# EQUIVARIANT VECTOR FIELDS ON THREE DIMENSIONAL REPRESENTATION SPHERES

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# ABSTRACT

EQUIVARIANT VECTOR FIELDS ON THREE DIMENSIONAL REPRESENTATION SPHERES

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Let $G$ be a finite group and $V$ be an orthogonal four-dimensional real representation space of $G$ where the action of $G$ is non-free. We give necessary and sufficient conditions for the existence of a $G$-equivariant vector field on the representation sphere of $V$ in the cases $G$ is the dihedral group, the generalized quaternion group and the semi-dihedral group in terms of decomposition of $V$ into irreducible representations. In the case $G$ is abelian, where the solution is already known, we give a more elementary solution.

Keywords: Equivariant Vector Field, Strong Euler Characteristic, Equivariant Euler Characteristic, Dihedral Group

# ÖZ

ÜÇ BOYUTLU TEMSİL KÜRELERİNDE EKUVARYANT VEKTÖR ALANLARI

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$G$ sonlu bir grup, $V$ ise $G$'nin etkisinin serbest olmadığı $G$'nin dört-boyutlu ortogonal gerçel bir temsil uzayı olsun. $G$'nin dihedral grup, genelleştirilmiş kuaterniyon grup ve semi-dihedral grup olması durumlarında, $V$'nin temsil küresinin üstünde $G$-ekuvaryant bir vektör alanı var olması için gerekli ve yeterli koşulları $V$'nin indirgenemez temsillere ayrışımı cinsinden veriyoruz. Ayrıca çözümün bilindiği $G$'nin abeliyen olması halinde, basit alternatif bir çözüm de sunmaktayız.

Anahtar Kelimeler: Ekuvaryant Vektör Alanı, Kuvvetli Euler Karakteristiği, Ekuvaryant Euler Karakteristiği, Dihedral Grup

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# CHAPTER 1

# INTRODUCTION

Let $G$ be a topological group. If we have a $G$-action on a manifold $M$, then we have an induced $G$-action on the tangent bundle $TM$ of $M$. Thus, we can consider vector fields which are $G$-equivariant as sections of $TM$. We call them $G$-equivariant vector fields or simply $G$-fields.

Let $G$ be a compact Lie group and let $S(V)$ be the unit sphere of a real orthogonal representation space $V$ of $G$. Then what can we say about the existence of nonzero $G$-fields on $S(V)$? Clearly, the number of linearly independent nonzero vector fields is always greater than or equal to the number of linearly independent nonzero $G$-fields. Consequently, if there is no nonzero vector field on $S(V)$, then there is no nonzero $G$-field on $S(V)$.

If $G$ acts trivially on $V$, then the existence problem for $G$-fields corresponds to the existence problem for ordinary vector fields. It was solved by the following theorem due to Hopf [9] in 1926.

**Theorem 1.0.1 (Poincare-Hopf, [13])** *Let $M$ be a smooth compact manifold and $v$ a smooth vector field on $M$. If $M$ has a boundary $\partial M$, then we require that $v$ points outward at all boundary points. Then the sum of the indexes of the zeros of $v$ is equal to the Euler characteristic of $M$.*

Thus, there is a nonzero vector field on $S^n$ if and only if $n$ is odd. Hopf [10] and Eckmann [7] proved that if $n = (2a+1)2^{c+4d}$ and $0 \leq c \leq 3$, then there are $\rho(n) - 1$ vector fields on $S^{n-1}$ where $\rho(n) = 8d + 2^c$ is the Hurwitz-Radon number. In 1962, Adams [1] showed that this is the maximal number of linearly independent vector fields.

If $G$ is a finite group and acts freely on the unit sphere $S(V)$ of a real orthogonal representation space $V$, it was shown by Becker [3] that, under mild hypotheses, the $G$-field number of $V$ depends only on $G$ and the real dimension of $V$. Here, $G$-field number of $V$ is the maximal number of linearly independent $G$-fields on $S(V)$.

Some concrete conditions were given by Namboodiri [14] for the existence of equivariant vector fields under mild fixed point conditions when the action of $G$ is not free. He also found the $G$-field number for finite group $G$ under some restrictions on the dimension of the real representation space. Part of his results answers the problem of existence when $G$ is finite abelian. However, his results for arbitrary finite groups only apply if $\dim V^G$ is at least 3. Existence of $k$-many $G$-fields on $S(V)$ clearly implies the existence of $k$-many vector fields on $S(V^H)$ for any subgroup $H$ of $G$, where $V^H = \{x \in V \mid hx = x \text{ for all } h \in H\}$ is the $H$-fixed point set of $V$. If $\dim V^G = 3$, then $S(V^G) \approx S^2$ which does not have a nonzero vector field. So, there is no $G$-field. If $\dim V^G > 3$ and $V$ is four-dimensional, then $G$ acts trivially on $S^3$ which is not an interesting case. Therefore, we will consider the existence problem for some finite non-abelian groups, namely the dihedral group, the generalized quaternion group and the semi-dihedral group. We will give the conditions for the existence of $G$-fields on the unit spheres of real orthogonal representations $V$ of these groups where $\dim V = 4$. The results are based on the decomposition of $V$ into irreducible real representations. In some cases, equivariant analogue of the Euler characteristic by Costenoble and Waner [5] is used.

We must note that Namboodiri actually constructed equivariant vector fields explicitly using Clifford algebras [14]. However, in general, the $G$-field number can be larger than the number of $G$-fields he constructed. In fact, there are cases these two numbers are not equal.

Our main results are given as Theorem 1.0.2, Theorem 1.0.3 and Theorem 1.0.4 which are about the dihedral group, the generalized quaternion group and the semi-dihedral group respectively. We give the statements below.

Let $a$ and $b$ be the generators of the dihedral group $\mathrm{D}_n$. As given in detail in Chapter

5, $\mathrm{D}_n$ has the following complete set of irreducible real representations.

It has the trivial real representation $r_0$ or $V_0$. For $n$ even, it has one-dimensional nontrivial representations $r_1$, $r_2$, and $r_3$ where $r_1(a) = 1$, $r_1(b) = -1$, $r_2(a) = -1$, $r_2(b) = 1$, and $r_3(a) = r_3(b) = -1$. For $n$ odd, $r_1$ with the same definition is the only one-dimensional nontrivial representation. We also have two-dimensional irreducible real representations $V_k$ where $a$ acts as rotation by $2k\pi/n$, $b$ acts as reflection with respect to $x$ axis on the plane and $1 \leq k < n/2$.

**Theorem 1.0.2** *Let $G$ be $\mathrm{D}_n$ and $V$ be a real orthogonal representation space of $G$ where $\dim V = 4$. Then there is a $G$-field on $S(V) \approx S^3$ if and only if the decomposition of $V$ into irreducible real representations of $G$ is one of the following:*

1. $2r_i \oplus 2r_j$ *where* $0 \leq i \leq j \leq 3$ *for n even and* $0 \leq i \leq j \leq 1$ *for n odd.*
2. $V_{k_1} \oplus V_{k_2}$ *where* $\gcd(n, k_1) = \gcd(n, k_2)$.

The theorems for the generalized quaternion group and the semi-dihedral group are similar. Let $a$ and $b$ be the generators of the generalized quaternion group $\mathrm{Q}_{2^{n+1}}$ and the semi-dihedral group $\mathrm{SD}_{2^n}$. Both groups have the irreducible one-dimensional real representations $r_0$, $r_1$, $r_2$ and $r_3$ with the same definitions. Let $m$ be $2^{n-1}$ and $2^{n-2}$ for $\mathrm{Q}_{2^{n+1}}$ and $\mathrm{SD}_{2^n}$ respectively. Then they have two-dimensional irreducible real representations $V_k$ where $a$ acts as rotation by $2k\pi/m$, $b$ acts as reflection with respect to $x$ axis on the plane and $1 \leq k < m/2$.

A significant difference from the dihedral group case is that the generalized quaternion group and the semi-dihedral group have four-dimensional irreducible real representations $U_l$ of quaternionic type and complex type respectively.

**Theorem 1.0.3** *Let $G$ be $\mathrm{Q}_{2^{n+1}}$ and $V$ be a real orthogonal representation space of $G$ where $\dim V = 4$. Then there is a $G$-field on $S(V) \approx S^3$ if and only if the decomposition of $V$ into irreducible real representations of $G$ is one of the following:*

1. $2r_i \oplus 2r_j$ *where* $0 \leq i \leq j \leq 3$,
2. $V_{k_1} \oplus V_{k_2}$ *where* $\gcd(2^{n-1}, k_1) = \gcd(2^{n-1}, k_2)$,

3. $U_l$.

**Theorem 1.0.4** *Let $G$ be $\mathrm{SD}_{2^n}$ and $V$ be a real orthogonal representation space of $G$ where $\dim V = 4$. Then there is a $G$-field on $S(V) \approx S^3$ if and only if the decomposition of $V$ into irreducible real representations of $G$ is one of the following:*

1. $2r_i \oplus 2r_j$ *where* $0 \leq i \leq j \leq 3$,
2. $V_{k_1} \oplus V_{k_2}$ *where* $\gcd(2^{n-2}, k_1) = \gcd(2^{n-2}, k_2)$,
3. $U_l$.

The outline of the thesis is as follows. In Chapter 2, we cover some basic definitions and facts about elementary representation theory and $G$-complexes. In Chapter 3, we give the definition of $G$-vector field and existence results of Costenoble and Waner [5]. In Chapter 4, we give simpler conditions for the existence of $G$-fields for finite abelian groups. In Chapter 5, we inspect decompositions of four-dimensional real representation spaces of the dihedral group in order to find existence results for $G$-fields. In one particular case, we use strong Euler characteristic. In Chapter 6 and Chapter 7, we utilize the techniques that we used in Chapter 5 to find similar results for the generalized quaternion group and the semi-dihedral group respectively.

# CHAPTER 2

# PRELIMINARIES

## 2.1 Elementary Representation Theory

In this section, we will cover some definitions and facts of representation theory. We assume $\mathbb{K}$ denotes one of the fields $\mathbb{R}$ (real numbers) or $\mathbb{C}$ (complex numbers).

**Definition 2.1.1 (Linear Representation, [17])** *A representation of $G$ on the finite dimensional vector space $V$ over $\mathbb{K}$ is a continuous action*

$$\rho : G \times V \to V$$

*of $G$ on $V$ such that the left translation $L_g : v \mapsto \rho(g, v)$ is a $\mathbb{K}$-linear map for each $g \in G$. $V$ is called the representation space of $G$. The dimension of a representation is defined to be the dimension $\dim_{\mathbb{K}} V$ of the representation space.*

We may sometimes call the representation space $V$ as representation $V$. Depending on $\mathbb{K}$, a representation of $G$ over $\mathbb{K}$ is called a real or complex representation respectively.

It is easy to show that $L_g$ is an automorphism of $V$ with inverse $L_{g^{-1}}$ for each $g \in G$. Alternatively, a representation is a continuous homomorphism $l : G \to Aut_{\mathbb{K}}(V)$, $g \mapsto L_g$ [4, 2]. If we choose a basis for $V$ of dimension $n$, then $Aut_{\mathbb{K}}(V)$ is isomorphic to $\mathrm{GL}(n, \mathbb{K})$.

