HEAT AND MASS TRANSFER PROBLEM AND SOME APPLICATIONS

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ABSTRACT

HEAT AND MASS TRANSFER PROBLEM AND SOME APPLICATIONS

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Numerical solutions of mathematical modelizations of heat and mass transfer in cubical and cylindrical reactors of solar adsorption refrigeration systems are studied. For the resolution of the equations describing the coupling between heat and mass transfer, Bubnov-Galerkin method is used. An exact solution for time dependent heat transfer in cylindrical multilayered annulus is presented. Separation of variables method has been used to investigate the temperature behavior. An analytical double series relation is proposed as a solution for the temperature distribution, and Fourier coefficients in each layer are obtained by solving some set of equations related to thermal boundary conditions at inside and outside of the cylinder.

Keywords: Heat and mass transfer, Separation of variables, Bubnov-Galerkin method, Cylindrical reactor, Gauss-Jordan elimination

ÖZ

ISI VE KÜTLE TRANSFERİ PROBLEMİ VE BAZI UYGULAMALAR

Kılıç, İlker
Doktora, Fizik Bölümü
Tez Yöneticisi : Prof. Dr. Ramazan Sever

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Solar adsorpsiyonlu soğutma sistemlerinin kübik ve silindirik reaktörleri içerisindeki ısı ve kütle transferinin matematiksel modellemelerinin nümerik çözümü incelenmiştir. Isı ve kütle transferi arasındaki eşleşmeyi açıklayan denklemlerin çözümleri için Bubnov-Galerkin yöntemi kullanılmıştır. Silindirik çok katmanlı dairesel halka içerisindeki zamana bağlı ısı transferinin kesin (analitik) bir çözümü sunulmuştur. Sıcaklık davranışını incelemek için degişkenlere ayırma yöntemi kullanılmıştır. Sıcaklık dağılımının bir çözümü olarak analitik bir çift-seri denklemi önerilmiştir ve silindirin içindeki ve dışındaki termal sınır şartları ile ilgili bazı denklem setlerini çözerek, her katmandaki Fourier katsayıları elde edilmiştir.

Anahtar Kelimeler: Isı ve kütle transferi, Değişkenlere ayırma , Bubnov-Galerkin metodu, Silindirik reaktör, Gauss-Jordan eliminasyonu

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CHAPTER 1

INTRODUCTION

Together with the environmental problems and decrease in fossil-fuel resources, studies focused on renewable energy sources have been accelerated. Solar energy seems to be an alternative solution for these issues. In this study, we will analyze some specific components of solar energy systems numerically and analytically and try to understand the nature of heat and mass transfer issue in these parts of the system. Thus, we may have an idea about the energy consumption needs and we might be able to develop some strategies for using solar energy effectively.

Solid adsorption cooling machines constitute very attractive solutions to recover important amount of industrial waste, to heat medium temperature and to use renewable energy sources such as solar energy. The development of the technology of these machines can be carried out by experimental studies and by mathematical modelization. This latter method allows saving time and money because it is more supple to use to simulate the variation of different parameters [1]. Chapter 2 is about the numerical solution of a mathematical modelization of heat and mass transfer in cubical reactor of solar adsorption cooling machine. The adsorption cooling machines consist essentially of an evaporator, a condenser and a reactor containing a porous medium, which is in our case the activated carbon reacting by adsorption with ammonia. (The reactor is heated by solar energy and contains a porous medium constituted of activated carbon reacting by adsorption with ammonia).

