ON ALGEBRAIC FUNCTION FIELDS WITH CLASS NUMBER THREE

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

DİLEK BUYRUK

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

FEBRUARY 2011

Approval of the thesis:

ON ALGEBRAIC FUNCTION FIELDS WITH CLASS NUMBER THREE

submitted by **DİLEK BUYRUK** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Mathematics Department, Middle East Technical University** by,

Prof. Dr. Canan Özgen
Dean, Graduate School of **Natural and Applied Sciences**

Prof. Dr. Zafer Nurlu
Head of Department, **Mathematics**

Prof. Dr. Mehpare Bilhan
Supervisor, **Mathematics Department, METU**

Prof. Dr. Ferruh Özbudak
Co-supervisor, **Mathematics Department, METU**

Examining Committee Members:

Prof. Dr. Halil İbrahim Karakaş
Banking Department, Başkent University

Prof. Dr. Mehpare Bilhan
Mathematics, METU

Prof. Dr. Hurşit Önsiper
Mathematics, METU

Prof. Dr. Ersan Akyıldız
Mathematics, METU

Assoc. Prof. Dr. Cem Güneri
Mathematics, Sabancı University

Date:

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: DİLEK BUYRUK

Signature :

ABSTRACT

ON ALGEBRAIC FUNCTION FIELDS WITH CLASS NUMBER THREE

Buyruk, Dilek
Ph.D., Department of Mathematics
Supervisor : Prof. Dr. Mehpare Bilhan
Co-Supervisor : Prof. Dr. Ferruh Özbudak

February 2011, 128 pages

Let $K/\mathbb{F}_q$ be an algebraic function field with full constant field $\mathbb{F}_q$ and genus g. Then the divisor class number h_K of $K/\mathbb{F}_q$ is the order of the quotient group, $D_K^0/P(K)$, degree zero divisors of K over principal divisors of K. The classification of the function fields K with $h_K = 1$ is done by MacRea, Leitzel, Madan and Queen and the classification of the extensions with class number two is done by Le Brigand. Determination of the necessary and the sufficient conditions for a function field to have class number three is done by Hülya Töre.

Let $k := \mathbb{F}_q(T)$ be the rational function field over the finite field $\mathbb{F}_q$ with q elements. For a polynomial $N \in \mathbb{F}_q[T]$, we construct the N^{th} cyclotomic function field K_N. Cyclotomic function fields were investigated by Carlitz, studied by Hayes, M. Rosen, M. Bilhan and many other mathematicians. Classification of cyclotomic function fields and subfields of cyclotomic function fields with class number one is done by Kida, Murabayashi, Ahn and Jung. Also the classification of function fields with genus one and classification of those with class number two is done by Ahn and Jung.

In this thesis, we classified all algebraic function fields and subfields of cyclotomic

function fields over finite fields with class number three.

Keywords: function fields, L-polynomial, class number, cyclotomic function field, abelian extensions.

ÖZ

SINIF SAYISI ÜC OLAN CEBİRSEL FONKSİYON CİSİMLERİ ÜZERİNE

Buyruk, Dilek
Doktora, Matematik Bölümü
Tez Yöneticisi : Prof. Dr. Mehpare Bilhan
Ortak Tez Yöneticisi : Prof. Dr. Ferruh Özbudak

Şubat 2011, 128 sayfa

$K/\mathbb{F}_q$, $\mathbb{F}_q$ sabit cismine sahip, cinsi g olan cebirsel bir fonksiyon cismi olsun. Bu durumda, $K/\mathbb{F}_q$'nun sınıf sayısı h_K, $D_K^0/P(K)$ sınıf grubunun, yani, K'nın derecesi sıfır olan divizörler grubunun temel (principal) divizörler grubuna bölünmesiyle elde edilen bölüm grubunun eleman sayısına eşittir. Sınıf sayısı bir olan K fonksiyon cisimlerinin sınıflandırılması MacRea, Leitzel, Madan ve Queen tarafından yapılmıştır. Sınıf sayısı iki olanların sınıflandırılması ise Le Brigand tarafından yapılmıştır. Bir fonksiyon cisminin sınıf sayısının üç olabilmesi için gerek ve yeter koşulların saptanması ise Hülya Töre tarafından yapılmıştır.

$k := \mathbb{F}_q(T)$, q elemanlı $\mathbb{F}_q$ sonlu cismi üzerindeki rasyonel bir fonksiyon cismi olsun. $N \in \mathbb{F}_q[T]$ polinomu için, $N-$inci "cyclotomic" fonksiyon cismi K_N inşa edilir. "Cyclotomic" fonksiyon cisimleri Carlitz tarafından inşa edilmiş, Hayes, M. Rosen, M. Bilhan ve diğer bir çok matematikçi tarafından çalışılmıştır. Sınıf sayısı bir olan "cyclotomic" fonksiyon cisimlerinin ve "cyclotomic" fonksiyon cisimlerinin alt cisimlerinin sınıflandırılması Kida, Murabayashi, Ahn ve Jung tarafından yapılmıştır. Ayrıca cinsi bir olan fonksiyon cisimlerinin sınıflandırılması ve sınıf sayısı iki olanların

sınıflandırması Ahn ve Jung tarafından yapılmıştır.

Bu tezde, sınıf sayısı üç olan bütün cebirsel fonksiyon cisimlerini ve "cyclotomic" fonksiyon cisimlerinin alt cisimlerini sınıflandırdık.

Anahtar Kelimeler: fonksiyon cisimleri, L polinomu, sınıf sayısı, "cyclotomic" fonksiyon cisimleri, değişmeli genişlemeler.

To My Family

ACKNOWLEDGMENTS

I would like to acknowledge my supervisor Prof. Dr. Mehpare Bilhan for her patience, support, guidance and encouragement throughout this thesis. I am very grateful to her for teaching me algebraic function fields, providing me significant documentation for my researh, listening me for hours, accompanying me throughout my study, a critical reading of my thesis and valuable suggestions in correcting my errors. I am also very pleased as I had a chance of being her final Ph.D student in her academic life at METU. Also, I want to express my gratitude to my co-supervisor Prof. Dr. Ferruh Özbudak for his great suggestions, supervision and providing me a version of MAGMA which was essential for my calculations. I also like to thank the other members of my Committee, Prof. Dr. Halil İbrahim Karakaş, Prof. Dr. Hurşit Önsiper, Prof Dr. Ersan Akyıldız and Assoc. Prof. Dr. Cem Güneri for their assistance.

It is a great pleasure to acknowledge a special dept of my gratitude to Prof. Dr. Hurşit Önsiper and Prof. Dr. Gilles Lachaud for their assistance in solving my problems on algebraic geometry. I also thank Prof. Dr. Dominique Le Brigand and Prof. Dr. Micheal Rosen for sending me their articles in order to use in my study.

I want to thank all my instructors, friends and all staff members in our department for making my METU life easy by helping me in various ways.

Finally, I would like to thank my family for their support, attention and encouragement.

TABLE OF CONTENTS

ABSTRACT iv

ÖZ vi

ACKNOWLEDGMENTS ix

TABLE OF CONTENTS x

LIST OF TABLES xii

CHAPTERS

1 INTRODUCTION 1

2 PRELIMINARIES 5

3 QUADRATIC FUNCTION FIELDS WITH CLASS NUMBER THREE 12

3.1 Elliptic and hyperelliptic function fields 12

3.2 Determination of elliptic function fields 15

3.3 Determination of hyperelliptic function fields 17

4 NON-QUADRATIC FUNCTION FIELDS WITH CLASS NUMBER THREE 33

4.1 Results for the non-hyperelliptic case 33

4.2 Projective spaces and complete linear systems 38

4.3 Proof of Theorem 4.1 45

4.3.1 Genus 3 and $q = 2$ 45

4.3.2 Genus 4 53

4.3.3 Genus 5 75

4.3.4 Genus 6 79

5 CYLOTOMIC FUNCTION FIELDS WITH CLASS NUMBER THREE 80

5.1 Construction of cyclotomic function fields 80

5.2 Character groups 82

5.3 Cyclotomic function fields and their subfields 83

5.4 Genus one case: 85

5.5 Genus two case: 97

5.5.1 $q = 2$ 97

5.5.2 $q = 3$ 98

APPENDICES

A Complete Intersection of Three Quadrics in Projective Space $\mathbb{P}^4(\mathbb{F}_2)$ 102

A.1 Non-degenerate Case: $l_1 : xy + zt + u^2 = 0$ 102

A.2 Degenerate Case $l_1 : xy + zt = 0$, ... 120

A.3 Degenerate Case $l_1 : xy + z^2 + zt + t^2 = 0$, ... 121

REFERENCES 126

VITA 127

LIST OF TABLES

TABLES

Table 2.1 Formulas for the S_i functions 9

Table 2.2 Formulas for the coefficients of the L-polynomials 9

Table 2.3 Formulas for the coefficients of the L-polynomials when $q = 2$ 10

Table 3.1 Complete squares in mod $p_i(x)$ 21

CHAPTER 1

INTRODUCTION

Let $\mathbb{F}_q$ be the finite field with q elements, and $K/\mathbb{F}_q$ an algebraic function field of one variable having $\mathbb{F}_q$ for full constant field with genus g. The principal divisors $P(K)$ of K is a subgroup of the abelian group D_K^0 of degree zero divisors of K. Then the quotient group $D_K^0/P(K)$ is a finite abelian group. Its order h_K is called the divisor class number of $K/\mathbb{F}_q$. It is well-known that $h_K = L(1)$, where L is the numerator polynomial of the zeta function of K.

In 1971, MacRae gave in [21] the quadratic function fields which have a place of degree one with ideal class number one. He showed that there is only one imaginary quadratic field if the characteristic of the finite field is not 2 and three if the characteristic is 2. The study on the algebraic function fields K over finite fields with $h_K = 1$ was done by Madan and Queen in [23] in 1972. They showed that when class number is one, $q = 2$ and genus is at most 4, $q = 3$ and genus is at most 2 or $q = 4$ and genus is at most 1. For $q = 2$, they derived necessary and sufficient conditions for class number to be one. When genus is three, they gave two examples to illustrate the given necessary and sufficient conditions. The question of the existence of fields of genus 4 with class number one was left open. They also proved that up to isomorphism, there is exactly one quadratic function field of class number one which has no place of degree one. The classification of algebraic function fields over finite fields with class number one is finished by Leitzel, Madan and Queen in [19] in 1975. They proved that there is no field of genus 4 over the field of 2 elements with class number one. They determined that, up to isomorphism, there are precisely two function fields of genus 3 with class number one when $q = 2$. They found possible cases for a field to have class number two and in each case, they derived necessary and sufficient conditions for that function field. They determined that up to isomorphism, there are eight imaginary quadratic function fields with class number two.

The divisor class number two problem for algebraic function fields was studied by Le Brigand. In [18] in 1996, up to isomorphism, she classified all quadratic algebraic function fields K with $h_K = 2$. She determined that, up to isomorphism, there are 13 imaginary function fields with ideal class number two. She proved that, up to isomorphism, there are 11 quadratic function fields over finite fields with divisor class number two. The classification of the non-quadratic function fields with class number two is done by Le Brigand in [17]. She proved that, up to isomorphism, there are only eight non-quadratic function fields K such that $h_K = 2$.

Let $k := \mathbb{F}_q(T)$ be the rational function field over the finite field $\mathbb{F}_q$ with q elements, and $\mathbb{A} := \mathbb{F}_q[T]$ the ring of polynomials. For $N \in \mathbb{A}$, there exists a field extension K_N, called the N^{th} cyclotomic function field K_N. It is an analogue of the classical cyclotomic number fields. These function fields were investigated by L. Carlitz in 1935 in [10].

In [13], Hayes developed the theory of cyclotomic function fields. Considering constant field extensions and wild ramification at the infinite place of k, he constructed the maximal abelian extension of k. By applying the Carlitz Theory with $1/T$ instead of T and $(1/T)^{\nu+1}$ instead of N, he constructed the fields F_ν and he defined the fixed field of F_ν under $\mathbb{F}_q^*$ as L_ν. Then the maximal abelian extension A of k is the composite $E.K_T.L_\infty$, where E is the union of all constant field extensions of k, K_T is the union of all cyclotomic function fields and L_∞ is the union of all fields L_ν.

In [8], Bilhan gave a proof for an analogue of the Kronecker-Weber theorem for rational function fields. That is, she proved that every finite abelian extension K of k is contained in a composite $N = k_n.K_M.L_\nu$, where k_n is a constant field extension of degree n, M is a non-zero polynomial in $\mathbb{A}$ and ν is a non-negative integer. This is called a (ν, n, M)-extension.

In [22], Madan proved that the class number of a function field $F/\mathbb{F}_q$ divides the class number of a function field E whenever E is a finite abelian Galois extension of F. If F is an abelian extension of k, then by the analogue [8] of the Kronecker-Weber theorem, F is contained in some (ν, n, M)-extension and by the above result the class number of F divides the class number of the (ν, n, M)-extension.

Let K be a subfield of a cyclotomic function field. By the conductor of K, we mean the monic polynomial $N \in \mathbb{A}$ such that K_N is the smallest cyclotomic function field containing K. K_N^+ denotes the maximal subfield of K_N such that the infinite place splits completely. Let $K^+ = K \cap K_N^+$ be the maximal real subfield of K. We say that K is a real extension of k if $K = K^+$ and imaginary otherwise. An imaginary extension K of k is called totally imaginary if $K^+ = k$.

In [16], Kida and Murabayashi determined all cyclotomic function fields and their maximal real subfields with divisor class number one, based on the previous results of Madan, Queen, Armitage and Macrae. They also determined which of these abelian extensions have genus one. In [2], Ahn and Jung determined all subfields of cyclotomic function fields with divisor class number one when $q \neq 2$, and all imaginary abelian extensions with relative divisor class number one. In [1], the same authors determined all subfields of cyclotomic function fields with genus one when $q \neq 2$. Moreover, in [3], they determined all subfields of cyclotomic function fields with divisor class number two and they also gave the generators of such fields, explicitly.

In this thesis, we classified all algebraic function fields and all subfields of cyclotomic function fields over finite fields with class number three. First we gave some preliminaries and facts on the theory of algebraic function fields. Since the extension is not a rational function field, genus is at least one. Then, we showed that when class number is three, then $q = 2$ and genus is at most 6, or $q = 3$ and genus is at most 3, or $q = 4$ and genus is at most 2, or $q = 5$ or 7 and genus is 1. We also gave the results calculated in [33] on the necessary and sufficient conditions for class number to be three.

In Chapter 3, we remarked some theorems presented in [17] in order to use in the determination of quadratic extensions. Then we classified the quadratic extensions over finite fields with class number three. Up to isomorphism, we determined that there are exactly 5 elliptic function fields and 10 hyper-elliptic function fields over finite fields with class number three.

In Chapter 4, we classified the non-quadratic extensions over finite fields with class number three. Up to isomorphism, we determined that there are at most 4 non-hyperelliptic function fields when genus is three and 58 non-hyperelliptic extensions when genus is four. We also proved that, up to isomorphism, there are at most 155 function fields of genus five with class number three as the complete intersection of three quadrics in $\mathbb{P}^4(\mathbb{F}_2)$. These are given in Appendix A. We also proved that for the given necessary and sufficient conditions [33], the numerator of the zeta function has a root with module different from $1/\sqrt{q}$, when genus is six and class number is three.

In the last chapter, we assume K is a finite abelian extension of k contained in a cyclotomic function field. In this chapter, we determine all subfields of cyclotomic function fields with class number three. First, we give some necessary and sufficient conditions for a subfield of a cyclotomic function field to have class number three. Then we conclude that if a subfield of a cyclotomic function field has class number three, then its genus is one or two. In section 2, we classify subfields of cyclotomic function fields of genus one. In section 3, we classify the subfields of genus two.

CHAPTER 2

PRELIMINARIES

Let $\mathbb{F}_q$ be the finite field with $q = p^n$ elements where p is a prime number. An algebraic function field K of one variable over $\mathbb{F}_q$ is an extension field $K \supset \mathbb{F}_q$ such that K is a finite extension of $\mathbb{F}_q(x)$ for some $x \in K$ which is transcendental over $\mathbb{F}_q$.

A valuation ring of the function field $K/\mathbb{F}_q$ is a subring $\mathcal{O} \subseteq K$ with the following properties:

(i) $\mathbb{F}_q \subsetneqq \mathcal{O} \subsetneqq K$ and

(ii) for any $z \in K$, either $z \in \mathcal{O}$ or $z^{-1} \in \mathcal{O}$.

It is a well-known fact that a valuation ring $\mathcal{O}$ is a local ring with unique maximal ideal $P = \mathcal{O} \setminus \mathcal{O}^*$ where $\mathcal{O}^*$ is the group of units of $\mathcal{O}$.

Theorem 2.1 (I.1.6, [32]) *Let $\mathcal{O}$ be a valuation ring of the function field $K/\mathbb{F}_q$ and P be its unique maximal ideal. Then*

(i) P is a principal ideal.

(ii) If $P = t\mathcal{O}$ then for any $0 \neq z \in K$ has a unique representation of the form $z = t^n u$ for some $n \in \mathbb{Z}$ and $u \in \mathcal{O}^*$.

(iii) $\mathcal{O}$ is a principal ideal domain.

A place of K is the maximal ideal P in some valuation ring $\mathcal{O}$ of K. Any element $t \in P$ such that $P = t\mathcal{O}$ is called a *prime element* for P. The set of all places of K is denoted by $\mathbb{P}_K$.

A discrete valuation of $K/\mathbb{F}_q$ is a function $v : K \to \mathbb{Z} \cup \{\infty\}$ with the following properties:

(i) $v(x) = \infty \Leftrightarrow x = 0$.

(ii) $v(xy) = v(x) + v(y)$ for any $x, y \in K$.

(iii) $v(x + y) \geq min\{v(x), v(y)\}$ for any $x, y \in K$.

(iv) There exists an element $z \in K$ with $v(x) = 1$.

(v)$v(a) = 0$ for any non-zero $a \in \mathbb{F}_q$.

Let $P \in \mathbb{P}_K$ and let $t \in P$ be a prime element. By Theorem 2.1, any $0 \neq z \in K$, has a unique representation $z = t^n u$ with $u \in \mathcal{O}_P^*$ and $n \in \mathbb{Z}$. To the place P, we associate a function $v_P : K \to \mathbb{Z} \cup \{\infty\}$ defined by $v_P(z) := n$ and $v_P(0) := \infty$. Clearly, this function is a discrete valuation of $K/\mathbb{F}_q$. For an arbitrary $w \in K$, $v_P(w) \neq 0$ only for finitely many $P \in \mathbb{P}_K$.

We say P is a zero(pole) of $x \in K$ if $x \in P (x \notin \mathcal{O}_P)$. The constant field is naturally embedded in the finite *residue field* $F_P = \mathcal{O}_P/P$ of P. The *degree* of P is defined as $degP := [F_P : \mathbb{F}_q]$.

The free abelian group D_K on the set $\mathbb{P}_K$ is called the divisor group of K. The map $deg : \mathbb{P}_K \to \mathbb{Z}$ is extended to D_K by linearity. The kernel of this map, the group of divisors of K of degree 0, is denoted by D_K^0. For each $x \in K^*$, its principal divisor is defined by $(x) := \sum_{P \in \mathbb{P}_K} v_P(x)P$. Then $(x) = (x)_0 - (x)_\infty$ where $(x)_0$, the zero divisor of x, and $(x)_\infty$, the pole divisor of x are positive divisors.

Theorem 2.2 *Let $x \in K \setminus \mathbb{F}_q$. Then $deg(x)_0 = deg(x)_\infty = [K : \mathbb{F}_q(x)]$.*

Definition 2.0.1 *Let D_K^0 denote the group of divisors of degree 0 and $P(K)$ denote the group of principal divisors of K. Clearly $P(K) \subseteq D_K^0$ and the order of the quotient group $D_K^0/P(K)$ is called the class number of K and it is finite. It is denoted by h_K.*

■

Definition 2.0.2 *For a divisor $D \in D_K$, we define the Riemann-Roch space associ-*

ated to D by

$$L(D) := \{x \in K^* : (x) \geq -D\} \cup \{0\}.$$

■

$L(D)$ is a finite dimensional vector space over $\mathbb{F}_q$ and $dimD := dim_{\mathbb{F}_q} L(D)$. The genus of K is defined as the nonnegative integer $g_K := max\{degD - dimD + 1 : D \in D_K\}$.

Theorem 2.3 (Riemann-Roch Theorem) *Let W be a canonical divisor of $K/\mathbb{F}_q$. Then, for any $A \in D_K$,*

$$dimA = degA + 1 - g + dim(W - A).$$

Thus for a canonical divisor W, we have $degW = 2g - 2$ and $dimW = g$.

The power series

$$Z_K(t) := \sum_{n=0}^{\infty} A_n t^n \in \mathbb{Z}[[t]],$$

with $A_n :=\mid \{D \in D_K : D \geq 0, degD = n\} \mid$, is called the zeta function of K. $Z_K(t) = L_K(t)/(1-t)(1-qt)$ where the numerator polynomial $L_K(t) \in \mathbb{Z}[t]$ is of degree $2g_K$. The numerator polynomial (*L-polynomial*) $L_K(t)$ satisfies the functional equation

$$L_K(t) = q^{g_K} t^{2g_K} L_K(1/qt).$$

Also, $L_K(1) = h_K$ is the class number of K.

Let $L_K(t) = 1 + a_1 t + a_2 t^2 + \cdots + a_{2g} t^{2g}$ denote the L-polynomial of K. Using the functional equation, we have the following relations among the coefficients of the L-polynomial of K:

$$a_{2g} = q^g, a_{2g-1} = q^{g-1} a_1, a_{2g-2} = q^{g-2} a_2, ..., a_{g+1} = q a_{g-1}.$$

By Theorem V.1.15 and Theorem V.2.1 of [32], $L(t)$ factors in $\mathbb{C}[t]$ in the form

$$L_K(t) = \prod_{i=1}^{2g} (1 - \alpha_i t),$$

where $|\alpha_i| = \sqrt{q}$.

Assume $h_K = 3$, then

$$3 = L_K(1) = \prod_{i=1}^{2g} |1 - \alpha_i| \geq \prod_{i=1}^{2g} (1 - |\alpha_i|) = (1 - \sqrt{q})^{2g}.$$

If $g = 0$, then $h_K = 1$. As $h_K = 3$, we must have $g \geq 1$ and $3 = L_K(1) \geq (1 - \sqrt{q})^2$.

Lemma 2.0.3 *$h_K = 3$ implies $g \geq 1$ and $2 \leq q \leq 7$.*

Lemma 2.0.4 *Let $2 \leq q \leq 7$. If $h_K = 3$ then one of the following conditions is satisfied:*

(i) $q = 2,\ 1 \leq g \leq 6$,

(ii) $q = 3,\ 1 \leq g \leq 3$,

(iii) $q = 4,\ 1 \leq g \leq 2$,

(iv) $q = 5, 7,\ g = 1$.

Proof. Let $\overline{K} = K\mathbb{F}_{q^{2g-1}}$ be the constant field extension of K of degree $2g - 1$. Let N_i denote the number of degree i places of $\overline{K}$. Then,

$$L_{\overline{K}}(t) = \prod_{i=1}^{2g} (1 - \beta_i t) \tag{2.1}$$

is the L-polynomial of $\overline{K}$ where $\beta_i \in \mathbb{C}$ and $|\beta_i| = q^{(2g-1)/2}$ for $i = 1, 2, ..., 2g$. On the other hand,

$$L_{\overline{K}}(t) = 1 + a_1 t + \cdots + a_{2g} t^{2g} \tag{2.2}$$

where $a_1 = N_1 - (q^{2g-1} + 1)$. Combining Equation 2.1 and Equation 2.2 , $a_1 = \beta_1 + \cdots + \beta_{2g}$. Thus, using the triangular inequality, $N_1 \geq q^{2g-1} + 1 - 2gq^{(2g-1)/2}$. Since N_1 comes from primes of K of degree dividing $2g-1$, the total number of positive divisors of K of degree $2g - 1$ is at least $(q^{2g-1} + 1 - 2gq^{(2g-1)/2})/(2g - 1)$. By Lemma V.1.4 of [32], the number of positive divisors of K of degree $2g-1$ is $h_K(q^g - 1)/(q - 1)$. Hence, $h_K > 3$ if

$$(q - 1)(q^{2g-1} + 1 - 2g.q^{\frac{2g-1}{2}}) > 3(q^g - 1)(2g - 1).$$

Using this inequality and Lemma 2.0.3, the result follows. ■

Lemma 2.0.5 *Let n_i denote the number of prime divisors of K of degree $i \in \mathbb{Z}^+$. If $h_K = 3$, then $n_1 \leq 3$.*

Proof.

Assume $n_1 > 3$. Let P_1, P_2, P_3, P_4 be distinct prime divisors of K of degree one. Then $P_1 - P_1, P_1 - P_2, P_1 - P_3, P_1 - P_4 \in Div^0(K)/P(K)$. However $|Div^0(K)/P(K)| = h_K$. Thus $P_1 - P_i \equiv P_1 - P_j \pmod{P(K)}$ for $i \neq j$. That is $P_i - P_j = (x) \in P(K)$ for some $x \in K \setminus \{0\}$. Then $[K : \mathbb{F}_q[x]] = deg(x)_0 = 1$ where $(x)_0$ denotes the zero divisor of x by Theorem I.4.11 of [32]. Then K is rational and $h_K = 1$, which is a contradiction. Thus $n_1 \leq 3$. ■

The symmetric functions S_i of the roots can be expressed in terms of n_i's (for $i = 1, 2, ..., 6$) as in the following table:

Table 2.1: Formulas for the S_i functions

$-S_1$	$n_1 - (q + 1)$
$-S_2$	$n_1 + 2n_2 - (q^2 + 1)$
$-S_3$	$n_1 + 3n_3 - (q^3 + 1)$
$-S_4$	$n_1 + 2n_2 + 4n_4 - (q^4 + 1)$
$-S_5$	$n_1 + 5n_5 - (q^5 + 1)$
$-S_6$	$n_1 + 2n_2 + 3n_3 + 6n_6 - (q^6 + 1)$

Then the coefficients a_i of the L-polynomial can be calculated by the following formulas:

Table 2.2: Formulas for the coefficients of the L-polynomials

a_1	$-S_1$
a_2	$\frac{-S_2+S_1^2}{2}$
a_3	$\frac{-S_1^3+3S_1S_2-2S_3}{6}$
a_4	$\frac{S_1^4-6S_1^2S_2+8S_1S_3+3S_2^2-6S_4}{24}$
a_5	$\frac{-S_1^5+10S_1^3S_2-15S_1S_2^2-20S_1^2S_3+20S_2S_3+30S_1S_4-24S_5}{120}$
a_6	$\frac{S_1^6-15S_1^4S_2+45S_1^2S_2^2-15S_2^3+40S_1^3S_3-120S_1S_2S_3-90S_1^2S_4+40S_3^2+90S_2S_4+144S_1S_5-120S_6}{720}$

In particular, when $q = 2$, we have the following formulas for the coefficients of L-polynomials:

Table 2.3: Formulas for the coefficients of the L-polynomials when $q = 2$

n_1	a_1	a_2	a_3	a_4	a_5
$n_1 = 0$	-3	$n_2 + 2$	$n_3 - 3n_2$	$n_4 - 3n_3 + \frac{n_2^2+5n_2}{2}$	$n_5 - 3n_4 + 2n_3 + n_2n_3 - \frac{3n_2^2+3n_2}{2}$
$n_1 = 1$	-2	n_2	$n_3 - 2n_2$	$n_4 - 2n_3 + \frac{n_2^2+n_2}{2}$	$n_5 - 2n_4 + n_2n_3 - (n_2^2 + n_2)$
$n_1 = 2$	-1	$n_2 - 1$	$n_3 - n_2 - 1$	$n_4 - n_3 - 1 + \frac{n_2^2-n_2}{2}$	$n_5 - n_4 - n_3 + n_2n_3 - 1 - \frac{n_2^2+3n_2}{2}$
$n_1 = 3$	0	$n_2 - 1$	$n_3 - 2$	$n_4 - 3 + \frac{n_2^2-n_2}{2}$	$n_5 - n_3 + n_2n_3 - 2n_2 - 4$

n_1	a_6
$n_1 = 0$	$n_6 - 3n_5 + 2n_4 + \frac{n_3+n_3^2}{2} + n_2n_4 - 3n_2n_3 + \frac{8n_2+9n_2^2+n_2^3}{6}$
$n_1 = 1$	$n_6 - 2n_5 + n_2n_4 - 2n_2n_3 + \frac{n_3^2+n_3}{2} + \frac{n_2^3+3n_2^2+2n_2}{6}$
$n_1 = 2$	$n_6 - n_5 - n_4 + n_2n_4 - n_2n_3 + \frac{n_3^2-n_3}{2} + \frac{n_2^3-7n_2}{6} - 1$
$n_1 = 3$	$n_6 - n_4 + n_4n_2 + \frac{n_3^2-3n_3}{2} + \frac{n_2^3-19n_2}{6} - 5$

Theorem 2.4 *[[33], Main Theorem] Let $K/\mathbb{F}_q$ be a function field of genus g. Then $h_K = 3$ if and only if one of the following conditions holds:*

(1) $g = 1$, $2 \leq q \leq 7$ and $n_1 = 3$.

(2) $g = 2$, $q = 2$ and $2n_2 + n_1^2 + n_1 = 10$

((i) $n_1 = 0$,$n_2 = 5$ or

(ii) $n_1 = 1, n_2 = 4$ or

(iii) $n_1 = 2, n_2 = 2$).

(3)$g = 2$, $q = 3$ and $2n_2 + n_1^2 + n_1 = 12$

((i)$n_1 = 0, n_2 = 6$ or

(ii) $n_1 = 1, n_2 = 5$ or

(iii) $n_1 = 2, n_2 = 3$).

(4)$g = 2$, $q = 4$, $h_K > 3$.

(5) $g = 3$, $q = 2$ and

(i) $n_1 = 0, n_3 = 3$,$n_2 \leq 13$ or

(ii) $n_1 = 1, n_2 + n_3 = 4$ *or*

(iii)$n_1 = 2, n_3 + 2n_2 = 3$.

(6) $g = 3,\ q = 3$, $h_K > 3$.

(7) $g = 4,\ q = 2$ *and*

(i) $n_1 = 0, 2n_4 + n_2^2 - 3n_2 = 6$ *and*

$$\left\{ \begin{array}{lll} n_2 = 0, & n_3 \le 11, & n_4 = 3 \text{ or} \\ n_2 = 1, & n_3 \le 8, & n_4 = 4 \text{ or} \\ n_2 = 2, & n_3 \le 6, & n_4 = 4 \text{ or} \\ n_2 = 3, & n_3 \le 3, & n_4 = 3 \text{ or} \\ n_2 = 4, & n_3 = 0, & n_4 = 1 \end{array} \right.$$

or

(ii) $n_1 = 1, 2n_4 + 2n_3 + n_2^2 - n_2 = 8$ *and*

$$\left\{ \begin{array}{lll} n_2 = 0, & n_3 \le 3, & 1 \le n_4 \le 4 \text{ or} \\ n_2 = 1, & n_3 = 0, & n_4 = 4 \end{array} \right.$$

(8) $g = 5,\ q = 2$ *and* $n_1 = 0, n_5 - 2n_3 + n_2 n_3 = 3,\ n_5 \neq 0$.

(9) $g = 6,\ q = 2$ *and* $n_1 = 0, n_6 - 2n_4 + (n_3 + n_3^2)/2 = 3,\ n_2 = 0,\ n_5 \le 6$.

CHAPTER 3

QUADRATIC FUNCTION FIELDS WITH CLASS NUMBER THREE

3.1 Elliptic and hyperelliptic function fields

Proposition 3.1 *[[18],Lemma 2.8 and Proposition 2.9] Assume* $char\mathbb{F}_q = 2$.

(1) Let $K/\mathbb{F}_q$ *be a quadratic function field (i.e,* $[K : \mathbb{F}_q(x)] = 2$ *for some* $x \in K$ *which is transcendental over* $\mathbb{F}_q$ *) of genus* $g \geq 2$. *Then* $K/\mathbb{F}_q$ *is a hyperelliptic function field and there exist* $x, y \in K$ *such that* $K = \mathbb{F}_q(x, y)$ *and*

$$y^2 + h(x)y = f(x)$$

with $h, f \in \mathbb{F}_q[x]$ *such that all zeros of* h *are simple zeros of* f *and*

$$deg(h) \leq g \text{ and } deg(f) = 2g + 1$$

or

$$deg(h) = g + 1 \text{ and } deg(f) \leq 2g + 2.$$

(2) Let $K = \mathbb{F}_q(x, y)$ *and* $y^2 + h(x)y = f(x)$ *for some* $x, y \in K$. *Then the places of* $\mathbb{F}_q(x)/\mathbb{F}_q$ *which ramify in* $K/\mathbb{F}_q$ *are*

all zeros of $h(x)$ *if* $deg(h) = g + 1$,

all zeros of $h(x)$ *and the pole of* (x) *if* $deg(h) \leq g$.

(3) Let $K/\mathbb{F}_q$ *be a hyperelliptic function field such that* $K = \mathbb{F}_q(x, y)$ *and* $y^2 + h(x)y =$

$f(x)$ for some $x, y \in K$ where

$$f(x) = \sum_{i=0}^{2m+1} b_i x^i + b x^{2m+2},$$

$$h(x) = \sum_{i=0}^{m} a_i x^i + a x^{m+1}.$$

(i) Let $Q \in \mathbb{F}_q[x]$ be a monic irreducible polynomial of degree r, c a root of Q in k_r, the extension of degree r of $\mathbb{F}_q$. We set $k_0 = \mathbb{F}_2$ and denote by tr_{k_r/k_0} the trace of k_r over k_0. Let α be the finite place of $\mathbb{F}_q(x)/\mathbb{F}_q$ associated to Q.

- *α ramifies if and only if Q divides $h(x)$.*
- *α splits if and only if $gcd(Q, h) = 1$ and $tr_{k_r/k_0}(f(c)/h(c)^2) = 0$.*
- *α is inert if and only if $gcd(Q, h) = 1$ and $tr_{k_r/k_0}(f(c)/h(c)^2) = 1$.*

(ii) Let β denote the infinite place of $\mathbb{F}_q(x)/\mathbb{F}_q$.

- *β ramifies if and only if $deg(h) \leq m$.*
- *β splits if and only if $deg(h) = m + 1$ and $t^2 + at + b$ is reducible over $\mathbb{F}_q[t]$.*
- *β is inert if and only if $deg(h) = m+1$ and t^2+at+b is irreducible over $\mathbb{F}_q[t]$.*

Proposition 3.2 *[[18], Lemma 2.6 and Proposition 2.7] Assume char$\mathbb{F}_q \neq 2$. Let $K/\mathbb{F}_q$ be a quadratic function field of genus $g \geq 2$. Then $K/\mathbb{F}_q$ is a hyperelliptic function field and there exist $x, y \in K$ such that $K = \mathbb{F}_q(x, y)$ and*

$$y^2 = f(x)$$

with a square-free polynomial $f \in \mathbb{F}_q[x]$ of degree $2g + 1$ or $2g + 2$.

(i) Let $Q \in \mathbb{F}_q[x]$ be a monic irreducible polynomial. Let α be the finite place of $\mathbb{F}_q(x)/\mathbb{F}_q$ associated to Q.

- *α ramifies if and only if Q divides f.*
- *α splits if and only if $gcd(Q, f) = 1$ and $f(x)$ is a square mod Q in $\mathbb{F}_q[x]$.*
- *α is inert if and only if $gcd(Q, f) = 1$ and $f(x)$ is not a square mod Q in $\mathbb{F}_q[x]$.*

(ii) Let β denote the infinite place of $\mathbb{F}_q(x)/\mathbb{F}_q$.

- *β ramifies if and only if $deg(f) = 2g + 1$.*
- *β splits if and only if $deg(f) = 2g + 2$ and the leading coefficient of f is a square in $\mathbb{F}_q$.*
- *β is inert if and only if $deg(f) = 2g + 2$ and the leading coefficient of f is not a square in $\mathbb{F}_q$.*

Proposition 3.3 *Let $K/\mathbb{F}_q$ be an elliptic function field, then there exists a place of K of degree one. Let P be a place of degree one of $K/\mathbb{F}_q$. There exist $x, y \in K$ such that $K = \mathbb{F}_q(x, y)$ and*

(1) If $charK = 2$, $y^2 + h(x)y + f(x) = 0$ where $h, f \in \mathbb{F}_q[x]$, $h(x) = ax + b$ is non-zero, f is monic and $deg(f) = 3$.

(2) If $charK \neq 2$, $y^2 + f(x) = 0$ where $f \in \mathbb{F}_q[x]$, f is monic, square-free and $deg(f) = 3$.

P is the unique place over the infinite place of $\mathbb{F}_q(x)/\mathbb{F}_q$. When $charK = 2$, a finite place Q of $\mathbb{F}_q(x)$ is ramified if the associated polynomial divides $h(x)$ and $f(x)$. When $charK \neq 2$, a finite place Q is ramified if the associated polynomial divides $f(x)$.

Proof. Let n_i denote the number of prime divisors of K of degree i. Since $g_K = 1$, $n_1 = h_K \geq 1$. Let P be a place of K of degree one. Then, by Riemann-Roch Theorem, $dim(2P) = deg(2P) - (g - 1)$. Thus $dim(2P) = 2$. Similarly, $dim(3P) = 3$. Let $\{1, x\}$ be a basis of $L(2P)$. Take $y \in L(3P) \setminus L(2P)$. Then $\{1, x, y\}$ is a basis for $L(3P)$ and $S = \{1, x, x^2, x^3, y, y^2, xy\}$ is a subset of $L(6P)$ of order 7. As the dimension of $L(6P)$ is 6, the elements of S are linearly dependent over $\mathbb{F}_q$. That is, there exist $a_i \in \mathbb{F}_q$ for $i = 1, ..., 7$ such that

$$a_1 y^2 + (a_2 + a_3 x)y + (a_4 x^3 + a_5 x^2 + a_6 x + a_7) = 0.$$

By Theorem 2.2, $[K : \mathbb{F}_q(x)] = 2$ and $[K : \mathbb{F}_q(y)] = 3$. Then, we may assume $a_1 = 1$ and $a_4 = 1$. Since $(x)_\infty = 2P$, P is the unique place above the infinite place of $\mathbb{F}_q(x)/\mathbb{F}_q$.

