proposed the well known overconstrained mechanisms most of which are recognized with their names.

Overconstrained mechanisms have many advantages. They are mobile using fewer links and joints than it is expected. For example, an ordinary closed loop spatial mechanism with revolute and prismatic joints must have at least seven links to be mobile. Overconstrained mechanisms can be mobile with four, five or six links [13]. The decrease in the number of links and joints result in decrease in the cost, complexity and weight of the mechanisms. Another advantage of the overconstrained mechanisms is that they are more rigid and robust. These properties make overconstrained mechanisms desirable.

Overconstrained mechanisms have been studied extensively by the mechanism researchers. The research activities regarding the overconstrained mechanisms can be divided into three main groups.

The first group of research activities focuses on generating a mobility formula valid for the overconstrained mechanisms. In literature, methods for the DOF calculation of mechanisms can be grouped into two categories: Those in the first category are based on setting up the kinematic constraint equations and their rank calculation for a given position of the mechanism with specific joint location. The ones in the second category do not need to develop the set of constraint equations. The former methods are valid for all kinds of mechanisms without exception; however, writing loop closure equations and performing position/velocity analysis by using analytical tools (screw system theory, linear algebra, affine geometry, Lie algebra, etc) is needed. Therefore, these formulas are not considered to be practical [5]. Rico et al. [15], Huang et al. [7], Zhao et al. [22], Fayet [4], Baker [1], Waldron [21] are the main contributors of this group. The later methods are much more practical. In this group, the proposed formulas of DOF are explicit relationships between structural parameters of the mechanism: the number of links and joints, the motion/constraint parameters of the joints and of the mechanism [5]. These formulas do not need kinematic constraint equations and are suitable for quick calculation of DOF whereas, they are not valid for overconstrained mechanisms. The most known and most commonly used DOF formula is the Kutzbach-Grübler formula. Dobrovolski and Hunt also proposed methods to compute DOF of mechanisms. However, these formulas do not give correct results for overconstrained mechanisms. A general formula for a quick calculation of DOF of all kind of mechanisms is not proposed yet. For calculation of mobility of overconstrained mechanisms, the former category is preferred by the researchers due to their precision. Gogu [5], discussed the general validity of the degree of freedom formulas proposed by many researches. In this thesis, mobility calculation of overconstrained mechanisms is not focused. Only the general DOF formula is used to emphasize that the mechanism is overconstrained.

The second group of research activities is the investigation of various overconstrained mechanisms. In literature it is inevitable to encounter too many researches in which a new type of overconstrained mechanism introduced. This is because an analytical approach for the construction of overconstrained mechanisms has not been introduced yet. Most of these overconstrained mechanisms are discovered intuitively. Very few of them are introduced as a result of analytical mathematic methods. Waldron used closure equations in order to state 4 link overconstrained mechanisms. Waldron 1979 [19], stated that all 4 link overconstrained mechanisms with lower pairs are determined. Unfortunately, the study was restricted to 4 link mechanisms and also linkages with mobility greater than one were excluded. Therefore, the overconstrained mechanisms determined by Waldron were only spatial overconstrained mechanisms. Pamidi, Soni and Dukkipati in 1971 [14] have also used a similar method to Waldron's. Although in 1973 they expand the type of joints in their study, it was restricted to 5 link mechanisms. Lee and Yan in 1993 [10] studied 6R mechanisms by using matrix loop equations and they stated that there are only three types of movable spatial 6R mechanisms. Mavroidis and Roth in 1994 [12], discussed the difficulty to know the existence of other overconstrained mechanisms. They classified known overconstrained mechanisms in four classes according to general common characteristics of all mechanisms belonging in this class: (i) symmetric mechanisms, (ii) Bennett based mechanisms, (iii) combined special geometry mechanisms and (iv) mechanisms derived by overconstrained manipulators. Overconstrained manipulators are the 6 joint manipulators that have less than 6 DOF for their end-effectors motion. Then using synthetic methods, they proposed new overconstrained mechanisms. They studied 6 link mechanisms.

The last group of research activities composes of the studies for the development of a method in the force analysis of overconstrained mechanisms. In terms of force analysis, overconstrained mechanisms show statically indeterminate characteristics. The total number of unknown forces and moments of an overconstrained mechanism is always more than the number of available equations. So the unknowns cannot be determined by the force analysis methods involving rigid bodies only. By relaxing the rigidity assumption and using stress-strain relationships it is possible to increase the number of equations and obtain a solution. However, this process is not practical. Therefore, researchers investigate a practical but efficient solution of the problem. Himmetoğlu and Özgören in 2000 [6], introduce a method to eliminate force indeterminacy encountered in the overconstrained mechanisms without using stress-strain relationships. They added extra joints to mechanisms in order to increase the equation number. Each unactuated single-DOF joint added to the system brings in 6 more equations and 5 more unknowns, thus 1 equation surplus.

This thesis contributes to the second and third groups of the relevant literature mentioned above. Chapter 2 of the thesis is about the second group. It involves a study on how an overconstrained mechanism can be constructed. Four methods are considered: (i) the analytical method, (ii) the method of cognates, (iii) the method of combining identical modules, (iv) the method of extending an overconstrained mechanism. Although nothing is novel with these methods, they are discussed in order to emphasize their significance in the construction of the overconstrained mechanisms. Chapter 3 of the thesis is about the third group of the relevant literature. It introduces a method in order to eliminate the indeterminacy in the unknown joint reaction forces and moments. This method is directly related to the construction phase of the mechanism. At the time of the assembly process, preloading is applied to some links and as a result the indeterminacy is eliminated in favour of the intended usage of the mechanism. An optimization process is performed in order to decide the amount of preloading.

This thesis is restricted to the planar overconstrained mechanisms only. Planar mechanisms are simpler than spatial mechanisms. However when overconstrained mechanisms are considered, planar mechanisms do not form a simple research field. This is because most of the overconstrained mechanisms are spatial. The possibility of obtaining an overconstrained planar mechanism is less than the possibility of obtaining an overconstrained spatial mechanism.

CHAPTER 2

KINEMATIC ANALYSIS

Kinematic analysis is performed to investigate the motion characteristics of the mechanisms for given geometrical parameters. Motion characteristics comprise displacement, velocity and acceleration knowledge of any point on a moving body and path traced by a point on any link of the mechanism.

In this chapter, the ways of generating overconstrained planar mechanisms are investigated. Four methods are considered: (i) the analytical method, (ii) the method of cognates, (iii) the method of combining identical modules, (iv) the method of extending an overconstrained mechanism with extra links. In the analytical method, by using velocity characteristics of a mechanism, the conditions that make it overconstrained are obtained. In the method of cognates, the cognate mechanisms that trace identical coupler curves are connected to each other in order to generate an overconstrained mechanism. In the method of combining identical modules, as the name implies, identical modules such as scissors like linkages are successively connected in order to generate an overconstrained mechanism. In the method of extending an overconstrained mechanism with extra links, a new overconstrained mechanism is generated by adding extra links and joints to an already existing overconstrained mechanism.

The details of these methods are explained in the sequel.

2.1 Analytical Method

This method utilizes the velocity analysis of the mechanisms. In order to apply this method, some of the structural characteristics of the mechanism should be determined primarily. Firstly, the number of links and joints and the joint types should be determined for the overconstrained mechanism to be generated. After the determination of these structural parameters, a closed kinematic chain should be constructed. A kinematic chain is a series of links connected by kinematic pairs. The chain is said to be closed if every link is connected to at least two other links [11], [18]. For the same number of links and joints and the joint types, it may be possible to construct several different kinematic chains. So, in order to proceed with the method, a closed kinematic chain should be selected out of the possible ones. After the selection of the closed kinematic chain, the loop closure equations for every independent loop in the kinematic chain should be written. The equations that describe the closure of the loops are known as the loop closure equations [18]. The number of the unknown joint variables should be equal to the number of the scalar loop closure equations. By differentiating the loop closure equations with respect to time, the velocity loop equations are obtained.

Any of the loop closure equations of a mechanism can be written as the following vector equation:

$$\overrightarrow{r_1} + \overrightarrow{r_2} + \overrightarrow{r_3} + \overrightarrow{r_4} + \cdots + \overrightarrow{r_n} = 0$$

where $\overrightarrow{r_k}$ is the kth vector in the loop.

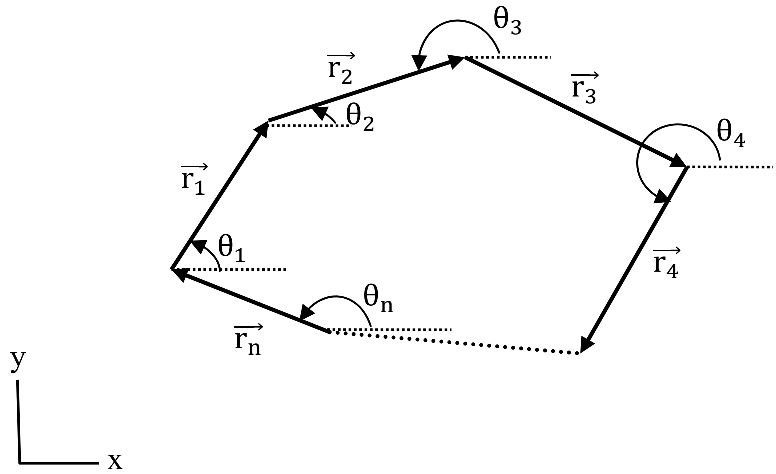


Figure 1: Vectorial Representation of a Closed Kinematic Chain

The preceding vector equation can be written into two scalar equations for planar mechanisms. That is,

$$r_1 \cos\theta_1 + r_2 \cos\theta_2 + r_3 \cos\theta_3 + r_4 \cos\theta_4 + \cdots + r_n \cos\theta_n = 0$$

$$r_1 \sin\theta_1 + r_2 \sin\theta_2 + r_3 \sin\theta_3 + r_4 \sin\theta_4 + \cdots + r_n \sin\theta_n = 0$$

where θ_k is the angle between the x axis and the vector $\overrightarrow{r_k}$ and r_k is the magnitude of $\overrightarrow{r_k}$.

For every independent loop these scalar equations should be written so that the total number of such scalar equations is equal to the number of unknowns.

In order to obtain velocity loop equations, the loop closure equations are to be differentiated with respect to time. Thus, the velocity loop equations are obtained as:

$$\begin{aligned}\dot{r}_1 \cos\theta_1 - r_1\dot{\theta}_1 \sin\theta_1 + \dot{r}_2 \cos\theta_2 - r_2\dot{\theta}_2 \sin\theta_2 + \dot{r}_3 \cos\theta_3 - r_3\dot{\theta}_3 \sin\theta_3 \\ + \dot{r}_4 \cos\theta_4 - r_4\dot{\theta}_4 \sin\theta_4 + \cdots + \dot{r}_n \cos\theta_n - r_n\dot{\theta}_n \sin\theta_n = 0\end{aligned}$$

$$\begin{aligned}\dot{r}_1 \sin\theta_1 + r_1\dot{\theta}_1 \cos\theta_1 + \dot{r}_2 \sin\theta_2 + r_2\dot{\theta}_2 \cos\theta_2 + \dot{r}_3 \sin\theta_3 + r_3\dot{\theta}_3 \cos\theta_3 \\ + \dot{r}_4 \sin\theta_4 + r_4\dot{\theta}_4 \cos\theta_4 + \cdots + \dot{r}_n \sin\theta_n + r_n\dot{\theta}_n \cos\theta_n = 0\end{aligned}$$

The links of the mechanisms are assumed to be rigid. Therefore, their shapes remain the same throughout the motion. Thus, the change of magnitude of vectors with respect to time is zero. Hence, the equations become;

$$-r_1\dot{\theta}_1\sin\theta_1 - r_2\dot{\theta}_2\sin\theta_2 - r_3\dot{\theta}_3\sin\theta_3 - r_4\dot{\theta}_4\sin\theta_4 - \cdots - r_n\dot{\theta}_n\sin\theta_n = 0$$

$$r_1\dot{\theta}_1\cos\theta_1 + r_2\dot{\theta}_2\cos\theta_2 + r_3\dot{\theta}_3\cos\theta_3 + r_4\dot{\theta}_4\cos\theta_4 + \cdots + r_n\dot{\theta}_n\cos\theta_n = 0$$

As seen, each pair of these scalar velocity loop equations is linear in the rates of the joint variables. Such a pair can be written as the following matrix equation:

$$\begin{bmatrix} -r_1\sin\theta_1 & -r_2\sin\theta_2 & -r_3\sin\theta_3 & -r_4\sin\theta_4 & \dots & -r_n\sin\theta_n \\ r_1\cos\theta_1 & r_2\cos\theta_2 & r_3\cos\theta_3 & r_4\cos\theta_4 & \dots & r_n\cos\theta_n \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} = [0]$$

All such equations written for the independent loops can be combined into the following overall matrix equation:

$$[A][\dot{x}] = [B][\dot{y}]$$

Here, in general,

[A] is a pxn coefficient matrix,

[B] is a pxm coefficient matrix,

$[\dot{x}]$ is an nx1 matrix of the unknown joint variable rates,

$[\dot{y}]$ is an mx1 matrix of the specified (i.e. known) joint variable rates.

m is the mobility (degree of freedom) of the system.

p is the number of scalar loop closure equations.

Before proceeding with the analytical method, the notion of a general mechanical system should be introduced. Then, its association with the coefficient matrix [A] should be considered. In general, a mechanical system is composed of mobile and immobile subsystems. If the mechanical system contains at least one mobile subsystem, it is called a mobile mechanical system. If it consists of only immobile subsystems then it is called an immobile mechanical system. An immobile subsystem is a substructure. Whereas, a mobile subsystem is a submechanism. If the geometric characteristics of a substructure can be selected in a special way, it may become a submechanism. Such a submechanism is called an overconstrained submechanism. A mechanical system that contains overconstrained submechanisms is called an overconstrained mechanism.

If a mechanical system is mobile, without any overconstrained submechanism, then [A] will be a full-rank matrix such that p = n. Furthermore, if the system is not in a dead center position with respect to $[\dot{y}]$, $[\dot{x}]$ can be found as

$$[\dot{x}] = [A]^{-1}[B][\dot{y}]$$

However, if the mechanical system contains an overconstrained submechanism, then det[A] becomes zero and the number of independent scalar equations reduces to $p' = p - r$. As a consequence, the mobility of the system increases to $m' = m + r$. In other words, m' elements of $[\dot{x}]$ must be additionally specified and incorporated into $[\dot{y}]$. In such a case, the velocity equation can be rearranged as

$$[A'][\dot{x}'] = [B'][\dot{y}']$$

Now, $p' = n' = n - m'$ and

$[A']$ is the new $n'xn'$ coefficient matrix,

$[B']$ is the new $n'xm'$ coefficient matrix,

$[\dot{x}']$ is the $n'x1$ matrix of the new unknown joint variable rates,

$[\dot{y}']$ is the $m'x1$ matrix of the new specified (i.e. known) joint variable rates. Note that m' is the increased mobility (new degree of freedom) of the system.

Hence, if the system is not in a dead center position with respect to $[\dot{y}']$, $[\dot{x}']$ can be found as

$$[\dot{x}'] = [A']^{-1}[B'][\dot{y}']$$

The preceding analysis implies that an overconstrained mechanism can be generated if a substructure in a mechanical system can be converted into an overconstrained submechanism. The conditions of this conversion can be obtained by considering the conversion of a single structure into a single overconstrained mechanism.

For a structure with $m \leq 0$, $[\dot{y}] = [0]$. Therefore, the velocity equation becomes

$$[A][\dot{x}] = [0]$$

In this structure, [A] is pxn and $[\dot{x}]$ is nx1. This structure can be converted into an overconstrained mechanism if the number of independent scalar equations can be reduced permanently by choosing the geometric features in a special way. If this number can thus be reduced to $n' = p - r$, then the mobility of the system will be $m' = m + r$ and the velocity equation can be written as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here, $[A_{11}]$ is a full-rank $n'xn'$ matrix, $[\dot{x}_1]$ is an $n'x1$ vector of unknown joint variable rates, and $[\dot{x}_2]$ is an $m'x1$ vector of the known joint variable rates, which are specified as the inputs of the mobilized structure with $DOF = m'$. Due to the rank deficiency, the second row of the preceding equation can be disregarded and $[\dot{x}_1]$ can be determined in terms of $[\dot{x}_2]$ as follows, provided that $\det[A_{11}] \neq 0$ due to a dead center position

$$[\dot{x}_1] = [A_{11}]^{-1}[A_{12}][\dot{x}_2]$$

The analytical approach described above is clarified by means of the following three examples.

2.1.1 Example I: A Parallelogram Mechanism (an overconstrained mechanism with 5 links and 6 revolute joints)

In this example, a planar, overconstrained mechanism with five links and six revolute joints will be generated from a structure with a similar kinematic chain. This kinematic chain with arbitrary geometric features is illustrated in figure 2.

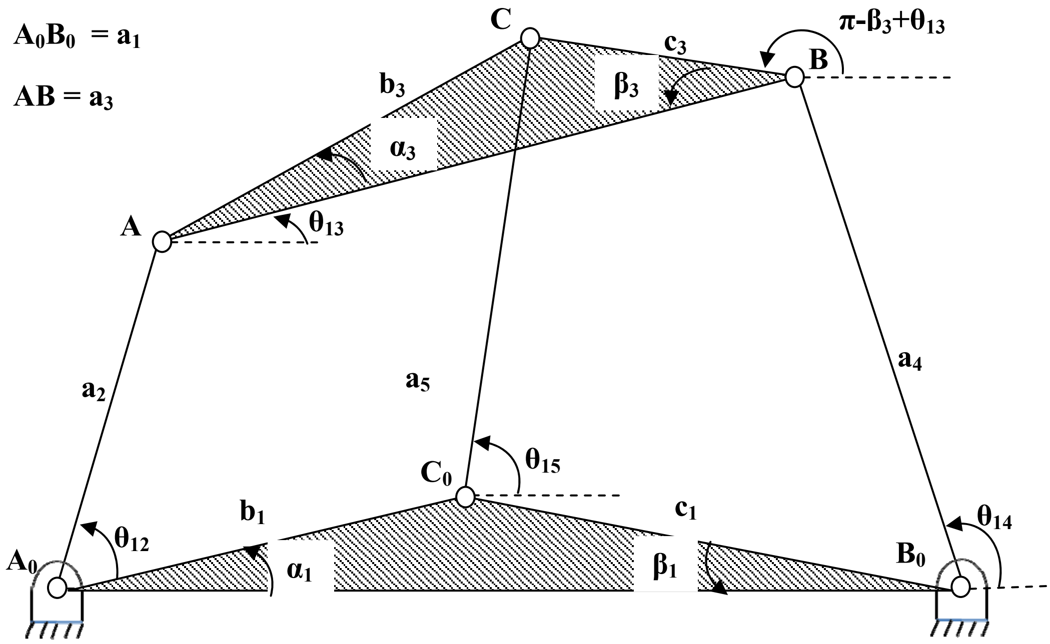


Figure 2: A structure with 5 links and 6 revolute joints

The loop closure equations for the considered mechanical system are

$$\overrightarrow{A_0A} + \overrightarrow{AC} + \overrightarrow{CC_0} + \overrightarrow{C_0A_0} = 0$$

$$\overrightarrow{C_0C} + \overrightarrow{CB} + \overrightarrow{BB_0} + \overrightarrow{B_0C_0} = 0$$

Using complex exponential notation the above equations can be written as;

$$a_2e^{i\theta_{12}}+b_3e^{i(\alpha_3+\theta_{13})} - a_5e^{i\theta_{15}} - b_1e^{i\alpha_1}=0$$

$$a_5e^{i\theta_{15}} + c_3e^{i(-\beta_3+\theta_{13})} - a_4e^{i\theta_{14}} - c_1e^{i(-\beta_1)}=0$$

Expanding the equations and equating real and imaginary parts separately the four equations below are obtained:

$$a_2 \cos\theta_{12} + b_3 \cos(\alpha_3 + \theta_{13}) - a_5 \cos\theta_{15} - b_1 \cos\alpha_1 = 0$$

$$a_2 \sin\theta_{12} + b_3 \sin(\alpha_3 + \theta_{13}) - a_5 \sin\theta_{15} - b_1 \sin\alpha_1 = 0$$

$$a_5 \cos\theta_{15} + c_3 \cos(-\beta_3 + \theta_{13}) - a_4 \cos\theta_{14} - c_1 \cos(-\beta_1) = 0$$

$$a_5 \sin\theta_{15} + c_3 \sin(-\beta_3 + \theta_{13}) - a_4 \sin\theta_{14} - c_1 \sin(-\beta_1) = 0$$

The corresponding velocity equations are obtained as

$$-a_2 \sin\theta_{12}\dot{\theta}_{12} - b_3 \sin(\alpha_3 + \theta_{13})\dot{\theta}_{13} + a_5 \sin\theta_{15}\dot{\theta}_{15} = 0$$

$$a_2 \cos\theta_{12}\dot{\theta}_{12} + b_3 \cos(\alpha_3 + \theta_{13})\,\dot{\theta}_{13} - a_5 \cos\theta_{15}\,\dot{\theta}_{15} = 0$$

$$-a_5 \sin\theta_{15}\,\dot{\theta}_{15} - c_3 \sin(-\beta_3 + \theta_{13})\dot{\theta}_{13} + a_4 \sin\theta_{14}\,\dot{\theta}_{14} = 0$$

$$a_5 \cos\theta_{15}\dot{\theta}_{15} + c_3 \cos(-\beta_3 + \theta_{13})\dot{\theta}_{13} - a_4 \cos\theta_{14}\dot{\theta}_{14} = 0$$

Writing in matrix form;

$$\begin{bmatrix} -a_2 \sin\theta_{12} & -b_3 \sin(\alpha_3 + \theta_{13}) & 0 & a_5 \sin\theta_{15} \\ a_2 \cos\theta_{12} & b_3 \cos(\alpha_3 + \theta_{13}) & 0 & -a_5 \cos\theta_{15} \\ 0 & -c_3 \sin(-\beta_3 + \theta_{13}) & a_4 \sin\theta_{14} & -a_5 \sin\theta_{15} \\ 0 & c_3 \cos(-\beta_3 + \theta_{13}) & -a_4 \cos\theta_{14} & a_5 \cos\theta_{15} \end{bmatrix} \begin{bmatrix} \dot{\theta}_{12} \\ \dot{\theta}_{13} \\ \dot{\theta}_{14} \\ \dot{\theta}_{15} \end{bmatrix} = [0]$$

In order to have non-zero velocities; which is the indication of mobility; the determinant of the coefficient matrix must be zero permanently. That is,

$$\begin{vmatrix} -a_2 \sin\theta_{12} & -b_3 \sin(\alpha_3 + \theta_{13}) & 0 & a_5 \sin\theta_{15} \\ a_2 \cos\theta_{12} & b_3 \cos(\alpha_3 + \theta_{13}) & 0 & -a_5 \cos\theta_{15} \\ 0 & -c_3 \sin(-\beta_3 + \theta_{13}) & a_4 \sin\theta_{14} & -a_5 \sin\theta_{15} \\ 0 & c_3 \cos(-\beta_3 + \theta_{13}) & -a_4 \cos\theta_{14} & a_5 \cos\theta_{15} \end{vmatrix} = 0$$

By using a symbolic manipulation software, such as mathcad, the equation below is obtained:

$$a_2a_4a_5\{b_3[\sin(\theta_{15}-\theta_{14})\sin(-\theta_{12}+\theta_{13}+\alpha_3)] \\ +c_3[\sin(\theta_{12}-\theta_{15})\sin(\beta_3-\theta_{13}+\theta_{14})]\}=0$$

The solution of this equation can be obtained by considering the following possibilities.