In recent years, considerable attention has been paid to adsorption refrigeration systems, which are regarded as environmentally friendly alternatives to conventional vapour compression refrigeration systems, since they can use refrigerants that do not contribute to ozone layer depletion and global warming. In addition, the adsorption systems have the benefits of simpler control, no vibration and lower operation costs, if compared with mechanical vapour compression systems and, in comparison with the absorption systems, they do not need a solution pump or rectifier for the refrigerant, do not present corrosion problems due to the working pairs normally used, they are less sensitive to shocks and to the installation position [2] and they could be operated with no-moving parts [3]. Furthermore, refrigeration as a solar energy application is particularly attractive because of (i) the non-dependence on conventional power and (ii) the near coincidence of peak cooling loads with the solar energy availability. A schematic diagram of the solar powered continuous adsorption refrigeration system is presented by El Fadar et al. [4]. Chapter 3 is about the numerical solution of a mathematical modelization of heat and mass transfer in cylindrical reactor of solar adsorption refrigeration system (a cooling machine). For the resolution of the equations describing the coupling between heat and mass transfer, we use Bubnov-Galerkin method. We choose an appropriate (basis) function, hence find a transcendent equation which leads to a system of N differential equations strongly non linear. Such systems are handled either by Gauss-Seidel iterations, or by some numerical methods. As the basis function, we proposed first kind, zeroth order Bessel function (J_0) for the Bubnov-Galerkin solution. This function provides us with a good convergence to the exact solution (as time $\to \infty$), and the numerical solution.

In modern engineering applications, multilayered components are widely used due to the advantage of combining physical, mechanical, and thermal properties of different materials. Many of these applications require a detailed knowledge of transient temperature and heat-flux distribution within the component layers. Both analytical and numerical techniques may be used to solve such problems. Because of unavailability or mathematical complexity of exact solutions, numerical solutions are usually preferred in practice. However, in an era of numerical modeling and simulation, there is still a need for simple, accurate and physically meaningful analytical models. Rather limited use of analytical solutions should not diminish their merit over numerical ones; since exact solutions, if available, provide an insight into the governing physics of the problem, which is typically missing in any numerical solution. Moreover, analyzing closed-form solutions to obtain optimal design options for any particular application of interest is relatively simpler. In addition, exact solutions find their applications in validating and comparing various numerical algorithms to help improve computational efficiency of computer codes that currently rely on numerical techniques [5]. In Chapter 4, we present an exact solution for time dependent heat transfer in cylindrical multilayered annulus.

Separation of variables method has been used to investigate the temperature behavior. An analytical double series relation is presented as a solution for the temperature distribution, and Fourier coefficients in each layer are obtained by solving some set of equations related to thermal boundary conditions at inside and outside of the cylinder. Thermal continuity and heat flux continuity between each layer is considered, as well. The method of Gauss-Jordan elimination will be used to solve some set of equations. During our study, we verified analytical calculations comparing them with their numerical counterparts whenever it is necessary and possible.

The thesis is organized as follows: Chapter 2 is basically an application of Bubnov-Galerkin method in a cubically shaped reactor of a solar adsorption cooling machine. Chapter 3 is a complementary of this work, however this time, about a cylindrical shaped system. At this point, we will use the experience and basic knowledge gained during the study of the previous system. The last physical system that we will be dealing with is a cylindrical multilayered annulus; an analytical approach will be sought for transient heat equations.

CHAPTER 2

NUMERICAL STUDY OF HEAT AND MASS TRANSFER IN CUBICAL REACTOR

2.1 Cubical Reactor

This study is about the numerical solution of a mathematical modelization of heat and mass transfer in cubical reactor of solar adsorption cooling machine. The adsorption cooling machines consist essentially of an evaporator, a condenser and a reactor (which is the object of this work) containing a porous medium, which is in our case the activated carbon reacting by adsorption with ammonia. (The reactor is heated by solar energy and contains a porous medium constituted of activated carbon reacting by adsorption with ammonia). The figure is given below [1].

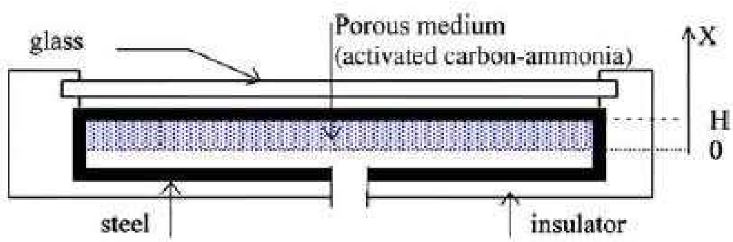

Figure 2.1: Sketch of the solar reactor studied

The principle can be described as follows;

When the adsorbent (at temperature T) is in exclusive contact with vapour of adsorbate (at pressure P), an amount m of adsorbate is trapped inside the micro-pores in an almost liquid state. This adsorbed mass m, is a function of T and P according to a divariant equilibrium $m = f(T, P)$. Moreover, at constant pressure, m decreases as T increases, and at constant adsorbed mass P increases with T. This makes it possible to imagine an ideal refrigerating cycle consisting of a period of heating/desorption/condensation followed by a period of cooling/adsorption/evaporation.