(1) When $charK = 2$, we have $K = \mathbb{F}_q(x, y)$ where

$$y^2 + h(x)y + f(x) = 0$$

with $h(x) = ax + b$ and $f(x) = x^3 + cx^2 + dx + e$ for $a, b, c, d, e \in \mathbb{F}_q$. Let Q be a finite place of $\mathbb{F}_q(x)$ which is ramified in $K/\mathbb{F}_q(x)$. Let c be a root of the associated polynomial of Q. Then $y^2 + h(c)y + f(c) = 0$ has a unique solution in y if and only if $h(c) = 0$. Thus $c \in \mathbb{F}_q$ and since finitely many places of $\mathbb{F}_q(x)$ are ramified, $h(x)$ is non-zero. Using the substitution $y \to y + f(c)^{q+2}$, we have $y^2 + h(x)y + f(x) + f(c) + f(c)^{q/2}h(x) = 0$. Let $\overline{f(x)} = f(x) + f(c) + f(c)^{q/2}h(x)$. Then $y^2 + h(x)y + \overline{f(x)} = 0$ where $\overline{f(c)} = 0$ and $\overline{f(x)}$ is a monic polynomial of degree three.

(2) When $charK \neq 2$, let $\overline{y} = (y + h(x)2^{-1})$ and $\overline{f(x)} = f(x) + h(x)^2$, then $\overline{y}^2 + \overline{f(x)} = 0$ where $deg\overline{f(x)} = 3$ and $\overline{f(x)}$ is monic. Let Q be a finite place of $\mathbb{F}_q(x)$ which is ramified in K and let c be a root of the associated polynomial of Q. Then $y^2 = \overline{f(c)}$ has a unique solution if and only if $\overline{f(c)} = 0$. Hence the result follows. Assume $\overline{f(x)}$ is not squarefree, then $y^2/(x+c)^2 + x + d = 0$ for some $c, d \in \mathbb{F}_q$. Substituting $y/(x+c) \to y$, $y^2 + x + d = 0$ and $K = \mathbb{F}_q(y)$, which contradicts $g_K = 1$. ■

3.2 Determination of elliptic function fields

Theorem 3.4 *Let $K/\mathbb{F}_q$ be a quadratic function field with class number 3 and $g = 1$. (That is, $[K : \mathbb{F}_q(x)] = 2$ for some $x \in K$, transcendental over $\mathbb{F}_q$.) Then there exist $x, y \in K$ such that $K = \mathbb{F}_q(x, y)$ satisfying one of the followings:*

(1) $y^2 + y = x^3$ for $q = 2$.

(2) $y^2 + x^3 + 2x^2 + 2x + 2 = 0$ for $q = 3$.

(3) $y^2 + \alpha y + x^3 = 0$ for $q = 4$ where $<\alpha>= \mathbb{F}_4^*$.

(4) $y^2 + x^3 + 4x + 2 = 0$ for $q = 5$.

(5) $y^2 + x^3 + 4 = 0$ for $q = 7$.

Proof.

1. Let $q = 2$. By Proposition 3.3, $K = \mathbb{F}_2(x, y)$ where $y^2 + h(x)y + f(x) = 0$ and $h(x) = ax + b$, $f(x) = x^3 + cx^2 + dx + e$ for $a, b, c, d, e \in \mathbb{F}_2$ and P_∞ is totally ramified in $K/\mathbb{F}_q(x)$. Since $h_K = n_1$ and $h_K = 3$, (x) and $(x + 1)$ are also ramified or one of them splits and the other one is inert. If both of them are ramified, then $x^2 + x$ divides $h(x)$, which contradicts that $deg(h) \leq 1$. Thus $deg(h) = 0$ and $h(x) = 1$. We assume (x) splits and $(x + 1)$ is inert. Then $y^2 + y + f(1) = 0$ has no rational solution, i.e. $f(1) = 1$. Since $y^2 + y + f(0) = 0$ has 2 distinct roots, $f(0) = 0$. We have $y^2 + y + x^3 + cx^2 + cx = 0$ for $c \in \mathbb{F}_2$. Using the substitution $x \to x + c$, we have $y^2 + y + x^3 + c = 0$. Substituting $y \to y + cx$ and $x \to x + c$, we have

$$y^2 + y + x^3 = 0. \tag{3.1}$$

2. Let $q = 3$. By Proposition 3.3, $K = \mathbb{F}_3(x, y)$ where $y^2 + f(x) = 0$ with $f(x) = x^3+cx^2+dx+e$ for $a, b, c, d, e \in \mathbb{F}_2$ and P_∞ is totally ramified in $K/\mathbb{F}_q(x)$. Since $h_K = n_1$, two of finite places of $\mathbb{F}_3(x)$ of degree one are ramified and the last one is inert or one of them splits and two of them are inert. Since f is squarefree, the first case is not possible. Assume (x) splits and $(x + 1)$ and $(x + 2)$ are inert. Then $y^2 + f(0) = 0$ has 2 rational distinct solutions, $y^2 + f(1) = 0$ and $y^2 + f(2) = 0$ have no solution. That is, $f(0) = 2$, $f(1) = 1$ and $f(2) = 1$. Hence,

$$y^2 + x^3 + 2x^2 + 2x + 2 = 0. \tag{3.2}$$

3. For $q = 4$, by Proposition 3.3, $K = \mathbb{F}_4(x, y)$ with $y^2 + h(x)y + f(x) = 0$ where $h(x) = ax + b$, $f(x) = x^3 + cx^2 + dx + e$ for $a, b, c, d, e \in \mathbb{F}_4$, and P_∞ is totally ramified in $K/\mathbb{F}_q(x)$. Since $h_K = n_1$, either two of the places of degree one are ramified or one of them splits and the others are inert. If two of them are ramified, then their product divides $h(x)$, which contradicts that $deg(h) \leq 1$. Thus $deg(h) = 0$. We assume (x) splits and $(x + i)$ is inert when $i \in \mathbb{F}_4^*$. Then $y^2 + h(i)y + f(i) = 0$ has no rational solution for $i \in \mathbb{F}_4^*$, Up to isomorphism $y \to ay + b$ for $a \in \mathbb{F}_4^*$ and $b \in \mathbb{F}_4$, we assume $h(x) = \alpha$ where $\alpha^2 + \alpha + 1 = 0$ and $f(x) = x^3 + cx^2 + dx$ for $c, d \in \mathbb{F}_4$. Up to isomorphism, we have

$$y^2 + \alpha y + x^3 = 0. \tag{3.3}$$

4. Let $q = 5$. By Proposition 3.3, $K = \mathbb{F}_5(x, y)$ with $y^2 + f(x) = 0$ where $f(x) = x^3 + cx^2 + dx + e$ for $a, b, c, d, e \in \mathbb{F}_2$, and P_∞ is totally ramified in $K/\mathbb{F}_q(x)$. Since $h_K = n_1$, two of the finite places of $\mathbb{F}_5(x)$ of degree one are ramified and the others are inert or one of them splits and four of them are inert. Since f is squarefree, the first case is not possible. Assume $(x+3)$ splits and the others are inert. Then $y^2+f(3) = 0$ has 2 rational distinct solutions, and $y^2+f(i) = 0$ has no solution for $i = 0, 1, 2, 4$. That is, $f(3) = 1$ or 4, $f(i) = 2$ or 3 for $i = 0, 1, 2, 4$. Hence,

$$y^2 + x^3 + 4x + 2 = 0. \tag{3.4}$$

5. Let $q = 7$. Similar to the case $q = 3$, we assume (x) splits and the others are inert. Then we have a unique solution. That is,

$$y^2 + x^3 + 4 = 0. \tag{3.5}$$

■

3.3 Determination of hyperelliptic function fields

We have finished the case 1 of Theorem 2.4. Now, under the assumption $K/\mathbb{F}_q$ is a quadratic function field, we examine the other cases of Theorem 2.4.

Theorem 3.5 *Let $K/\mathbb{F}_q$ be a quadratic function field of genus g with class number 3. $g \geq 2$ if and only if there exist $x, y \in K$ such that $K = \mathbb{F}_q(x, y)$ with*

1. *(for $q = 2$ and $g = 2$)*

   (i) $y^2 + y = x^5 + x^4 + 1$ where $L(t) = 4t^4 - 4t^3 + 4t^2 - 2t + 1$ or

   (ii) $y^2 + (x^3 + x + 1)y = x^3(x^3 + x + 1)$ where $L(t) = 4t^4 - 2t^3 + t^2 - t + 1$ or

   (iii) $y^2 + (x^3 + x^2 + 1)y = x^3(x^3 + x^2 + 1)$ where $L(t) = 4t^4 - 2t^3 + t^2 - t + 1$.

2. *(for $q = 3$ and $g = 2$)*

   (i) $y^2 = 2x^6 + x^2 + 2$ where $L(t) = 9t^4 - 12t^3 + 9t^2 - 4t + 1$ or

(ii) $y^2 = 2x^6 + x^5 + 2x^4 + x^3 + 2x^2 + x + 2$ *where* $L(t) = 9t^4 - 12t^3 + 9t^2 - 4t + 1$ *or*

(iii) $y^2 = x^5 + x^3 + x + 2$ *where* $L(t) = 9t^4 - 9t^3 + 5t^2 - 3t + 1$ *or*

(iv) $y^2 = 2x^6 + x^5 + x^4 + 2x^3 + x^2 + 1$ *where* $L(t) = 9t^4 - 6t^3 + t^2 - 2t + 1$.

3. *(for* $q = 2$ *and* $g = 3$*)*

   (i) $y^2 + y = x^7 + x^6 + 1$ *where* $L(t) = 8t^6 - 8t^5 + 4t^4 - 2t^3 + 2t^2 - 2t + 1$.

4. *(for* $q = 2$ *and* $g = 4$*)*

   (i) $y^2 + (x^5 + x^2 + 1)y = (x^5 + x^2 + 1)(x^5 + x^3 + 1)$ *where* $L(t) = 16t^8 - 24t^7 + 20t^6 - 14t^5 + 9t^4 - 7t^3 + 5t^2 - 3t + 1$ *or*

   (ii) $y^2 + (x^5 + x^2 + 1)y = (x^5 + x^2 + 1)(x^5 + x^4 + x^3 + x + 1)$ *where* $L(t) = 16t^8 - 24t^7 + 20t^6 - 18t^5 + 15t^4 - 9t^3 + 5t^2 - 3t + 1$.

Proof. By Theorem 2.4, we have the following cases:

1. Let $q = 2$ and $g = 2$. By Theorem 2.4, we have three cases:

   (i)Let $n_1 = 0$ and $n_2 = 5$. Then P_∞ is inert in $K/\mathbb{F}_q(x)$. By Proposition 3.1, $K = \mathbb{F}_q(x, y)$ for some $x, y \in K$ and

   $$y^2 + h(x)y = f(x),$$

   where $h(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, $f(x) = b_6x^6 + \cdots + b_0 \in \mathbb{F}_q[x]$ and $t^2 + a_3t + b_6$ is irreducible over $\mathbb{F}_2$. Then $b_6 = a_3 = 1$. Since $n_1 = 0$, all places of degree one of $\mathbb{F}_q(x)$ are inert. Since $n_2 = 5$, $(x^2 + x + 1)$ splits. Then $x, x + 1, x^2 + x + 1$ do not divide $h(x)$, i.e. $h(x)$ is irreducible of degree 3. Up to isomorphism, $h(x) = x^3 + x + 1$. Then $f(x) = h(x)g(x)$, where $g(x)$ is a polynomial of degree 3. Since $f(1) = f(0) = 1$, $g(1) = g(0) = 1$ and $g(x)$ is irreducible of degree 3 different from $h(x)$ by Proposition 3.1. That means $g(x) = x^3 + x^2 + 1$. Let c be a root of $Q = x^2 + x + 1$. Since $(x^2 + x + 1)$ splits, $tr_{k_2/k_0}(f(c)/h(c)^2) = 0$. However,

   $$\begin{aligned} tr_{k_2/k_0}(f(c)/h(c)^2) &= c^2g(c) + cg(c^2) \\ &= c + c^2 \\ &= 1, \end{aligned}$$

which is a contradiction.

(ii) Let $n_1 = 1$ and $n_2 = 4$. By Proposition 3.1, $K = \mathbb{F}_q(x, y)$ for some $x, y \in K$ and

$$y^2 + h(x)y = f(x),$$

where $h(x), f(x) \in \mathbb{F}_q[x]$.

Since $n_1 = 1$, only one of the places of $\mathbb{F}_q(x)$ of degree one is ramified and the others are inert. Up to isomorphism, we assume P_∞ is ramified. Since $n_2 = 4$, $(x^2 + x + 1)$ splits. By Proposition 3.1, $deg(h) \leq 2$ and $deg(f) = 5$ and the only ramified prime of $\mathbb{F}_q(x)$ of degree one is P_∞. $deg(h) = 1$ or 2 implies $h(x) = x, x+1, x^2, (x+1)^2, x(x+1)$ or $x^2 + x + 1$. Since the associated finite places are not ramified, this is not possible by part (2) and (3) of Proposition 3.1. Thus $h(x) = 1$ and $deg(f) = 5$ such that $f(0) = f(1) = 1$. Let

$$f(x) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

Since $f(0) = f(1) = 1$, $a_0 = 1$ and $a_4 + a_3 + a_2 + a_1 = 1$. Let c be a root of $Q = x^2 + x + 1$. Then $tr_{k_2/k_0}(f(c)/h(c)^2) = 0$, by part (3) of Proposition 3.1.

$$tr_{k_2/k_0}(f(c)/h(c)^2) = f(c) + f(c^2).$$

Hence $f(c) + f(c^2) = 0$. We have

$$\begin{aligned}
0 &= (c^5 + a_4c^4 + a_3c^3 + a_2c^2 + a_1c + 1) + (c^{10} + a_4c^8 + a_3c^6 + a_2c^4 + a_1c^2 + 1) \\
&= (c^2 + a_4c + a_3 + a_2c^2 + a_1c + 1) + (c + a_4c^2 + a_3 + a_2c + a_1c^2 + 1) \\
&= 1 + a_4 + a_2 + a_1.
\end{aligned}$$

We have $1 = a_4 + a_2 + a_1$ and $a_4 + a_3 + a_2 + a_1 = 1$, then $a_3 = 0$. We have 4 possibilities for $f(x)$. These are $x^5 + x^4 + x^2 + x + 1$, $x^5 + x^4 + 1$, $x^5 + x^2 + 1$ or $x^5 + x + 1$. But up to isomorphism $f(x)$ is unique and $K = \mathbb{F}_2(x, y)$ where

1.(i)

$$y^2 + y = x^5 + x^4 + 1, \tag{3.6}$$

for some $x, y \in K$.

(iii) Let $n_1 = 2$ and $n_2 = 2$. Then we have two possibilities:

• Two of the primes of $\mathbb{F}_2(x)$ of degree one are ramified and the last one is inert. Up to isomorphism, assume P_∞ and $(x+a)$ are ramified for $a \in \mathbb{F}_2$. Since $deg(h) \leq 2$ for this case, $h(x) = x + a$ or $(x+a)^2$. Thus (x^2+x+1) either is inert or splits. That is $n_2 = 1$ or 3. However $n_2 = 2$ and hence, there exists no solution for this case.

• One of the places of degree one splits and the others are inert.($n_2 \geq 2$) We may assume (x) splits. Since n_2 is exactly 2, (x^2+x+1) is inert. Since P_∞ is inert, $deg(h) = 3$ and $deg(f) = 6$ by part (2) and (3) of Proposition 3.1. $x, x+1, x^2+x+1$ do not divide $h(x)$, then $h(x)$ is irreducible of degree 3, $f(0) = 0$, $f(1) = 1$ and $h(x) \mid f(x)$. Then $f(x) = g(x)h(x)$ where $g(x) = x^3 + e_2x^2 + e_1x \in \mathbb{F}_2[x]$ ($f(0) = 0$ implies $g(0) = 0$ and $f(1) = 1$ implies $1 + e_2 + e_1 = 1$).

<u>Let $h(x) = x^3 + x + 1$.</u> Since (x^2+x+1) is inert,

$$tr_{k_2/k_0}(f(c)/h(c)^2) = tr_{k_2/k_0}(g(c)/h(c)) = 1.$$

$h(c) = c^3 + c + 1 = c$ and $h(c)^{-1} = c^2$. Thus

$$\begin{aligned}
1 &= c^2g(c) + (c^2)^2g(c^2) \\
&= c^2(c^3 + e_2c^2 + e_1c) + c^4(c^6 + e_2c^4 + e_1c^2) \\
&= (c^2 + c) + e_2(c + c^2) + e_1(1+1) \\
&= 1 + e_2.
\end{aligned}$$

Then $e_2 = 0$ and $e_3 = 1$. That is, $K = \mathbb{F}_2(x, y)$ satisfying the following equation:

1.(ii)

$$y^2 + (x^3 + x + 1)y = x^3(x^3 + x + 1). \tag{3.7}$$

<u>Let $h(x) = x^3 + x^2 + 1$.</u> Since (x^2+x+1) is inert,

$$\begin{aligned}
tr_{k_2/k_0}(f(c)/h(c)^2) &= tr_{k_2/k_0}(g(c)/h(c)) \\
&= 1.
\end{aligned}$$

$h(c) = c^3 + c^2 + 1 = c^2$ and $h(c)^{-1} = c$. Thus

$$\begin{aligned}
1 &= cg(c) + c^2 g(c^2) \\
&= c(c^3 + e_2c^2 + e_1c) + c^2(c^6 + e_2c^4 + e_1c^2) \\
&= (c + c^2) + e_2(1 + 1) + e_1(c^2 + c) \\
&= 1 + e_1.
\end{aligned}$$

Then $e_1 = 0$ and $e_3 = 1$. That is, $K = \mathbb{F}_2(x, y)$ satisfying the following equation:

1.(iii)

$$y^2 + (x^3 + x^2 + 1)y = x^3(x^3 + x^2 + 1) \tag{3.8}$$

2. Let $q = 3$ and $g = 2$. By Theorem 2.4 we have the following cases:

(i)Let $n_1 = 0$ and $n_2 = 6$. Then P_∞ and all finite places of degree one are inert. By Proposition 3.2, $f(x) = 2x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + 2$ and $f(1) = f(2) = 2$, which implies $a_2 + a_4 = 1$ and $a_1 + a_3 + a_5 = 0$. Let $p_1(x) = x^2 + 1$, $p_2(x) = x^2 + 2x + 2$, $p_3(x) = x^2 + x + 2$. Since $n_2 = 6$, we have the following two possibilities:

• Two of the places of degree two are ramified and one of them is inert. Assume (p_1) and (p_2) are ramified and (p_3) is inert. Then $f(x) = p_1(x).p_2(x).g(x)$ where $deg(g) = 2$. But f is square-free and $p_3, x, x+1, x+2$ does not divide f. Hence there exists no solution for this case.

• One of the places of degree one splits and the others are inert. Up to isomorphism, we may assume $(p_1(x))$ splits and the others are inert.

Table 3.1: Complete squares in mod $p_i(x)$

Mod $p_1(x)$	Mod $p_2(x)$	Mod $p_3(x)$
$1^2 \equiv 1$	$1^2 \equiv 1$	$1^2 \equiv 1$
$2^2 \equiv 1$	$2^2 \equiv 1$	$2^2 \equiv 1$
$x^2 \equiv 2$	$x^2 \equiv x + 1$	$x^2 \equiv 2x + 1$
$(x+1)^2 \equiv 2x$	$(x+1)^2 \equiv 2$	$(x+1)^2 \equiv x + 2$
$(x+2)^2 \equiv x$	$(x+2)^2 \equiv 2x + 2$	$(x+2)^2 \equiv 2$
$(2x)^2 \equiv 2$	$(2x)^2 \equiv x + 1$	$(2x)^2 \equiv 2x + 1$
$(2x+2)^2 \equiv 2x$	$(2x+2)^2 \equiv 2$	$(2x+2)^2 \equiv x + 2$
$(2x+1)^2 \equiv x$	$(2x+1)^2 \equiv 2x + 2$	$(2x+1)^2 \equiv 2$

Then

$$\begin{aligned} f(x) &\equiv 2.(2)^3 + a_5.2^2x + a_4.2^2 + a_3.2x + a_2.2 + a_1.x + 2(Mod\ p_1(x)) \\ &\equiv (a_5 + 2a_3 + a_1)x + (a_4 + 2a_2)(Mod\ p_1(x)). \end{aligned}$$

Since f(x) is a square Mod $p_1(x)$ by Proposition 3.2, $a_4+2a_2 = 0$ or $a_5+2a_3+a_1 = 0$ by the given table. As $(p_2(x))$ and $(p_3(x))$ are inert, we find 4 solutions. Up to isomorphism, we have two solutions. That is, there exist $x, y \in K$ such that $K = \mathbb{F}_q(x, y)$ satisfying one of the following equations:

2.(i)

$$y^2 = 2x^6 + x^2 + 2 \ or \tag{3.9}$$

2.(ii)

$$y^2 = 2x^6 + x^5 + 2x^4 + x^3 + 2x^2 + x + 2. \tag{3.10}$$

(ii)Let $n_1 = 1$ and $n_2 = 5$. Then one of the primes of degree one is ramified. Up to isomorphism, assume P_∞ is ramified and the others are inert, that is $n_2 \geq 3$. Since $n_2 = 5$, two of the places of degree two are ramified and the last one is inert or one of them splits and the others are inert.

• Let two of the places of degree two be ramified. Assume (p_i) and (p_j) are ramified. Since P_∞ is ramified, $degf = 5$ and $f(x) = p_i(x)p_j(x)g(x)$ where $deg(g) = 1$. This implies that there exists a ramified finite prime of degree one, which contradicts $n_1 = 1$.

• Let one of the places of degree two split and the others be inert. Since P_∞ is ramified, we deduce that $deg(f) = 5$ by Proposition 3.2. Since $p_i(x)$ does not divide $f(x)$ for all i, $f(b) \neq 0$ for all $b \in \mathbb{F}_3$ and f is square-free, $f(x)$ is monic irreducible of degree 5. That is

$$f(x) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

$f(0) = f(1) = f(2) = 2$ implies $a_0 = 2$, $a_4 + a_2 = 0$ and $a_3 + a_1 = 2$. Among 9 possibilities for $f(x)$, we get 3 isomorphic function fields for this case. Up to isomorphism, $K = \mathbb{F}_3(x, y)$ for some $x, y \in K$ satisfying the following equation:

2.(iii)

$$y^2 = x^5 + x^3 + x + 2. \tag{3.11}$$

(iii)Let $n_1 = 2$ and $n_2 = 3$. Then two of the primes of degree one are ramified and the others are inert or one of them splits and the others are inert.

• Let two of the places of degree one be ramified and the others be inert, that is $n_2 \geq 2$. But $n_2 = 3$. Thus one of the places of degree two is ramified. Assume (p_i) is the ramified prime of degree two.

<u>If P_∞ is inert,</u> then $deg(f) = 6$ by Proposition 3.2. Let $(x + a)$ and $(x + b)$ be two ramified primes of degree one. Thus $f(x) = p_i(x)(x + a)(x + b)g(x)$, where $deg(g) = 2$. Since f is square-free, only two of the finite places of degree one are ramified and p_j does not divide f for $j \neq i$, there exists no solution for $g(x)$.

<u>If P_∞ is ramified,</u> $degf = 5$. Let $(x + a)$ be the other ramified prime. We have $f(x) = p_i(x)(x+a)g(x)$ where $deg(g) = 2$. Using a similar argument, there exist no solution for $g(x)$.

• Let one of the places of degree one split. Up to isomorphism, we may assume (x) splits and the others are inert, that is $n_2 \geq 3$. But $n_2 = 3$. Hence all places of degree two are inert and $deg(f) = 6$. Let

$$f(x) = a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

Since P_∞ is inert, $a_6 = 2$ and since (x) splits, $a_0 = 1$. Also $f(1) = f(2) = 2$ implies $a_4 + a_2 = 2$ and $a_1 + a_3 + a_5 = 0$.

$$\begin{aligned} f(x) &\equiv 2.2^3 + a_5 2^2 x + a_4 2^2 + a_3 2x + a_2 2 + a_1 x + 1(Mod\ x^2 + 1) \\ &\equiv (a_5 + 2a_3 + a_1)x + (2 + a_4 + 2a_2)(Mod\ x^2 + 1), \end{aligned}$$

which is not a square Mod $x^2 + 1$. $(a_5 + 2a_3 + a_1) \neq 0$ and $(2 + a_4 + 2a_2) \neq 0$. Since $a_4 + a_2 = 2$, we have $a_2 \neq 2$ and $a_4 \neq 0$. Similarly $a_3 \neq 0$. As all the places of degree two are inert, up to isomorphism $K = \mathbb{F}_3(x, y)$ for some $x, y \in K$ satisfying the following equation:

2.(iv)

$$y^2 = 2x^6 + x^5 + x^4 + 2x^3 + x^2 + 1. \tag{3.12}$$

3. Let $g = 3$ and $q = 2$. By Theorem 2.4, we have the following cases:

(a) Let $n_1 = 0$ and $n_3 = 3$. By Proposition 3.1, $K = \mathbb{F}_2(x, y)$ where

$$y^2 + h(x)y = f(x)$$

for some $x, y \in K$ such that $deg(h) = 4$ and $deg(f) \leq 8$. Since there exist at least one ramified prime of degree three, say $(p(x))$, we have $h(x) = p(x)g(x)$ where $deg(g) = 1$. That means there exists a finite ramified prime of degree one and $n_1 \geq 1$, which is a contradiction.

(b) Let $n_1 = 1$ and $n_2 + n_3 = 4$. Then one of the primes of $\mathbb{F}_2(x)$ of degree one is ramified. Up to isomorphism, assume P_∞ is ramified. Then $K = \mathbb{F}_2(x, y)$ where

$$y^2 + h(x)y = f(x)$$

for some $x, y \in K$ such that $deg(h) \leq 3$ and $deg(f) = 7$. Also, (x) and $(x + 1)$ are inert and hence $n_2 = 2, 3$ or 4.

• Let $n_2 = 2$ and $n_3 = 2$. Since $deg(h) \leq 3$, one of the primes of degree three splits and the other one is inert. Also $(x^2 + x + 1)$ is inert. Hence there exist no finite ramified prime of degree ≤ 3 and $h(x) = 1$.

Let $f(x) = x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$. Since $f(0) = f(1) = 1$, $a_0 = 1$ and $a_6 + a_5 + a_4 + a_3 + a_2 + a_1 = 1$. Let n, m be roots of $x^3 + x + 1$ and $x^3 + x^2 + 1$, respectively. Then $n^7 = m^7 = 1$ and let c, c^2 be roots of $x^2 + x + 1 (c^3 = 1)$. Then

$$\begin{aligned}
tr_{k_2/k_0}(f(c)/h(c)^2) &= f(c) + f(c^2) \\
&= (c^7 + c^{14}) + a_6(c^6 + c^{12}) + a_5(c^5 + c^{10}) + a_4(c^4 + c^8) \\
&+ a_3(c^3 + c^6) + a_2(c^2 + c^4) + a_1(c + c^2) + (1 + 1) \\
&= 1 + a_5 + a_4 + a_2 + a_1 \\
&= 1.
\end{aligned}$$

This implies $a_6 + a_3 = 1$ and $a_5 + a_4 + a_2 + a_1 = 0$

$\star$ Up to isomorphism $x \to x+1$, we assume $(x^3 + x + 1)$ is inert and $(x^3 + x^2 + 1)$

splits. By part (3) of Proposition 3.1, we have respectively:

$$tr_{k_3/k_0}(f(n)/h(n)^2) = 1$$
$$tr_{k_3/k_0}(f(m)/h(m)^2) = 0$$
$$f(n)+f(n^2) + f(n^4) = 1$$
$$f(m)+f(m^2) + f(m^4) = 0$$

That is, we have explicitly:

$$(n^7 + n^{14} + n^{28}) + a_6(n^6 + n^{12} + n^{24}) + a_5(n^5 + n^{10} + n^{20}) + a_4(n^4 + n^8 + n^{16})+$$
$$a_3(n^3 + n^6 + n^{12}) + a_2(n^2 + n^4 + n^8) + a_1(n + n^2 + n^4) + (1 + 1 + 1) = 1$$
$$(m^7 + m^{14} + m^{28}) + a_6(m^6 + m^{12} + m^{24}) + a_5(m^5 + m^{10} + m^{20}) + a_4(m^4 + m^8 + m^{16})+$$
$$a_3(m^3 + m^6 + m^{12}) + a_2(m^2 + m^4 + m^8) + a_1(m + m^2 + m^4) + (1 + 1 + 1) = 0$$

$$a_6 + a_5 + a_3 = 1$$
$$a_4 + a_2 + a_1 = 0$$

We also have $a_6 + a_3 = 1$ and $a_5 + a_4 + a_2 + a_1 = 0$. Then $a_5 = 0$. We have eight equations satisfying the given conditions. Up to isomorphism, there exists a unique solution, that is, $K = \mathbb{F}_2(x, y)$ for some $x, y \in K$ where

3.(i)

$$y^2 + y = x^7 + x^6 + 1, \tag{3.13}$$

• Let $n_2 = 3$ and $n_3 = 1$. Since $n_1 = 1$ and $n_2 = 3$, $(x^2 + x + 1)$ is ramified and since $n_3 = 1$ one of the degree 3 primes is ramified. Hence their associated polynomials divide $h(x)$ and $deg(h) \geq 5$. But $deg(h) \leq g = 3$. So there exists no solution.

• Let $n_2 = 4$ and $n_3 = 0$. We have $deg(h) \leq 3$ and both of the primes of degree three are inert. Also, $(x^2 + x + 1)$ splits. Hence there exists no finite ramified prime of degree ≤ 3 and $h(x) = 1$.

Let $f(x) = x^7+a_6x^6+a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0$. Since $f(0) = f(1) = 1$, we have $a_0 = 1$ and $a_6 + a_5 + a_4 + a_3 + a_2 + a_1 = 1$. Let n, n^2, n^4 be roots of

x^3+x+1, then $n^7=1$. Let m, m^2, m^4 be roots of x^3+x^2+1, then $m^7=1$. Let c, c^2 be roots of x^2+x+1 ($c^3=1$). Then

$$\begin{aligned}
tr_{k_2/k_0}(f(c)/h(c)^2) &= f(c)+f(c^2)\\
&= 1+a_5+a_4+a_2+a_1\\
&= 0.
\end{aligned}$$

This implies $a_6+a_3=0$ and $a_5+a_4+a_2+a_1=1$.

Since both (x^3+x+1) and (x^3+x^2+1) are inert,

$$\begin{aligned}
tr_{k_3/k_0}(f(n)/h(n)^2) &= 1\\
tr_{k_3/k_0}(f(m)/h(m)^2) &= 1\\
f(n)+f(n^2)+f(n^4) &= 1\\
f(m)+f(m^2)+f(m^4) &= 1
\end{aligned}$$

More explicitly,

$$\begin{aligned}
&(n^7+n^{14}+n^{28})+a_6(n^6+n^{12}+n^{24})+a_5(n^5+n^{10}+n^{20})+a_4(n^4+n^8+n^{16})\\
&\quad +a_3(n^3+n^6+n^{12})+a_2(n^2+n^4+n^8)+a_1(n+n^2+n^4)+(1+1+1)=1\\
&(m^7+m^{14}+m^{28})+a_6(m^6+m^{12}+m^{24})+a_5(m^5+m^{10}+m^{20})+a_4(m^4+m^8+m^{16})\\
&\quad +a_3(m^3+m^6+m^{12})+a_2(m^2+m^4+m^8)+a_1(m+m^2+m^4)+(1+1+1)=1
\end{aligned}$$

Then $a_1+a_2+a_4=1$ and $a_6+a_5+a_3=1$. We also have $a_6+a_3=0$. Thus $a_3=a_6$ and $a_5=1$. $a_5+a_4+a_2+a_1=1$ implies $a_4+a_2+a_1=0$. which contradicts the first equation.

(c) Let $n_1=2$ and $2n_2+n_3=3$. Then one of the primes of $\mathbb{F}_2(x)$ of degree one splits and the others are inert or two of the primes of degree one are ramified.

• Let one of the primes of $\mathbb{F}_2(x)$ of degree one split and the others be inert, that is $n_2 \geq 2$. Since $2n_2+n_3=3$, we have $n_3 \leq -1$, which is not possible.

• Let two of the primes of degree one be ramified. Up to isomorphism, assume P_∞ and (x) are ramified. Then $deg(h) \leq 3$ and $deg(f)=7$. Since $(x+1)$ is inert and the extension is quadratic, $n_2 \geq 1$. But $2n_2 + n_3 = 3$ implies $n_2 \leq 1$. Thus $n_2 = 1$ and $n_3 = 1$. That means (x^2+x+1) is inert and one of the primes of degree three is ramified and the associated polynomial divides $h(x)$, but we also have x divides $h(x)$. Then $deg(h) \geq 4$, which is a contradiction.

4. Let $q = 2$ and $g = 4$. By Theorem 2.4 we have the following cases:

(a) Let $n_1 = 0, 2n_4 + n_2^2 - 3n_2 = 6$. By Proposition 3.1, $K = \mathbb{F}_2(x, y)$ where

$$y^2 + h(x)y = f(x)$$

for some $x, y \in K$ such that $deg(h) = 5$ and $deg(f) = 10$. Necessarily, $n_2 = 3$ or 4.

<ins>For $n_2 = 4$,</ins> $2n_4 + 16 - 12 = 6$, $n_4 = 1$. For this case (x^2+x+1) is ramified and one of the degree 4 places, say $(p(x))$ is ramified and $(x^2 + x + 1)p(x) \mid h(x)$. Then $deg(h) \geq 6$, which is not possible.

<ins>Hence $n_2 = 3$,</ins> $(x^2 + x + 1)$ is inert and $n_4 = 3$. One of the degree 4 places of K is over $(x^2 + x + 1)$ and two of them come from places of $\mathbb{F}_q(x)$ of degree four. If two of the places of degree four are ramified, then $deg(h) \geq 8$, which is a contradiction. Thus one of the places of degree four splits and the others are inert. Then $h(x)$ is irreducible of degree 5 and $f(x) = h(x)g(x)$ where $deg(g) = 5$. Let

$$g(x) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

As $f(0) = f(1) = 1$, we have $g(0) = g(1) = 1$. So $a_0 = 1$ and $a_4+a_3+a_2+a_1 = 1$.

We have 6 irreducible polynomials of degree 5. These are

(1) $x^5 + x^2 + 1$,

(2) $x^5 + x^4 + x^3 + x^2 + 1$,

(3) $x^5 + x^4 + x^2 + x + 1$,

(4) $x^5 + x^4 + x^3 + x + 1$,

(5) $x^5 + x^3 + x^2 + x + 1$,

(6) $x^5 + x^3 + 1$.

Up to isomorphisms $x \to x+1$ and $x \to 1/x$, $h(x)$ is x^5+x^2+1. Then $f(x) = (x^5+x^2+1)(x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+1)$. Let c and c^2 be roots of x^2+x+1. Then, $c^3=1$ and $h(c)=1=h(c^2)$.

Since the associated place is inert,

$$
\begin{aligned}
& tr_{k_2/k_0}(f(c)/h(c)^2) = f(c)+f(c^2) \\
& \quad = h(c)g(c)+h(c^2)g(c^2) \\
& \quad = g(c)+g(c^2) \\
& \quad = (c^5+a_4c^4+a_3c^3+a_2c^2+a_1c+1)+(c^{10}+a_4c^8+a_3c^6+a_2c^4+a_1c^2+1) \\
& \quad = (c^2+c)+a_4(c+c^2)+a_3(1+1)+a_2(c^2+c)+a_1(c+c^2)+(1+1) \\
& \quad = 1+a_4+a_2+a_1 \\
& \quad = 1.
\end{aligned}
$$

Thus $a_3 = 1$.

Let a, b, d be roots of x^4+x+1, $x^4+x^3+x^2+x+1$ and x^4+x^3+1 respectively. Then

$h(a) = a+1$, $a^{15}=1$, $h(a)^{-1}=a^{11}$ and $f(a)/h(a)^2 = g(a)/h(a) = a^{11}g(a)$,

$h(b) = b^2$, $b^5=1$, $h(b)^{-1}=b^3$ and $f(b)/h(b)^2 = g(b)/h(b) = b^3g(b)$,

$h(d) = d^8$, $d^{15}=1$, $h(d)^{-1}=d^7$ and $f(d)/h(d)^2 = g(d)/h(d) = d^8g(d)$.

$$
\begin{aligned}
tr_{k_4/k_0}(f(a)/h(a)^2) &= a^{11}g(a)+a^{22}g(a^2)+a^{44}g(a^4)+a^{88}g(a^8) \\
&= a_2+a_1.
\end{aligned}
$$

$$
\begin{aligned}
tr_{k_4/k_0}(f(b)/h(b)^2) &= b^3g(b)+b^6g(b^2)+b^{12}g(b^4)+b^{24}g(b^8) \\
&= a_5+a_4+1+a_1+1 \\
&= 1+a_4+a_1.
\end{aligned}
$$

$$
\begin{aligned}
tr_{k_4/k_0}(f(d)/h(d)^2) &= d^7g(d)+d^{14}g(d^2)+d^{28}g(d^4)+d^{56}g(d^8) \\
&= 1+a_2+a_1.
\end{aligned}
$$

Since one of the primes of degree four splits and the others are inert, we have the following cases:

First case: $a_1 + a_2 = 0$. Then $1 + a_2 + a_1 = 1 = 1 + a_4 + a_1$. So $a_4 = a_1 = a_2$. However, $1 + a_4 + a_2 + a_1 = 1$, hence $a_4 = a_1 = a_2 = 0$ and there exist $x, y \in K$ such that $K = \mathbb{F}_2(x, y)$ where

4.(i)

$$y^2 + (x^5 + x^2 + 1)y = (x^5 + x^2 + 1)(x^5 + x^3 + 1). \tag{3.14}$$

Second case: $a_1 + a_2 = 1$. Then $a_2 + a_1 = 1 + a_4 + a_1$ and $a_4 = a_1 = a_2 + 1$. However, $1 + a_4 + a_2 + a_1 = 1$, hence $1 = a_1 = a_4$, $a_2 = 0$ and there exist $x, y \in K$ such that $K = \mathbb{F}_2(x, y)$ where

4.(ii)

$$y^2 + (x^5 + x^2 + 1)y = (x^5 + x^2 + 1)(x^5 + x^4 + x^3 + x + 1). \tag{3.15}$$

We remark that the function field satisfying Equation 3.15 is not isomorphic to the one which satisfies Equation 3.14, since their L-polynomials are different.

(b) Let $n_1 = 1, 2n_4 + 2n_3 + n_2^2 - n_2 = 8$ and $n_2 = 0$ or 1. Since the extension is quadratic, two of the places of degree one are inert and n_2 is at least 2, which is a contradiction.

