DYNAMIC SWITCHING TIMES FOR SEASON AND SINGLE TICKETS IN SPORTS AND ENTERTAINMENT WITH TIME DEPENDENT DEMAND RATES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
INDUSTRIAL ENGINEERING

AUGUST 2011

Approval of the thesis:

DYNAMIC SWITCHING TIMES FOR SEASON AND SINGLE TICKETS IN SPORTS AND ENTERTAINMENT WITH TIME DEPENDENT DEMAND RATES

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ABSTRACT

DYNAMIC SWITCHING TIMES FOR SEASON AND SINGLE TICKETS IN SPORTS AND ENTERTAINMENT WITH TIME DEPENDENT DEMAND RATES

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August 2011, 46 pages

The most important market segmentation in sports and entertainment industry is the competition between customers that buy bundle and single tickets. A common selling practice is starting the selling season with bundle ticket sales and switching to selling single tickets later on. The aim of this practice is to increase the number of customers that buy bundles, to create a fund before the season starts and to increase the load factor of the games with low demand. In this thesis, we investigate the effect of time dependent demand on dynamic switching times and the potential revenue gain over the case where the demand rate is assumed to be constant with time.

Keywords: Stopping times, revenue management, bundling, Non-homogeneous Poisson processes

ÖZ

SPOR VE EĞLENCE SEKTÖRÜNDE ZAMANA BAĞLI DEĞİŞKEN TALEP DURUMUNDA SEZON VE TEKLİ BİLETLER ARASINDAKİ DİNAMİK GEÇİŞ ZAMANLARI

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Ağustos 2011, 46 sayfa

Spor ve eğlence sektöründe en önemli pazar bölümlemesi sezon veya kombine bilet alan müşteriler ile tek bir maç / organizasyon için bilet alan müşteriler arasındaki yarışmadır. Yaygın olarak gözlemlenen bir satış yöntemi bilet satış dönemine müşterilere sezon veya kombine bilet satışı ile başlanması, biletlerin tekli (diğer oyun / organizasyonlardan bağımsız) olarak satılmasına ise daha sonra geçilmesidir. Bu yöntemle sezon bileti alan müşteri sayısının artırılması, sezon başlamadan mali kaynak elde edilmesi ve talebin az olduğu organizasyonların da doluluk oranının arttırılması amaçlanır. Bu çalışmada talebin zamana bağlı olarak değişmesinin tekli bilet satışına geçiş zamanları üzerindeki etkisi ve sabit talep varsayımı üzerine getirebileceği potansiyel gelir miktarı incelenmiştir.

Anahtar Kelimeler: Durdurma zamanları, gelir yönetimi, paketleme, homojen olmayan Poisson süreçleri

to my family

ACKNOWLEDGMENTS

My sincere thanks belong to my great advisor Dr. Serhan Duran. His guidance and support have a great impact on my thesis work. Without his knowledge and insights this work could not be completed. I would also like to thank Dr. Mustafa Alp Ertem, Dr. Sinan Gürel, Dr. İsmail Serdar Bakal and Ertan Yakıcı for their time and effort in serving on my committee. In addition, I would like to thank TUBITAK for their financial support for this work.

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CHAPTER 1

INTRODUCTION

There are several fundamental decision problems that every seller faces regarding how to price, how many to allocate to a specific customer group, when to make a specific service or a specific product available etc., and there are several uncertainties involved in such decisions. Revenue Management (RM) is the application of analytical tools that mathematically assists to determine those decisions and reduces uncertainties in order to maximize profitability. More specifically, revenue management deals with three basic categories of decisions. The first category is the structural decisions in which selling format and differentiation mechanisms such as market segmentations are determined. The second category is about pricing; pricing strategies across product groups are determined in this level. Allocation of the capacity to different market segments and timing decisions are included in the third category of Revenue Management which is called quantity decisions.

Importance of these decision categories varies according to the type of business. Although Revenue Management has made its outstanding reputation upon the successful application in the airline industry, it can be applied to any other industry where making decisions about three categories mentioned above is critical. Sports & Entertainment (S&E) industry is one of the areas that revenue management can be applied appreciably when three basic properties of the services or products of S&E industry are regarded. First of all, the services or products of S&E industry are perishable since tickets have no values after the event takes place. Moreover, there is always a limited capacity for the service or product since events are held in venues like a stadium or a theater hall, which have limited capacity. In addition, customers for S&E products can be segmented into different types of customers.

One of the major market segmentation utilized commonly in S&E industry is that some customers prefer to purchase a season ticket to all events during the season, while others prefer to purchase single ticket to an event. There are lots of intangible values associated with the season tickets. These intangibles include sense of belonging, personal seat license, guaranteeing of finding an available seat which may be sold out soon, etc. Season ticket holders are also more beneficial to company since company may benefit from early customer commitment which provides early cash flow and season ticket holders are likely to renew their tickets for consecutive years.

Most of the times, season tickets are offered starting with the beginning of the selling period, then single ticket sales are allowed later on in the selling period before the performance period begins. Selling the bundle ticket first permits the firm to sell the premium quality seats to season ticket buyers since season ticket buyers, most of the time, are lucrative fans who are interested in season tickets. Also allocating better seats to season ticket buyers encourages the customers to purchase bundled tickets.

In this work, we study the specific question of timing the switch from bundle tickets to single tickets when time-dependent customer arrivals are observed and compare the results with the case where customer arrivals are assumed to be constant over time. The main feature of the problem we study is that some limited capacity can be sold as bundled tickets or single tickets as well within the selling period. To maximize profitability, a limited capacity must be shared optimally between bundle and single ticket customers with the usage of an optimal switch time.

For our studies, we will consider in our problem that there are two events in the performance period. Selling period starts with the selling of bundle tickets. We consider Non-homogenous Poisson arrivals for bundle ticket customers since arrivals for bundle tickets may change due to various reasons. After that selling of bundle tickets is ceased at the switch time and selling of single ticket is started and continues till the end of selling season which is the performance time of the first event. After the switch, customer arrivals split into two independent Non-homogenous Poisson process with time-dependent demand rates for each event. The problem is illustrated in Figure 1.1. Our ultimate policy determines the switch times from bundle tickets to single tickets dynamically according to a threshold set which consists of pairs of

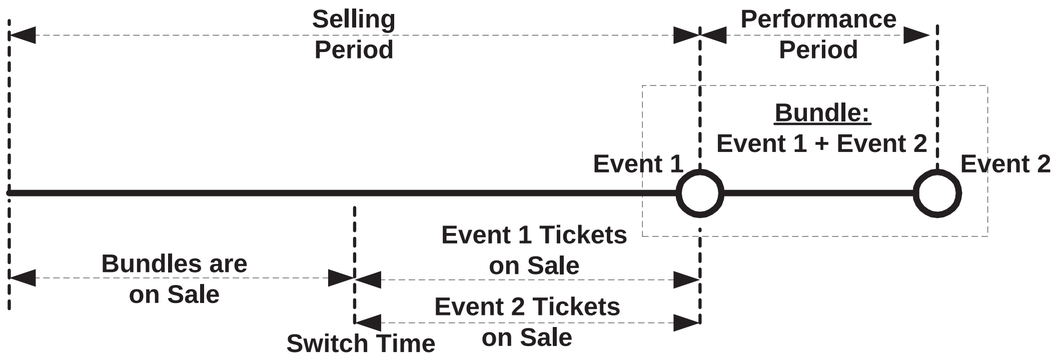

Figure 1.1: Event Time Line

system time and remaining inventory.

Remainder of this study is organized as follows. In Chapter 2, we review the related literature and discuss recent studies about Revenue Management and switching times and their application in S&E industry. In Chapter 3, we first present the basic assumptions about the problem in Section 3.1 and we discuss the optimal revenue and impact of the delaying switch on revenue in Section 3.2 and 3.3. Then, in Section 3.4, we present a method to calculate optimal switching times and in Section 3.1, we talk about simplifications of this method that will be used in numerical calculations. A brief explanation of the usage of our policy is presented in Section 3.6. Next, numerical analysis are conducted in Chapter 4. In Section 4.1 of Chapter 4, we discuss the structure of optimal switching times for different demand rate schemes and compare them with the case where demand rates are assumed to be constant. After that, simulation studies are conducted and % improvement on revenue is discussed in Section 4.2. Finally, brief summary of our results are presented in Chapter 5.

CHAPTER 2

LITERATURE REVIEW

Revenue Management is the application of mathematical tools that help to determine decisions about capacity allocation, pricing across customer segments etc. in order to maximize profit. Traditional Revenue Management applications have been initiated in the airline industry. In airline industry, variable cost per passenger is very small. Therefore, revenue management practices in airline industry generally include booking policies and pricing strategies across fare classes. The value of early commitment to discount fair booking is discussed in Littlewood [22] and extension to multiple fare classes in Belobaba [1]. Moreover, McGill [24], Curry [7], Wollmer [33] and Brumelle and McGill [5] provided the techniques to determine optimal booking limits for single leg flights.

However, in airline industry the problem is highly dependent on the origin and destination of the flight and there may be multiple paths for the same origin-destination pair. Minimum cost network flow formulation considering multiple paths for origin-destination problem is described in Glover et al. [17]. For the last twenty years airline Revenue Management has evolved from single leg control to origin-destination control, which requires big investment in information systems. Excellent return from these investments on RM information systems is presented in Smith [28]. The bid-price approach which is discussed in Williamson [32], Talluri and van Ryzin [29] appears to be the most promising practice due to its ease of use and improved rate of return. Latest studies in airline RM include Bertsimas and Popescu [4] and Karaesmen and Ryzin [19].

Successful application of RM in airline industry has attracted other industries such as car rental and hotels. Dramatic improvements on revenue upon application of an RM program in car rental industry are stated in Geraghty and Johnson [16] and application in hotel management is considered in Feng and Gallego [12] and Bitran and Gilbert [2]. However, there are many other industries that present several RM-type problems. Sports & Entertainment (S&E) industry is an area whose RM-type problems are not addressed entirely. Ho [18] presents six characteristics of S&E environment which are completely relevant to RM. In S&E industry problems, customers are single individuals that are likely to purchase multiple tickets during a season, while in the airline industry customers are groups of individuals that purchase a single ticket on a plane. Therefore group sale in airline is not a key element in contrast to S&E industry. Thus, the focus in S&E industry is the timing of bundle and single ticket sales rather than determining the price and seat allocation as in airline industry. However, there are limited studies on group sales such as Yuen [35] and Farley [11].

Most of the papers in airline RM and retail industries are focused on pricing as a function of time. Gallego [15] studies pricing of a set stock of products to be sold by a deadline and use intensity control to identify optimal prices as a function of the stock level and remaining time. Nevertheless, most of the organizations in S&E industries announce the prices before the selling season starts and keep them constant throughout the selling period which is known as price stickiness (Courty [6]). Therefore timing of availability of different product groups is more applicable than timing of a price chance of airline RM applications.

Among airline Revenue Management studies, Feng and Gallego [13] is one of the most related study to our work. In that paper, they study the optimal dynamic switching time from one fare class to another fare class. Fare classes are predetermined and they assume that demands are stochastic. Their eventual procedure is a set of threshold values which consist of remaining inventory and corresponding time pairs. Constant demand rates assumption and restriction to single price change at a time are relaxed in Feng and Gallego [12]. Their focus of study is the timing of a price change in order to sell out the fixed inventory over a finite time horizon so as to maximize revenue. However, focus of our study is the timing of capacity allocation to different customer group and this is the main difference in our work.

Some organizations in S&E industries announce switching time from bundle ticket sales to single ticket sale in advance before the beginning of the selling period. Drake et al. [8] deal with that kind of static switching time problem. However, demand realizations in real word may not be as smooth as considered in static switching problem. Therefore, Most organizations in (S&E) industry prefers to select their switching times dynamically and determine the timing of promotion according to previous demand realization. For that reason, in our work, we study the case where switching time is selected dynamically after observing demand realization.

The most relevant study in the literature is carried out in Duran [9] where dynamic switching is considered. Their ultimate procedure finds a set of threshold values which depends on time and corresponding remaining inventory. However, they consider homogenous Poisson process with the assumption that demand rates are constant over the time. Moreover, in that paper, they study the option of two switches case as an extension to Duran [10]. In our work, we consider Non-homogenous Poisson process with time-dependent demand rates for both bundle and single ticket. Although mathematical effort needed is more intense, we believe that it is necessary in order to match the real life situation.

There are also several works in other disciplines which are related to revenue improvement in SE industry. Numerous papers have studied pricing problem within a venue. However, they did not consider bundling in order to maximize revenue. For example, Leslie [21] and Rosen and Rosenfield [26] study ticket pricing that maximize revenue according to different seat qualities. Venkatesh and Kamakura [30], McAfee et al. [23] and Salinger [27] are the most relevant studies in the economics and marketing literature. However, these papers focus on the pricing of the bundles, not the timing of decisions.

CHAPTER 3

MODEL AND ASSUMPTIONS

3.1 Assumptions

In S&E industry, season tickets are hardly sold after performance period begins Duran [10]. Moreover, switching from bundle to single ticket is made within the selling period. Therefore, we concentrate on a selling period which begins with the start of the selling of bundle tickets at $t = 0$ and ends with the start of the first event at $t = T$, where $T \in \mathbb{R}^+$. Also, let $M \in \mathbb{Z}^+$ number of available seats for sale for each event. We assume that prices for both bundle ticket and single ticket are announced before the selling period begins and remain constant during the selling period, which is the real practice for most S&E organizations. At the beginning of the selling period, bundle tickets are offered at price p_B and then single tickets are offered later on after the switch at price p_i for $i = 1, 2$ since we also assume that there are two event in the performance period.