The model is mainly described by four equations;

$$\rho c_m \frac{\partial T}{\partial t} - Q(T) = K_e \frac{\partial^2 T}{\partial x^2} \tag{2.1}$$

Continuity of the heat flux at the interface adsorbent - metallic wall:

$$h(Tp - T(H)) = K_e \frac{\partial T}{\partial x}\bigg|_{x=H} \tag{2.2}$$

Adiabatic condition at the back of the solid adsorbent ($x = 0$):

$$\frac{\partial T}{\partial x}\bigg|_{x=0} = 0 \tag{2.3}$$

Initial condition; The bed is assumed to be at the uniform temperature:

$$T(x, 0) = T_{A'} \tag{2.4}$$

We make the following change of variable to eliminate the non-homogeneity of equations Eq. (2.2) and Eq. (2.4):

$$T^* = T - T_p \tag{2.5}$$

Eqs. (2.1-2.4) become:

$$\rho c_m \frac{\partial T^*}{\partial t} - K_e \frac{\partial^2 T^*}{\partial x^2} - Q + \rho c_m \frac{\partial T_p}{\partial t} = 0 \tag{2.6}$$

$$\frac{\partial T^*}{\partial x}\bigg|_{x=0} = 0 \tag{2.7}$$

$$\frac{\partial T^*}{\partial x}\bigg|_{x=H} = -\frac{h}{K_e} T^*(H) \tag{2.8}$$

$$T^*(x, 0) = 0 \tag{2.9}$$

For the resolution of the equations describing the coupling between heat and mass transfer, ”Bubnov-Galerkin” method is adapted. That is; the solution is assumed by the form:

$$T^*(x, t) = \sum_{i=1}^{N} a_i(t) \varphi_i(x) \tag{2.10}$$

We choose an appropriate (base)function φ_i and then find the coefficients $a_i(t)$ such that Eq. (2.10) satisfies Eqs. (2.6-2.9).

If we choose $\varphi_i(x)$ of the form:

$$\varphi_i(x) = \cos(\omega_i x) \tag{2.11}$$

and satisfying the boundary conditions, then putting Eq. (2.11) in Eq. (2.8) we obtain

$$(\omega_i H) tan(\omega_i H) = Bi \qquad i = 1, ...N \tag{2.12}$$

where Bi is the number of Biot: Bi=hH/K_e

The pulsations ω_i $(i = 1, ..N)$ are solutions of the transcendent equation Eq. (2.12).

The coefficients $a_i(t)$ are found by requiring that the scalar product of Eq. (2.6) and the weighting function $\varphi_j(x)$ $(j = 1, ...N)$ over the solution region is zero:

$$\sum_{i=1}^{N} \frac{da_i}{dt} < \varphi_i, \varphi_j > - \sum_{i=1}^{N} a_i < \frac{K_e}{\rho c_m} \frac{d^2\varphi}{dx^2}, \varphi_j > + < \frac{dT_p}{dt} - \frac{Q}{\rho c_m}, \varphi_j >= 0 \tag{2.13}$$

$$j = 1, ...N$$

where

$$< u, v >= \int_0^H u(x), v(x) dx$$

On matrix notation this differential equation is equivalent to

$$[A] \frac{d}{dt}[a] + [B][a] = [F]$$

This represents a system of N differential equations strongly non linear. In order to simplify this system of differential equations, and obtain a clear convergence analysis, we should deal with dimensionless variables. This is what we do in next section.