(a) $\sin(\theta_{15}-\theta_{14})=0$ and $\sin(\theta_{12}-\theta_{15})=0 \Rightarrow$

$$\theta_{15}=\theta_{14} \text{ or } \theta_{15}=\theta_{14}\pm\pi$$

$$\theta_{12}=\theta_{15} \text{ or } \theta_{12}=\theta_{15}\pm\pi$$

This result implies that link 5, link 4 and link 2 must always be parallel to each other.

(b) $\sin(-\theta_{12}+\theta_{13}+\alpha_3)=0$ and $\sin(\theta_{12}-\theta_{15})=0\Rightarrow$

$$\theta_{12}=(\theta_{13}+\alpha_3) \text{ or } \theta_{12}=(\theta_{13}+\alpha_3)\pm\pi$$

$$\theta_{12}=\theta_{15} \text{ or } \theta_{12}=\theta_{15}\pm\pi$$

This result implies that AC must be parallel to A_0A while C_0C is also parallel to A_0A. Therefore, this solution cannot be permanent. So, it is not valid.

(c) $\sin(\theta_{15}-\theta_{14})=0$ and $\sin(\beta_3-\theta_{13}+\theta_{14})=0\Rightarrow$

$$\theta_{15}=\theta_{14} \text{ or } \theta_{15}=\theta_{14}\pm\pi$$

$$\theta_{13}=(\theta_{14}+\beta_3) \text{ or } \theta_{13}=(\theta_{14}+\beta_3)\pm\pi$$

This result implies CB must be parallel to B_0B while C_0C is also parallel to B_0B. Therefore, this solution is not valid either as in case (b).

(d) $\sin(-\theta_{12}+\theta_{13}+\alpha_3)=0$ and $\sin(\beta_3-\theta_{13}+\theta_{14})\Rightarrow$

$$\theta_{12}=(\theta_{13}+\alpha_3) \text{ or } \theta_{12}=(\theta_{13}+\alpha_3)\pm\pi$$

$$\theta_{13} = (\theta_{14} + \beta_3) \text{ or } \theta_{13} = (\theta_{14} + \beta_3) \pm \pi$$

This result implies AC must be parallel to A_0A while CB is parallel to B_0B. Therefore, this solution is not valid either as in case (b) and (c).

Among the four possibilities considered above, only the possibility (a) may be satisfied permanently. The additional conditions for its permanent satisfaction can be obtained from the loop closure equations. With $\theta_{12}=\theta_{14}=\theta_{15}$, the loop closure equations become:

$$a_2 \cos\theta_{12} + b_3 \cos(\alpha_3 + \theta_{13}) - a_5 \cos\theta_{12} - b_1 \cos\alpha_1 = 0$$

$$a_2 \sin\theta_{12} + b_3 \sin(\alpha_3 + \theta_{13}) - a_5 \sin\theta_{12} - b_1 \sin\alpha_1 = 0$$

$$a_5 \cos\theta_{12} + c_3 \cos(-\beta_3 + \theta_{13}) - a_4 \cos\theta_{12} - c_1 \cos(-\beta_1) = 0$$

$$a_5 \sin\theta_{12} + c_3 \sin(-\beta_3 + \theta_{13}) - a_4 \sin\theta_{12} - c_1 \sin(-\beta_1) = 0$$

These can be written as;

$$\cos\theta_{12}\,(a_2 - a_5) + b_3 \cos(\alpha_3 + \theta_{13}) - b_1 \cos\alpha_1 = 0$$

$$\sin\theta_{12}\,(a_2 - a_5) + b_3 \sin(\alpha_3 + \theta_{13}) - b_1 \sin\alpha_1 = 0$$

$$\cos\theta_{12}\,(a_5 - a_4) + c_3 \cos(-\beta_3 + \theta_{13}) - c_1 \cos(-\beta_1) = 0$$

$$\sin\theta_{12}\,(a_5 - a_4) + c_3 \sin(-\beta_3 + \theta_{13}) - c_1 \sin(-\beta_1) = 0$$

For any arbitrary value of θ_{12}, which is selected here as the input variable, these equations should be satisfied in order to obtain a permanently overconstrained mechanism. The only solution is:

$$a_2 = a_4 = a_5\ ,\ \ b_1 = b_3\ ,\ \ \alpha_3+\theta_{13} = \alpha_1\ ,\ \ c_1 = c_3\ \ ,\ \text{and}\ \ \beta_3+\theta_{13} = \beta_1$$

These results imply that $\theta_{13}=0$.

The generated overconstrained mechanism is shown in figure 3. Note that, although the general DOF formula gives m=0, the actual DOF of the mechanism is $m' = 1$.

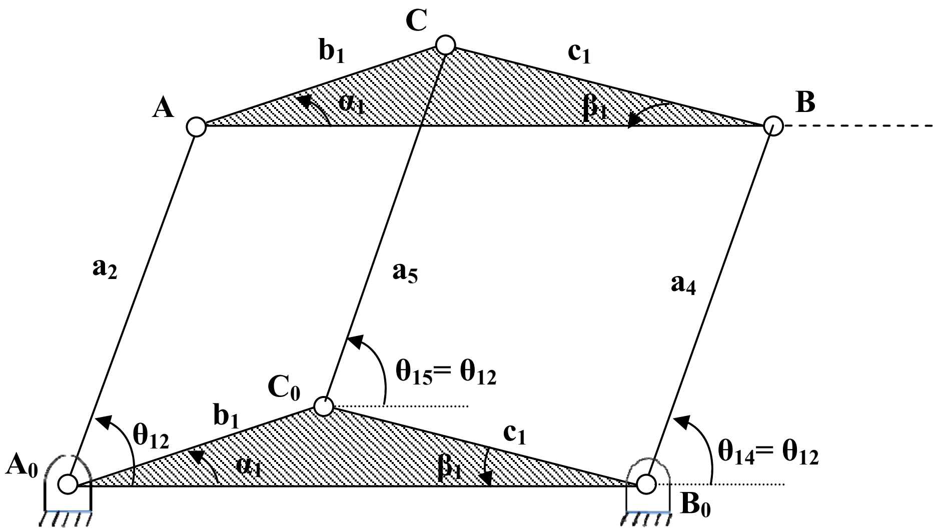


Figure 3: An overconstrained mechanism with 5 links and 6 revolute joints

For the specially selected geometric features of the mobile overconstrained mechanical system, the coefficient matrix becomes;

$$\begin{bmatrix} -a_2 \sin\theta_{12} & -b_1 \sin\alpha_1 & 0 & a_2 \sin\theta_{12} \\ a_2 \cos\theta_{12} & b_3 \cos\alpha_1 & 0 & -a_2 \cos\theta_{12} \\ 0 & -c_1 \sin(-\beta_1) & a_2 \sin\theta_{12} & -a_2 \sin\theta_{12} \\ 0 & c_1 \cos(-\beta_1) & -a_2 \cos\theta_{12} & a_2 \cos\theta_{12} \end{bmatrix}$$

As seen the last column becomes the linear combination of the first and third columns. Thus, the matrix has become rank deficient. That is, the rank of the matrix has decreased to 3. Previously, it was 4. The difference between the ranks gives the actual DOF gained by the mechanical system. Previously, the actual DOF was 0, the mechanical system was not mobile. With the special geometric features, the actual DOF becomes 1 and the system becomes mobile.

2.1.2 Example II: A Double Slider Mechanism (an overconstrained mechanism with 5 links and 4 revolute and 2 prismatic joints)

In this example, a planar, overconstrained mechanism with five links and four revolute and two prismatic joints will be generated from a structure with a similar kinematic chain. This kinematic chain with arbitrary geometric features is illustrated in figure 4.

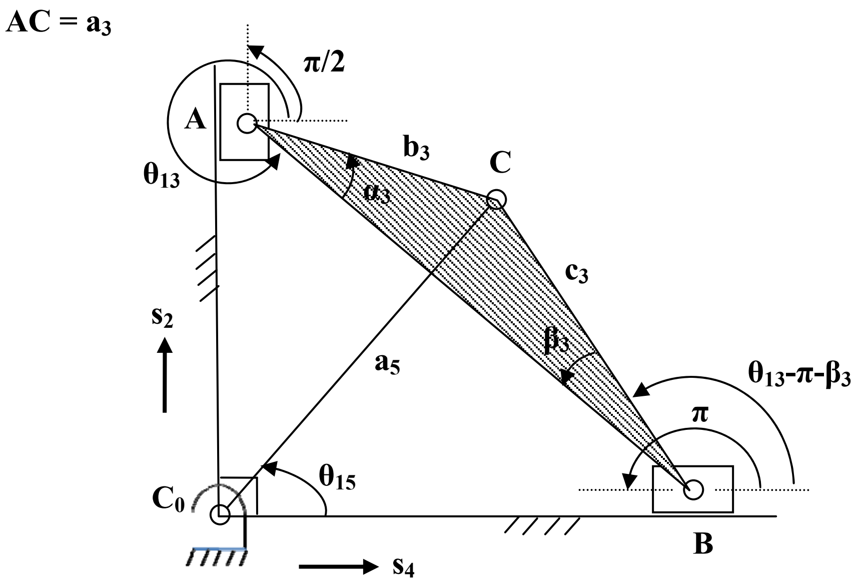


Figure 4: A structure with 5 links and 4 revolute and 2 prismatic joints

The loop closure equations for the considered mechanical system are

$$\overrightarrow{C_0A} + \overrightarrow{AC} + \overrightarrow{CC_0} = 0$$

$$\overrightarrow{C_0C} + \overrightarrow{CB} + \overrightarrow{BC_0} = 0$$

Using complex exponential notation the above equations can be written as;

$$s_2 e^{i(\frac{\pi}{2})} + b_3 e^{i(\alpha_3+\theta_{13})} - a_5 e^{i\theta_{15}} = 0$$

$$a_5 e^{i\theta_{15}} + c_3 e^{i(-\beta_3+\theta_{13})} + s_4 e^{i\pi} = 0$$

Expanding the equations and equating real and imaginary parts separately the four equations are obtained:

$$b_3 \cos(\alpha_3 + \theta_{13}) - a_5 \cos\theta_{15} = 0$$

$$s_2 + b_3 \sin(\alpha_3 + \theta_{13}) - a_5 \sin\theta_{15} = 0$$

$$a_5 \cos\theta_{15} + c_3 \cos(-\beta_3 + \theta_{13}) - s_4 = 0$$

$$a_5 \sin\theta_{15} + c_3 \sin(-\beta_3 + \theta_{13}) = 0$$

The corresponding velocity equations are obtained as

$$-b_3 \sin(\alpha_3 + \theta_{13})\dot{\theta}_{13} + a_5 \sin\theta_{15}\dot{\theta}_{15} = 0$$

$$\dot{s_2} + b_3 \cos(\alpha_3 + \theta_{13})\,\dot{\theta}_{13} - a_5 \cos\theta_{15}\,\dot{\theta}_{15} = 0$$

$$-a_5 \sin\theta_{15}\,\dot{\theta}_{15} - c_3 \sin(-\beta_3 + \theta_{13})\dot{\theta}_{13} - \dot{s_4} = 0$$

$$a_5 \cos\theta_{15}\,\dot{\theta}_{15} + c_3 \cos(-\beta_3 + \theta_{13})\dot{\theta}_{13} = 0$$

Writing in matrix form;

$$\begin{bmatrix} 0 & -b_3 \sin(\alpha_3 + \theta_{13}) & 0 & a_5 \sin\theta_{15} \\ 1 & b_3 \cos(\alpha_3 + \theta_{13}) & 0 & -a_5 \cos\theta_{15} \\ 0 & -c_3 \sin(-\beta_3 + \theta_{13}) & -1 & -a_5 \sin\theta_{15} \\ 0 & c_3 \cos(-\beta_3 + \theta_{13}) & 0 & +a_5 \cos\theta_{15} \end{bmatrix} \begin{bmatrix} \dot{s}_2 \\ \dot{\theta}_{13} \\ \dot{s}_4 \\ \dot{\theta}_{15} \end{bmatrix} = [0]$$

In order to have non-zero velocities; which is the indication of mobility; the determinant of the coefficient matrix must be zero permanently. That is,

$$\begin{vmatrix} 0 & -b_3 \sin(\alpha_3 + \theta_{13}) & 0 & a_5 \sin\theta_{15} \\ 1 & b_3 \cos(\alpha_3 + \theta_{13}) & 0 & -a_5 \cos\theta_{15} \\ 0 & -c_3 \sin(-\beta_3 + \theta_{13}) & -1 & -a_5 \sin\theta_{15} \\ 0 & c_3 \cos(-\beta_3 + \theta_{13}) & 0 & +a_5 \cos\theta_{15} \end{vmatrix} = 0$$

By using a symbolic manipulation software, such as mathcad, the following equation is obtained:

$$a_5[-b_3 \cos(\theta_{15})\sin(\theta_{13}+\alpha_3) - c_3 \sin(\theta_{15})\cos(-\beta_3+\theta_{13})] = 0$$

If $b_3 = c_3$, $\cos(\theta_{15}) = \cos(-\beta_3+\theta_{13})$ and $-\sin(\theta_{13}+\alpha_3) = \sin(\theta_{15})$ the equation is satisfied.

$$\alpha_3 = -\beta_3$$

$$\theta_{15} = \beta_3 - \theta_{13}$$

These equations should be satisfied permanently. The additional conditions for their permanent satisfaction can be obtained from the loop closure equations. With $\theta_{12}=\theta_{14}=\theta_{15}$, the loop closure equations become:

$$b_3 \cos(-\beta_3+\theta_{13}) - a_5 \cos(-\beta_3+\theta_{13}) = 0$$

$$s_2 + b_3 \sin(-\beta_3+\theta_{13}) - a_5 \sin(-\beta_3+\theta_{13}) = 0$$

$$a_5 \cos(-\beta_3+\theta_{13}) + b_3 \cos(-\beta_3+\theta_{13}) - s_4 = 0$$

$$-a_5 \sin(-\beta_3+\theta_{13}) + b_3 \sin(-\beta_3+\theta_{13}) = 0$$

For any arbitrary value of θ_{13}, which is selected here as the input variable, these equations should be satisfied in order to obtain a permanently overconstrained mechanism. The only solution is:

$$a_5 = b_3 = c_3 \text{ and } \alpha_3 = \beta_3 = 0$$

$$\text{So, } \theta_{15} = -\theta_{13}$$

The generated overconstrained mechanism is shown in figure 5. Note that, although the general DOF formula gives m=0, the actual DOF of the mechanism is $m' = 1$.

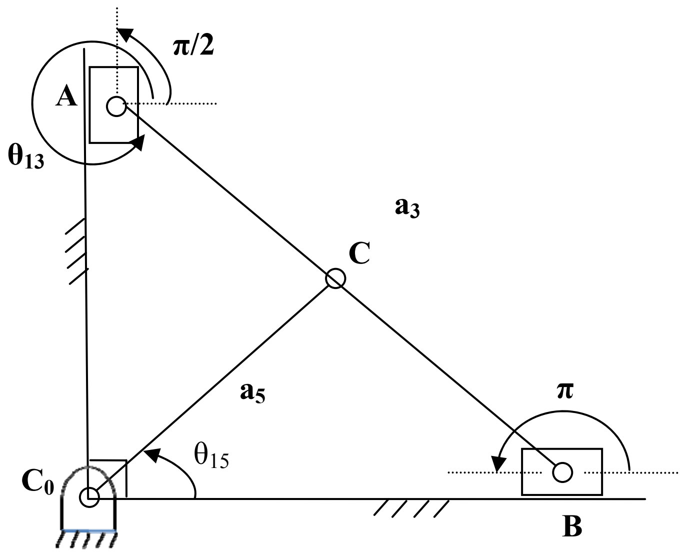


Figure 5: An overconstrained mechanism with 5 links and 4 revolute and 2 prismatic joints

The mechanical system in figure 5 with special geometric characters is mobile, thus it is a mechanism. Although according to the general DOF formula m=0, the actual freedom of the mechanism is $m' = 1$.

For specially selected geometric characters of the mobile overconstrained mechanical system the coefficient matrix becomes;

$$\begin{bmatrix} 0 & -a_5\sin(\theta_{13}) & 0 & -a_5\sin(\theta_{13}) \\ 1 & a_5\cos(\theta_{13}) & 0 & -a_5\cos(\theta_{13}) \\ 0 & -a_5\sin(\theta_{13}) & -1 & +a_5\sin(\theta_{13}) \\ 0 & a_5\cos(\theta_{13}) & 0 & +a_5\cos(\theta_{13}) \end{bmatrix}$$

By using mathcad, the rank of the matrix is find to be equal to 3. Previously, it was 4. The difference between the ranks gives the actual DOF gained by the mechanical system. Previously, the actual DOF was 0; the mechanical system was not mobile. Finally, the actual DOF is 1 and the system is mobile.

2.1.3 Example III: A Parallelogram Mechanism (an overconstrained mechanism with 6 links and 8 revolute joints)

This example is given in order to discuss more about the rank reduction of the coefficient matrix [A] in an immobile mechanical system when it is converted into a mobile overconstrained mechanical system.

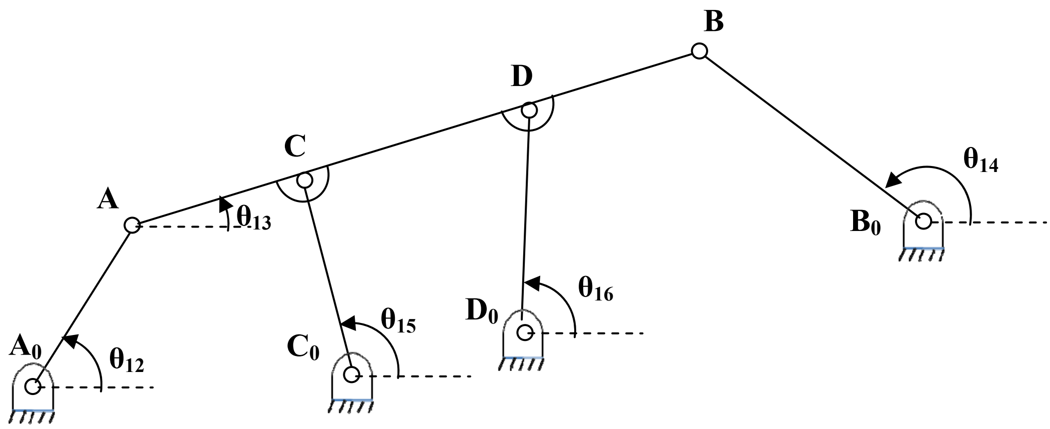


Figure 6: A structure with 6 links and 8 revolute joints

The immobile mechanical system in figure 6 is the extended version of the immobile mechanical system with 6 links and five revolute joints. One more link, D_0D is added into the system.

The mechanical system in figure 6 with arbitrarily selected geometric features is immobile. In other words, according to the general DOF formula, it is a structure with m = -1.

There are 3 independent loops; so, 6 independent scalar equations can be written. The number of variables is 5. Thus, the coefficient matrix is 6x5. When the mechanical system is made mobile with 1 DOF, one of the variables becomes the input of the system. Then, the remaining 4 variables are to be determined. In order to determine these 4 unknowns 4 independent equations are sufficient. Therefore, the rank of the coefficient matrix of the mobile overconstrained mechanical system has to be 4. In other words, two of the six equations become dependent and thus can be disregarded.

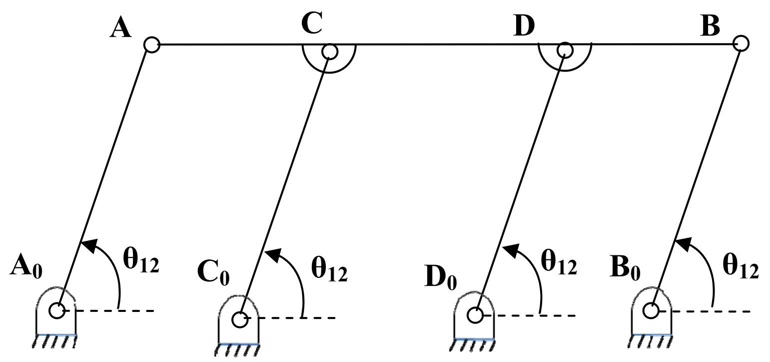


Figure 7: An overconstrained mechanism with 6 links and 8 revolute joints

The mechanical system in figure 7 with special geometric features is mobile. Although according to the general DOF formula m=-1, the actual DOF of the mechanism is $m' = 1$.

2.2 Method of Cognates

The name *cognate* was first introduced by Hartenberg and Denavit for alternative four-bar linkages that trace identical coupler-curves. The property was already introduced by Roberts-Chebyshev theorem. Roberts-Chebyshev theorem states that three different planar 4R linkages trace identical coupler curves. These three linkages are referred to as coupler curve cognates [11]. Hartenberg and Denavit [23] also extended the notion to special forms of six-bar linkages. Soni and Harrisberger [17] studied planar six-bar and spatial four-bar cognates. Dijksman [3] distinguishes between (a) curve cognates, (b) timed-curve cognates, (c) coupler-cognates, (d) timed-coupler cognates and (e) function cognates. Curve-cognates are those that generate the same (coupler) curve by a coupler-point attached to a moving body of the mechanism. Curve-cognates that additionally show the same functional relationship between coupler-point-position and the position of the input-crank, are called timed-curve cognates. Coupler-cognates are alternative mechanisms with the same kinematic chain and a common coupler-plane as well as a common frame. With respect to coupler-cognates, timed coupler cognates additionally have the same functional relationship between the position coordinates of the plane and the position of the input-crank. Function-cognates are cognate mechanisms that produce the same functional relationship between the in- and output angle [3].

The occurrence of cognates in planar mechanism kinematics and their association with overconstrained mechanisms have been studied extensively.

Firstly, starting from Roberts-Chebyshev theorem, cognate-overconstrained mechanism relationship indicated, then some examples are given to overconstrained mechanisms derived from cognates.