5. Let $q = 2$ and $g = 5$. By Theorem 2.4, we have $n_1 = 0, n_5 - 2n_3 + n_2 n_3 = 3$ and $n_5 > 0$. By Proposition 3.1, $K = \mathbb{F}_2(x, y)$ where

$$y^2 + h(x)y = f(x)$$

for some $x, y \in K$ such that $deg(h) = 6$ and $deg(f) = 12$. Since all places of $\mathbb{F}_q(x)$ of degree one are inert and the extension is quadratic, we have $n_2 = 3, 4$ or 5.

• For $n_3 = 0$, $n_5 = 3$, that is at least one of the places of degree five is ramified, say $(p(x))$. Then $h(x) = p(x)g(x)$ and $deg(g) = 1$, which implies that there exists a ramified prime of degree one and $n_1 \geq 1$. This is not possible.

• For $n_3 = 1$, $5 = n_5 + n_2$. Then there exist a ramified place of degree three, say $(q(x))$. Let $n_2 = 4$, then $n_5 = 1$ and we have a ramified place $(p(x))$ of degree five. Then $q(x)p(x) \mid h(x)$ and $deg(h) \geq 8$, which is not possible. Since $n_5 > 0$, we have $n_2 \neq 5$. Thus $n_2 = 3$.

For $n_2 = 3$, we have $n_5 = 2$. Then (x^2+x+1) is inert, one of the places of degree three is ramified, say $(q(x))$ and the other one is inert. One of the places of degree five splits and the others are inert. Then $h(x) = q(x)^2$ and $f(x) = q(x)g(x)$ where $deg(g) = 9$. Up to isomorphism, assume $q(x) = x^3 + x + 1$. Let

$$f(x) = (x^3+x+1)(x^9+a_8x^8+a_7x^7+a_6x^6+a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0)$$

Since $f(1) = f(0) = 1$, we have $a_0 = 1$ and

$$a_8 + a_7 + a_6 + a_5 + a_4 + a_3 + a_2 + a_1 = 1.$$

Let c be a root of $x^2 + x + 1$. Then $h(c)^2 (= (c^3 + c + 1)^2)^2 = c$ and $h(c)^{-2} = c^2$. Since $(x^2 + x + 1)$ is inert, we have

$$\begin{aligned} tr_{k_2/k_0}(f(c)/h(c)^2) &= tr_{k_2/k_0}(c^2(c^3 + c + 1)g(c)) \\ &= tr_{k_2/k_0}(g(c)) \\ &= a_8 + a_7 + a_5 + a_4 + a_2 + a_1 \\ &= 1. \end{aligned}$$

Then $a_6 + a_3 = 0$.

Let m be a root of $x^3 + x^2 + 1$, $h(m) = (m^2 + m)^2$. Since $(x^3 + x^2 + 1)$ is inert,

$$\begin{aligned} tr_{k_3/k_0}(f(m)/h(m)^2) &= tr_{k_3/k_0}((m^2 + m)g(m)/(m^2 + m)^4) \\ &= tr_{k_3/k_0}(g(m)/(m)^4) \\ &= a_8 + a_6 + a_5 + a_4 + a_1 \\ &= 1. \end{aligned}$$

Let s,t,u,v,w,j be roots of $x^5 + x^2 + 1, x^5 + x^3 + 1, x^5 + x^4 + x^3 + x^2 + 1, x^5 + x^4 + x^3 + x + 1, x^5 + x^4 + x^2 + x + 1, x^5 + x^3 + x^2 + x + 1$ respectively. Since one of the associated places splits and the others are inert,

$$\begin{aligned} tr_{k_5/k_0}(f(t)/h(t)^2) &= tr_{k_5/k_0}((t^{26})g(t)/t^{11}) \\ &= tr_{k_5/k_0}(t^{15}g(t)) \\ &= a_7 + a_6 + a_5 + a_4 + a_3 \\ &= a_7 + a_5 + a_4, \end{aligned}$$

as $a_6 + a_3 = 0$.

$$\begin{aligned} tr_{k_5/k_0}(f(v)/h(v)^2) &= tr_{k_3/k_0}((v^{17})g(v)/v^6) \\ &= tr_{k_3/k_0}(v^{11}g(v)) \\ &= 1 + a_7 + a_5 + a_4 \end{aligned}$$

Hence, $tr_{k_5/k_0}(f(t)/h(t)^2) + 1 = tr_{k_5/k_0}(f(v)/h(v)^2)$, that is, either $(x^5 + x^3 + 1)$ or $(x^5 + x^4 + x^3 + x + 1)$ splits and the others are inert.

$$\begin{aligned} tr_{k_5/k_0}(f(u)/h(u)^2) &= tr_{k_5/k_0}((u^{16})g(u)/u^2) \\ &= tr_{k_5/k_0}(u^{14}g(u)) \\ &= a_5 + a_3 + a_2 + 1 \\ &= 1. \end{aligned}$$

$$\begin{aligned} tr_{k_5/k_0}(f(j)/h(j)^2) &= tr_{k_3/k_0}((j^{10})g(j)/j^9) \\ &= tr_{k_3/k_0}(jg(j)) \\ &= a_6 + a_5 + a_2 \\ &= 1. \end{aligned}$$

As $a_3 = a_6$, we have $a_2 + a_3 + a_5 + 1 = a_2 + a_5 + a_3$, which is not possible.

• For $n_3 = 2$, we have $7 = n_5 + 2n_2$. For $n_2 \geq 4$, we get $n_5 \leq -1$, which is not possible. Then $n_2 = 3$ and $n_5 = 1$. There exists a ramified place of degree 5, say $(p(x))$ and $h(x) = p(x)g(x)$ where $deg(g) = 1$ and we have a ramified place of degree 1. That is $n_1 \geq 1$.

• For $n_3 = 3$, we get $9 = n_5 + 3n_2$. If $n_2 \geq 3$, then $n_5 \leq 0$, which is impossible. On the other hand, the extension is quadratic and $n_1 = 0$. That implies $n_2 \geq 3$. Thus we have no solution.

• For $n_3 \geq 4$, we have $3 = n_5 + n_3(n_2 - 2) \geq n_5 + 4(n_2 - 2) \geq n_5 + 4$, as $n_2 \geq 3$. That means $n_5 \leq -1$.

We deduce that there exists no quadratic function field with class number 3 for $q = 2$ and $g = 5$.

6. Let $q = 2$ and $g = 6$. By Theorem 2.4, we have $n_1 = 0$, $n_2 = 0$ and $n_6 - 2n_4 + n_2 n_4 + \frac{n_3^2 + n_3}{2} = 3$. Since all places of $k(x)$ of degree one are inert, n_2 must be at least 3. But $n_2 = 0$ and hence we have no quadratic extension with class number 3 for this case.

$\blacksquare$

CHAPTER 4

NON-QUADRATIC FUNCTION FIELDS WITH CLASS NUMBER THREE

4.1 Results for the non-hyperelliptic case

In this case, we assume K is a non-hyperelliptic function field of genus g with class number 3. Then $g \geq 3$ and $n_2 \leq 3$. By Theorem 2.4, we have the following cases:

(A) $g = 3$, $q = 2$ and

(i) $n_1 = 0, n_3 = 3$, $n_2 \leq 11$ or

(ii) $n_1 = 1, n_2 + n_3 = 4$ or

(iii) $n_1 = 2, n_3 + 2n_2 = 3$.

(B) $g = 4$, $q = 2$ and

(i) $n_1 = 0, 2n_4 + n_2^2 - 3n_2 = 6$ or

(ii) $n_1 = 1, 2n_4 + 2n_3 + n_2^2 - n_2 = 8$.

(C) $g = 5$, $q = 2$ and

$n_1 = 0, n_5 - 2n_3 + n_2 n_3 = 3$, $n_5 \geq 1$.

(D) $g = 6$, $q = 2$ and

$n_1 = 0, n_6 - 2n_4 + \frac{n_3 + n_3^2}{2} = 3$, $n_2 = 0$, $n_5 \leq 6$.

Theorem 4.1 *Let $K/\mathbb{F}_q$ be a non-hyperelliptic function field of genus g with class number 3. If $g \neq 5$, then there exist $x, y \in K$ such that $K = \mathbb{F}_q(x, y)$ satisfying one of the following equations:*

(A) *(for $q = 2$ and $g = 3$)*

(1) $y^3 + y + x^4 + x + 1 = 0$ *with* $L(t) = 8t^6 - 8t^5 + 4t^3 - 2t + 1$.

(2) $y^4 + y^3 + (x^3 + 1)y + x^4 + x + 1 = 0$ *with* $L(t) = 8t^6 - 8t^5 + 2t^4 + t^3 + t^2 - 2t + 1$.

(3) $y^4 + y^3 + xy^2 + (x^3 + x + 1)y + (x^4 + x + 1) = 0$ *with* $L(t) = 8t^6 - 8t^5 + 4t^4 - 2t^3 + 2t^2 - 2t + 1$.

(4) $y^3 + x^2y + x^4 + x^3 + x = 0$ *with* $L(t) = 8t^6 - 4t^5 - 2t^4 + 2t^3 - t^2 - t + 1$.

(B) *(for $q = 2$ and $g = 4$)*

(I-1) $x^3y^3 + x^3y^2 + x^3 + x^2y^3 + x^2y^2 + x + y^3 + y^2 + 1 = 0,$ *with* $L(t) = 16t^8 - 24t^7 + 12t^6 + 6t^5 - 11t^4 + 3t^3 + 3t^2 - 3t + 1$.

(I-2) $x^3y^3 + x^3y^2 + x^3 + x^2y^3 + xy^2 + x + y^3 + y^2 + 1 = 0,$ *with* $L(t) = 16t^8 - 24t^7 + 12t^6 + 6t^5 - 11t^4 + 3t^3 + 3t^2 - 3t + 1$.

(I-3) $x^3y^3 + x^3y^2 + x^3 + x^2y^2 + xy^3 + x + y^3 + y^2 + 1 = 0,$ *with* $L(t) = 16t^8 - 24t^7 + 8t^6 + 12t^5 - 15t^4 + 6t^3 + 2t^2 - 3t + 1$.

(I-4) $x^3y^3 + x^3y^2 + x^3 + x^2y^3 + x^2y^2 + x^2y + xy + x + y^3 + y^2 + 1 = 0,$ *with* $L(t) = 16t^8 - 24t^7 + 8t^6 + 12t^5 - 15t^4 + 6t^3 + 2t^2 - 3t + 1$.

(I-5) $x^3y^3 + x^3y^2 + x^3 + x^2y^3 + x^2y + xy^2 + xy + x + y^3 + y^2 + 1 = 0,$ *with* $L(t) = 16t^8 - 24t^7 + 8t^6 + 12t^5 - 15t^4 + 6t^3 + 2t^2 - 3t + 1$.

(I-6) $x^3y^3 + x^3y^2 + x^3 + x^2y^2 + x^2y + xy^3 + xy^2 + x + y^3 + y^2 + 1 = 0,$ *with* $L(t) = 16t^8 - 24t^7 + 12t^6 + 6t^5 - 11t^4 + 3t^3 + 3t^2 - 3t + 1$.

(I-7) $x^3y^3 + x^3y^2 + x^3 + x^2y^3 + x^2y^2 + x^2 + xy^2 + xy + y^3 + y^2 + 1 = 0,$ *with* $L(t) = 16t^8 - 24t^7 + 12t^6 + 6t^5 - 11t^4 + 3t^3 + 3t^2 - 3t + 1$.

(I-8) $x^3y^3 + x^3y^2 + x^3 + x^2y^2 + x^2y + x^2 + xy^3 + xy + y^3 + y^2 + 1 = 0,$ *with* $L(t) = 16t^8 - 24t^7 + 8t^6 + 12t^5 - 15t^4 + 6t^3 + 2t^2 - 3t + 1$.

(II-1) $y^3 + (x^4 + x^2 + 1)y + (x^6 + x^4 + x^3 + x^2 + 1) = 0$, *with* $L(t) = 16t^8 - 24t^7 + 12t^6 + 4t^5 - 8t^4 + 2t^3 + 3t^2 - 3t + 1$.

(II-2) $y^3 + x^4y + x^6 + x^4 + x = 0$, *with* $L(t) = 16t^8 - 16t^7 + 4t^5 - 2t^4 + 2t^3 - 2t + 1$.

(II-3) $y^3 + (x^6 + x)y^2 + x^4y + x^3 + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 12t^6 + 4t^5 - 8t^4 + 2t^3 + 3t^2 - 3t + 1$.

(II-4) $y^3 + xy^2 + y^2 + x^4y + x^2y + x^6 + x^3 + x^2 + 1 = 0$, *with* $L(t) = 16t^8 - 16t^7 + 4t^5 - 2t^4 + 2t^3 - 2t + 1$.

(III-1) $y^6 + xy^5 + (x^3 + x + 1)y^3 + x^2y^2 + (x^5 + x^3)y + x^6 + x^2 + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 12t^6 - 2t^4 + 3t^2 - 3t + 1$.

(III-2) $y^6 + xy^5 + (x^3 + x + 1)y^3 + x^2y^2 + (x^5 + x^3 + x^2 + x)y + x^6 + x + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 12t^6 - 2t^4 + 3t^2 - 3t + 1$.

(III-3) $y^6 + xy^5 + (x^3 + 1)y^3 + (x^2 + x)y^2 + (x^5 + x^3 + x^2 + x)y + x^6 + x^4 + x^3 + x + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 12t^6 - 2t^4 + 3t^2 - 3t + 1$.

(III-4) $y^6 + xy^5 + x^3y^3 + (x^2 + x + 1)y^2 + (x^5 + x^3)y + x^6 + x^4 + x^3 + x + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 12t^6 - 2t^4 + 3t^2 - 3t + 1$.

(III-5) $y^6 + xy^5 + xy^4 + (x^3 + 1)y^3 + (x^3 + x + 1)y^2 + (x^5 + x^2 + x + 1)y + x^6 + x^5 + x^3 + x^2 + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 12t^6 - 2t^4 + 3t^2 - 3t + 1$.

(III-6) $y^6 + xy^5 + xy^4 + (x^3 + x)y^3 + (x^3 + x^2 + x + 1)y^2 + (x^5 + x^3 + x)y + x^6 + x^5 + x^2 + x + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 12t^6 - 2t^4 + 3t^2 - 3t + 1$.

(III-7) $y^6 + xy^5 + xy^4 + (x^3 + x)y^3 + (x^3 + x^2 + x)y^2 + (x^5 + x^3 + x^2 + 1)y + x^6 + x^5 + x^3 + x^2 + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 12t^6 - 2t^4 + 3t^2 - 3t + 1$.

(III-8) $y^6 + xy^5 + xy^4 + (x^3 + x + 1)y^3 + (x^3 + x^2 + 1)y^2 + (x^5 + x^3 + x^2 + x + 1)y + x^6 + x^5 + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 12t^6 - 2t^4 + 3t^2 - 3t + 1$.

(III-9) $y^6 + xy^5 + (x^3 + 1)y^3 + xy^2 + x^5y + x^6 + x^3 + x^2 + x + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 16t^6 - 10t^5 + 8t^4 - 5t^3 + 4t^2 - 3t + 1$.

(III-10) $y^6 + xy^5 + (x^3 + 1)y^3 + xy^2 + (x^5 + x^2 + x)y + x^6 + x + 1 = 0$, *with* $L(t) =$

$16t^8 - 24t^7 + 16t^6 - 8t^5 + 5t^4 - 4t^3 + 4t^2 - 3t + 1$.

(III-11) $y^6 + xy^5 + (x^3+x+1)y^3 + x^2y^2 + (x^5+x^3+x^2+x)y + x^6 + x^3 + x^2 + x + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 16t^6 - 8t^5 + 5t^4 - 4t^3 + 4t^2 - 3t + 1$.

(III-12) $y^6 + xy^5 + (x^3+x)y^3 + (x^2+x)y^2 + (x^5+x^3+x+1)y + x^6 + x + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 16t^6 - 10t^5 + 8t^4 - 5t^3 + 4t^2 - 3t + 1$.

(III-13) $y^6 + xy^5 + (x^3+x)y^3 + x^2y^2 + (x^5+x^3+x^2+x+1)y + x^6 + x^3 + x^2 + x + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 16t^6 - 6t^5 + 2t^4 - 3t^3 + 4t^2 - 3t + 1$.

(III-14) $y^6 + xy^5 + (x^3+1)y^3 + (x^2+x)y^2 + (x^5+x^3)y + x^6 + x^4 + x^3 + x + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 16t^6 - 8t^5 + 5t^4 - 4t^3 + 4t^2 - 3t + 1$.

(III-15) $y^6 + xy^5 + (x^3+1)y^3 + x^2y^2 + (x^5+x^3+x)y + x^6 + x^4 + x^3 + x + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 16t^6 - 6t^5 + 2t^4 - 3t^3 + 4t^2 - 3t + 1$.

(III-16) $y^6 + xy^5 + x^3y^3 + x^2y^2 + (x^5+x^3+x^2+1)y + x^6 + x^4 + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 16t^6 - 6t^5 + 2t^4 - 3t^3 + 4t^2 - 3t + 1$.

(III-17) $y^6 + xy^5 + x^3y^3 + x^2y^2 + (x^5+x^3+x+1)y + x^6 + x^4 + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 16t^6 - 10t^5 + 8t^4 - 5t^3 + 4t^2 - 3t + 1$.

(III-18) $y^6 + xy^5 + x^3y^3 + x^2y^2 + (x^5+x^3+x+1)y + x^6 + x^4 + x^3 + x^2 + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 16t^6 - 8t^5 + 5t^4 - 4t^3 + 4t^2 - 3t + 1$.

(III-19) $y^6 + xy^5 + (x^3+1)y^3 + (x^2+1)y^2 + (x^5+x^3+x+1)y + x^6 + x^4 + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 16t^6 - 6t^5 + 2t^4 - 3t^3 + 4t^2 - 3t + 1$.

(III-20) $y^6 + xy^5 + (x^3+1)y^3 + (x^2+x+1)y^2 + (x^5+x^3+x^2+x+1)y + x^6 + x^4 + x^3 + x^2 + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 16t^6 - 6t^5 + 2t^4 - 3t^3 + 4t^2 - 3t + 1$.

(III-21) $y^6 + xy^5 + y^4 + x^3y^3 + (x^2+x)y^2 + (x^5+x)y + x^6 + x^4 + x^3 + x + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 16t^6 - 10t^5 + 8t^4 - 5t^3 + 4t^2 - 3t + 1$.

(III-22) $y^6 + xy^5 + y^4 + (x^3+1)y^3 + (x^2+1)y^2 + x^5y + x^6 + x^4 + x^3 + x + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 16t^6 - 8t^5 + 5t^4 - 4t^3 + 4t^2 - 3t + 1$.

(III-23) $y^6 + xy^5 + xy^4 + (x^3+x+1)y^3 + (x^3+x^2)y^2 + (x^5+x^3)y + x^6 + x^5 + x^2 + x + 1 = 0$,

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 8t^5 + 5t^4 - 4t^3 + 4t^2 - 3t + 1.$

(III-24) $y^6 + xy^5 + xy^4 + (x^3 + x + 1)y^3 + (x^3 + x^2 + x)y^2 + (x^5 + x^3 + x)y + x^6 + x^5 + x^2 + x + 1 = 0,$ *with* $L(t) = 16t^8 - 24t^7 + 16t^6 - 6t^5 + 2t^4 - 3t^3 + 4t^2 - 3t + 1.$

(III-25) $y^6 + xy^5 + xy^4 + (x^3 + x)y^3 + (x^3 + x^2 + x)y^2 + (x^5 + x^3 + x + 1)y + x^6 + x^5 + 1 = 0,$ *with* $L(t) = 16t^8 - 24t^7 + 16t^6 - 6t^5 + 2t^4 - 3t^3 + 4t^2 - 3t + 1.$

(III-26) $y^6 + xy^5 + xy^4 + (x^3 + x)y^3 + (x^3 + x^2 + x)y^2 + (x^5 + x^3 + x + 1)y + x^6 + x^5 + x^3 + x^2 + 1 = 0,$ *with* $L(t) = 16t^8 - 24t^7 + 16t^6 - 10t^5 + 8t^4 - 5t^3 + 4t^2 - 3t + 1.$

(III-27) $y^6 + xy^5 + xy^4 + (x^3 + x + 1)y^3 + (x^3 + x^2 + x + 1)y^2 + (x^5 + x^3 + x + 1)y + x^6 + x^5 + x^3 + x^2 + 1 = 0,$ *with* $L(t) = 16t^8 - 24t^7 + 16t^6 - 8t^5 + 5t^4 - 4t^3 + 4t^2 - 3t + 1.$

(III-28) $y^6 + xy^5 + xy^4 + (x^3 + x)y^3 + (x^3 + x^2 + x)y^2 + (x^5 + x^3 + x^2 + 1)y + x^6 + x^5 + x^2 + x + 1 = 0,$ *with* $L(t) = 16t^8 - 24t^7 + 16t^6 - 10t^5 + 8t^4 - 5t^3 + 4t^2 - 3t + 1.$

(III-29) $y^6 + xy^5 + xy^4 + (x^3 + x + 1)y^3 + (x^3 + x^2 + 1)y^2 + (x^5 + x^3 + x^2 + x + 1)y + x^6 + x^5 + x^2 + x + 1 = 0,$ *with* $L(t) = 16t^8 - 24t^7 + 16t^6 - 6t^5 + 2t^4 - 3t^3 + 4t^2 - 3t + 1.$

(III-30) $y^6 + xy^5 + (x + 1)y^4 + (x^3 + x + 1)y^3 + (x^3 + x^2 + 1)y^2 + (x^5 + x)y + x^6 + x^5 + x^2 + x + 1 = 0,$ *with* $L(t) = 16t^8 - 24t^7 + 16t^6 - 6t^5 + 2t^4 - 3t^3 + 4t^2 - 3t + 1.$

(III-31) $y^6 + xy^5 + (x + 1)y^4 + (x^3 + 1)y^3 + (x^3 + x + 1)y^2 + (x^5 + x^3 + x)y + x^6 + x^5 + x^2 + x + 1 = 0,$ *with* $L(t) = 16t^8 - 24t^7 + 16t^6 - 10t^5 + 8t^4 - 5t^3 + 4t^2 - 3t + 1.$

(III-32) $y^6 + xy^5 + (x^3 + 1)y^3 + xy^2 + (x^5 + x^2 + x)y + x^6 + x^3 + 1 = 0,$ *with* $L(t) = 16t^8 - 24t^7 + 20t^6 - 16t^5 + 12t^4 - 8t^3 + 5t^2 - 3t + 1.$

(III-33) $y^6 + xy^5 + x^3y^3 + y^2 + (x^5 + x)y + x^6 + x + 1 = 0,$ *with* $L(t) = 16t^8 - 24t^7 + 20t^6 - 16t^5 + 12t^4 - 8t^3 + 5t^2 - 3t + 1.$

(III-34) $y^6 + xy^5 + (x^3 + x + 1)y^3 + (x^2 + x)y^2 + (x^5 + x^3 + x)y + x^6 + x^3 + 1 = 0,$ *with* $L(t) = 16t^8 - 24t^7 + 20t^6 - 16t^5 + 12t^4 - 8t^3 + 5t^2 - 3t + 1.$

(III-35) $y^6 + xy^5 + (x^3 + x)y^3 + x^2y^2 + (x^5 + x^3 + 1)y + x^6 + x + 1 = 0,$ *with* $L(t) = 16t^8 - 24t^7 + 20t^6 - 16t^5 + 12t^4 - 8t^3 + 5t^2 - 3t + 1.$

(III-36) $y^6 + xy^5 + y^4 + (x^3 + 1)y^3 + (x^2 + x + 1)y^2 + (x^5 + x)y + x^6 + x^4 + x^3 + x^2 + 1 = 0,$

with $L(t) = 16t^8 - 24t^7 + 20t^6 - 16t^5 + 12t^4 - 8t^3 + 5t^2 - 3t + 1$.

(III-37) $y^6 + xy^5 + xy^4 + x^3y^3 + (x^3+1)y^2 + (x^5+x)y + x^6 + x^5 + x^3 + x + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 20t^6 - 16t^5 + 12t^4 - 8t^3 + 5t^2 - 3t + 1$.

(III-38) $y^6 + xy^5 + xy^4 + (x^3+1)y^3 + (x^3+1)y^2 + (x^5+x^2+1)y + x^6 + x^5 + x^3 + x^2 + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 20t^6 - 16t^5 + 12t^4 - 8t^3 + 5t^2 - 3t + 1$.

(III-39) $y^6 + xy^5 + xy^4 + (x^3+1)y^3 + (x^3+x^2+1)y^2 + (x^5+x^3+x^2+1)y + x^6 + x^5 + x^4 + x^2 + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 20t^6 - 16t^5 + 12t^4 - 8t^3 + 5t^2 - 3t + 1$.

(III-40) $y^6 + xy^5 + (x+1)y^4 + (x^3+x+1)y^3 + (x^3+x^2)y^2 + (x^5+x+1)y + x^6 + x^5 + x^3 + x^2 + 1 = 0$, *with* $L(t) = 16t^8 - 24t^7 + 20t^6 - 16t^5 + 12t^4 - 8t^3 + 5t^2 - 3t + 1$.

(III-41) $y^6 + xy^5 + (x^3+x+1)y^3 + x^2y^2 + (x^5+x^3+x^2)y + x^6 + x^2 + x + 1 = 0$, *with* $L(t) = 16t^8 - 16t^7 + 4t^6 - 4t^5 + 5t^4 - 2t^3 + t^2 - 2t + 1$.

(III-42) $y^6 + xy^5 + (x^3+x+1)y^3 + (x^2+x+1)y^2 + (x^5+x^3)y + x^6 + x^3 + x = 0$, *with* $L(t) = 16t^8 - 16t^7 + 4t^6 - 4t^5 + 5t^4 - 2t^3 + t^2 - 2t + 1$.

(III-43) $y^6 + xy^5 + (x^3+1)y^3 + (x^2+x+1)y^2 + (x^5+x^3+x^2)y + x^6 + x^4 + x^3 + x + 1 = 0$, *with* $L(t) = 16t^8 - 16t^7 + 4t^6 - 4t^5 + 5t^4 - 2t^3 + t^2 - 2t + 1$.

(III-44) $y^6 + xy^5 + y^4 + (x^3+1)y^3 + (x^2+x)y^2 + x^5y + x^6 + x^4 + x^3 + x + 1 = 0$, *with* $L(t) = 16t^8 - 16t^7 + 4t^6 - 4t^5 + 5t^4 - 2t^3 + t^2 - 2t + 1$.

(III-45) $y^6 + xy^5 + xy^4 + x^3y^3 + x^3y^2 + (x^5+x^2+x+1)y + x^6 + x^5 + 1 = 0$, *with* $L(t) = 16t^8 - 16t^7 + 4t^6 - 4t^5 + 5t^4 - 2t^3 + t^2 - 2t + 1$.

(III-46) $xy^4 + xy^3 + (x^3+x^2+1)y^2 + (x^3+x+1)y + x^5 + x^3 + x^2 + x + 1 = 0$, *with* $L(t) = 16t^8 - 16t^7 + 4t^6 - 4t^5 + 5t^4 - 2t^3 + t^2 - 2t + 1$.

4.2 Projective spaces and complete linear systems

For a field $\mathbf{k}$, the projective n-space, $\mathbb{P}^n(\mathbf{k})$, is defined by

$$\mathbb{P}^n(\mathbf{k}) := (\mathbf{k}^{n+1} \setminus \{0_{\mathbf{k}^{n+1}}\})/\sim$$

with the equivalence relation $(x_0 : x_1 : \cdots : x_n) \sim (\lambda x_0 : \lambda x_1 : \cdots : \lambda x_n)$ for $x_i \in \mathbf{k}$ where $i = 0, ..., n$ such that at least one of $x_i \neq 0$ and $\lambda \in \mathbf{k} \setminus \{0_{\mathbf{k}}\}$.

Let $\mathbb{P}^n(\mathbf{k})$ be the projective n-space over a field $\mathbf{k}$. For each set S of homogeneous polynomials $f \in \mathbf{k}[x_0, ..., x_n]$, define the zero-locus of S to be the set of points in $P(\mathbf{k}, n)$ on which the functions in S vanish:

$$Z(S) := \{x \in \mathbb{P}^n(\mathbf{k}) : f(x) = 0 \textit{ for all } f \in S\}$$

A subset V of $\mathbb{P}^n(\mathbf{k})$ is called a projective algebraic set if $V = Z(S)$ for some S. A projective set is said to be irreducible if it cannot be written as a union of two projective algebraic sets. An irreducible projective algebraic set is called a projective variety.

Given a subset V of $\mathbb{P}^n(\mathbf{k})$, let $I(V)$ be the ideal of $\mathbf{k}[x_0, ..., x_n]$, generated by all homogeneous polynomials vanishing on V. For any projective algebraic set V, the coordinate ring of V is the quotient of the polynomial ring by this ideal. That is,

$$\mathbf{k}[V] = \mathbf{k}[x_0, ..., x_n]/I(V).$$

It is an integral domain and its quotient field is the function field of V denoted by $\mathbf{k}(V)$. The dimension of V is the transcendence degree of $\mathbf{k}(V)$ over $\mathbf{k}$.

A hypersurface in a projective space of dimension n is an algebraic set defined by a single equation F = 0, for a homogeneous polynomial F in the homogeneous coordinates.

Definition 4.2.1 ([12], Ex. 2.16 of § II) *A variety Y of dimension r in $\mathbb{P}^n(\mathbf{k})$ is a complete intersection if it can be written as the intersection of $n - r$ hypersurfaces.* ■

Definition 4.2.2 ([24], V.3-3.67) *For a divisor D, we denote by $|D|$ the set of all positive divisors which are equivalent to D. This is called a complete linear system.* ■

The projectivization $\mathbb{P}(V)$ for a vector space V over $\mathbb{F}_q$ is the set of 1-dimensional subspaces of V, and if V has dimension $n+1$, then $\mathbb{P}(V)$ can be put into a one-to-one correspondence with the projective n-space $\mathbb{P}^n(\mathbb{F}_q)$.

Taking the vector space $\mathbb{P}(L(D))$, we define

$$S : \mathbb{P}(L(D)) \to |D|$$

by sending the span of a function $f \in L(D)$ to the divisor $div(f) + D$. Since $div(af) = div(f)$ for $a \in \mathbb{F}_q^*$, S is well-defined.

A linear system is a subset of a complete linear system $|D|$, which corresponds via the map S to a linear subspace of $\mathbb{P}(L(D))$.

Theorem 4.2 ([32], V.1.4) *For a divisor D, $card(|D|) = (q^{dim(D)} - 1)/(q - 1)$.*

Definition 4.2.3 ([24], V.4.1) *Let X be an algebraic curve. A rational map $\phi : X \to \mathbb{P}^n(k)$ is a map of the form*

$$\phi = [g_0 : g_1 : \cdots : g_n]$$

where $g_0, g_1, ..., g_n$ are in the function field K associated to X and $g_0, g_1, ...g_n$ have the property that for every rational place $P \in X$ at which $g_0, g_1, ...g_n$ are well-defined and not all zero,

$$\phi(P) = [g_0(P) : \cdots : g_n(P))].$$

■

The rational map ϕ is not necessarily a function on all of X. However, it is sometimes possible to evaluate $\phi(P)$ at a rational place P of V where some g_i is not well-defined.

Definition 4.2.4 *A rational map $\phi = [f_0 : \cdots : f_n] : X \to \mathbb{P}^n(k)$ is regular (or defined) at a rational place P of the associated function field K if there is a function $g \in K$ such that*

(i) each gf_i is regular at P and

(ii) for some i, $(gf_i)(P) \neq 0$.

If such a g exists, we set $\phi(P) = [(gf_0)(P).....(gf_n)(P)]$.

It may be necessary to take different g's for different points. A rational map which is regular at every point is called a morphism.

■

Let $\phi : X \to \mathbb{P}^n(k)$ be a rational map to projective space. To every such rational map we associate a linear system:

Write $\phi = [f_0 : \cdots : f_n]$ where each f_i are in the function field K associated to X. Let $D = -min_i\{div(f_i)\}$ be the inverse of the minimum divisors of f_i. Therefore, for a rational place $P \in X$, we have that $-D(P)$ is the minimum among the orders of the f_i at P, so $-D(P) \leq v_P(f_i)$ for each i.

Therefore, $-D \leq div(f_i)$ for all i, we have $f_i \in L(D)$ for each i. Let V_f be the k-linear space of the functions $\{f_i\}$, that is, the set of all linear combinations $\sum_i a_i f_i$ with $a_i \in k$. We have $V_f \subset L(D)$ is a linear subspace of $L(D)$.

Lemma 4.2.5 ([24], V.4.4) *The linear system $|\phi| := \{div(g) + D | g \in V_f\}$ is well-defined, independent of the choice of the function $\{f_i\}$ used to define ϕ.*

Lemma 4.2.6 ([24], V.4.6) *Let $\phi : X \to \mathbb{P}^n(k)$ be a regular map, then for every rational place $P \in X$, there exists a divisor $E \in |\phi|$ which does not have P in its support. In other words, there is no point of X which is contained in every divisor of the linear system $|\phi|$.*

Definition 4.2.7 ([12], § II.7) *A point $P \in X$ is a base point of a complete linear system $|D|$ if $P \in supp(E)$ for all $E \in |D|$. A linear system is base point free if it has no base point.* ■

Theorem 4.3 *A complete linear system $|D|$ is base point free if and only if $dim(D - P) = dim(D) - 1$ for any rational place $P \in K$.*

Proof. ($\Leftarrow$) Assume $|D|$ is not base point free, that is, there exists a place P of K of degree one such that $P \in supp(E)$ for all $E \in |D|$. Let $x \in L(D)$, then $(x) = D_1 - D$ for a positive divisor $D_1 \in |D|$. So $P \in supp(D_1)$, that is, $D_1 = P + D_2$ for a positive divisor D_2. Thus $(x) = D_2 + P - D$ and then $x \in L(D - P)$. We have $L(D) = L(D - P)$ which contradicts $dim(D - P) = dim(D) - 1$. Hence $|D|$ is base point free.

($\Rightarrow$) Let $|D|$ be base point free, then there exists $E \in |D|$ such that $P \notin supp(E)$ and $(x) = E - D$ for some $x \in K$. So $(x) \ngeq P - D$, that is, $dim(D-P) < dim(D)$. On the other hand, by Riemann-Roch Theorem

$dim(D-P) = deg(D) - g + dim(W-D+P)$ *and* $dim(D) = deg(D) - g + 1 + dim(W-D)$,

for a canonical divisor W. Then

$$dim(D) - dim(D-P) = 1 + dim(W-D) - dim(W-D+P).$$

Since $dim(W-D) - dim(W-D+P) \leq 0$, $dim(D) - 1 \leq dim(D-P)$. Hence, we have the equality. ■

Corollary 4.4 *Let W be a canonical divisor of a nonhyperelliptic function field $K/\mathbb{F}_q$. Then $|W|$ is base-point-free.*

Proof. Let P be a rational place of a nonhyperelliptic function field K. Since P is positive, $dim(P) \geq 1$. Since K is nonhyperelliptic, the genus g of K is at least three. By Clifford's Theorem, $dim(P) \leq 1 + deg(P)/2 (= 3/2)$ as $0 \leq deg(P) \leq 2g-2$. Then $dim(P) = 1$ and by the Riemann-Roch Theorem,

$$\begin{aligned} dim(W-P) &= 2g - 3 + 1 - g + dim(P) \\ &= g - 1 \\ &= dim(W) - 1 \end{aligned}$$

By Theorem 4.3, $|W|$ is base point free. ■

Proposition 4.5 ([24], V.4.15) *Let $Q \in |D|$ be a base-point-free linear system of projective dimension n on an algebraic curve X. Then, there is a regular map $\phi : X \to \mathbb{P}^n(k)$ such that $Q = |\phi|$. Moreover, ϕ is unique up to the choice of coordinates in $\mathbb{P}^n(k)$.*

Therefore, we have a one to one correspondence

$$\left\{ \begin{array}{c} \text{base-point-free} \\ \text{linear system of} \\ \text{dimension n on X} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{regular maps } \phi : X \to \mathbb{P}^n(\mathbb{F}_2) \\ \text{with non-degenerate image,} \\ \text{up to linear coordinate changes} \end{array} \right\}$$

Lemma 4.2.8 ([24], V.4.17) *Let X be an algebraic curve and let D be a divisor on X with $|D|$ base-point-free. Fix a rational place $P \in X$. Then there is a basis $f_0, f_1, ..., f_n$ for $L(D)$ such that $v_P(f_0) = -D(P)$ and $v_P(f_i) > -D(P)$ for all $i \geq 1$.*

Proposition 4.6 ([24], V.4.18) *Let X be an algebraic curve and let D be a divisor on X with $|D|$ base-point-free. Fix distinct rational places P, Q in X. Then*

$$\phi_D(P) = \phi_D(Q) \Leftrightarrow L(D - P - Q) = L(D - P) = L(D - Q).$$

Hence, ϕ_D is one to one if and only if for every pair of distinct rational places P and Q on X, we have $dim(D - P - Q) = dim(D) - 2$.

When the map ϕ_D is an isomorphism onto its image, we say that it is an embedding.

A divisor D such that $|D|$ has no base point and ϕ_D is an embedding is called a very ample divisor.

Thus by Proposition 4.6, a divisor D, such that $|D|$ has no base point, is very ample if and only if $dim(D) = 2 + dim(D - A)$ where A is a positive divisor of degree two.

Every divisor of degree at least 2g+1 is very ample. That is, if D is a divisor of degree $2g + 1$, A is a positive divisor of degree two and B is a positive divisor of degree one, then by Riemann-Roch Theorem,

$$dim(D) = g + 2 = dim(D - A) + 2 \text{ and } dim(D) = g + 2 = dim(D - B) + 1.$$

Proposition 4.7 ([12], IV.5.2) *The canonical divisor is very ample if and only if the function field is not hyperelliptic.*

Proof. Let W be a canonical divisor of a function field $K/\mathbb{F}_q$. Then $|W|$ is base point free.

($\Rightarrow$) Let K be hyperelliptic. Then there exists $x \in K \setminus \mathbb{F}_q$ such that $[K : \mathbb{F}_q(x)] = 2$ where $(x)_\infty = A$ is a positive divisor of K of degree two. Thus $dim(A) \geq 2$ and by Clifford's theorem, $dim(D) \leq 2$. Thus it is exactly 2 and by Riemann-Roch Theorem,

$$dim(A) - dim(W - A) = 2 + 1 - g.$$

So

$$dim(W - A) = g - 1 \neq g - 2 = dim(W) - 2$$

and W is not very ample.