**Definition 2.1.2** *A continuous homomorphism $G \to \mathrm{GL}(n, \mathbb{K})$ is called a matrix representation of $G$.*

Thus, a representation $V$ of $G$ with $\dim_{\mathbb{K}} V = n$ corresponds to a matrix representation which assigns each $g \in G$ a matrix in $\mathrm{GL}(n, \mathbb{K})$.

**Definition 2.1.3 ([2])** *Let $V$ and $W$ be two representations of $G$ over $\mathbb{K}$. An isomorphism $f : V \to W$ is a $\mathbb{K}$-linear map which is equivariant and has an inverse. If $V$ and $W$ are isomorphic, then we will call them equivalent.*

*Let $A$ and $B$ be matrix representations. Then they are isomorphic if there exists an invertible matrix $T$ such that*

$$TA(g) = B(g)T \text{ for all } g \in G.$$

**Definition 2.1.4 (Irreducible Representation, [2])** *If a subspace $U$ of a representation space $V$ is $G$-invariant, then it is called a subrepresentation of $V$. A nonzero representation $V$ is called irreducible if $\{0\}$ and $V$ itself are the only subrepresentations of $V$.*

**Example 2.1.5** *One-dimensional representations are irreducible.*

If $V$ and $W$ are representations, then their direct sum $V \oplus W$ is a representation with the diagonal action $g(v, w) = (gv, gw)$.

**Proposition 2.1.6 ([2])** *Let $G$ be a compact group. If $V$ is a subrepresentation of $U$, then there is a complementary subrepresentation $W$ such that $U = V \oplus W$. Each representation is a direct sum of irreducible subrepresentations.*

**Proof.** See Proposition 3.18 and Theorem 3.20 of [2]. ∎

**Theorem 2.1.7 (Schur's Lemma, [2])** *Let $G$ be a group and let $V$ and $W$ be irreducible representations of $G$. Then:*

1. *A $G$-map $V \to W$ is either zero or an isomorphism.*
2. *If $\mathbb{K} = \mathbb{C}$, then every $G$-map $f : V \to V$ has the form $f(v) = \lambda v$ for some $\lambda \in G$.*

**Proof.** See Proposition 3.22 of [2]. ■

Let $\mathrm{Irr}(G, \mathbb{K})$ denote the complete set of irreducible pairwise nonisomorphic representations of $G$ over $\mathbb{K}$, that is, each irreducible representation of $G$ over $\mathbb{K}$ is isomorphic to exactly one element of $\mathrm{Irr}(G, \mathbb{K})$. We denote direct product of $n$ copies of $V \in \mathrm{Irr}(G, \mathbb{K})$ by $nV$. Let $V_i$ run through $\mathrm{Irr}(G, \mathbb{K})$ and let $V = U_1 \oplus \cdots \oplus U_n$ be a decomposition of $V$ into the direct sum of irreducible representations. We let $m_i$ denote the number of representations in $\{U_1, \ldots, U_n\}$ which are isomorphic to $V_i$. Then $V$ is isomorphic to $\bigoplus_i m_i V_i$ and $m_i$'s are unique by the following theorem.

**Theorem 2.1.8 ([2])** *Let $V_i$ run through $\mathrm{Irr}(G, \mathbb{K})$ and let $m_i$ and $n_i$ be nonnegative integers. If $\bigoplus_i m_i V_i$ is equivalent to $\bigoplus_i n_i V_i$, then $m_i = n_i$ for all $i$.*

**Proof.** See Proposition 3.24 of [2]. ■

**Proposition 2.1.9** *An irreducible complex representation of an abelian Lie group $G$ is one-dimensional.*

**Proof.** See Proposition 1.13 of [4]. ■

**Definition 2.1.10 (Character, [2, 4])** *Let $V$ be a representation space of $G$. The function $\chi_V : G \to \mathbb{K},\ g \mapsto Tr(L_g)$ is called the character of $V$. Here, $Tr(L_g)$ is the trace of the linear map $L_g : V \to V,\ v \mapsto gv$. If the representation is irreducible, then its character is called an irreducible character.*

**Theorem 2.1.11 ([4])** *A representation is determined by its character up to isomorphism.*

**Proof.** See Theorem 4.12 of [4]. ■

**Example 2.1.12** *Character of a one-dimensional matrix representation is the representation itself. Thus, if two one-dimensional matrix representations are isomorphic, then they are the same.*

If $V$ and $W$ are representations, then the tensor product $V \otimes W$ is a representation with the action $g(v \otimes w) = gv \otimes gw$.

**Proposition 2.1.13 ([4])** *Let $V$ and $W$ be irreducible complex representations of $G$ and $H$ respectively. Then $V \otimes W$ is an irreducible representation of $G \times H$. Moreover, any irreducible representation of $G \times H$ is of this tensor product form.*

**Proof.** See Proposition 4.14 of [4]. ■

**Definition 2.1.14 ([17])** *Let $V$ be a representation space of $G$ over $\mathbb{K}$ with a $G$-invariant inner product, that is, $\langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle$ for all $g \in G$ and $v_1, v_2 \in V$. Then we can consider orthogonal or unitary representations for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ respectively. In the case of a matrix representation, an orthogonal or a unitary representation is a homomorphism $G \to \mathrm{O}(n)$ or $G \to \mathrm{U}(n)$ respectively.*

**Definition 2.1.15 (Representation Sphere, [17])** *Assume $G$ acts on a representation space $V$ by orthogonal (or unitary) transformations. Then the unit sphere*

$$S(V) = \{v \in V \mid \langle v, v \rangle = 1\}$$

*of $V$ is $G$-invariant. Thus, there is an induced action of $G$ on $S(V)$. The $G$-spaces of this type are called representation spheres.*

For a representation space $S(V)$ of $G$, we have $S(V^H) = S(V)^H$ for any subgroup $H$ of $G$. Here, $V^H$ is the $H$-fixed point set $\{v \in H \mid hv = v \ \forall h \in H\}$.

**Definition 2.1.16 (Structure Map, [4])** *Let $V$ be a complex representation space of $G$. If a $G$-map $j : V \to V$ satisfies the following conditions, then it is called a structure map on $V$.*

1. *$j$ is conjugate linear, that is, $j(zv) = \overline{z}(jv)$ for $z \in \mathbb{C}$.*
2. *$j^2 = \pm 1$.*

*If $j^2 = 1$, then $j$ is called a real structure map. Otherwise, if $j^2 = -1$, then $j$ is called a quaternionic structure map.*

A complex representation space is of real (respectively quaternionic) type if it admits a real (respectively quaternionic) structure map. If it is not self-conjugate, i.e., $V \ncong \overline{V}$, then it is of complex type. We can determine the type of an irreducible complex representation with the help of its character.

**Proposition 2.1.17 ([4])** *Let V be an irreducible complex representation of a compact group G and $\chi$ be its character. Then*

$$\int \chi(g^2)\,\mathrm{d}g = \begin{cases} 1 & \Leftrightarrow V \text{ is of real type,} \\ 0 & \Leftrightarrow V \text{ is of complex type,} \\ -1 & \Leftrightarrow V \text{ is of quaternionic type.} \end{cases}$$

Here, $\int f\,\mathrm{d}g$ is the invariant (Harr-) integral [4]. If $G$ is finite group, then instead of $\int \chi(g^2)\,\mathrm{d}g$ we can write $\frac{1}{|G|}\sum \chi(g^2)$ [16].

**Proof.** See Proposition 6.8 of [4] and Proposition 39 of [16]. ■

Given a complex representation, we can find the corresponding real representation depending on the type of the given complex representation. We define a map $r^{\mathbb{C}}_{\mathbb{R}}$ which we will call the restriction map or realification. Given a complex representation space $V$, $r^{\mathbb{C}}_{\mathbb{R}}V$ is $V$ viewed as a real vector space with the same $G$-action. We define $e^{\mathbb{C}}_{\mathbb{R}}$ by $e^{\mathbb{C}}_{\mathbb{R}}(V) = \mathbb{C} \otimes_{\mathbb{R}} V$ for a given real representation space V.

We partition $\mathrm{Irr}(G, \mathbb{C})$ into three disjoint sets; $\mathrm{Irr}(G, \mathbb{C})_{\mathbb{R}}$, $\mathrm{Irr}(G, \mathbb{C})_{\mathbb{C}}$ and $\mathrm{Irr}(G, \mathbb{C})_{\mathbb{H}}$ which denote the sets of real type, complex type and quaternionic type representations in $\mathrm{Irr}(G, \mathbb{C})$ respectively. We can also define $\mathrm{Irr}(G, \mathbb{R})_{\mathbb{K}}$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$, that is, we partition of $\mathrm{Irr}(G, \mathbb{R})$ into real type, complex type and quaternionic type representations respectively using complex representations.

**Definition 2.1.18 ([4])** *Let G be a compact group. Let $U \in \mathrm{Irr}(G, \mathbb{R})$. Then:*

1. *If $e^{\mathbb{C}}_{\mathbb{R}}(U) = V$ and V is of real type, then $U \in \mathrm{Irr}(G, \mathbb{R})_{\mathbb{R}}$.*

2. *If $U = r^{\mathbb{C}}_{\mathbb{R}}(V)$ and V is of complex type, then $U \in \mathrm{Irr}(G, \mathbb{R})_{\mathbb{C}}$.*

3. *If $U = r^{\mathbb{C}}_{\mathbb{R}}(V)$ and V is of quaternionic type, then $U \in \mathrm{Irr}(G, \mathbb{R})_{\mathbb{H}}$.*

**Proposition 2.1.19 ([4])** *Let G be a compact group. Then:*

1. $V \in \mathrm{Irr}(G, \mathbb{C})_{\mathbb{R}} \Rightarrow r_{\mathbb{R}}^{\mathbb{C}} V = U \oplus U, \quad U \in \mathrm{Irr}(G, \mathbb{R})_{\mathbb{R}}.$

2. $V \in \mathrm{Irr}(G, \mathbb{C})_{\mathbb{C}} \Rightarrow r_{\mathbb{R}}^{\mathbb{C}} V = U = r_{\mathbb{R}}^{\mathbb{C}} \overline{V}, \quad U \in \mathrm{Irr}(G, \mathbb{R})_{\mathbb{C}}.$

3. $V \in \mathrm{Irr}(G, \mathbb{C})_{\mathbb{H}} \Rightarrow r_{\mathbb{R}}^{\mathbb{C}} V = U, \quad U \in \mathrm{Irr}(G, \mathbb{R})_{\mathbb{H}}.$

**Proof.** See Proposition 6.6 of [4]. ■

**Proposition 2.1.20 ([16])** *Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible complex representation of a finite group G of dimension n with character $\chi$. Then there are three cases.*

1. *All values of $\chi$ are real, and $\rho$ is realizable by a real representation $\rho_0$, that is, $V = V_0 \oplus iV_0 = \mathbb{C} \otimes_{\mathbb{R}} V_0$ where $V_0$ is a G-invariant real subspace of V. The representation $\rho_0$ is irreducible with character $\chi$.*

2. *One of the values of $\chi$ is not real. By realification, $\rho$ defines an irreducible representation of dimension $2n$ with character $\chi + \overline{\chi}$.*

3. *All values of $\chi$ are real, but $\rho$ is not realizable over $\mathbb{R}$. By realification, $\rho$ defines an irreducible representation of dimension $2n$ and with character.*

*A complex representation $\rho$ is of real, complex or quaternionic type if and only if it is of case 1, 2 or 3 respectively.*