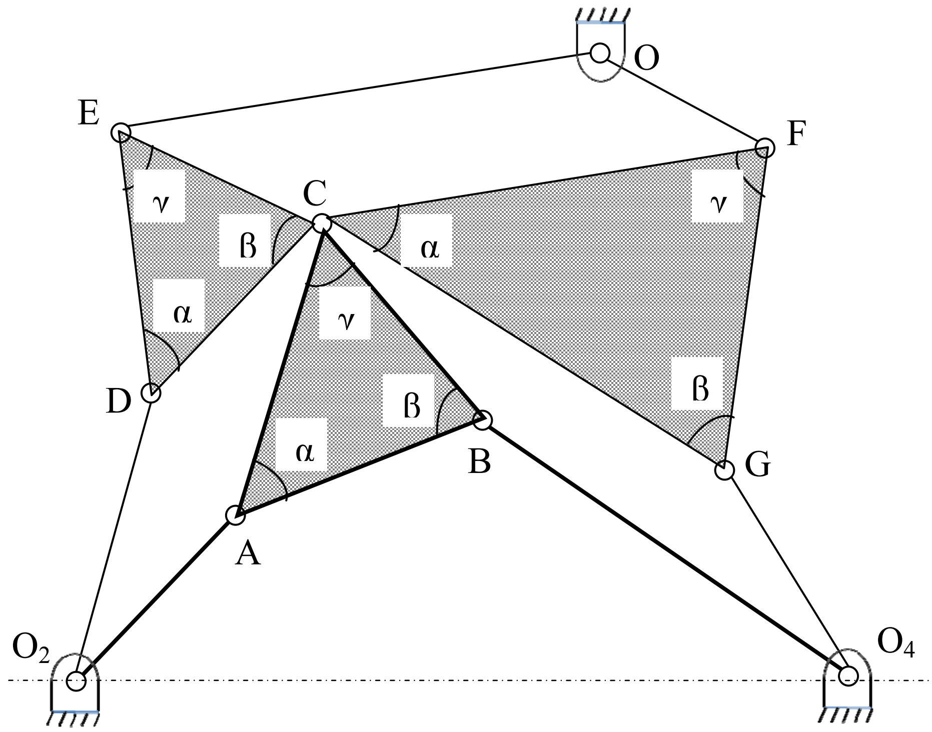


Figure 8: Cognates of Four-Bar

In figure 8, O_2ABO_4 is a four-bar linkage with a coupler curve C. O_2ACD, OECF and O_4GCB parallelograms and ABC, DCE and CGF similar triangles are drawn. As a result the two other four-bars O_2DEO and O_4GFO which also generate the same coupler curve for the coupler point C are obtained. This mechanism is an overconstrained mechanism. According to general DOF formula m=-1 The redundant constrained can be indicated as:

$$O_2O = O_2D + DE + EO$$

$$= AC + DC\frac{DE}{DC}e^{i\alpha} + CF$$

$$= AB\frac{AC}{AB}e^{i\alpha} + O_2A\frac{DE}{DC}e^{i\alpha} + CG\frac{CF}{CG}e^{i\alpha}$$

$$= (AB + O_2A + BO_4)\frac{AC}{AB}e^{i\alpha}$$

$$O_2O = O_2O_4\frac{AC}{AB}e^{i\alpha}$$

This equation shows that the point O is stationary (there is no variable parameter). If the pivot is not fixed, nothing will be changed; the mechanism will make exactly the same motion. If this pivot is not connected to the fixed link the mechanism become constrained with $m' = 1$. [11].

Moreover, focusing on the angular velocities some other overconstrained mechanisms can be obtained.

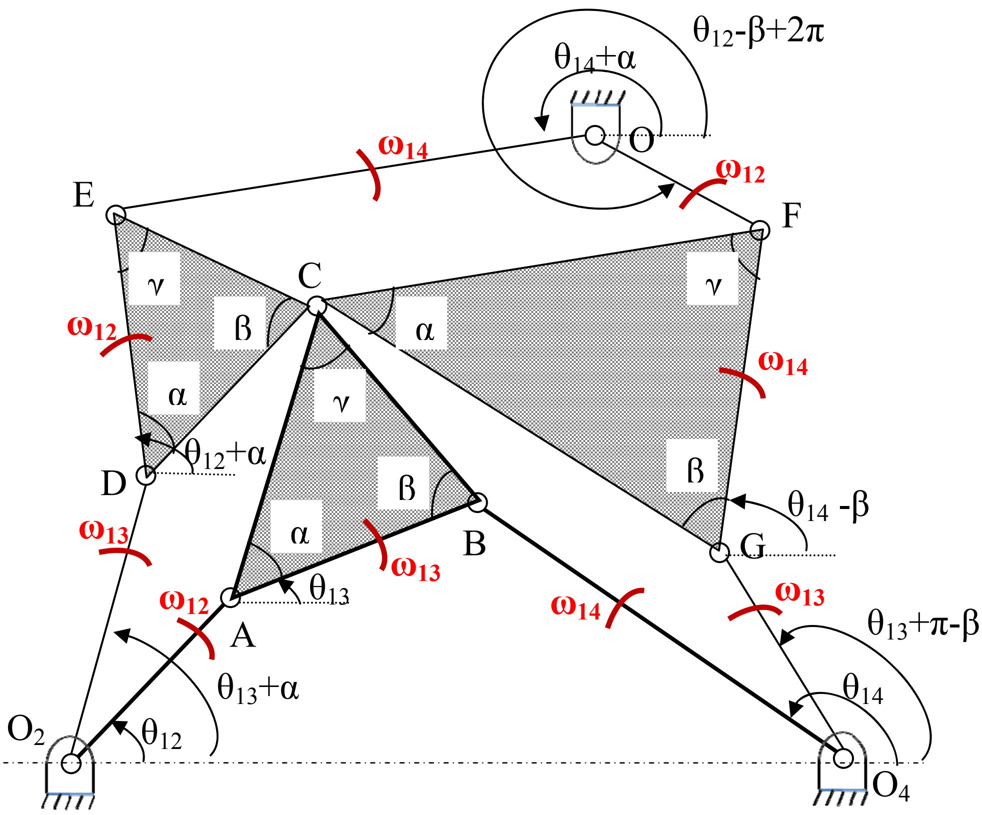


Figure 9: Angular Velocities of Four-Bar Cognates

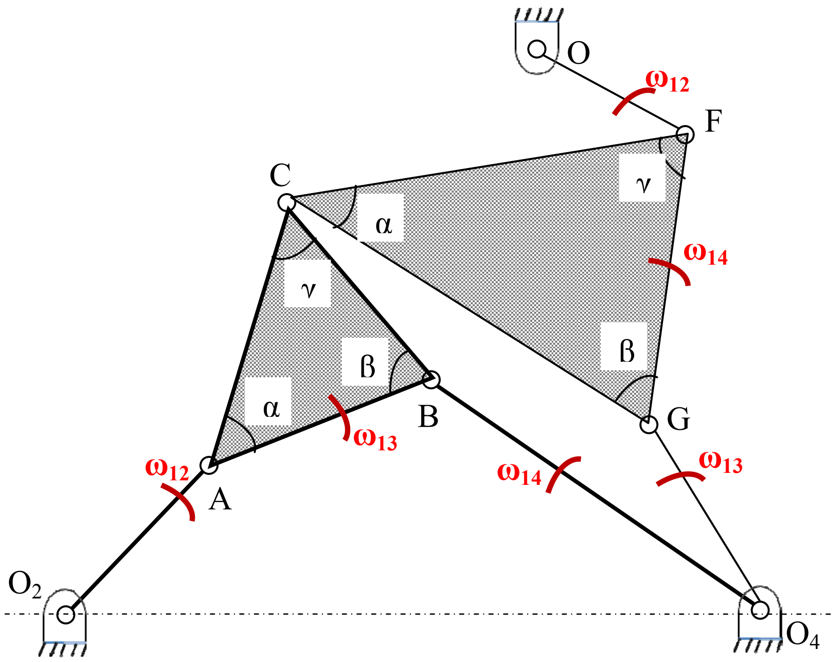


Figure 10: Right-hand Cognate of Four-Bar

Figure 10 shows the four-bar O_2ABO_4 and its right hand cognate O_4GFO. The cognate O_4GFO can be translated without rotation, as a rigid body, so that O coincides with O_2.

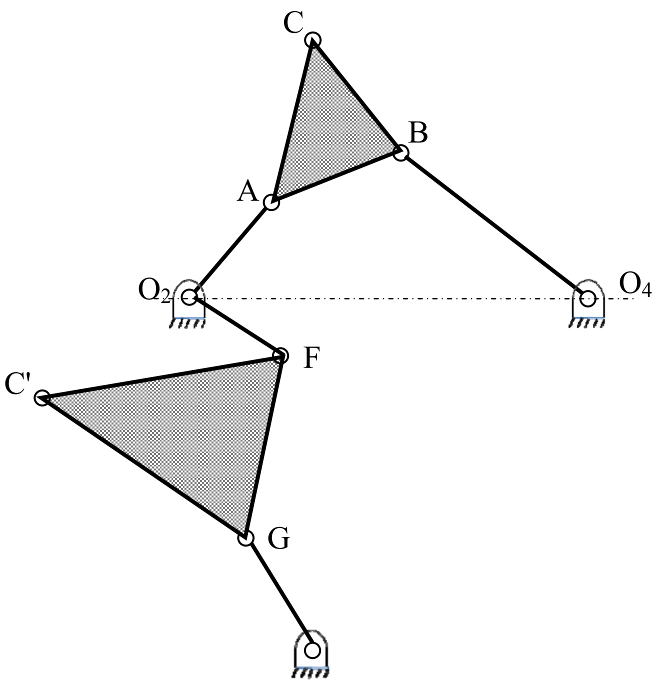


Figure 11: Shifted Right-hand Cognate of Four-bar

The angular velocities of links O_2A and O_2F are the same; thus, these links can be connected to each other rigidly. Additionally, the paths of C and C' are parallel to each other. The distance between these points always remains the same so, a rigid link can be added to connect C and C'.

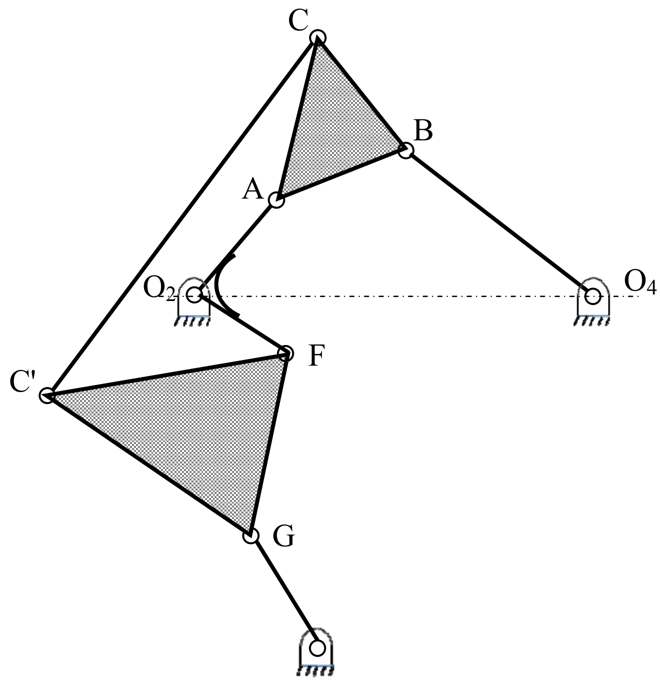


Figure 12: An Overconstrained Mechanism Constructed Using Cognates

Mechanism in figure 12, is an overconstrained mechanism with m=0 according to general DOF formula. The presence of link CC' does not contribute the motion of the mechanism. If this link is removed the mechanism will make exactly the same motion.

Dijksman [3] found that six-bar, 7R, Watt II linkage has a double infinity of function cognates.. Simionescu [16] also reached the same result by using a different technique. Furthermore, Simionescu constituted overconstrained linkages by using 7R Watt II mechanism and its cognates. Similarly, via 3RT3R Watt II mechanism and its cognates, overconstrained mechanisms had been constructed in the same study.

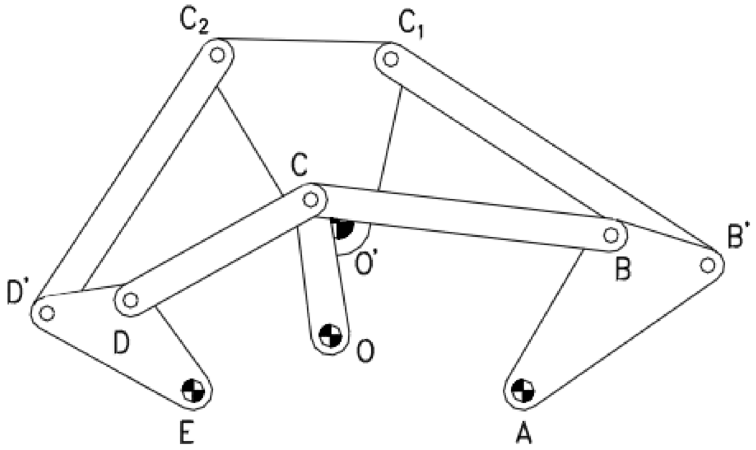


Figure 13: An Overconstrained Mechanism Obtained by 7R Watt II Mechanism with one of its Function Cognates

The overconstrained mechanism in figure 13 is obtained by merging a reference dimensional configuration 7R Watt II mechanism with one of its function cognates [16].

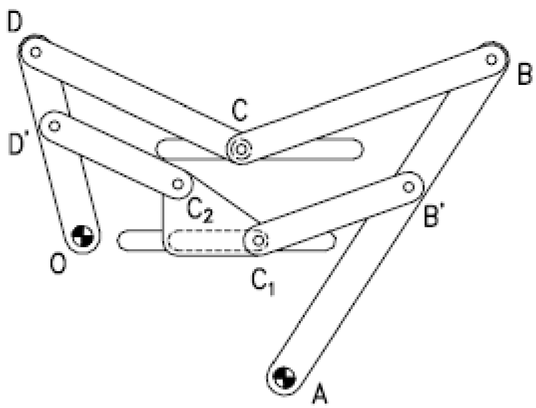


Figure 14: An Overconstrained Mechanism Obtained by 3RT3R Watt II Mechanism with one of its Function Cognates

The overconstrained mechanism in figure 14 is obtained by merging a reference dimensional configuration 3RT3R Watt II mechanism with one of its function cognates [16].

2.3 Method of Combining Identical Modules

Overconstrained mechanisms can be generated by combining identical modules without obstructing the basic motion of the modules. This process may not be feasible for all module types. In order to combine the modules, primarily the basic motion should be studied extensively.

Assume a scissor element seen in figure 13. For regular scissor mechanisms, the intermediate links have three collinear hinges and two of such element are joined at the mid joint such that the resulting regular scissor element is symmetric with respect to the horizontal [9]

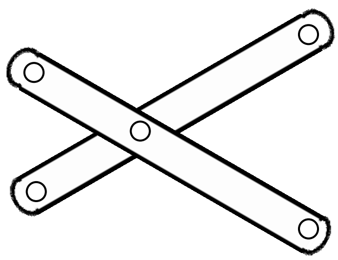


Figure 15: Regular Scissor Element

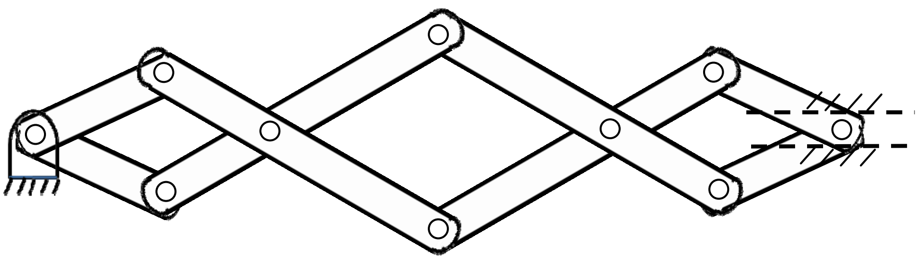


Figure 16: An Overconstrained Mechanism obtained by combining Regular Scissor Elements

By combining regular scissor elements the overconstrained mechanism in figure 16, is obtained. According to general DOF formula m=0, but the actual freedom of the mechanism is $m' = 1$.

When the element is symmetric with respect to the vertical instead of the horizontal the so-called polar scissor element is obtained [9].

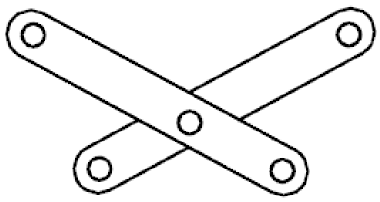


Figure 17: Polar Scissor Element

Combination of polar scissor elements is also possible.

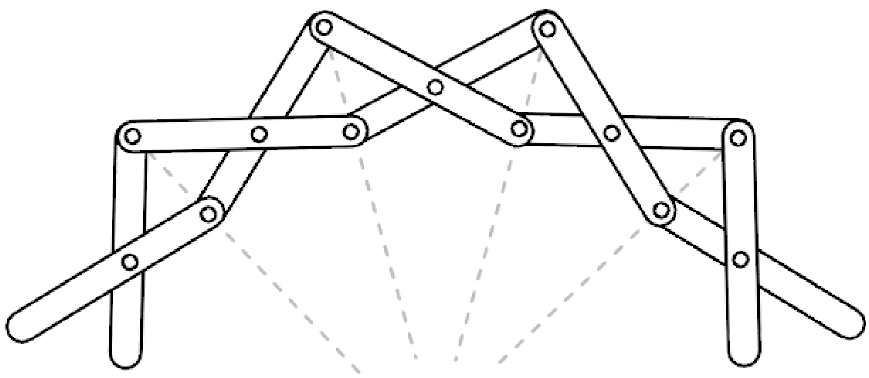


Figure 18: Combination of Polar Scissor Elements

Similar to combined regular scissor elements, this combination can also be used to construct overconstrained mechanisms.

Scissor elements with hinges that are not collinear are called angulated elements and can also be used to construct overconstrained mechanisms.

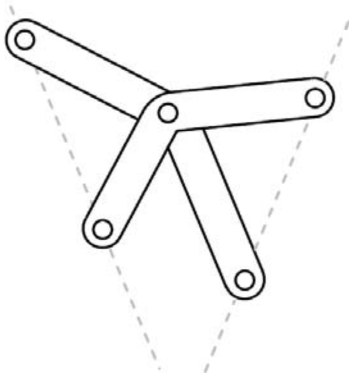


Figure 19: Angulated Scissor Element

Combination of angulated scissor elements is also possible.

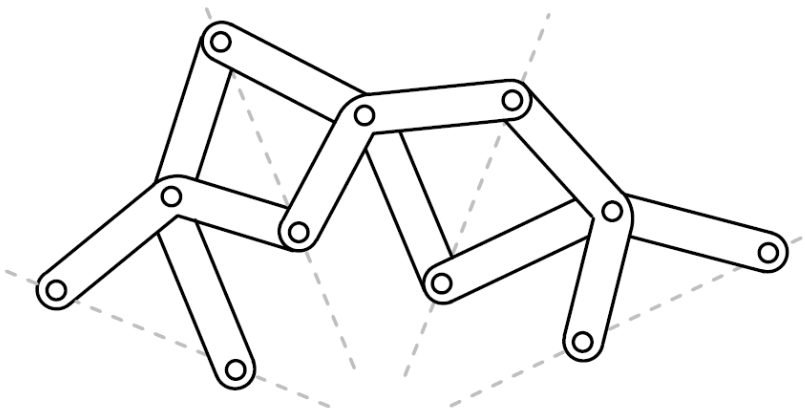


Figure 20: Combination of Angulated Scissor Elements

This combination can also be used to construct overconstrained mechanisms

An overconstrained mechanism can also be generated by combining gear pairs.

In Figure 21 a planetary gear train with one planet gear is shown. A simple gear train consists of a sun gear (S) in the center, a planet gear (P), a planet carrier or arm (C), and an internal or ring gear (R) is called as planetary gear train or epicyclic gear train. In these gear trains, one or more gears are carried on a rotating planet carrier rather than on a shaft that rotates on a fixed axis.

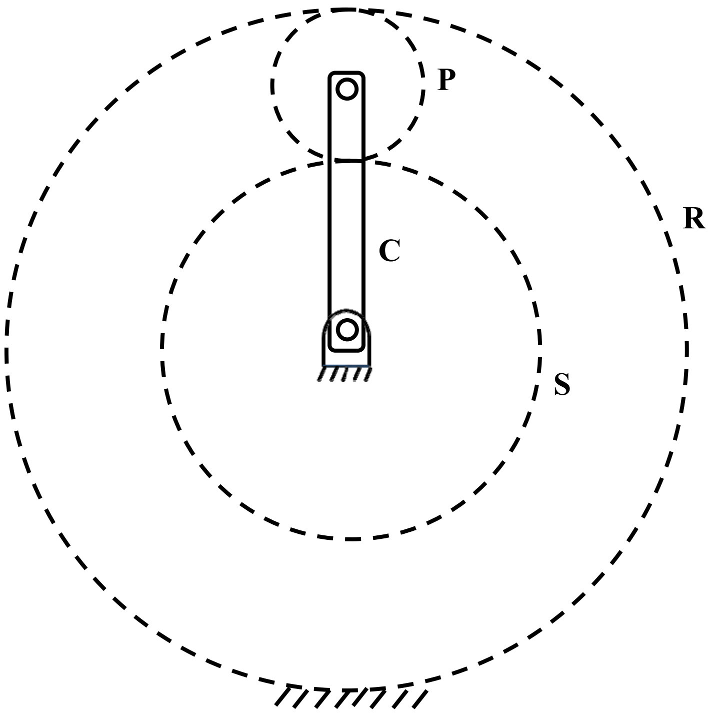


Figure 21: A Planetary Gear Train

The freedom of the planetary gear train shown in figure 21 is 1. This mechanism may become overconstrained by the addition of planet gears.

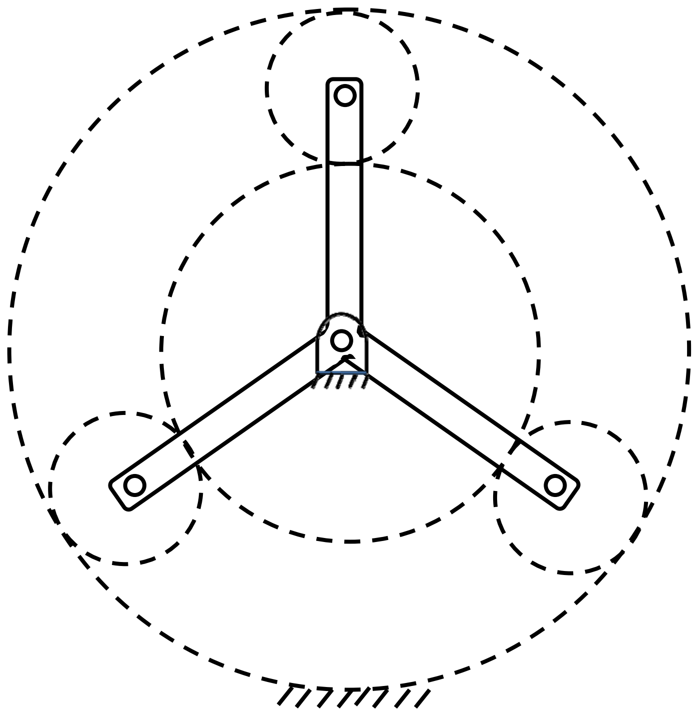


Figure 22: An Overconstrained Planetary Gear Train

In figure 22, two more planet gears are combined to the mechanism symmetrically around the sun gear. Although the symmetry is not necessary, it is used to balance out the centrifugal forces. The additional planet gears are identical to the first one. The motion of the basic mechanism, mechanism in figure 21, does not change. For large force transmission this combined planetary gear train is preferred as it has better force transmission characteristics.