($\Leftarrow$) Assume W is not very ample, then there exists a positive divisor A of degree two such that $dim(W) - dim(W - A) \neq 2$, that is $dim(W - A) \neq g - 2$. But $1 \leq dim(A) \leq 2$ and $dim(A) - dim(W - A) = 3 - g$. Thus $dim(W - A) = g - 1$ and $dim(A) = 2$. That is, there exists $x \in L(A) \setminus \mathbb{F}_q$. Then $[K : \mathbb{F}_2(x)] = 2$ and K is hyperelliptic. ■

Let X be an algebraic curve of genus 3 or more. Let K be a canonical divisor on X. Then the complete linear system $|K|$ is base-point-free and the associated map

$$\phi_K : X \to \mathbb{P}^{g-1}(\mathbb{F}_q)$$

is defined.

Proposition 4.8 ([24], VII.2.1) *Let X be an algebraic curve (a projective variety of dimension one) of genus $g \geq 3$. Then the canonical map is an embedding if and only if X is not hyperelliptic. If X is not hyperelliptic, the canonical map embeds X into $\mathbb{P}^{g-1}(\mathbb{F}_2)$ as a smooth projective curve of degree $2g - 2$.*

Theorem 4.9 *Two non-hyperelliptic function fields are isomorphic if and only if their canonical models are isomorphic under an automorphism of the projective space $\mathbb{P}^{g-1}(k)$.*

Proof. See [17] Page 157. ■

Definition 4.2.9 *If K/k has a divisor D of degree m such that $dim(D) = 2$, $|D|$ is called a complete linear system of* <u>*type g^1_m*</u>. ■

The cardinality of $|D|$ is $q + 1$ by Theorem 4.2, that is, there exist $q + 1$ equivalent positive divisors of degree m. Let A and B be two of them. Then there exists $x \in K$ such that $A - B = (x)$ and $[K : k(x)] = m$.

4.3 Proof of Theorem 4.1

Let $\mathcal{X}$ be a non-hyperelliptic smooth projective curve defined and absolutely irreducible over $\mathbb{F}_q$ whose function field is $K/\mathbb{F}_q$. We denote by W any positive divisor of the canonical class of $K/\mathbb{F}_q$. The set $|W|$ of all positive canonical divisors is a complete linear system with no base points.

Since $K/\mathbb{F}_q$ is not hyperelliptic, the linear system $|W|$ induces an embedding ϕ from $\mathcal{X}$ to $\mathbb{P}^{g-1}(\mathbb{F}_q)$,

$$\phi : \mathcal{X} \to \mathbb{P}^{g-1}(\mathbb{F}_q)$$

and the curve $\mathcal{C} = \phi(\mathcal{X})$ is a normal smooth curve of degree $2g-2$ whose function field is $K/\mathbb{F}_q$. Then $\mathcal{C}$ is the canonical model of $K/\mathbb{F}_q$.

4.3.1 Genus 3 and $q = 2$

The canonical model $\mathcal{C}$ of K/k is a smooth plane quartic curve in $\mathbb{P}^2(\mathbb{F}_2)$. Two non-quadratic function fields of genus 3 are $\mathbb{F}_2$-isomorphic if and only if their canonical models are transformed one to the other under an automorphism of the projective plane $\mathbb{P}^2(\mathbb{F}_2)$. Since the canonical class is cut out on the curve by the rational lines of $\mathbb{P}^2(\mathbb{F}_2)$, we can find $x, y \in K$ such that $(x) = (X) - (Z) = D_1 - D, y = (Y) - (Z) = D_2 - D$ for canonical divisors D_1, D_2, D such that $\{1, x, y\}$ is a basis of $L(D)$ and $K = \mathbb{F}_2(x, y)$ with the affine equation $F(x, y) = 0$ of $\mathcal{C}$.

Let W denote a canonical divisor of K. Then

$$deg(W) = 4, dim(W) = 3.$$

(i) Let $n_1 = 0$ and $n_3 = 3$.

For $n_1 = 0$ and $n_2 \leq 2$,

There are seven positive integral divisors in the canonical linear system $|W|$ by Theorem 4.2 and each of them is of degree four. Since $n_1 = 0$ and $n_2 \leq 2$, then there exist at most two places P, Q of K of degree two. So $2P$, $2Q$ and $P + Q$ are non-prime positive divisors of K of degree four. Since the extension is not quadratic, $P \nsim Q$.

Since $h_K = 3$, the order of the class $\overline{P-Q}$ is three in $D_K^0/P(K)$. That is, $2P \nsim 2Q$. Clearly $2P \nsim P+Q \nsim 2Q$. So at most one of the positive divisors in the canonical class is not a prime divisor, that is, at least six of the positive divisors of the canonical class are places of K of degree four. Let P be one of these places and consider $L(P)$. Let $\{1, x, y\}$ be a basis of this space. Since $dimL(4P) = 14$ by Riemann-Roch Theorem, the subset $S = \{1, x, x^2, x^3, x^4, y, y^2, y^3, y^4, xy, xy^2, xy^3, x^2y, x^3y, x^2y^2\}$ of $L(4P)$ of cardinality 15 is linearly dependent over $\mathbb{F}_2$. That is,

$$ey^4 + (a_0 + a_1x)y^3 + (b_0 + b_1x + b_2x^2)y^2 + (c_0 + c_1x + c_2x^2 + c_3x^3)y \tag{4.1}$$

$$+(d_0 + d_1x + d_2x^2 + d_3x^3 + d_4x^4) = 0$$

where $e, a_i, b_i, c_i, d_i \in \mathbb{F}_2$ for all i. Since $[K : \mathbb{F}_2(x)] = 4 = [K : \mathbb{F}_2(y)]$, $e = 1 = d_4$ and the homogeneous polynomial associated with Equation 4.1 is

$$f(X,Y,Z) = Y^4 + (a_0Z + a_1X)Y^3 + (b_0Z^2 + b_1XZ + b_2X^2)Y^2+ \tag{4.2}$$

$$(c_0Z^3 + c_1XZ^2 + c_2X^2Z + c_3X^3)Y + (d_0Z^4 + d_1XZ^3 + d_2X^2Z^2 + d_3X^3Z + X^4) = 0.$$

Since the canonical class is cut out on the curve by the rational lines of $\mathbb{P}^2(\mathbb{F}_2)$, we choose $x, y \in K$ such that $(x) = (X) - (Z)$ and $(y) = (Y) - (Z)$ where $(Z) = P$, $(X) = F_1$ and $(Y) = F_2$ are places in the canonical class. Since $(Z) = P$ and

$$f(X,Y,0) = Y^4 + a_1XY^3 + b_2X^2Y^2 + c_3X^3Y + X^4 = 0,$$

$a_1 + b_2 + c_3 = 1$. Since $(X) = F_1$ and

$$f(0,Y,Z) = Y^4 + a_0ZY^3 + b_0Z^2Y^2 + c_0Z^3Y + d_0Z^4 = 0,$$

$d_0 = 1$ and $a_0 + b_0 + c_0 = 1$.

We checked all possibilities with Magma and we got no solution with $h_K = 3$ and $g_K = 3$.

For $n_1 = 0$ and $n_2 = 3$: Let Q_1, Q_2, Q_3 be places of K of degree two. Since the extension is non-quadratic, $Q_1 + Q_2 \nsim Q_1 + Q_3 \nsim Q_2 + Q_3 \nsim Q_1 + Q_2$ and $2Q_1 \nsim 2Q_2 \nsim 2Q_3 \nsim 2Q_1$. Then five of the positive divisors in the canonical class are places. Let P be one of these places and consider $L(P)$. Let $\{1, x, y\}$ be a basis of this space. Since $dimL(4P) = 14$, the subset $S = \{1, x, x^2, x^3, x^4, y, y^2, y^3, y^4, xy, xy^2, xy^3, x^2y, x^3y, x^2y^2\}$ of $L(4P)$ of cardinality 15 is linearly dependent over $\mathbb{F}_2$. That is,

$$ey^4 + (a_0 + a_1x)y^3 + (b_0 + b_1x + b_2x^2)y^2 + (c_0 + c_1x + c_2x^2 + c_3x^3)y \qquad (4.3)$$

$$+(d_0 + d_1x + d_2x^2 + d_3x^3 + d_4x^4) = 0.$$

where $e, a_i, b_i, c_i, d_i \in \mathbb{F}_2$ for all i.

Similar to the above remark, $e = 1 = d_4$. The homogeneous polynomial associated with Equation 4.3 is

$$f(X, Y, Z) = Y^4 + (a_0Z + a_1X)Y^3 + (b_0Z^2 + b_1XZ + b_2X^2)Y^2+ \qquad (4.4)$$

$$(c_0Z^3 + c_1XZ^2 + c_2X^2Z + c_3X^3)Y + (d_0Z^4 + d_1XZ^3 + d_2X^2Z^2 + d_3X^3Z + X^4) = 0$$

Since the canonical class is cut out on the curve by the rational lines of $\mathbb{P}^2(\mathbb{F}_2)$, we choose $x, y \in K$ such that $(x) = (X) - (Z)$ and $(y) = (Y) - (Z)$ where $(Z) = P$, $(X) = F_1$ are places in the canonical class and $(Y) = 2Q_i$ for $0 \leq i \leq 3$. Since $(Z) = P$ and

$$f(X, Y, 0) = Y^4 + a_1XY^3 + b_2X^2Y^2 + c_3X^3Y + X^4 = 0,$$

$a_1 + b_2 + c_3 = 1$. Since $(X) = F_1$ and

$$f(0, Y, Z) = Y^4 + a_0ZY^3 + b_0Z^2Y^2 + c_0Z^3Y + d_0Z^4 = 0,$$

$d_0 = 1$ and $a_0 + b_0 + c_0 = 1$.

Since $(Y) = 2Q_i$ and

$$f(X, 0, Z) = X^4 + d_3ZX^3 + d_2Z^2X^2 + d_1Z^3X + Z^4 = 0,$$

$d_3 = 0 = d_1$ and $d_2 = 1$.

We checked all possibilities with Magma and we got no solution with $h_K = 3$ and $g_K = 3$.

(ii) Let $n_1 = 1, n_2 + n_3 = 4$.

For $n_2 = 0, n_3 = 4$: Let P be the unique place of $\mathbb{F}_2(x)$ of degree one. Since $\mid W \mid$ is base point free and W is very ample, $dim(W - P) = 2$ and $dim(W - 2P) = 1$.

Since $n_2 = 0$ and $L(W - 2P) \neq \{0\}$, $W - 2P \sim 2P$ and $4P$ is canonical. Then $\mid W - P \mid = \{3P, D_1, D_2\}$ where D_1, D_2 are mutually equivalent places of K of degree 3 such that $W \sim D_1 + P$ and $W \sim D_2 + P$. Thus $dim(3P) = 2$ and $dim(4P) = 3$. Let $\{1, x\}$ and $\{1, x, y\}$ be bases of $L(3P)$ and $L(4P)$, respectively. Then the set

$$S = \{1, x, x^2, x^3, x^4, y, xy, x^2y, y^2, xy^2, y^3\} \subseteq L(12P)$$

of cardinality 11 is linearly dependent over $\mathbb{F}_2$, since $dim(12P) = 10$. Then

$$a_0y^3 + (b_0 + b_1x)y^2 + (c_0 + c_1x + c_2x^2)y + (d_0 + d_1x + d_2x^2 + d_3x^3 + d_4x^4) = 0,$$

for some $a_0, b_0, b_1, c_0, c_1, c_2, d_0, d_1, d_2, d_3, d_4 \in \mathbb{F}_2$, not all of which are zero. Since $[K : \mathbb{F}_2(x)] = deg((x)_\infty) = 3$ and $[K : \mathbb{F}_2(y)] = deg((y)_\infty) = 4$, we have $a_0 = 1$ and $d_4 = 1$. Using the substitution $y \to y + (b_0 + b_1x)$, we have

$$y^3 + (e_0 + e_1x + e_2x^2)y + (f_0 + f_1x + f_2x^2 + f_3x^3 + x^4) = 0,$$

where $e_0, e_1, e_2, f_0, f_1, f_2, f_3 \in \mathbb{F}_2$. Since $(x) = \bar{D} - 3P$ where $\bar{D}$ is a place of degree three, $x = 0$ implies $y^3 + e_0y + f_0$ is irreducible of degree three, that is, $e_0 = f_0 = 1$. Similarly, $(y) = \tilde{D} - 4P$ where $\tilde{D}$ is a place of degree four. Then $y = 0$ implies $1 + f_1x + f_2x^2 + f_3x^3 + x^4$ is irreducible of degree four. Up to isomorphism, it is $x^4 + x + 1$ or $x^4 + x^3 + 1$. Hence we have a unique solution, that is,

(A-1)

$$y^3 + y + x^4 + x + 1 = 0, \tag{4.5}$$

where $L(t) = 8t^6 - 8t^5 + 4t^3 - 2t + 1$.

For $n_2 = 1, n_3 = 3$:

Let P be the unique place of K of degree one and let Q be the unique place of K of degree two. We have $dim(W - P) = 2$ and $dim(W - 2P) = 1$, that is, there exists a positive divisor D of degree 3 such that $W \sim D + P$ and $W - 2P \sim 2P$ or Q. If $2P \sim Q$, then the extension is quadratic which is a contradiction. Thus $\overline{2P - Q}$ is a nonzero element of D_K^0/P_K. Thus $4P - 2Q$ is not principal and $6P \sim 3Q$. We have $4P$ or $2P + Q$ is canonical.

Let $4P$ be canonical, then $dim(3P) = 2$ and $dim(4P) = 3$. Let $\{1, x\}$ and $\{1, x, y\}$ be bases of $L(3P)$ and $L(4P)$, respectively. Then the set

$$S = \{1, x, x^2, x^3, x^4, y, xy, x^2y, y^2, xy^2, y^3\}$$

in $L(12P)$ of cardinality 11 is linearly dependent over $\mathbb{F}_2$, since $dim(12P) = 10$. Then

$$a_0y^3 + (b_0 + b_1x)y^2 + (c_0 + c_1x + c_2x^2)y + (d_0 + d_1x + d_2x^2 + d_3x^3 + d_4x^4) = 0,$$

where $a_0, b_0, b_1, c_0, c_1, c_2, d_0, d_1, d_2, d_3, d_4 \in \mathbb{F}_2$. Since $[K : \mathbb{F}_2(x)] = 3$ and $[K : \mathbb{F}_2(y)] = 4$, $a_0 = 1$ and $d_4 = 1$. Using the substitution $y \to y + (b_0 + b_1x)$, we have

$$y^3 + (e_0 + e_1x + e_2x^2)y + (f_0 + f_1x + f_2x^2 + f_3x^3 + x^4) = 0,$$

where $e_0, e_1, e_2, f_0, f_1, f_2, f_3 \in \mathbb{F}_2$.

$(x) = \bar{D} - 3P$ where $\bar{D}$ is a positive divisor of degree three. Since $x \in K \setminus \mathbb{F}_2$ and $2P \nsim Q$, $\overline{D}$ is a place of degree three. Then $x = 0$ implies $y^3 + e_0y + f_0$ is irreducible of degree three, that is, $e_0 = f_0 = 1$. Also, $(y) = \tilde{D} - 4P$ where $\tilde{D}$ is a positive divisor of degree four. Since $2P \nsim Q$, $4P \nsim 2Q$ and $y \in L(4P) \setminus L(3P)$, $\tilde{D}$ is a place and up to isomorphism, $1 + f_1x + f_2x^2 + f_3x^3 + x^4$ is $x^4 + x + 1$ or $x^4 + x^3 + 1$. In any case, we have no solution.

Let $2P + Q$ be canonical, then $dim(P + Q) = 2$ and $dim(2P + Q) = 3$. Then, $\mid W - P \mid = \{P+Q, D_1, D_2\}$ where D_1, D_2 are places of degree three and $\mid W \mid = \{2P+Q, P + D_1, P + D_2, F_1, F_2, F_3, F_4\}$ where F_i are places of degree four for $i = 1, 2, 3, 4$. Choose x,y in K such that $(x) = 2P + D - F_1$ and $(y) = F_2 - F_1$. Then $\{1, x, y\}$ is a basis of $L(F_1)$ and $K = \mathbb{F}_2(x, y)$. Then the set

$$S = \{1, x, x^2, x^3, x^4, y, y^2, y^3, y^4, xy, xy^2, xy^3, x^2y, x^2y^2, x^3y\}$$

of cardinality 15 is a subset of $L(4F_1)$ whose dimension is 14. Then it is linearly dependent over $\mathbb{F}_2$, that is

$$a_0y^4 + (a_1 + a_2x)y^3 + (a_3 + a_4x + a_5x^2)y^2 + (a_6 + a_7x + a_8x^2 + a_9x^3)y$$

$$+(a_{10} + a_{11}x + a_{12}x^2 + a_{13}x^3 + a_{14}x^4) = 0,$$

where $a_i \in \mathbb{F}_2$ for $i = 0, 1, ..., 14$. Since $[K : \mathbb{F}_2(x)] = 4$, $a_0 = 1$ and $y = 0$ implies $a_{10} + a_{11}x + a_{12}x^2 + a_{13}x^3 + a_{14}x^4$ is $x^4 + x + 1$ or $x^4 + x^3 + 1$. $x = 0$ implies $y^4 + a_1y^3 + a_3y^2 + a_6y + 1$ is divisible by $y + 1$, that is $a_6 = a_1 + a_3$. Up to coordinate change of projective space, we have 6 isomorphic solutions for this case. Then $K = \mathbb{F}_2(x, y)$ such that

(A-2)

$$y^4 + y^3 + (x^3 + 1)y + x^4 + x + 1 = 0, \tag{4.6}$$

where $L(t) = 8t^6 - 8t^5 + 2t^4 + t^3 + t^2 - 2t + 1$.

For $n_2 = 2, n_3 = 2$:

Let P be the unique place of $\mathbb{F}_2(x)$ of degree one and let Q_1, Q_2 be the places of $\mathbb{F}_2(x)$ of degree two. Then $dim(W - P) = 2$ and $dim(W - 2P) = 1$, that is, there exist two positive divisors D_1, D_2 of degree 3 mutually equivalent such that $W \sim D_i + P$ for $i = 1, 2$ and $W - 2P \sim 2P$ or Q_1 or Q_2. If $2P \sim Q_i$ for $i = 1, 2$ or $Q_1 \sim Q_2$, then the extension is quadratic, which is a contradiction. Then one of $4P$, $2P + Q_1$ and $2P + Q_2$ is canonical.

Let $4P$ be canonical, then $dim(3P) = 2$ and $dim(4P) = 3$. Let $\{1, x\}$ and $\{1, x, y\}$ be basis of $L(3P)$ and $L(4P)$, respectively. Then the set

$$S = \{1, x, x^2, x^3, x^4, y, xy, x^2y, y^2, xy^2, y^3\} \subseteq L(12P)$$

of cardinality 11 is linearly dependent over $\mathbb{F}_2$, since $dim(12P) = 10$. Then

$$a_0y^3 + (b_0 + b_1x)y^2 + (c_0 + c_1x + c_2x^2)y + (d_0 + d_1x + d_2x^2 + d_3x^3 + d_4x^4) = 0,$$

for some $a_0, b_0, b_1, c_0, c_1, c_2, d_0, d_1, d_2, d_3, d_4 \in \mathbb{F}_2$. Since $[K : \mathbb{F}_2(x)] = 3$ and $[K : \mathbb{F}_2(y)] = 4$, we have $a_0 = 1$ and $d_4 = 1$. Using the substitution $y \to y + (a_0 + a_1x)$, we have

$$y^3 + (e_0 + e_1x + e_2x^2)y + (f_0 + f_1x + f_2x^2 + f_3x^3 + x^4) = 0,$$

where $e_0, e_1, e_2, f_0, f_1, f_2, f_3 \in \mathbb{F}_2$. We have no solution for this case.

Let $2P + Q_i$ for $i = 1$ or 2 be canonical. We may assume $2P + Q_1$ is canonical, then $dim(P + Q_1) = 2$ and $dim(2P + Q_1) = 3$. Then, $\mid W - P \mid = \{P + Q_1, D_1, D_2\}$ where D_1, D_2 are places of degree three and $\mid W \mid = \{2P + Q_1, P + D_1, P + D_2, 2Q_2, F_1, F_2, F_3\}$ where F_i are places of degree four for $i = 1, 2, 3$. Choose x,y in K such that $(x) = 2P + Q_1 - F_1$ and $(y) = F_2 - F_1$. Then $\{1, x, y\}$ is a basis of $L(F_1)$ and $K = \mathbb{F}_2(x, y)$. Then the set $S = \{1, x, x^2, x^3, x^4, y, y^2, y^3, y^4, xy, xy^2, xy^3, x^2y, x^2y^2, x^3y\}$ of cardinality 15 is a subset of $L(4F_1)$ whose dimension is 14. Then it is linearly dependent over $\mathbb{F}_2$, that is

$$a_0y^4 + (a_1 + a_2x)y^3 + (a_3 + a_4x + a_5x^2)y^2 + (a_6 + a_7x + a_8x^2 + a_9x^3)y$$

$$+(a_{10}+a_{11}x+a_{12}x^2+a_{13}x^3+a_{14}x^4)=0,$$

where $a_i \in \mathbb{F}_2$ for $i=0,1,...,14$. Since $[K:\mathbb{F}_2(x)]=4$, we have $a_0=1$. $y=0$ implies $a_{10}+a_{11}x+a_{12}x^2+a_{13}x^3+a_{14}x^4$ is x^4+x+1 or x^4+x^3+1. $x=0$ implies $y^4+a_1y^3+a_3y^2+a_6y+1$ is divisible by $y+1$, that is $a_6=a_1+a_3$. We have 4 isomorphic solutions for this case. Then $K=\mathbb{F}_2(x,y)$ such that

(A-3)

$$y^4+y^3+xy^2+(x^3+x+1)y+(x^4+x+1)=0, \tag{4.7}$$

where $L(t)=8t^6-8t^5+4t^4-2t^3+2t^2-2t+1$.

(iii) Let $n_1=2, n_3+2n_2=3$. Then $n_2=0$ or 1. If $n_2=1$, then we have 4 distinct positive divisors of degree two. Since $h_K=3$, at least two of them are equivalent, which contradicts that the extension is non-quadratic. Then $n_2=0, n_3=3$. Let P_1, P_2 be places of $\mathbb{F}_2(x)$ of degree one. Since $\mid W \mid$ is base point free, $dim(W-P_1)=2$ and $dim(W-2P_1)=1$, that is, there exists a positive divisor D of degree 3 such that $W \sim D+P_1$ and $W-2P_1 \sim 2P_1$ or $2P_2$. If $P_1 \sim P_2$, then K is a rational function field, which is a contradiction. But $3P_1 \sim 3P_2$. We have $4P_1 \sim P_1+3P_2$, $4P_2 \sim P_2+3P_1$ and $2P_1+2P_2$ are positive divisors of degree four which are not prime. Then $4P_i$ for $i=1$ or 2 or $2P_1+2P_2$ is canonical.

Let $4P_i$ be canonical for $i=1$ or 2. Without loss of generality, we may assume $4P_1$ is canonical. Then, $\mid W-P_1 \mid = \{3P_1, 3P_2, D\}$ where D is a place of degree three and $\mid W \mid = \{4P_1, P_1+3P_2, P_1+D, F_1, F_2, F_3, F_4\}$ where F_i are places of degree four for $i=1,2,3,4$. Let $\{1,x\}$ and $\{1,x,y\}$ be bases of $L(3P_1)$ and $L(4P_1)$, respectively. Then the set $S=\{1,x,x^2,x^3,x^4,y,xy,x^2y,y^2,xy^2,y^3\} \subseteq L(12P)$ of cardinality 11 is linearly dependent over $\mathbb{F}_2$, since $dim(12P)=10$. Then

$$a_0y^3+(b_0+b_1x)y^2+(c_0+c_1x+c_2x^2)y+(d_0+d_1x+d_2x^2+d_3x^3+d_4x^4)=0,$$

for some $a_0,b_0,b_1,c_0,c_1,c_2,d_0,d_1,d_2,d_3,d_4 \in \mathbb{F}_2$. Since $[K:\mathbb{F}_2(x)]=3$ and $[K:\mathbb{F}_2(y)]=4$, $a_0=1$ and $d_4=1$. Using the substitution $y \to y+(a_0+a_1x)$, we have

$$y^3+(e_0+e_1x+e_2x^2)y+(f_0+f_1x+f_2x^2+f_3x^3+x^4)=0,$$

where $e_0,e_1,e_2,f_0,f_1,f_2,f_3 \in \mathbb{F}_2$. Up to isomorphism $x \to x+1$, we have a unique solution, that is $K=\mathbb{F}_2(x,y)$ such that

(A-4)

$$y^3 + x^2y + x^4 + x^3 + x = 0, \tag{4.8}$$

where $L(t) = 8t^6 - 4t^5 - 2t^4 + 2t^3 - t^2 - t + 1$.

Let $2P_1 + 2P_2$ be canonical. Then, $\mid W - P_1 \mid = \{P_1 + 2P_2, D_1, D_2\}$ where D_1, D_2 are places of degree three and $\mid W \mid = \{2P_1 + 2P_2, P_1 + D_1, P_1 + D_2, F_1, F_2, F_3, F_4\}$ where F_i are places of degree four for $i = 1, 2, 3, 4$. Choose x,y in K such that $(x) = P_1 + D_2 - F_1$ and $(y) = 2P_1 + 2P_2 - F_1$. Then $\{1, x, y\}$ is a basis of $L(F_1)$ and $K = \mathbb{F}_2(x, y)$. Then the set $S = \{1, x, x^2, x^3, x^4, y, y^2, y^3, y^4, xy, xy^2, xy^3, x^2y, x^2y^2, x^3y\}$ of cardinality 15 is a subset of $L(4F_1)$ whose dimension is 14. Then it is linearly dependent over $\mathbb{F}_2$, that is

$$a_0y^4 + (a_1 + a_2x)y^3 + (a_3 + a_4x + a_5x^2)y^2 + (a_6 + a_7x + a_8x^2 + a_9x^3)y$$

$$+(a_{10} + a_{11}x + a_{12}x^2 + a_{13}x^3 + a_{14}x^4) = 0,$$

where $a_i \in \mathbb{F}_2$ for $i = 0, 1, ..., 14$. Since $[K : \mathbb{F}_2(x)] = 4$,we have $a_0 = 1$. Also, $y = 0$ implies $a_{10} + a_{11}x + a_{12}x^2 + a_{13}x^3 + a_{14}x^4$ is $x^2(x^2+1)$, then $a_{14} = 1$, $a_{13} = 0$, $a_{12} = 1$, $a_{11} = 0$ and $a_{10} = 0$. Moreover, $x = 0$ implies $y^4 + a_1y^3 + a_3y^2 + a_6y$ is $(y^3 + y + 1)y$ or $(y^3 + y^2 + 1)y$, that is $a_6 = 1$ and $a_1 + a_3 = 1$. We have

$$y^4 + (a_1 + a_2x)y^3 + (1 + a_1 + a_4x + a_5x^2)y^2 + (1 + a_7x + a_8x^2 + a_9x^3)y$$

$$+(x^2 + x^4) = 0.$$

Using Magma, we get no function field satisfying the given conditions.

4.3.2 Genus 4

Let $g_K = 4$ and $q = 2$. By Theorem 2.4, $h = 3$ if and only if one of the following cases occurs:

$$\text{(a) } n_1 = 0 \text{ and } \left\{ \begin{array}{lll} n_2 = 0 & n_3 \leq 11 & n_4 = 3 \textit{ or} \\ n_2 = 1 & n_3 \leq 8 & n_4 = 4 \textit{ or} \\ n_2 = 2 & n_3 \leq 6 & n_4 = 4 \textit{ or} \\ n_2 = 3 & n_3 \leq 3 & n_4 = 3 \textit{ or} \\ n_2 = 4 & n_3 = 0 & n_4 = 1 \end{array} \right.$$

$$\text{(b) } n_1 = 1 \text{ and } \left\{ \begin{array}{lll} n_2 = 0 & n_3 \leq 3 & 1 \leq n_4 \leq 4 \textit{ or} \\ n_2 = 1 & n_3 = 0 & n_4 = 4 \end{array} \right.$$

Lemma 4.3.1 ([17], Lemma 5.1) *Let $K/\mathbb{F}_q$ be a non-hyperelliptic function field of genus 4 and let $\mathcal{C}$ be the canonical model of $K/\mathbb{F}_q$. The curve $\mathcal{C}$ is the complete intersection in $\mathbf{P}^3(k)$ of a unique rational absolutely irreducible quadric surface $\mathcal{D}$ and a rational cubic surface $\mathcal{S}$. If we denote by N the number of linear systems g_3^1 of $K/\mathbb{F}_q$, then $0 \leq N \leq 2$ and*

(1) $N = 0$ if and only if $\mathcal{C}$ lies on an elliptic quadric.

(2) $N = 1$ if and only if $\mathcal{C}$ lies on a cone.

(3) $N = 2$ if and only if $\mathcal{C}$ lies on a hyperbolic quadric.

Since g_3^1 contains three positive divisors of degree 3. For $q = 2$ and $g = 4$, the existence of g_3^1 implies

(a') $n_1 = 0$ and $n_3 \geq 3$

(b') $n_1 = 1$ and $n_2 = 0$, $n_3 = 2$ or 3, $1 \leq n_4 \leq 4$.

Let A_n denote the number of degree n positive divisors of K. By [7], for $0 \leq n \leq 2g-2$,

$$A_n = q^{n+1-g} A_{2g-2-n} + h_K \frac{q^{n+1-g} - 1}{q - 1}$$

Let $n = 6, q = 2$ and $g = 4$, then $A_6 = 29$.

• **Assume there exist two g_3^1.** Since the class number is 3, n_3 is 6 or 7.

• Let $n_3 = 7$. Then we have the following cases:

(a1) $n_1 = 0$ and $n_2 = 0, n_3 = 7, n_4 = 3$,

(a2) $n_1 = 0$ and $n_2 = 1, n_3 = 7, n_4 = 4$.

Let $D_{3,1}^+ = \{T_1, T_2, T_3\}$ (T_i are mutually equivalent for $i = 1,2,3$) and let $D_{3,2}^+ = \{T_4, T_5, T_6\}$ (T_j are mutually equivalent for $j = 4,5,6$ and T_i are not equivalent to T_j for $i = 1,2,3$ and $j = 4,5,6$). Since $n_1 = 0$, all T_i are places of degree 3. But $n_3 = 7$, hence there exists a place T_7 such that T_7 is not equivalent to T_i for $i = 1,2,...,6$.

(a1) Since $n_1 = 0 = n_2$, there exist 6 groups of mutually equivalent non-prime divisors of degree 6. These are

(1) $T_i + T_j$, $i,j = 1,2,3$. (6)

(2) $T_i + T_j$, $i,j = 4,5,6$ (6)

(3) $T_i + T_j$, $i = 1,2,3$ and $j = 4,5,6$. (9)

(4)$2T_7$. (1)

(5)$T_i + T_7$, $i = 1,2,3$. (3)

(6) $T_i + T_7$, $i = 4,5,6$. (3)

Since $h = 3$, a divisor of a group cannot be equivalent to a divisor in another group, except the following possibilities:

(i) a divisor of (1) can be equivalent to a divisor of (6).

(ii) a divisor of (2) can be equivalent to a divisor of (5).

(iii) a divisor of (3) can be equivalent to a divisor of (4).

Since $A_6 = 29$, we have a unique place of degree 6. That means we have at most 11 positive divisors of degree 6 mutually equivalent among the divisors contained in these groups. But the canonical class contains 15 positive divisors of degree 6 which are equivalent to each other. This is not possible.

(a2) Let $n_1 = 0$, $n_2 = 1$, $n_3 = 7$ and $n_4 = 4$. Let P be a place of degree 2, $\{T_1, ...T_7\}$ be places of degree 3 and $\{Q_1, ...Q_4\}$ be places of degree 4. Then $S = \{2T_1, ...2T_7, T_1 + T_2, ..., T_6 + T_7, 3P, P + Q_1, ..., P + Q_4\}$ of order 33 is a set of positive divisors of degree 6. But $A_6 = 29$, which is a contradiction.

• Then $n_3 = 6$. Then we have the following cases:

(b1) $n_1 = 0$, $n_2 = 0$, $n_3 = 6$ and $n_4 = 3$,

(b2) $n_1 = 0$, $n_2 = 1$, $n_3 = 6$ and $n_4 = 4$,

(b3) $n_1 = 0$, $n_2 = 2$, $n_3 = 6$ and $n_4 = 4$.

Let $D^+_{3,1} = \{T_1, T_2, T_3\}$ (T_i are mutually equivalent for $i = 1, 2, 3$) and $D^+_{3,2} = \{T_4, T_5, T_6\}$ (T_j are mutually equivalent for $j = 4, 5, 6$ and T_i are not equivalent to T_j for $i = 1, 2, 3$ and $j = 4, 5, 6$). Since $n_1 = 0$, all T_i are of degree 3.

(b1) Since $n_1 = 0 = n_2$, there exist 3 groups of mutually equivalent non-prime divisors of degree 6. These are

(1) $T_i + T_j$, $i, j = 1, 2, 3$. (6)

(2) $T_i + T_j$, $i, j = 4, 5, 6$ (6)

(3) $T_i + T_j$, $i = 1, 2, 3$ and $j = 4, 5, 6$. (9)

Since $h = 3$, a divisor belonging to one of the above groups cannot be equivalent to a divisor in another group and since $A_6 = 29$, there exist 8 places of degree 6 of K. By Theorem 4.2, the canonical class contains 15 mutually equivalent positive divisors of degree 6. Thus the only possibility is that the divisors in the group (3) are canonical.

(b2) We have $n_1 = 0, n_2 = 1, n_3 = 6$ and $n_4 = 4$. Let P denote the place of K of degree 2 and let $\{Q_1, ...Q_4\}$ be the set of places of K of degree 4. Then there exist 3 groups of mutually equivalent non-prime divisors of degree 6. These are

(1) $T_i + T_j$, $i, j = 1, 2, 3$. (6)

(2) $T_i + T_j$, $i, j = 4, 5, 6$ (6)

(3) $T_i + T_j$, $i = 1, 2, 3$ and $j = 4, 5, 6$. (9)

Since $h = 3$, a divisor of a group cannot be equivalent to a divisor in another group. $3P, P+Q_1, P+Q_2, P+Q_3$ and $P+Q_4$ are also positive divisors of degree 6. So we have 26 positive non-prime divisors of degree 6. Since $A_6 = 29$, there exist three places of degree 6 of K. Since canonical class contains 15 mutually equivalent positive divisors of degree 6. Thus the only possibility is that divisors of the group 3 are canonical.

(b3) We have $n_1 = 0, n_2 = 2, n_3 = 6$ and $n_4 = 4$. Let P, Q denote the places of K of degree 2 and let $\{Q_1, ...Q_4\}$ be the set of places of K of degree 4. Then $\{2T_1, ...2T_6, T_1 + T_2, ..., T_5 + T_6, 3P, 3Q, 2P + Q, P + 2Q, P + Q_1, ..., Q + Q_4\}$ of order 33 is a set of positive divisors of degree 6. Since $A_6 = 29$, this case is not possible.

To find a solution for (b1) and (b2): Using the argument above, $T_1 + T_4$ is canonical for both cases. Since $T_1 \sim T_2$ and $T_4 \sim T_5$, $dim(T_1)), dim(T_4)) \geq 2$. By Clifford's Theorem, $dim(T_i) \leq 1+1/2(deg(T_i)) = 5/2$ for $i = 1, 4$. Thus $dim(T_1) = dim(T_4) = 2$. Take $x \in L(T_1) \setminus \mathbb{F}_2$ and $y \in L(T_4) \setminus \mathbb{F}_2$. Then $\{1, x, y, xy\}$ is a basis for $L(T_1 + T_4)$. Also $1, x, x^2, x^3, y, y^2, y^3, xy, ..., x^3y^3$ are in $L(3T_1+3T_3)$ which has dimension 15. Thus they are linearly dependent. That is there exist $a_i, b_i, c_i, d_i \in \mathbb{F}_2$ for $i = 0, 1, 2, 3$ such that

$$(a_3x^3 + a_2x^2 + a_1x + a_0)y^3 + (b_3x^3 + b_2x^2 + b_1x + b_0)y^2 + (c_3x^3 + c_2x^2 + c_1x + c_0)y$$
$$+(d_3x^3 + d_2x^2 + d_1x + d_0) = 0.$$

Since $n_1 = 0$ for both cases, the curve has no rational point. That is for $x = 0$, $a_0y^3 + b_0y^2 + c_0y + d_0$ is irreducible of degree 3. Then it is of the form

$y^3 + y + 1$ or $y^3 + y^2 + 1$.

For $y = 0$, $d_3x^3 + d_2x^2 + d_1x + d_0$ is irreducible of degree 3. Then it is of the form

$x^3 + x + 1$ or $x^3 + x^2 + 1$.

Since the infinite place of $k(x)$ is inert in K,

$1 + b_3 + c_3 + d_3 = 1$.

Since the infinite place of $k(y)$ is inert, $a_3x^3 + a_2x^2 + a_1x + a_0$ is irreducible over $\mathbb{F}_2$. So, it is $x^3 + x + 1$ or $x^3 + x^2 + 1$.

For $x = 1 = y$, sum of all coefficients is 1. Then up to isomorphism, the curve has one of the following forms:

$$(x^3+x+1)y^3+(b_3x^3+b_2x^2+b_1x+b_0)y^2+((1-b_3)x^3+c_2x^2+c_1x+(1-b_0))y+(x^3+x^2+1),$$

where $b_2 + b_1 + c_2 + c_1 = 1$

$$(x^3+x+1)y^3+(b_3x^3+b_2x^2+b_1x+b_0)y^2+((1-b_3)x^3+c_2x^2+c_1x+(1-b_0))y+(x^3+x+1),$$

where $b_2 + b_1 + c_2 + c_1 = 0$

$$(x^3+x+1)y^3+(b_3x^3+b_2x^2+b_1x+b_0)y^2+((b_3)x^3+c_2x^2+c_1x+(1-b_0))y+(x^2+x+1).$$

where $b_2 + b_1 + c_2 + c_1 = 0$

Among 256 possibilities, we get 48 solutions.