We assume there is a perfect market segmentation between bundles and singles and the arrival process for bundles up to the switching time is independent of switch time. Our study focuses a realistic practice where customer arrivals for both bundle and single tickets are time-dependent. Therefore, we assume that there is a corresponding Non-homogenous Poisson demand with time-dependent demand rates for each ticket group: $N_B(s)$, $0 \leq s \leq t$, with known time-dependent demand rate $\lambda_B(t)$ for the bundled events; $N_1(s)$, $0 \leq s \leq t$, with known time-dependent demand rate $\lambda_1(t)$, and $N_2(s)$, $0 \leq s \leq t$, with known time-dependent demand rate $\lambda_2(t)$ for the two single events, respectively. In addition, we assume that demand rates $\lambda_i(t)$ for $i = B, 1, 2$

are linearly changing or constant over time. Thus, the major objective of our study is to find an optimal method to allocate M seats among bundle and single ticket buyers within a time period of T when time-dependent customer arrivals with $\lambda_i(t)$ (i=B,1,2) are observed.

Revenue rates for bundle and single tickets are defined as $r_B(t) = \lambda_B(t)p_B$ and $r_i(t) = \lambda_i(t)p_i$, $i = 1, 2$. We assume that at the beginning of the selling period, the revenue rates for the bundle is higher than the sum of revenue rates of the single tickets, i.e., $r_B(0) > r_1(0) + r_2(0)$ and $r_B(t) - r_1(t) + r_2(t)$ is a non-decreasing function. Otherwise, switching immediately would be optimal for all states.

Since we assumed a two-event selling horizon, both of the events considered must be appropriate to be bundled. Therefore, demand rate for the bundled tickets is larger than the individual events' demand rates ($\lambda_B(t) > \lambda_i(t)$, $i = 1, 2$). Moreover, we assume that if the event demand rates increase (decrease) over time, increase (decrease) rate is higher for singles (bundles), i.e., $\lambda_B(t) - \lambda_i(t)$, $i = 1, 2$ is non-increasing in t.

3.2 The Dynamic Timing Problem

Initially, we will start with the calculation of the total expected revenue from ticket sales for a specific switching time. Then, we will study the effects of delaying the switch by comparing different switching options. Specifically, we will compare two situations; switching immediately at time t or switching later at time τ with the help of the marginal gain (or loss) expression. In order to quantify this gain (or loss), we will define a generator function and use this quantification to discover the characteristic of the optimal time to switch. Subsequently, we will develop a function that allows us to compute the optimal switching times for each (time, left seat) pair. Finally, we will use this function to show the structure of the optimal switching times.

For a switching time τ such that $\tau \in \mathcal{T}$, where $\mathcal{T}$ is the set of switching times satisfying $t \leq \tau \leq T$, we assume that only bundle tickets will be sold in the period from beginning of the selling period up to the switching time τ and only single tickets will be sold in the period from switching time τ to the end of the selling period. Therefore, expected total revenue from ticket sale over the entire time horizon $[t, T]$ is composed of two main parts. The first part is the revenue from bundle ticket sales up to the switching time τ with n seats available for sale and it is given by $B(t, n)$:

$$B(t, n) = E[p_B((N_B(\tau) - N_B(t)) \wedge n)],$$

where $(x \wedge y)$ is the function whose value is the minimum of the x and y. The second part is the revenue from single ticket sales from switching at time τ to the end of the selling period T and it is given by $S(\tau, n(\tau))$:

$$S(\tau, n(\tau)) = p_1 E[(N_1(T) - N_1(\tau)) \wedge n(\tau)] + p_2 E[(N_2(T) - N_2(\tau)) \wedge n(\tau)],$$

where $n(\tau) = [n - N_B(\tau) + N_B(t)]^+$, where $x^+ = \max\{0, x\}$. $n(\tau)$ is the function that indicates the remaining inventory for singles after the switch. This function also reflects the distribution of the capacity usage between single and bundle tickets.

So far how to calculate the revenue from bundle ticket sales $B(t, n)$ and the revenue from single ticket sales $S(\tau, n(\tau))$ is stated. Total revenue for a specific switching time τ over the selling period $[t, T]$ can be calculated by adding these two main terms together as $E[p_B((N_B(\tau) - N_B(t)) \wedge n) + S(\tau, n(\tau))]$. Furthermore, taking the supremum of this function over all stopping times $\tau \epsilon \mathcal{T}$ will give us the optimal expected total revenue from bundle and single ticket sales over $[t, T]$ with n remaining seats and it is given by $V(t, n)$:

$$V(t, n) = \sup_{\tau \in \mathcal{T}} E[p_B((N_B(\tau) - N_B(t)) \wedge n) + S(\tau, n(\tau))]. \tag{3.1}$$

3.3 Delaying the Switch

It is possible to investigate the effects of the delaying the switch to a later time on $V(t, n)$ by comparing different switching options. For example if the switch is made immediately at time t, the expected revenue will be $S(t, n)$. However, instead of switching immediately, switch may be delayed to a later time τ ($t \leq \tau \leq T$) and the expected revenue in this case will be $E[p_B((N_B(\tau) - N_B(t)) \wedge n)] + S(\tau, n(\tau))$. If we can compare these two switching options, then we can decide whether delaying the switch is beneficial or not. To make this comparison, we start with the evaluation of the changes within an infinitesimal time interval. We define the infinitesimal generator $\mathcal{G}$ with respect to the Non-homogenous Poisson process (NPP) for bundles $(t, N_B(t))$ for a uniformly bounded function $g(t,n)$. Infinitesimal generator $\mathcal{G}$ is a stochastic partial differential operator which is defined as:

$$\mathcal{G}g(t,n) = \lim_{\Delta t \to 0} \frac{E[g(t+\Delta t, n(t+\Delta t)) - g(t,n)]}{(t+\Delta t) - t} \tag{3.2}$$

we know that:

$$\begin{aligned} n(t+\Delta t) &= n(t) - (N_B(t+\Delta t) - N_B(\Delta t)) && (3.3)\\ &= n - (N_B(t+\Delta t) - N_B(\Delta t)) && (3.4) \end{aligned}$$

therefore

$$\mathcal{G}g(t,n) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[g(t+\Delta t, n - (N_B(t+\Delta t) - N_B(\Delta t))) - g(t,n)]. \tag{3.5}$$

Remember the assumption that customer arrivals for both bundle and single tickets occur in a nonuniform fashion. In such a case, probability that a customer buys an i-type ticket ($i = 1, 2, B$) in the time interval $(t, t+h]$ is $\lambda_i(t)h + o(h)$. In our case we want to investigate the changes from time t to $t + \Delta t$ while Δt approaches to 0.

$$\begin{aligned} \mathcal{G}g(t,n) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[g(t+\Delta t, n - (N_B(t+\Delta t) - N_B(\Delta t))) - g(t,n)] && (3.6)\\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \sum_{j=0}^{\infty} [g(t+\Delta t, (n-j)^+) - g(t,n)] P(N_B(t+\Delta t) = k + j | N_B(t) = k) \end{aligned}$$

where $N_B(t+\Delta t) - N_B(\Delta t) = j; \quad j = 1, 2, 3...$

If we use the probability values for $j = 1, 2, 3...$, Equation (3.6) can be simplified much further. From the definition of Non-homogenous Poisson process [20], we know that;

$$\begin{aligned} P[N_B(t+\Delta t) = k | N_B(t) = k] &= 1 - \lambda_B(t)\Delta t + o(\Delta t),\\ P[N_B(t+\Delta t) = k+1 | N_B(t) = k] &= \lambda_B(t)\Delta t + o(\Delta t),\\ P[N_B(t+\Delta t) = k+j | N_B(t) = k] &= o(\Delta t) \;\; for \;\; j \geq 2. \end{aligned}$$

Substituting these values into Equation (3.6):

$$\begin{aligned} \mathcal{G}g(t,n) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \{ & [g(t+\Delta t, n-0) - g(t,n)](1 - \lambda_B(t)\Delta t + o(\Delta t)) + [g(t+\Delta t, n-1)\\ & - g(t,n)](\lambda_B(t)\Delta t + o(\Delta t)) + [g(t+\Delta t, n-2) - g(t,n)]o(\Delta t)\\ & + [g(t+\Delta t, n-3) - g(t,n)]o(\Delta t) + [g(t+\Delta t, n-4) - g(t,n)]o(\Delta t)\\ & + (\cdot)o(\Delta t) + \;...\; + (\cdot)o(\Delta t)\}. \end{aligned}$$

Collecting the terms into three groups; having $\lambda(t)$ multipliers, having $o(\Delta t)$ multipliers and other terms, we obtain;

$$\begin{aligned}
\mathcal{G}g(t,n) &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}[(g(t+\Delta t,n)-g(t,n))+\lambda_B(t)\Delta t(g(t+\Delta t,n-1) \\
&\quad -g(t+\Delta t,n))+(\cdot)o(\Delta t)] \\
&= \lim_{\Delta t\to 0}[\frac{g(t+\Delta t,n)-g(t,n)}{\Delta t}+\lambda_B(t)(g(t+\Delta t,n-1)-g(t+\Delta t,n)) \\
&\quad +\frac{(\cdot)o(\Delta t)}{\Delta t}].
\end{aligned}$$

Finally we have:

$$\mathcal{G}g(t,n) = \frac{\partial g(t,n)}{\partial t}+\lambda_B(t)[g(t,n-1)-g(t,n)].$$

When we apply this generator function $\mathcal{G}$ to the revenue function of only single ticket sales $S(t,n)$ together with the revenue function from bundle ticket sales $B(t,n) = E[p_B((N_B(\tau)-N_B(t))\wedge n)] = \lambda_B(t)p_B$, we obtain the marginal gain (or loss) expression as:

$$\begin{aligned}
\mathcal{G}[S(t,n)+B(t,n)] = \frac{\partial S(t,\ n)}{\partial t} &+ \lambda_B(t)[S(t,n-1)-S(t,n)]+\lambda_B(t)p_B \\
&+ \mathcal{G}[p_B((N_B(\tau)-N_B(t))\wedge n)]
\end{aligned}$$

it can be written as:

$$\mathcal{G}S(t,n)+\lambda_B(t)p_B = \frac{\partial S(t,\ n)}{\partial t}+\lambda_B(t)[S(t,n-1)-S(t,n)]+\lambda_B(t)p_B.$$

Note that marginal gain(or loss) expression is composed of three parts. The first part is $\frac{\partial S(t,n)}{\partial t}$ which is the revenue loss from single ticket sales due to elapsed time. The second part is $\lambda_B(t)[S(t,n-1)-S(t,n)]$ which is the revenue loss again from single ticket sales, but due to the decrease in inventory. The third and last part is $\lambda_B(t)p_B$ which is the revenue gain from bundle ticket sales within infinitesimal delay time since bundle ticket sale is active in this delay time.

The net marginal gain (or loss) expression will be one of the main terms in our analysis. Behavior of our final optimal switching times expression will strongly be dependent on the behavior of the net marginal gain expression. Therefore it is beneficial to investigate the properties of the net marginal gain (or loss) in detail. Following lemma will focus on this term more closely.

Lemma 3.3.1. *The net marginal gain from delaying for* $0 \le t \le T$ *can be written as*

$$\begin{aligned}\mathcal{G}S(t,n) + \lambda_B(t)p_B = (r_B(t) - r_1(t) - r_2(t)) \quad &+ \quad p_1(\lambda_1(t) - \lambda_B(t))P[N_1(T) - N_1(t) \ge n] \\ &+ \quad p_2(\lambda_2(t) - \lambda_B(t))P[N_2(T) - N_2(t) \ge n],\end{aligned}$$

and is increasing in both n and t, when $\lambda_B(t) > \lambda_i(t) \ \ (i = 1,2) \ \ \forall t$

Proof of Lemma 3.3.1. We will begin the proof by stating the simplification of some general terms, and then we will use these simplification in our proof. Let $\Lambda_i(t) = \int_0^t \lambda_i(s)ds$ From the properties of NPP we know that:

$$P[N_i(T) - N_i(t) = k] = \frac{e^{-(\Lambda_i(T)-\Lambda_i(t))}(\Lambda_i(T) - \Lambda_i(t))^k}{k!}. \tag{3.7}$$

Thus derivative of $P[N_i(T) - N_i(t) \ge k]$ with respect to t can be written as:

$$\begin{aligned}\frac{\partial P[N_i(T) - N_i(t) \ge k]}{\partial t} &= \frac{\partial}{\partial t}(1 - P[N_i(T) - N_i(t) < k]) \\ &= \frac{\partial}{\partial t}(1 - \frac{e^{-(\Lambda_i(T)-\Lambda_i(t))}(\Lambda_i(T) - \Lambda_i(t))^{k-1}}{(k-1)!} \\ &\quad - \frac{e^{-(\Lambda_i(T)-\Lambda_i(t))}(\Lambda_i(T) - \Lambda_i(t))^{k-2}}{(k-2)!} \\ &\quad\quad \vdots \\ &\quad - \frac{e^{-(\Lambda_i(T)-\Lambda_i(t))}(\Lambda_i(T) - \Lambda_i(t))^{0}}{(0)!}).\end{aligned}$$