**Proof.** See Proposition 39 of [16]. ■

### 2.2 Equivariant *CW*-Complexes

**Definition 2.2.1 ([17])** *Let n be a nonnegative integer. Let A be a G-space. Given a family $(H_j \mid j \in J)$ of closed subgroup $H_j$ of G and G-maps*

$$\varphi_j : G/H_j \times S^{n-1} \longrightarrow A, \;\; j \in J,$$

*we consider pushouts of G-spaces*

$$\begin{array}{ccc} \coprod_{j\in J} G/H_j \times S^{n-1} & \xrightarrow[\varphi]{} & A \\ \bigcap & & \Big\downarrow i \\ \coprod_{j\in J} G/H_j \times D^n & \xrightarrow[\phi]{} & X. \end{array}$$

*Such that* $\varphi \mid G/H_j \times S^{n-1} = \varphi_j$ *and* $\phi \mid G/H_j \times D^n = \phi_j$*. If such pushout exist and* $i$ *is a closed embedding, we use the following terminology:* $X$ *is obtained from* $A$ *by (simultaneously) attaching the family of (equivariant) n-cells* $(G/H_j \times D^n \mid j \in J)$ *of type* $(G/H_j \mid j \in J)$*. We call* $\phi(G/H_j \times D^n)$ *a closed n-cell of type* $G/H_j$ *and* $\phi(G/H_j \times \mathring{D}^n)$ *an open n-cell of type* $G/H_j$*. Moreover,* $\phi(G/H_j \times S^{n-1})$ *is called the boundary of* $\phi(G/H_j \times D^n)$*. The map*

$$(\phi_j, \varphi_j) : (G/H_j \times D^n, G/H_j \times S^{n-1}) \to (X, A)$$

*is called the characteristic map of the corresponding n-cell and* $\varphi_j$ *is called the attaching map.*

**Definition 2.2.2 (Equivariant *CW*-Decomposition, [17])** *Suppose* $(X, A)$ *is a pair of G-spaces with A being a Hausdorff space. An equivariant CW-decomposition of* $(X, A)$ *consists of a filtration* $(X_n \mid n \in \mathbb{Z})$ *of X such that the following holds:*

1. $A \subset X_0$*;* $A = X_n$ *for* $n < 0$*;* $X = \cup X_n$*.*
2. *For each* $n \geq 0$ *the space* $X_n$ *is obtained from* $X_{n-1}$ *by attaching n-cells.*
3. *X carries the colimit topology with respect to* $(X_n)$*, i.e.* $B \subset X$ *is closed if and only if* $B \cap X_n$ *is closed in* $X_n$ *for all n.*

**Definition 2.2.3 (Equivariant *CW*-Complex, [17])** *If* $(X_n)$ *is an equivariant CW-decomposition of* $(X, A)$*, then* $(X, A)$ *is called a relative equivariant CW-complex. If* $A = \emptyset$*, then X is called an equivariant CW-complex. The subspace* $X_n$ *is called the n-skeleton of* $(X, A)$*. The cells of* $(X_n, X_{n-1})$ *are called the n-cells of* $(X, A)$*.*

Instead of $G$-equivariant $CW$-complex, we will use the term $G$-complex. The following Definition 2.2.4 and Proposition 2.2.5 enable us to construct a $G$-complex from a $CW$-complex that satisfies certain properties. A more general identification can be found in the paper by Matumoto [12].

**Definition 2.2.4 (Cellular Action, [17])** *Let $X$ be a $CW$-complex and $G$ be a discrete group. $G$ acts cellularly (or cell preserving) on $X$ if the following holds:*

1. *If $e$ is an open cell of $X$ then $ge$ is also an open cell of $X$ for all $g \in G$.*
2. *If $ge = e$ then $gx = x$ for any point $x$ in $e$.*

**Proposition 2.2.5 ([17])** *Let $X$ be a $CW$-complex and $G$ act cellularly on $X$. Then $X$ is a $G$-complex with $n$-skeleton $X_n$.*

**Proof.** See Proposition 1.15 in Chapter 2 of [17] ■

**Example 2.2.6 ([11])** *Let $G$ be a finite group and $V$ be an orthogonal $G$ representation. The representation sphere $S(V)$ has the following $G$-complex structure. Let $X$ be the convex hull of*

$$\{\pm ge_i \mid g \in G, 1 \leq i \leq m\},$$

*where $\{e_i \mid 1 \leq i \leq m\}$ is an orthonormal basis for $V$. Then its boundary $\partial X$ is $G$-homeomorphic to $S(V)$ by radial projection. So we have a simplicial complex structure on $\partial X$ where $g$ acts simplicially. If we take its first barycentric subdivision, then $g$ induces the identity map on $e$ for any simplex $e$ with $ge = e$. Thus, it is a $G$-complex by Proposition 2.2.5.*

# CHAPTER 3

# *G*-VECTOR FIELDS

## 3.1 Definitions

In this section, we give some basic definitions and general results about $G$-vector fields.

**Definition 3.1.1 ($G$-Manifold, [17])** *Let $G$ be a Lie group and $M$ be a smooth manifold. If the action $G \times M \to M$, $(g, m) \mapsto gm$ is a smooth map, then it is called a smooth action. A manifold along with a smooth action is called a smooth $G$-manifold.*

**Proposition 3.1.2** *Let $M$ be a smooth $G$-manifold. Then the tangent bundle $TM$ of $M$ has an induced $G$-action.*

**Proof.** Let $M$ be a smooth $G$-manifold. Since the action of $G$ is smooth on $M$, left translation $L_g : M \to M$, $p \mapsto gp$, is a smooth map for each $g \in G$. $L_g$ induces a smooth map $dL_g : TM \to TM$ such that the following diagram commutes.

$$\begin{array}{ccc} TM & \xrightarrow{dL_g} & TM \\ {\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ M & \xrightarrow{L_g} & M \end{array}$$

Then we define the action of $g \in G$ on $u_p \in T_pM \subset TM$ by $dL_g(u_p)$. ■

**Definition 3.1.3 ($G$-Vector Field)** *A vector field $v : M \to TM$ on $M$ is a $G$-vector field if $v(gx) = gv(x)$ for all $x \in M$ and $g \in G$.*

For simplicity, we will use the term $G$-field to refer to $G$-vector fields.

## 3.2 The Strong Euler Characteristic

In this section, we give the existence results of Costenoble and Waner [5].

**Definition 3.2.1 ([5])** *Let $X$ be a finite $G$-complex. The strong equivariant Euler characteristic of $X$ is $t(X)$ which is the equivariant transfer associated with the fibration $X \to *$. That is, $t(X)$ is the composite*

$$S \to DX^+ \wedge X^+ \to DX^+ \wedge X^+ \wedge X^+ \to S \wedge X^+,$$

*where $DX^+$ is the equivariant Spanier-Whitehead dual of $X^+$ and the second map is given by the diagonal on $X$. We consider this stable $G$-map from $S^0$ to $X^+$ as an element of $\hat{\pi}_0^G(X^+)$, the equivariant stable 0th homotopy group of $X$, i.e., the group of equivariant stable maps from $S^0$ to $X^+$.*

Instead of the definition above, we will use Proposition 3.2.5 since it simplifies the computation of the strong Euler characteristic for a $G$-complex.

**Definition 3.2.2** *Let $H$ be a subgroup of $G$. We will denote the normalizer of $H$ in $G$ by $NH$ or $N_G H$. The Weyl group $WH$ of $H$ is defined by $NH/H$.*

**Definition 3.2.3 ($G$-Homotopy, [17])** *Let $f_0, f_1 : X \to Y$ be $G$-maps. Then they are called $G$-homotopic if there exists a continuous $G$-map $F : X \times [0,1] \to Y$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. Here, we define action of $G$ on $X \times [0,1]$ with the diagonal action where $[0,1]$ has the trivial action. Each map $f_t : x \mapsto F(x,t)$ is then a $G$-map. $F$ is called a $G$-homotopy from $f_0$ to $f_1$.*

**Proposition 3.2.4 ([5])** *If $X$ is a $G$-space then $\hat{\pi}_0^G(X^+)$ is the free abelian group generated by the equivalence classes of $G$-maps $x : G/H \to X$ for those subgroups $H$ such that $WH$ is finite. A map $x$ is equivalent to a map $x' : G/H' \to X$ if there is a $G$-homeomorphism $\xi : G/H \to G/H'$ such that $x' \circ \xi \simeq x$.*

We will denote by $[x]$ the equivalence class of $x : G/H \to X$.

**Proposition 3.2.5 ([5])** *Let $X$ be a finite $G$-complex with cells $G/H_i \times e^{n_i} \to X$. Let $x_i : G/H_i \to G/H_i \times e^{n_i} \to X$ be the composite with the inclusion of an orbit. Then*

$$t(X) = \sum_i (-1)^{n_i}[x_i] \in \hat{\pi}_0^G(X^+) .$$

We have two simple observations.

**Proposition 3.2.6** *Let $X$ be a finite $G$-CW complex and $G$ a finite group. If $t(X) = 0$ then the ordinary Euler characteristic of $X$ is also 0.*

**Proof.** If $t(X) = 0$, then we have

$$t(X) = \sum_i (-1)^{n_i}[x_i] = 0 .$$

If $[x]$=$[x']$ then $|G/H| = |G/H'|$ by Proposition 3.2.4. Thus,

$$\sum_i (-1)^{n_i}|G/H_i| = 0.$$

Since $G$ is finite, $X$ is also a $CW$-complex [12]. Thus, the above sum is its Euler characteristic. ■

**Proposition 3.2.7** *Let $X$ be a finite $G$-complex which is composed of even dimensional cells $G/H_i \times e^{n_i} \to X$, $i \in I$, and odd dimensional cells $G/H_j \times e^{n_j} \to X$, $j \in J$. If $t(X) = 0$ then $|I| = |J|$.*

**Proof.** Assume $t(X) = 0$ then

$$t(X) = \sum_{i \in I}[x_i] - \sum_{j \in J}[x_j] = 0 .$$

As the equivalence classes are the generators of a free abelian group, an equivalence class corresponding to an even dimensional cell can only cancel with an equivalence class corresponding to an odd dimensional cell. Thus $|I| = |J|$. ■

**Theorem 3.2.8 ([5])** *Let $M$ be a smooth compact $G$-manifold. Then $M$ has a nonzero tangent vector field that is outward normal on $\partial M$ if and only if $t(M) = 0$.*

**Proof.** See [5, Theorem 3.5]. ■

# CHAPTER 4

# $G$-FIELDS ON $S^3$ FOR FINITE ABELIAN GROUPS

$G$-field number for finite abelian groups are given by Namboodiri [14]. His results are general results for all spheres and he uses K-theory techniques to prove them. When $S(V)$ is three-dimensional, it is possible to prove the existence of $G$-fields on $S(V)$ by more elementary arguments. The statement and the proof is based on the decomposition of $V$ into irreducible real representations of $G$. The main existence theorem for the finite abelian group $G$ is the following.