Second way (Geometric Solution): If there exist two g_3^1, then $n_3 \geq 6$ and the canonical model $\mathcal{C}$ lies on a hyperbolic quadric. The equation of the hyperbolic quadric $\mathcal{D}$ is

$$f(X,Y,Z,T) = XY + ZT = 0.$$

Let

$$\begin{aligned} s(X,Y,Z,T) = {} & c_1X^2Z + c_2X^2T + c_3XZ^2 + c_4XT^2 + c_5XZT + c_6Y^2Z \\ & +c_7Y^2T + c_8YZ^2 + c_9YZT + c_{10}YT^2 + c_{11}ZT^2 + c_{12}Z^2T + c_{13}Z^3 \\ & +c_{14}T^3 + c_{15}X^3 + c_{16}Y^3 = 0 \end{aligned}$$

be the general form of the cubic surface. Then $\mathcal{C}$ is the intersection of these surfaces. $\mathcal{D}$ has 9 rational points. These are

$$P_1 = (0,0,0,1), P_2 = (0,0,1,0), P_3 = (0,1,0,0), P_4 = (0,1,0,1), P_5 = (0,1,1,0),$$

$$P_6 = (1,0,0,0), P_7 = (1,0,0,1), P_8 = (1,0,1,0), P_9 = (1,1,1,1).$$

Since $n_1 = 0$, none of P_i satisfies the cubic equation. So we have

$$c_{13} = c_{14} = c_{15} = c_{16} = 1,$$

$$c_1 + c_3 = 1, c_2 + c_4 = 1, c_6 + c_8 = 1, c_7 + c_{10} = 1,$$

$$c_5 + c_9 + c_{11} + c_{12} = 1.$$

Among 128 possibilities, we get 48 solutions for the cubic surface with class number 3 as we found above. Up to coordinate change of projective space, we get 8 equations:

(1) $x^3 + xz^2 + xt^2 + y^3 + y^2t + yz^2 + z^3 + z^2t + t^3 = 0,$

(2) $x^3 + xz^2 + xt^2 + y^3 + y^2t + yz^2 + yzt + z^3 + t^3 = 0,$

(3) $x^3 + xz^2 + xt^2 + y^3 + y^2z + y^2t + z^3 + z^2t + t^3 = 0,$

(4) $x^3 + xz^2 + xzt + xt^2 + y^3 + y^2t + yz^2 + z^3 + z^2t + zt^2 + t^3 = 0,$

(5) $x^3 + xz^2 + xzt + xt^2 + y^3 + y^2t + yz^2 + yzt + z^3 + zt^2 + t^3 = 0,$

(6) $x^3 + xz^2 + xzt + xt^2 + y^3 + y^2z + y^2t + yzt + z^3 + z^2t + t^3 = 0,$

(7) $x^3 + x^2t + xz^2 + y^3 + y^2t + yz^2 + yzt + z^3 + z^2t + zt^2 + t^3 = 0,$

(8) $x^3 + x^2t + xz^2 + xzt + y^3 + y^2z + y^2t + z^3 + z^2t + zt^2 + t^3 = 0.$

Let U be the open set of $\mathbf{P}^3(k)$ defined by $T \neq 0$. Let V be the open set in $\mathbf{P}^2(k)$ consisting of the elements of the form (x, y, z) where $z \neq 0$. We define a morphism from V to U such that

$$(x : y : 1) \to (x : y : xy : 1),$$

to get a plane model for K/k. That is $K = k(x, y)$ where $x, y \in K$ satisfying one of the following equations:

B.(I-1)

$$x^3y^3 + x^3y^2 + x^3 + x^2y^3 + x^2y^2 + x + y^3 + y^2 + 1 = 0, \tag{4.9}$$

with $L(t) = 16t^8 - 24t^7 + 12t^6 + 6t^5 - 11t^4 + 3t^3 + 3t^2 - 3t + 1.$

B.(I-2)

$$x^3y^3 + x^3y^2 + x^3 + x^2y^3 + xy^2 + x + y^3 + y^2 + 1 = 0, \tag{4.10}$$

with $L(t) = 16t^8 - 24t^7 + 12t^6 + 6t^5 - 11t^4 + 3t^3 + 3t^2 - 3t + 1$.

B.(I-3)

$$x^3y^3 + x^3y^2 + x^3 + x^2y^2 + xy^3 + x + y^3 + y^2 + 1 = 0, \tag{4.11}$$

with $L(t) = 16t^8 - 24t^7 + 8t^6 + 12t^5 - 15t^4 + 6t^3 + 2t^2 - 3t + 1$.

B.(I-4)

$$x^3y^3 + x^3y^2 + x^3 + x^2y^3 + x^2y^2 + x^2y + xy + x + y^3 + y^2 + 1 = 0, \tag{4.12}$$

with $L(t) = 16t^8 - 24t^7 + 8t^6 + 12t^5 - 15t^4 + 6t^3 + 2t^2 - 3t + 1$.

B.(I-5)

$$x^3y^3 + x^3y^2 + x^3 + x^2y^3 + x^2y + xy^2 + xy + x + y^3 + y^2 + 1 = 0, \tag{4.13}$$

with $L(t) = 16t^8 - 24t^7 + 8t^6 + 12t^5 - 15t^4 + 6t^3 + 2t^2 - 3t + 1$.

B.(I-6)

$$x^3y^3 + x^3y^2 + x^3 + x^2y^2 + x^2y + xy^3 + xy^2 + x + y^3 + y^2 + 1 = 0, \tag{4.14}$$

with $L(t) = 16t^8 - 24t^7 + 12t^6 + 6t^5 - 11t^4 + 3t^3 + 3t^2 - 3t + 1$.

B.(I-7)

$$x^3y^3 + x^3y^2 + x^3 + x^2y^3 + x^2y^2 + x^2 + xy^2 + xy + y^3 + y^2 + 1 = 0, \tag{4.15}$$

with $L(t) = 16t^8 - 24t^7 + 12t^6 + 6t^5 - 11t^4 + 3t^3 + 3t^2 - 3t + 1$.

B.(I-8)

$$x^3y^3 + x^3y^2 + x^3 + x^2y^2 + x^2y + x^2 + xy^3 + xy + y^3 + y^2 + 1 = 0, \tag{4.16}$$

with $L(t) = 16t^8 - 24t^7 + 8t^6 + 12t^5 - 15t^4 + 6t^3 + 2t^2 - 3t + 1$.

• **Assume there exists unique g^1_3.** Let $g^1_3 = \{T_1, T_2, T_3\}$. Then by Theorem 2.4, we have the following possibilities:

$$\text{(a”) } n_1 = 0 \text{ and } \{ \begin{array}{lll} n_2 = 0 & 3 \le n_3 \le 5 & n_4 = 3 \ \textit{or} \\ n_2 = 1 & 3 \le n_3 \le 5 & n_4 = 4 \ \textit{or} \\ n_2 = 2 & 3 \le n_3 \le 5 & n_4 = 4 \ \textit{or} \\ n_2 = 3 & n_3 = 3 & n_4 = 3 \end{array}$$

(b") $n_1 = 1$ and $\{ \ n_2 = 0 \quad n_3 = 2 \ or \ 3 \quad 1 \le n_4 \le 4$

Since n_1 is at most one and for $n_1 = 1, n_2 = 0$, at least two of T_i are prime. Since we have unique g_3^1, the curve lies on a cone. We denote by L_i the rational line passing thorough the vertex of the cone and containing T_i for $i = 1, 2, 3$. Then the rational plane $P_{i,j}$ containing L_i and L_j cuts the curve at $T_i + T_j$, which is a canonical divisor. But $T_i + T_j \sim 2T_i$ for $i, j = 1, 2, 3$. Since at least two of T_i are prime, there exists a place T_i of degree 3 such that $dim(T_i) = 2$ and $dim(2T_i) = 4$.

Let T be a place of degree 3 of K/k such that $dim(T) = 2$ and $dim(2T) = 4$. Let $\{1, x\}$ be a basis of $L(T)$. Then $[K : k(x)] = 3$. Since $(x)_\infty = T$, $(1/x)$ is inert in $K/k(x)$. Since $dim(2T) = 4$, there exists $y \in L(2T) \backslash L(T)$ such that $\{1, x, x^2, y\}$ is a basis for $L(2T)$. We have $(y)_\infty = 2T$ and $k(x, y) = K$. Also

$$\{1, x, x^2, x^3, ...x^6, y, yx, ..yx^4, y^2, xy^2, x^2y^2, y^3\}$$

are in $L(6T)$(16 elements). But $dim(6T) = 15$. Hence they are linearly dependent. That is

$$y^3 + \phi_2(x)y^2 + \phi_4(x)y + \phi_6(x) = 0,$$

where $\phi_i(x) \in k[x]$ of degree $\le$ i for $i = 2, 4, 6$. Set $v = y + \phi_2(x)$. Then $k(x, y) = k(x, v)$ and

$$v^3 + \psi_4(x)v + \psi_6(x) = 0, (*)$$

where $\psi_i(x) \in k[x]$ and $deg\psi_i \le i$ for $i = 4, 6$. Set $u = 1/x$. Then multiplying (*) by u^6 and setting $t = u^2v$, we have

$$t^3 + (a_4 + a_3u + a_2u^2 + a_1u^3 + a_0u^4)t + (b_6 + \cdots b_0u^6) = 0.(**)$$

Since the infinite place of $k(x)/k$ corresponds to (u) (and since $n_1 \le 1$), (u) is inert and $t^3 + a_4t + b_6 \in k[t]$ is irreducible and $a_4 = b_6 = 1$. We have the following solutions:

(1) $t^3 + t + (u^5 + u^2 + 1) = 0$, $(n_1 = 1, n_2 = 0, n_3 = 2, n_4 = 2)$

(2) $t^3 + t + (u^5 + u^4 + u^2 + u + 1) = 0$, $(n_1 = 1, n_2 = 0, n_3 = 2, n_4 = 2)$

(3) $t^3 + (u^4 + 1)t + (u^6 + u^2 + u + 1) = 0$, $(n_1 = 1, n_2 = 0, n_3 = 2, n_4 = 2)$

(4) $t^3+(u^4+1)t+(u^6+u^5+u^4+1)=0$, $(n_1=1, n_2=0, n_3=2, n_4=2)$

(5) $t^3+(u^4+u^2+1)t+(u^6+u^3+u^2+u+1)=0$, $(n_1=0, n_2=1, n_3=5, n_4=4)$

(6) $t^3+(u^4+u^2+1)t+(u^6+u^4+u^3+u^2+1)=0$, $(n_1=0, n_2=1, n_3=5, n_4=4)$

(7) $t^3+(u^4+u^2+1)t+(u^6+u^5+u^4+u^3+1)=0$, $(n_1=0, n_2=1, n_3=5, n_4=4)$.

Geometric Solution(Second way): The canonical model $\mathcal{C}$ lies on a cone. The equation of the cone $\mathcal{D}$ is

$$f(X,Y,Z,T)=X^2+ZT=0.$$

Then the canonical model $\mathcal{C}$ of K/k is the intersection of the cone and a cubic surface. $\mathcal{D}$ has 7 rational points. These are

$$P_1=(0,0,0,1), P_2=(0,0,1,0), P_3=(0,1,0,0), P_4=(0,1,0,1), P_5=(0,1,1,0),$$
$$P_6=(1,0,1,1), P_7=(1,1,1,1).$$

Let

$$s(X,Y,Z,T)=c_1XYT+c_2XYZ+c_3XY^2+c_4XZ^2+c_5XT^2+c_6XZT+c_7Y^2Z+c_8Y^2T$$
$$+c_9YZ^2+c_{10}YZT+c_{11}YT^2+c_{12}ZT^2+c_{13}Z^2T+c_{14}Z^3+c_{15}T^3+c_{16}Y^3=0.$$

be the general form of the cubic surface.

There exist 72 cubic surface satisfying the above conditions. The general form of the elements of the subgroup M of $PGL(\mathbf{F}_2,4)$, fixing the cone is

$$\begin{pmatrix} 1 & 0 & d_1d_3 & d_2d_4 \\ a_1 & 1 & a_2 & a_3 \\ 0 & 0 & d_1 & d_2 \\ 0 & 0 & d_3 & d_4 \end{pmatrix},$$

where $a_i, d_i \in \mathbb{F}_2$ satisfying $d_1d_4+d_2d_3=1$. Up to isomorphism with respect to M, we get 4 equations for the cubic surface :

(1) $xzt+y^3+yz^2+yzt+yt^2+z^3+z^2t+zt^2+t^3=0$,

(2) $xt^2+y^3+yz^2+z^3+z^2t=0$,

(3) $xy^2 + xzt + y^3 + y^2t + yz^2 + z^3 + t^3 = 0$,

(4) $xy^2 + xzt + y^3 + y^2t + yz^2 + yzt + z^3 + zt^2 + t^3 = 0$.

Let U be the open set of $\mathbf{P}^3(k)$ defined by $T \neq 0$. Let V be the open set in $\mathbf{P}^2(k)$ consisting of the elements of the form (x, y, z) where $z \neq 0$. We define a morphism from V to U such that

$$(x : y : 1) \to (x : y : x^2 : 1),$$

to get a plane model for K/k. Then, $K = k(x, y)$ where $x, y \in K$ satisfying one of the following equations:

B.(II-1)

$$x^6 + x^4y + x^4 + x^3 + x^2y + x^2 + y^3 + y + 1 = 0, \tag{4.17}$$

with $L(t) = 16t^8 - 24t^7 + 12t^6 + 4t^5 - 8t^4 + 2t^3 + 3t^2 - 3t + 1$.

B.(II-2)

$$x^6 + x^4y + x^4 + x + y^3 = 0, \tag{4.18}$$

with $L(t) = 16t^8 - 16t^7 + 4t^5 - 2t^4 + 2t^3 - 2t + 1$.

B.(II-3)

$$x^6 + x^4y + x^3 + xy^2 + y^3 + y^2 + 1 = 0, \tag{4.19}$$

with $L(t) = 16t^8 - 24t^7 + 12t^6 + 4t^5 - 8t^4 + 2t^3 + 3t^2 - 3t + 1$.

B.(II-4)

$$x^6 + x^4y + x^3 + x^2y + x^2 + xy^2 + y^3 + y^2 + 1 = 0, \tag{4.20}$$

with $L(t) = 16t^8 - 16t^7 + 4t^5 - 2t^4 + 2t^3 - 2t + 1$.

- **Assume there exists no g_3^1**

In this case, since $h = 3$, by Theorem 2.4, we must have the following possibilities:

$$(\bar{a})\ n_1 = 0 \text{ and } \left\{ \begin{array}{llll} n_2 = 0 & n_3 \leq 3 & n_4 = 3 & or \\ n_2 = 1 & n_3 \leq 3 & n_4 = 4 & or \\ n_2 = 2 & n_3 \leq 3 & n_4 = 4 & or \\ n_2 = 3 & n_3 \leq 3 & n_4 = 3 & or \\ n_2 = 4 & n_3 = 0 & n_4 = 1 & \end{array} \right.$$

$$(\bar{b})\ n_1 = 1 \text{ and } \left\{ \begin{array}{llll} n_2 = 0 & n_3 \leq 2 & 1 \leq n_4 \leq 4 & or \\ n_2 = 1 & n_3 = 0 & n_4 = 4 & \end{array} \right.$$

Let $\mathcal{C} = \mathcal{D} \cap \mathcal{S}$ be the canonical model of $K/\mathbb{F}_2$. Then we choose $(X : Y : Z : T) \in \mathbf{P}^3(k)$ such that $\mathcal{D}$ is given by

$$XY + Z^2 + ZT + T^2 = 0.$$

$\mathcal{D}$ has 5 rational points. These are

$$P_1 = (1, 0, 0, 0), P_2 = (0, 1, 0, 0), P_3 = (1, 1, 0, 1), P_4 = (1, 1, 1, 0), P_5 = (1, 1, 1, 1).$$

Let α be a root of $x^2 + x + 1$. Then the places of degree 2 of $\mathcal{D}$ are

$$Q_1 = (\alpha : \alpha^2 : 0 : 1) + (\alpha^2 : \alpha : 0 : 1),$$

$$Q_2 = (\alpha : \alpha^2 : 1 : 0) + (\alpha^2 : \alpha : 1 : 0),$$

$$Q_3 = (\alpha : \alpha^2 : 1 : 1) + (\alpha^2 : \alpha : 1 : 1),$$

$$Q_4 = (0 : 1 : 1 : \alpha) + (0 : 1 : 1 : \alpha^2),$$

$$Q_5 = (1 : 0 : 1 : \alpha) + (1 : 0 : 1 : \alpha^2),$$

$$Q_6 = (0 : 0 : 1 : \alpha) + (0 : 0 : 1 : \alpha^2),$$

$$Q_7 = (0 : 1 : \alpha : 1) + (0 : 1 : \alpha^2 : 1),$$

$$Q_8 = (0 : \alpha^2 : \alpha : 1) + (0 : \alpha : \alpha^2 : 1),$$

$$Q_9 = (1 : 0 : \alpha : 1) + (1 : 0 : \alpha^2 : 1),$$

$$Q_{10} = (\alpha^2 : 0 : \alpha : 1) + (\alpha : 0 : \alpha^2 : 1).$$

Then we may assume that the equation of the cubic surface $\mathcal{S}$ is

$$\begin{aligned} s(X,Y,Z,T) = {} & c_1X^2Z + c_2X^2T + c_3XZ^2 + c_4XT^2 + c_5XZT + c_6Y^2Z \\ & + c_7Y^2T + c_8YZ^2 + c_9YZT + c_{10}YT^2 + c_{11}ZT^2 + c_{12}Z^2T + c_{13}Z^3 \\ & + c_{14}T^3 + c_{15}X^3 + c_{16}Y^3 = 0. \end{aligned}$$

Case 1: $n_1 = 0, n_2 = 0$**.** For $n_1 = 0$, none of P_i's satisfies $\mathcal{S}$. Thus we have the following equations:

$$c_{16} = 1, c_{15} = 1, c_{14} = 1 + c_{10} + c_7 + c_4 + c_2,$$

$$c_{13} = 1 + c_8 + c_6 + c_3 + c_1, c_{12} = 1 + c_{11} + c_9 + c_5.$$

Also, none of Q_i's satisfies $\mathcal{S}$. Using Magma, we have no function field with class number 3.

Case 2: $n_1 = 0, n_2 \geq 1$**.** One of Q_i's satisfies $\mathcal{S}$.

Let G be the group of permutations of five rational points P_i of $\mathcal{D}$. Then G is isomorphic to S_5.

For $n_1 = 0, n_2 = 1$, there exist 120 cubic surfaces such that the intersection of $\mathcal{D}$ and one of these cubic surfaces $\mathcal{S}$ gives a a function field of genus 4 with class number 3. Up to isomorphism, we have 8 solutions for the cubic equation satisfying the given conditions:

(1) $x^3 + xzt + y^3 + yt^2 + z^3 = 0$, $(n_1 = 0, n_2 = 1, n_3 = 3, n_4 = 4)$

(2) $x^3 + xzt + y^3 + y^2t + yzt + z^3 + zt^2 = 0$, $(n_1 = 0, n_2 = 1, n_3 = 3, n_4 = 4)$

(3) $x^3 + xt^2 + y^3 + y^2t + yzt + z^3 + z^2t + zt^2 + t^3 = 0$, $(n_1 = 0, n_2 = 1, n_3 = 3, n_4 = 4)$

(4) $x^3 + xt^2 + y^3 + y^2t + yz^2 + z^2t + t^3 = 0$, $(n_1 = 0, n_2 = 1, n_3 = 3, n_4 = 4)$

(5) $x^3 + x^2t + y^3 + y^2z + yz^2 + yzt + yt^2 + z^3 + z^2t + zt^2 + t^3 = 0$, $(n_1 = 0, n_2 = 1, n_3 = 3, n_4 = 4)$

(6) $x^3 + x^2t + xzt + y^3 + y^2t + yz^2 + yzt + yt^2 + z^2t = 0$, $(n_1 = 0, n_2 = 1, n_3 = 3, n_4 = 4)$

(7) $x^3+x^2t+xzt+y^3+y^2z+yt^2+z^2t+zt^2+t^3=0$, $(n_1=0, n_2=1, n_3=3, n_4=4)$

(8) $x^3+x^2t+xzt+y^3+y^2z+yz^2+yzt+z^3+zt^2=0$. $(n_1=0, n_2=1, n_3=3, n_4=4)$

Let U be the open set of $\mathbf{P}^3(\mathbb{F}_2)$ defined by $Y \neq 0$. Let V be the open set in $\mathbf{P}^2(k)$ consisting of the elements of the form (x, y, z) where $z \neq 0$. We define a morphism from V to U such that

$$(x : y : 1) \to (x^2+xy+y^2 : 1 : y : x),$$

to get a plane model for K/k. Then $K = k(x, y)$ satisfying one of the following equations:

That is $K = \mathbb{F}_2(x, y)$ such that

B.(III-1)

$$y^6+xy^5+(x^3+x+1)y^3+x^2y^2+(x^5+x^3)y+x^6+x^2+1=0, \tag{4.21}$$

with $L(t) = 16t^8 - 24t^7 + 12t^6 - 2t^4 + 3t^2 - 3t + 1$.

B.(III-2)

$$y^6+xy^5+(x^3+x+1)y^3+x^2y^2+(x^5+x^3+x^2+x)y+x^6+x+1=0, \tag{4.22}$$

with $L(t) = 16t^8 - 24t^7 + 12t^6 - 2t^4 + 3t^2 - 3t + 1$.

B.(III-3)

$$y^6+xy^5+(x^3+1)y^3+(x^2+x)y^2+(x^5+x^3+x^2+x)y+x^6+x^4+x^3+x+1=0, \tag{4.23}$$

with $L(t) = 16t^8 - 24t^7 + 12t^6 - 2t^4 + 3t^2 - 3t + 1$.

B.(III-4)

$$y^6+xy^5+x^3y^3+(x^2+x+1)y^2+(x^5+x^3)y+x^6+x^4+x^3+x+1=0, \tag{4.24}$$

with $L(t) = 16t^8 - 24t^7 + 12t^6 - 2t^4 + 3t^2 - 3t + 1$.

B.(III-5)

$$y^6+xy^5+xy^4+(x^3+1)y^3+(x^3+x+1)y^2+(x^5+x^2+x+1)y+x^6+x^5+x^3+x^2+1=0, \tag{4.25}$$

with $L(t) = 16t^8 - 24t^7 + 12t^6 - 2t^4 + 3t^2 - 3t + 1$.

B.(III-6)

$$y^6 + xy^5 + xy^4 + (x^3 + x)y^3 + (x^3 + x^2 + x + 1)y^2 + (x^5 + x^3 + x)y + x^6 + x^5 + x^2 + x + 1 = 0, \tag{4.26}$$

with $L(t) = 16t^8 - 24t^7 + 12t^6 - 2t^4 + 3t^2 - 3t + 1$.

B.(III-7)

$$y^6 + xy^5 + xy^4 + (x^3 + x)y^3 + (x^3 + x^2 + x)y^2 + (x^5 + x^3 + x^2 + 1)y + x^6 + x^5 + x^3 + x^2 + 1 = 0, \tag{4.27}$$

with $L(t) = 16t^8 - 24t^7 + 12t^6 - 2t^4 + 3t^2 - 3t + 1$.

B.(III-8)

$$y^6 + xy^5 + xy^4 + (x^3 + x + 1)y^3 + (x^3 + x^2 + 1)y^2 + (x^5 + x^3 + x^2 + x + 1)y + x^6 + x^5 + 1 = 0, \tag{4.28}$$

with $L(t) = 16t^8 - 24t^7 + 12t^6 - 2t^4 + 3t^2 - 3t + 1$.

For $n_1 = 0, n_2 = 2$, we have 360 cubic surfaces such that the intersection of $\mathcal{D}$ and one of these cubic surfaces $\mathcal{S}$ gives a a function field of genus 4 with class number 3. Up to isomorphism, we have 23 solutions for the cubic surface.

(1) $x^3 + y^3 + y^2t + yt^2 + z^3 + z^2t + t^3 = 0$, $(n_1 = 0, n_2 = 2, n_3 = 1, n_4 = 4)$

(2) $x^3 + y^3 + y^2t + yzt + z^3 + z^2t + zt^2 = 0$, $(n_1 = 0, n_2 = 2, n_3 = 2, n_4 = 4)$

(3) $x^3 + xzt + y^3 + y^2t + yzt + yt^2 + z^3 + zt^2 + t^3 = 0$, $(n_1 = 0, n_2 = 2, n_3 = 2, n_4 = 4)$

(4) $x^3 + xzt + y^3 + y^2z + y^2t + yzt + z^2t = 0$, $(n_1 = 0, n_2 = 2, n_3 = 1, n_4 = 4)$

(5) $x^3 + xzt + y^3 + y^2z + y^2t + yzt + yt^2 + zt^2 + t^3 = 0$, $(n_1 = 0, n_2 = 2, n_3 = 3, n_4 = 4)$

(6) $x^3 + xt^2 + y^3 + y^2t + z^3 + z^2t + t^3 = 0$, $(n_1 = 0, n_2 = 2, n_3 = 2, n_4 = 4)$

(7) $x^3 + xt^2 + y^3 + y^2t + yzt + z^3 + t^3 = 0$, $(n_1 = 0, n_2 = 2, n_3 = 3, n_4 = 4)$

(8) $x^3 + xt^2 + y^3 + y^2z + zt^2 = 0$, $(n_1 = 0, n_2 = 2, n_3 = 3, n_4 = 4)$

(9) $x^3 + xt^2 + y^3 + y^2z + yzt = 0$, $(n_1 = 0, n_2 = 2, n_3 = 1, n_4 = 4)$

(10) $x^3 + xt^2 + y^3 + y^2z + yzt + yt^2 + t^3 = 0$, $(n_1 = 0, n_2 = 2, n_3 = 2, n_4 = 4)$

(11) $x^3 + xt^2 + y^3 + y^2z + yz^2 + yzt + z^3 = 0$, $(n_1 = 0, n_2 = 2, n_3 = 3, n_4 = 4)$

(12) $x^3 + xt^2 + y^3 + y^2z + yz^2 + yzt + yt^2 + z^3 + z^2t + zt^2 + t^3 = 0$, $(n_1 = 0, n_2 = 2, n_3 = 3, n_4 = 4)$

(13) $x^3 + xz^2 + xzt + xt^2 + y^3 + y^2t + yzt + z^2t + t^3 = 0$, $(n_1 = 0, n_2 = 2, n_3 = 1, n_4 = 4)$

(14) $x^3 + xz^2 + xzt + xt^2 + y^3 + y^2t + yz^2 + z^3 + t^3 = 0$, $(n_1 = 0, n_2 = 2, n_3 = 2, n_4 = 4)$

(15) $x^3 + x^2t + xzt + y^3 + y^2t + yt^2 + z^3 = 0$, $(n_1 = 0, n_2 = 2, n_3 = 2, n_4 = 4)$

(16) $x^3 + x^2t + xzt + y^3 + y^2t + yzt + yt^2 + z^3 + z^2t = 0$, $(n_1 = 0, n_2 = 2, n_3 = 3, n_4 = 4)$

(17) $x^3 + x^2t + xzt + y^3 + y^2z + yzt + z^2t = 0$, $(n_1 = 0, n_2 = 2, n_3 = 3, n_4 = 4)$

(18) $x^3 + x^2t + xzt + y^3 + y^2z + yzt + yt^2 + z^2t + t^3 = 0$, $(n_1 = 0, n_2 = 2, n_3 = 1, n_4 = 4)$

(19) $x^3 + x^2t + xzt + y^3 + y^2z + yz^2 + yzt + yt^2 + z^3 + z^2t + t^3 = 0$, $(n_1 = 0, n_2 = 2, n_3 = 2, n_4 = 4)$

(20) $x^3 + x^2t + xzt + y^3 + y^2z + y^2t + yt^2 + z^2t + zt^2 = 0$, $(n_1 = 0, n_2 = 2, n_3 = 1, n_4 = 4)$

(21) $x^3 + x^2t + xzt + y^3 + y^2z + y^2t + yz^2 + yzt + yt^2 + z^3 + zt^2 = 0$, $(n_1 = 0, n_2 = 2, n_3 = 3, n_4 = 4)$

(22) $x^3 + x^2t + xz^2 + y^3 + y^2t + yz^2 + yzt + yt^2 + z^3 = 0$, $(n_1 = 0, n_2 = 2, n_3 = 3, n_4 = 4)$

(23) $x^3 + x^2t + xz^2 + xzt + y^3 + y^2t + yz^2 + yzt + yt^2 + z^3 + z^2t = 0$. $(n_1 = 0, n_2 = 2, n_3 = 1, n_4 = 4)$

Using the morphism

$$(x : y : 1) \to (x^2 + xy + y^2 : 1 : y : x),$$

we get a plane model for K/k. Then $K = \mathbb{F}_2(x, y)$ satisfying one of the following equations:

B.(III-9)

$$y^6 + xy^5 + (x^3 + 1)y^3 + xy^2 + x^5y + x^6 + x^3 + x^2 + x + 1 = 0, \tag{4.29}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 10t^5 + 8t^4 - 5t^3 + 4t^2 - 3t + 1$.

B.(III-10)

$$y^6 + xy^5 + (x^3+1)y^3 + xy^2 + (x^5+x^2+x)y + x^6 + x + 1 = 0, \tag{4.30}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 8t^5 + 5t^4 - 4t^3 + 4t^2 - 3t + 1$.

B.(III-11)

$$y^6 + xy^5 + (x^3+x+1)y^3 + x^2y^2 + (x^5+x^3+x^2+x)y + x^6 + x^3 + x^2 + x + 1 = 0, \tag{4.31}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 8t^5 + 5t^4 - 4t^3 + 4t^2 - 3t + 1$.

B.(III-12)

$$y^6 + xy^5 + (x^3+x)y^3 + (x^2+x)y^2 + (x^5+x^3+x+1)y + x^6 + x + 1 = 0, \tag{4.32}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 10t^5 + 8t^4 - 5t^3 + 4t^2 - 3t + 1$.

B.(III-13)

$$y^6 + xy^5 + (x^3+x)y^3 + x^2y^2 + (x^5+x^3+x^2+x+1)y + x^6 + x^3 + x^2 + x + 1 = 0, \tag{4.33}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 6t^5 + 2t^4 - 3t^3 + 4t^2 - 3t + 1$.

B.(III-14)

$$y^6 + xy^5 + (x^3+1)y^3 + (x^2+x)y^2 + (x^5+x^3)y + x^6 + x^4 + x^3 + x + 1 = 0, \tag{4.34}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 8t^5 + 5t^4 - 4t^3 + 4t^2 - 3t + 1$.

B.(III-15)

$$y^6 + xy^5 + (x^3+1)y^3 + x^2y^2 + (x^5+x^3+x)y + x^6 + x^4 + x^3 + x + 1 = 0, \tag{4.35}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 6t^5 + 2t^4 - 3t^3 + 4t^2 - 3t + 1$.

B.(III-16)

$$y^6 + xy^5 + x^3y^3 + x^2y^2 + (x^5+x^3+x^2+1)y + x^6 + x^4 + 1 = 0, \tag{4.36}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 6t^5 + 2t^4 - 3t^3 + 4t^2 - 3t + 1$.

B.(III-17)

$$y^6 + xy^5 + x^3y^3 + x^2y^2 + (x^5 + x^3 + x + 1)y + x^6 + x^4 + 1 = 0, \tag{4.37}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 10t^5 + 8t^4 - 5t^3 + 4t^2 - 3t + 1$.

B.(III-18)

$$y^6 + xy^5 + x^3y^3 + x^2y^2 + (x^5 + x^3 + x + 1)y + x^6 + x^4 + x^3 + x^2 + 1 = 0, \tag{4.38}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 8t^5 + 5t^4 - 4t^3 + 4t^2 - 3t + 1$.

B.(III-19)

$$y^6 + xy^5 + (x^3 + 1)y^3 + (x^2 + 1)y^2 + (x^5 + x^3 + x + 1)y + x^6 + x^4 + 1 = 0, \tag{4.39}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 6t^5 + 2t^4 - 3t^3 + 4t^2 - 3t + 1$.

B.(III-20)

$$\begin{aligned} y^6 + xy^5 &+ (x^3 + 1)y^3 + (x^2 + x + 1)y^2 + (x^5 + x^3 + x^2 + x + 1)y \\ &+ x^6 + x^4 + x^3 + x^2 + 1 = 0, \end{aligned} \tag{4.40}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 6t^5 + 2t^4 - 3t^3 + 4t^2 - 3t + 1$.

B.(III-21)

$$y^6 + xy^5 + y^4 + x^3y^3 + (x^2 + x)y^2 + (x^5 + x)y + x^6 + x^4 + x^3 + x + 1 = 0, \tag{4.41}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 10t^5 + 8t^4 - 5t^3 + 4t^2 - 3t + 1$.

B.(III-22)

$$y^6 + xy^5 + y^4 + (x^3 + 1)y^3 + (x^2 + 1)y^2 + x^5y + x^6 + x^4 + x^3 + x + 1 = 0, \tag{4.42}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 8t^5 + 5t^4 - 4t^3 + 4t^2 - 3t + 1$.

B.(III-23)

$$\begin{aligned} y^6 + xy^5 &+ xy^4 + (x^3 + x + 1)y^3 + (x^3 + x^2)y^2 + (x^5 + x^3)y \\ &+ x^6 + x^5 + x^2 + x + 1 = 0, \end{aligned} \tag{4.43}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 8t^5 + 5t^4 - 4t^3 + 4t^2 - 3t + 1$.

B.(III-24)

$$y^6 + xy^5 + xy^4 + (x^3 + x + 1)y^3 + (x^3 + x^2 + x)y^2 + (x^5 + x^3 + x)y + x^6 + x^5 + x^2 + x + 1 = 0, \tag{4.44}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 6t^5 + 2t^4 - 3t^3 + 4t^2 - 3t + 1$.

B.(III-25)

$$y^6 + xy^5 + xy^4 + (x^3 + x)y^3 + (x^3 + x^2 + x)y^2 + (x^5 + x^3 + x + 1)y + x^6 + x^5 + 1 = 0, \tag{4.45}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 6t^5 + 2t^4 - 3t^3 + 4t^2 - 3t + 1$.

B.(III-26)

$$y^6 + xy^5 + xy^4 + (x^3 + x)y^3 + (x^3 + x^2 + x)y^2 + (x^5 + x^3 + x + 1)y + x^6 + x^5 + x^3 + x^2 + 1 = 0, \tag{4.46}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 10t^5 + 8t^4 - 5t^3 + 4t^2 - 3t + 1$.

B.(III-27)

$$y^6 + xy^5 + xy^4 + (x^3 + x + 1)y^3 + (x^3 + x^2 + x + 1)y^2 + (x^5 + x^3 + x + 1)y + x^6 + x^5 + x^3 + x^2 + 1 = 0, \tag{4.47}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 8t^5 + 5t^4 - 4t^3 + 4t^2 - 3t + 1$.

B.(III-28)

$$y^6 + xy^5 + xy^4 + (x^3 + x)y^3 + (x^3 + x^2 + x)y^2 + (x^5 + x^3 + x^2 + 1)y + x^6 + x^5 + x^2 + x + 1 = 0, \tag{4.48}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 10t^5 + 8t^4 - 5t^3 + 4t^2 - 3t + 1$.

B.(III-29)

$$y^6 + xy^5 + xy^4 + (x^3 + x + 1)y^3 + (x^3 + x^2 + 1)y^2 + (x^5 + x^3 + x^2 + x + 1)y + x^6 + x^5 + x^2 + x + 1 = 0, \tag{4.49}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 6t^5 + 2t^4 - 3t^3 + 4t^2 - 3t + 1$.

B.(III-30)

$$y^6+xy^5 + (x + 1)y^4 + (x^3 + x + 1)y^3 + (x^3 + x^2 + 1)y^2+ \\ (x^5 + x)y + x^6 + x^5 + x^2 + x + 1 = 0, \tag{4.50}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 6t^5 + 2t^4 - 3t^3 + 4t^2 - 3t + 1$.

B.(III-31)

$$y^6 + xy^5 + (x + 1)y^4 + (x^3 + 1)y^3 + (x^3 + x + 1)y^2+ \\ (x^5 + x^3 + x)y + x^6 + x^5 + x^2 + x + 1 = 0, \tag{4.51}$$

with $L(t) = 16t^8 - 24t^7 + 16t^6 - 10t^5 + 8t^4 - 5t^3 + 4t^2 - 3t + 1$.

For $n_1 = 0, n_2 = 3,$ we have 80 solutions. Up to isomorphism, we have 9 cubic surfaces satisfying the given conditions:

(1) $x^3 + y^3 + yzt + z^3 + z^2t + zt^2 + t^3 = 0$, $(n_1 = 0, n_2 = 3, n_3 = 1, n_4 = 3)$

(2) $x^3 + y^3 + y^2t + yz^2 + yzt = 0$, $(n_1 = 0, n_2 = 3, n_3 = 1, n_4 = 3)$

(3) $x^3 + xzt + y^3 + yzt + z^3 + z^2t + t^3 = 0$, $(n_1 = 0, n_2 = 3, n_3 = 1, n_4 = 3)$

(4) $x^3 + xzt + y^3 + y^2z + y^2t = 0$, $(n_1 = 0, n_2 = 3, n_3 = 1, n_4 = 3)$

(5) $x^3+xz^2+xzt+xt^2+y^3+yz^2+yzt+yt^2+z^3+z^2t+t^3 = 0$, $(n_1 = 0, n_2 = 3, n_3 = 1, n_4 = 3)$

(6) $x^3 + x^2t + y^3 + y^2t + yz^2 + yzt + t^3 = 0$, $(n_1 = 0, n_2 = 3, n_3 = 1, n_4 = 3)$

(7) $x^3 + x^2t + y^3 + y^2z + yz^2 + yt^2 + z^3 + zt^2 + t^3 = 0$, $(n_1 = 0, n_2 = 3, n_3 = 1, n_4 = 3)$

(8) $x^3 + x^2t + xt^2 + y^3 + y^2z + yz^2 + yt^2 + z^3 + zt^2 = 0$, $(n_1 = 0, n_2 = 3, n_3 = 1, n_4 = 3)$

(9) $x^3 + x^2t + xz^2 + y^3 + y^2z + yzt + yt^2 + z^3 + t^3 = 0$. $(n_1 = 0, n_2 = 3, n_3 = 1, n_4 = 3)$

Using the morphism $(x : y : 1) \to (x^2 + xy + y^2 : 1 : y : x)$, $K = \mathbb{F}_2(x, y)$ satisfying one of the following equations:

B.(III-32)

$$y^6 + xy^5 + (x^3+1)y^3 + xy^2 + (x^5+x^2+x)y + x^6 + x^3 + 1 = 0, \tag{4.52}$$

with $L(t) = 16t^8 - 24t^7 + 20t^6 - 16t^5 + 12t^4 - 8t^3 + 5t^2 - 3t + 1$.

B.(III-33)

$$y^6 + xy^5 + x^3y^3 + y^2 + (x^5+x)y + x^6 + x + 1 = 0, \tag{4.53}$$

with $L(t) = 16t^8 - 24t^7 + 20t^6 - 16t^5 + 12t^4 - 8t^3 + 5t^2 - 3t + 1$.