Taking derivatives of each term in the left hand side we have:

$$\begin{aligned}&\frac{\partial P[N_i(T) - N_i(t) \ge k]}{\partial t} = \\ &\quad -\lambda_i(t)P[N_i(T) - N_i(t) = k-1] \quad + \quad \cancel{\lambda_i(t)P[N_i(T) - N_i(t) = k-2]} \\ &\quad \cancel{-\lambda_i(t)P[N_i(T) - N_i(t) = k-2]} \quad + \quad \cancel{\lambda_i(t)P[N_i(T) - N_i(t) = k-3]} \\ &\quad \cancel{-\lambda_i(t)P[N_i(T) - N_i(t) = k-3]} \quad + \quad \cancel{(....)} \\ &\quad - \quad \cancel{(....)} \quad + \quad \cancel{(....)} \\ &\quad - \quad \cancel{(....)} \quad + \quad \cancel{\lambda_i(t)P[N_i(T) - N_i(t) = 0]} \\ &\quad \cancel{-\lambda_i(t)P[N_i(T) - N_i(t) = 0]}.\end{aligned}$$

After eliminations we have:

$$\frac{\partial P[N_i(T) - N_i(t) \ge k]}{\partial t} = -\lambda_i(t)P[N_i(T) - N_i(t) = k-1], \tag{3.8}$$

therefore, we may write

$$\frac{\partial \sum_{k=1}^{n} P[N_i(T) - N_i(t) \geq k]}{\partial t} = -\lambda_i(t) P[N_i(T) - N_i(t) \leq n-1]. \tag{3.9}$$

Also note that $E[(N_i(T) - N_i(t)) \wedge n] = \sum_{k=1}^{n} P[N_i(T) - N_i(t) \geq k]$, we can express $S(t,n)$ for $n \geq 1$ as

$$S(t,n) = p_1 \sum_{k=1}^{n} P[N_1(T) - N_1(t) \geq k] + p_2 \sum_{k=1}^{n} P[N_2(T) - N_2(t) \geq k]. \tag{3.10}$$

Using Equations (3.9) and (3.10):

$$\begin{aligned}
\mathcal{G}S(t,n) &= \frac{\partial S(t,n)}{\partial t} + \lambda_B(t)[S(t,n-1) - S(t,n)] \\
&= -\lambda_1(t)p_1 P[N_1(T) - N_1(t) \leq n-1] - \lambda_2(t)p_2 P[N_2(T) - N_2(t) \leq n-1] \\
&\quad -\lambda_B(t)p_1 P[N_1(T) - N_1(t) \geq n] - \lambda_B(t)p_2 P[N_2(T) - N_2(t) \geq n] \\
&= -\lambda_1(t)p_1(1 - P[N_1(T) - N_1(t) \geq n]) - \lambda_2(t)p_2(1 - P[N_2(T) - N_2(t) \geq n]) \\
&\quad -\lambda_B(t)p_1 P[N_1(T) - N_1(t) \geq n] - \lambda_B(t)p_2 P[N_2(T) - N_2(t) \geq n] \\
&= -\lambda_1(t)p_1 - \lambda_2(t)p_2 + p_1(\lambda_1(t) - \lambda_B(t))P[N_1(T) - N_1(t) \geq n] \\
&+ p_2(\lambda_2(t) - \lambda_B(t))P[N_2(T) - N_2(t) \geq n].
\end{aligned}$$

Finally we have:

$$\begin{aligned}
\mathcal{G}S(t,n) + \lambda_B(t)p_B = r_B(t) - r_1(t) - r_2(t) &+ p_1(\lambda_1(t) - \lambda_B(t))P[N_1(T) - N_1(t) \geq n] \\
&+ p_2(\lambda_2(t) - \lambda_B(t))P[N_2(T) - N_2(t) \geq n].
\end{aligned}$$

Obviously, $\mathcal{G}S(t,n) + \lambda_B(t)p_B$ is an increasing function in n. Defining the following functions as:

$$\begin{aligned}
R(t) &= r_B(t) - r_1(t) - r_2(t), \\
k_i(t,n) &= P[N_i(T) - N_i(t) \geq n],
\end{aligned}$$

we may write the net marginal gain (or loss) expression simply as

$$\begin{aligned}
\mathcal{G}S(t,n) + \lambda_B(t)p_B &= R(t) + p_1(\lambda_1(t) - \lambda_B(t))k_1(t,n) + p_2(\lambda_2(t) - \lambda_B(t))k_2(t,n) \\
&= R(t) - [p_1(\lambda_B(t) - \lambda_1(t))k_1(t,n) + p_2(\lambda_B(t) - \lambda_2(t))k_2(t,n)],
\end{aligned}$$

Also defining $f(t,n) = [p_1(\lambda_B(t) - \lambda_1(t))k_1(t,n) + p_2(\lambda_B(t) - \lambda_2(t))k_2(t,n)]$, marginal gain (or loss) expression simply turns into:

$$\mathcal{G}S(t,n) + \lambda_B(t)p_B = R(t,n) - f(t,n).$$

For any fixed n, we have the following marginal gain (or loss) values at the beginning and at the end of the selling period:

- at $t = 0$, $\quad \mathcal{G}S(0, n) + \lambda_B(0)p_B = R(0) - f(0, n),$
- at $t = T$, $\quad \mathcal{G}S(T, n) + \lambda_B(T)p_B = R(T) - \cancelto{0}{f(T, n)} = R(T).$

Remember the assumption that $R(t) = r_B(t) - r_1(t) - r_2(t)$ is a non-decreasing function on $[0, T]$. Also $f(0, n)$ is a positive value for any n. Therefore following inequalities hold

$$R(0) - f(0, n) < R(0) \leq R(t),$$

Thus it is easy to compare the net marginal gain (or loss) values at the beginning of the selling period at $t = 0$ and at the end of the selling period at $t = T$ as:

$$\mathcal{G}S(0, n) + \lambda_B(0)p_B = R(0) - f(0, n) < R(T) = \mathcal{G}S(T, n) + \lambda_B(T)p_B,$$

Since $f(t, n)$ is the product of two positive and non-increasing functions; $\lambda_B(t) - \lambda_i(t)$ and $k(t, n)$, is also positive and non-increasing in t on $[t, T)$. Thus marginal gain (or loss) expression $\mathcal{G}S(t, n) + \lambda_B(t)p_B$ is an increasing function in t starting from the value $R(0) - f(0, n)$ at t=0 and reaching to a higher value $R(T)$ at $t = T$. $\blacksquare$

In general, a martingale system is a stochastic process where the value of a future observation is equal to the value of the present observed value even if values of all observations up to the present observation are known. This means that past observations have no effect in prediction of the future observations.

If the random variable of a martingale is discrete-time stochastic process, it is called as a discrete-time martingale. Assume that X_n is an observation of a discrete-time martingale at time n, $n = 1, 2, 3...$. Therefore $E(X_{n+1}|X_1, ..., X_n) = X_n$ and $E(X_{n+1} - X_n|X_1, ..., X_n) = 0$. Similarly, if the random variable of a martingale is continuous-time stochastic process, it is called as a continuous-time martingale. Let Y_t is an observation of a continuous-time martingale at time t. Then, $E(Y_s|\{Y_\tau, \tau \leq t\}) = Y_t, \ \forall t \leq s$. This expresses the property that the conditional expectation of an observation at time s, given all the observations up to time t, is equal to the observation at time t for $s \geq t$.

Assume that $Z(t)$ is the random variable of a continuous-time stochastic process, then $\mathcal{G}Z(t)$ is the derivative of $Z(t)$. If $E[Z(s) - Z(t)] - E[\int_t^s \mathcal{G}Z(u)du] = 0$ for $s \geq t$, this system is a continuous-time martingale. Therefore, we can say that the following two expressions are martingales for any $s \geq t$, which is also stated in Dynkin's Lemma [25] :

$$\begin{aligned} m_1(s) &= S(s, n(s)) - S(t, n) - \int_t^s \mathcal{G}S(u, n(u))du, && (3.11) \\ m_2(s) &= p_B((N_B(s) - N_B(t)) \wedge n) - \int_t^s \lambda_B(t) p_B I_{\{n(u)>0\}} du, && (3.12) \end{aligned}$$

where $I_{\{n(u)>0\}}$ is the indicator function. We also know from properties of martingales that expected value of an observation at some time s is equal to the expected vale at starting time t. Therefore we have $E[m_1(s)] = E[m_2(s)] = 0$. This leads us to:

$$\begin{aligned} S(s, n(s)) - S(t, n) &= E \int_t^s \mathcal{G}S(u, n(u))du, && (3.13) \\ E[p_B((N_B(s) - N_B(t)) \wedge n)] &= E \int_t^s \lambda_B(t) p_B I_{\{n(u)>0\}} du. && (3.14) \end{aligned}$$

It is stated in Doob's Optional Stopping Theory [31] that the expected value of a martingale at a stopping time is equal to its initial expected value or expected value at any deterministic time. Thus we may replace s in (3.13) and (3.14) with any stopping time $\tau \geq t$. So we may write

$$\begin{aligned} S(\tau, n(\tau)) - S(t, n) &= E \int_t^\tau \mathcal{G}S(u, n(u))du, && (3.15) \\ E[p_B((N_B(\tau) - N_B(t)) \wedge n)] &= E \int_t^\tau \lambda_B(t) p_B I_{\{n(u)>0\}} du. && (3.16) \end{aligned}$$

Adding Equations (3.15) and (3.16) together we obtain:

$$\begin{aligned} E[p_B((N_B(\tau) - N_B(t)) \wedge n)] + S(\tau, n(\tau)) &- S(t, n) && (3.17) \\ &= E \int_t^\tau [\mathcal{G}S(u, n(u)) + \lambda_B(t) p_B I_{\{n(u)>0\}}] du. \end{aligned}$$

We see that the left-hand side of (3.17) is the expected revenue gained over $S(t, n)$ by delaying the switch from t to τ. As shown in the right-hand side, this can be quantified by using $\mathcal{G}$. Thus from Equation (3.17) we can conclude that delaying the switch from t to a later time τ is beneficial if $E \int_t^\tau [\mathcal{G}S(u, n(u)) + \lambda_B(t) p_B I_{\{n(u)>0\}}] du > 0$. Furthermore if we take the supremum of both sides in (3.17) over all stopping times $t \leq \tau \leq T$, we get:

$$V(t, n) = S(t, n) + \sup_{t \leq \tau \leq T} E \int_t^\tau [\mathcal{G}S(u, n(u)) + \lambda_B(t) p_B I_{\{n(u)>0\}}] du. \qquad (3.18)$$

At this point we may introduce

$$\widetilde{V}(t,n) = \sup_{t\le\tau\le T} E\int_t^{\tau}[\mathcal{G}S(u,n(u)) + \lambda_B(t)p_B I_{\{n(u)>0\}}]du. \tag{3.19}$$

Thus, from Equations (3.18) and (3.19) we have $V(t,n) = S(t,n) + \widetilde{V}(t,n)$. This implies that the optimal revenue over $[t,T]$ consists of two parts. The first part is $S(t,n)$, which stands for the revenue from immediately switching and selling only single tickets until the end of the selling period. The second part is $\widetilde{V}(t,n)$, which stands for the additional revenue from delaying the switch to a later time. Since $\widetilde{V}(t,n)$ is also given by

$$\widetilde{V}(t,n) = \sup_{t\le\tau\le T} E[p_B((N_B(\tau)-N_B(t))\wedge n) + S(\tau,n(\tau))] - S(t,n), \tag{3.20}$$

it is obvious that $\widetilde{V}(t,n) \ge 0$ for any $0 \le t \le T$ and $0 \le n \le M$. In particular, $\widetilde{V}(t,0) = 0$ for all $0 \le t \le T$ and $\widetilde{V}(T,n) = 0$ for all $0 \le n \le M$. Moreover, Equation (3.20) indicates that when $\widetilde{V}(t,n) = 0$, delaying the switch further is not optimal, whereas $\widetilde{V}(t,n) > 0$ implies a revenue potential from delaying the switch.

At this point, we introduce $\overline{V}(t,n)$ to compute $\widetilde{V}(t,n)$. When the conditions in Theorem 3.3.2 are satisfied $\overline{V}(t,n)$ will be identical to $\widetilde{V}(t,n)$ and can be derived recursively.

Theorem 3.3.2. *Suppose there exists a function $\overline{V}(t,n)$ such that $\overline{V}(t,n)$ is continuous and differentiable with right continuous derivatives in [0,T] for each fixed n. In addition, if $\overline{V}(t,n)$ satisfies:*

(i) $\overline{V}(t,n) \ge 0,\;\; 0 \le t \le T$ *and* $0 \le n \le M$*;*

(ii) $\overline{V}(T,n) = 0$ *for* $0 \le n \le M$ *and* $\overline{V}(t,0) = 0$ *for* $0 \le t \le T$*;*

(iii) $\overline{V}(t,n) = 0 \Rightarrow \mathcal{G}(\overline{V}+S)(t,n) + \lambda_B(t)p_B \le 0$*,* $0 \le t \le T$ *and* $0 \le n \le M$*;*

(iv) $\overline{V}(t,n) > 0 \Rightarrow \mathcal{G}(\overline{V}+S)(t,n) + \lambda_B(t)p_B = 0$*,* $0 \le t \le T$ *and* $0 \le n \le M$*;*

then $\overline{V}(t,n) = \widetilde{V}(t,n)$*.*

The first two conditions in the list are the non-negativity property and the boundary conditions of $\overline{V}$. Also, since $\widetilde{V}(t,n)$ determines whether it is optimal to switch immediately or not, the conditions for $\overline{V}(t,n)$ to be positive or zero are also crucial and are listed in conditions (*iii*) and (*iv*).