The degree of freedom of the overconstrained planetary gear train is m=-1 by the general DOF formula but the actual freedom is still $m' = 1$. Although this planetary gear train is overconstrained, it is mobile. By addition of more planets the mechanism may become more and more overconstrained.

2.4 Method of Extending an Overconstrained Mechanism with Extra Links

Once an overconstrained mechanism is generated it can be extended in order to obtain various other overconstrained mechanisms. In other words, by adding extra links and joints to an already existing overconstrained mechanism, a new overconstrained mechanism can be generated.

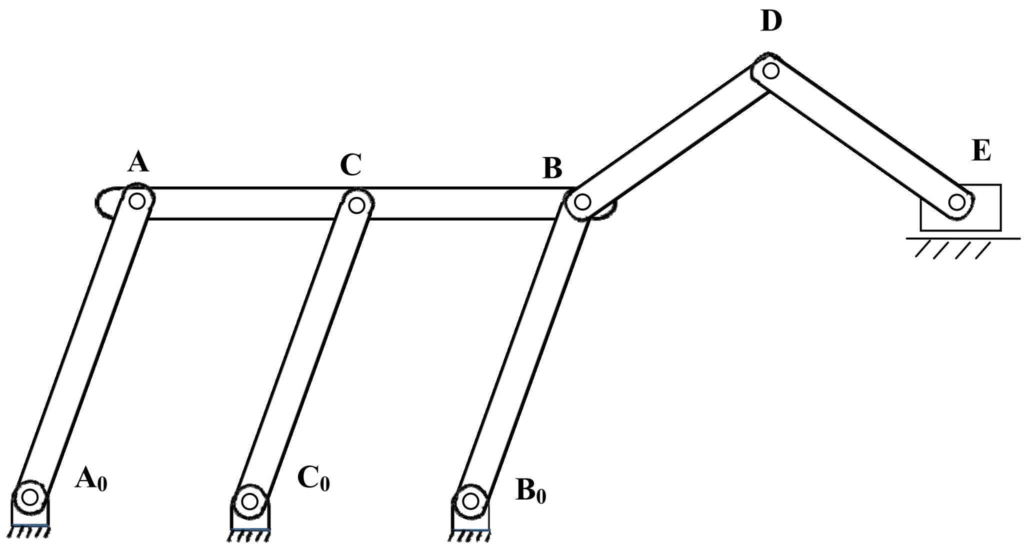


Figure 23: A Mobile Overconstrained Extended Parallelogram

In figure 23, an overconstrained parallelogram mechanism (A_0ABB_0) is extended with a slider crank mechanism (BDE). The actual DOF of the system is $m' = 2$. However, the general DOF formula gives m=1.

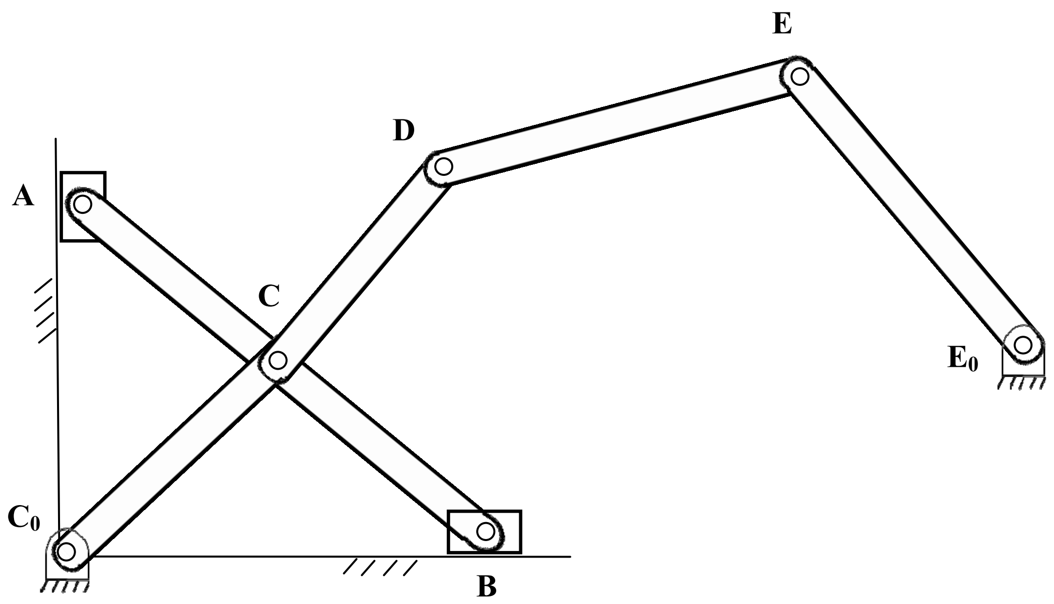


Figure 24: A Mobile Overconstrained Extended Double Slider

In figure 24, a overconstrained double slider mechanism (AC_0B) is extended with a four bar mechanism ($CDEE_0$). The actual DOF of the system is $m' = 2$. However, the general DOF formula again gives m=1.

CHAPTER 3

FORCE ANALYSIS

A mechanism is designed to transmit force and motion. Firstly a motion characteristic of a mechanism is designated, and then the sizes and shapes of its links and joints are determined to finalize the design. To do this, information about the forces and moments acting on the links of mechanisms is required.

In the force analysis of mechanisms, if they do not move at very high speeds, they can be assumed to be in pseudo-static equilibrium. In other words, in the pseudo-static force analysis, the inertia forces and moments are not considered because of their negligible values. Thus, this analysis method is based on writing the static equilibrium equations. Free body diagrams are used to indicate all the known and unknown forces and moments clearly. For each body, free body diagrams are drawn and unknowns are identified. Then, the static equilibrium equations are written for all the moving links. The total number of unknown forces and moments should be equal to the number of static force equilibrium equations in order to find the unknowns. In a regular mechanism, which is not overconstrained, this is the case. That is, the number of unknowns is equal to the number of static force equilibrium equations. Therefore, all of the unknowns can be determined.

On the other hand, in overconstrained mechanisms total number of unknown forces and moments is always more than the number of static force equilibrium equations.

Therefore, performing force analysis in overconstrained mechanisms is troublesome. The difference between the number of unknowns and equations is equal to the difference between actual mobility of the mechanism and mobility calculated by general DOF formula. By using stress-strain relationships it is possible to increase the number of equations and obtain a solution. However, this process needs intensive effort and consumes too much time. On the other hand, all unknowns can be written in terms of minimum number of unknowns and by giving a range of values to this minimum number of unknowns it is possible to see the relationship between the joint reaction forces and moments. This practically accessed knowledge can be used in design phase of the mechanism. In this part of the study a systematic procedure for the mentioned approach is described and also applied to some sample mechanisms.

If m is the number of static equilibrium equations and n is the total number of unknown forces and moments, all other unknowns can be written in terms of any n-m number of unknowns. For example, if there are five static equilibrium equations but seven unknowns, all other unknowns can be written in terms of any two of the unknowns. It is not possible to describe all unknowns with less than two unknowns. So the minimum number of unknowns in terms of which other unknowns can be written is two. A range of values can be given to the unknowns in terms of which other unknowns are written. So that, the values of all other unknowns correspond to the given values can be calculated. This process helps the designer to perceive the relationship between the unknowns. Moreover, the influence of change in the input can also be noticed via this process. The designer can use this knowledge in design phase to construct a more reliable overconstrained mechanism. It is possible to observe under which circumstances all the joint reaction forces/moments take smaller values and accordingly to decide giving preloading to some of the links while assembling the mechanism. Most probably all the forces/moments will not take their smallest values simultaneously. In such a case, an optimization should be performed to decide most suitable assembling mode.

In this thesis, the method described for the force analysis of overconstrained mechanisms is based on three main assumptions:

i) The preloaded link is so less stiff than the other links that it can assumed as the only non-rigid link in the mechanism.

ii) The clearances in the joints are negligible.

iii) The preloaded link should be a two force member, based on the assumption that its inertia effects are negligible.

When a link is assembled with preloading (compression or tension) it will act as a spring. If the link is assembled with initial compression, it will act as a compression spring; if it is assembled in tension it will act as tension/extension spring. For springs Hooke's law states that;

$$F = -kx$$

x is the displacement of the spring's end from its equilibrium position

F is the restoring force exerted by the spring on that end

k is the spring constant

As the links are rigid, the distance between the two end points of the preloaded link is constant. Therefore, the displacement of the so called spring's end is also constant. This result in constant force on the spring. Via this process one of the potential unknowns can be eliminated previously.

To clarify mentioned force analysis method five examples are given. Different loading and operating conditions of similar mechanism is also discussed to emphasize their effects on force analysis.

3.1 Example I: A Parallelogram Mechanism (an overconstrained mechanism with 5 links and 6 revolute joints) Loaded with Torque T_{12}

As mentioned in Chapter 2.1 this parallelogram mechanism is an overconstrained mechanism with actual mobility $m' = 1$. while mobility obtained from general DOF formula is m=0.

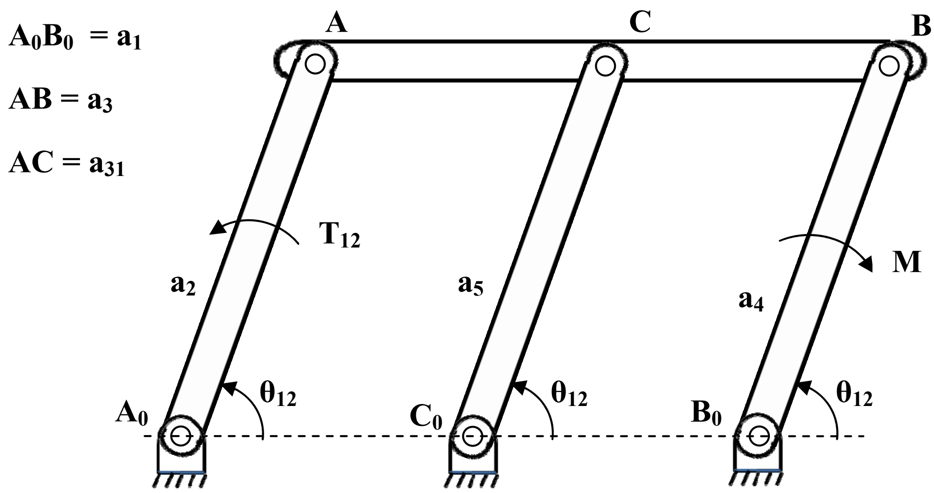


Figure 25: An Overconstrained Parallelogram Mechanism Loaded with Torque

$a_2 = a_4 = a_5 = 80 \text{ mm}$

$a_1 = a_3 = 120 \text{ mm}$

$a_{31} = 60 \text{ mm}$

$T_{12} = 300 \text{ Nmm}$

In this case, the mechanism is designed to transmit moment. The input torque is T_{12}. while the output torque is M. The operation range of θ_{12} is assumed to be 0° to 90°

The free body diagrams and static equilibrium equations are given.

Link 2: Two force & a moment member

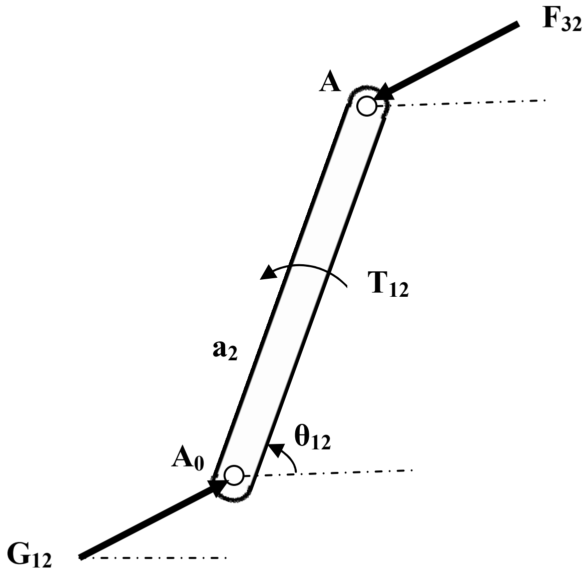


Figure 26: Free Body Diagram of link 2 belongs to the Mechanism in Figure 25

$$G_{12} = -F_{32}$$

$$-T_{12} - F_{32,x} \sin\theta_{12}\, a_2 + F_{32,y} \cos\theta_{12} a_2 = 0 \ \ (\sum M_{A_0} = 0) \qquad (3.1.1)$$

where $F_{32,x}$ is the x component of the force F_{32} while $F_{32,y}$ is the y component of the same force.

Link 3: Three force member

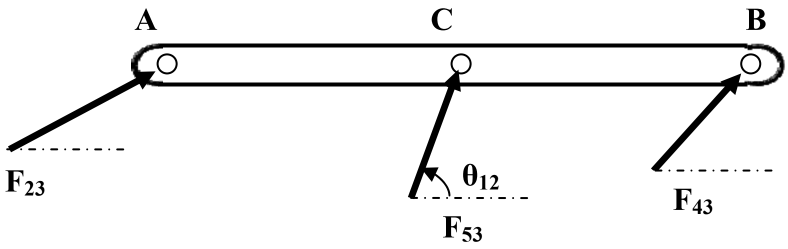


Figure 27: Free Body Diagram of link 3 belongs to the Mechanism in Figure 25

$$F_{23,x} + F_{53}\cos\theta_{12} + F_{43,x} = 0 \quad (\sum F_x = 0) \tag{3.1.2}$$

$$F_{23,y} + F_{53}\sin\theta_{12} + F_{43,y} = 0 \quad (\sum F_y = 0) \tag{3.1.3}$$

$$-F_{53}\sin\theta_{12}\, a_{31} - F_{43,y}\, a_3 = 0 \quad (\sum M_A = 0) \tag{3.1.4}$$

where $F_{23,x}$ is the x component of the force F_{23} while $F_{23,y}$ is the y component of the same force and $F_{43,x}$ is the x component of the force F_{43} while $F_{43,y}$ is the y component of the same force.

Link 4: Two force & a moment member

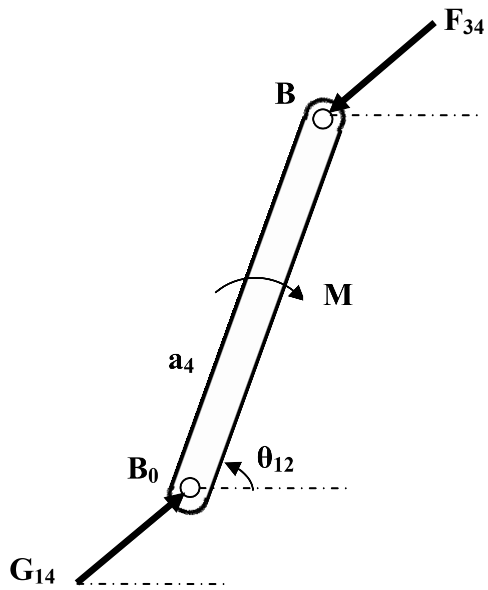


Figure 28: Free Body Diagram of link 4 belongs to the Mechanism in Figure 25

$G_{14} = -F_{34}$

$$M - F_{34,x}\sin\theta_{12}\,a_4 + F_{34,y}\cos\theta_{12}a_4 = 0 \ \ (\sum M_{B_0} = 0) \qquad (3.1.5)$$

where $F_{34,x}$ is the x component of the force F_{34} while $F_{34,y}$ is the y component of the same force.

Link 5: Two force member

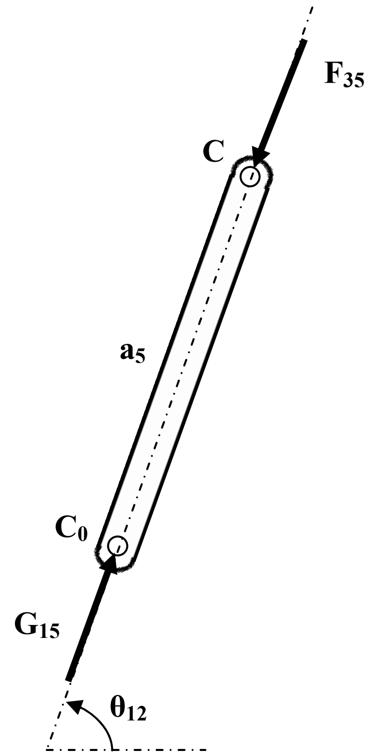


Figure 29: Free Body Diagram of link 5 belongs to the Mechanism in Figure 25

$G_{15} = -F_{35}$

Due to action-reaction (Newton's third law):

$F_{23} = -F_{32}$

$F_{53} = -F_{35}$

$F_{43} = -F_{34}$

Unknowns; $F_{23,x}$, $F_{23,y}$, F_{53} , $F_{43,x}$, $F_{43,y}$, M

Equations;

$$-T_{12} - F_{32,x} \sin\theta_{12}\, a_2 + F_{32,y} \cos\theta_{12} a_2 = 0 \tag{3.1.1}$$

$$F_{23,x} + F_{53} \cos\theta_{12} + F_{43,x} = 0 \tag{3.1.2}$$

$$F_{23,y} + F_{53} \sin\theta_{12} + F_{43,y} = 0 \tag{3.1.3}$$

$$-F_{53} \sin\theta_{12}\, a_{31} - F_{43,y}\, a_3 = 0 \tag{3.1.4}$$

$$M - F_{34,x} \sin\theta_{12}\, a_4 + F_{34,y} \cos\theta_{12} a_4 = 0 \tag{3.1.5}$$

The number of unknowns is six while the number of equations is five. As the unknown quantity is more than the equations, the solution cannot be obtained. All other unknowns can be written in terms of one of the unknown. Link 5 is the only two force member in this mechanism; therefore, all other unknowns are written in terms of F_{53}.

$$F_{43,y} = -\frac{F_{53} \sin\theta_{12}\, a_{31}}{a_3}$$

$$F_{34,y} = -F_{43,y}$$

$$F_{23,y} = -F_{43,y} - F_{53} \sin\theta_{12}$$

$$F_{32,y} = -F_{23,y}$$

$$F_{32,x} = \frac{F_{32,y} \cos\theta_{12} a_2 - T_{12}}{\sin\theta_{12}\, a_2}$$

$$F_{23,x} = -F_{32,x}$$

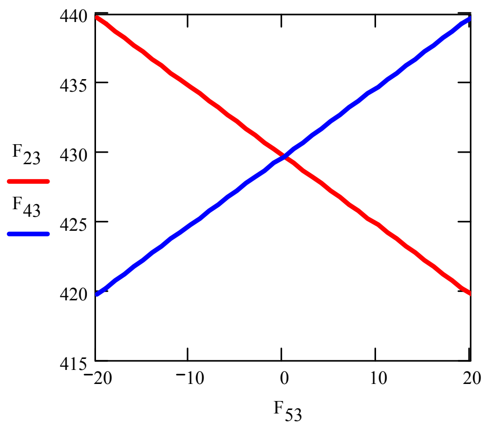
$$F_{43,x} = -F_{23,x} - F_{53} \cos\theta_{12}$$

$$F_{34,x} = -F_{43,x}$$

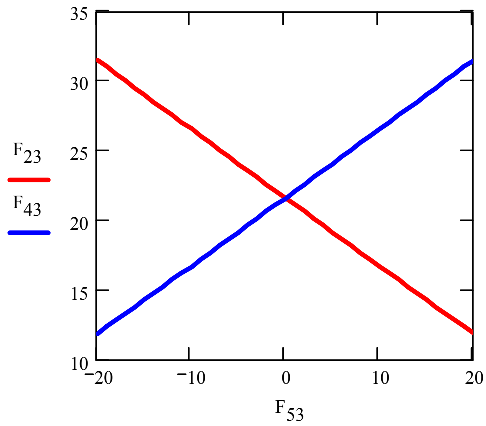
$$M = F_{34,x} \sin\theta_{12}\, a_4 - F_{34,y} \cos\theta_{12} a_4$$

By using virtual work method M value can be calculated. The output moment is equal to the input moment and it is independent of the θ_{12} value.