**Theorem 4.0.9** *Let $G$ be a finite abelian group. Let $V$ be a four-dimensional real orthogonal representation space of $G$. Then there is a $G$-field on $S(V) \approx S^3$ if and only if the real representation of $G$ on $V$ is the direct sum of two rotations.*

Let $G$ be a finite abelian group. Then $G$ is isomorphic to

$$\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k} .$$

So, we can write any $g \in G$ as $(g_1^{a_1}, g_2^{a_2}, \ldots, g_k^{a_k})$ where $g_l$ is a generator of $\mathbb{Z}_{m_l}$. Since $G$ is abelian, its irreducible complex representations are one-dimensional. Complex representations of $\mathbb{Z}_{m_l}$ are $t_l^{s_l}$ defined by

$$g_l^a v = e^{a2\pi i s_l/m_l} v \text{ for} s_l = 0, 1, \ldots, m_l - 1 \text{ [16]}.$$

If we use Proposition 2.1.13, then the irreducible representations of $G$ are

$$t_1^{s_1} \otimes t_2^{s_2} \otimes \cdots \otimes t_k^{sk}$$

defined by

$$\begin{aligned} gv &= (g_1^{a_1}, g_2^{a_2}, \ldots, g_k^{a_k})v \\ &= e^{a_1 2\pi i s_1/m_1} e^{a_2 2\pi i s_2/m_2} \cdots e^{a_k 2\pi i s_k/m_k} v\,. \end{aligned}$$

Its character is $e^{a_1 2\pi i s_1/m_1} \cdots e^{a_k 2\pi i s_k/m_k}$. It is real valued for all $g \in G$ if each $e^{2\pi i s_l/m_l}$ is real valued, i.e., $s_l = 0$ or $m_l$ is even and $s_l = m_l/2$. Those with real valued character for all $g \in G$ correspond to one-dimensional real representations $r$ defined by

$$gv = (g_1^{a_1}, g_2^{a_2}, \ldots, g_k^{a_k})v = (-1)^{2a_1 s_1/m_1}(-1)^{2a_2 s_2/m_2} \ldots (-1)^{2a_k s_k/m_k} v.$$

Thus, for a given $g \in G$, $r$ is either a reflection or identity and $2r$ can be considered as a rotation by $\pi$ or 0 respectively. Otherwise, the complex representation corresponds to a two-dimensional real irreducible representation $R$ defined by

$$gv = (g_1^{a_1}, g_2^{a_2}, \ldots, g_k^{a_k})v = \begin{pmatrix} \cos\theta_g & -\sin\theta_g \\ \sin\theta_g & \cos\theta_g \end{pmatrix} v,$$

where $\theta_g = a_1 2\pi s_1/m_1 + a_2 2\pi s_2/m_2 + \ldots + a_k 2\pi s_k/m_k$ and its character is $2\cos\theta_g$. Thus, R is a rotation by angle $\theta_g$.

**Definition 4.0.10** *We define $i_x$ to be the map $\bigoplus^n \mathbb{R}^2 \to \bigoplus^n \mathbb{R}^2$ such that*

$$i_x : (x_1, x_2) \mapsto (-x_2, x_1)$$

*on each $\mathbb{R}^2$. That is, $i_x$ is multiplication by the complex number $i$ on $\mathbb{C}^n$ considered in $\bigoplus^n \mathbb{R}^2$.*

**Lemma 4.0.11** *Let $G$ be a finite group and $(V, p)$ be a real orthogonal representation of dimension four. Let $r_1$ and $r_2$ denote one-dimensional (not necessarily distinct) representations of $G$. Let the decomposition of $V$ be isomorphic to the direct sum of one-dimensional real representation spaces. Then there is a nonzero $G$-field on $S(V) \approx S^3$ if and only if $p \cong 2r_1 \oplus 2r_2$.*

**Proof.** Let $p_0$ denote the trivial representation. If there exist non-trivial representations, we let $p_i$ denote non-trivial one-dimensional representations where $i = 1, 2, 3, 4$. Then, we inspect each possible decomposition of $p$. However, we should note that some of the cases does not apply to some groups.

1. The case $p = 4p_0$ : In this case, we have trivial action of $G$ on $V$. Since the non-equivariant Euler characteristic of $S^3$ is 0, there is a nonzero $G$-field on $S(V)$.

2. The case $p = p_1 \oplus 3p_0$ : $S(V^G)$ is $S^2$ which has no nonzero vector field. So $S(V)$ does not admit a nonzero $G$-field.

3. The case $p = 2p_1 \oplus 2p_0$ : In this case, $\mathrm{i}_x$ is a nonzero $G$-field on $S(V)$.

4. The case $p = p_1 \oplus p_2 \oplus 2p_0$ : Assume $r_1 \neq r_2$ then there is an $h \in G$ such that $r_1(h) \neq r_2(h)$. Without loss of generality, assume $r_1(h) = 1$ then $S(V^{<h>})$ is $S^2$. So $S(V)$ does not admit a nonzero $G$-field.

5. The case $p = p_1 \oplus p_2 \oplus p_3 \oplus p_0$ : $S(V^G)$ is $S^0$ which has no nonzero vector field. So $S(V)$ does not admit a nonzero $G$-field.

6. The case $p = p_1 \oplus p_2 \oplus p_3 \oplus p_4$ : Assume $p_1, p_2, p_3, p_4$ are all distinct representations. Then, there is an $h \in G$ such that $(p_1(h), p_2(h)) \neq (p_3(h), p_4(h))$. Without loss of generality, cases are the following:

   (a) $p_1(h) = 1$ and $p_2(h) = p_3(h) = p_4(h) = -1$ then $S(V^{<h>})$ is $S^0$,

   (b) $p_1(h) = p_2(h) = p_3(h) = 1$ and $p_4(h) = -1$ then $S(V^{<h>})$ is $S^2$,

   (c) $p_1(h) = p_2(h) = 1$ and $p_3(h) = p_4(h) = -1$ then there is an $h'$ such that $p_1(h') \neq p_2(h')$. Without loss of generality, assume $p_1(h') = 1$ then $S(V^{<h,h'>})$ is $S^0$.

   Thus, in any case, $S(V)$ does not admit a nonzero $G$-field.

7. The case $p = p_1 \oplus p_2 \oplus 2p_3$: Assume $p_1, p_2, p_3$ are distinct representations. There is an $h \in G$ such that $r_1(h) \neq r_2(h)$. Without loss of generality, assume $r_1(h) = 1$ then $S(V^{<h>})$ is $S^2$ or $S^0$. So $S(V)$ does not admit a nonzero $G$-field.

8. The case $p = 2p_1 \oplus 2p_2$ : In this case, $\mathrm{i}_x$ is a nonzero $G$-field.

9. The case $p = p_1 \oplus 3p_2$ : Since $p_1 \neq p_2$, there is an $h \in G$ such that $p_1(h) \neq p_2(h)$. If $p_1(h) = 1$ then $S(V^{<h>})$ is $S^0$. Otherwise, $r_2(h) = 1$ and $S(V^{<h>})$ is $S^2$. In any case, $S(V)$ does not admit a nonzero $G$-field.

10. The case $p = 4p_1$ : In this case, $i_x$ is a nonzero $G$-field on $S(V)$.

■

**Lemma 4.0.12** *Let $G$ be a group. If $V$ and $W$ are realification of complex representations then $i_x$ is a nonzero $G$-field on $S(V \oplus W)$.*

**Proof.** Assume $A_g$ and $B_g$ are corresponding matrices of $V$ and $W$ respectively. Since $V$ and $W$ are realification of complex representations, $A_g$ and $B_g$ correspond to some complex matrices. Now, consider a complex number $z = a + bi$ as an entry of a complex matrix, then

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

So, $A_g \oplus B_g$ commutes with the matrix of $i_x$ which is the complex identity matrix times complex $i$ considered on $\mathbb{R}$. ■

**Proof of Theorem 4.0.9.** In Lemma 4.0.11, we covered all the possible combinations of four-dimensional real orthogonal representations which can be decomposed into a direct sum of one-dimensional representations. Let $r_1$ and $r_2$ denote one-dimensional irreducible real representations and $R_1$ and $R_2$ two-dimensional irreducible real representations of finite abelian group $G$. Then the remaining cases are the following.

1. The case $R_1 \oplus R_2$ : $i_x$ is a $G$-field which directly follows from Lemma 4.0.12.
2. The case $2r_1 \oplus R_1$ : $2r_1$ can be considered as the realification of a complex representation. Thus, $i_x$ is a $G$-field.
3. The case $r_1 \oplus r_2 \oplus R_1$ : $S(V^G)$ is $S^0$ which does not admit a nonzero vector field. Thus, there is no $G$-field.

Therefore there is a $G$-field if the representation is $2r_1 \oplus 2r_2$, $R_1 \oplus R_2$ or $2r_1 \oplus R_1$ which can be considered as the direct sum of two rotations. ■

# CHAPTER 5

# $G$-FIELDS ON $S^3$ FOR DIHEDRAL GROUP ACTIONS

In this chapter, we will give the proof of Theorem 1.0.2 about the existence of $G$-fields on the representation sphere $S(V)$ of a four-dimensional real orthogonal representation space $V$ of $G$, where $G$ is the dihedral group $\mathrm{D}_n$.

## 5.1 Real Representations of the Dihedral Group

The dihedral group of order $2n$ generated by elements $a$ and $b$ has the presentation

$$\mathrm{D}_n = \langle a, b \mid a^n = b^2 = 1,\ ab = ba^{-1} \rangle.$$

The elements of $\mathrm{D}_n$ are $1, a, a^2, \ldots, a^{n-1}, b, ab, a^2b, \ldots, a^{n-1}b$. It has the following complete set of pairwise nonisomorphic irreducible complex representations [16].

One-dimensional irreducible complex representations when $n$ is even:

- $q_0(a) = 1, q_0(b) = 1,$
- $q_1(a) = 1, q_1(b) = -1,$
- $q_2(a) = -1, q_2(b) = 1,$
- $q_3(a) = -1, q_3(b) = -1.$

One-dimensional irreducible complex representations when $n$ is odd:

- $q_0(a) = 1, q_0(b) = 1,$

- $q_1(a) = 1, q_1(b) = -1$.

Two-dimensional irreducible complex representation:

$$\rho_k(a) = \begin{pmatrix} \omega^k & 0 \\ 0 & \omega^{-k} \end{pmatrix}, \qquad \rho_k(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $k$ is an integer such that $1 \leq k < n/2$ and $\omega = e^{2\pi i/n}$.

Now, we will use Proposition 2.1.17 to determine the types of the complex representations given above. We note that $(a^m b)^2 = a^m b a^m b = b a^{-m} a^m b = b^2 = 1$.

For even $n$, since $q_j((a^m)^2) = 1$ and $q_j((a^m b)^2) = q_j(1) = 1$ for $j = 0, 1, 2, 3$, we have

$$\frac{1}{|G|} \sum_{g \in G} q_j(g^2) = \frac{1}{2n} 2n = 1.$$

For odd $n$, the above sum has the same value for $j = 0, 1$. Hence, the one-dimensional complex representations are of real type for even and odd values of $n$.

If we let $\chi_k$ denote the character of $\rho_k$, then

$$\chi_k((a^m)^2) = \chi_k(a^{2m}) = 2\cos\frac{2k(2m)\pi}{n} = 2\cos\frac{4km\pi}{n}$$

and

$$\chi_k((a^m b)^2) = \chi_k(1) = 2.$$

Consider $z = e^{i\phi}$ such that $z \neq 1$ and $z^n = 1$. Then $\sum_{m=0}^{n-1} z^m = (z^n - 1)/(z - 1) = 0$. As a result, if $n\phi = 2\pi l$ for $0 < \phi < 2\pi$ and some integer $l$, then $\sum_{m=0}^{n-1} \cos m\phi = 0$. Consequently,

$$\begin{aligned} \frac{1}{|G|} \sum_{g \in G} \chi_k(g^2) &= \frac{1}{2n} \left( \sum_{m=0}^{n-1} \chi_k((a^m)^2) + \sum_{m=0}^{n-1} \chi_k((a^m b)^2) \right) \\ &= \frac{1}{2n} \left( \sum_{m=0}^{n-1} 2\cos\frac{4km\pi}{n} + n\chi(1) \right) = \frac{1}{2n}(0 + 2n) = 1. \end{aligned}$$

So, the two-dimensional irreducible complex representations are also of real type.