Proof of Theorem 3.3.2. The proof is similar to Duran [10]. We will begin the proof by assuming that there exists a function satisfying the conditions in the theorem. Then, we will show that $\overline{V}$ is equal to $\widetilde{V}$. For $s \geq t$. From Dynkin's Lemma, we know that following equation is a martingale:

$$m(s) = \overline{V}(s, n(s)) \quad - \quad \overline{V}(t, n) - \int_t^s \mathcal{G}\overline{V}(u, n(u))du.$$

As we previously stated the expected value of a martingale at any time s is equal to its expected value at the starting time t. Hence, we have $Em(s) = 0$. Further, by the optional stopping theorem, for any discreet stopping time $\tau \geq t$ we have

$$E[\overline{V}(\tau, n(\tau)] \quad - \quad E\int_t^\tau \mathcal{G}\overline{V}(u, n(u))du = \overline{V}(t, n), \tag{3.21}$$

$$\begin{aligned} E[\overline{V}(\tau, n(\tau))] \quad - \quad & E\int_t^\tau [\mathcal{G}(\overline{V} + S)(u, n(u)) + \lambda_B(t)p_B I_{\{n(u)>0\}}]du \qquad (3.22) \\ = \quad & \overline{V}(t, n) - E\int_t^\tau [\mathcal{G}S(u, n(u)) + \lambda_B(t)p_B I_{\{n(u)>0\}}]du. \end{aligned}$$

If we subtract $E\int_t^\tau [\mathcal{G}S(u, n(u)) + \lambda_B(t)p_B I_{\{n(u)>0\}}]du$ from both sides of (3.21), the left-hand side of the resulting term, given by (3.22), is always positive by conditions (i), (iii) and (iv). Therefore,

$$\overline{V}(t, n) \geq E\int_t^\tau [\mathcal{G}S(u, n(u)) + \lambda_B(t)p_B I_{\{n(u)>0\}}]du.$$

Since $\overline{V}(t, n)$ is greater than or equal to each term in the right-hand side of Equation (3.19) for any τ, it is also greater than or equal to the supremum of those terms over all τ, which is $\widetilde{V}(t, n)$ in Equation (3.19). Hence, we conclude that $\overline{V}(t, n) \geq \widetilde{V}(t, n)$ for any stopping time $\tau \geq t$.

To prove that $\overline{V}(t, n) \leq \widetilde{V}(t, n)$, we will define a specific stopping time. Let σ be defined as $\sigma = \inf\{t \leq s \leq T : \overline{V}(s, n(s)) = 0\}$. Note that σ is well-defined because $\overline{V}(T, \cdot) = 0$. Replacing τ in Equation (3.22) with σ, we obtain

$$\begin{aligned} E[\overline{V}(\sigma, n(\sigma))] \quad - \quad & E\int_t^\sigma [\mathcal{G}(\overline{V} + S)(u, n(u)) + \lambda_B(t)p_B I_{\{n(u)>0\}}]du \qquad (3.23) \\ = \quad & \overline{V}(t, n) - E\int_t^\sigma [\mathcal{G}S(u, n(u)) + \lambda_B(t)p_B I_{\{n(u)>0\}}]du. \end{aligned}$$

The definition of σ implies that $\overline{V}(\sigma, n(\sigma)) = 0$, and the definition of σ and condition (iv) together imply that $\mathcal{G}(\overline{V} + S)(u, n(u)) + \lambda_B p_B = 0$ for all $u \in [t, \sigma]$. Therefore the left-hand side of (3.23) is zero and we have

$$\overline{V}(t,n) \quad = \quad E\int_t^{\sigma} [\mathcal{G}S(u,n(u)) + \lambda_B(t)p_B I_{\{n(u)>0\}}]du \leq \widetilde{V}(t,n).$$

The inequality follows from the fact that the left-hand side of the inequality is the right-hand side of Equation (3.19) for a specific stopping time, and $\widetilde{V}(t,n)$ is the supremum over all stopping times τ in that equation. Hence, $\overline{V}(t,n) = \widetilde{V}(t,n)$. ■

3.4 Calculation of Optimal (x_n, n) Pairs

In the previous section we have shown the existence of the alternate function $\overline{V}(t,n)$. In this section we will develop a formal procedure to calculate $\overline{V}(t,n)$ for any (t,n) pairs. From condition (iv), we know that $\mathcal{G}\overline{V}(t,n) = -\mathcal{G}S(t,n) - \lambda_B(t)p_B$ when $\overline{V}(t,n) > 0$. Applying the infinitesimal generator $\mathcal{G}$ to $\overline{V}(t,n)$, we get the differential equation

$$\frac{\partial\overline{V}(t,n)}{\partial t} - \lambda_B(t)\overline{V}(t,n) = -[\lambda_B(t)\overline{V}(t,n-1) + \mathcal{G}S(t,n) + \lambda_B(t)p_B], \tag{3.24}$$

By the integrating factor approach [3], solution of this differential equation is straightforward. For convenience let
$[\lambda_B(t)\overline{V}(t,n-1) + \mathcal{G}S(t,n) + \lambda_B(t)p_B] = A(t,n)$ and $\Lambda_B(t) = \int_0^t \lambda_B(s)ds$. Thus:

$$\frac{\partial\overline{V}(t,n)}{\partial t} - \lambda_B(t)\overline{V}(t,n) = -A(t,n). \tag{3.25}$$

Multiply both side by $e^{-\Lambda_B(t)}$ in order to transform Equation (3.25)into a form that integration can be performed,

$$e^{-\Lambda_B(t)}\frac{\partial\overline{V}(t,n)}{\partial t} - e^{-\Lambda_B(t)}\lambda_B(t)\overline{V}(t,n) = -e^{-\Lambda_B(t)}A(t,n), \tag{3.26}$$

It is easy to see that left hand side of the Equation (3.26) is the derivative of the product of $e^{-\Lambda_B(t)}$ and $\overline{V}(t,n)$. So

$$\frac{\partial(e^{-\Lambda_B(t)}\overline{V}(t,n))}{\partial t} \quad = \quad -e^{-\Lambda_B(t)}A(t,n), \tag{3.27}$$

$$e^{-\Lambda_B(t)}\overline{V}(t,n) \quad = \quad \int -e^{-\Lambda_B(t)}A(t,n)dt, \tag{3.28}$$

$$\overline{V}(t,n) \quad = \quad e^{\Lambda_B(t)}\int [-e^{-\Lambda_B(t)}A(t,n)]dt. \tag{3.29}$$

Note that Equation (3.29) is an indefinite integral equation. Using the boundary conditions on $\overline{V}(t,n)$, we can convert Equation (3.29) into a definite form as:

$$\overline{V}(t,n) = \overline{V}(T,n) - e^{\Lambda_B(t)} \int_t^T [-e^{-\Lambda_B(u)} A(u,n)]du \tag{3.30}$$

From condition (ii), we know that $\overline{V}(T,\cdot) = 0$. Substituting $\overline{V}(T,\cdot) = 0$ into Equation (3.30), we obtain:

$$\overline{V}(t,n) = e^{\Lambda_B(t)} \int_t^T e^{-\Lambda_B(u)} A(u,n)du \tag{3.31}$$

$$\overline{V}(t,n) = e^{\Lambda_B(t)} \int_t^T e^{-\Lambda_B(u)} [\lambda_B(u)\overline{V}(u,n-1) + \mathcal{GS}(u,n) + \lambda_B(u)p_B]du \tag{3.32}$$

$\overline{V}(t,n)$ can be calculated if $\overline{V}(t,n-1)$ is known. From condition (ii) we know that $\overline{V}(t,0) = 0$. Therefore, $\overline{V}(t,n)$ can be recursively determined for any (t,n) pairs starting from $\overline{V}(t,0) = 0$. The formal procedure for calculation $\overline{V}(t,n)$ is given in Theorem 3.4.1. This theorem also provides a formulation for the optimal switching time (x_n) for each possible unsold inventory level n.

Theorem 3.4.1. *For all* $1 \le n \le M$ *and* $\lambda_B(t) > \lambda_i(t)$, *for* $i = 1, 2$, *the switching-time thresholds* $\{x_n\}$ *and* $\overline{V}(t,n)$ *is recursively determined by*

$$\overline{V}(t,n) = \begin{cases} e^{\Lambda_B(t)} \int_t^T e^{-\Lambda_B(s)} A(s,n)ds & if\ t > x_n, \\ 0 & otherwise. \end{cases} \tag{3.33}$$

where

$$\begin{aligned} x_n &= \inf\{0 \le t \le T : \ e^{\Lambda_B(t)} \int_t^T e^{-\Lambda_B(s)} A(s,n)ds > 0\}, \\ x_1 &\ge x_2 \ge \cdots \ge x_n, \\ A(t,n) &= \mathcal{GS}(t,n) + \lambda_B(t)p_B + \lambda_B \overline{V}(t,n-1),\ 0 \le t \le T, \\ \overline{V}(t,0) &= 0,\ 0 \le t \le T. \end{aligned}$$

Proof of Theorem 3.4.1. The proof is similar to Duran [10]. We will prove by induction on n that $\overline{V}(t,n)$ which is determined by Theorem 3.4.1 satisfies condition (i)-(iv). Therefore we will prove that $\overline{V}(t,n)$ which is calculated by Theorem 3.4.1 is equivalent to $\widetilde{V}(t,n)$. From condition (ii) we know that when $n = 1$, $\overline{V}(t,n-1) = 0$. Also from Lemma 2 we know that $\mathcal{GS}(t,1) + \lambda_B(t)p_B$ is a non-decreasing function in t. We require that for $t \le x_1$ the following inequalities must hold:

$$A(t,1) = \mathcal{GS}(t,1) + \lambda_B(t)p_B \le \mathcal{GS}(x_1,1) + \lambda_B(x_1)p_B \le 0.$$

As we already said the first inequality is from the non-decreasing property of $\mathcal{G}S(t,1)+\lambda_B(t)p_B$ in t. The second inequality results from the fact that if $\mathcal{G}S(x_1,1)+\lambda_B(x_1)p_B > 0$, then $e^{\Lambda_B(t)}\int_t^T e^{-\Lambda_B(s)}A(s,n)ds > 0$, which contradicts the definition of x_1. Hence, for $t \le x_1$ (or $\overline{V}(t,1)=0$ by the definition of $\overline{V}$)

$$\begin{aligned}\mathcal{G}(\overline{V}+S)(t,1)+\lambda_B(t)p_B &= \frac{\partial\overline{V}(t,1)}{\partial t}+\lambda_B(t)[\overline{V}(t,0)-\overline{V}(t,1)]+\mathcal{G}S(t,1)+\lambda_B(t)p_B\\ &= \mathcal{G}S(t,1)+\lambda_B(t)p_B = A(t,1)\le 0.\end{aligned}$$

Thus, condition (iii) is satisfied when $n=1$ and $t\le x_1$ (or $\overline{V}(t,1)=0$).

When $t > x_1$ (or $\overline{V}(t,1)>0$) we have that

$$\begin{aligned}\mathcal{G}(\overline{V}+S)(t,1)+\lambda_B(t)p_B &= \frac{\partial\overline{V}(t,1)}{\partial t}+\lambda_B(t)[\overline{V}(t,0)-\overline{V}(t,1)]+\mathcal{G}S(t,1)+\lambda_B(t)p_B\\ &= \frac{\partial\overline{V}(t,1)}{\partial t}-\lambda_B(t)\overline{V}(t,1)+\mathcal{G}S(t,1)+\lambda_B(t)p_B. \qquad (3.34)\end{aligned}$$

We previously stated that applying the infinitesimal generator $\mathcal{G}$ to $\overline{V}(t,n)$, we get the following differential equation as in Equation(3.24)

$$\frac{\partial\overline{V}(t,n)}{\partial t}-\lambda_B(t)\overline{V}(t,n) = -\lambda_B(t)\overline{V}(t,n-1)-\mathcal{G}S(t,n)-\lambda_B(t)p_B, \qquad (3.35)$$

$$\frac{\partial\overline{V}(t,n)}{\partial t} = \lambda_B(t)\overline{V}(t,n)-\lambda_B(t)\overline{V}(t,n-1)-\mathcal{G}S(t,n)-\lambda_B(t)p_B, \qquad (3.36)$$

Since $\overline{V}(t,0)=0$, for $n=1$ we have:

$$\frac{\partial\overline{V}(t,1)}{\partial t} = \lambda_B(t)\overline{V}(t,1)-\mathcal{G}S(t,1)-\lambda_B(t)p_B. \qquad (3.37)$$

Substituting (3.37) into (3.34), we get $\mathcal{G}(\overline{V}+S)(t,1)+\lambda_B(t)p_B = 0$ when $\overline{V}(t,1) > 0$. Therefore, condition (iv) is satisfied when $n=1$. Moreover, we have $\overline{V}(t,1)\ge\overline{V}(t,0)=0$ by the definition of x_1 (there exists a time t such that $\overline{V}(t,1)>0$ if $x_1>0$).

Now assume that the following statements hold for $n\le k<M$: there exist k time thresholds with $T\ge x_1\ge\cdots\ge x_k\ge 0$ such that $\overline{V}(t,n)$ is derived from Equation (3.33) and satisfies conditions (i)-(iv), and the inequality $\overline{V}(t,n)\ge\overline{V}(t,n-1)$ holds for $n=1\ldots k$.