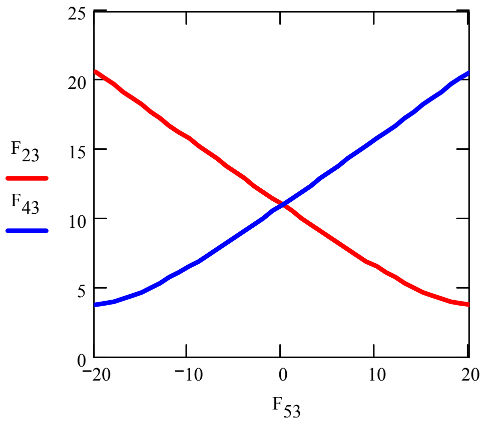
For a given range of F_{53} values (-20 N to 20 N) F_{53} versus F_{43} and F_{23} graphs are drawn in figure 30 for different θ_{12} values with a 10° increment. It should be noted that as at θ_{12}=0° singularity occurs the first graph is drawn at θ_{12}=0.5°



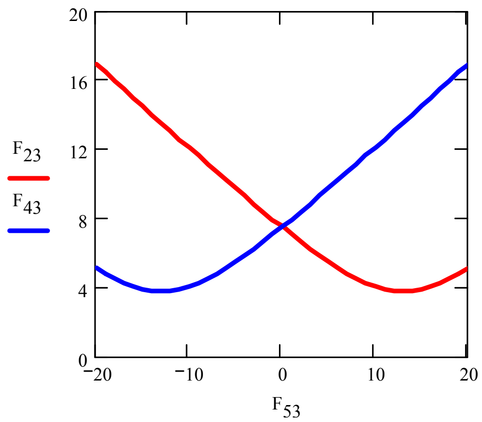
θ_{12}=0.5



θ_{12}=10°



θ_{12}=20°



θ_{12}=30°

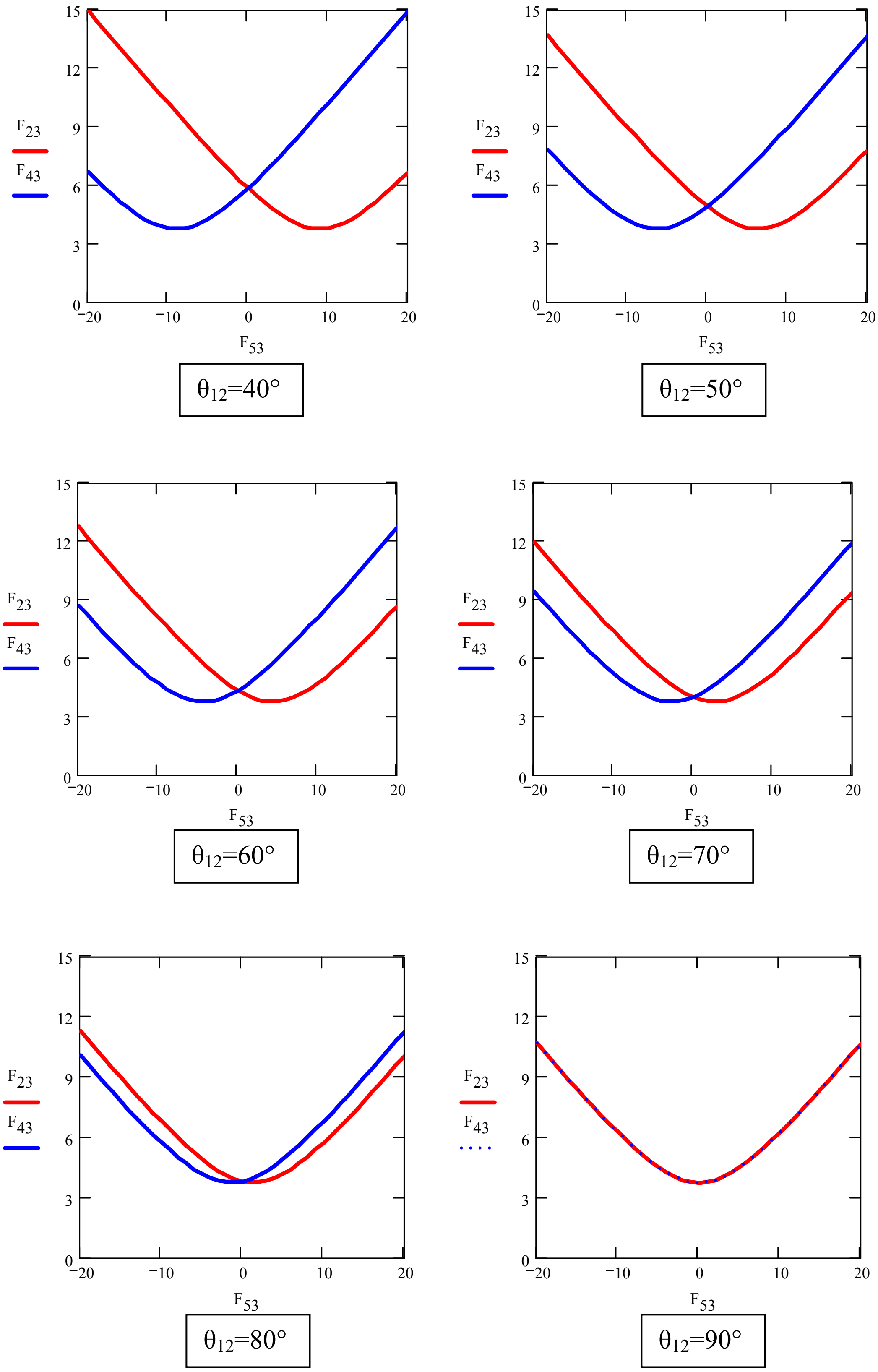


Figure 30: F_{23} and F_{43} versus F_{53} Graphs for Different θ_{12} Values of Example 3.1

Figure 30 shows that F_{23} is the symmetric of F_{43} with respect to $F_{53}=0$ axis. Therefore, abs(F_{23}) values for negative F_{53} are replaced with abs(F_{43}) values for positive F_{53} and vice versa. It should be noted that the positive values of F_{53} imply the direction of this force is as given in free body diagrams and the negative values imply the direction is just the opposite. In other words, the positive values of F_{53} show that link 5 is in compression and the negative values of F_{53} show link 5 is in tension.

In this case in order to progress, the designer should determine whether link 2 or link 4 is more critical. For example, if link 2 is more critical for some reason, the designer can decide to assemble link 5 with initial compression because F_{23} take smaller values for positive F_{53} values and the positive values of F_{53} imply that link 5 is in compression. Now the problem is demoted to decide the value of the initial compression. For the minimum values of F_{23}, F_{43} takes relatively higher values especially up to $\theta_{12}=60°$ as shown in figure 30. Choosing F_{53} values that make F_{23} minimum may not be an appropriate approach especially for smaller θ_{12} values. In order to restrain F_{43} at acceptable values, selecting F_{53} value between zero and giving minimum F_{23} is more suitable. The table 1 is arranged accordingly by using MathCAD. The values for $\theta_{12}=0°$ and $\theta_{12}=90°$ are accepted as extremums and disregarded in order to get more reliable results.

Table 1: Minimum Forces for Example 3.1

θ_{12}	F_{53} (N)	abs(F_{23}) (N)	abs(F_{43}) (N)
10°	6	18,648	24,555
20°	6	8,210	13,821
30°	6	5,126	10,209
40°	5	4,236	7,914
50°	3	4,096	5,971
60°	2	3,927	4,907
70°	3	3,752	4,719
80°	1	3,754	3,926

Now the final F_{53} value can be calculated by using the equation:

$$F_{53} = \frac{\sum_{i=1}^{n=8} (F_{53})_i}{n}$$

$$F_{53} = \frac{(F_{53})_1 + (F_{53})_2 + (F_{53})_3 + (F_{53})_4 + (F_{53})_5 + (F_{53})_6 + (F_{53})_7 + (F_{53})_8}{8}$$

$$F_{53} = 4 \text{ N}$$

As a result, link 5 can be assembled with 4 N initial compression. The F_{23} and F_{43} values for F_{53}=4 N needed to be checked before finalizing the design. If corresponding F_{23} and F_{43} values are acceptable the design can be finalized.

Table 2: Final Forces for Example 3.1

θ_{12}	F_{53} (N)	abs(F_{23}) (N)	abs(F_{43}) (N)
10°	4	19,629	23,568
20°	4	9,111	12,862
30°	4	5,854	9,286
40°	4	4,490	7,477
50°	4	3,921	6,368
60°	4	3,754	5,604
70°	4	3,803	5,038
80°	4	3,982	4,598

3.2 Example II: A Parallelogram Mechanism (an overconstrained mechanism with 5 links and 6 revolute joints) Loaded with Force F

$A_0B_0 = a_1$

$AB = a_3$

$AC = a_{31}$

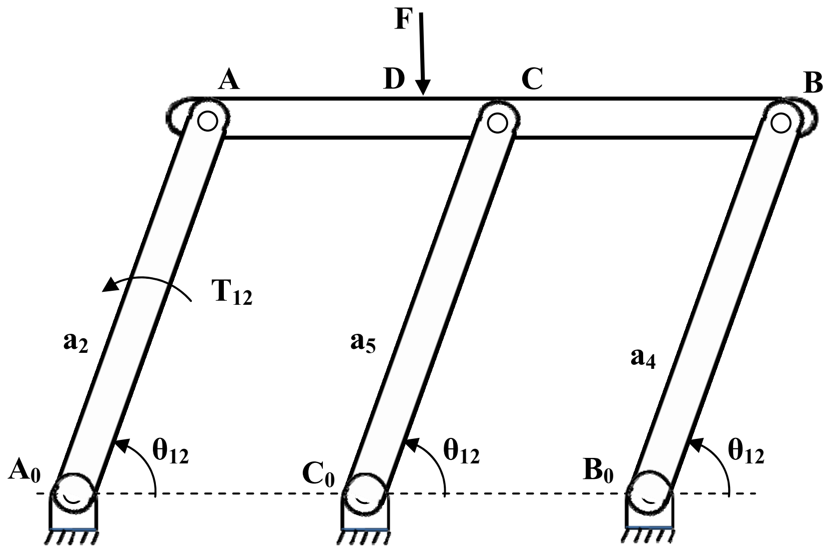


Figure 31: An Overconstrained Parallelogram Mechanism Loaded with an External Force

$a_2 = a_4 = a_5 = 80 \text{ mm}$

$a_1 = a_3 = 120 \text{ mm}$

$F = 5 \text{ N}$

$a_{31} = 60 \text{ mm}$

$AD = 45 \text{ mm}$

In this case, the mechanism is designed to carry load F. The input is torque $T_{12.}$ The operation range of θ_{12} is assumed to be 0° to 90°

The free body diagrams and static equilibrium equations are given.

Link 2: Two force & a moment member

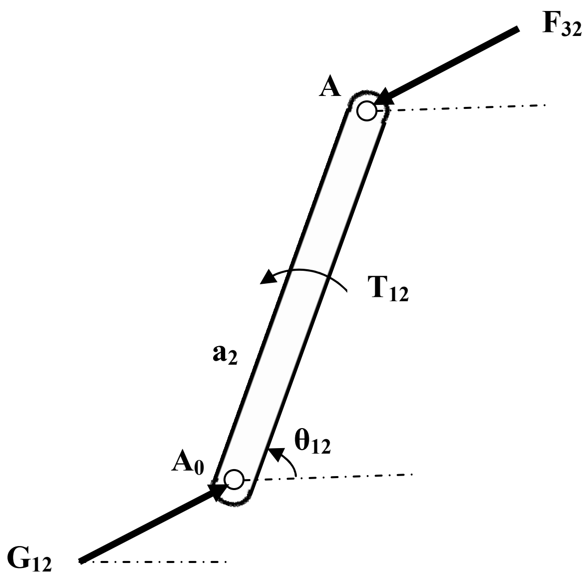


Figure 32: Free Body Diagram of link 2 belongs to the Mechanism in Figure 31

$$G_{12} = -F_{32}$$

$$-T_{12} - F_{32,x} \sin\theta_{12}\, a_2 + F_{32,y} \cos\theta_{12} a_2 = 0 \quad (\sum M_{A_0} = 0) \tag{3.2.1}$$

where $F_{32,x}$ is the x component of the force F_{32} while $F_{32,y}$ is the y component of the same force.

Link 3: Three force member

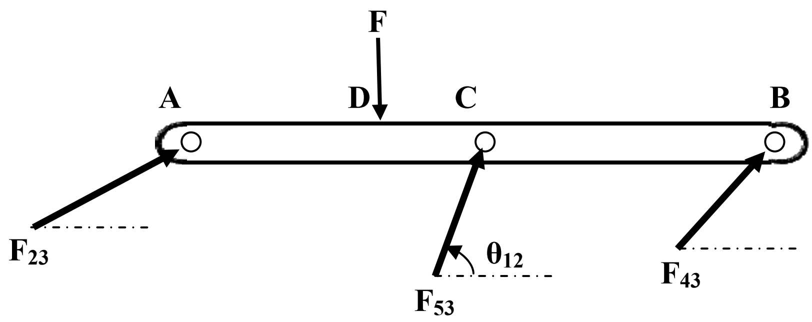


Figure 33: Free Body Diagram of link 3 belongs to the Mechanism in Figure 31

$$F_{23,x} + F_{53} \cos\theta_{12} + F_{43} \cos\theta_{12} = 0 \quad (\sum F_x = 0) \tag{3.2.2}$$

$$F_{23,y} + F_{53} \sin\theta_{12} + F_{43} \sin\theta_{12} - F = 0 \quad (\sum F_y = 0) \tag{3.2.3}$$

$$-F_{53} \sin\theta_{12}\, a_{31} - F_{43} \sin\theta_{12}\; a_3 + F\,\frac{3a_{31}}{4} = 0 \quad (\sum M_A = 0) \tag{3.2.4}$$

where $F_{23,x}$ is the x component of the force F_{23} while $F_{23,y}$ is the y component of the same force.

Link 4: Two force member

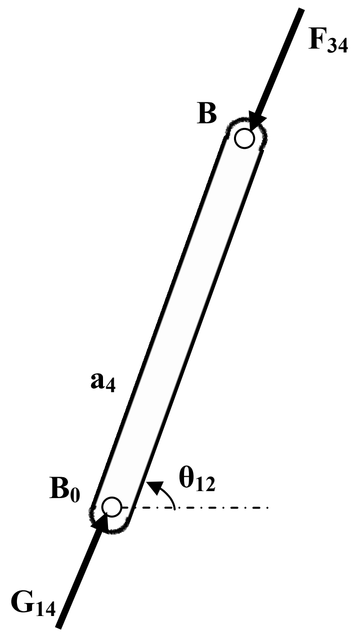


Figure 34: Free Body Diagram of link 4 belongs to the Mechanism in Figure 31

$$G_{14} = -F_{34}$$

Link 5: Two force member

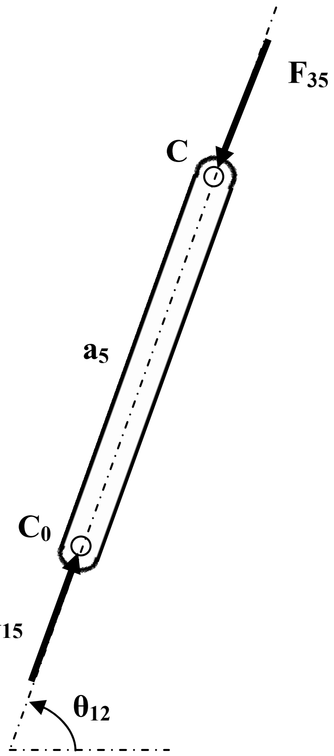


Figure 35: Free Body Diagram of link 5 belongs to the Mechanism in Figure 31

$G_{15} = -F_{35}$

Due to action-reaction (Newton's third law):

$F_{23} = -F_{32}$

$F_{53} = -F_{35}$

$F_{43} = -F_{34}$

Unknowns; $T_{12}, F_{23,x}, F_{23,y}, F_{53}, F_{43}$

Equations;

$$-T_{12} - F_{32,x} \sin\theta_{12} a_2 + F_{32,y} \cos\theta_{12} a_2 = 0 \tag{3.2.1}$$

$$F_{23,x} + F_{53} \cos\theta_{12} + F_{43} \cos\theta_{12} = 0 \tag{3.2.2}$$

$$F_{23,y} + F_{53} \sin\theta_{12} + F_{43} \sin\theta_{12} - F = 0 \tag{3.2.3}$$

$$-F_{53} \sin\theta_{12} a_{31} - F_{43} \sin\theta_{12}\ a_3 + F\frac{3a_{31}}{4} = 0 \tag{3.2.4}$$

The number of unknowns is five while the number of equations is four. As the unknown quantity is more than the equations, the solution cannot be obtained. All other unknowns can be written in terms of one of the unknown. Link 5 is a two force member and in this example all other unknowns are written in terms of F_{53}.

$$F_{43} = \frac{F\frac{3a_{31}}{4} - F_{53} \sin\theta_{12} a_{31}}{\sin\theta_{12} a_3}$$

$$F_{23,y} = F - F_{53} \sin\theta_{12} - F_{43} \sin\theta_{12}$$

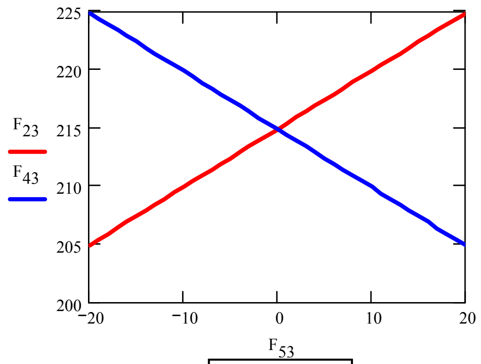
$$F_{32,y} = -F_{23,y}$$

$$F_{23,x} = -F_{53} \cos\theta_{12} - F_{43} \cos\theta_{12}$$

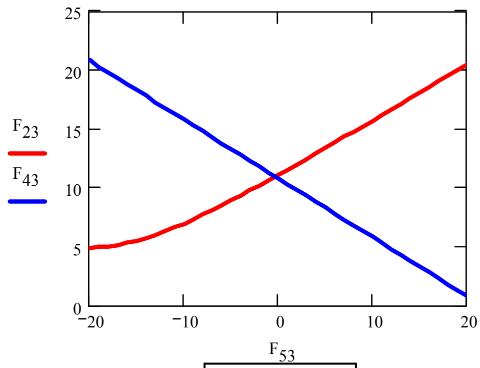
$$F_{32,x} = -F_{23,x}$$

$$T_{12} = -F_{32,x} \sin\theta_{12} a_2 + F_{32,y} \cos\theta_{12} a_2$$

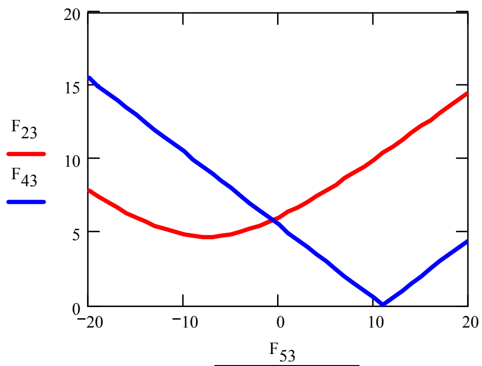
For a given range of F_{53} values (-20 N to 20 N) F_{53} versus F_{43} and F_{23} graphs are drawn in figure 36 for different θ_{12} values with a 10° increment. It should be noted that as at θ_{12}=0° singularity occurs the first graph is drawn at θ_{12}=0.5°



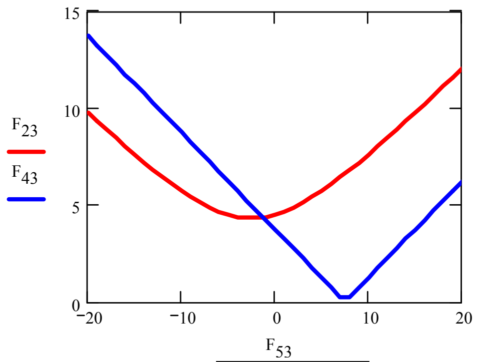
θ_{12}=0.5



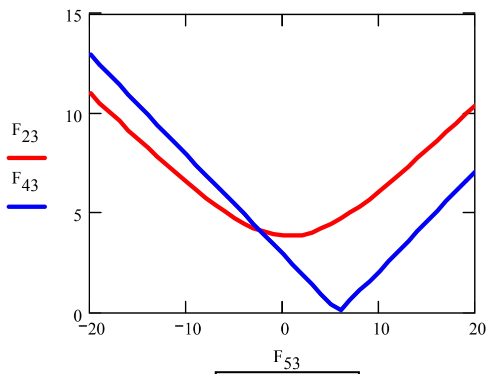
θ_{12}=10°



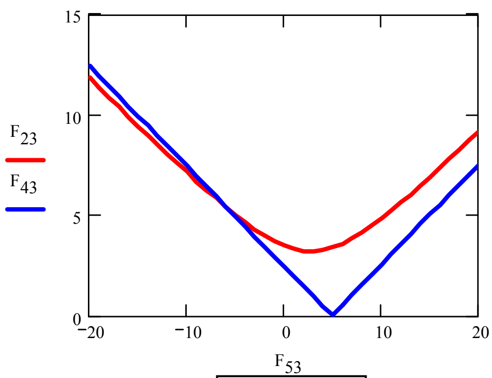
θ_{12}=20°



θ_{12}=30°



θ_{12}=40°



θ_{12}=50°

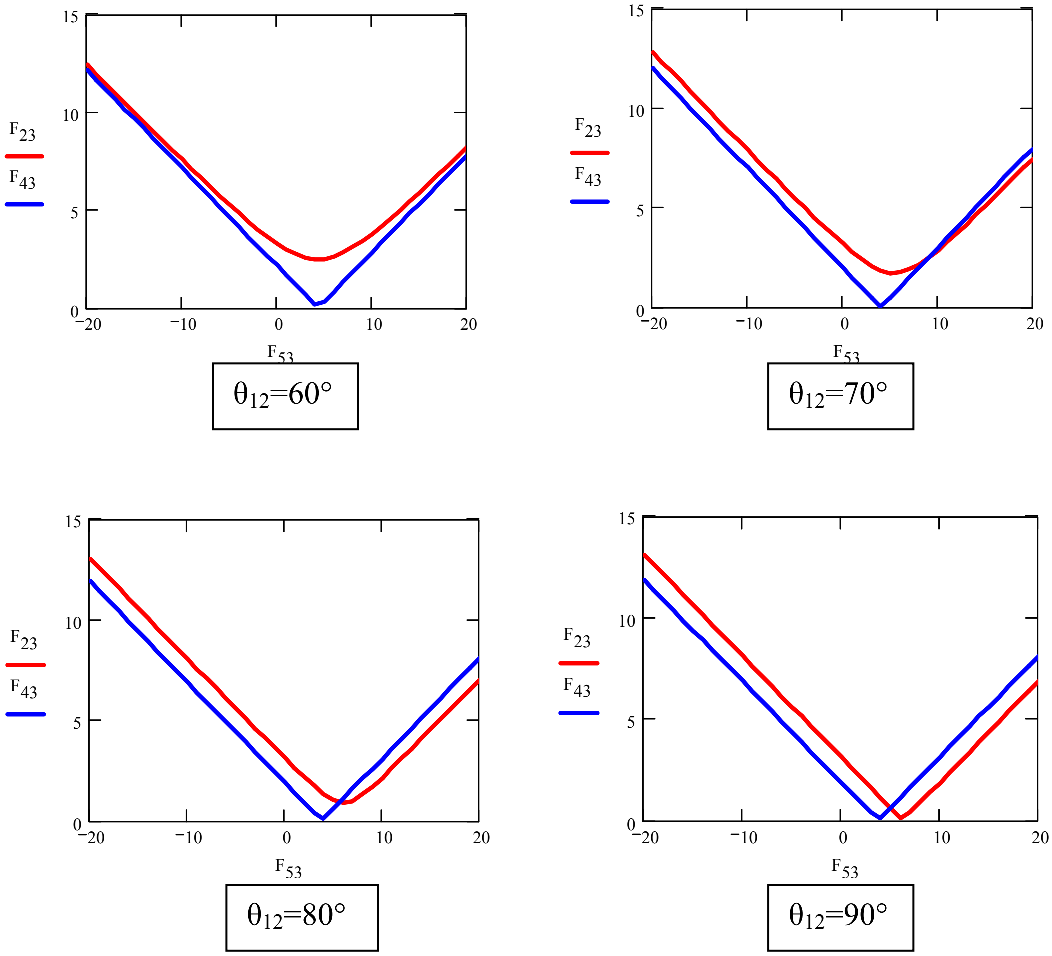


Figure 36: F_{23} and F_{43} versus F_{53} Graphs for Different θ_{12} Values of Example 3.2

Figure 36 shows that there is no symmetry between F_{23} and F_{43} in this case. For higher θ_{12} values it is obviously seen from the graphs that for positive F_{53} values both F_{23} and F_{43} take their minimum values. The positive values of F_{53} imply the direction of this force is as given in free body diagrams and the negative values imply the direction is just the opposite. In other words, the positive values of F_{53} show that link 5 is in compression and the negative values of F_{53} show link 5 is in tension.