B.(III-34)

$$y^6 + xy^5 + (x^3+x+1)y^3 + (x^2+x)y^2 + (x^5+x^3+x)y + x^6 + x^3 + 1 = 0, \tag{4.54}$$

with $L(t) = 16t^8 - 24t^7 + 20t^6 - 16t^5 + 12t^4 - 8t^3 + 5t^2 - 3t + 1$.

B.(III-35)

$$y^6 + xy^5 + (x^3+x)y^3 + x^2y^2 + (x^5+x^3+1)y + x^6 + x + 1 = 0, \tag{4.55}$$

with $L(t) = 16t^8 - 24t^7 + 20t^6 - 16t^5 + 12t^4 - 8t^3 + 5t^2 - 3t + 1$.

B.(III-36)

$$\begin{aligned} y^6 + xy^5 + y^4 + (x^3+1)y^3 + (x^2+x+1)y^2 + (x^5+x)y \\ + x^6 + x^4 + x^3 + x^2 + 1 = 0, \end{aligned} \tag{4.56}$$

with $L(t) = 16t^8 - 24t^7 + 20t^6 - 16t^5 + 12t^4 - 8t^3 + 5t^2 - 3t + 1$.

B.(III-37)

$$y^6 + xy^5 + xy^4 + x^3y^3 + (x^3+1)y^2 + (x^5+x)y + x^6 + x^5 + x^3 + x + 1 = 0, \tag{4.57}$$

with $L(t) = 16t^8 - 24t^7 + 20t^6 - 16t^5 + 12t^4 - 8t^3 + 5t^2 - 3t + 1$.

B.(III-38)

$$\begin{aligned} y^6 + xy^5 + xy^4 + (x^3+1)y^3 + (x^3+1)y^2 + (x^5+x^2+1)y \\ + x^6 + x^5 + x^3 + x^2 + 1 = 0, \end{aligned} \tag{4.58}$$

with $L(t) = 16t^8 - 24t^7 + 20t^6 - 16t^5 + 12t^4 - 8t^3 + 5t^2 - 3t + 1$.

B.(III-39)

$$y^6 + xy^5 + xy^4 + (x^3+1)y^3 + (x^3+x^2+1)y^2 + (x^5+x^3+x^2+1)y + x^6 + x^5 + x^4 + x^2 + 1 = 0, \tag{4.59}$$

with $L(t) = 16t^8 - 24t^7 + 20t^6 - 16t^5 + 12t^4 - 8t^3 + 5t^2 - 3t + 1$.

B.(III-40)

$$y^6 + xy^5 + (x+1)y^4 + (x^3+x+1)y^3 + (x^3+x^2)y^2 + (x^5+x+1)y + x^6 + x^5 + x^3 + x^2 + 1 = 0, \tag{4.60}$$

with $L(t) = 16t^8 - 24t^7 + 20t^6 - 16t^5 + 12t^4 - 8t^3 + 5t^2 - 3t + 1$.

Case 3:($n_1 = 1$**)** In this case, there exist 120 solutions. Up to coordinate change of the projective space, the cubic surface satisfies one of the following equations:

(1) $x^3 + xzt + y^3 + y^2t + yt^2 + z^3 + zt^2 = 0$, $(n_1 = 1, n_2 = 1, n_3 = 0, n_4 = 4)$

(2) $x^3 + xzt + y^2t + yz^2 + z^3 + z^2t + t^3 = 0$, $(n_1 = 1, n_2 = 1, n_3 = 0, n_4 = 4)$

(3) $x^3 + xt^2 + y^3 + y^2t + yz^2 + z^3 + z^2t + zt^2 + t^3 = 0$, $(n_1 = 1, n_2 = 1, n_3 = 0, n_4 = 4)$

(4) $x^3 + xz^2 + xzt + xt^2 + y^3 + y^2t + z^3 + z^2t + t^3 = 0$, $(n_1 = 1, n_2 = 1, n_3 = 0, n_4 = 4)$

(5) $x^3 + x^2t + y^3 + y^2z + yzt + zt^2 = 0$, $(n_1 = 1, n_2 = 1, n_3 = 0, n_4 = 4)$

(6) $x^2t + xzt + y^3 + y^2z + y^2t + yz^2 + yzt + yt^2 + t^3 = 0$. $(n_1 = 1, n_2 = 1, n_3 = 0, n_4 = 4)$

By the morphism $(x : y : 1) \to (x^2 + xy + y^2 : 1 : y : x)$, we have $K = K(x, y)$ satisfying one of the following equations:

B.(III-41)

$$y^6 + xy^5 + (x^3+x+1)y^3 + x^2y^2 + (x^5+x^3+x^2)y + x^6 + x^2 + x + 1 = 0, \tag{4.61}$$

with $L(t) = 16t^8 - 16t^7 + 4t^6 - 4t^5 + 5t^4 - 2t^3 + t^2 - 2t + 1$.

B.(III-42)

$$y^6 + xy^5 + (x^3+x+1)y^3 + (x^2+x+1)y^2 + (x^5+x^3)y + x^6 + x^3 + x = 0, \tag{4.62}$$

with $L(t) = 16t^8 - 16t^7 + 4t^6 - 4t^5 + 5t^4 - 2t^3 + t^2 - 2t + 1$.

B.(III-43)

$$y^6 + xy^5 + (x^3+1)y^3 + (x^2+x+1)y^2 + (x^5+x^3+x^2)y + x^6 + x^4 + x^3 + x + 1 = 0, \tag{4.63}$$

with $L(t) = 16t^8 - 16t^7 + 4t^6 - 4t^5 + 5t^4 - 2t^3 + t^2 - 2t + 1$.

B.(III-44)

$$y^6 + xy^5 + y^4 + (x^3+1)y^3 + (x^2+x)y^2 + x^5y + x^6 + x^4 + x^3 + x + 1 = 0, \tag{4.64}$$

with $L(t) = 16t^8 - 16t^7 + 4t^6 - 4t^5 + 5t^4 - 2t^3 + t^2 - 2t + 1$.

B.(III-45)

$$y^6 + xy^5 + xy^4 + x^3y^3 + x^3y^2 + (x^5+x^2+x+1)y + x^6 + x^5 + 1 = 0, \tag{4.65}$$

with $L(t) = 16t^8 - 16t^7 + 4t^6 - 4t^5 + 5t^4 - 2t^3 + t^2 - 2t + 1$.

B.(III-46)

$$xy^4 + xy^3 + (x^3+x^2+1)y^2 + (x^3+x+1)y + x^5 + x^3 + x^2 + x + 1 = 0, \tag{4.66}$$

with $L(t) = 16t^8 - 16t^7 + 4t^6 - 4t^5 + 5t^4 - 2t^3 + t^2 - 2t + 1$.

4.3.3 Genus 5

Let $g_K = 5$ and $q = 2$. By Theorem 2.4, we have $n_1 = 0, n_5 - 2n_3 + n_2n_3 = 3$ and $n_5 \neq 0$. Since the extension is non-quadratic, $n_2 \leq 3$. Thus one of the following conditions holds:

$(\overline{a})$ $n_1 = 0, n_2 = 0, n_5 = 3 + 2n_3$,

$(\overline{b})$ $n_1 = 0, n_2 = 1, n_5 = 3 + n_3$,

$(\overline{c})$ $n_1 = 0, n_2 = 2, n_5 = 3$,

$(\overline{d})$ $n_1 = 0, n_2 = 3, n_5 = 3 - n_3, 0 \leq n_3 \leq 2$.

Let A_n denote the order of the set of positive divisors of K of degree n. Assume K has a positive divisor M of degree 3. By Clifford's Theorem, $dim(M)$ is at most 2. By Lemma V.1.4 of [32], $A_3 \leq 3(2^2 - 1)$. Hence n_3 is at most 9. Similarly, $A_4 \leq 3.(2^3 - 1)$ and n_4 is at most 21. Considering the possibilities for n_i for $i = 1, ..., 5$, we have 723 distinct L-polynomials. Calculating the roots of L-polynomials for each case in Magma, except the following cases, we find that L-polynomial has a root with absolute value different from $1/\sqrt{2}$. So, it remains only the following cases to study:

(1) $n_2 = 0, n_3 = 0, n_4 = 8$,

(2) $n_2 = 0, n_3 = 1, n_4 = 5$,

(3) $n_2 = 0, n_3 = 1, n_4 = 6$,

(4) $n_2 = 0, n_3 = 2, n_4 = 2, n_6 = 6$,

(5) $n_2 = 0, n_3 = 2, n_4 = 3, n_6 = 3$,

(6) $n_2 = 0, n_3 = 3, n_4 = 1, n_5 = 9, n_7 = 18$,

(7) $n_2 = 1, n_3 = 0, n_4 = 3$,

(8) $n_2 = 1, n_3 = 0, n_4 = 4$,

(9) $n_2 = 1, n_3 = 1, n_4 = 0, n_5 = 4$,

(10) $n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4$.

• If K has a g^1_3, then it has a trisecant in the canonical embedding and it can be represented as a plane quintic with one node by Exercise 5.5 of § IV of [12]. We assume K has a g^1_3. Since $n_1 = 0$, g^1_3 is a set of order 3, consisting of mutually equivalent places of K of degree 3. Then $n_3 \geq 3$ and we are in case (6):

$$n_1 = 0, n_2 = 0, n_3 = 3, n_4 = 1, n_5 = 9, n_7 = 18. \tag{4.67}$$

Let $g^1_3 = \{P_i \in \mathbb{P}_K : degP_i = 3, i = 1, 2, 3\}$ and let $A_5 := \{D \in Div(K) : D \geq 0, deg(D) = 5\}$. Since $n_1 = n_2 = 0$, all divisors in A_5 are places of K of degree 5. Then, $A_5 = \{Q_i \in \mathbb{P}_K : degQ_i = 5, i = 1, ..., 9\}$. By Lemma V.1.4 of [32], $\mid Q_i \mid = 2^{dim(Q_i)} - 1$. Since $\mid A_5 \mid = 9$, $\mid Q_i \mid = 1, 3$ or 7. Hence, we assume Q_1, Q_2, Q_3 are in three distinct divisor classes. Since $h_K = 3$, $P_1 + Q_i$ is in the canonical class for some $i = 1, 2, 3$. Let W be a canonical divisor of K and $P_1 + Q_1$ be equivalent to W. Then by Riemann-Roch Theorem,

$$dim(Q_1) = deg(Q_1) - (g - 1) + dim(P_1).$$

That is, $dim(Q_1) = 3$. Let $\{1, x, y\}$ be a basis of $L(Q_1)$. Since $[K : \mathbb{F}_2(x)] = [K : \mathbb{F}_2(y)] = 5$, $\mathbb{F}_2(x, y)$ is K or $\mathbb{F}_2(x) = \mathbb{F}_2(y)$. The second case implies $y = (ax + b)/(cx + d)$ where $ad - bc \neq 0$, $a, b, c, d \in \mathbb{F}_2$. Clearly, this is not possible and so $K = \mathbb{F}_2(x, y)$. Let $S := \{x^i y^j : 0 \leq i + j \leq 6, j \leq 5, i, j \in \mathbb{Z}\}$. Then, S is a set of cardinality 27 in $L(6Q_1)$ whose dimension is 26. So S is linearly dependent on $\mathbb{F}_2$, that is,

$$(a_0 + a_1 x)y^5 + (b_0 + b_1 x + b_2 x^2)y^4 + (c_0 + c_1 x + c_2 x^2 + c_3 x^3)y^3 + (d_0 + \cdots + d_4 x^4)y^2 + \tag{4.68}$$

$$(e_0 + \cdots + e_5 x^5)y + (f_0 + \cdots + f_6 x^6) = 0,$$

where $a_i, b_i, c_i, d_i, e_i, f_i \in \mathbb{F}_2$. Since $n_1 = n_2 = 0$, $y = 0$ implies any polynomial of degree less than or equal to 2 does not divide $f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + f_5 x^5 + f_6 x^6$ and $x = 0, y = 1$ implies $a_0 + b_0 + c_0 + d_0 + e_0 = 0$. Using Magma, we have seen that the equation 4.68 does not give a function field satisfying the conditions in $(*)$ and hence K does not have a g^1_3.

• If K does not have a g^1_3, then by Exercise 5.5 of § IV.5 of [12], its canonical model in $P^4(\mathbb{F}_2)$ is a complete intersection of three quadric hypersurfaces.

Let $PGL(\mathbb{F}_2, 5)$ be the general linear group of 5×5 invertible matrices whose entries are in $\mathbb{F}_2$.

$$\begin{aligned} \mid PGL(\mathbb{F}_2, 5) \mid &= (2^5 - 1)(2^5 - 2)(2^5 - 4)(2^5 - 8)(2^5 - 16) \\ &= 9999360. \end{aligned}$$

Let S be the set of nonzero quadrics with the indeterminates x,y,z,t,u over $\mathbb{F}_2$. $\mid S \mid = 2^{15} - 1$. By Theorem 6.30 of [20], up to isomorphism, there exists a unique nondegenerate quadric, that is, $k_0 : xy + zt + u^2 = 0$. Let G be a subgroup of $PGL(\mathbb{F}_2, 5)$ fixing k_0. Then $\mid G \mid = 720$. Thus the number of nondegenerate quadrics is $PGL(\mathbb{F}_2, 5)/G = 13888$. Nonzero degenerate quadrics are isomorphic to one of the following quadrics, that is, if s is a nonzero degenerate quadric, then there exists $\sigma \in PGL(\mathbb{F}_2, 5)$ such that $\sigma(s)$ is equal to one of the following quadrics:

$k_1 : xy+zt = 0$, 1152 automorphisms of $PGL(\mathbb{F}_2, 5)$ fix k_1 and $GL(\mathbb{F}_2, 5)/1152 = 8680$ quadrics are isomorphic to k_1.

$k_2 : xy+zt+z^2+t^2 = 0$, 1920 automorphisms of $PGL(\mathbb{F}_2, 5)$ fix k_2 and $GL(\mathbb{F}_2, 5)/1920 = 5208$ quadrics are isomorphic to k_2.

$k_3 : xy+z^2 = 0$, 2304 automorphisms of $PGL(\mathbb{F}_2, 5)$ fix k_3 and $GL(\mathbb{F}_2, 5)/2304 = 4340$ quadrics are isomorphic to k_3.

$k_4 : xy = 0$, 21504 automorphisms of $PGL(\mathbb{F}_2, 5)$ fix k_4 and 465 quadrics are isomorphic to k_4.

$k_5 : xy + x^2 + y^2 = 0$, 64512 automorphisms of $PGL(\mathbb{F}_2, 5)$ fix k_5 and 155 quadrics are isomorphic to k_5.

$k_6 : x^2 = 0$, 322560 automorphisms of $PGL(\mathbb{F}_2, 5)$ fix k_6 and 31 quadrics are isomorphic to k_6.

Sum of all the above numbers of quadrics gives the order of S, as expected.

We are interested in the intersection of three linearly independent quadrics l_1, l_2 and l_3, satisfying the following conditions:

$$n_1 = 0, n_2 \leq 1, n_3 \leq 3, n_4 \leq 8, n_5 \leq 9. \tag{4.69}$$

Let at least one of l_i be nondegenerate. Assume l_1 is k_0. Then there exist $2^{15} - 1$ possibilities for l_2. Up to isomorphims, we have 152 possibilities for l_2. That is, for a given quadric l_2, if G_1 is a subgroup of G, fixing l_2, then $\mid G \mid / \mid G_1 \mid$ quadrics are isomorphic to l_2 while G_1 fixes l_1.

Let all of l_i be degenerate and at least one of them be degenerate in four indeterminates. Then it is k_1 or k_2. If l_1 is k_1, then there exist $2^{15} - 1 - 13888$ possibilities for l_2. Up to isomorphims, we have 112 possibilities for l_2. If l_1 is k_2, again, we have $2^{15} - 1 - 13888$ possibilities for l_2. Up to isomorphims, we have 88 possibilities for l_2.

If at least one of them is degenerate in three indeterminates and the others are degenerate in at most three indeterminates, we assume l_1 is k_3. Then there exist $2^{15} - 1 - 13888 - 860 - 5208$ possibilities for l_2. Up to isomorphims, we have 59 possibilities for l_2.

If at least one of them is degenerate in two indeterminates and the others are degenerate in at most two indeterminates, then the first one is k_4 or k_5. If l_1 is k_4, there exist $465 + 155 + 31$ possibilities for l_2. Up to isomorphims, we have 14 values for l_2. If l_1 is k_5, we have the same number of possibilities for l_2. Up to isomorphims, we have 9 possibilities for l_2.

If all of the quadrics are degenerate in at most one indeterminate, we assume l_1 is k_6. We have 31 possibilities for l_2. Up to isomorphims, we have 2 values for l_2.

For nondegenerate case, we have 556 distinct solutions for l_3 satisfying the given condition. Up to coordinate change of the projective space $\mathbb{P}^4(\mathbb{F}_2)$, we have 125 solutions which are given in Appendix A and for the degenerate quadric $l_1 : xy+zt = 0$, we have 32 values for l_3 with class number three. First we eliminate the nondegenerate quadrics among them, then up to coordinate change, there exist 5 solutions, given in Appendix A. For the degenerate quadric $l_1 : xy+zt+z^2+t^2 = 0$, we have 424 solutions for l_3. Among these values for l_3, we eliminate the nondegenerate quadrics and the quadrics which are isomorphic to $xy + zt = 0$, up to change of coordinates. Then up to isomorphism of the projective space, we get 25 solutions. These are presented in

Appendix A, as well. For other cases, checking all $2^{15}-1$ possible values for l_3 in magma, we get no solution satisfying 4.69.

4.3.4 Genus 6

Let $g_K = 6$ and $q = 2$. We have $n_1 = 0$, $n_6 - 2n_4 + \frac{n_3+n_3^2}{2} = 3$, $n_5 \leq 6$ and $n_2 = 0$. We have $A_3 \leq 9$ and $A_4 \leq 21$ by Clifford's Theorem and Lemma V.1.4 of [32]. Hence n_4 is at most 21 and n_3 is at most 9. Then we have 1540 distinct L-polynomials. Using Magma, we calculate the roots of them. For each case L-polynomial has a root with absolute value different from $1/\sqrt{2}$. Hence there exists no non-quadric function field of genus 6 with class number 3.

CHAPTER 5

CYLOTOMIC FUNCTION FIELDS WITH CLASS NUMBER THREE

5.1 Construction of cyclotomic function fields

Let $k = \mathbb{F}_q(t)$ be the rational function field over the finite field $\mathbb{F}_q$ where q is a prime power and let $\mathbb{A} = \mathbb{F}_q[t]$ be the ring of polynomials. Let $\bar{k}$ be the algebraic closure of k. We define an action of $\mathbb{A}$ on the additive group $\bar{k}^+$ of $\bar{k}$

$\mathbb{A} \times \bar{k}^+ \to \bar{k}^+$

$(N(t), u) \to u^{N(t)}$.

We take $\Phi, \mu_t \in End(\bar{k}^+)$ defined by $\Phi(u) = u^q$ and $\mu_t(u) = tu$. Using these endomorphisms, we define $u^{N(t)} := N(\Phi + \mu_t)(u)$. Clearly $u^a = a.u$ for all $a \in \mathbb{F}_q$.

Let $N(t) = a_d t^d + a_{d-1}t^{d-1} + \cdots + a_1 t + a_0$, then

$$u^{N(t)} = (a_d(\Phi + \mu_t)^d + \cdots + a_1(\Phi + \mu_t) + a_0 I)(u).$$

With the help of this action, we consider $\bar{k}^+$ as an $\mathbb{A}$-module.

If we assume $deg(N) = d$,

$$u^{N(t)} = \sum_{i=0}^{d} \binom{N(t)}{i} u^{q^i},$$

where $\binom{N(t)}{i}$ is a polynomial in $\mathbb{A}$ of degree $(d-i)q^i$ satisfying the following properties:

(i) $\binom{N(t)}{0} = N(t)$,

(ii) $\binom{N(t)}{d}$ is the leading coefficient of $N(t)$.

To extend this polynomial for all $i \in \mathbb{Z}$, we define

(iii) $\binom{N(t)}{i} = 0$ for $i > d$ or $i < 0$.

Moreover,

$\binom{aN(t)+bM(t)}{i} = a\binom{N(t)}{i} + b\binom{M(t)}{i}$ for $M(t), N(t) \in \mathbb{A}$ and $a, b \in \mathbb{F}_q$,

and $\binom{t^{d+1}}{i} = t\binom{t^d}{i} + \binom{t^d}{i-1}^q$.

Since $char(k) = p$ and q is a p-power and $\frac{d}{du}(u^{N(t)}) = N(t)$ is nonzero, $u^{N(t)}$ is a separable polynomial in u of degree q^d over $\mathbb{A}$.

Let Λ_N be the set of roots of $u^{N(t)}$. For $N \neq 0$, Λ_N is a finite cyclic $\mathbb{A}$-module with q^d elements and $\Lambda_N \cong \mathbb{A}/N$ as an $\mathbb{A}$-module and $K_N = k(\Lambda_N)$ is called the cyclotomic function field associated to N with the constant field $\mathbb{F}_q$. Let K_N^+ be its maximal real subfield , that is, the maximal subfield of K_N in which infinite prime ∞ splits completely.

Example 5.1.1 *Let $N(t) = 2t + 1 \in \mathbb{F}_3[t]$ be a polynomial, then*

$$\begin{aligned} u^{N(t)} &= (2(\Phi + \mu_t) + I)(u) \\ &= 2(u^3 + tu) + u \\ &= u(2u^2 + 2t + 1) \end{aligned}$$

$$u^{N(t)} = 0 \Rightarrow u = 0 \ or \ u = \pm\sqrt{-t - 1/2}.$$

That is $K_N = \mathbb{F}_3(t)(\sqrt{-t-1/2})$ which is a totally imaginary extension of k.

Proposition 5.1 (1.4, [13]) *Let $N = a\prod P^n$ be a factorization of the polynomial $N \in \mathbb{A}$ into the powers of monic irreducible polynomials. Then*

$$\Lambda_N = \sum_{P|N} \Lambda_{P^n}$$

where Λ_{P^n} is the set of the roots of $u^{P^n(t)}$ and the sum is direct as a sum of $\mathbb{A}-$modules.

Proposition 5.2 (1.5, [13]) *If $N = P^n$ where P is an irreducible polynomial in $\mathbb{A}$, then Λ_N is a cyclic $\mathbb{A}$-module.*

Definition 5.1.2 *If $N \in \mathbb{A}$, $N \neq 0$, then* $\underline{\Phi(N)}$ *is the order of the group of units of* $\mathbb{A}/(N)$. ■

The cyclic $\mathbb{A}$-module Λ_N has exactly $\Phi(N)$ generators. If λ is a generator of Λ_N, then $\Lambda_N = \{\lambda^M | M \in \mathbb{A}\}$. Clearly $\Lambda_N =< \lambda^M >$ if and only if $(M, N) = 1$.

Proposition 5.3 (2.2, [13]) *Suppose $N = P^n$ where P is a monic irreducible polynomial in t with $degP = d$. Then every prime divisor of K except P and the infinite place ∞ are unramified in K_N, and the ramification index of P is $\Phi(N) = q^{dn} - q^{d(n-1)}$.*

Proposition 5.4 (2.4, [13]) *If $N = P^n$, where P is a monic irreducible in $\mathbb{A}$, then $f(u) = u^{P^n}/u^{P^{n-1}}$ is an Eisenstein polynomial over $\mathbb{A}$ at P.*

Theorem 5.5 (3.1, [13]) *Let $N \in \mathbb{A}$, $N \neq 0$. Then ∞ is tamely ramified in K_N/k.*

Theorem 5.6 (3.2, [13]) *Let $N = P^n$ where P is a monic irreducible polynomial in $\mathbb{A}$ with $degP = d$. Then, the infinite place ∞ splits into $\Phi(N)/(q-1)$ prime divisors in K_N. The ramification index $e(\infty, K_N/k)$ is $q-1$ and the degree of inertia $f(\infty, K_N/k)$ is 1.*

5.2 Character groups

Definition 5.2.1 *Let M be a polynomial in $\mathbb{A}$. A* <u>Dirichlet character</u> modulo *M is a function from $\mathbb{A} \to \mathbb{C}$ such that*

(i) $\chi(A + BM) = \chi(A)$ for all $A, B \in \mathbb{A}$.

(ii) $\chi(A)\chi(B) = \chi(AB)$ for all $A, B \in \mathbb{A}$.

(iii) $\chi(A) \neq 0$ if and only if $(A, M) = 1$.

■

A Dirichlet character modulo M induces a homomorphism from $(\mathbb{A}/M\mathbb{A})^* \to \mathbb{C}^*$ and conversely, given such a homomorphism, there is a unique corresponding Dirichlet character. The trivial Dirichlet character χ_0 is defined by $\chi_0(A) = 1$ if $(A, M) = 1$ and $\chi_0(A) = 0$ if $(A, M) \neq 1$. Since for any $A \in \mathbb{A}$ such that $(A, M) = 1$, $A^k \equiv 1$ modulo M for some $k \in \mathbb{Z}^+$ and $\chi(A^k) = 1$, we deduce that $\chi(A)$ is a root of unity.

There are exactly $\Phi(M)$ Dirichlet characters modulo M. This number is equal to the order of the group $(\mathbb{A}/M\mathbb{A})^*$. Let X_M be the set of Dirichlet characters modulo M. If $\chi, \psi \in X_M$, we define their product $\chi\psi$ by the formula $\chi\psi(A) = \chi(A)\psi(A)$. By this product, X_M is a group. The identity of this group is the trivial character χ_0. The inverse of a character is given by $\chi^{-1}(A) = \chi(A)^{-1}$ if $(A, M) = 1$, and $\chi^{-1}(A) = 0$ if $(A, M) \neq 1$.

Moreover, X_M is isomorphic to $(\mathbb{A}/M\mathbb{A})^*$.

If $\chi \in X_M$, let $\overline{\chi}(A) := \overline{\chi(A)}$, the complex conjugate of $\chi(A)$. Since $\chi(A)$ is a root of unity or 0, $\overline{\chi} = \chi^{-1}$.

Proposition 5.7 (4.2, [26]) *Let χ and ψ be two Dirichlet characters modulo M and A and B two elements of $\mathbb{A}$ relatively prime to M. Then we have the orthogonality relations:*

(i) $\sum_A \chi(A)\overline{\psi(A)} = \Phi(M)\delta(\chi, \psi)$.

(ii) $\sum_\chi \chi(A)\overline{\chi(B)} = \Phi(M)\delta(A, B)$.

The first sum is over any set of representatives for $\mathbb{A}/M\mathbb{A}$ and the second sum is over all Dirichlet characters modulo M. By definition, $\delta(\chi, \psi) = 0$ if $\chi \neq \psi$ and 1 if $\chi = \psi$. Similarly, $\delta(A, B) = 0$ if $A \neq B$ and 1 if $A = B$.

5.3 Cyclotomic function fields and their subfields

The infinite prime divisor of k associated to $(1/T)$ is denoted by ∞. Throughout this chapter, we assume K is a finite abelian extension of k such that $K \subseteq K_N$ for some $N \in \mathbb{A}$. Then $K^+ = K \cap K_N^+$ is the maximal real subfield of K.

Definition 5.3.1 *The <u>conductor</u> of K is the monic polynomial N such that K_N is the smallest cyclotomic function field containing K.* ■

Definition 5.3.2 *K/k is called a real extension, if $J_K(:= Gal(K/K^+) =< 1 >$, and imaginary otherwise. If $J_K = Gal(K/k)$, then K is called a totally imaginary extension.*

■

Let K be a subfield of a cyclotomic function field. Let us denote by X_K the character group of $Gal(K/k)$. For a fixed monic irreducible polynomial $Q \in \mathbb{F}_q[T]$, let $Y_K = \{\chi \in X_K : \chi(Q) \neq 0\}$ and $Z_K = \{\chi \in X_K : \chi(Q) = 1\}$. By Chapter 3 of [34], we have $[Y_K : Z_K] = f((Q), K/k)$, the inertia degree of the associated place of Q in K/k and $\mid Z_K \mid = g(Q, K/k)$, the number of primes of K lying above (Q).

Definition 5.3.3 *Let F'/K be a finite separable extension of F/K. Then the divisor*

$$D(F'/F) := \sum_{P \in \mathbb{P}_F} \sum_{P'|P} d(P'|P)P'$$

is called the different of F'/F. If $P'|P$ is tamely ramified, then $d(P'|P) = e(P'|P) - 1$. Otherwise

$$d(P'|P) = \sum_{n=0}^{\infty} (|G^0(P', F'/F)| - [G^0(P', F'/F) : G^n(P', F'/F)]),$$

where $G^n(P', F'/F)$ denotes the nth upper ramification group of P' in F'. ■

Theorem 5.8 (Hurwitz Genus Formula) *Let L/K be a finite abelian extension with the same constant field. Then we have*

$$2g_L - 2 = (2g_K - 2)[L : K] + deg(D(L/K))$$

where $D(L/K)$ is the different of L/K. Especially $g_L \geq g_K$.

In this chapter, we determine all subfields of the cyclotomic function fields with class number three. Using Theorem 2.4,

(i) $q = 2$, $1 \leq g_K \leq 6$,

(ii)$q = 3$, $1 \leq g_K \leq 2$,

(iii)$q = 5, 7$, $g_K = 1$.

Let $S_\infty(K)$ denote the set of prime divisors of K lying above ∞. Then $n_1 \geq |S_\infty(K)| = [K^+ : k] \geq 1$. For $q = 2$, $K = K^+$, then $n_1 \geq |S_\infty(K)| = [K : k] \geq 2$ in this case. Then by Theorem 2.4,

(A) $g_K = 1$, ($q = 2, 3, 5, 7$, $n_1 = 3$),

(B) $g_K = 2$, ($q = 2$, $n_1 = 2, n_2 = 2$) or ($q = 3$, $n_1 = 1, n_2 = 5$ or $n_1 = 2, n_2 = 3$),

(C) $g_K = 3$, $q = 2$, $n_1 = 2, n_3 + 2n_2 = 3$.

For the last case, $2 = n_1 \geq |S_\infty(K)| = [K : k] \geq 2$, that is, the extension is quadratic and ∞ splits. That implies (T) and $(T+1)$ are inert, then $n_2 \geq 2$ and $n_3 \leq -1$, which is not possible. Hence we skip this case.

5.4 Genus one case:

Proposition 5.9 *Let K/k be a real extension of genus 1 with class number 3, then $n_1 = 3 \geq [K : k] \geq 2$ and K satisfies one of the following cases:*

(i) $q = 2$, K is a quadratic extension of k with conductor P^4, $degP = 1$,

(ii) $q = 7$, $K = k(\sqrt[3]{P})$, $degP = 3$,

(iii) $q = 3, 5, 7$, $K = k(\sqrt{P_1 P_2})$, $degP_1 = 1, degP_2 = 3$.

Proof. Let K/k be a real extension of genus 1 with class number 3. Then $K = K^+$ and $3 \geq [K : k] \geq 2$ and by Theorem 3.3 and 3.4 of [1], K satisfies one of the following conditions:

if $[K : k]$ is a p-power where $p = char(k)$,

(a) $p = 2$, K/k is a quadratic extension, $cond(K) = P^4$, $degP = 1$,

(b) $p = 2$, K/k is a quartic extension, $cond(K) = P^3$, $degP = 1$,

($|n_1 \geq S_\infty(K)| = [K : k] = 4$, which contradicts $h_K = 3$)

(c) $p = 2$, K/k is a quadratic extension, $cond(K) = P^2$, $degP = 2$.

$(|S_\infty(K)| = [K : k] = 2 = h_K - 1$. That means there exits a ramified finite prime of degree 1, which is not possible.)

(d) $p = 3$, K/k is a cubic extension, $cond(K) = P^3$, $degP = 1$,

$(|S_\infty(K)| = [K : k] = 3$ and since P is ramified $h_K = n_1 \geq 4)$

(e) $p = 2$, K/k is quadratic or biquadratic $cond(K) = P_1^2P_2^2$, $degP_i = 1$.

(Similarly, $n_1 \geq 4$ for this case.)

if $[K : k]$ is relatively prime to p where $p = char(k)$,

(a') $K = k(\sqrt[3]{P})$, $degP = 3$, $q \equiv 1 \pmod 3$,

(b') $K = k(\sqrt{P})$, $degP = 4$, q odd,

(That means $h_K = n_1$ is even.)

(c') $K = k(\sqrt{-P_1}\sqrt[4]{P_2})$, $degP_1 = 1, degP_2 = 2$, $q \equiv 1 \pmod 4$,

$(n_1 \geq |S_\infty(K)| = [K : k] = 4.)$

(d') $K = k(\sqrt[3]{-P_1^2P_2^2})$, $degP_1 = 1, degP_2 = 2$, $q \equiv 1 \pmod 3$,

$(|S_\infty(K)| = [K : k] = 3$, but P_1 is totally ramified and $n_1 \geq 4$.)

(e') $K = k(\sqrt[3]{-P_3^2P_1}, \sqrt[3]{-P_3^2P_2})$ or $k(\sqrt[3]{P_1P_2P_3})$, $degP_i = 1$, $q \equiv 1 \pmod 3$,

$(n_1 \geq |S_\infty(K)| = [K : k] \geq 9$ for the first case. For the second case, $|S_\infty(K)| = [K : k] = 3$, but $n_1 \geq 6$.)

(f') $K = k(\sqrt{P_1P_3}, \sqrt[3]{-P_2P_3^2})$, $degP_i = 1$, $q \equiv 1 \pmod 6$,

$(n_1 \geq |S_\infty(K)| = [K : k] \geq 4.$)

(g') $K = k(\sqrt{P_1P_3}, \sqrt[4]{P_2P_3^3})$ or $k(\sqrt[4]{P_1^2P_2P_3})$, $degP_i = 1$, $q \equiv 1 \pmod 4$,

$(n_1 \geq |S_\infty(K)| = [K : k] \geq 4.)$

(h')$K = k(\sqrt{P_1P_2})$, $degP_1 = 1, degP_2 = 3$, q odd,

(i')$K = k(\sqrt{P_1P_2})$ or $k(\sqrt{P_1}, \sqrt{P_2})$, $degP_i = 2$, q odd,

(In this case n_1 is even)

(j') $K = k(\sqrt{P_1P_2}, \sqrt{P_3})$ or $k(\sqrt{P_1P_2P_3})$, $degP_1 = degP_2 = 1, degP_3 = 2$, q odd,

$(n_1 \geq 4)$

(k') $K = k(\sqrt{P_1P_2}, \sqrt{P_1P_3}, \sqrt{P_1P_4})$ or $k(\sqrt{P_1P_2}, \sqrt{P_3P_4})$ or $k(\sqrt{P_1P_2P_3P_4})$, $degP_i = 1$, q odd,

$(n_1 \geq 4\ )$

Hence, the result follows. ■

Using the conditions of the previous proposition, we get the following result:

Theorem 5.10 *Let K/k be a real extension of genus 1 with class number 3. Then K is one of the following function fields up to isomorphism ($x \to x + a$, $a \in \mathbb{F}_q^*$):*

(1) $q = 3$, $K = k(y)$ such that $y^2 = T(T^3 + 2T^2 + T + 1)$ or $y^2 = T(T^3 + T^2 + T + 2)$ with $L(t) = 3t^2 - t + 1$

(2) $q = 5$, $K = k(y)$ satisfying one of the following equations:

(i)$y^2 = T(T^3 - T^2 - T - 1)$,

(ii)$y^2 = T(T^3 + 2T^2 + T + 3)$,

(iii)$y^2 = T(T^3 + 3T^2 + T + 2)$,

(iv) $y^2 = T(T^3 + T^2 - T + 1)$.

For each one, $L(t) = 5t^2 - 3t + 1$.

(3)$q = 7$, $K = k(y)$ such that $y^2 = T(T^3+2)$ or $y^2 = T(T^3+5)$ with $L(t) = 7t^2-5t+1$.

(4)$q = 2$, $K = k(y)$ such that $y^2 + y = 1/T^3$ with $L(t) = 2t^2 + 1$

(5) $q = 7$, $K = k(y)$ such that $y^3 = T^3 + 3$ or $y^3 = T^3 + 4$ with $L(t) = 7t^2 - 5t + 1$.

Proof. Clearly, K satisfies one of the conditions of Proposition 5.9:

(i)$q = 2$, K is a quadratic extension of k with conductor P^4, $degP = 1$:

Then $|S_\infty(K)| = [K:k] = 2 = n_1 - 1$ and one of the finite places of k of degree one is ramified. Up to isomorphism, let (T) be ramified. Using the previous results, up to isomorphism, $K = \mathbb{F}_2(x,y)$ with $y^2 + y = x^3$, where ∞ is ramified and (x) splits in $K/\mathbb{F}_2(x)$. Using the substitution, $x \to 1/T$, we get $K = k(y)$ with $y^2 + y = 1/T^3$ where ∞ splits and (T) is ramified in K/k.

By Hurwitz Genus Formula, we check our result:

$$0 = -2[K:k] + deg(Diff(K/k)).$$

That is, $d((T), K/k) = 4$ and $v_T(u) = -3$, where $u = 1/T^3$. Since $u \neq w^2 - w$ for any $w \in \mathbb{F}_2(T)$, by Artin-Schreier extension, $y^2 + y = u$ where (T) is totally ramified in K/k.

(ii) $q = 7$, $K = k(\sqrt[3]{P})$, $degP = 3$:

Let X_{K_P}, X_K be the character groups of K_P and K, respectively. $X_{K_P} \cong (\mathbb{A}/P)^*$ is a cyclic group of order $7^3 - 1$. Let χ be a generator, then $X_K =< \chi^a >$ for some integer a and the order of χ^a is $[K:k] = 3$. Hence we may assume $X_K =< \chi^{114} >$. Assume $Y_K = \{\chi \in X_K : \chi(Q) \neq 0\}$ and $Z_K = \{\chi \in X_K : \chi(Q) = 1\}$ for an irreducible polynomial Q. Then, we have $[Y_K : Z_K] = f((Q), K/k)$, the inertia degree of the associated place of Q in K/k and $\mid Z_K \mid = g(Q, K/k)$, the number of primes of K lying above (Q). Since $|S_\infty(K)| = [K:k] = 3$, none of the finite places of k of degree one splits in K/k. That is, we have

$$\chi^{114}(T+a) \neq 1$$

for all $a \in \mathbb{F}_7$. Up to isomorphism $T \to T + \alpha, \alpha \in \mathbb{F}_7^*$, we have 16 possibilities for P. Among them, for $P = T^3 + 3$ and $T^3 + 4$, the result follows.

Let $P = T^3 + 3$, then $X_{K_P} =< \chi >$ and $T + 1$ is a primitive element of $(\mathbb{F}_7(T)/P)^*$.