For $n = k + 1$ we have that

$$\begin{aligned} A(t, k+1) = \mathcal{G}S(t, k+1) + \lambda_B(t)p_B + \lambda_B(t)\overline{V}(t, k) &\geq \mathcal{G}S(t, k) + \lambda_B(t)p_B \\ &+ \lambda_B(t)\overline{V}(t, k-1) = A(t, k), \end{aligned}$$

since $\mathcal{G}S(t, k) + \lambda_B(t)p_B$ and $\overline{V}(t, k)$ are increasing in k by the induction assumption. This implies that

$$e^{\Lambda_B(t)} \int_t^T e^{-\Lambda_B(s)} A(s, k+1) ds \geq e^{\Lambda_B(t)} \int_t^T e^{-\Lambda_B(s)} A(s, k) ds.$$

Together with Equation (3.33), this implies $\overline{V}(t, k+1) \geq \overline{V}(t, k)$ and $x_k \geq x_{k+1}$.

For $t \leq x_{k+1}$ (or $\overline{V}(t, k+1) = 0$),

$$\begin{aligned} &\mathcal{G}(\overline{V} + S)(t, k+1) + \lambda_B(t)p_B \\ &\quad = \frac{\partial \overline{V}(t, k+1)}{\partial t} + \lambda_B(t)[\overline{V}(t, k) - \overline{V}(t, k+1)] \\ &\qquad + \mathcal{G}S(t, k+1) + \lambda_B(t)p_B \\ &\quad = \mathcal{G}S(t, k+1) + \lambda_B(t)p_B + \lambda_B(t)\overline{V}(t, k) = A(t, k+1) \\ &\quad \leq A(x_{k+1}, k+1) \leq 0. \end{aligned}$$

Note that $\overline{V}(t, k) = \overline{V}(t, k+1) = 0$ since $t \leq x_{k+1} \leq x_k$. The first inequality follows from $\mathcal{G}S(t, k+1) + \lambda_B(t)p_B$ being increasing in t, and the second inequality follows from the fact that if $A(x_{k+1}, k+1) > 0$ then this will contradict the definition of x_{k+1}. Therefore, condition (*iii*) is satisfied, when $t \leq x_{k+1}$ (or $\overline{V}(t, k+1) = 0$). For $t > x_{k+1}$ (or $\overline{V}(t, k+1) > 0$),

$$\begin{aligned} \mathcal{G}(\overline{V} &+ S)(t, k+1) + \lambda_B(t)p_B \\ &= \frac{\partial \overline{V}(t, k+1)}{\partial t} + \lambda_B(t)[\overline{V}(t, k) - \overline{V}(t, k+1)] + \mathcal{G}S(t, k+1) + \lambda_B(t)p_B \\ &= -A(t, k+1) + \lambda_B(t)\overline{V}(t, k+1) + \lambda_B(t)[\overline{V}(t, k) - \overline{V}(t, k+1)] + \mathcal{G}S(t, k+1) \\ &\quad + \lambda_B(t)p_B = 0. \end{aligned}$$

Therefore condition (*iv*) is satisfied when $t > x_{k+1}$ (or $\overline{V}(t, k+1) = 0$).

For $n = k+1$ we showed that conditions (*i*)-(*iv*) hold. Thus the function $\overline{V}(t, k)$ that is determined by the proposed procedure, is equal to $\widetilde{V}(t, k)$. Further the switching time thresholds (x_n) are monotonically non-increasing in n. ■

3.5 Details for the Approximation of the $\overline{V}$ Function

Although we have demonstrated the calculation of $\overline{V}(t,n)$ values for each (t,n) pair, we will use an approximation using discrete time intervals for computational study. The details of the approximation using discrete time intervals are given below. For any $1 \le n \le M$, and $x_n < t < T$ with some $\delta > 0$ such that $t + \delta \le T$, we have

$$\begin{aligned}
\overline{V}(t,n) &= e^{\Lambda_B(t)} \int_t^T e^{-\Lambda_B(u)} A(u,n)du \\
&= e^{\Lambda_B(t)} \int_{t+\delta}^T e^{-\Lambda_B(u)} A(u,n)du + e^{\Lambda_B(t)} \int_t^{t+\delta} e^{-\Lambda_B(u)} A(u,n)du \\
&= e^{-(\Lambda_B(t+\delta)-\Lambda_B(t))}[e^{\Lambda_B(t+\delta)} \int_{t+\delta}^T e^{-\Lambda_B(u)} A(u,n)du] + e^{\Lambda_B(t)} \int_t^{t+\delta} e^{-\Lambda_B(u)} A(u,n)du \\
&\cong e^{-(\Lambda_B(t+\delta)-\Lambda_B(t))}\overline{V}(t+\delta,n) + \int_t^{t+\delta} e^{-(\Lambda_B(u)-\Lambda_B(t))} A(u,n)du
\end{aligned}$$

$$\begin{aligned}
\overline{V}(t,n) &\cong e^{-(\Lambda_B(t+\delta)-\Lambda_B(t))}\overline{V}(t+\delta,n) \\
&+ \int_t^{t+\delta} e^{-(\Lambda_B(u)-\Lambda_B(t))}[\mathcal{G}S(u,n) + \lambda_B(t)p_B + \lambda_B(t)\overline{V}(u,n-1)]du
\end{aligned}$$

$$\begin{aligned}
\overline{V}(t,n) &\cong e^{-(\Lambda_B(t+\delta)-\Lambda_B(t))}\overline{V}(t+\delta,n) + \int_t^{t+\delta} e^{-(\Lambda_B(u)-\Lambda_B(t))}[\frac{(t,\ n)}{\partial t} + \lambda_B(t)[S(t,n-1) \\
&- S(t,n)] + \lambda_B(t)p_B + \lambda_B(t)\overline{V}(u,n-1)]du
\end{aligned}$$

For a small time interval from t to $t+\delta$, we may take $S(t,n)$,$\lambda_B(t)p_B$ and $\overline{V}(t,n)$ as constants, defining the function $\theta(t,\delta) = e^{-(\Lambda_B(t+\delta)+\Lambda_B(t))}$

$$\begin{aligned}
\overline{V}(t,n) &\cong \overline{V}(t+\delta,n)\theta(t,\delta) + (1-\theta(t,\delta))p_B + (1-\theta(t,\delta))\overline{V}(t,n-1) \\
&+ (1-\theta(t,\delta))[S(t,n-1) - S(t,n)] + \theta(t,\delta)[S(t+\delta,n) - S(t,n)].
\end{aligned}$$

Therefore $\overline{V}(t,n)$ can be estimated by

$$\overline{V}(t,n) \cong (\overline{V}+S)(t+\delta,n)\theta(t,\delta) + (1-\theta(t,\delta))[p_B + (\overline{V}+S)(t,n-1)] - S(t,n).$$

If the selling horizon T is divided into a large number K of intervals of length δ, we obtain

$$\overline{V}(k\delta,n) \cong (\overline{V}+S)((k+1)\delta,n)\theta(t,\delta) + (1-\theta(t,\delta))[p_B + (\overline{V}+S)(k\delta,n-1)] - S(k\delta,n).$$

3.6 Usage of Optimal Switching Thresholds

We have demonstrated in our analysis that for any remaining inventory level $n \in M$, there exists a corresponding optimal switch time x_n . If the unsold inventory level n is reached at time t where $t > x_n$, delaying the switch is optimal since there is an additional revenue from delaying the switch to a later time, that is, $\overline{V}(t, n) > 0$ for $t > x_n$. Intuition behind this is as follows: when there is n remaining inventory, singles can only sell out this n items of inventory in more than $T - x_n$ times. Therefore, there is not enough time for singles to sell out the remaining inventory when $t > x_n$ since $T - t < T - x_n$ when $t > x_n$. Conversely, If the unsold inventory level n is reached at time t where $t \leq x_n$, switching immediately is optimal since there is no additional revenue from delaying the switch to a later time, that is, $\overline{V}(t, n) = 0$ for $t \leq x_n$. A more detailed interpretation of the meaning of switch-time thresholds and their behaviors under different demand rates schemes are discussed in the numerical study section.

3.7 Extension to $n > 2$ Events

Although we specifically focused on the case where there are only two events in the performance season in our work, number of events in a season can be extended to more than two. However, this extension complicates the model and presents several problems to be addressed. First of all, in our analysis we assumed that selling period ends with the start of the first event. However, this assumption become harder to justify in the case where there are more than two events since the time period between the first event and the last event gets larger demand for upcoming events matters. Moreover, scheduling becomes another important factor to be considered. Consider a sequence of events which will be performed in a season. Most of the time, some events are high-demand events compared to the others and specific order of those high-demand games in the sequence effects the optimal solution. Therefore, scheduling should also be included in the model, which complicates the mathematics and algorithm. Another important problem to be considered in the case where there are more than two events in a performance period is about bundling. Particularly, which events to be included in the bundle and possibility of multiple bundle that consist of mini bundles should be considered in the case where there are more than two events in a season.

CHAPTER 4

NUMERICAL STUDY

In the previous section we developed a procedure to find the optimal time to switch from bundle ticket sales to single ticket sale. In this section we will present several computational analysis regarding different situations. First, we will study the structure of optimal switching times for different demand rates schemas and then compare the calculated optimal switching times for various demand rate schemas with the case demand rate is assumed to be constant. By this way intuition behind the procedure can be realized more clearly. After this, we will study percentage improvement on revenue over constant demand rate assumption case for different demand rates schemes. Thus, the value of recognizing the fact that demand rates depend on time and utilizing the time dependent demand rates in switching time calculation will be observed.

4.1 Optimal Switching Times and Their Behavior for Different Demand Rates

In order to calculate optimal switching times, we will use the approximation shown in Chapter 3. Then, $\overline{V}$ and optimal switching times can be calculated using the following algorithm:

Algorithm

$$\begin{aligned} Let,\ \Delta A(k\delta, n) &= (\overline{V} + S)((k+1)\delta, n)\theta(t,\delta) \\ &\quad +(1-\theta(t,\delta))[p_B + (\overline{V} + S)(k\delta, n-1)] - S(k\delta, n). \end{aligned}$$

where $\theta(t,\delta) = e^{-(\Lambda_B(t+\delta)+\Lambda_B(t))}$

- **Step 0**: Initialize $\overline{V}(T,\cdot) = \overline{V}(K\delta,\cdot) = 0$ for all inventory levels. Set $n = 1$ and

$k = (K-1)$.

- **Step 1**: Calculate $\Delta A(k\delta, n)$.
- **Step 2**: Set $\overline{V}(k\delta, n) = (\Delta A(k\delta, n))^+$ and $k = k-1$.
  - if $k \neq -1$ and $\overline{V}(k\delta, n) \geq 0$, go to Step 1;
  - otherwise set $\overline{V}(j\delta, n) = 0$ for all $j < k-1$ and $n = n+1$.

For numerical studies we will consider two events for a season which will be held in a 120-ticket stadium. One game will be a high-demand game while the other will be a low-demand. We will also consider a selling horizon of 2 months. The price charged for a high-demand game seat will be \$200 and the price charged for a low-demand game will be \$50, constant over the selling horizon. However, if the seats are to be sold as bundles with one high-demand and one low-demand seat, the price will be \$220. We will study different demand rate schemes, i.e. we will study linearly decreasing demand rates, linearly increasing demand rates and their various combinations as well. Regardless of their behavior, arithmetic average of demand rates for all cases will be taken equal for two-month selling period. As the constant demand rate assumption case, we will consider 30 customers per month for high-demand game, 25 customers per month for low-demand game and 70 customers per month for bundle tickets on average. For example, if we are to investigate linearly increasing demand rate for high-demand game with an increase rate of 10, we will use $\lambda_H = 20 + 10t$. In such a situation, demand rate will start from 20 customers per month and will reach 40 customers per month linearly at the end of the selling horizon and provides 30 customers per month on average. Similarly, if we are to investigate linearly decreasing demand rate for bundle tickets with a decrease rate of 10, we will use $\lambda_B = 80 - 10t$. In this situation, demand rate will start from 80 customers per month and will drop to 60 customers per month linearly at the end of the selling horizon and provides 70 customers per month on average.

Initially, we will calculate switching thresholds assuming that demand rates are constant for all tickets as in Duran [10] using $\lambda_B = 70$, $\lambda_H = 30$, $\lambda_L = 25$ and show the mechanics and usage of switching threshold times. Then we will examine different demand rate schemes and their comparisons with constant demand rates assumptions.

Optimal switching times for constant demand rates assumption are provided in Table (4.1).