Thus, all irreducible real representations of $\mathrm{D}_n$ are of real type and of degree at most two. Then it has the following complete set of irreducible real representations.

One-dimensional irreducible real representations when $n$ is even:

- $r_0(a) = 1, r_0(b) = 1,$
- $r_1(a) = 1, r_1(b) = -1,$
- $r_2(a) = -1, r_2(b) = 1,$
- $r_3(a) = -1, r_3(b) = -1.$

These correspond to the irreducible complex representations $q_0, q_1, q_2$ and $q_3$ respectively. If $n$ is odd, $r_0$ and $r_1$ defined above are the only one-dimensional representations. We will also denote their respective representation spaces with $r_0$, $r_1$, $r_2$ and $r_3$. We may sometimes use $V_0$ instead of $r_0$ to denote the trivial real representation space.

Two-dimensional irreducible real representations:

$$a \mapsto \begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix}, \qquad b \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $\theta_k = 2\pi k/n$ and k is an integer such that $1 \leq k < n/2$. We will denote these irreducible real representation spaces with $V_k$. They correspond to the irreducible complex representation spaces of $\rho_k$

## 5.2 $G$-Complex Structure of Join

Although, we can use Example 2.2.6 to get a generic $G$-complex. It will be tedious to compute the strong Euler characteristic using the structure given in that example. Instead, we will use the fact that representation is given as $V_{k_1} \oplus V_{k_2}$.

**Definition 5.2.1 (Join, [8])** *Let $X$ and $Y$ be spaces, then the join $X * Y$ of $X$ and $Y$ is the quotient space $X \times Y \times [0,1]$ under the identifications $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$. If $X$ and $Y$ are CW-complexes, then $X * Y$ has a CW-complex structure. It contains the subspaces $X$ and $Y$ as subcomplexes and the remaining cells are the product cells of $X \times Y \times [0,1]$.*

**Example 5.2.2** ([8])

1. $S^0 * S^0$ is $S^1$.

2. Join of $n$ copies of $S^0$ is $S^{n-1}$.

3. $S^1 * S^1$ is $S^3$. In general, join of $S^n$ and $S^m$ is $S^{n+m+1}$.

**Lemma 5.2.3** *Let $X$ and $Y$ be $G$-spaces and $CW$-complexes. If $G$ acts cellularly on $X$ and $Y$, then $G$ acts cellularly on $X * Y$.*

**Proof.** Since $X$ and $Y$ are $CW$-complexes, we have a $CW$-complex structure on $X * Y$. This $CW$-complex structure contains $X$ and $Y$ as subcomplexes and the product cells of $X \times Y \times (0, 1)$. Since, $G$ already acts cellularly on subcomplexes $X$ and $Y$, we should check the product cells of $X \times Y \times (0, 1)$. We define the action of $G$ on $X \times Y \times (0, 1)$ with the diagonal action where $(0, 1)$ has the trivial $G$-action.

1. Let $e$ be an open cell in $X$, $f$ be an open cell in $Y$, and $g \in G$. Then

$$g(e \times f \times (0, 1)) = ge \times gf \times (0, 1).$$

   Since $G$ acts cellularly on $X$ and $Y$, $ge$ and $gf$ are open cells in $X$ and $Y$ respectively. So $g(e \times f \times (0, 1))$ is an open cell in $X \times Y \times (0, 1)$. Hence, it is an open cell in $X * Y$.

2. If $g(e \times f \times (0, 1)) = e \times f \times (0, 1)$, then

$$ge \times gf \times (0, 1) = e \times f \times (0, 1) \Rightarrow ge = e, \; gf = f.$$

   Because of the cellular action of $G$ on $X$ and $Y$, the maps induced by $g$ on the cells $e$ and $f$ are identity maps. Thus, the map induced by $g$ on the cell $e \times f \times (0, 1)$ is identity.

Therefore, $G$ acts cellularly on $X * Y$. ■

**Proposition 5.2.4** *Let $X$ and $Y$ be $G$-spaces and $CW$-complexes. If $G$ acts cellularly on $X$ and $Y$, then $X * Y$ is a $G$-complex.*

**Proof.** Lemma 5.2.3 implies that $G$ acts cellularly on $X * Y$. So $X * Y$ is a $G$-complex by Proposition 2.2.5. ■

## 5.3 Existence of $G$-Fields on $S(V)$

Let $V$ be a real orthogonal representation space of $\mathrm{D}_n$ of dimension four. In order to determine the existence of $G$-fields on $S(V) \approx S^3$, we want to inspect all the possible decompositions of $V$ into irreducible real representations of $\mathrm{D}_n$. We already inspected the representations which can be decomposed into a direct sum of one-dimensional representations in a more general setting in Lemma 4.0.11. The remaining cases are the following:

1. $2V_0 \oplus V_k$,
2. $V_0 \oplus r_m \oplus V_k$ where $m = 1, 2, 3$ for even $n$ or $m = 1$ for odd $n$,
3. $2r_m \oplus V_k$ where $m = 1, 2, 3$ for even $n$ or $m = 1$ for odd $n$,
4. $r_i \oplus r_j \oplus V_k$ where $1 \leq i < j \leq 3$ for even $n$,
5. $2V_k$,
6. $V_{k_1} \oplus V_{k_2}$ where $k_1 \neq k_2$.

We shall give the proof of Theorem 1.0.2 as a series of propositions and lemmas.

**Proposition 5.3.1** *Let $V$ be a real orthogonal representation space of $\mathrm{D}_n$ of dimension four. There is no nonzero $G$-field on $S(V) \approx S^3$ if the decomposition of $V$ into irreducible real representations of $\mathrm{D}_n$ is one of the following:*

1. $2V_0 \oplus V_k$,
2. $V_0 \oplus r_m \oplus V_k$ *where* $m = 1, 2, 3$ *for even $n$ or* $m = 1$ *for odd $n$,*
3. $2r_m \oplus V_k$ *where* $m = 1, 2, 3$ *for even $n$ or* $m = 1$ *for odd $n$,*
4. $r_i \oplus r_j \oplus V_k$ *where* $1 \leq i < j \leq 3$ *for even $n$.*

**Proof.**

1. $2V_0 \oplus V_k$:

   Since $b \cdot (x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -x_4)$, $S(V^{<b>})$ is $S^2$. Since $S^2$ has no nonzero vector field, $S(V)$ does not admit a nonzero $G$-field.

2. $V_0 \oplus r_m \oplus V_k$:

   $S(V^G)$ is $S^0$ which has no nonzero vector field. So $S(V)$ does not admit a nonzero $G$-field.

3. $2r_m \oplus V_k$:

   For $m = 1$ or 3 we have $b \cdot (x_1, x_2, x_3, x_4) = (-x_1, -x_2, x_3, -x_4)$ and for $m = 2$, $b \cdot (x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -x_4)$. $S(V^{<b>})$ is $S^0$ or $S^2$ respectively. Since neither has nonzero vector field, $S(V)$ does not admit a nonzero $G$-field.

4. $r_i \oplus r_j \oplus V_k$:

   a. $r_1 \oplus r_2 \oplus V_k$:

      This case is valid only if $n$ is even. Since generator $a$ acts as a rotation for $V_{k/n}$, either it fixes the whole $e_3e_4$ plane or it does not fix any point on the $e_3e_4$ plane. Also $a \cdot (x_1, x_2) = (x_1, -x_2)$ on the $e_1e_2$ plane. So $S(V^{<a>})$ is $S^0$ or $S^2$. Thus, $S(V)$ does not admit a nonzero $G$-field.

   b. $r_1 \oplus r_3 \oplus V_k$:

      It is similar to the above case.

   c. $r_2 \oplus r_3 \oplus V_k$ :

      In this case, $b \cdot (x_1, x_2, x_3, x_4) = (x_1, -x_2, x_3, -x_4)$, $a^2 \cdot (x_1, x_2) = (x_1, x_2)$ on the $e_1e_2$ plane and $a^2$ acts as a rotation on the $e_3e_4$ plane. Since the generator $a$ rotates with $2\pi k/n \in (0, \pi)$ , $a^2$ does not fix any point on the $e_3e_4$ plane then $S(V^{<b,a^2>})$ is $S^0$. As a result, $S(V)$ does not admit a nonzero $G$-field.

■

For the case $2V_k$, we need the following lemma.

**Lemma 5.3.2** *Let $G$ be a finite group and $V$ a real (matrix) representation. Let $I$ be* *the $n \times n$ identity matrix. Then the vector field $v : x \mapsto Jx$ where*

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

*is a $G$-vector field on $S(V \oplus V)$.*

**Proof.** Let $A_g$ be the corresponding matrix of $V$ for $g \in G$. Then

$$v(gx) = J\left(\begin{bmatrix} A_g & 0 \\ 0 & A_g \end{bmatrix} x\right) = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}\left(\begin{bmatrix} A_g & 0 \\ 0 & A_g \end{bmatrix} x\right) = \begin{bmatrix} 0 & -A_g \\ A_g & 0 \end{bmatrix} x$$

and

$$gv(x) = g(Jx) = \begin{bmatrix} A_g & 0 \\ 0 & A_g \end{bmatrix}\left(\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} x\right) = \begin{bmatrix} 0 & -A_g \\ A_g & 0 \end{bmatrix} x.$$

Then we have $v(gx) = gv(x)$. Thus $v$ is a nonzero $G$-field on $S(V \oplus V)$. ■

**Proposition 5.3.3** *If the representation has the decomposition $2V_k$, then the vector field $x \mapsto Jx$ where*

$$J = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

*is a nonzero $G$-field on $S(V)$.*

**Proof.** This is a direct consequence of Lemma 5.3.2. ■

Next, we examine the case $V_{k_1} \oplus V_{k_2}$.

**Proposition 5.3.4** *Let $G$ be $\mathrm{D}_n$ and $V$ be a real orthogonal representation space of $G$ of dimension four. Let $V_{k_1} \oplus V_{k_2}$ be the decomposition of $V$ into irreducible representations. Then there is a nonzero $G$-field on $S(V) \approx S^3$ if and only if we have* $\gcd(n, k_1) = \gcd(n, k_2)$.

Now, we will give the proof of Proposition 5.3.4. There is a nonzero $G$-field on $S(V)$ if $k_1 = k_2$ by Proposition 5.3.3. So, we assume that $k_1 \neq k_2$. Unlike the $k_1 = k_2$

case, we cannot find a linear equivariant vector field. We have $b \cdot (x_1, x_2, x_3, x_4) = (x_1, -x_2, x_3, -x_4)$. Assume that $a^l \cdot (x_1, x_2) = (x_1, x_2)$ on the $e_1e_2$ plane and $a^l \cdot (x_3, x_4) \neq (x_3, x_4)$ on the $e_3e_4$ plane. Since $a^l$ acts as a rotation, it means that it fixes all points on the $e_1e_2$ plane and only the origin on the $e_3e_4$ plane. Then $S(V^{<b,a^l>})$ is $S^0$ which implies that, $S(V)$ does not admit a nonzero $G$-field. So, we should restrict ourselves to the case $n/\gcd(n, k_1) = n/\gcd(n, k_2)$. Let $s = n/\gcd(n, k_1) = n/\gcd(n, k_2)$. We will compute the strong Euler characteristic to determine the existence of a nonzero $G$-field on $S(V) \approx S^3$.