We have

$\chi^{114}(T) = \chi(T^{114}) = \chi(T^6) = \chi(2) = \exp(4\pi i/3)$, $\chi^{114}(T+1) = \exp(2\pi i/3)$,

$\chi^{114}(T+2) = \exp(2\pi i/3)$, $\chi^{114}(T+3) = \chi((T+3)^{114}) = \chi(2) = \exp(4\pi i/3)$,

$\chi^{114}(T+4) = \chi(4) = \exp(2\pi i/3)$, $\chi^{114}(T+5) = \chi(2) = \exp(4\pi i/3) = \chi^{114}(T+6)$.

Hence all finite places of k of degree one are inert and

$$K = k(\sqrt[3]{T^3+3}). \tag{5.1}$$

Similarly, for $P = T^3 + 4$, $\chi(T + a) \neq 1$ for all $a \in \mathbb{F}_7$ and

$$K = k(\sqrt[3]{T^3+4}). \tag{5.2}$$

<u>(iii) $q = 3, 5, 7$, $K = k(\sqrt{P_1P_2})$, $degP_1 = 1, degP_2 = 3$:</u>

- $q = 3$

$|S_\infty(K)| = [K : k] = 2 = n_1 - 1$ and one of the finite places of k of degree one, say P_1 is ramified and all other places of degree one are inert, except ∞. WLOG, assume $P_1 = (T)$ and let the associated polynomial of P_2 be $T^3 + aT^2 + bT + c$ where $a, b, c \in \mathbb{F}_3$. Using the results of [18],

$$K = k(y),\ with\ y^2 = T(T^3 + aT^2 + bT + c). \tag{5.3}$$

Let $P_1P_2(T)$ denote the product of the associated polynomials of P_1 and P_2, respectively. $P_1P_2(1) = P_1P_2(2) = 2 \in \mathbb{F}_3^*\backslash\mathbb{F}_3^{*2}$ and P_2 is irreducible. By Proposition 3.2, we have $b = 1$, $a = 2$, $c = 1$ or $b = 1$, $a = 1$, $c = 2$. That is,

$$y^2 = T(T^3 + 2T^2 + T + 1)\ or\ y^2 = T(T^3 + T^2 + T + 2). \tag{5.4}$$

- $q = 5$

Similarly, $K = k(y)$ where $y^2 = P_1P_2$. Assume $P_1 = (T)$ and $P_2 = T^3 + aT^2 + bT + c$ where $a, b, c \in \mathbb{F}_5$. Also $P_1P_2(\alpha) = 2$ or 3 for $\alpha \in \mathbb{F}_5$. Checking all possibilities we have $K = k(y)$ satisfying one of the following equations:

$$y^2 = T(T^3 - T^2 - T - 1), \tag{5.5}$$

$$y^2 = T(T^3 + 2T^2 + T + 3), \tag{5.6}$$

$$y^2 = T(T^3 + 3T^2 + T + 2), \tag{5.7}$$

$$y^2 = T(T^3 + T^2 - T + 1). \tag{5.8}$$

- $q = 7$

$K = k(y)$ where $y^2 = P_1P_2$. Assume $P_1 = (T)$ and $P_2 = T^3 + aT^2 + bT + c$ where $a, b, c \in \mathbb{F}_7$. Also $P_1P_2(\alpha) = 3, 5$ or 6 for $\alpha \in \mathbb{F}_7$. Checking all possibilities we have $K = k(y)$ where

$$y^2 = T(T^3 + 2) \textit{ or } y^2 = T(T^3 + 5). \tag{5.9}$$

■

Remark 5.11 (Lemma 4.1, [1]) *Let K/k be an imaginary extension of k with $g_K = 1$. Then $g_{K^+} = 0$.*

Proposition 5.12 *Let K/k be a totally imaginary extension of genus 1 with class number 3, then K satisfies one of the following cases:*

(i) $q = 7$, $K = k(\sqrt[3]{P_1P_2})$, $degP_i = 1$,

(ii) $q = 7$, $K = k(\sqrt{-P_1}, \sqrt[3]{-P_2})$, $degP_i = 1$,

(iii) $q = 3, 5, 7$, $K = k(\sqrt{-P})$, $degP = 3$.

Proof. We have $|S_\infty(K)| = 1 = n_1 - 2$. By Theorem 4.2 of [1], K is one of the following function fields:

(a) $K = k(\sqrt[3]{-P^2})$, $degP = 2$, $q \equiv 1 \pmod 3$, ($|S_\infty(K)| = 1$ and none of finite places of k of degree one is ramified. If one of them splits, $n_1 \geq 4$. Otherwise, $n_1 = 1$.)

(b) $K = k(\sqrt[3]{P_1P_2})$, $degP_i = 1$, $q \equiv 1 \pmod 3$,

(c) $K = k(\sqrt{-P_1}\sqrt[4]{-P_2})$, $degP_i = 1$, $q \equiv 1 \pmod 4$,

(While P_2 is totally ramified, P_1 is not totally ramified, that is $n_1 = 2$ or $n_1 \geq 4$.)

(d) $K = k(\sqrt{-P_1}, \sqrt[3]{-P_2})$, $degP_i = 1$, $q \equiv 1 \pmod 6$,

(e) $K = k(\sqrt{-P})$, $degP = 3$, q odd,

(f) $K = k(\sqrt{-P_1P_2})$, $degP_1 = 1$, $degP_2 = 2$, q odd,

(Similarly, P_1 is totally ramified and $n_1 = 2$ or $n_1 \geq 4$.)

(g) $K = k(\sqrt{-P_1P_2P_3})$, $degP_i = 1$, q odd,

$(n_1 \geq 4.)$ ■

Theorem 5.13 *Let K/k be a totally imaginary extension of genus 1 with class number 3. Then,up to isomorphism ($x \to x+a$, $a \in \mathbb{F}_q^*$), K is one of the following function fields:*

(1) $q = 7$, $K = k(y)$ such that $y^3 = T(T+3)$ or $y^3 = T(T+4)$ with $L(t) = 7t^2 - 5t + 1$.

(2) $q = 7$, $K = k(y, z)$ such that $y^2 + T = 0$ and $z^3 + T + 4 = 0$ with $L(t) = 7t^2 - 5t + 1$.

(3) $q = 3$, $K = k(y)$ such that $y^2 + T^3 + 2T^2 + 1 = 0$ with $L(t) = 3t^2 - t + 1$.

(4) $q = 5$, $K = k(y)$ such that $y^2 + T^3 + 4T + 2 = 0$ or $y^2 + T^3 + 4T + 3 = 0$ with $L(t) = 5t^2 - 3t + 1$.

(5) $q = 7$, $K = k(y)$ such that $y^2 + T^3 + 3 = 0$ with $L(t) = 7t^2 - 5t + 1$.

Proof. Clearly, K satisfies one of the conditions of Proposition 5.12:

(i) $q = 7$, $K = k(\sqrt[3]{P_1P_2})$, $degP_i = 1$: $|S_\infty(K)| = 1$ and P_1 and P_2 are totally ramified. Then all of the other finite places of k of degree 1 are inert in K/k. Assume $P_1 = (T)$ and $P_2 = (T + a)$ for $a \in \mathbb{F}_7^*$. Let $X_{K_{P_i}}, X_K$ be the character groups of K_{P_i} and K, respectively. $X_{K_{P_i}} \cong (\mathbb{A}/P_i)^*$ is a cyclic group of order 6. Let χ_i be the generator of $X_{K_{P_i}}$, then $X_K =< \chi_1^2\chi_2^2 >$ where $\chi_i(3) = \exp(2\pi i/6)$.

- Let $a = 1$, then $\chi_1^2\chi_2^2(T + 3) = 1$ and $n_1 \geq 6$.

- Let $a = 2$, then $\chi_1^2\chi_2^2(T + 1) = 1$ and $n_1 \geq 6$.

- Let $a = 3$, then $\chi_1^2\chi_2^2(T+1) = \exp(10\pi i/3)$, $\chi_1^2\chi_2^2(T+2) = \exp(4\pi i/3)$,$\chi_1^2\chi_2^2(T+4) = \exp(2\pi i/3)$,$\chi_1^2\chi_2^2(T+5) = \exp(2\pi i/3)$,$\chi_1^2\chi_2^2(T+6) = \exp(2\pi i/3)$ and all of the places of degree one, except (T) and $(T + 3)$ are inert. That is,

$$K = k(y) \text{ where } y^3 = T(T + 3). \tag{5.10}$$

- Let $a = 4$, then $\chi_1^2\chi_2^2(T+1) = \exp(2\pi i/3)$, $\chi_1^2\chi_2^2(T+2) = \exp(2\pi i/3)$,$\chi_1^2\chi_2^2(T+3) = \exp(2\pi i/3)$,$\chi_1^2\chi_2^2(T+5) = \exp(10\pi i/3)$,$\chi_1^2\chi_2^2(T+6) = \exp(4\pi i/3)$ and all of the places of degree one, except (T) and $(T + 4)$ are inert. That is,

$$K = k(y) \text{ where } y^3 = T(T + 4). \tag{5.11}$$

• Let $a = 5$, then $\chi_1^2\chi_2^2(T+6) = 1$ and $n_1 \geq 6$.

• Let $a = 6$, then $\chi_1^2\chi_2^2(T+2) = 1$ and $n_1 \geq 6$.

<u>(ii) $q = 7$, $K = k(\sqrt{-P_1}, \sqrt[3]{-P_2})$, $degP_i = 1$:</u> Let $P_1 = (T)$ and $P_2 = (T+a)$ for $a \in \mathbb{F}_7^*$ Using the notation of part (i), $X_K =< \chi_1^3, \chi_2^2 >$. Since $n_1 = 3$, P_1 is inert in $k(\sqrt[3]{-P_2})/k$, P_2 splits in $k(\sqrt{-P_1})/k$ and all the other finite places of k of degree one do not split in K/k.

• Let $a = 1$, then $\chi_2^2(T) = 1$ and $n_1 \geq 4$.

• Let $a = 2$, then $\chi_1^3(T+1) = 1 = \chi_2^2(T+1)$ and $n_1 \geq 7$.

• Let $a = 3$, then $\chi_1^3(T+2) = 1 = \chi_2^2(T+2)$ and $n_1 \geq 7$.

• Let $a = 4$, then $\chi_2^2(T+1) = \exp(2\pi i/3)$, $\chi_2^2(T+2) = \exp(10\pi i/3)$,$\chi_1^3(T+3) = -1$,$\chi_1^3(T+5) = -1$,$\chi_2^2(T+6) = \exp(4\pi i/3)$. Also $\chi_1^3(T+4) = 1$ and $\chi_2^2(T) = \exp(\pi i/3)$ That is,

$$K = k(y, z) \text{ where } y^2 + T = 0 \text{ and } z^3 + T + 4 = 0. \tag{5.12}$$

• Let $a = 5$, then $\chi_1^3(T+4) = 1 = \chi_2^2(T+4)$ and $n_1 \geq 7$.

• Let a_6, then $\chi_2^2(T) = 1$ and $n_1 \geq 4$.

<u>(iii) $q = 3, 5, 7$, $K = k(\sqrt{-P})$, $degP = 3$:</u>

• Let $q = 3$. Up to isomorphism ($T \to T + a$, $a \in F_3^*$) there exist four possibilities for P. These are $T^3 + 2T + 1$, $T^3 + 2T + 2$, $T^3 + T^2 + 2$ and $T^3 + 2T^2 + 1$. Since $|S_\infty(K)| = 1$, one of the finite places of k of degree one splits and the others are inert in K/k.

(a) Let $P = T^3 + 2T + 1$, then $(\mathbb{F}_3(T)/P)^* =< T >$. Let $X_{K_P} =< \chi >$ where $| X_{K_P} |= 3^3 - 1$. $X_K =< \chi^a >$ for some $a \in \mathbb{Z}$ and $| X_K |= [K : k]$ which is equal to 2. Since the order of χ^a is 2, we may assume $X_K =< \chi^{13} >$ and $\chi^{13}(T) = \chi^{13}(T+1) = \chi^{13}(T+2) = -1$. Hence $n_1 = 1$.

(b) Let $P = T^3 + 2T + 2$, then $(\mathbb{F}_3(T)/P)^* =< -T >$. Let $X_{K_P} =< \chi >$, we have $X_K =< \chi^{13} >$ and $\chi^{13}(T) = \chi^{13}(T+1) = 1$. Thus $n_1 \geq 5$.

(c) Let $P = T^3 + T^2 + 2$, then $(\mathbb{F}_3(T)/P)^* =< -T >$. Let $X_{K_P} =< \chi >$, we have $X_K =< \chi^{13} >$ and $\chi^{13}(T) = \chi^{13}(T+1) = 1$. So $n_1 \geq 5$.

(d) Let $P = T^3 + 2T^2 + 1$, then $(\mathbb{F}_3(T)/P)^* =< T >$. Let $X_{K_P} =< \chi >$, we have $X_K =< \chi^{13} >$ and $\chi^{13}(T+2) = 1$ *and* $\chi^{13}(T) = \chi^{13}(T+1) = -1$. Hence $n_1 = 3$ and $K = k(y)$ where

$$y^2 + T^3 + 2T^2 + 1 = 0. \tag{5.13}$$

• Let $q = 5$. Up to isomorphism ($T \to T + a$, $a \in F_5^*$) we have 8 possibilities for P. These are T^3+T+1, T^3+T+4, T^3+3T+2, T^3+3T+3, T^3+2T+1, T^3+2T+4, T^3+4T+2 and T^3+4T+3. Since $|S_\infty(K)| = 1$, one of the finite places of k of degree one splits and others are inert in K/k. Let $X_{K_P} =< \chi >$ where $| X_{K_P} |= 5^3 - 1$. $X_K =< \chi^a >$ for some integer a and $| X_K |= [K : k]$ which is equal to 2. That is, order of χ^a is 2. So we may assume $X_K =< \chi^{62} >$.

(a) Let $P = T^3 + T + 1$, then $\chi^{62}(T) = \chi^{62}(T+2) = 1$ and $n_1 \geq 5$.

(b) For $P = T^3 + T + 4$, $\chi^{62}(T) = \chi^{62}(T+2) = 1$. Thus $n_1 \geq 5$.

(c) For $P = T^3 + 3T + 2$, $\chi^{62}(T+3) = \chi^{62}(T+4) = 1$ and $n_1 \geq 5$.

(d) Assume $P = T^3 + 3T + 3$, then $\chi^{62}(T+1) = \chi^{62}(T+2) = 1$ and $n_1 \geq 5$.

(e) Let $P = T^3 + 2T + 1$. Then $\chi^{62}(T) = \chi^{62}(T+2) = 1$ and $n_1 \geq 5$.

(f) For $P = T^3 + 2T + 4$, $\chi^{62}(T+1) = \chi^{62}(T) = 1$ and $n_1 \geq 5$.

(g) Let $P = T^3 + 4T + 2$. $\chi^{62}(T) = \chi^{62}(T+1) = \chi^{62}(T+3) = \chi^{62}(T+4) = -1$ and $\chi^{62}(T+2) = 1$. That is, $n_1 = 3$ and $K = k(y)$ where

$$y^2 + T^3 + 4T + 2 = 0. \tag{5.14}$$

(h) Let $P = T^3 + 4T + 3$. $\chi^{62}(T) = \chi^{62}(T+1) = \chi^{62}(T+2) = \chi^{62}(T+4) = -1$ and $\chi^{62}(T+3) = 1$. That is, $n_1 = 3$ and $K = k(y)$ where

$$y^2 + T^3 + 4T + 3 = 0. \tag{5.15}$$

• Let $q = 7$. Up to isomorphism ($T \to T + a$, $a \in F_7^*$) we have 16 possibilities for P. These are $T^3 + 2$, $T^3 + 3$, $T^3 + 4$, $T^3 + 5$, $T^3 + T + 1$, $T^3 + T + 6$, $T^3 + 2T + 1$,

T^3+2T+6, T^3+3T+2, T^3+3T+5, T^3+4T+1, T^3+4T+6, T^3+5T+2, T^3+5T+5, T^3+6T+2, T^3+6T+5. Since $|S_\infty(K)| = 1$, one of the finite places of k of degree one splits and others are inert in K/k. Let $X_{K_P} =< \chi >$ where $| X_{K_P} |= 7^3 - 1$. $X_K =< \chi^a >$ for some $a \in \mathbb{Z}$ and $| X_K |= [K : k]$ which is equal to 2. That is, order of χ^a is 2. Then we may assume $X_K =< \chi^{171} >$. Among them, the result follows for only $T^3 + 3$. That is, let $P = T^3 + T + 1$, then $(\mathbb{F}_7[T]/P)^* =< T + 1 >$.

$\chi^{171}(T+1) = \chi^{171}(T+2) = \chi^{171}(T+3) = \chi^{171}(T+4) = \chi^{171}(T+5) = \chi^{171}(T+6) = -1$ and $\chi^{171}(T) = \chi(T^{171}) = \chi(1) = 1$, that is $n_1 = 3$ and $K = k(y)$ where

$$y^2 + T^3 + 3 = 0. \tag{5.16}$$

■

Proposition 5.14 *Let K/k be an imaginary (not totally imaginary) extension of genus 1 with class number 3, then K satisfies one of the following cases:*

(i) $q = 7$, $K = k(\sqrt[3]{-P_1}, \sqrt[3]{-P_2})$, $degP_i = 1$,

(ii) $q = 7$, $K = k(\sqrt{-P_1}\sqrt[6]{-P_2})$, $degP_i = 1$.

Proof. We have $K \neq K^+ \neq k$ and $|S_\infty(K)| \geq 2$. By Theorem 4.4 and by Theorem 4.6 of [1], K is one of the following function fields:

(a) $K = k(\sqrt[4]{P})$, $degP = 2$, $q \equiv 1$ (Mod 4),

($|S_\infty(K)| = 2$, hence there exists $Q \in \mathbb{P}_k$ of degree 1 which is totally ramified. Then $Q|cond(K)$, which is a contradiction.)

(b) $K = k(\sqrt[3]{-P^1}, \sqrt[3]{-P_2})$, $degP_i = 1$, $q \equiv 1$ (mod 3),

(c) $K = k(\sqrt[3]{-P^1}, \sqrt[4]{P_1P_2^3})$, $degP_i = 1$, $q \equiv 1$ (mod 4),

(By the proof of Theorem 4.4 of [1], $|S_\infty(K)| = 4 > h_K$.)

(d) $K = k(\sqrt[3]{-P_1}\sqrt[6]{-P_2})$, $degP_i = 1$, $q \equiv 1$ (mod 6),

(By the proof of Theorem 4.4 of [1], $|S_\infty(K)| = 3$ and P_2 is totally ramified, then $n_1 \geq 4$.)

(e) $K = k(\sqrt{-P_1}, \sqrt[4]{-P_2})$, $degP_i = 1$, $q \equiv 1 \pmod 4$,

(By the proof of Theorem 4.4 of [1], $|S_\infty(K)| = 2$, hence there exists $Q \in \mathbb{P}_k$ of degree 1 which is totally ramified. Then $Q|cond(K)$, which is a contradiction.)

(f) $K = k(\sqrt[3]{-P_1}\sqrt[6]{-P_2})$, $degP_i = 1$, $q \equiv 1 \pmod 6$,

(g) $K = k(\sqrt[3]{-P}, \alpha)$, $degP = 1$, $q = 4^k$, $k \in \mathbb{Z}^+$, $\alpha \in K^+_{P^2}$, $[k(\alpha) : k] = 2$,

(For $q = 4^k$, $k \in \mathbb{Z}^+$, $h_K \neq 3$.)

(h) $K = k(\sqrt[3]{-P}, \beta)$, $degP = 1$, $q = 3^k$, $k \in \mathbb{Z}^+$, $\beta \in K^+_{P^2}$, $[k(\beta) : k] = 3$,

(By the proof of Theorem 4.4 of [1], $|S_\infty(K)| = 3$ and P is totally ramified and $n_1 \geq 4$.)

(i) $K = k(\sqrt{-P^1}, \sqrt{-P_2}, \sqrt{-P_3})$, $degP_i = 1$, q odd,

(By the proof of Theorem 4.4 of [1], $|S_\infty(K)| = 4 > h_K$.)

(j) $K = k(\sqrt{-P}, \sqrt{Q})$, $degP = 1$, $degQ = 2$,

(k) $K = k(\sqrt{-P}, \sqrt{P_1P_2})$, $degP = degP_i = 1$.

(For (j) and (k), $|S_\infty(K)| = 2$, hence there exists $Q' \in \mathbb{P}_k$ of degree 1 which is totally ramified. Then $Q'|cond(K)$, which is a contradiction.) ■

Theorem 5.15 *Let K/k be an imaginary (not totally imaginary) extension of genus 1 with class number 3. Then, up to isomorphism($x \to x + a$, $a \in \mathbb{F}_q^*$), K is one of the following function fields:*

(1) $q = 7$, $K = k(\sqrt[3]{-T}, \sqrt[3]{-(T+3)})$ with $L(t) = 7t^2 - 5t + 1$,

(2) $q = 7$, $K = k(\sqrt[3]{-T}, \sqrt[3]{-(T+4)})$ with $L(t) = 7t^2 - 5t + 1$,

(3)$q = 7$, $K = k(\sqrt{-(T+2)}\sqrt[6]{-T})$ with $L(t) = 7t^2 - 5t + 1$,

(4)$q = 7$, $K = k(\sqrt{-(T+5)}\sqrt[6]{-T})$ with $L(t) = 7t^2 - 5t + 1$.

Proof.K satisfies one of the conditions of Proposition 5.14:

(i) Let $q = 7$ and $K = k(\sqrt[3]{-P_1}, \sqrt[3]{-P_2})$ where $degP_i = 1$. Without loss of generality, we assume $P_1 = (T)$ and $P_2 = (T+a)$ for some $a \in \mathbb{F}_t^*$. Since $|S_\infty(K)| = 3$, all places of k of degree one do not split in the extension. Let $X_{K_{P_i}}$ denote the character group of K_{P_i} for $i = 1, 2$. Let $X_{K_{P_i}} =< \chi_i >$, then $o(\chi_i) = 6$ and $X_K =< \chi_1^2, \chi_2^2 >$. Since $(\mathbb{F}_7(T)/P_i)^* \cong \mathbb{F}_7^*$, we define χ_i such that $\chi_i(3) = \exp(2\pi i/6)$ for $i = 1, 2$.

For $a = 1$, $\chi_1^2(T+1) = 1$, then $n_1 \geq 6$.

For $a = 2$, $\chi_1^2(T+1) = \chi_2^2(T+1) = 1$, then $n_1 \geq 12$.

For $a = 3$, $\chi_2^2(T+1) = \exp(10\pi i/3) \neq 1$, $\chi_1^2(T+2) = \exp(8\pi i/6) \neq 1$, $\chi_1^2(T+3) = \exp(4\pi i/6) \neq 1$, $\chi_1^2(T+4) = \exp(8\pi i/3) \neq 1$, $\chi_2^2(T+5) = \exp(4\pi i/3) \neq 1$, $\chi_2^2(T+6) = \exp(4\pi i/6) \neq 1$. Hence none of them splits and

$$K = k(\sqrt[3]{-T}, \sqrt[3]{-(T+3)}). \tag{5.17}$$

For $a = 4$, $\chi_2^2(T+1) = \exp(8\pi i/3) \neq 1$, $\chi_1^2(T+2) = \exp(8\pi i/6) \neq 1$, $\chi_1^2(T+3) = \exp(4\pi i/6) \neq 1$, $\chi_1^2(T+4) = \exp(8\pi i/3) \neq 1$, $\chi_1^2(T+5) = \exp(10\pi i/3) \neq 1$, $\chi_2^2(T+6) = \exp(4\pi i/3) \neq 1$. Hence none of them splits and

$$K = k(\sqrt[3]{-T}, \sqrt[3]{-(T+4)}). \tag{5.18}$$

For $a = 5$, $\chi_1^2(T+6) = \chi_2^2(T+6) = 1$, then $n_1 \geq 12$.

For $a = 6$, $\chi_2^2(T) = 1$, then $n_1 \geq 6$.

(ii) $q = 7$, $K = k(\sqrt{-P_1}\sqrt[6]{-P_2})$, $degP_i = 1$. Without loss of generality, we assume $P_1 = (T+a)$ and $P_2 = (T)$ for some $a \in \mathbb{F}_t^*$. By the proof of Theorem 4.4 of [1], $|S_\infty(K)| = 2$. Since P_2 is totally ramified, $(T+b)$ does not split in the extension for all $b \in \mathbb{F}_7^*$. Let $X_{K_{P_i}}$ denote the character group of K_{P_i} for $i = 1, 2$. Let $X_{K_{P_i}} =< \chi_i >$, then $o(\chi_i) = 6$ and $X_K =< \chi_1^3\chi_2 >$. Since $(\mathbb{F}_7(T)/P_i)^* \cong \mathbb{F}_7^*$, we define χ_i such that $\chi_i(3) = \exp(2\pi i/6)$ for $i = 1, 2$.

For $a = 1$, $\chi_2(T+1) = 1$, then $n_1 \geq 6$.

For $a = 2$, $\chi_1^3\chi_2(T+1) = -1$, $\chi_1^3\chi_2(T+3) = \exp(\pi i/3)$, $\chi_1^3\chi_2(T+4) = \exp(4\pi i/3)$, $\chi_1^3\chi_2(T+5) = (-1)\exp(5\pi i/3)$, $\chi_1^3\chi_2(T+6) = (-1)$, and $\chi_2(T+2) = \exp(2\pi i/3)$.

Hence none of them splits and

$$K = k(\sqrt{-(T+2)\sqrt[6]{-T}}). \tag{5.19}$$

For $a = 3$, $\chi_1^3\chi_2(T+6) = 1$, then $n_1 \geq 9$.

For $a = 4$, $\chi_1^3\chi_2(T+1) = 1$, then $n_1 \geq 9$.

For $a = 5$, $\chi_1^3\chi_2(T+1) = -1$, $\chi_1^3\chi_2(T+2) = \exp(2\pi i/3)$, $\chi_1^3\chi_2(T+3) = (-1)\exp(\pi i/3)$, $\chi_1^3\chi_2(T+4) = (-1)\exp(4\pi i/3)$, $\chi_1^3\chi_2(T+6) = (-1)$, and $\chi_2(T+5) = \exp(5\pi i/3)$. Hence none of them splits and

$$K = k(\sqrt{-(T+5)\sqrt[6]{-T}}). \tag{5.20}$$

For $a = 6$, $\chi_1^3\chi_2(T+1) = 1$, then $n_1 \geq 9$.

$\blacksquare$

5.5 Genus two case:

5.5.1 $q = 2$

For $q = 2$, K is a real extension of k and $n_1 = 2 = n_2$. That implies $[K : k] = 2 = |S_\infty(K)|$. Thus any finite place of k of degree one is inert in K/k. Since $n_2 = 2$ and (T) and $(T+1)$ are inert, (T^2+T+1) is also inert. Then P does not divide $N := cond(K)$, when $degP \leq 2$. Assume $N = \prod_{i=1}^{r} P_i^{m_i}$, then $degP_i \geq 3$. By Hurwitz's Genus Formula for K/k, $deg(D(K/k)) = 6$. Since P_i are wildly ramified, $6 \geq 2(\sum_{i=1}^{r} degP_i)$. Equality holds if and only if $m_i = 2$ for all i. Hence $N = P^2$ where $degP = 3$.

Up to isomorphism $T \to T+1$, we assume $P = T^3+T+1$. Using Proposition 2.8 and Proposition 2.9 of [18], $K = k(y)$, where $y^2 + (T^3+T+1)y = (T^3+T+1)g(T)$ where $0 \neq g(T) \in \mathbb{F}_2[T]$ is of degree less than 4 and $g(0) = g(1) = 1$. Also let α be a root of T^2+T+1. Since (T^2+T+1) is inert, by Proposition 3.1, $g(\alpha)/\alpha + g(\alpha^2)/\alpha^2 = 1$. That implies $g(T) = 1$. Hence $K = k(y)$ where $y^2 + y = 1/(T^3+T+1)$.

Theorem 5.16 *Let $q = 2$ and K be an extension of k of genus 2 with class number 3. Then, up to isomorphism, $K = k(y)$ where $y^2 + y = 1/(T^3 + T + 1)$ and $L(t) = 4t^4 - 2t^3 + t^2 - t + 1$.*

5.5.2 $q = 3$

Theorem 5.17 *Let $q = 3$ and K be a real extension of k of genus 2 with class number 3. Then, up to isomorphism, $K = k(y)$ where $y^2 = T^6 + T^4 + T^3 + T^2 + 2T + 2$ or $y^2 = T^6 + T^4 + 2T^3 + T^2 + T + 2$ and for each case $L(t) = 9t^4 - 6t^3 + t^2 - 2t + 1$.*

Proof. Let K/k be a real extension, then $[K : K^+] = [K : k]$ divides $q - 1 = 2$. Hence, $[K : k]$ is quadratic and $|S_\infty(K)| = 2$. Thus we are in the case $n_1 = 2, n_2 = 3$. Since all finite places of k of degree one are inert, all finite places of k of degree two are also inert. That is, any place P of k of degree less than or equal to two does not divide the conductor N of K. Since the extension degree is prime to q, we may assume $N = \prod_{i=1}^{r} P_i$ where $P_i \in \mathbb{P}_k$. By Hurwitz's Genus formula for K/k, $\sum_{i=1}^{r} degP_i = 6$ where $degP_i \geq 3$. Thus $N = P_1P_2$ where $degP_i = 3$ or $N = P$ where $degP = 6$. Then by Theorem 2.5 and Lemma 2.6 of [18], $y^2 = N$ such that N is not a square modulo Q for a place Q of k of degree one or two. Considering each case, the result follows for only $N = T^6 + T^4 + T^3 + T^2 + 2T + 2$ and $N = T^6 + T^4 + 2T^3 + T^2 + T + 2$. ■

Theorem 5.18 *Let $q = 3$ and K be an imaginary extension of k of genus 2 with class number 3. Then, up to isomorphism, $K = k(y)$ where $y^2 = T^5 + T^3 + T + 2$ and $L(t) = 9t^4 - 9t^3 + 5t^2 - 3t + 1$.*

Proof. Let K/k be an imaginary extension, then $2 \leq [K : K^+]$ divides $q - 1 = 2$. That is, $[K : K^+] = 2$. Then by Theorem 2.4, we have

(i) $n_1 = 1$, $n_2 = 5$ or

(ii) $n_1 = 2$, $n_2 = 3$

(i) Let $n_1 = 1$ and $n_2 = 5$. Then $|S_\infty(K)| = 1$ and $K^+ = k$. That is K/k is quadratic, ∞ is ramified and all finite places of k of degree one are inert in the extension. Then we have two possibilities: either two of the places of k of degree two are ramified and the third one is inert or one of them splits and the others are inert. Extension degree is prime to q and we assume $N = \prod_{i=1}^{r} P_i$ where $P_i \in \mathbb{P}_k$. By Hurwitz's Genus formula for K/k, $\sum_{i=1}^{r} degP_i = 5$ where $degP_i \geq 2$. Thus $N = P$ where $degP = 5$. Then by Theorem 2.5 and Lemma 2.6 of [18], $y^2 = N$ such that N is not a square modulo Q for a place Q of k of degree one and N is a square modulo Q' for only one of the places Q' of k of degree two and it is not a square modulo Q'' where Q'' is a place of k of degree two different from Q'. There exist three distinct place in $\mathbb{P}_{\mathbb{F}_3(x)}$ of degree two. Up to isomorphism ($T \to T + a$, $a \in \mathbb{F}_3^*$), we may assume $Q' = T^2 + T + 2$. Then we have $N = T^5 + T^3 + T + 2$. That is,

$$K = k(y) \ where \ y^2 = T^5 + T^3 + T + 2. \tag{5.21}$$

(ii) Let $n_1 = 2$ and $n_2 = 3$. Then $|S_\infty(K)| = 1$ or 2.

• Assume $|S_\infty(K)| = 1$. Then K/k is quadratic, ∞ and one of the finite places P of k of degree one are ramified and the other places of k of degree one are inert in the extension. Up to isomorphism, let $P = (T)$. Then one of the places Q of k of degree two is ramified and the others are inert. Assume $N = P.Q \prod_{i=1}^{r} P_i$ where $P_i \in \mathbb{P}_k$ of degree greater than two . By Hurwitz's Genus formula for K/k, $degP + degQ + \sum_{i=1}^{r} degP_i = 5$ where $degP_i \geq 3$. That is, $\sum_{i=1}^{r} degP_i = 2$ and $degP_i \geq 3$, which is not possible.

• Assume $|S_\infty(K)| = 2$, then K/k is quartic. It is a well-known fact that $g_{K^+} \leq g_K$. For $g_{K^+} = 2$, by Hurwitz's Genus Formula, degree of the different of K/K^+ is -2. Since this is not reasonable, $g_{K^+} = 0$ or 1.

Let $g_{K^+} = 0$, that is $h_{K^+} = 1$. Then by Proposition 4.1 of [2], $K^+ \subseteq K_P^+$ with $degP = 2$ or $K^+ \subseteq K_{P_1P_2}^+$ with $degP_i = 1$.

Assume the first case holds.

⋆ If P is ramified in K/K^+, by Hurwitz's Genus formula for K/K^+,

$$2 = -4 + 2.deg\infty + degP + 2\sum_{i=1}^{r} degQ_i,$$

where $P \neq Q_i$ are places of k which are also ramified in K/K^+. Then $N = PQ$ where $degQ = 1$. If Q splits in K^+/k, then $n_1 \geq 4$, which is a contradiction. Let Q be inert in K^+/k. Since it is ramified in K/K^+ then there exists $\gamma \in \mathbb{P}_K$ lying over Q such that $deg\gamma = 2$. Since $n_2 = 3$ and γ and the place lying over P are of degree two, there exists a place Q' of k, different from P and Q, with $degQ' \leq 2$, which is ramified in K/k. That means Q' divides N, which is a contradiction.

$\star$ If P is not ramified in K/K^+, by Hurwitz's Genus formula for K/K^+,

$$2 = -4 + 2.deg\infty + 2\sum_{i=1}^{r} degQ_i,$$

where $P \neq Q_i$ are places of k which are ramified in K/K^+. Then $N = PQ$ where $degQ = 2$ or $N = PQ_1Q_2$ where $degQ_i = 1$. Since $n_1 = 2, n_2 = 3$, using an argument similar to above, there exists another ramified place Q' of k with $degQ' \leq 2$. Then Q' divides N, which is not possible.

Assume $K^+ \subseteq K^+_{P_1P_2}$ with $degP_i = 1$. Since $n_1 = |S_\infty(K)|$, P_i are inert in K/K^+. By Hurwitz's Genus formula for K/K^+,

$$2 = -4 + 2.deg\infty + 2\sum_{i=1}^{r} degQ_i,$$

where $P_i \neq Q_j$ are places of k which are ramified in K/K^+. We have $N = P_1P_2Q$ with $degQ = 2$ or $N = P_1P_2Q_1Q_2$ with $degQ_i = 1$. $N = P_1P_2Q_1Q_2$ with $degQ_i = 1$ implies $n_1 \geq 4$ or $n_2 \geq 4$, which is a contradiction. $N = P_1P_2Q$ with $degQ = 2$ implies $n_2 \geq 4$ or there exists another ramified place of degree less than or equal to two. Since both of them are not reasonable, we skip this case. Hence $g_{K^+} \neq 0$.

Let $g_{K^+} = 1$. It is known that $h_{K^+} \mid h_K = 3$, then $h_{K^+} = 1$ or 3. Since $[K^+ : k] = 2$ and $q = 3$, by Theorem 3.4 of [1], $K^+ = k(\sqrt{P_1P_2})$ with $degP_1 = 1$ and $degP_2 = 3$. Then $h_{K^+} = 3$. Up to isomorphism, by Theorem 5.10, $P_1 = (T)$ and $P_2 = (T^3 + 2T^2 + T + 1)$ or $P_2 = (T^3 + T^2 + T + 2)$.

Let $X_{K_{P_1}} =< \chi_1 >$ and $X_{K_{P_2}} =< \chi_2 >$, then $o(\chi_1) = 2$, $o(\chi_2) = 26$ and $X_{K^+} =< \chi_1\chi_2^{13} >$. Since $|X_K| = 4$ and $\chi_1\chi_2^{13} \in X_K$, $X_K =< \chi_1, \chi_2 >$.

$\star$ Let $P_2 = (T^3 + 2T^2 + T + 1)$. Then

$$\chi_1(T+1) = 1, \chi_2^{13}(T+1) = -1,$$

$$\chi_1(T+2) = -1, \chi_2^{13}(T+2) = 1.$$

That is $n_2 \geq 4$, which is not our case.

$\star$ Let $P_2 = (T^3 + T^2 + T + 2)$. Then $\chi_2^{13}(T) = 1$ and $n_1 \geq 4$, which is not true.

Thus $g_{K^+} \neq 1$ and there does not exist K such that $[K : k] = 4$ with $h_K = 3$ for $q = 3$ and $g_K = 2$.

$\blacksquare$

APPENDIX A

Complete Intersection of Three Quadrics in Projective Space $\mathbb{P}^4(\mathbb{F}_2)$

Let $K/\mathbb{F}_2$ be an algebraic function field of genus 5 with class number three. We proved in Chapter 4 that canonical model $\mathcal{C}$ of K is the complete intersection of three linearly independent quadrics l_1, l_2 and l_3 in $\mathbb{P}^4(\mathbb{F}_2)$ as they are listed below. In each case, we calculated the L-polynomial of the function field $K/\mathbb{F}_2$.

A.1 Non-degenerate Case: $l_1 : xy + zt + u^2 = 0$

(1) $l_2 : xz + z^2 + zu + t^2 + tu + u^2 = 0$, $l_3 : x^2 + y^2 + yu = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

Example: We find the function field of the complete intersection of quadrics in the non-degenerate case $l_1 : xy + zt + u^2 = 0$ given in (1). Let U be the open set of $\mathbf{P}^4(\mathbb{F}_2)$ defined by $z \neq 0$. Let V be the open set in $\mathbf{P}^3(k)$ consisting of the elements of the form (x, z, t, u) where $z \neq 0$. We define a morphism from V to U such that

$$(x : z : t : u) \to (1 + u + t^2 + tu + u^2 : y : 1 : t : u),$$

to get a plane model for K/k. Using $x^2 = yu + y^2$ in l_3, we have $x^4 = x^2yu + x^2y^2$. By l_1, $xy = t + u^2$. Using this one in the previous equation, $x^4 = x(t + u^2)u + (t^2 + u^4)$. But $x = 1 + u + t^2 + tu + u^2$. Then, we have $K = \mathbb{F}_2(t, u)$ such that

$$(1 + u + t^2 + tu + u^2)^4 + (1 + u + t^2 + tu + u^2)(t + u^2)u + (t^2 + u^4) = 0$$

with $L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1$.