Table 4.1: Selected Optimal Switching Times when $\lambda_B = 70$, $\lambda_H = 30$ and $\lambda_L = 25$.

sales	77	76	75	74	73	72	71	70	69
remained seats (n)	43	44	45	46	47	48	49	50	51
switch time (x_n)	0.272	0.232	0.196	0.156	0.120	0.084	0.044	0.01	0

For maximum profitability, we want to sell as much singles as possible because total price for singles are higher than bundle price. However, we also have to sell some bundles in order to completely sell out the inventory since demands only for singles themselves do not match the capacity. Our ultimate procedure dynamically finds the optimal time to switch from bundle ticket sales to single ticket sales according to the system state (n, t) in such a way that possible maximum singles are sold while some portion of capacity is initially sold as bundle in order to completely sell out the inventory. Consider the situation that 76 bundle tickets has already been sold. Therefore 44 seats remained as unsold inventory. For 44 unsold inventories, optimal

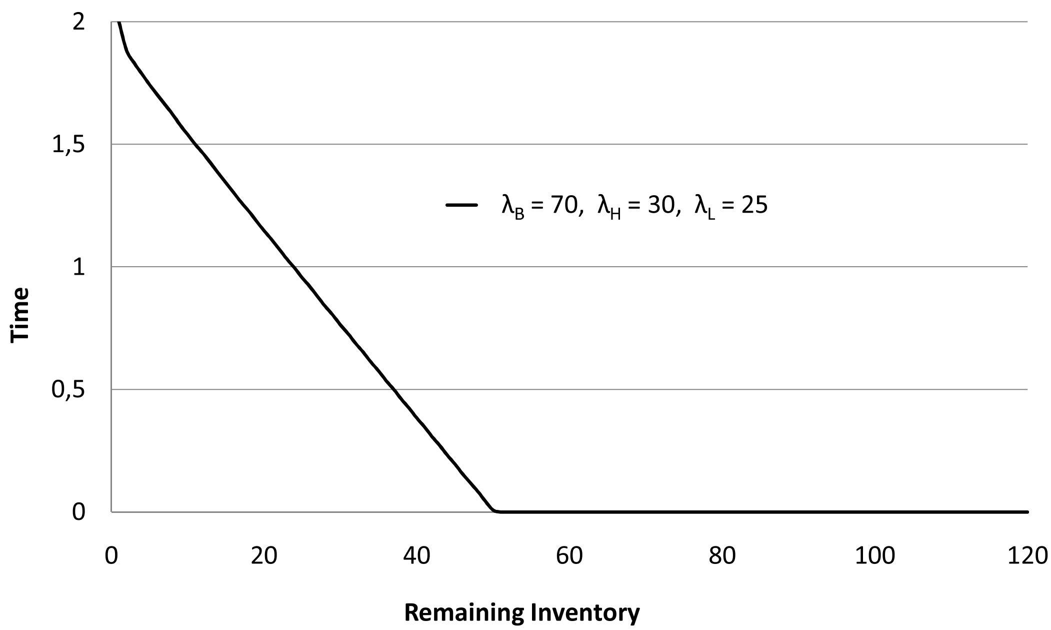

Figure 4.1: Optimal Switching Times when $\lambda_B = 70$, $\lambda_H = 30$ and $\lambda_L = 25$

switching time is calculated as 0.232 months.If the team sold 76 seats before time 0.232 ($t < x_{44}$), it is optimal to switch before 77^{th} customer arrive since remaining inventory can be completely sold out within remaining time by single ticket sales. In other words, there is no expected revenue from delaying the switch further ($\overline{V}(t, 44) = 0$).

All optimal threshold values are plotted in Figure 4.1. If the system state (t, n) falls below the threshold line then it is optimal to switch because there is possible expected revenue in delaying. Likewise, if the system state (t, n) is above the threshold line, it is not optimal to switch. For some remaining inventory level, for example 51, the optimal switching time is calculated as 0. Zero switching time until 69 seats are sold means that switching should not be considered before 69^{th} sale because it is not possible to sell out the remaining inventory by selling only singles.
We have stated the interpretation and usage of optimal switching times. However, demand rates may not be constant most of the times throughout the selling period in real life. Now, we will examine different demand rate schemes and compare them with constant demand rates assumption. Although there are 9 possible demand rates schemes and requirements of the theorem brings additional restriction on the cases to be considered, we will investigate the following six most probable demand rate schemes which have monotonic threshold values.

Case 1: All decreasing demand rates.
First demand rates scheme is "all decreasing demand rates" case. For some situations, demand rates for both games and for bundle ticket as well may decrease due to timing of the pricing, remained bad seats etc. In order to see the effect of the rate of change, two decreasing demand rates schemes with different slopes are investigated. Following two demand rates schemes are used:

- $\lambda_B = 80 - 10t$, $\lambda_H = 40 - 10t$, $\lambda_L = 30 - 5t$,
- $\lambda_B = 90 - 20t$, $\lambda_H = 50 - 20t$, $\lambda_L = 35 - 10t$.

Optimal switching thresholds for these two demand rate schemes and for constant demand rates are plotted in Figure 4.2.

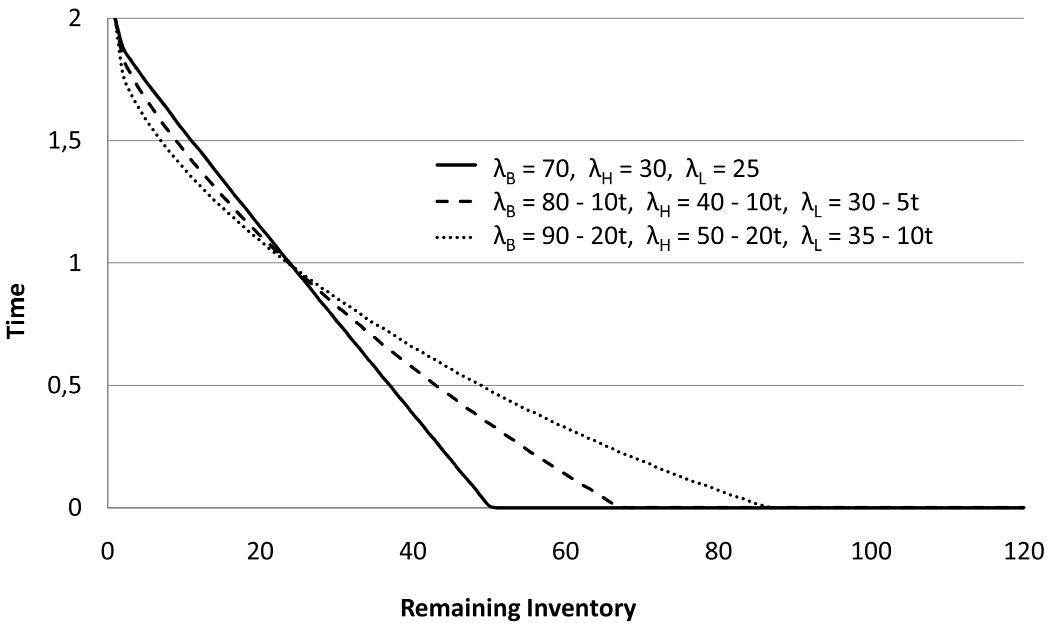

Figure 4.2: Optimal Switching Times for all Decreasing Demand Rates

In this case, all demand rates start from high values and decreases linearly as the time passes. At early times in the selling period, demand rates for both bundles and singles are relatively higher compared to the constant demand rates assumption case. Thus more customers for both bundle and single tickets arrive in early times compared to the constant demand rates case. Therefore, the sales portion that is intended to be sold as bundles (this portion is also dynamically changing according to system state (t, n)) to completely sell out the inventory are realized earlier. Consequently switching occurs more frequently at early times in the selling period. We see in the graph that areas under the threshold curves calculated for decreasing demand rate cases are greater than the area under the threshold line calculated for constant demand rate case. This means that switching occurs more frequently in these areas. Therefore, it is obvious that switching thresholds are underestimated at early times in the selling horizon if the decreasing demand rates are failed to be recognized. Moreover as the time goes by; all demand rates decrease, average inter-arrival times between events increase, that is, time is not in favor of single ticket sales. Therefore switching tends to become less frequent as time passes. This is the reason why the curve is concave up because deceasing demand rates bend the curves downwards. At later times throughout the selling period, demand rates for both bundle and single tickets are relatively lower compared to the constant demand rate case. Thus switching occurs less frequently in later times. That is why threshold curves for decreasing demands fall below the threshold line for constant demand assumption case. Accordingly, we may say that switching thresholds are overestimated at later times in the selling period again if the decreasing demand rate fact is not considered.

Case 2: All increasing demand rates
Our second case is all increasing demand rates case. On contrary to all decreasing demand rates case, for some situations, demand rates for both games and for bundles as well may increase due to timing of the events, advertisement policy etc. In order to see the effect of the rate of change, two increasing demand rates schemes with different slopes are investigated. Following two demand rates schemes are used:

- $\lambda_B = 60 + 10t$, $\lambda_H = 25 + 5t$, $\lambda_L = 20 + 5t$,
- $\lambda_B = 50 + 20t$, $\lambda_H = 20 + 10t$, $\lambda_L = 15 + 10t$.

Optimal switching thresholds for these two demand rate schemes and for constant demand rates are plotted in Figure 4.3.

In this case, demand rates start from low values and increases linearly as the time passes. On the contrary to all decreasing demand rates case, demand rates are relatively lower at early times in the selling period and relatively higher at later times compared to demand rates in constant demand rates assumption case. At early times, switching is less frequent than what is suggested by constant demand rates assumption. Thus, we can say that switching thresholds are overestimated in constant demand rates assumption at early times in the selling period. Demand rates increase with increasing time and so, switching tends to become more frequent, which is the reason why the curve is concave down. At later times demand rates are relatively greater and accordingly, switching is more frequent than what is suggested in constant demand rates assumption. Therefore, switching thresholds are underestimated by assuming constant demand rates at later times in the selling period if the demand rates are incorrectly considered to be constant.

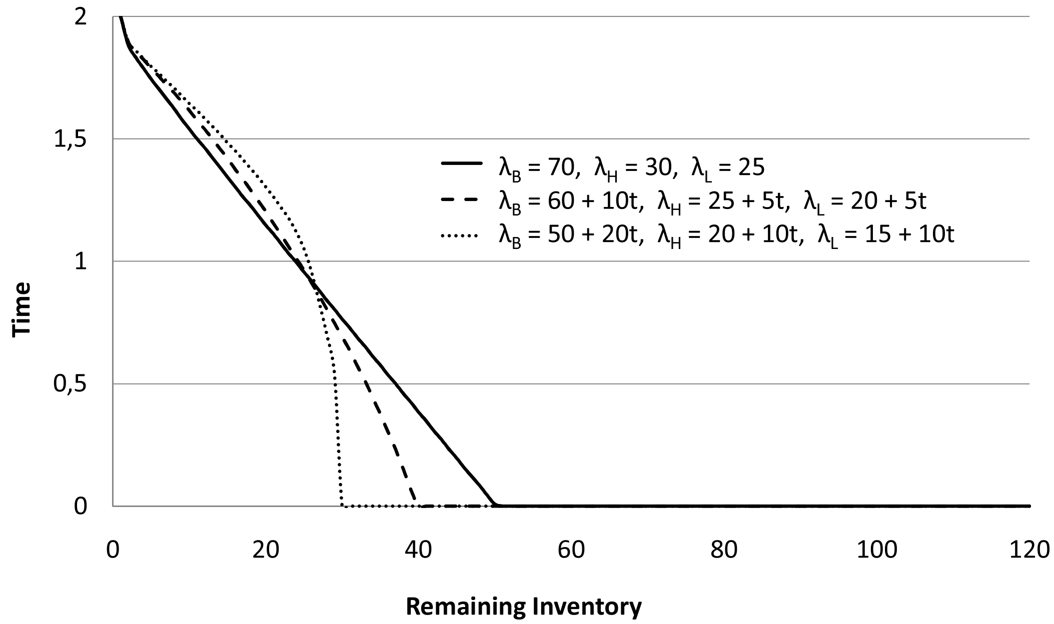

Figure 4.3: Optimal Switching Times for all Increasing Demand Rates

Case 3: Decreasing λ_B and λ_H, increasing λ_L

In some situations, demand rates for high-demand game may decrease due to high price, bad seats left etc. while demand rate for low-demand game may increase due to low price, time of the event etc. In such a situation demand rate for bundle also decreases but not as sharp as demand rates for high-demand game. In order to see the effect of the rate of change, two different schemes with different slopes are investigated. Following two demand rate schemes are used in this case:

- $\lambda_B = 75 - 5t$, $\lambda_H = 40 - 10t$, $\lambda_L = 20 + 5t$,
- $\lambda_B = 80 - 10t$, $\lambda_H = 50 - 20t$, $\lambda_L = 15 + 10t$.

Optimal switching thresholds for these two demand rate schemes and for constant demand rates are plotted in Figure 4.4. In this case λ_B and λ_H starts from high values and decreases linearly, but λ_L stars from a low value and increases linearly as the time goes by. For early times in the selling period, λ_B and λ_H are relatively higher

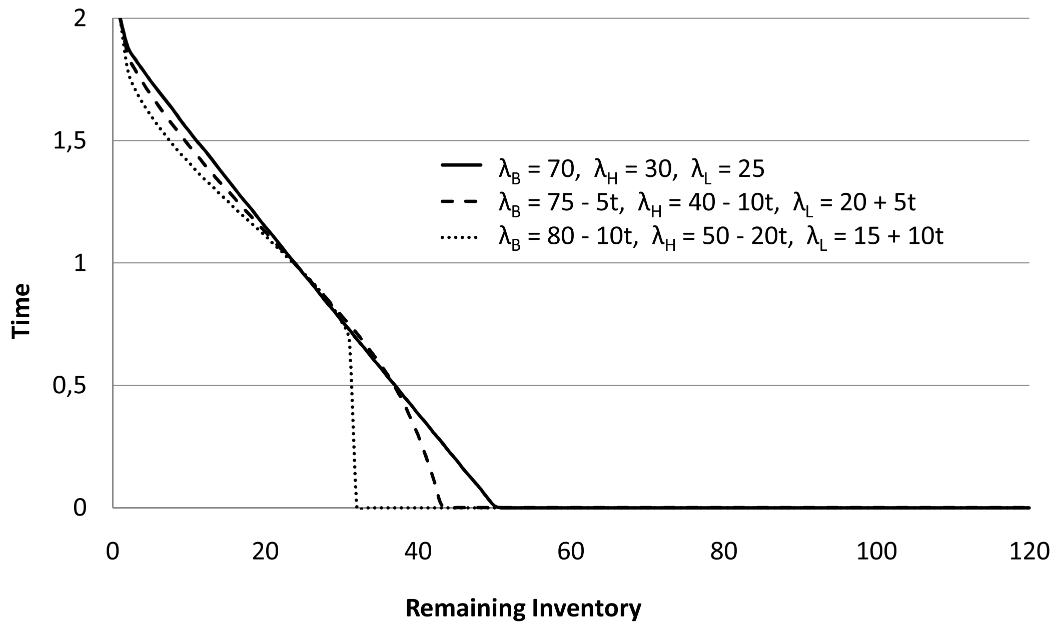

Figure 4.4: Optimal Switching Times for Decreasing λ_B and λ_H, Increasing λ_L Case

and λ_L is relatively lower at early limes compared to the constant demand assumption case. Hence more customers for both bundle and high-demand game tickets and fewer customers for low-demand game tickets arrive in early times compared to the constant demand rates case. Although we may expect more frequent switching at early times since λ_B and λ_H are relatively higher, switching times are smaller in early times due to the relatively lower demand for λ_L and we may say that λ_L is the critical demand rate at early times in the selling horizon. Here λ_L happens to be a limiting factor for switching because there is a potential risk for not being able to sell out all low-demand game tickets in the case of early switching. Similarly, at higher times λ_B and λ_H are relatively lower and thus, threshold curve behaves like the threshold curves in all decreasing demand rates case although λ_L is relatively higher. At this point we can conclude that critical demand rate has a considerable effect on the shape of the switching time thresholds curve.