Since our representation is the sum of two two-dimensional representations, we can use the fact that $S^3 = S^1 * S^1$ together with Lemma 5.2.3 to define a $G$-complex. We will denote $S^1$ in the $e_1e_2$ plane by $S^1_1$ and $S^1$ in the $e_3e_4$ plane by $S^1_2$. First, we will define $CW$-complexes for $S^1_1$ and $S^1_2$ on which $G$ acts cellularly. So, first consider $S^1_1$ and take the points $ge_1$. Then, in order to satisfy the second condition of the Proposition 2.2.5, we take the midpoints of those arcs formed by $ge_1$ on $S^1_1$. So, we have the points $ge_1$ and $g\overline{e_1}$ where $\overline{e_1}$ denotes the vertex following $e_1$ in the counter clockwise direction. On $S^1_1$, we have the (distinct) vertexes $a^l e_1$ and $a^l\overline{e_1}$, where $l = 0, 1, \ldots, s-1$. We note that, if $s$ is even $-a^{l_1}e_1$ is $a^{l_2}e_1$ for some $l_2$. If $s$ is odd $-a^{l_1}e_1$ is $a^{l_2}\overline{e_1}$ for some $l_2$. We apply the same argument to $S^1_2$. Similarly, we have the vertexes $a^l e_3$ and $a^l\overline{e_3}$, where $l = 0, 1, \ldots, s-1$ and $\overline{e_3}$ denotes the vertex following $e_3$ in the counter clockwise direction. Since $G$ acts cellularly on both $CW$-complexes, we defined for $S^1_1$ and $S^1_2$, $G$ also acts cellularly on $S^1_1 * S^1_2 = S^3$. Thus, we have a $G$-complex structure for $S^3$. However, for simplicity, we will use a simplicial complex based on the above definition. That is, we take the join of two polygons formed by the vertexes, given above, on $S^1_1$ and $S^1_2$ respectively. Similarly, it has the cellular action of $G$. Since the representation is orthogonal, it is $G$-homeomorphic to $S^3$ via radial projection.

In the case $n/\gcd(n, k_1) = s = n/\gcd(n, k_2)$, we have $\gcd(n, k_1) = \gcd(n, k_2)$. If we write $\gcd(n, k_i) = c_i n + d_i k_i$, then $1 = c_i s + d_i(k_i/\gcd(n, k_i))$. So, we can uniquely determine $d_i$ as the multiplicative inverse of $k_i/\gcd(n, k_i)$ modulo $s$ in the interval $(0, s)$ for $i = 1, 2$. Thus, the edge following $[e_1, \overline{e_1}]$ is $\left[\overline{e_1}, a^{d_1}e_1\right]$ on $S^1_1$ and the edge following $[e_3, \overline{e_3}]$ is $\left[\overline{e_3}, a^{d_2}e_3\right]$ on $S^1_2$. Furthermore, $a^{d_1}b\overline{e_1} = \overline{e_1}$ , $a^{d_2}b\overline{e_3} = \overline{e_3}$, and $a^{d_1}ba^t\overline{e_3} = a^t\overline{e_3}$ where t is $(d_1 - d_2)/2$ if $d_1 - d_2$ is even (if $s$ is even $d_1 - d_2$ is always even) or $(d_1 - d_2 + s)/2$ if $d_1 - d_2$ is odd.

Thus, our $G$-complex structure has the following $G$-cells.

- *0-cells(0.i)*: We have four 0-cells. These are formed by the vertexes on $e_1e_2$ plane and $e_3e_4$ plane.

  a. $G/\langle b, a^s\rangle \times e_1$,

  b. $G/\langle b, a^s\rangle \times e_3$,

  c. $G/\langle a^{d_1}b, a^s\rangle \times \overline{e_1}$,

  d. $G/\langle a^{d_1}b, a^s\rangle \times a^t\overline{e_3}$.

- *1-cells(1.i)*: We have two 1-cells. These are formed by the edges on $e_1e_2$ plane and $e_3e_4$ plane respectively.

  a. $G/\langle a^s\rangle \times [e_1, \overline{e_1}]$,

  b. $G/\langle a^s\rangle \times [e_3, \overline{e_3}]$.

- *1-cells(1.ii)*: We have four 1-cells formed by joining two vertexes, one from the $e_1e_2$ plane and the other from the $e_3e_4$ plane with the isotropy group generated by a reflection and a rotation.

  a. $G/\langle b, a^s\rangle \times [e_1, e_3]$,

  b. $G/\langle b, a^s\rangle \times [e_1, -e_3]$,

  c. $G/\langle a^{d_1}b, a^s\rangle \times [\overline{e_1}, a^t\overline{e_3}]$,

  d. $G/\langle a^{d_1}b, a^s\rangle \times [\overline{e_1}, -a^t\overline{e_3}]$.

- *1-cells(1.iii)*: We have $(2s-2)$-many 1-cells formed by joining two 0-cells one from the $e_1e_2$ plane and the other from the $e_3e_4$ plane with the isotropy group generated by a rotation.

  a. $G/\langle a^s\rangle \times [e_1, a^{d_2 i}e_3]$,

  b. $G/\langle a^s\rangle \times [e_1, a^{d_2 j}\overline{e_3}]$,

  c. $G/\langle a^s\rangle \times [\overline{e_1}, a^{d_2 i}a^t\overline{e_3}]$,

  d. $G/\langle a^s\rangle \times [\overline{e_1}, a^{d_2(j+1)}a^t e_3]$, where

- $i = 1, 2, \ldots, (s-1)/2$ and $j = 0, 1, \ldots, (s-3)/2$ if $s$ is odd.
- $i = 1, 2, \ldots, (s-2)/2$ and $j = 0, 1, \ldots, (s-2)/2$ if $s$ is even.

- *2-cells(2.i)*: We have $4s$-many 2-cells formed by either joining a 1-cell from the $e_1e_2$ plane and a 0-cell from the $e_3e_4$ plane or vice-versa.

  a. $G/\langle a^s\rangle \times [e_1, \overline{e_1}, a^l e_3]$,

  b. $G/\langle a^s\rangle \times [e_1, \overline{e_1}, a^l \overline{e_3}]$,

  c. $G/\langle a^s\rangle \times [e_1, a^l e_3, a^l \overline{e_3}]$,

  d. $G/\langle a^s\rangle \times [\overline{e_1}, a^l e_3, a^l \overline{e_3}]$,

  where $l = 0, 1, \ldots, s-1$.

- *3-cells(3.i)*: We have $2s$-many 3-cells formed by joining two 1-cells one from the $e_1e_2$ plane and the other from the $e_3e_4$ plane.

  a. $G/\langle a^s\rangle \times [e_1, \overline{e_1}, a^l e_3, a^l \overline{e_3}]$,

  b. $G/\langle a^s\rangle \times [e_1, \overline{e_1}, a^l b e_3, a^l b \overline{e_3}]$,

  where $l = 0, 1, \ldots, s-1$.

Assume we have a simplicial complex with the cellular action of $G$. Let $H$ be the isotropy group of the simplex $\sigma$ and $x$ be an point in $\sigma$. If $x$ is an interior point, then necessarily $x$ have the same isotropy group $H$. Otherwise, we require it to have the same isotropy group $H$. Now, let $y$ be a point on the boundary of $\sigma$ with the same isotropy group $H$. Since $gx$ and $gy$ are in $g\sigma$, $t(gy) + (1-t)(gx)$ is in $g\sigma$ for $t \in [0,1]$. We define maps $f_0, f_1 : G/H \to G/H \times \sigma$ such that $f_0(gH) = gx$ and $f_1(gH) = gy$. We define a homotopy $F : G/H \times [0,1] \to G/H \times \sigma$ from $f_0$ to $f_1$ by $(gH, t) \mapsto t(gy) + (1-t)(gx)$. Thus, $F(gH, 0) = gx$, $F(gh, 1) = gy$ and

$$\begin{aligned} F(g'(gH,t)) &= F(g'gH, t) = t(g'gy) + (1-t)(g'gx) \\ &= g'(t(gy)) + g'((1-t)(gx)) = g'(t(gy) + (1-t)(gx)) \\ &= g'F(gH, t). \end{aligned}$$

Hence, $F$ is a $G$-homotopy. Therefore, in order to show the equivalence of the equivalence classes of two maps, it suffices to show that their respective $G$-cells have the same isotropy group $H$ and one of the following holds.

1. One of the cells is on the boundary of the other.
2. Both cells are on the boundary of another cell with the same isotropy group $H$.

By cancellation of cells, we mean the cancellation of their corresponding equivalence classes in the sum given for strong Euler characteristic.

Using the $G$-homotopy above, we can say that

$$G/\langle b, a^s\rangle \times e_1, G/\langle b, a^s\rangle \times e_3,$$
$$G/\langle a^{d_1}b, a^s\rangle \times \overline{e_1}, G/\langle a^{d_1}b, a^s\rangle \times a^t\overline{e_3}$$

cancel with

$$G/\langle b, a^s\rangle \times [e_1, -e_3], G/\langle b, a^s\rangle \times [e_1, e_3],$$
$$G/\langle a^{d_1}b, a^s\rangle \times [\overline{e_1}, -a^t\overline{e_3}], G/\langle a^{d_1}b, a^s\rangle \times [\overline{e_1}, a^t\overline{e_3}]$$

respectively.

That is, $G$-cells in 0.i cancel with $G$-cells in 1.ii.

Consider the set of cells $G/\langle a^s\rangle \times [e_1, a^l e_3, a^l\overline{e_3}]$ where $l = 0, 1, \ldots, s-1$ from 2.i.c which is the same set of cells as $G/\langle a^s\rangle \times [e_1, a^{d_2 l}e_3, a^{d_2 l}\overline{e_3}]$ where $l = 0, 1, \ldots, s-1$. We partition this set of $G$-cells into three. We have:

2.i.c1. $G/\langle a^s\rangle \times [e_1, e_3, \overline{e_3}]$,

2.i.c2. $G/\langle a^s\rangle \times [e_1, a^{d_2 i}e_3, a^{d_2 i}\overline{e_3}]$, $i = 1, 2, \ldots, m$,

2.i.c3. $G/\langle a^s\rangle \times [e_1, a^{d_2 i}e_3, a^{d_2 i}\overline{e_3}]$, $i = s-1, s-2, \ldots, m+1$.

$[e_1, e_3, \overline{e_3}]$ and $[e_1, \overline{e_1}]$ are on the boundary of $[e_1, \overline{e_1}, e_3, \overline{e_3}]$ and all have the same isotropy group. Thus, first one $G/\langle a^s\rangle \times [e_1, e_3, \overline{e_3}]$ cancels with $G/\langle a^s\rangle \times [e_1, \overline{e_1}]$.