(2)$l_2 : xz + y^2 + zu + t^2 + tu = 0$, $l_3 : x^2 + yu + z^2 + t^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1$.

(3) $l_2 : xz+y^2+t^2+tu+u^2 = 0$, $l_3 : x^2+xz+xt+yz+yt+yu+z^2+zt+zu+u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1$.

(4) $l_2 : xz + y^2 + zu + t^2 + tu = 0$, $l_3 : x^2 + yu + z^2 + zt + t^2 + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1$.

(5) $l_2 : x^2+xz+y^2+yu+z^2+zt+zu+tu+u^2 = 0$, $l_3 : x^2+yu+z^2+zt+t^2+u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1$.

(6) $l_2 : x^2+xz+yu+z^2+zt+t^2+tu+u^2 = 0$, $l_3 : y^2+yu+z^2+zt+zu+t^2+u^2 = 0$,

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4$,

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1$.

(7) $l_2 : x^2 + xz + z^2 + t^2 + tu = 0$, $l_3 : x^2 + y^2 + yt + zu + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1$.

(8) $l_2 : x^2+xz+y^2+yu+z^2+zt+zu+tu+u^2 = 0$, $l_3 : yt+z^2+zu+t^2+tu+u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1$.

(9) $l_2 : x^2 + xz + y^2 + yu + z^2 + zt + tu = 0,\ l_3 : yt + z^2 + zt + zu + t^2 + u^2 = 0,$

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4,$

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(10) $l_2 : x^2+xz+y^2+yu+z^2+zt+zu+tu+u^2 = 0,\ l_3 : yt+z^2+zt+zu+t^2+u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(11) $l_2 : x^2 + xz + z^2 + t^2 + tu = 0,\ l_3 : y^2 + yt + z^2 + zt + zu + t^2 + u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(12) $l_2 : x^2+xz+y^2+yu+z^2+zt+zu+tu+u^2 = 0,\ l_3 : x^2+yt+z^2+zt+zu+t^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(13) $l_2 : xz+y^2+yu+z^2+zt+zu+t^2+tu+u^2 = 0,\ l_3 : x^2+y^2+yt+zt+zu+t^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(14) $l_2 : xz + y^2 + t^2 + tu + u^2 = 0,\ l_3 : x^2 + yt + z^2 + zt + zu + tu = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(15) $l_2 : xz + y^2 + zu + t^2 + tu = 0,\ l_3 : x^2 + y^2 + yt + yu + z^2 + zu + t^2 + u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(16) $l_2 : xz+y^2+yu+zt+zu+t^2+tu+u^2 = 0,\ l_3 : x^2+y^2+yt+yu+z^2+zu+t^2+u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(17) $l_2 : x^2 + xz + z^2 + t^2 + tu = 0,\ l_3 : x^2 + y^2 + yt + yu + z^2 + zt + u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(18) $l_2 : x^2 + xz + z^2 + t^2 + tu = 0,\ l_3 : y^2 + yt + yu + z^2 + zt + t^2 + tu + u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(19) $l_2 : xz + y^2 + yu + zt + t^2 + tu = 0,\ l_3 : x^2 + yz + z^2 + zt + t^2 + tu = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(20) $l_2 : xz+y^2+yu+zt+zu+t^2+tu+u^2 = 0,\ l_3 : x^2+yz+z^2+zt+zu+t^2+tu+u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(21) $l_2 : xz+yu+z^2+zt+zu+t^2+tu+u^2 = 0,\ l_3 : x^2+y^2+yz+zt+zu+t^2+tu+u^2 = 0,$

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4,$

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(22) $l_2 : xz + y^2 + zu + t^2 + tu = 0,\ l_3 : x^2 + y^2 + yz + yu + z^2 + tu = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(23) $l_2 : x^2 + xz + z^2 + t^2 + tu = 0,\ l_3 : y^2 + yz + yu + z^2 + zt + zu + t^2 + u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(24) $l_2 : x^2+xz+y^2+yu+z^2+zt+zu+tu+u^2 = 0$, $l_3 : y^2+yz+yu+z^2+zt+zu+t^2+u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(25) $l_2 : x^2 + xz + z^2 + t^2 + tu = 0$, $l_3 : x^2 + y^2 + yz + yu + z^2 + zt + zu = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(26) $l_2 : x^2 + xz + y^2 + yu + z^2 + zt + tu = 0$, $l_3 : x^2 + yz + yt + t^2 + tu + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(27) $l_2 : x^2 + xz + yu + z^2 + zt + t^2 + tu + u^2 = 0$, $l_3 : x^2 + y^2 + yz + yt + tu + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(28) $l_2 : xz + z^2 + zu + t^2 + tu + u^2 = 0$, $l_3 : x^2 + y^2 + yz + yt + t^2 + tu = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(29) $l_2 : xz + z^2 + zu + t^2 + tu + u^2 = 0$, $l_3 : x^2 + y^2 + yz + yt + zu + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(30) $l_2 : x^2 + xz + yu + z^2 + zt + t^2 + tu + u^2 = 0$, $l_3 : x^2 + y^2 + yz + yt + zu + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(31) $l_2 : x^2 + xz + y^2 + yu + z^2 + zt + tu = 0$, $l_3 : x^2 + yz + yt + yu + t^2 + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(32) $l_2 : xz + y^2 + zu + t^2 + tu = 0$, $l_3 : x^2 + yz + yt + yu + z^2 + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(33) $l_2 : x^2 + xz + yu + z^2 + zt + t^2 + tu + u^2 = 0$, $l_3 : x^2 + y^2 + yz + yt + yu + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(34) $l_2 : xz + z^2 + zu + t^2 + tu + u^2 = 0$, $l_3 : x^2 + y^2 + yz + yt + yu + t^2 + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(35) $l_2 : xz + y^2 + t^2 + tu = 0$, $l_3 : x^2 + yz + yt + yu + z^2 + tu + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(36) $l_2 : x^2 + xz + y^2 + yu + z^2 + zt + tu = 0$, $l_3 : x^2 + yz + yt + yu + zu + t^2 + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(37) $l_2 : x^2+xz+y^2+yu+z^2+zt+zu+tu+u^2 = 0$, $l_3 : x^2+yz+yt+yu+zu+t^2+u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(38) $l_2 : xz + z^2 + zu + t^2 + tu + u^2 = 0,\ l_3 : x^2 + y^2 + yz + yt + yu + zu + tu = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 5, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 4t^6 - 8t^5 + 2t^4 + t^3 + 2t^2 - 3t + 1.$

(39) $l_2 : xz + z^2 + zu + t^2 + tu + u^2 = 0,\ l_3 : x^2 + y^2 + yz + yt + yu + zt + u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(40) $l_2 : x^2 + xz + y^2 + yu + z^2 + zt + tu = 0,\ l_3 : x^2 + xu + y^2 + zt + t^2 + tu = 0,$

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4,$

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(41) $l_2 : x^2 + xz + z^2 + t^2 + tu = 0,\ l_3 : xu + y^2 + yu + z^2 + t^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(42) $l_2 : xz + z^2 + zu + t^2 + tu + u^2 = 0,\ l_3 : x^2 + xu + y^2 + yu + zu + t^2 + tu = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(43) $l_2 : xz + z^2 + zu + t^2 + tu + u^2 = 0,\ l_3 : x^2 + xu + y^2 + yu + zt + t^2 + u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(44) $l_2 : xz + y^2 + zu + t^2 + tu = 0,\ l_3 : x^2 + xu + y^2 + yu + z^2 + zt + u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(45) $l_2 : x^2 + xz + yu + z^2 + zt + zu + t^2 + tu = 0,\ l_3 : xu + y^2 + yt + z^2 + t^2 + u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(46) $l_2 : xz + y^2 + yu + zt + zu + t^2 + tu + u^2 = 0$, $l_3 : x^2 + xu + yt + z^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(47) $l_2 : x^2 + xz + yu + z^2 + zt + zu + t^2 + tu = 0$, $l_3 : x^2 + xu + y^2 + yt + tu = 0$,

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4,$

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(48) $l_2 : x^2 + xz + y^2 + yu + z^2 + zt + zu + tu + u^2 = 0$, $l_3 : xu + yt + z^2 + zu + t^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(49) $l_2 : x^2 + xz + y^2 + yu + z^2 + zt + zu + tu + u^2 = 0$, $l_3 : x^2 + xu + y^2 + yt + zu + t^2 = 0$,

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4,$

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(50) $l_2 : x^2 + xz + yu + z^2 + zt + t^2 + tu + u^2 = 0$, $l_3 : x^2 + xu + y^2 + yt + zt = 0$,

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4,$

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(51) $l_2 : x^2 + xz + y^2 + yu + z^2 + zt + zu + tu + u^2 = 0$, $l_3 : xu + yt + z^2 + zt + zu + t^2 + tu = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(52) $l_2 : x^2 + xz + y^2 + yu + z^2 + zt + zu + tu + u^2 = 0$, $l_3 : x^2 + xu + yt + z^2 + zt + zu + t^2 + tu + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(53) $l_2 : x^2 + xz + yu + z^2 + zt + zu + t^2 + tu = 0,\ l_3 : xu + y^2 + yt + yu + z^2 + t^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(54) $l_2 : xz + y^2 + yu + zt + zu + t^2 + tu + u^2 = 0,\ l_3 : x^2 + xu + yt + yu + z^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(55) $l_2 : x^2+xz+y^2+yu+z^2+zt+zu+tu+u^2 = 0,\ l_3 : xu+yt+yu+z^2+zu+t^2+tu+u^2 = 0,$

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4,$

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(56) $l_2 : x^2 + xz + y^2 + yu + z^2 + zt + zu + tu + u^2 = 0,\ l_3 : x^2 + xu + y^2 + yt + yu + zu + t^2 + tu + u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(57) $l_2 : x^2+xz+y^2+z^2+tu+u^2 = 0,\ l_3 : x^2+xu+y^2+yt+yu+zt+zu+t^2+u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(58) $l_2 : x^2 + xz + y^2 + yu + z^2 + zt + tu = 0,\ l_3 : x^2 + xu + y^2 + yz + yu + z^2 + t^2 = 0,$

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4,$

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(59) $l_2 : x^2+xz+y^2+yu+z^2+zt+zu+tu+u^2 = 0,\ l_3 : xu+y^2+yz+yu+z^2+t^2+tu = 0,$

$n_1=0, n_2=0, n_3=1, n_4=6, n_5=5,$

$L(t)=32t^{10}-48t^9+16t^8+4t^7+6t^6-11t^5+3t^4+t^3+2t^2-3t+1.$

(60) $l_2: x^2+xz+yu+z^2+zt+t^2+tu+u^2=0,\ l_3: x^2+xu+y^2+yz+yu+z^2+tu=0,$

$n_1=0, n_2=1, n_3=1, n_4=1, n_5=4,$

$L(t)=32t^{10}-48t^9+8t^8-8t^7+2t^6+5t^5+t^4-2t^3+t^2-3t+1.$

(61) $l_2: xz+y^2+zu+t^2+tu=0,\ l_3: x^2+xu+yz+yu+z^2+zu+t^2=0,$

$n_1=0, n_2=1, n_3=1, n_4=1, n_5=4,$

$L(t)=32t^{10}-48t^9+8t^8-8t^7+2t^6+5t^5+t^4-2t^3+t^2-3t+1.$

(62) $l_2: xz+z^2+zu+t^2+tu+u^2=0,\ l_3: x^2+xu+y^2+yz+yu+zt+t^2+tu=0,$

$n_1=0, n_2=0, n_3=1, n_4=6, n_5=5,$

$L(t)=32t^{10}-48t^9+16t^8+4t^7+6t^6-11t^5+3t^4+t^3+2t^2-3t+1.$

(63) $l_2: x^2+xz+z^2+t^2+tu=0,\ l_3: xu+y^2+yz+yu+z^2+zt+zu+t^2+tu+u^2=0,$

$n_1=0, n_2=0, n_3=1, n_4=6, n_5=5,$

$L(t)=32t^{10}-48t^9+16t^8+4t^7+6t^6-11t^5+3t^4+t^3+2t^2-3t+1.$

(64) $l_2: xz+z^2+zu+t^2+tu+u^2=0,\ l_3: x^2+xu+y^2+yz+yt+t^2+tu+u^2=0,$

$n_1=0, n_2=0, n_3=2, n_4=2, n_5=7,$

$L(t)=32t^{10}-48t^9+16t^8+8t^7-8t^6+5t^5-4t^4+2t^3+2t^2-3t+1.$

(65) $l_2: xz+z^2+zu+t^2+tu+u^2=0,\ l_3: x^2+xu+y^2+yz+yt+zu=0,$

$n_1=0, n_2=0, n_3=2, n_4=2, n_5=7,$

$L(t)=32t^{10}-48t^9+16t^8+8t^7-8t^6+5t^5-4t^4+2t^3+2t^2-3t+1.$

(66) $l_2: x^2+xz+z^2+t^2+tu=0,\ l_3: xu+y^2+yz+yt+z^2+zu+t^2+tu+u^2=0,$

$n_1=0, n_2=0, n_3=2, n_4=3, n_5=7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(67) $l_2 : x^2 + xz + z^2 + t^2 + tu = 0$, $l_3 : x^2 + xu + y^2 + yz + yt + z^2 + zu + tu + u^2 = 0$,

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4$,

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(68) $l_2 : xz + y^2 + yu + z^2 + zt + t^2 + tu + u^2 = 0$, $l_3 : x^2 + xu + yz + yt + zt + tu + u^2 = 0$,

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4$,

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(69) $l_2 : xz + z^2 + zu + t^2 + tu + u^2 = 0$, $l_3 : x^2 + xu + y^2 + yz + yt + zt + tu + u^2 = 0$,

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4$,

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(70) $l_2 : x^2 + xz + y^2 + z^2 + tu + u^2 = 0$, $l_3 : x^2 + xu + y^2 + yz + yt + z^2 + zt + zu + t^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(71) $l_2 : x^2 + xz + yu + z^2 + zt + zu + t^2 + tu = 0$, $l_3 : xu + y^2 + yz + yt + yu + t^2 = 0$,

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4$,

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(72) $l_2 : x^2 + xz + z^2 + t^2 + tu = 0$, $l_3 : x^2 + xt + y^2 + yu + z^2 + t^2 + u^2$,

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4$,

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(73) $l_2 : x^2 + xz + y^2 + yu + z^2 + zt + zu + tu + u^2 = 0$, $l_3 : x^2 + xt + yu + z^2 + t^2 + tu + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(74) $l_2 : xz + y^2 + yu + z^2 + zt + zu + t^2 + tu + u^2 = 0$, $l_3 : x^2 + xt + y^2 + yu + tu = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1$.

(75) $l_2 : x^2 + xz + y^2 + yu + z^2 + zt + zu + tu + u^2 = 0$, $l_3 : x^2 + xt + y^2 + yu + t^2 + tu = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1$.

(76) $l_2 : x^2 + xz + y^2 + yu + z^2 + zt + zu + tu + u^2 = 0$, $l_3 : x^2 + xt + yu + z^2 + zt + t^2 + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1$.

(77) $l_2 : xz + z^2 + zu + t^2 + tu + u^2 = 0$, $l_3 : x^2 + xt + y^2 + yu + zt = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1$.

(78) $l_2 : xz + y^2 + yu + zt + t^2 + tu + u^2 = 0$, $l_3 : x^2 + xt + y^2 + yu + z^2 + zt + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1$.

(79) $l_2 : xz + y^2 + yt + yu + zu + t^2 + u^2 = 0$, $l_3 : x^2 + xt + y^2 + yu + z^2 + zt + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1$.

(80) $l_2 : xz + yu + z^2 + zt + t^2 + tu = 0$, $l_3 : x^2 + xt + y^2 + yt + zu + t^2 + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1$.

(81) $l_2 : x^2 + xz + y^2 + yu + z^2 + zt + zu + tu + u^2 = 0$, $l_3 : xt + yt + z^2 + zu + t^2 + tu + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(82) $l_2 : x^2+xz+y^2+yu+z^2+zt+zu+tu+u^2 = 0$, $l_3 : x^2+xt+y^2+yt+yu+zu+t^2 = 0,$

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4,$

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(83) $l_2 : xz+y^2+yu+z^2+zt+t^2+tu+u^2 = 0$, $l_3 : x^2+xt+y^2+yt+yu+zu+t^2+u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(84) $l_2 : xz + y^2 + yu + zt + t^2 + tu = 0$, $l_3 : x^2 + xt + yt + yu + z^2 + zt + u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(85) $l_2 : x^2 + xz + z^2 + t^2 + tu = 0$, $l_3 : xt + y^2 + yz + z^2 + zt + zu = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(86) $l_2 : x^2 + xz + z^2 + t^2 + tu = 0$, $l_3 : x^2 + xt + y^2 + yz + z^2 + zt + zu + t^2 = 0,$

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4,$

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(87) $l_2 : x^2+xz+y^2+yu+z^2+zt+zu+tu+u^2 = 0$, $l_3 : xt+y^2+yz+yu+z^2+t^2+u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(88) $l_2 : x^2+xz+y^2+yu+z^2+zt+zu+tu+u^2 = 0$, $l_3 : x^2+xt+yz+yu+t^2+u^2 = 0,$

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4,$

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(89) $l_2 : xz + y^2 + zu + t^2 + tu = 0,\ l_3 : x^2 + xt + yz + yu + z^2 + zu + t^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(90) $l_2 : x^2+xz+yu+z^2+zt+zu+t^2+tu = 0,\ l_3 : x^2+xt+y^2+yz+yu+zu+t^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(91) $l_2 : xz+y^2+yu+zt+zu+t^2+tu+u^2 = 0,\ l_3 : x^2+xt+y^2+yz+yu+z^2+zu+u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(92) $l_2 : xz + y^2 + t^2 + tu + u^2 = 0,\ l_3 : x^2 + xt + yz + yu + z^2 + zt + zu + t^2 + u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(93) $l_2 : x^2 + xz + yu + z^2 + zt + t^2 + tu + u^2 = 0,\ l_3 : x^2 + xt + y^2 + yz + yu + z^2 + zt + zu + t^2 + tu + u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(94) $l_2 : xz + z^2 + zu + t^2 + tu + u^2 = 0,\ l_3 : x^2 + xt + y^2 + yz + yt + zt + t^2 + tu = 0,$

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4,$

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(95) $l_2 : x^2 + xz + y^2 + yu + z^2 + zt + tu = 0,\ l_3 : x^2 + xt + y^2 + yz + yt + z^2 + zt + zu + t^2 + tu + u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(96) $l_2 : xz+z^2+zu+t^2+tu+u^2 = 0$, $l_3 : x^2+xt+y^2+yz+yt+yu+zt+t^2+u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(97) $l_2 : xz+yu+zt+t^2+tu = 0$, $l_3 : x^2+xt+y^2+yz+yt+yu+z^2+zt+zu+tu+u^2 = 0$,

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4$,

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(98) $l_2 : xz+y^2+t^2+tu+u^2 = 0$, $l_3 : x^2+xt+y^2+yz+yt+yu+z^2+zt+zu+t^2+tu = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(99) $l_2 : x^2+xz+y^2+yu+z^2+zt+zu+tu+u^2 = 0$, $l_3 : x^2+xt+xu+yu+z^2+t^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(100) $l_2 : xz+y^2+yu+z^2+zt+zu+t^2+tu+u^2 = 0$, $l_3 : x^2+xt+xu+y^2+yu+u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(101) $l_2 : x^2+xz+y^2+yu+z^2+zt+zu+tu+u^2 = 0$, $l_3 : x^2+xt+xu+y^2+yu+t^2+u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(102) $l_2 : x^2+xz+y^2+z^2+tu+u^2 = 0$, $l_3 : x^2+xt+xu+yu+z^2+zt+zu+t^2+tu+u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(103) $l_2 : x^2+xz+y^2+yu+z^2+zt+zu+tu+u^2 = 0$, $l_3 : x^2+xt+xu+y^2+yt+zu+t^2 = 0$,

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4$,

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(104) $l_2 : xz + y^2 + t^2 + tu + u^2 = 0$, $l_3 : x^2 + xt + xu + yz + z^2 + zt + t^2 + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(105) $l_2 : x^2 + xz + yu + z^2 + zt + t^2 + tu + u^2 = 0$, $l_3 : x^2 + xt + xu + y^2 + yz + z^2 + zt + zu + t^2 + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(106) $l_2 : x^2+xz+yu+z^2+zt+zu+t^2+tu = 0$, $l_3 : x^2+xt+xu+y^2+yz+yu+t^2+u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(107) $l_2 : x^2 + xz + y^2 + yu + z^2 + zt + tu = 0$, $l_3 : x^2 + xt + xu + y^2 + yz + yt + z^2 + zt + zu + t^2 + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(108) $l_2 : x^2 + xz + yu + z^2 + zt + t^2 + tu + u^2 = 0$, $l_3 : x^2 + xt + xu + y^2 + yz + yt + yu + z^2 + zt + t^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7$,

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(109) $l_2 : x^2 + xz + y^2 + yu + z^2 + zt + tu = 0$, $l_3 : x^2 + xt + xu + y^2 + yz + yt + yu + z^2 + zt + zu + t^2 + tu + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(110) $l_2 : xz + y^2 + zu + t^2 + tu = 0$, $l_3 : x^2 + xz + y^2 + yu + z^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(111) $l_2 : xz+z^2+zu+t^2+tu+u^2 = 0$, $l_3 : x^2+xz+y^2+yu+z^2+zu+t^2+tu+u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(112) $l_2 : x^2 + xz + z^2 + t^2 + tu = 0$, $l_3 : x^2 + xz + y^2 + yt + yu + zt + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(113) $l_2 : x^2+xz+y^2+yu+z^2+zt+zu+tu+u^2 = 0$, $l_3 : x^2+xz+y^2+yt+yu+zt+t^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(114) $l_2 : x^2 + xz + z^2 + t^2 + tu = 0$, $l_3 : xz + y^2 + yt + yu + zt + t^2 + tu + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(115) $l_2 : xz + y^2 + yu + zt + t^2 + tu = 0$, $l_3 : x^2 + xz + y^2 + yz + yu + z^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(116) $l_2 : xz + yu + z^2 + zt + zu + t^2 + tu + u^2 = 0$, $l_3 : x^2 + xz + y^2 + yz + yu + z^2 = 0$,

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4,$

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(117) $l_2 : xz + y^2 + yu + zt + zu + t^2 + tu + u^2 = 0$, $l_3 : x^2 + xz + y^2 + yz + yu + z^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(118) $l_2 : x^2 + xz + z^2 + t^2 + tu = 0$, $l_3 : xz + y^2 + yz + yu + zt + zu + t^2 + tu = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(119) $l_2 : x^2 + xz + z^2 + t^2 + tu = 0$, $l_3 : x^2 + xz + y^2 + yz + yu + zt + zu + tu + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(120) $l_2 : xz + y^2 + zt + zu + t^2 = 0$, $l_3 : x^2 + xz + xu + y^2 + yt + yu + z^2 + zt + t^2 + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(121) $l_2 : x^2 + xz + yu + z^2 + zt + zu + t^2 + tu = 0$, $l_3 : x^2 + xz + xu + y^2 + yt + yu + zt + zu + tu + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(122) $l_2 : xz + y^2 + yu + zt + zu + t^2 + tu + u^2 = 0$, $l_3 : x^2 + xz + xu + y^2 + yt + yu + z^2 + zt + zu + t^2 + tu + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(123) $l_2 : xz + yu + z^2 + zt + t^2 + tu = 0$, $l_3 : x^2 + xz + xu + y^2 + yz + yu + z^2 + tu + u^2 = 0$,

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4,$

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(124) $l_2 : x^2 + xz + yu + z^2 + zt + zu + t^2 + tu = 0$, $l_3 : x^2 + xz + xu + y^2 + yz + yt + z^2 + zt + zu + tu = 0$,

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4,$

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(125) $l_2 : xz+y^2+yu+z^2+zt+t^2+tu+u^2 = 0$, $l_3 : x^2+xz+xt+yt+z^2+zu+tu+u^2 = 0$,

$n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4,$

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

A.2 Degenerate Case $l_1 : xy + zt = 0$,

(1) $l_2 : xz+xu+y^2+yu+zt+t^2+u^2 = 0$, $l_3 : x^2+xu+y^2+yt+z^2+zu+t^2+tu+u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(2) $l_2 : x^2+xz+xu+y^2+yu+zt+u^2 = 0$, $l_3 : xu+y^2+yt+z^2+zt+zu+t^2+tu = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(3) $l_2 : xz + xu + y^2 + yu + z^2 + zt + t^2 = 0$, $l_3 : x^2 + xu + yz + zu + t^2 + tu + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(4) $l_2 : xz+xu+y^2+yu+zt+t^2+u^2 = 0$, $l_3 : x^2+xt+y^2+yz+yu+z^2+zu+tu = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(5) $l_2 : xz+xu+y^2+yu+zt+t^2+u^2 = 0$, $l_3 : x^2+xt+xu+yz+z^2+zu+t^2+tu+u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

A.3 Degenerate Case $l_1 : xy + z^2 + zt + t^2 = 0$,

(1) $l_2 : x^2 + xu + y^2 + yu + z^2 + zt + t^2 + tu + u^2 = 0$, $l_3 : yz + yu + t^2 + tu + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

Example: We find the function field of the quadrics in the degenerate case l_1 : $xy + zt + z^2 + t^2 = 0$. Let U be the open set of $\mathbf{P}^4(\mathbb{F}_2)$ defined by $y \neq 0$. Let V be the open set in $\mathbf{P}^3(k)$ consisting of the elements of the form (x, z, t, u) where $y \neq 0$. We define a morphism from V to U such that

$$(x : z : t : u) \to (z^2 + zt + t^2 : 1 : z : t : u),$$

to get a plane model for K/k. Using $x = z^2 + zt + t^2$ in l_3, we have $z = u + t^2 + tu + u^2$. By l_2, we have $K = \mathbb{F}_2(t, u)$ such that

$$u^8 + u^5 + (t^4 + t^2)u^4 + (t^2 + t + 1)u^3 + (t^4 + t^2)u^2 + (t^4 + t^3 + 1)u +$$
$$(t^8 + t^6 + t^3 + t^2 + 1) = 0$$

with $L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(2) $l_2 : x^2 + xz + xu + yt + z^2 + zt + zu + t^2 + u^2 = 0$, $l_3 : y^2 + zt + zu + t^2 + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 5, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 4t^6 - 8t^5 + 2t^4 + t^3 + 2t^2 - 3t + 1.$

(3) $l_2 : xu + y^2 + yt + zu + t^2 = 0$, $l_3 : x^2 + y^2 + yz + zt + zu + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 5, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 4t^6 - 8t^5 + 2t^4 + t^3 + 2t^2 - 3t + 1.$

(4) $l_2 : x^2 + xu + yt + z^2 + zu + t^2 + u^2 = 0$, $l_3 : x^2 + y^2 + yz + zt + zu + t^2 + u^2 = 0$,

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 5, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 4t^6 - 8t^5 + 2t^4 + t^3 + 2t^2 - 3t + 1.$

(5)$l_2 : x^2+xz+xu+y^2+yz+yu+z^2+zt+u^2 = 0$, $l_3 : y^2+yz+yt+z^2+zu+t^2+u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 5, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 4t^6 - 8t^5 + 2t^4 + t^3 + 2t^2 - 3t + 1.$

(6) $l_2 : x^2+xz+xu+yt+z^2+zt+zu+t^2+u^2 = 0$, $l_3 : y^2+yz+yt+z^2+zu+t^2+u^2 = 0,$

$n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4,$

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(7) $l_2 : x^2+xz+xu+yu+z^2+zu+tu+u^2 = 0$, $l_3 : x^2+y^2+yz+yt+yu+zt+zu+t^2+u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 5, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 4t^6 - 8t^5 + 2t^4 + t^3 + 2t^2 - 3t + 1.$

(8) $l_2 : x^2 + xz + xu + yt + z^2 + zt + zu + t^2 + u^2 = 0$, $l_3 : y^2 + yz + yt + yu + zt + zu + t^2 + tu + u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(9) $l_2 : x^2+xz+xu+yt+z^2+zt+zu+t^2+u^2 = 0$, $l_3 : xu+y^2+yt+yu+t^2+tu = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(10) $l_2 : xu + y^2 + yt + zu + t^2 = 0$, $l_3 : x^2 + xu + yz + yt + zt + t^2 + u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 5, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 4t^6 - 8t^5 + 2t^4 + t^3 + 2t^2 - 3t + 1.$

(11) $l_2 : xz + xu + y^2 + yt + z^2 + zt + zu + t^2 + u^2 = 0$, $l_3 : x^2 + xu + y^2 + yz + yt +$

$yu + zu + t^2 + u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(12) $l_2 : x^2{+}xu{+}yt{+}z^2{+}zu{+}t^2{+}u^2 = 0,\ l_3 : x^2{+}xt{+}y^2{+}yt{+}z^2{+}zt{+}zu{+}t^2{+}tu = 0,$

$n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4,$

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(13) $l_2 : x^2{+}xu{+}y^2{+}yu{+}z^2{+}zt{+}t^2{+}tu{+}u^2 = 0,\ l_3 : x^2{+}xt{+}yt{+}yu{+}zt{+}zu{+}t^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(14) $l_2 : xu + y^2 + yt + zu + t^2 = 0,\ l_3 : x^2 + xt + y^2 + yz + zu + t^2 + u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(15) $l_2 : x^2{+}xz{+}xu{+}yu{+}z^2{+}zu{+}tu{+}u^2 = 0,\ l_3 : x^2{+}xt{+}y^2{+}yz{+}z^2{+}zu{+}tu = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(16) $l_2 : xu + y^2 + yt + zu + t^2 = 0,\ l_3 : x^2 + xt + y^2 + yz + yt + zu + t^2 + tu + u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(17) $l_2 : x^2{+}xz{+}xu{+}yt{+}z^2{+}zt{+}zu{+}t^2{+}u^2 = 0,\ l_3 : xt{+}xu{+}y^2{+}yu{+}z^2{+}zu{+}tu = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(18) $l_2 : x^2 + xz + xu + yu + z^2 + zu + tu + u^2 = 0,\ l_3 : xt + xu + y^2 + yt + tu = 0,$

$n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 4,$

$L(t) = 32t^{10} - 48t^9 + 8t^8 - 8t^7 + 2t^6 + 5t^5 + t^4 - 2t^3 + t^2 - 3t + 1.$

(19) $l_2 : x^2{+}xu{+}y^2{+}yu{+}z^2{+}zt{+}t^2{+}tu{+}u^2 = 0$, $l_3 : x^2{+}xt{+}xu{+}y^2{+}yz{+}z^2{+}tu = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(20) $l_2 : x^2{+}xu{+}yt{+}z^2{+}zu{+}t^2{+}u^2 = 0$, $l_3 : xz{+}y^2{+}yu{+}z^2{+}zt{+}zu{+}t^2{+}tu{+}u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(21) $l_2 : x^2{+}xu{+}yt{+}z^2{+}zu{+}t^2{+}u^2 = 0$, $l_3 : xz{+}y^2{+}yt{+}yu{+}z^2{+}zu{+}t^2{+}tu{+}u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 6t^6 + 2t^5 - 3t^4 + 2t^3 + 2t^2 - 3t + 1.$

(22) $l_2 : x^2{+}xu{+}y^2{+}yu{+}z^2{+}zt{+}t^2{+}tu{+}u^2 = 0$, $l_3 : x^2{+}xz{+}y^2{+}yz{+}yu{+}z^2{+}zu = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(23) $l_2 : x^2{+}xz{+}xu{+}yu{+}z^2{+}zu{+}tu{+}u^2 = 0$, $l_3 : x^2{+}xz{+}y^2{+}yz{+}yt{+}z^2{+}zt{+}t^2{+}tu = 0,$

$n_1 = 0, n_2 = 0, n_3 = 2, n_4 = 2, n_5 = 7,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 8t^7 - 8t^6 + 5t^5 - 4t^4 + 2t^3 + 2t^2 - 3t + 1.$

(24) $l_2 : xu + y^2 + yt + zu + t^2 = 0$, $l_3 : x^2 + xz + xu + y^2 + yu + zu + u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 6, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 6t^6 - 11t^5 + 3t^4 + t^3 + 2t^2 - 3t + 1.$

(25) $l_2 : x^2{+}xu{+}yt{+}z^2{+}zu{+}t^2{+}u^2 = 0$, $l_3 : xz{+}xt{+}y^2{+}yu{+}z^2{+}zt{+}zu{+}tu{+}u^2 = 0,$

$n_1 = 0, n_2 = 0, n_3 = 1, n_4 = 5, n_5 = 5,$

$L(t) = 32t^{10} - 48t^9 + 16t^8 + 4t^7 + 4t^6 - 8t^5 + 2t^4 + t^3 + 2t^2 - 3t + 1.$

REFERENCES

[1] J.Ahn, H.Jung, Determination of all subfields of cyclotomic function fields with genus one, *Commun. Korean Math. Soc.* **20** pp. 259-273, (2005).

[2] J.Ahn, H.Jung, Divisor class number one problem for abelian extensions over rational function fields, *J. Algebra* **310** pp. 1-14, (2007).

[3] J.Ahn, H.Jung, Determination of all subfields of cyclotomic function fields with divisor class number two, *Commun. Korean Math. Soc.* **22**, 2, pp. 163-171, (2007).

[4] O. Alkam, M. Bilhan, Class number of (ν, n, M)-Extensions, Bull. Austral. Math. Soc. **63**, pp.21-34, (2001).

[5] B. Angles, On the Class Group Problems for Function Fields, *J. Number Theory* **70**, pp.146-159, (1998).

[6] S.Bae, J.Ahn, H.Jung, Class Numbers of Some Abelian Extensions of Rational Function Fields, *Math. Comp.* **73**, 245, pp. 377-386, (2003).

[7] S. Ballet, D. Le Brigand, On The Existence of Non-special Divisors of Degree g and $g-1$ in Algebraic Function Fields over $\mathbb{F}_q$, *J. Number Theory* **116**, pp. 293-310, (2006).

[8] M.Bilhan, Arithmetic Progressions of Polynomials over Finite Fields, *Number Theory and Its Applications-Lecture Notes in Pure and Applied Mathematics,* 204, (1999), p.1-21.

[9] W. Bosna, J. Cannon, C. Playoust, The MAGMA algebra system I: The user language, *Journal of Symbolic Computation* **22,** issue 3-4 (Sept/Oct 1997) Special issue on computational algebra and number theory: proceedings of the first MAGMA conference, pp.235-265.

[10] L.Carlitz, On Certain Functions Connected with Polynomials in a Galois Field, *Duke Math. J.* **1**, 137, (1935).

[11] R.Clement, The Genus Field of an Algebraic Function Field, *J. Number Theory* **40**, pp. 359-375, (1992).

[12] R.Hartshorne, *Algebraic Geometry* Springer, New York (1977).

[13] D.R. Hayes, Explicit Class Field Theory for Rational Function Fields, *Trans. Amer. Math. Soc.* **189**, 77, (1974).

[14] J. W. P. Hirschfeld, *Projective Geometries over Finite Fields,* Oxford University Press, New York (1998).

[15] J. W. P. Hirschfeld, *Finite Projective Spaces of Three Dimensions,* Oxford University Press, New York (1985).

[16] M.Kida, N.Murabayashi, Cyclotomic function fields and divisor class number one, *Tokyo J. Math.* **14**, 1, pp.45-56, (1991).

[17] D. Le Brigand, Classification of algebraic function fields with divisor class number two, *Finite Fields Appl.* **2**, pp. 153-172, (1996).

[18] D. Le Brigand, Quadratic algebraic function fields with ideal class number two. In: R. Pellikaan, M. Perret and S. Vl du , Editors, Arithmetic,Geometry and Coding Theory, de Gruyter, Berlin, pp. 105-126, (1996).

[19] J. Leitzel, M. Madan and C. Queen, Algebraic function fields with small class number, *J. Number Theory* **7**, pp. 11-27, (1975).

[20] R Lidl and H. Niederreiter, *Finite Fields,* Cambridge University Press, Cambridge (2000).

[21] R. E. MacRae, On unique factorization in certain rings of algebraic functions, *J. Algebra* **17**, pp. 243-261, (1971).

[22] M. Madan, On Class Numbers in Fields of Algebraic Functions, *Arch. Math.* **XXI**, 167, (1970).

[23] M. Madan and C. Queen, Algebraic function fields of class number one, *Acta Arith.* **20**, pp. 423-432, (1972).

[24] R. Miranda, *Algebraic Curves and Riemann Surfaces,* Amer. Math. Soc. **5** (1995).

[25] H. Niederreiter and C. Xing, *Rational Points on Curves over Finite Fields: Theory and Applications,* Cambridge University Press, Cambridge (2001).

[26] M. Rosen, *Number Theory in Function Fields,* Springer-Verlag, New York (2002).

[27] M. Rosen, The Hilbert Class Field in Function Fields, *Expo. Math.* **5**, pp. 365-378, (1987).

[28] S. Semirat, Cyclotomic Function Fields with Ideal Class Number One, *J. Algebra.* **236**, pp. 376-395, (2001).

[29] J. Serre, *Linear Representation of Finite Groups,* Springer-Verlag, New York (1977).

[30] J. Serre, *Local Fields,* Springer-Verlag, New York (1979).

[31] J. H. Silverman, *The Arithmetic of Elliptic Curves,* Springer-Verlag, New York (1986).

[32] H. Stichtenoth, *Algebraic Function Fields and Codes,* Springer, New York (1993).

[33] H. Töre, *Class numbers of Algebraic Function fields,* Phd Thesis, Hacettepe University, Ankara (1983) .

[34] L. Washington, *Introduction to cyclotomic fields,* Graduate Texts in Mathematics 83, Springer-Verlag, New York, 1997.

VITA

PERSONAL INFORMATION

Surname, Name: BUYRUK, Dilek

Nationality: Turkish(TC)

Date and Place of Birth: 12 July 1979, Bolu

Marital Status: Single

Phone: +90 312 210 22 92

Fax: +90 312 210 79 59

email: wwwfbe@metu.edu.tr

EDUCATION

Degree	Institution	Year of Graduation
MS	AIBU Mathematics	2005
BS	Bilkent University Mathematics	2003
High School	Hafsa Sultan High School, Ankara	1997

WORK EXPERIENCE

Year	Place	Enrollment
2003-present	AIBU Mathematics	Research Assistant

FOREIGN LANGUAGE

Advanced English

PUBLICATIONS

D. Yılmaz and D. Buyruk, ”Modules Over Prüfer Domains Which Satisfy Radical Formula”, Glasgow Journal, 49, 127-131, (2007)