In this case, the designer already knows to assemble link 5 with initial compression. Again the problem is demoted to decide the value of the initial compression. If, for some reason, one of the link is more critical than the other, the minimum values of force on that link should be focused. If the links are equally critical, values between their minimum cases should be considered.

Assuming link 2 and link 4 are equally critical an initial compression value i.e. F_{53}, between F_{53} value corresponding to minimum F_{23} and F_{53} value corresponding to minimum F_{43} should be decided.

$$x = \sqrt{(\min(F_{23}) - F_{23})^2 + (\min(F_{43}) - F_{43})^2}$$

For each θ_{12} values x values are calculated and minimum x value is determined. For minimum x values corresponding F_{53}, F_{23} and F_{43} values are tabulated in table 3. Similar to example 3.1 the values for θ_{12}=0° and θ_{12}=90° are accepted as extremes and disregarded in order to get more reliable results.

Table 3: Minimum Forces for Example 3.2

θ_{12}	F_{53}	abs(F_{23})	abs(F_{43})
10°	5	13,369	8,298
20°	6	8,242	2,482
30°	5	5,728	1,250
40°	5	4,419	0,417
50°	5	3,403	0,052
60°	4	2,505	0,165
70°	4	1,849	0,005
80°	4	1,340	0,096

Similar to former example, the final F_{53} value can be calculated by using the equation:

$$F_{53} = \frac{\sum_{i=1}^{n=8} (F_{53})_i}{n}$$

$$F_{53} = \frac{(F_{53})_1 + (F_{53})_2 + (F_{53})_3 + (F_{53})_4 + (F_{53})_5 + (F_{53})_6 + (F_{53})_7 + (F_{53})_8}{8}$$

$F_{53} = 4{,}75$ N

As a result, link 5 can be assembled with 4,75 N initial compression. The F_{23} and F_{43} values for F_{53}=4,75 N needed to be checked before finalizing the design. If corresponding F_{23} and F_{43} values are acceptable the design can be finalized.

Table 4: Final Forces for Example 3.2

θ_{12}	F_{53} (N)	abs(F_{23}) (N)	abs(F_{43}) (N)
10°	4,75	13,253	8,423
20°	4,75	7,737	3,107
30°	4,75	5,647	1,375
40°	4,75	4,358	0,542
50°	4,75	3,364	0,073
60°	4,75	2,509	0,210
70°	4,75	1,741	0,380
80°	4,75	1,082	0,471

3.3 Example III: A Double Slider Mechanism (an overconstrained mechanism with 5 links and 4 revolute and 2 prismatic joints) Loaded with Force F_{14}

As mentioned in Chapter 2.1 this double slider mechanism is an overconstrained mechanism with actual mobility $m' = 1$. while mobility obtained from Grübler formula is m=0.

$$\mathbf{C_0C = a_5}$$

$$\mathbf{AB = a_3}$$

$$\mathbf{AC = a_{31}}$$

$$\mathbf{\frac{AB}{2} = AC = C_0C}$$

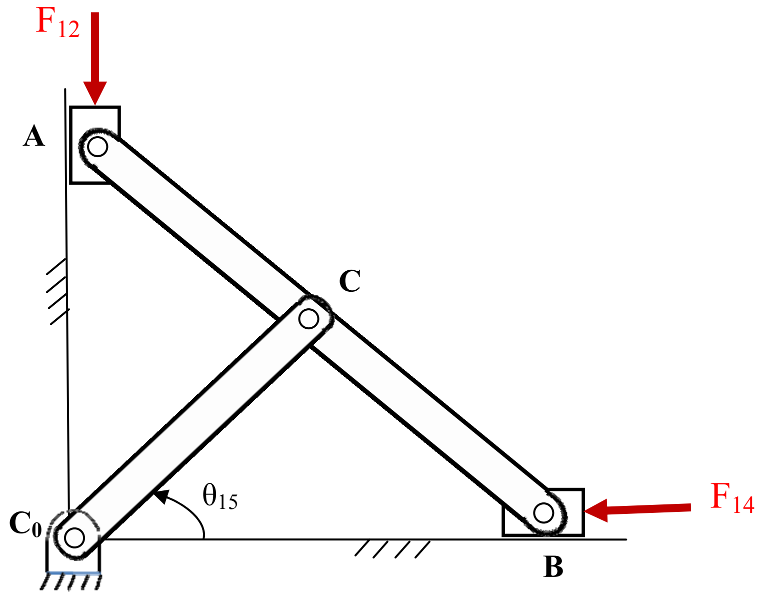


Figure 37: An Overconstrained Double Slider Mechanism Loaded with Force F_{14}

$F_{14} = 5$ N

$a_5 = 60$ mm

$a_3 = 120$ mm

In this case the mechanism is designed to transmit force. The mechanism is in static equilibrium with two external forces. The operation range of θ_{12} is assumed to be 0° to 90°

The free body diagrams and static equilibrium equations are given.

Link 2: Three force member

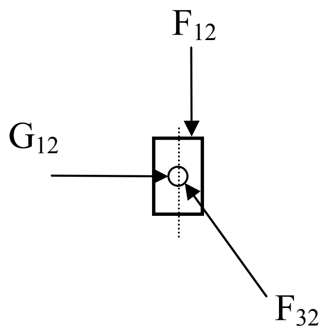


Figure 38: Free Body Diagram of link 2 belongs to the Mechanism in Figure 37

$$G_{12} - F_{32,x} = 0 \tag{3.3.1}$$

$$-F_{12} + F_{32,y} = 0 \tag{3.3.2}$$

where $F_{32,x}$ is the x component of the force F_{32} while $F_{32,y}$ is the y component of the same force.

Link 5: Two force member

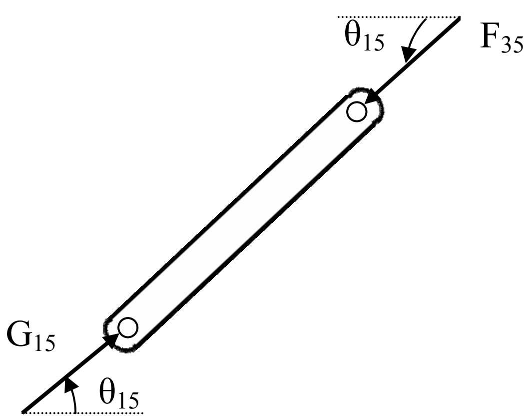


Figure 39: Free Body Diagram of link 5 belongs to the Mechanism in Figure 37

$G_{15} = -F_{35}$

Link 3: Three force member

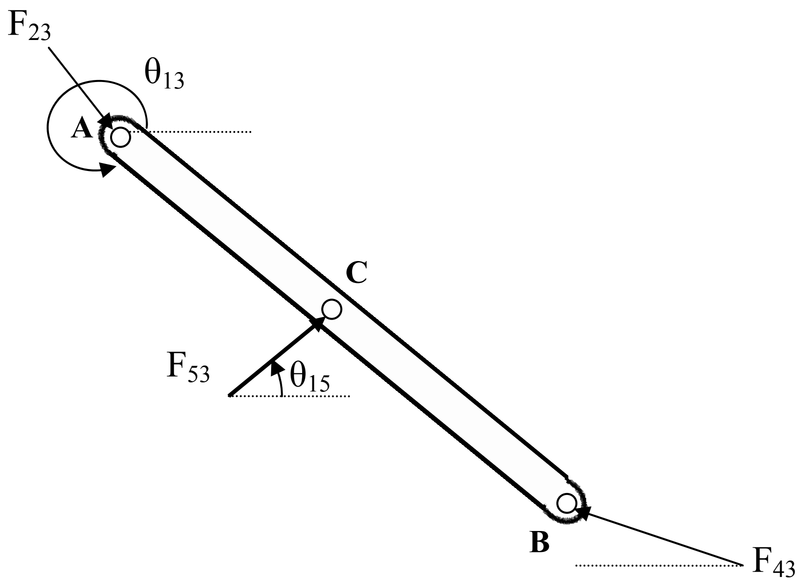


Figure 40: Free Body Diagram of link 3 belongs to the Mechanism in Figure 37

$$-F_{23,y} + F_{53} \sin \theta_{15} + F_{43,y} = 0 \tag{3.3.3}$$

$$F_{23,x} + F_{53} \cos \theta_{15} - F_{43,x} = 0 \tag{3.3.4}$$

$$-F_{53} \sin \theta_{15} \cos(2\pi - \theta_{13}) a_{31} - F_{53} \cos \theta_{15} \sin(2\pi - \theta_{13}) a_{31}$$

$$-F_{43,y} \cos(2\pi - \theta_{13}) a_3 + F_{43,x} \sin(2\pi - \theta_{13})\, a_3 = 0 \tag{3.3.5}$$

where $F_{23,x}$ is the x component of the force F_{23} while $F_{23,y}$ is the y component of the same force and $F_{43,x}$ is the x component of the force F_{43} while $F_{43,y}$ is the y component of the same force.

Link 4: Three force member

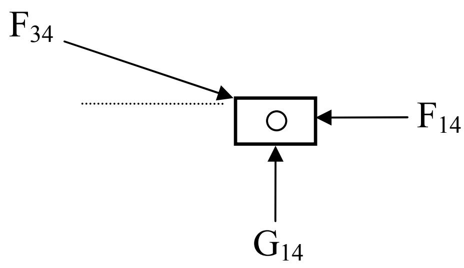


Figure 41: Free Body Diagram of link 4 belongs to the Mechanism in Figure 37

$$G_{14} - F_{34,y} = 0 \tag{3.3.6}$$

$$-F_{14} + F_{34,x} = 0 \tag{3.3.7}$$

where $F_{34,x}$ is the x component of the force F_{34} while $F_{34,y}$ is the y component of the same force.

Due to action-reaction (Newton's third law):

$F_{23} = -F_{32}$

$F_{53} = -F_{35}$

$F_{43} = -F_{34}$

Unknowns; $F_{23,x}$, $F_{23,y}$, F_{53} , $F_{43,x}$, $F_{43,y}$, F_{12} , G_{12}, G_{14}

Equations;

$$G_{12} - F_{32,x} = 0 \tag{3.3.1}$$

$$-F_{12} + F_{32,y} = 0 \tag{3.3.2}$$

$$-F_{23,y} + F_{53} \sin\theta_{15} + F_{43,y} = 0 \tag{3.3.3}$$

$$F_{23,x} + F_{53} \cos\theta_{15} - F_{43,x} = 0 \tag{3.3.4}$$

$$-F_{53} \sin\theta_{15} \cos(2\pi - \theta_{13})a_{31} - F_{53} \sin\theta_{15} \cos(2\pi - \theta_{13})a_{31}$$

$$-F_{43,y} \cos(2\pi - \theta_{13})a_3 + F_{43,x} \sin(2\pi - \theta_{13})\, a_3 = 0 \tag{3.3.5}$$

$$G_{14} - F_{34,y} = 0 \tag{3.3.6}$$

$$-F_{14} + F_{34,x} = 0 \tag{3.3.7}$$

The number of equations is seven while the number of unknowns is eight. As the unknown quantity is more than the equations, the solution cannot be obtained. All other unknowns can be written in terms of one of the unknown. Link 5 is the only two force member in this mechanism; therefore, all other unknowns are written in terms of F_{53}.

$$F_{34,x} = F_{14}$$

$$F_{43,x} = -F_{34,x}$$

$$F_{23,x} = -F_{53} \cos\theta_{15} + F_{43,x}$$

$$F_{32,x} = -F_{23,x}$$

$$G_{12} = F_{32,x}$$

$$F_{43,y} = \frac{-F_{53}\sin\theta_{15}\cos(2\pi - \theta_{13})a_{31} - F_{53}\cos\theta_{15}\sin(2\pi - \theta_{13})a_{31} + F_{43,x}\sin(2\pi - \theta_{13})\,a_3}{\cos(2\pi - \theta_{13})a_3}$$

$$F_{34,y} = -F_{43,y}$$

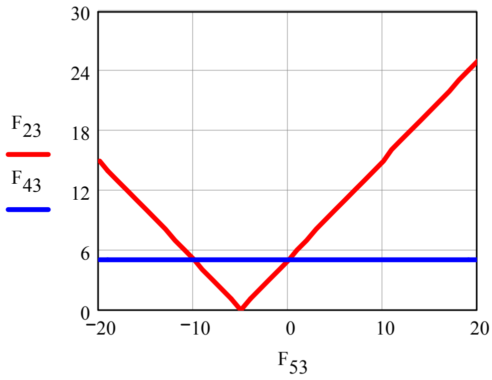
$$G_{14} = F_{34,y}$$

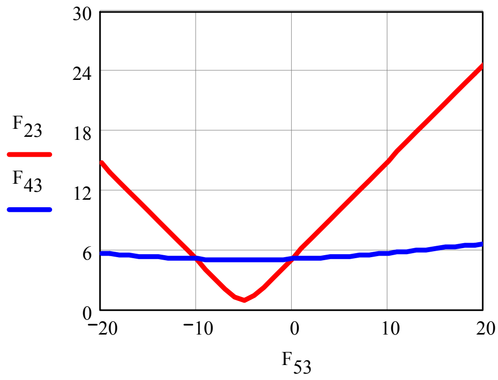
$$F_{23,y} = F_{53}\sin\theta_{15} + F_{43,y}$$

$$F_{32,y} = -F_{23,y}$$

$$F_{12} = F_{32,y}$$

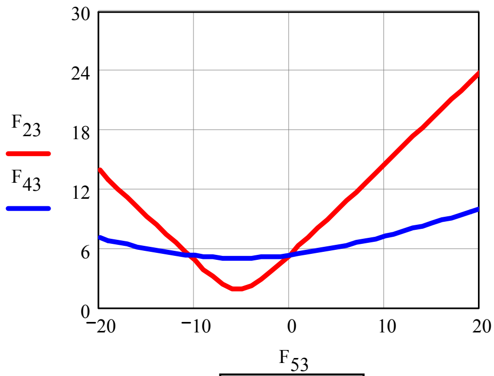
For the given range of F_{53} values obtained values of F_{43}, F_{23}, F_{12}, G_{12} and G_{14} are given in figures 42 and 43. Figure 42 shows F_{53} versus F_{43} and F_{23} values and figure 43 shows F_{53} versus F_{12}, G_{12} and G_{14} values. All graphs are drawn in figure 42 and in figure 43 for different θ_{15} values with a 10° increment. It should be noted that as at θ_{15}=90° singularity occurs the last graphs are drawn at θ_{15}=89.5°

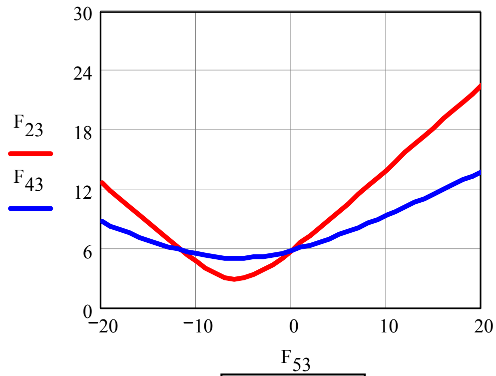




θ_{15}=0°

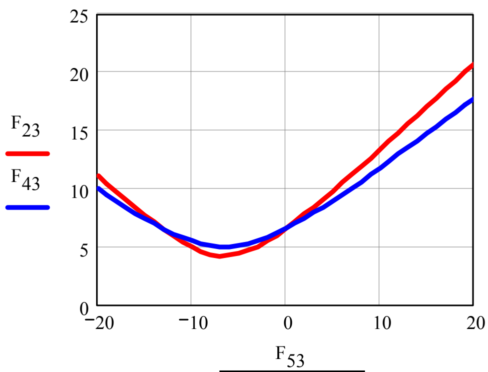
θ_{15}=10°

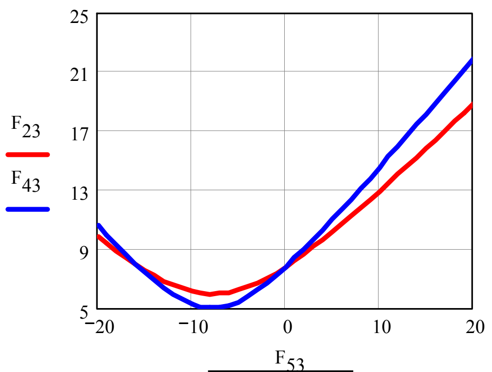




θ_{15}=20°

θ_{15}=30°





θ_{15}=40°

θ_{15}=50°

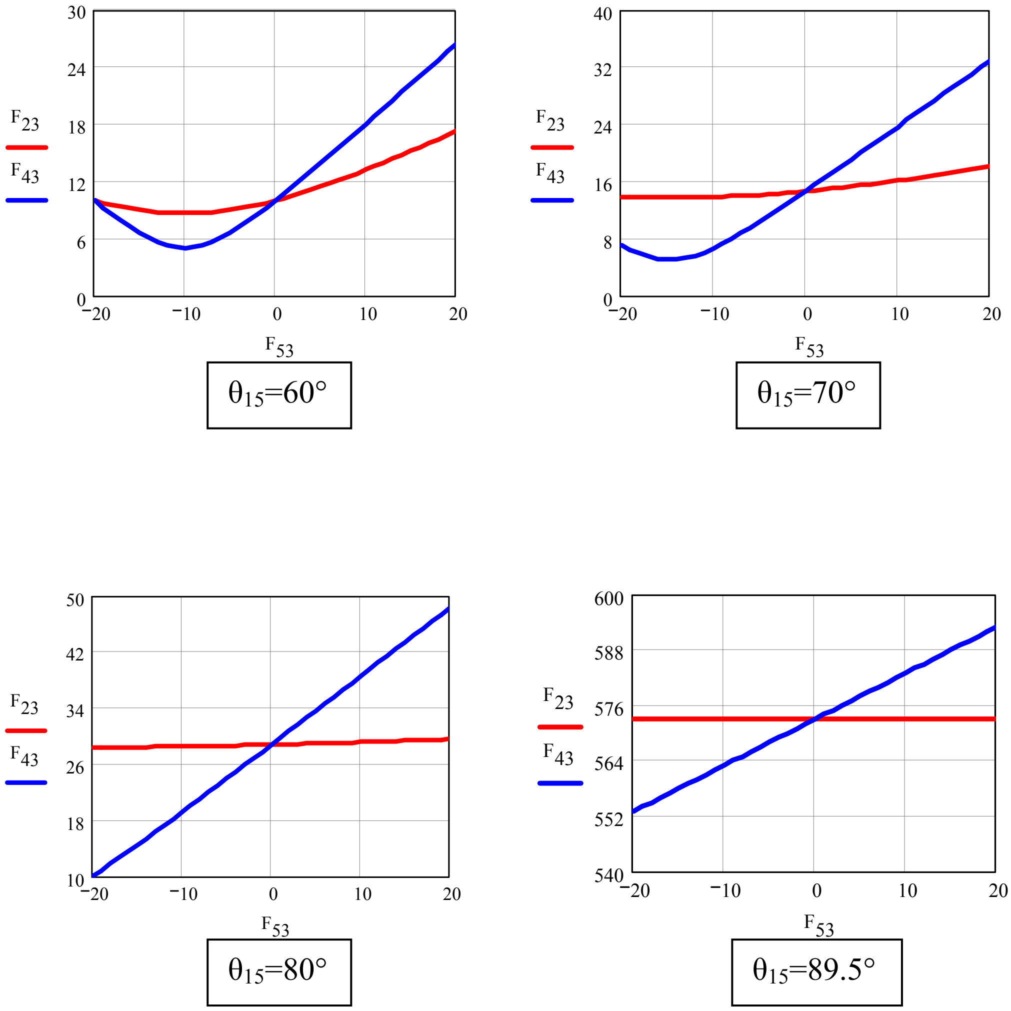
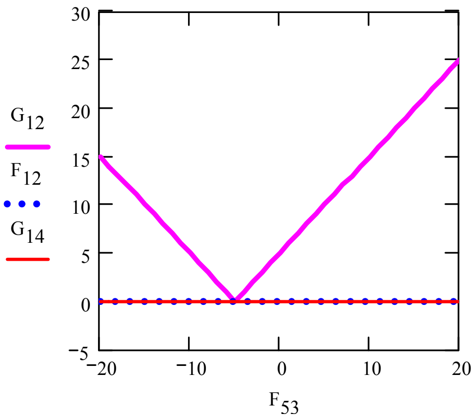
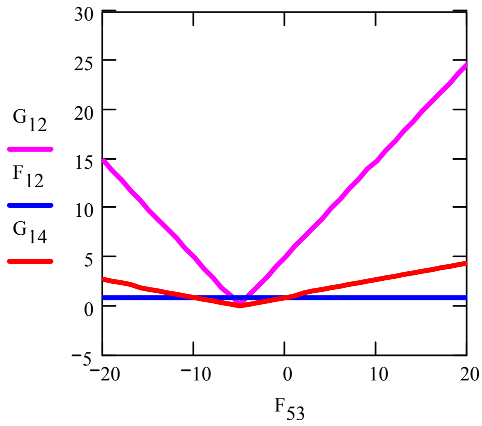


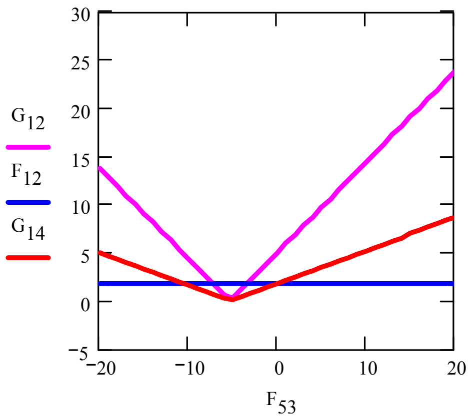
Figure 42: F_{23} and F_{43} versus F_{53} Graphs for Different θ_{12} Values of Example 3.3