If we rewrite the third set of $G$-cells (2.i.c3) above in a different way using the following operation:

$$\begin{aligned}
& a^{d_2 s} b \cdot G/\langle a^s\rangle \times [e_1, a^{d_2 i} e_3, a^{d_2 i}\overline{e_3}] \\
&= G/\langle a^s\rangle \times [a^{d_2 s} b e_1, a^{d_2 s} a^{-d_2 i} b e_3, a^{d_2 s} a^{-d_2 i} b\overline{e_3}] \\
&= G/\langle a^s\rangle \times [e_1, a^{d_2(s-i)} e_3, a^{d_2(s-i)} a^{-d_2} a^{d_2} b\overline{e_3}] \\
&= G/\langle a^s\rangle \times [e_1, a^{d_2(s-i)} e_3, a^{d_2(s-i-1)}\overline{e_3}]
\end{aligned}$$

Setting $j = s - i - 1$, we have

$$G/\langle a^s\rangle \times [e_1, a^{d_2(j+1)} e_3, a^{d_2 j}\overline{e_3}], \;\; j = 0, 1, \ldots, s - m - 2.$$

If $s$ is odd, we choose $m$ to be $(s-1)/2$ so $s - m - 2 = (s-3)/2$ . If $s$ is even, we choose $m$ to be $(s-2)/2$ then $s - m - 2 = (s-2)/2$.

Thus, the second (2.i.c2) and the third (2.i.c3) sets of the partition of 2.i.c cancel with $G/\langle a^s\rangle\times[e_1, a^{d_2 i} e_3]$ and $G/\langle a^s\rangle\times[e_1, a^{d_2 j}\overline{e_3}]$ respectively, where $i = 1, 2, \ldots, (s-1)/2$ and $j = 0, 1, \ldots, (s-3)/2$ if $s$ is odd, or $i = 1, 2, \ldots, (s-2)/2$ and $j = 0, 1, \ldots, (s-2)/2$ if $s$ is even.

That is, $G$-cells of 2.i.c cancel with $G$-cells in 1.i.a, 1.iii.a, 1.iii.b.

Now, consider the set of cells $G/\langle a^s\rangle \times [\overline{e_1}, a^l e_3, a^l\overline{e_3}]$ where $l = 0, 1, \ldots, s-1$ from 2.i.d which are the same set of cells as $G/\langle a^s\rangle\times[\overline{e_1}, a^{d_2 l} a^t e_3, a^{d_2 l} a^t\overline{e_3}]$, $l = 0, 1, \ldots, s-1$. We partition this set of $G$-cells into three. We have:

2.i.d1. $G/\langle a^s\rangle \times [\overline{e_1}, a^t e_3, a^t\overline{e_3}]$,

2.i.d2. $G/\langle a^s\rangle \times [\overline{e_1}, a^{d_2(j+1)} a^t e_3, a^{d_2(j+1)} a^t\overline{e_3}]$, $\;\; j = 0, 1, \ldots, m$,

2.i.d3. $G/\langle a^s\rangle \times [\overline{e_1}, a^{d_2(j+1)} a^t e_3, a^{d_2(j+1)} a^t\overline{e_3}]$, $\;\; j = s-2, s-3, \ldots, m+1$.

We can write $G/\langle a^s\rangle \times [e_3, \overline{e_3}]$ as $G/\langle a^s\rangle \times [a^t e_3, a^t\overline{e_3}]$. So, it cancels with the first one $G/\langle a^s\rangle \times [\overline{e_1}, a^t e_3, a^t\overline{e_3}]$.

If we rewrite the third set of $G$-cells (2.i.d3) in a different way using the following operation.

$$
\begin{aligned}
&a^{d_2 s}a^{d_1}b\cdot G/\langle a^s\rangle\times[\overline{e_1},a^{d_2(j+1)}a^te_3,a^{d_2(j+1)}a^t\overline{e_3}]\\
&=G/\langle a^s\rangle\times[a^{d_2 s}a^{d_1}b\overline{e_1},a^{d_2 s}a^{d_1}ba^{d_2(j+1)}a^te_3,a^{d_2 s}a^{d_1}ba^{d_2(j+1)}a^t\overline{e_3}]\\
&=G/\langle a^s\rangle\times[\overline{e_1},a^{d_2 s}a^{d_1}a^{d_2(-j-1)}a^{-t}be_3,a^{d_2 s}a^{d_2(-j-1)}a^{d_1}ba^t\overline{e_3}]\\
&=G/\langle a^s\rangle\times[\overline{e_1},a^{d_1}a^{d_2(s-j-1)}a^{-t}e_3,a^{d_2(s-j-1)}a^t\overline{e_3}]
\end{aligned}
$$

Setting $i=s-j-1$, we have

$$G/\langle a^s\rangle\times[\overline{e_1},a^{d_1}a^{d_2 i}a^{-t}e_3,a^{d_2 i}a^t\overline{e_3}],\ i=1,2,\ldots,s-m-2.$$

If $s$ is odd, we choose $m$ to be $(s-3)/2$ so $s-m-2=(s-1)/2$ . If $s$ is even, we choose $m$ to be $(s-2)/2$ then $s-m-2=(s-2)/2$.

Thus, the second (2.i.d2) and the third (2.i.d3) sets of the partition of 2.i.d cancel with $G/\langle a^s\rangle\times[\overline{e_1},a^{d_2(j+1)}a^te_3]$ and $G/\langle a^s\rangle\times[\overline{e_1},a^{d_2 i}a^t\overline{e_3}]$ respectively where if $s$ is odd, $j=0,1,\ldots,(s-3)/2$ and $i=1,2,\ldots,(s-1)/2$ or if $s$ is even, $j=0,1,\ldots,(s-2)/2$ and $i=1,2,\ldots,(s-2)/2$.

That is, $G$-cells of 2.i.d cancel with $G$-cells in 1.i.b, 1.iii.d, 1.iii.c.

Since we have $a^lbe_3=a^le_3$, the $G$-cells from 2.i (2.i.a and 2.i.b)

$$G/\langle a^s\rangle\times[e_1,\overline{e_1},a^le_3],G/\langle a^s\rangle\times[e_1,\overline{e_1},a^l\overline{e_3}]$$

cancel with the cells

$$G/\langle a^s\rangle\times[e_1,\overline{e_1},a^lbe_3,a^lb\overline{e_3}],G/\langle a^s\rangle\times[e_1,\overline{e_1},a^le_3,a^l\overline{e_3}]$$

from 3.i respectively where $l=0,1,\ldots,s-1$.

That is, $G$-cells in 2.i.a and 2.i.b cancel with $G$-cells in 3.i.b and 3.i.a.

Hence, $G$-cells in 2.i cancel with $G$-cells in 1.i, 1.iii and 3.i and $G$-cells in 0.i cancel with $G$-cells in 1.ii. As a result, the equivariant Euler characteristic is 0. Thus there is a nonzero $G$-field on $S(V)\approx S^3$.

This completes the proof of Proposition 5.3.4.

**Proof of Theorem 1.0.2.** The proof follows from Lemma 4.0.11, Proposition 5.3.1 and Proposition 5.3.4 ■

# CHAPTER 6

# $G$-FIELDS ON $S^3$ FOR QUATERNION GROUP ACTIONS

In this chapter, we will give the proof of Theorem 1.0.3 about the existence of $G$-fields on the representation sphere $S(V)$ of a four-dimensional real orthogonal representation space $V$ of $G$, where $G$ is the generalized quaternion group $Q_{2^{n+1}}$.

## 6.1 Real Representations of the Quaternion Group

The dicyclic group of order $4n$ generated by elements $a$ and $b$ has the presentation

$$\mathrm{Dic}_n = \left\langle a, b \mid a^{2n} = 1, a^n = b^2, ab = ba^{-1} \right\rangle.$$

The elements of $\mathrm{Dic}_n$ are $1, a, \ldots, a^{2n-1}, b, ab, \ldots, a^{2n-1}b$. The generalized quaternion group of order $2^{n+1}$ is

$$Q_{2^{n+1}} = \left\langle a, b \mid a^{2^n} = 1, a^{2^{n-1}} = b^2, ab = ba^{-1} \right\rangle.$$

So, it can be considered as the dicyclic group with parameter $2^{n-1}$. The dicyclic group, $\mathrm{Dic}_n$, has the following complete set of pairwise nonisomorphic irreducible complex representations [6].

One-dimensional irreducible complex representations when $n$ is odd:

- $q_0(a) = 1, q_0(b) = 1,$
- $q_1(a) = 1, q_1(b) = -1,$
- $q_2(a) = -1, q_2(b) = i,$

- $q_3(a) = -1, q_3(b) = -i$.

One-dimensional irreducible complex representations when $n$ is even:

- $q_0(a) = 1, q_0(b) = 1$,
- $q_1(a) = 1, q_1(b) = -1$,
- $q_2(a) = -1, q_2(b) = 1$,
- $q_3(a) = -1, q_3(b) = -1$.

Two-dimensional irreducible complex representations:

$$\rho_k(a) = \begin{pmatrix} \omega^k & 0 \\ 0 & \omega^{-k} \end{pmatrix}, \qquad \rho_k(b) = \begin{pmatrix} 0 & (-1)^k \\ 1 & 0 \end{pmatrix}$$

where $1 \leq k < n$ and $\omega = e^{i\pi/n}$.

Now, we can proceed to determine the types of these irreducible representations. We note that $(a^m b)^2 = a^m b a^m b = b a^{-m} a^m b = b^2$. We can find the types of these complex representations with the help of Proposition 2.1.17. We have

$$\frac{1}{|G|}\sum_{g\in G}\chi(g^2) = \frac{1}{4n}\left(\sum_{m=0}^{2n-1}\chi(a^{2m}) + \sum_{m=0}^{2n-1}\chi((a^m b)^2)\right) = \frac{1}{4n}\left(\sum_{m=0}^{2n-1}\chi(a^{2m}) + 2n\chi(b^2)\right).$$

Type of one-dimensional complex representations when $n$ is odd:

$q_i(a^{2m}) = 1$ for $i = 0, 1, 2, 3$ and $m = 0, 1, 2, \ldots, 2n-1$

- For $q_0$ and $q_1$ we have $q_i(b^2) = 1$, the sum is $(1/4n)(2n + 2n) = 1$. Then these representations are of real type.
- For $q_2$ and $q_3$ we have $q_i(b^2) = -1$, the sum is $(1/4n)(2n - 2n) = 0$. Then these representations are of complex type.

Type of one-dimensional complex representations when $n$ is even:

$q_i(a^{2m}) = 1$ and $q_i(b^2) = 1$ for i=0,1,2,3 and $m = 0, 1, 2, \ldots, 2n-1$

- For $q_0, q_1, q_2$ and $q_3$, the sum is $(1/4n)(2n+2n) = 1$. Then these representations are of real type.

Type of two-dimensional complex representations:

Let $\chi$ denote the character of $\rho_k$. Then $\chi(a^{2m}) = 2\cos(2mk\pi/n)$ and $\chi(b^2) = 2(-1)^k$. Since $(a^m b)^2 = b^2$, we have

$$\sum_{m=0}^{2n-1} \chi((a^m b)^2) = \sum_{m=0}^{2n-1} \chi(b^2) = 2n\chi(b^2) = 2n(2(-1)^k) = 4n(-1)^k.$$

Also,

$$\sum_{m=0}^{2n-1} \chi(a^{2m}) = \sum_{m=0}^{n-1} \chi(a^{2m}) + \sum_{m=n}^{2n-1} \chi(a^{2m}) = 2\sum_{m=0}^{n-1} \chi(a^{2m}) = 2\sum_{m=0}^{n-1} 2\cos\frac{2mk\pi}{n} = 0.$$

Then using Proposition 2.1.17 we determine their types:

$$\frac{1}{|G|}\sum_{g\in G} \chi(g^2) = \frac{1}{4n}\left(\sum_{m=0}^{2n-1} \chi(a^{2m}) + \sum_{m=0}^{2n-1} \chi(a^{2m})\right) = \frac{1}{4n}\left(0 + 4n(-1)^k\right) = (-1)^k.$$

So, $\rho_k$ is of real type if k is even and of quaternionic type if k is odd.