In this case the threshold curves always fall under the threshold line which is determined by constant demand rates assumption. Therefore, switching should be less frequent than what is suggested by constant demand rates assumption both at early and later times. Consequently, it is clear that switching thresholds are overestimated by constant demand rates assumption.

Case 4: Increasing λ_B and λ_H, decreasing λ_L

In contrast to Case 3, in some cases, demand rates for both bundle and high-demand game may increase in time while demand rate for low-demand game may decrease in time. Again, two different schemes with different slopes are used as follows:

- $\lambda_B = 65 + 5t$, $\lambda_H = 25 + 5t$, $\lambda_L = 35 - 10t$,
- $\lambda_B = 60 + 10t$, $\lambda_H = 20 + 10t$, $\lambda_L = 45 - 20t$.

Optimal switching thresholds for these two demand rate schemes and for constant demand rates are plotted in Figure 4.5. In this case, we see the dominant effect of

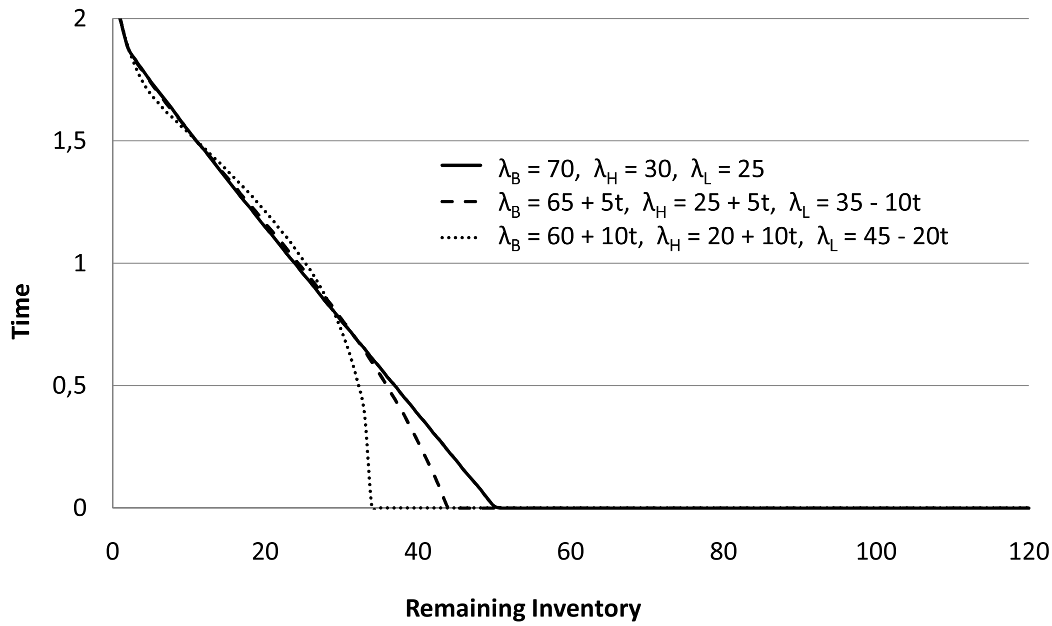

Figure 4.5: Optimal Switching Times for Increasing λ_B and λ_H, Decreasing λ_L Case

low values of λ_B and λ_H at early times and dominant effect of low value of λ_L at later times. At middle times, high values of demand makes the switching slightly more frequent. For such a case, optimal switching thresholds are overestimated both at early and later times and slightly underestimated at middle times by constant demand rates assumption.

Case 5: Constant λ_B, decreasing λ_H and increasing λ_L.
It is possible for some cases that demand for bundle tickets may be constant while demands for individual games may decrease in time. Again, two different schemes with different slopes are used as follows:

- $\lambda_B = 70$, $\lambda_H = 40 - 10t$, $\lambda_L = 15 + 10t$,
- $\lambda_B = 70$, $\lambda_H = 50 - 20t$, $\lambda_L = 5 + 20t$.

Optimal switching thresholds for these two demand rate schemes and for constant demand rates are plotted in Figure 4.6.

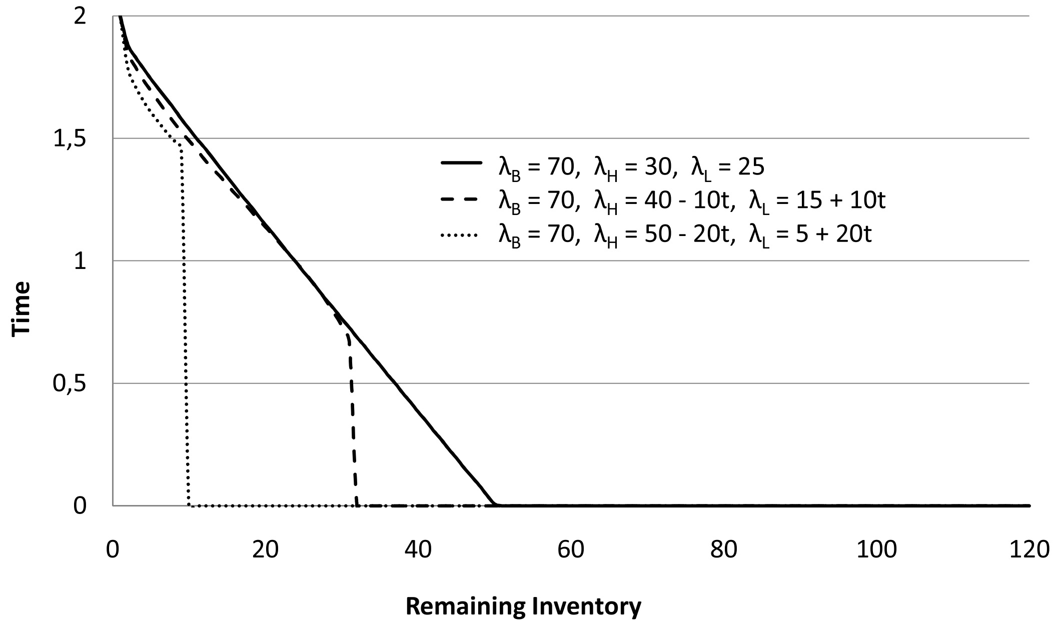

Figure 4.6: Optimal Switching Times for Constant λ_B, Decreasing λ_H and Increasing λ_L

In this case λ_B is constant throughout the selling horizon. Thus λ_B has no considerable effect on the shape of the switching threshold curves. Threshold values are strongly affected by the behavior of λ_H and λ_L. At early times in the selling period, switching is much less frequent than what is suggested by constant demand rates assumption due to very low values of λ_L. At later times, switching is again less frequent but not that much as at early times. Accordingly, threshold values are significantly underestimated at early times and moderately underestimated in later times when constant demand rates are assumed for such a demand rate profile.

Case 6: Increasing λ_B constant λ_H and λ_L

For some cases, demand rate for bundle may increase in time due to its lower price, loyalty, or other advantageous provided to bundle-ticket-buyers etc. while demand rates for single tickets may remain constant in time. Nevertheless it may be a rare case, it is beneficial to investigate this case in order to understand the behavior of optimal switching thresholds.

- $\lambda_B = 60 + 10t$, $\lambda_H = 30$, $\lambda_L = 25$,
- $\lambda_B = 50 + 20t$, $\lambda_H = 30$, $\lambda_L = 28$.

Optimal switching thresholds for these two demand rate schemes and for constant demand rates are plotted in Figure 4.7. In this case, λ_H and λ_L are constant throughout the selling season but λ_B is increasing in time. Since switching is mainly related with the amount of singles that can be sold after the switch, switching threshold curve is similar to the threshold line for all constant demand rates assumption case

After evaluating different demand rate schemes, it can be concluded that switching thresholds are mainly effected by the behavior of demand rates for single tickets. Moreover, when demand rates for singles are higher than their constant counterparts, switching occur more frequently whereas it tends to be less frequent when single demand rates are lower. Also, critical demand rate limits the behavior of the threshold curve as explained in Case 3. Furthermore note that the difference between threshold curves calculated for time-dependent demand rates and the threshold line calculated for constant demand rates is changing with time. This means that effect of constant demand rates assumption may vary according to the realization path. Also, we see that all threshold values intersect at the same point in each demand rate scheme.

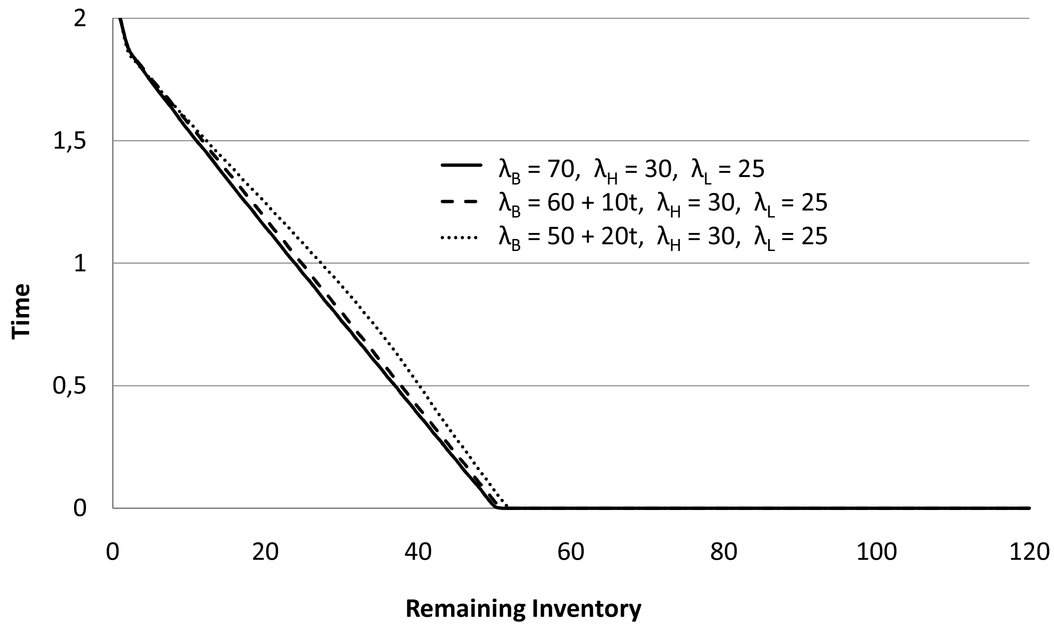

Figure 4.7: Optimal Switching Times for Increasing λ_B Constant λ_H and λ_L Case

4.2 Simulation studies

Impact of the dynamic switching times calculated with constant demand rates assumption on revenue improvement is demonstrated by a numerical experiment in Duran [10]. Average revenue from dynamic switching policy is compared with the average revenues from two possible industry practices in which switching times are determined before selling season begins. The first comparison is with a naive approach in which the middle point of the selling horizon is selected as the switching time. The second comparison is with the optimal approach, where the best static switch time is selected using enumeration over all possible switching times using Equation 3.1. In order to see the impact of different demand realizations, comparisons of dynamic switching policy with static ones are made over ten different scenarios. For each scenario, 100 random sample paths for the bundle and single ticket customers are generated. The results are summarized in Figure 4.8

The percentage improvement on revenue from dynamic switching over the static case can be between 2.5-4% when the static switching time is arbitrarily selected to be

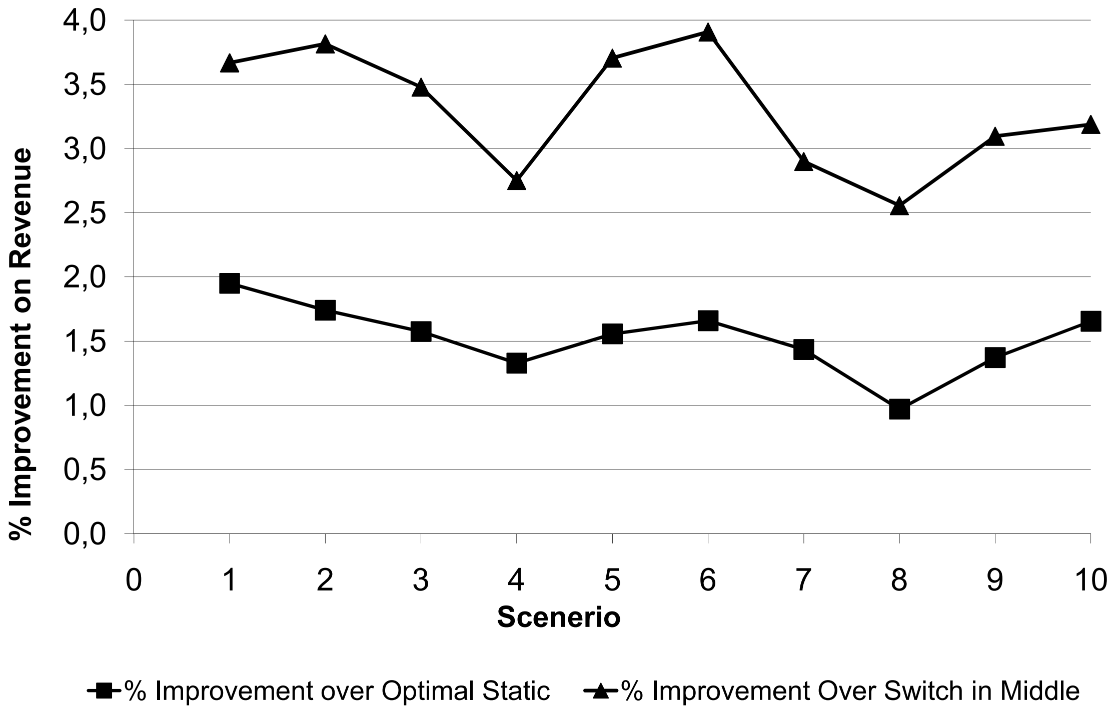

Figure 4.8: % Improvement over Static when Constant Demand Rates are Assumed

the middle point of the selling period (i.e., 1 month) or 1-2% when the optimal static switch time is used.