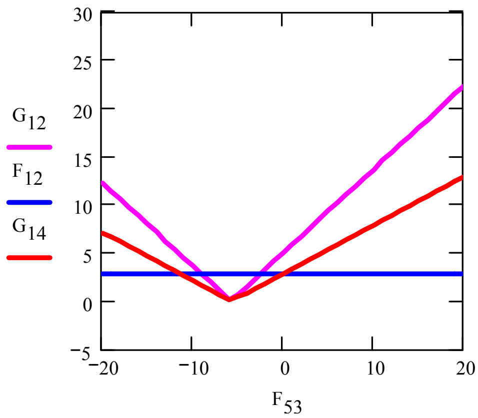
θ_{15}=0°



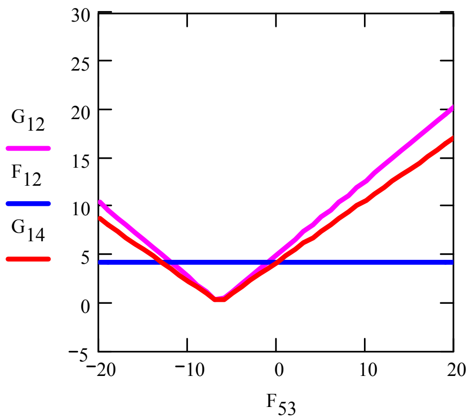
θ_{15}=10°



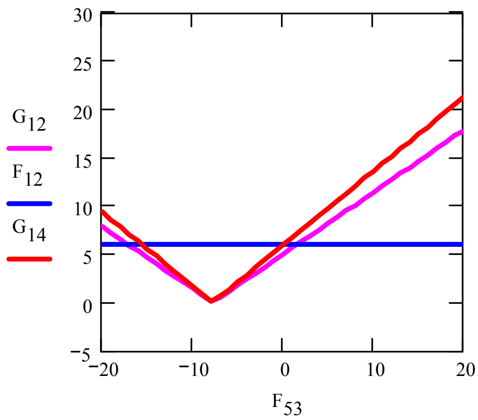
θ_{15}=20°



θ_{15}=30°



θ_{15}=40°



θ_{15}=50°

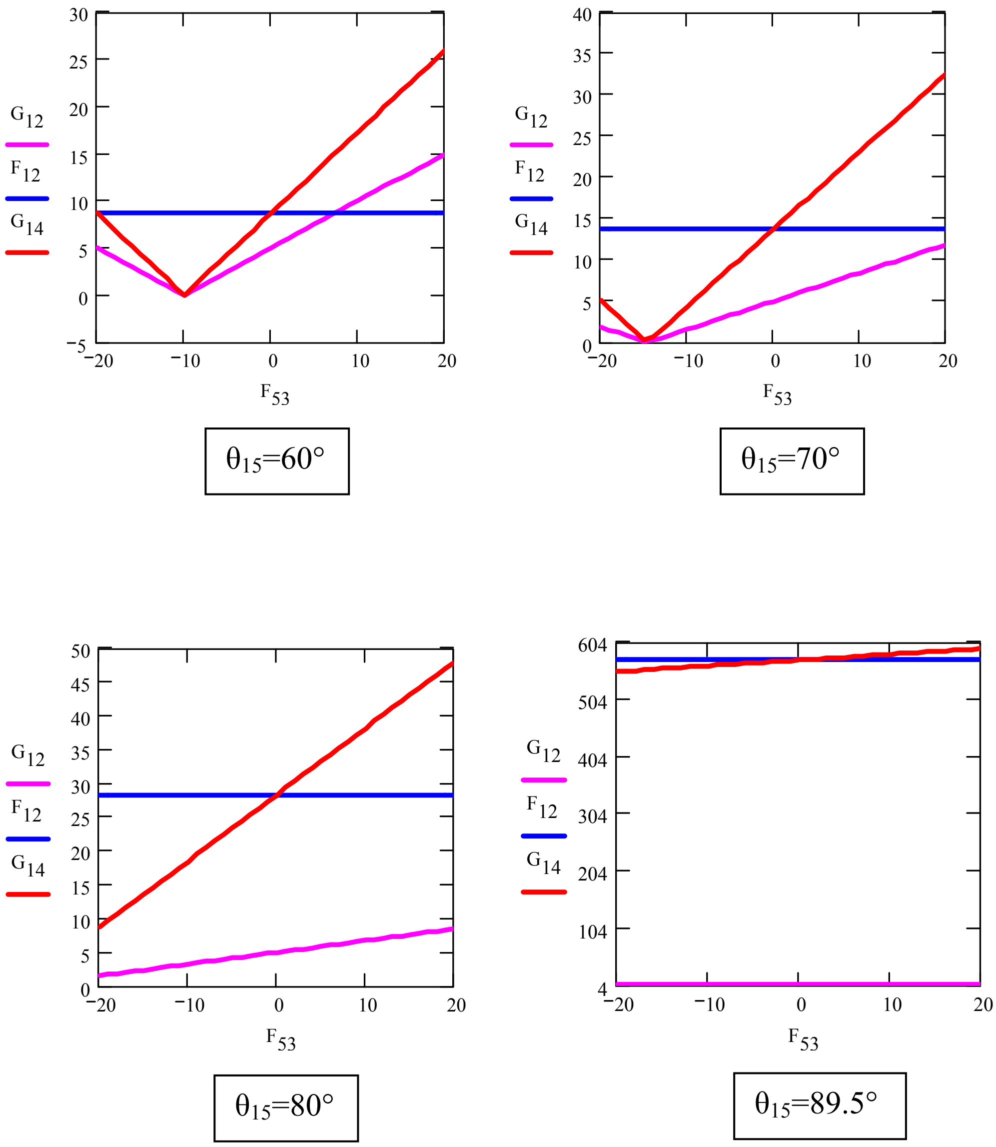


Figure 43: F_{12}, G_{12} and G_{14} versus F_{53} Graphs for Different θ_{12} Values of Example 3.3

Figure 42 shows that there is no symmetry between F_{23} and F_{43} in this case. For intermediate values of θ_{12} it is obviously seen from the graphs that for negative F_{53} values both F_{23} and F_{43} take their minimum values. The negative values of F_{53} imply the direction is just the opposite of shown in free body diagrams. In other words, the negative values of F_{53} show that link 5 is in tension while the positive values of F_{53} show link 5 is in compression.

In this case, the designer already knows to assemble link 5 with initial tension. Again the problem is demoted to decide the value of the initial tension. If, for some reason, one of the link is more critical than the other, the minimum values of force on that link should be focused. If the links are equally critical, values between their minimum cases should be considered.

Assuming link 2 and link 4 are equally critical an initial tension value i.e. F_{53}, between F_{53} value corresponding to minimum F_{23} and F_{53} value corresponding to minimum F_{43} should be decided.

$$x = \sqrt{(\min(F_{23}) - F_{23})^2 + (\min(F_{43}) - F_{43})^2}$$

For each θ_{12} values x values are calculated and minimum x value is determined. For minimum x values corresponding F_{53}, F_{23} and F_{43} values are tabulated in table 5. Similar to example 3.1 and 3.2 the values for θ_{12}=0° and θ_{12}=90° are accepted as extremes and disregarded in order to get more reliable results.

Table 5: Minimum Forces for Example 3.3

θ_{12}	F_{53}	abs(F_{23})	abs(F_{43})
10°	-5	0,885	5
20°	-5	1,845	5,001
30°	-6	2,893	5,001
40°	-7	4,211	5,009
50°	-8	5,960	5,003
60°	-10	8,660	5
70°	-15	13,738	5,013
80°	-20	28,397	10

F_{12} is independent of F_{53} values. It is constant for each θ_{12} value but if G_{12} and G_{14} are assumed to be as important as F_{23} and F_{43} they can be also implemented into x equation. However, for this example implementation of G_{12} and G_{14} into the equation does not affect the result because they also take minimum values for the F_{53} values in table 5.

Similar to former examples, the final F_{53} value can be calculated by using the equation:

$$F_{53} = \frac{\sum_{i=1}^{n=8} (F_{53})_i}{n}$$

$$F_{53} = \frac{(F_{53})_1 + (F_{53})_2 + (F_{53})_3 + (F_{53})_4 + (F_{53})_5 + (F_{53})_6 + (F_{53})_7 + (F_{53})_8}{8}$$

$F_{53} = -9{,}5$ N

As a result, link 5 can be assembled with 9,5 N initial tension. The F_{23} and F_{43} values for F_{53}=-9,5 N needed to be checked before finalizing the design. If corresponding F_{23} and F_{43} values are acceptable the design can be finalized.

Table 6: Final Forces for Example 3.3.

θ_{12}	F_{53} (N)	abs(F_{23}) (N)	abs(F_{43}) (N)
10°	-9,5	4.444	5.059
20°	-9,5	4.328	5.200
30°	-9,5	4.330	5.336
40°	-9,5	4.774	5.353
50°	-9,5	6.061	5.171
60°	-9,5	8.664	5.019
70°	-9,5	13.849	6,938
80°	-9	28.554	19,648

Table 7: Final Forces for Example 3.3

θ_{12}	F_{53} (N)	abs(G_{12}) (N)	abs(G_{14}) (N)	abs(F_{12}) (N)
10°	-9,5	4,356	0,768	0,882
20°	-9,5	3,927	1,429	1,820
30°	-9,5	3,227	1,863	2,887
40°	-9,5	2,277	1,911	4,195
50°	-9,5	1,106	1,319	5,959
60°	-9,5	0,250	0,433	8,660
70°	-9,5	1,751	4,810	13,737
80°	-9,5	3,350	19,001	28,356

3.4 Example IV: A Double Slider Mechanism (an overconstrained mechanism with 5 links and 4 revolute, 2 prismatic joints) Loaded with Force F

$$\mathbf{C_0C = a_5}$$

$$\mathbf{AB = a_3}$$

$$\mathbf{AC = a_{31}}$$

$$\mathbf{\frac{AB}{2} = AC = C_0C}$$

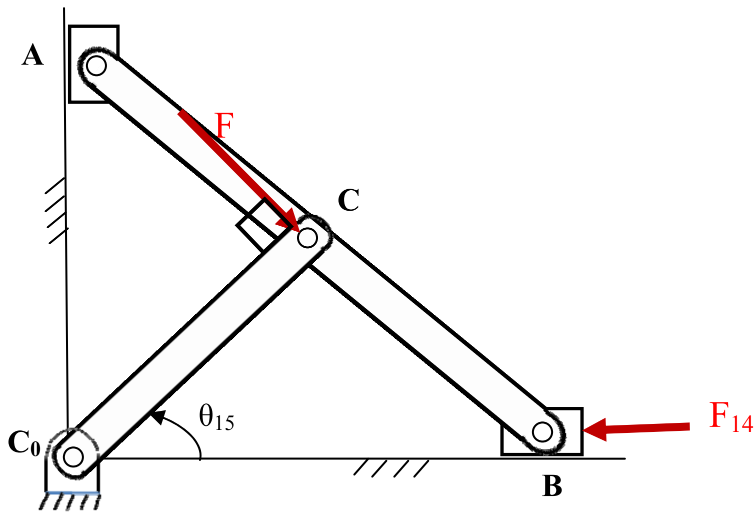


Figure 44: An Overconstrained Double Slider Mechanism Loaded with F

$F = 10\,\mathrm{N}$

$a_5 = 60\,\mathrm{mm}$

$a_3 = 120\,\mathrm{mm}$

In this case the mechanism is designed to cut materials with a cutting tool on point C. F is the reaction force of the material and it is in static equilibrium with F_{14}. The operation range of θ_{15} is assumed to be 0° to 90°

Link 2: Three force member

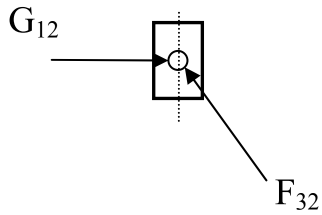


Figure 45: Free Body Diagram of link 2 belongs to the Mechanism in Figure 44

$$G_{12} - F_{32,x} = 0 \quad (3.4.1)$$

$$F_{32,y} = 0 \quad (3.4.2)$$

Link 5: Two force member

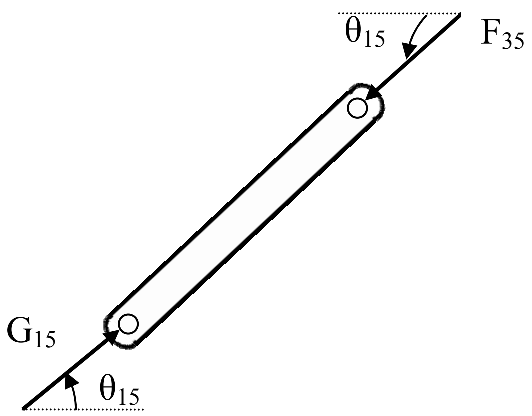


Figure 46: Free Body Diagram of link 5 belongs to the Mechanism in Figure 44

$$G_{15} = -F_{35}$$

Link 4: Three force member

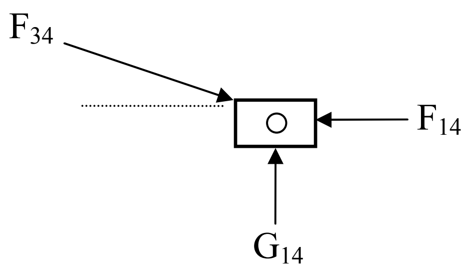


Figure 47: Free Body Diagram of link 4 belongs to the Mechanism in Figure 44

$$G_{14} - F_{34,y} = 0 \tag{3.4.6}$$

$$-F_{14} + F_{34,x} = 0 \tag{3.4.7}$$

Link 3: Three force member

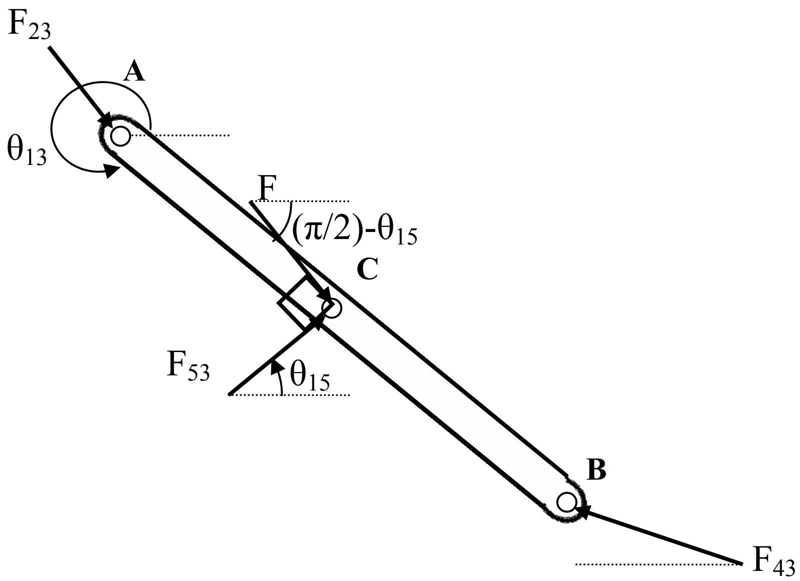


Figure 48: Free Body Diagram of link 3 belongs to the Mechanism in Figure 44

$$-F_{23,y} + F_{53}\sin\theta_{15} + F_{43,y} - F\sin(\tfrac{\pi}{2} - \theta_{15}) = 0 \tag{3.4.3}$$

$$F_{23,x} + F_{53}\cos\theta_{15} - F_{43,x} + F\cos(\tfrac{\pi}{2} - \theta_{15}) = 0 \tag{3.4.4}$$

$$-F_{23,y}\cos(2\pi - \theta_{13})a_{31} + F_{23,x}\sin(2\pi - \theta_{13})a_{31}$$

$$-F_{43,y}\cos(2\pi - \theta_{13})a_3 + F_{43,x}\sin(2\pi - \theta_{13})\,a_3 = 0 \tag{3.4.5}$$

Due to action-reaction (Newton's third law):

$F_{23} = -F_{32}$

$F_{53} = -F_{35}$

$F_{43} = -F_{34}$

Unknowns; $F_{23,x}$, $F_{23,y}$, F_{53} , $F_{43,x}$, $F_{43,y}$, F_{14} , G_{12}, G_{14}

Equations;

$$G_{12} - F_{32,x} = 0 \tag{3.4.1}$$

$$F_{32,y} = 0 \tag{3.4.2}$$

$$-F_{23,y} + F_{53}\sin\theta_{15} + F_{43,y} - F\sin(\frac{3\pi}{2} + \theta_{15}) = 0 \tag{3.4.3}$$

$$F_{23,x} + F_{53}\cos\theta_{15} - F_{43,x} + F\cos(\frac{3\pi}{2} + \theta_{15}) = 0 \tag{3.4.4}$$

$$-F_{23,y}\cos(2\pi - \theta_{13})a_{31} + F_{23,x}\sin(2\pi - \theta_{13})a_{31}$$

$$-F_{43,y}\cos(2\pi - \theta_{13})a_3 + F_{43,x}\sin(2\pi - \theta_{13})\,a_3 = 0 \tag{3.4.5}$$

$$G_{14} - F_{34,y} = 0 \tag{3.4.6}$$

$$-F_{14} + F_{34,x} = 0 \tag{3.4.7}$$

The number of equations is seven while the number of unknowns is eight. As the unknown quantity is more than the equations , the solution cannot be obtained. . All other unknowns can be written in terms of one of the unknown. Link 5 is the only two force member in this mechanism; therefore, all other unknowns are written in terms of F_{53}.

$$F_{32,y} = 0$$

$$F_{23,y} = 0$$

$$F_{43,y} = -F_{53}\sin\theta_{15} + F\sin(\frac{3\pi}{2} + \theta_{15})$$

$$F_{34,y} = -F_{43,y}$$

$$G_{14} = F_{34,y}$$

$$F_{23,x} = -F_{53}\cos\theta_{15} + F_{43,x} - F\cos(\frac{3\pi}{2} + \theta_{15})$$

$$F_{23,x} = \frac{F_{43,y}\cos(2\pi - \theta_{13})a_3 - F_{43,x}\sin(2\pi - \theta_{13})\,a_3}{\sin(2\pi - \theta_{13})\,a_{31}}$$

combining the two equation above;

$$F_{43,x} = (\frac{a_{31}}{a_{31} + a_3})(F_{53}\cos\theta_{15} + F\cos\left(\frac{3\pi}{2} + \theta_{15}\right) + \frac{F_{43,y}\cos(2\pi - \theta_{13})a_3}{\sin(2\pi - \theta_{13})\,a_{31}})$$

now $F_{23,x}$ can also be found

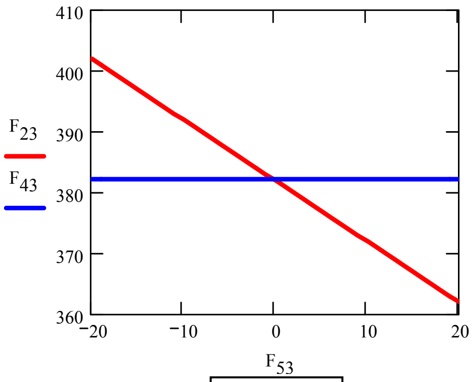
$$F_{32,x} = -F_{23,x}$$

$$G_{12} = F_{32,x}$$

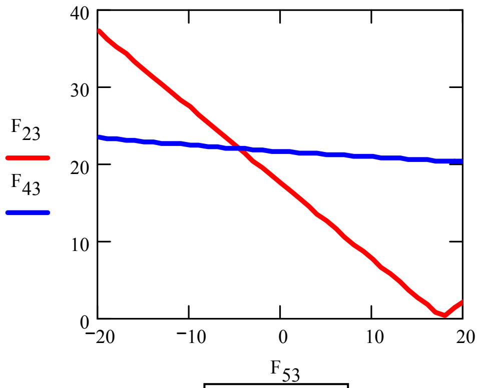
$$F_{34,x} = -F_{43,x}$$

$$F_{14} = F_{34,x}$$

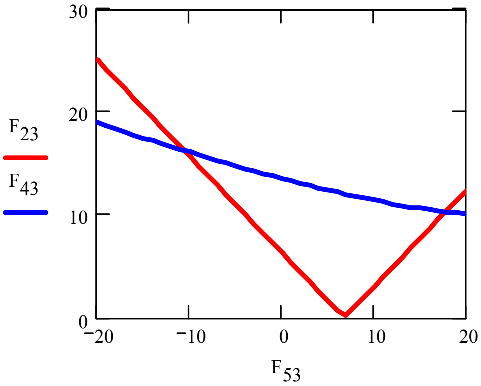
For the given range of F_{53} values obtained values of F_{43}, F_{23}, F_{14}, G_{12} and G_{14} are given in figures 49 and 50. Figure 49 shows F_{53} versus F_{43} and F_{23} values and figure 50 shows F_{53} versus F_{14}, G_{12} and G_{14} values. All graphs are drawn in figure 49 and in figure 50 for different θ_{15} values with a 10° increment. It should be noted that as at θ_{15}=0° singularity occurs the first graphs are drawn at θ_{15}=0.5°