Using the fact that $Q_{2^{n+1}}$ is the dicyclic group with parameter $2^{n-1}$, we have the following complete set of irreducible real representations of $Q_{2^{n+1}}$.

One-dimensional irreducible real representations:

They correspond to $q_0, q_1, q_2$ and $q_3$ (for even $n$ in dicyclic group case) respectively.

- $r_0(a) = 1, r_0(b) = 1$,
- $r_1(a) = 1, r_1(b) = -1$,
- $r_2(a) = -1, r_2(b) = 1$,
- $r_3(a) = -1, r_3(b) = -1$.

We also denote their corresponding representation spaces with $r_0, r_1, r_2$ and $r_3$ respectively.

Two-dimensional irreducible real representation spaces $V_k$:

They correspond to the complex representation spaces of $\rho_{2k}$ and defined by:

$$a \mapsto \begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix}, \qquad b \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $1 \leq k \leq (2^{n-1} - 2)/2$ and $\theta_k = 2\pi k/2^{n-1}$ provided that $n \geq 3$.

Four-dimensional irreducible real representations spaces $U_k$:

They correspond to the two-dimensional complex representations of quaternionic type. Thus, they are realifications of the complex representation spaces of $\rho_k$ for odd $k$ where $1 \leq k < 2^{n-1}$.

## 6.2 Existence of $G$-Fields on $S(V)$

Let $V$ be a four-dimensional real representation space of $Q_{2^{n+1}}$. We want to inspect all the possible decompositions of $V$ and their actions on $S(V) \approx S^3$. We inspected the representations which can be decomposed into a direct sum of one-dimensional representations in a more general case. We have the irreducible real representation spaces with the similar actions; $V_0 = r_0, r_1, r_2, r_3$ and $V_k$ as in the dihedral group case for even $n$. Also, these irreducible real representations of $Q_{2^{n+1}}$ have the following properties for any element $x$ of the representation space.

1. $b^2 x = x$,
2. $a^{2^{n-1}} x = x$,
3. $abx = ba^{-1}x$.

Thus, we have no equivariant vector field in the following cases:

- $2V_0 \oplus V_k$,
- $V_0 \oplus r_m \oplus V_k$, $2r_m \oplus V_k$,
- $r_1 \oplus r_2 \oplus V_k$, $r_1 \oplus r_3 \oplus V_k$, $r_2 \oplus r_3 \oplus V_k$.

Similar to the $2V_k$ case for the dihedral group, we have the same equivariant vector field in same case for the quaternion group. Since $U_k$ is the realification of a two-dimensional complex representation of quaternionic type, the vector field $i_x$ is equivariant.

Since the action on $V_k$ has the above properties, the remaining case $V_{k_1} \oplus V_{k_2}$ is similar to the dihedral group case. When $\gcd(2^{n-1}, k_1) \neq \gcd(2^{n-1}, k_2)$, there is no nonzero $G$-field. When $\gcd(2^{n-1}, k_1) = \gcd(2^{n-1}, k_2)$, we see that there exists a nonzero $G$-field using the strong Euler characteristic.

As a result, the proof of Theorem 1.0.3 is similar to the proof of Theorem 1.0.2.

# CHAPTER 7

# $G$-FIELDS ON $S^3$ FOR SEMI-DIHEDRAL GROUP ACTIONS

In this chapter, we will give the proof of Theorem 1.0.4 about the existence of $G$-fields on the representation sphere $S(V)$ of a four-dimensional real orthogonal representation space $V$ of $G$, where $G$ is the semi-dihedral group $\mathrm{SD}_{2^n}$.

## 7.1 Real Representations of the Semi-Dihedral Group

The semi-dihedral group of order $2^n$, for $n \geq 4$, is given by

$$\mathrm{SD}_{2^n} = \left\langle a, b \mid a^{2^{n-1}} = b^2 = 1, bab = a^{2^{n-2}-1} \right\rangle.$$

These $2^n$ elements are $1, a, \ldots, a^{2^{n-1}-1}, b, ab, \ldots, a^{2^{n-1}-1}b$. It has the following complete set of irreducible complex representations [15].

One-dimensional irreducible complex representations:

- $q_0(a) = 1, q_0(b) = 1,$
- $q_1(a) = 1, q_1(b) = -1,$
- $q_2(a) = -1, q_2(b) = 1,$
- $q_3(a) = -1, q_3(b) = -1.$

Two-dimensional irreducible complex representations:

Consider the set

$$C = \Big\{1, 2, 3, \ldots, 2^{n-3}, 2^{n-3}+2, 2^{n-3}+4, \ldots, 2^{n-2}, \\ 2^{n-2}+1, 2^{n-2}+3, \ldots, 2^{n-2}+(2^{n-3}-1)\Big\}.$$

For $k \in C - \{2^{n-2}\}$, we have two-dimensional irreducible complex representations

$$\rho_k(a) = \begin{pmatrix} \omega^k & 0 \\ 0 & \omega^{(2^{n-2}-1)k} \end{pmatrix}, \qquad \rho_k(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $\omega = e^{i\pi/2^{n-2}}$. The character of $\rho_k$ is

$$\chi_k(a^m b) = 0, \chi_k(a^m) = \begin{cases} 2\cos\frac{km\pi}{2^{n-2}} & \text{if } km \text{ is even,} \\ 2i\sin\frac{km\pi}{2^{n-2}} & \text{if } km \text{ is odd.} \end{cases}$$

We can write the irreducible real representations since we know the irreducible complex representations. Firstly, all of the one-dimensional complex representations are clearly of real type. Thus, we have one-dimensional real representations:

- $r_0(a) = 1, r_0(b) = 1,$
- $r_1(a) = 1, r_1(b) = -1,$
- $r_2(a) = -1, r_2(b) = 1,$
- $r_3(a) = -1, r_3(b) = -1.$

Note that, since the order of the generator $a$ is $2^{n-1}$,

$$\begin{aligned} (a^m b)^2 &= (a^m b)(a^m b) = a^m(ba^m b) = a^m(bab)^m \\ &= a^m a^{m(2^{n-2}-1)} = a^{m(2^{n-2})} \\ &= \begin{cases} 1 & \text{if } m \text{ is even,} \\ a^{2^{n-2}} & \text{if } m \text{ is odd.} \end{cases} \end{aligned}$$

Then the character at $(a^m b)^2$ is

$$\chi_k((a^m b)^2) = \begin{cases} \chi_k(1) = 2 & \text{if m is even,} \\ \chi_k(a^{2^{n-2}}) = 2 & \text{if m is odd and k is even,} \\ \chi_k(a^{2^{n-2}}) = -2 & \text{if m is odd and k is odd.} \end{cases}$$

If $k$ is even, then

$$\sum_{m=0}^{2^{n-1}-1} \chi((a^m b)^2) = 2^{n-1}2 = 2^n.$$

If $k$ is odd, then

$$\sum_{m=0}^{2^{n-1}-1} \chi((a^m b)^2) = 0,$$

since $\chi((a^m b)^2)$ is -2 for odd $m$, 2 for even $m$ and we have even number of summands.

$$\begin{aligned}\sum_{m=0}^{2^{n-1}-1} \chi(a^{2m}) = \sum_{m=0}^{2^{n-2}-1} \chi(a^{2m}) + \sum_{m=2^{n-2}}^{2^{n-1}-1} \chi(a^{2m}) &= 2\sum_{m=0}^{2^{n-2}-1} \chi(a^{2m}) \\ = 2\sum_{m=0}^{2^{n-2}-1} 2\cos\frac{2km\pi}{2^{n-2}} = 4\sum_{m=0}^{2^{n-2}-1} \cos\frac{2km\pi}{2^{n-2}} &= 0.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{|G|}\sum \chi(g^2) &= \frac{1}{2^n}\left(\sum_{m=0}^{2^{n-1}-1} \chi(a^{2m}) + \sum_{m=0}^{2^{n-1}-1} \chi((a^m b)^2)\right) = \frac{1}{2^n}\left(0 + \sum_{m=0}^{2^{n-1}-1} \chi((a^m b)^2)\right) \\ &= \begin{cases} 1 & \text{if k is even,} \\ 0 & \text{if k is odd.} \end{cases}\end{aligned}$$

If $k$ is even $\rho_k$ is of real type. Indeed, when $k$ is even, we have:

$$\rho_k(a) = \begin{pmatrix} \omega^k & 0 \\ 0 & \omega^{-k} \end{pmatrix}, \qquad \rho_k(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which has the corresponding irreducible real representation space $V_k$ defined by

$$a \mapsto \begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix}, \qquad b \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $\theta_k = k\pi/2^{n-2}$, $k \in C - \{2^{n-2}\}$ and $k$ is even. Moreover, if we substitute $k/2$ for $k$, then $k$ runs through $1, 2, \ldots, 2^{n-3} - 1$ and $\theta_k$ becomes $2k\pi/2^{n-2}$.

Otherwise, if $k$ is odd we have four-dimensional irreducible real representations $U_k$ of complex type which are realifications of the representation spaces of $\rho_k$ for $k \in C - \{2^{n-2}\}$ and $k$ is odd.

## 7.2 Existence of $G$-Fields on $S(V)$

Let $V$ be a real orthogonal representation space of $\mathrm{SD}_{2^n}$. We want to inspect all the possible decompositions of $V$, so that, we can determine the existence of $G$-fields on $S(V) \approx S^3$. We inspected the representations that can be decomposed into a direct sum of one-dimensional representations in a more general case. We have the similar irreducible real representations $V_0 = r_0, r_1, r_2, r_3$ and $V_k$ as in the dihedral group case for even $n$. Furthermore, these irreducible real representations of $\mathrm{SD}_{2^n}$ have the following properties for any element $x$ of the representation space.

1. $b^2 x = x$,
2. $a^{2^{n-2}} x = x$,
3. $abx = ba^{-1} x$.

Hence, we have no equivariant vector field in the following cases:

- $2V_0 \oplus V_k$,
- $V_0 \oplus r_m \oplus V_k$, $2r_m \oplus V_k$,
- $r_1 \oplus r_2 \oplus V_k$, $r_1 \oplus r_3 \oplus V_k$, $r_2 \oplus r_3 \oplus V_k$.

Similar to the $2V_k$ case for the dihedral group, we have the same equivariant vector field $J$ in the same case for the semi-dihedral group. Since $U_k$ is the realification of a two-dimensional complex representation of complex type, the vector field $i_x$ is equivariant.

Since the action on $V_k$ has the above properties, the remaining case $V_{k_1} \oplus V_{k_2}$ is similar to the dihedral group case. When $\gcd(2^{n-2}, k_1) \neq \gcd(2^{n-2}, k_2)$, there is no nonzero $G$-field. When $\gcd(2^{n-2}, k_1) = \gcd(2^{n-2}, k_2)$, we see that there exists a nonzero $G$-field using the strong Euler characteristic.

Therefore, the proof of Theorem 1.0.4 is similar to the proof of Theorem 1.0.2.

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