Now we will investigate the first five time-dependent demand rate cases and find the percentage improvement over constant demand rates assumption case. In order to do that, we will generate two switching threshold sets for each case; one with time dependent demand rates and one with constant demand rates. Then we will simulate the sale horizon. For simulation, we will basically investigate what would the revenue be if we used the switching thresholds which are determined by constant demand rate assumption instead of using threshold values based on time dependent demand rates while customers are arriving according to the time-dependent rates. To do that, we write a program in Java and use Intel Core i3 2.53 Ghz processer. Also, we use predefined Java random number generator for our analysis. In each iteration, we will generate a random inter-arrival sets for both bundle and single tickets. First, we will calculate expected revenues using switching threshold values which are determined for time dependent demand rates. Then, we will calculate the expected revenue using the same random inter arrival set but utilizing the switching threshold values which are determined for constant demand rate assumption. Lastly, we will find the % improvement comparing these two revenue values and continue with next iteration. In this way, we guarantee that same random realization path that assumes time-dependent customer arrivals is taken for both case in each iteration, which makes the comparison more reliable. For each case which are evaluated in Section 4.1 we will make ten thousand replication using random inter arrivals. The result is summarized in Figure 4.9

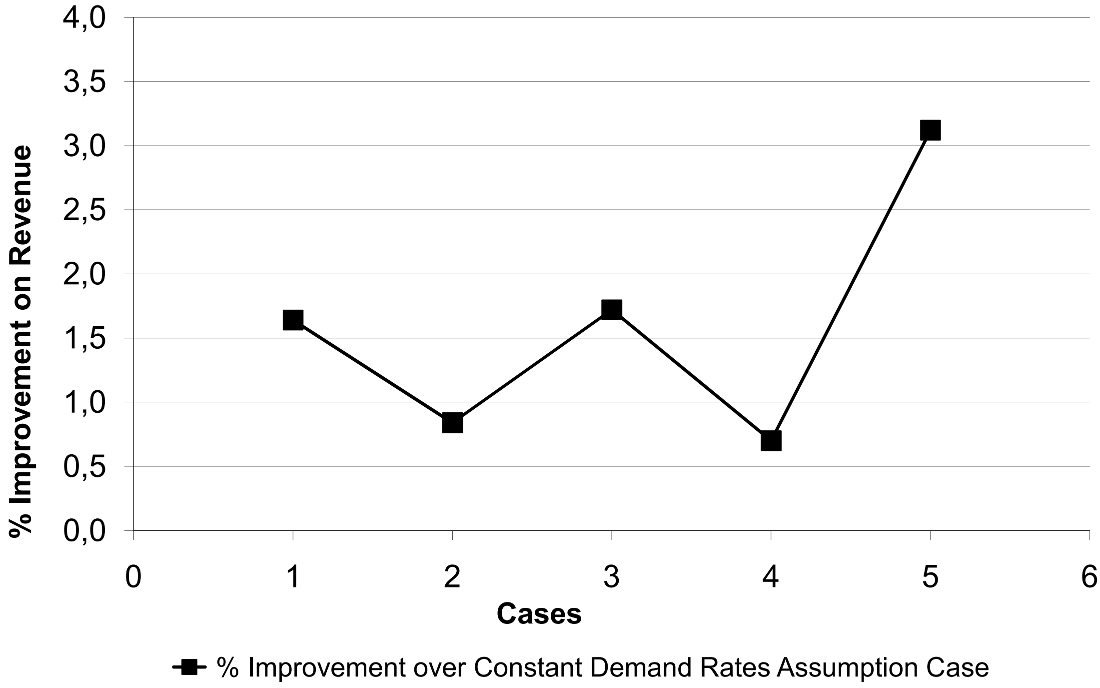

Figure 4.9: % Improvement over Constant Demand Rates Assumption Case

We find that revenue gain is between 0.8% and 2.5% varying according to demand rate schemes. Also remember that this gain is over the optimal revenue which is calculated using constant demand rates which is also a dynamic switching method. Moreover, we can conjecture due to the results in Duran [10] that % improvement on expected revenue over optimal static practice will be 3 - 6 % when dynamic switching with time dependent demand rates policy is utilized. Also note that this improvement is the minimum improvement because comparison is made with the optimal static case. Realization path is another factor that has an impact on revenue improvement.

According to the figures in Section 4.1, difference between switching threshold profiles are greater at early times and at later times. Therefore, we can expect higher improvements for different early demand realizations. As an example for such a situation, the case when 40 early demand is realized within 0.5 months is evaluated. The result are presented in Figure 4.10.

Figure 4.10: % Improvement on Revenue when 40 Early Demand is Realized within 0.5 Months

In this case % improvement over constant demand rates assumption is greater for the first four cases due to the greater difference between switching times at earlier times in the selling period. However for the 5^{th} case it is smaller because in 5^{th} case there is a problem of selling out the event and that is the reason why switching is much less frequent at early times. Another important factor that has a considerable effect on revenue improvement is the difference between single ticket prices and bundled ticket price, $p_1 + p_2 - p_B$. For our base case, price difference is $p_1 + p_2 - p_B = \$200 + \$50 - \$220 = \$30$. Keeping all system parameters same as in our base case, we can see the effect of price difference by changing bundle ticket price p_B. Percentage improvement for each cases for bundle ticket price $p_B = \$210$, bundle ticket price $p_B = \$230$ and its comparison with the base case where bundle ticket price $p_B =$

$220 are presented in Figure 4.11. We see that percentage improvement on revenue is greater for larger difference between single ticket prices and bundle ticket price. Similarly, percentage improvement on revenue is smaller for smaller price difference between bundled tickets and single tickets.

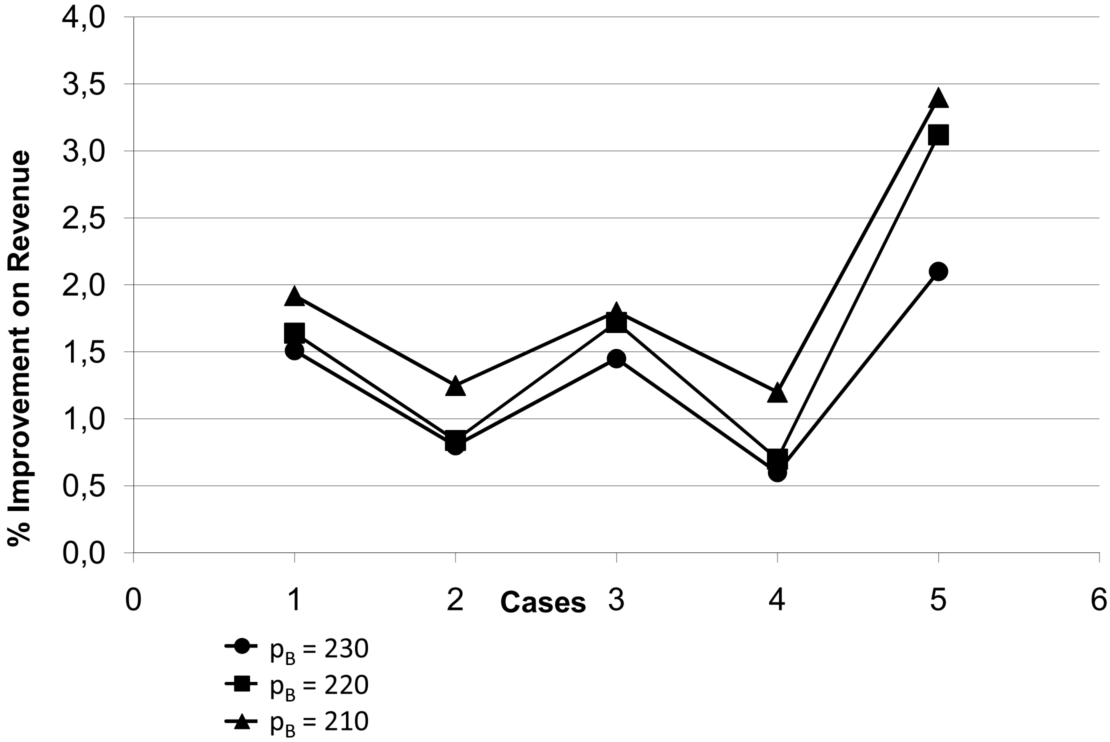

Figure 4.11: % Improvement on Revenue for Different Bundle Prices

CHAPTER 5

CONCLUSIONS

Revenue management has been recognized as one of the most flourishing application of Operation Research (OR). Most of the revenue management practices deal with the allocation of the right inventory to the right market segment at the right time for an optimal price in order to maximize profitability. Therefore, revenue management can be applied to every industry where number of fundamental decisions about inventory, timing, pricing, etc. is needed although it has gained mostly its reputation upon successful application in airline industry.

In this study, we have worked on the application of revenue management in Sports & Entertainment industry, which is a fairly new practice. We have studied the problem of switching from bundle tickets to single tickets and particularly comparison of the case where time dependent demand rates are utilized to the case where demand rates are assumed to be constant while time-dependent customer arrivals are realized. Major characteristics of the problem incorporate that a limited capacity is shared between bundle ticket buyers and single ticket buyers. The capacity allocation is made according to a switch-time after which bundle split into multiple simultaneous Non-homogenous Poisson processes with time-dependent demand rates for singles.

In Chapter 3, we showed that for any inventory level n, there exist a time x_n after which switching from bundle ticket sales to single ticket sales is not optimal. Therefore, the optimal switch times constitute a set of thresholds defined by the remaining inventory and the time spent in the selling period. Usage of the policy is as follows: After each sale, the current time is compared to the time threshold for the corresponding remaining inventory to determine if the switch should be made immediately or not. Moreover, we have illustrated that threshold are decreasing in remaining inventory n and monotonically increasing in t as suggested in Duran [10].

In Chapter 4, we have worked on several numerical experiments considering different situations. Initially, we have studied the structure of optimal switching times for various demand rates schemes in six cases and compared them with the case where demand rates were assumed to be constant. We illustrated that when time-dependent demand rates are used; switching thresholds form a curve rather than being a line which is suggested by constant demand rates assumption due to varying characteristics of demand rates in time. We have also demonstrated that when demand rates are higher than their constant demand rates counterpart in a specific time interval, threshold values are underestimated by constant demand rates assumption in that specific time interval as in Case 1 at early times in the selling period. Likewise, we have demonstrated that when demand rates are lower than their constant demand rates counterpart in a specific time interval, threshold values are overestimated by constant demand rates assumption in that specific time interval as in Case 2 at early times in the selling period. We can generalize this as: if the demand rates are incorrectly considered lower than their actual values, switching threshold would be overestimated and if the demand rates are incorrectly considered higher than their actual values, switching thresholds would be underestimated. Furthermore, it is possible that a demand rate may drop to critical values and that demand rate turns out to be a critical demand rate. Behavior of the switching thresholds may be limited by the critical demand as in Case 3. In addition, behavior of the threshold curve is strongly dependent on the behavior of demand rates for singles as shown in Case 6.

In simulation studies, we have expressed the value of utilizing the dynamic switching threshold policy by calculating % improvement on revenue. We have found that revenue gain can be between 0.8% and 2.5% (varying according to demand rate schemes) over constant demand rate assumption. Along with the results of Duran [10], we conjuncture that % improvement on revenue over static switch may be 3 - 6 % when dynamic switching with time dependent demand rates policy is utilized. Finally, we have illustrated that usage of time-dependent demand rates instead of assuming that they are constant can be more beneficial in the case of different demand realizations.

In this work, we have studied the dynamic switching problem considering two-event for a season. It would be an interesting study to evaluate the cases where there are more than two events for a season. In such a case, the decision of which events to include into the bundle is also an interesting question (Yakıcı [34]). Moreover, mini-bundle and bundle combinations can be offered to the customers and timing of such products is another interesting problem to work on.

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