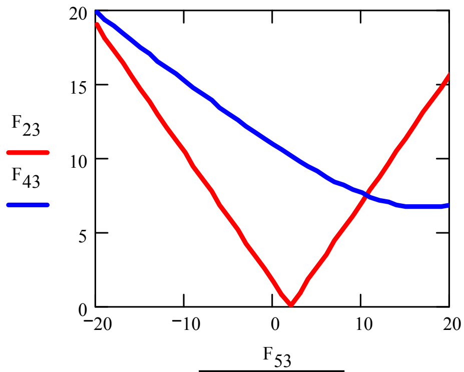
θ_{15}=0°



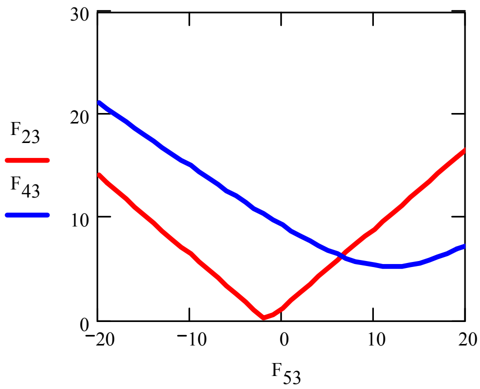
θ_{15}=10°



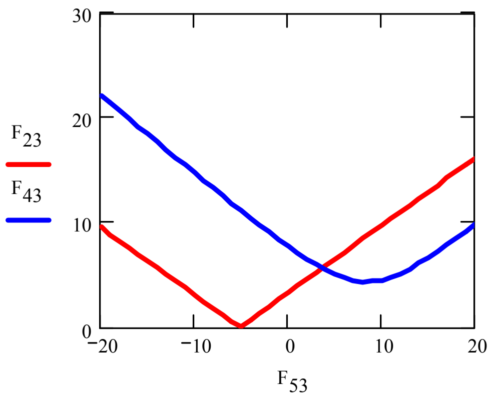
θ_{15}=20°



θ_{15}=30°



θ_{15}=40°



θ_{15}=50°

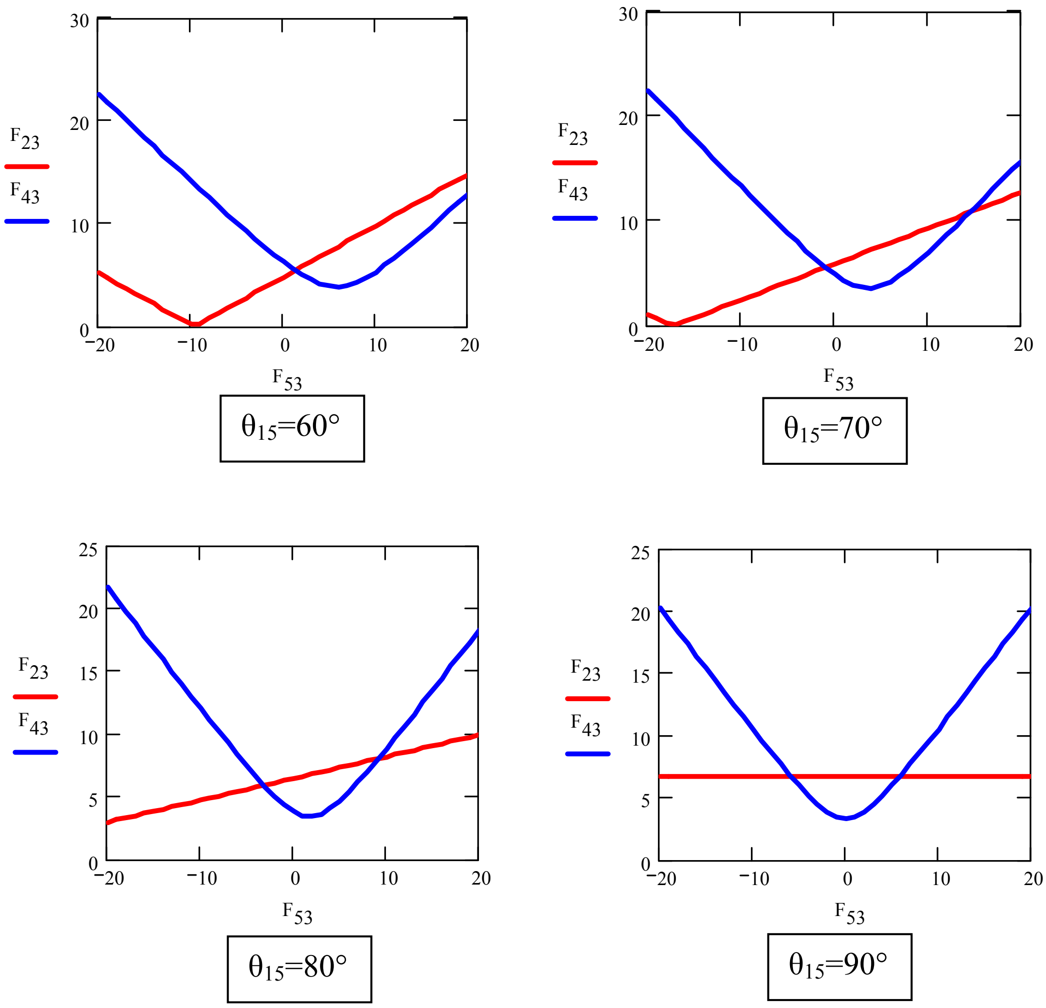
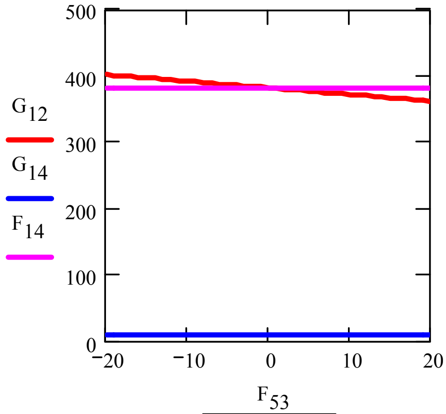
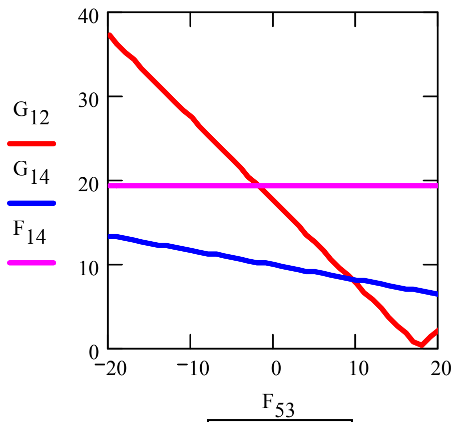


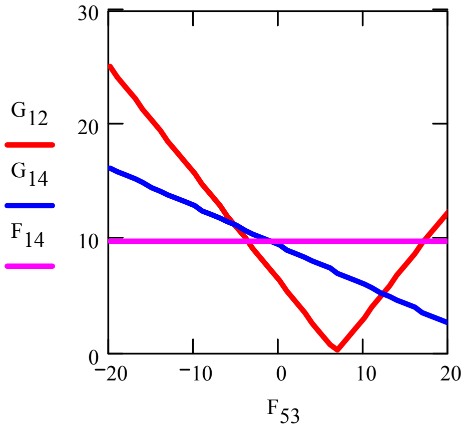
Figure 49: F_{23} and F_{43} versus F_{53} Graphs for Different θ_{12} Values of Example 3.4



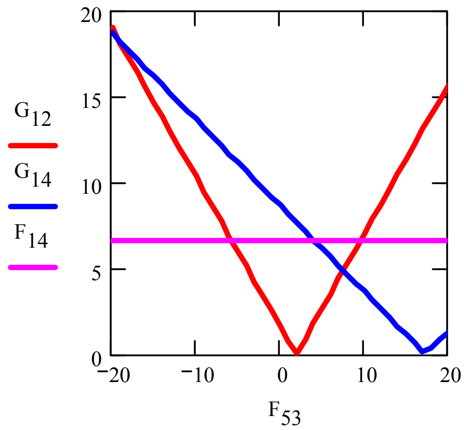
$\theta_{15}=0°$



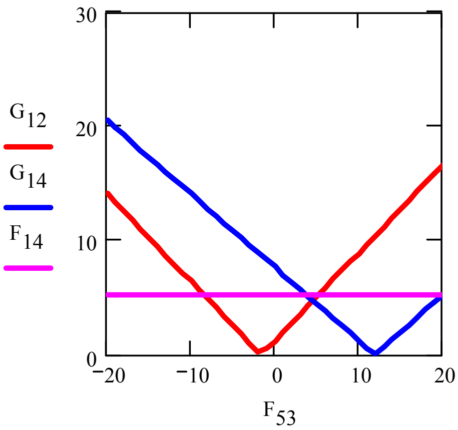
$\theta_{15}=10°$



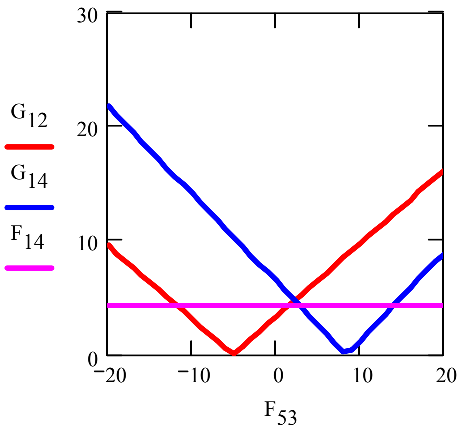
$\theta_{15}=20°$



$\theta_{15}=30°$



$\theta_{15}=40°$



$\theta_{15}=50°$

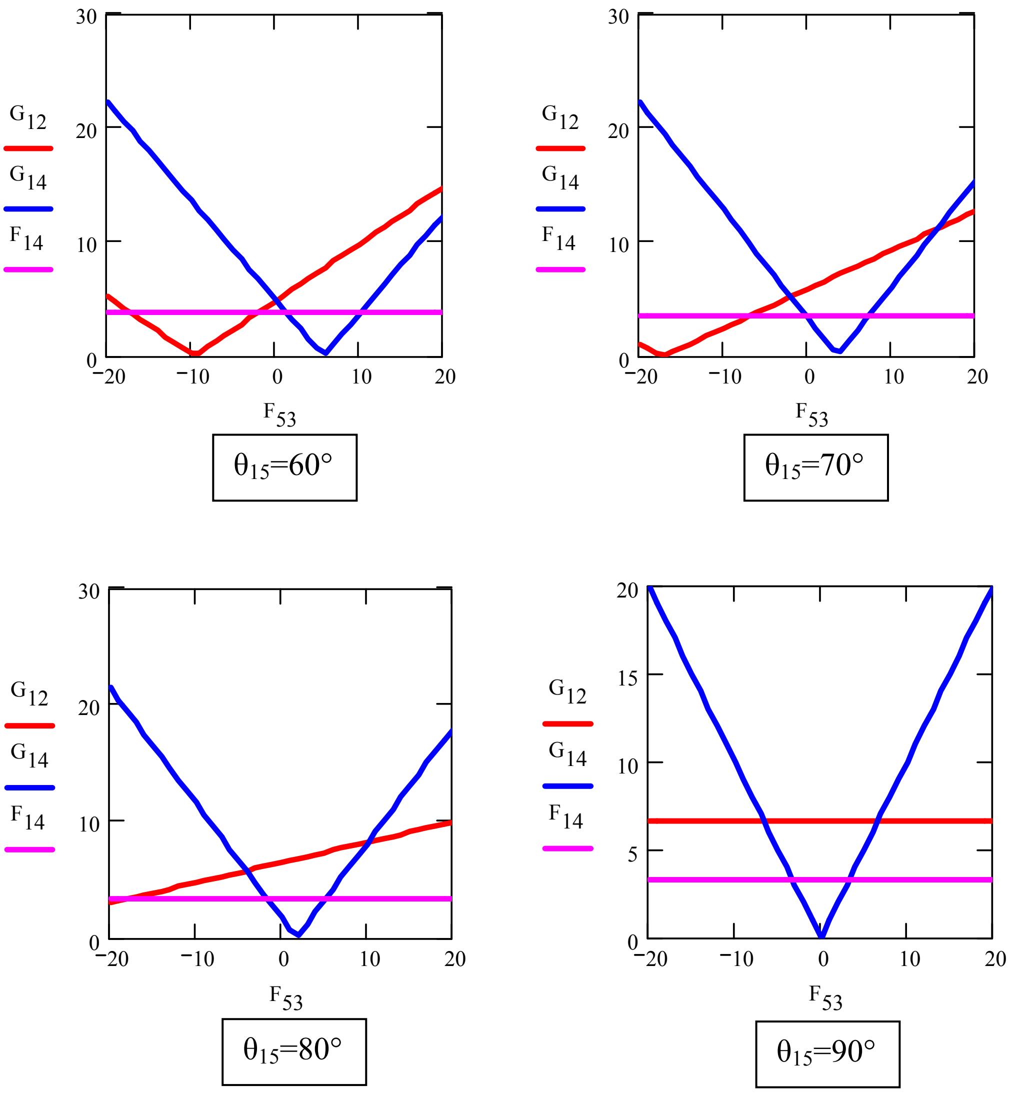


Figure 50: F_{14}, G_{14} and G_{12} versus F_{53} Graphs for Different θ_{12} Values of Example 3.4

Figure 49 shows that there is no symmetry between F_{23} and F_{43} in this case. For lower values of θ_{12} it is seen from the graphs that for positive F_{53} values both F_{23} and F_{43} take their minimum values. However, for higher values of θ_{12}, F_{23} take its minimum value when F_{53} is negative while F_{43} take its minimum value when F_{53} is positive. Therefore, in this case, the designer should decide whether to assemble link 5 in compression or in tension and the value of initial loading simultaneously.

Similar to other cases, the positive values of F_{53} imply the direction of this force is as given in free body diagrams and the negative values imply the direction is just the opposite. In other words, the positive values of F_{53} show that link 5 is in compression and the negative values of F_{53} show link 5 is in tension.

If, for some reason, one of the link is more critical than the other, the minimum values of force on that link should be focused. If the links are equally critical, values between their minimum cases should be considered.

Assuming link 2 and link 4 are equally critical an initial loading value i.e. F_{53}, between F_{53} value corresponding to minimum F_{23} and F_{53} value corresponding to minimum F_{43} should be decided.

$$x = \sqrt{(\min(F_{23}) - F_{23})^2 + (\min(F_{43}) - F_{43})^2}$$

For each θ_{12} values x values are calculated and minimum x value is determined. For minimum x values corresponding F_{53}, F_{23} and F_{43} values are tabulated in table 8. Similar to former examples the values for θ_{12}=0° and θ_{12}=90° are accepted as extremes and disregarded in order to get more reliable results.

Table 8: Minimum Forces for Example 3.4

θ_{12}	F_{53}	abs(F_{23})	abs(F_{43})
10°	18	0,267	20,339
20°	7	0,252	12,001
30°	3	0,931	9,783
40°	2	2,774	8,218
50°	0	3,309	7,762
60°	-1	4,311	7,016
70°	-1	5,508	5,621
80°	-1	6,290	4,343

F_{12} is independent of F_{53} values. It is constant for each θ_{12} value. G_{12} is equal to $F_{32,x}$ and G_{14} is equal to $F_{34,y}$. If G_{12} and G_{14} are assumed to be as important as F_{23} and F_{43} they can be also implemented into x equation. However, for this example from figure 46 and figure 47 it is obvious that, the behaviour of G_{12} is similar to F_{23} and the behaviour of G_{14} is similar to F_{43}. Therefore, such an implementation is not needed as the impact is negligible.

As aforementioned, the final F_{53} value can be calculated by using the equation:

$$F_{53} = \frac{\sum_{i=1}^{n=8} K_i (F_{53})_i}{n}$$

$$F_{53} = \frac{(F_{53})_1 + (F_{53})_2 + (F_{53})_3 + (F_{53})_4 + (F_{53})_5 + (F_{53})_6 + (F_{53})_7 + (F_{53})_8}{8}$$

$F_{53} = 3{,}375$ N

As a result, link 5 can be assembled with 3,375 N initial compression. The F_{23} and F_{43} values for F_{53}=3,375 N needed to be checked before finalizing the design. If corresponding F_{23} and F_{43} values are acceptable the design can be finalized.

Table 9: Final Forces for Example 3.4.

θ_{12}	F_{53} (N)	abs(F_{23}) (N)	abs(F_{43}) (N)
10°	3,375	14,136	21,314
20°	3,375	3,154	12,764
30°	3,375	1,256	9,647
40°	3,375	3,828	7,553,
50°	3,375	5,478	5,805
60°	3,375	6,499	4,374
70°	3,375	7,004	3,556
80°	3,375	7,049	3,738

Table 10: Final Forces for Example 3.4.

θ_{12}	F_{53} (N)	abs(G_{12}) (N)	abs(G_{14}) (N)	abs(F_{12}) (N)
10°	3,375	14,136	9,262	19,196
20°	3,375	3,154	8,243	9,746
30°	3,375	1,256	6,973	6,667
40°	3,375	3,828	5,491	5,186
50°	3,375	5,478	3,842	4,351
60°	3,375	6,499	2,077	3,849
70°	3,375	7,004	0,249	3,547
80°	3,375	7,049	1,587	3,385

3.5 Example V: A Parallelogram Mechanism (an overconstrained mechanism with 6 links and 8 revolute joints) Loaded with Torque T_{12}

This example is given as a further consideration about the preloaded link selection in overconstrained mechanisms. As discussed in chapter 2.1.3, the overconstrained mechanism in figure 51 has actual $m' = 1$. However, by the general DOF formula m=-1 is obtained.

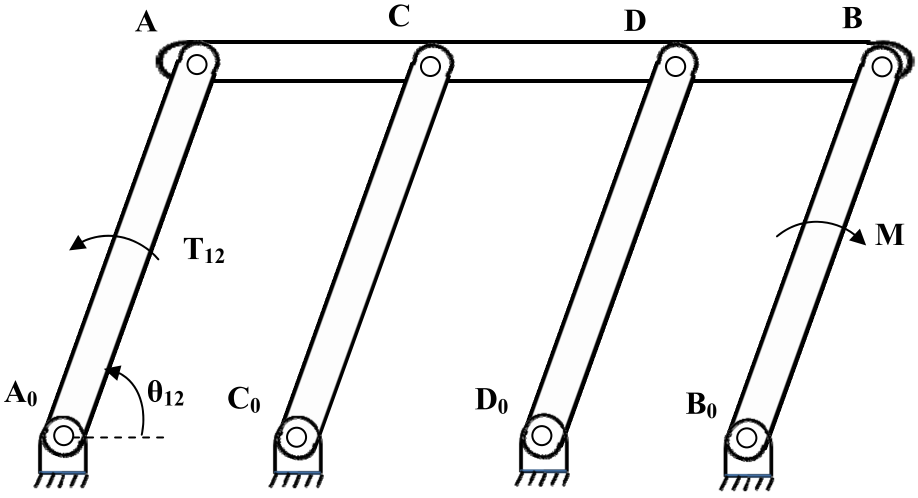


Figure 51: An Overconstrained Parallelogram Mechanism Loaded with Torque

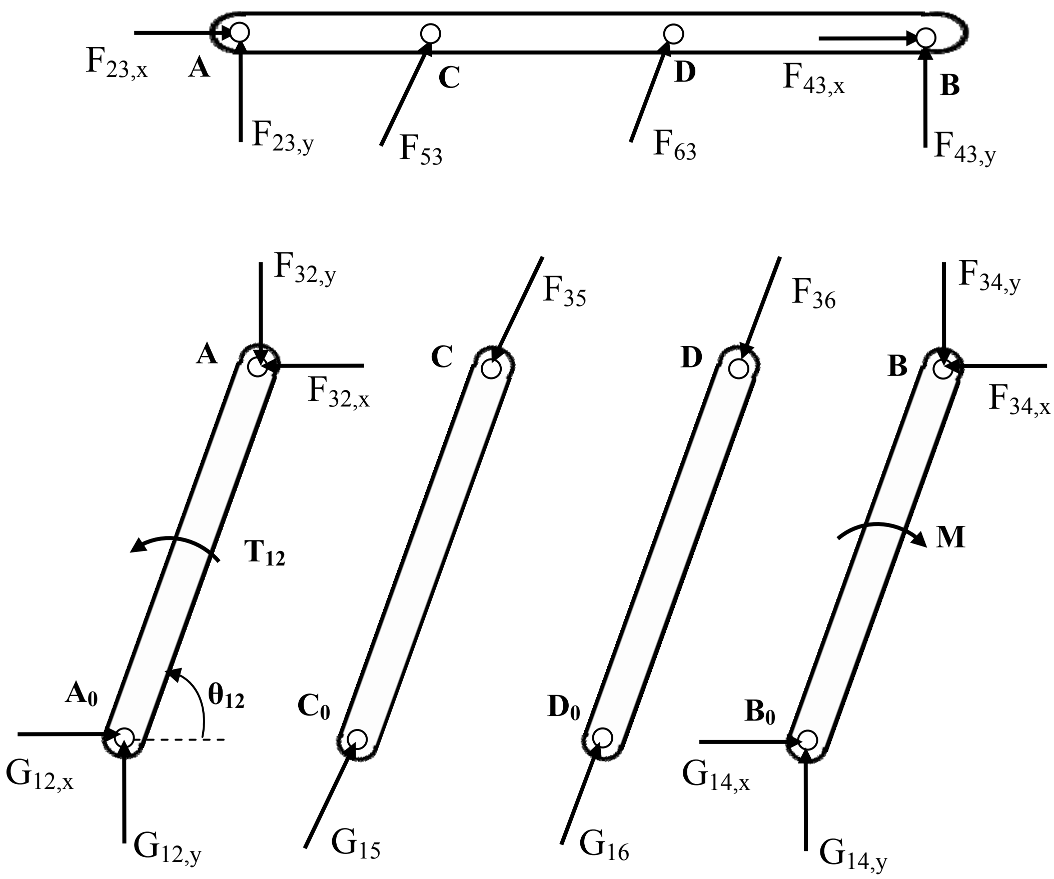


Figure 52: Free body diagrams of the mechanism in figure 51

In figure 52, it is obviously seen that,

$$F_{32} = -G_{12}$$

$$F_{35} = -G_{15}$$

$$F_{36} = -G_{16}$$

$$F_{34} = -G_{14}$$

Due to action-reaction (Newton's third law):

$$F_{23} = -F_{32}$$

$$F_{53} = -F_{35}$$

$$F_{63} = -F_{36}$$

$$F_{43} = -F_{34}$$

So the unknowns are; $F_{23,x}$, $F_{23,y}$, $F_{43,x}$, $F_{43,y}$, F_{53}, F_{63}, M. The number of independent static force equilibrium equations is 5, however the number of unknowns is 7. Therefore, all other unknowns can be written in terms of 2 of the unknowns. As link 5 and link 6 are the only two force members of the mechanism. So, they can be selected as the links to be preloaded and all the other unknowns can be expressed in terms of F_{53} and F_{63}.

CHAPTER 4

DISCUSSION AND CONCLUSION

In this thesis kinematic and force analyses of overconstrained mechanisms are performed. The study is restricted to planar overconstrained mechanisms only. However, the investigated methods and approaches could be extended to the spatial overconstrained mechanisms as well.

In chapter 1, general information about overconstrained mechanisms, previous studies, open problems in the literature and motivation are given.

In chapter 2, overconstrained mechanisms are investigated in terms of kinematic analysis. Indeed, in that chapter what performed is not the kinematic analysis of existing overconstrained mechanisms but the methods to generate overconstrained mechanisms by using kinematics. Four different methods for the generation of overconstrained mechanisms are discussed. The first one, analytical method, is applicable to all kinds of planar overconstrained mechanisms; however, obtaining a mobile overconstrained mechanisms using the results of this method is not simple. On the other hand, such a mobile overconstrained mechanism, for the given link number, joint number and joint type, may never be exist. The second one, the method of cognates, is a practical way of mobile overconstrained mechanism generation. If one or more cognates of any mechanism is known, a mobile overconstrained mechanism can always be constructed using the cognates. Various cognate types, such as coupler curve cognates, function cognates can be used for this purpose. However, cognate linkages are a different extensive research topic within mechanism researches. Although many methods are improved for the derivation of cognates, a general procedure applicable to all kinds of linkages has not been introduced yet. Nevertheless, if cognates of a mechanism are known, a mobile overconstrained mechanism can always be generated using cognates. The third one, the method of combining identical modules, is another practical way of mobile overconstrained mechanism constitution. If combination of identical modules without blocking the basic motion of the module is possible, a mobile overconstrained mechanism can be generated via this way. This basic motion can be a straight line or circular motion or any kind of motion. The only necessary condition is that the basic motion of the module should not be blocked. Via the last method, the method of extending an overconstrained mechanism with extra links, a new overconstrained mechanism can be generated by adding extra links and joints to an already existing overconstrained mechanism.

In chapter 3, a method for the force analysis of overconstrained mechanisms is introduced. This analysis should not be performed to find the forces on any overconstrained mechanism which is already constructed. On the contrary, that should be performed before the assembly process of the mechanism. Overconstrained mechanisms show statically indeterminate characteristics. In order to cope with the indeterminacy, applying preloading at assembly phase is suggested in the method. Prior to preloading, the forces and moments are written in terms of each other. Then, the amount of preloading is decided such that the other joint reaction forces/moments kept in minimum. An optimization process is performed to determine the minimum point. This preloading may be compression or tension. The direction of the preloading is decided as well as the amount of preloading. Finally, the designated preloading is applied at assembly phase of the mechanism. The method described in chapter 3 is based on three main assumptions: i) The preloaded link is so less stiff than the other links that it can assumed as the only non-rigid link in the mechanism. ii) The clearances in the joints are negligible. iii) The preloaded link should be a two force member, based on the assumption that its inertia effects are negligible.

A future work can be the application of the analytical method to different groups of link-joint combinations in order to generate new overconstrained mechanisms. As another future work, the methods presented in this study can be extended to spatial overconstrained mechanisms.

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