2.1.1 Convergence Study

The quantities ρ, c_m and Q depend strongly on the temperature and the pressure, hence the problem is not linear. For a clear convergence analysis, the quantities ρ, c_m and Q can be considered as constants. Then by using the following dimensionless quantities:

$$\theta = \frac{T - T_p}{T_o}, \qquad \bar{x} = \frac{x}{H} \qquad \tau = \frac{t\lambda}{\rho c_m H^2} \qquad Bi = \frac{hH}{\lambda} \qquad \Psi = \frac{H^2}{T_o \lambda}(Q + \rho c_m \frac{dT_p}{dt})$$

Eqs. (2.1-2.4) become:

$$\frac{\partial\theta}{\partial\tau} - \frac{\partial^2\theta}{\partial\bar{x}^2} - \Psi = 0 \qquad \frac{\partial\theta}{\partial\bar{x}}(\tau, 0) = 0 \tag{2.14}$$

$$\frac{\partial\theta}{\partial\bar{x}}(\tau, 1) = -Bi\theta(\tau, 1) \qquad \theta(0, \bar{x}) = 0 \tag{2.15}$$

Also, for simplification, the boundary condition T_p is considered as a linear function of time ($T_p = \zeta t + T_{A'}$), the dimensionless source term Ψ becomes

$$\Psi = \frac{H^2}{\lambda}(Q + \rho c_m \zeta) \tag{2.16}$$

In this case Eq. (2.13) is written as:

$$\sum_{i=1}^{N} \frac{da_i}{d\tau} < \varphi_i,\varphi_j > -\sum_{i=1}^{N} a_i < \frac{d^2\varphi}{d\bar{x}^2}, \varphi_j > - < \Psi, \varphi_j >= 0 \tag{2.17}$$

As time goes to infinity the time derivative term $\frac{da_i}{d\tau}$ goes to zero in order to have a physical solution (for temperature). This means the term $\frac{\partial\theta}{\partial\tau}$ also goes to zero. Using this fact and solving equations Eq. (2.14) and Eq. (2.15) simultaneously, we get the exact (analytical) solution for θ. For the sake of simplicity in convergence study, we take $\Psi = 1$ and $Bi = 10$. Then the exact solution comes out as

$$\theta(\infty, \bar{x}) = -\frac{\bar{x}^2}{2} + \frac{3}{5} \tag{2.18}$$

As $N > 1$, Eq. (2.17) will have many non-linear terms which contain $a(t)$ and da/dt with different coefficients. Such system of coupled non-linear equations can be solved by using Gauss-Seidel iterations. We solved them using Mathematica program, instead. As we use bigger N's, the numerical solutions get closer to the analytical result obtained for θ. As an example, see Table 2.1; [1]

Table 2.1: Comparison between the Bubnov-Galerkin solution and the exact solution ($Bi = 10, \Psi = 1, \tau = +\infty$)

x/H	Bubnov-Galerkin Solution						Exact Solution
	$N = 1$	$N = 2$	$N = 4$	$N = 6$	$N = 8$	$N = 10$	
0.0	0.6181	0.5968	0.5996	0.5999	0.5999	0.5999	0.6000
0.2	0.5930	0.5792	0.5802	0.5800	0.5799	0.5799	0.5800
0.4	0.5198	0.5230	0.5199	0.5199	0.5200	0.5199	0.5200
0.6	0.4045	0.4225	0.4198	0.4199	0.4200	0.4199	0.4199
0.8	0.2563	0.2766	0.2805	0.2801	0.2799	0.2799	0.2800
1.0	0.0874	0.0958	0.0990	0.0996	0.0998	0.0999	0.1000

2.1.2 Base function $\varphi_i(x) = cos(\omega_i x) + (\omega_i x) sin(\omega_i x)$

In order to see whether we can have a better convergence of the data (to the exact values), we tried some other test functions instead of $\varphi_i(x) = \cos(\omega_i x)$.

Note that using a different $\varphi_i(x)$ gives us transcendental equations different than Eq. (2.12). Let's use $\varphi_i(x) = cos(\omega_i x) + (\omega_i x) sin(\omega_i x)$ instead of $\varphi_i(x) = cos(\omega_i x)$. In this case our transcendental equation is of the form;

$$(\omega_i H)tan(\omega_i H) = -\frac{K_e}{hH}(\omega_i^2 H^2 + \frac{hH}{K_e}) \tag{2.19}$$

Noting that $Bi = \frac{hH}{K_e}$, we have;

$$(\omega_i H)tan(\omega_i H) = -\frac{1}{Bi}(\omega_i^2 H^2 + Bi) \tag{2.20}$$

We use the roots ($\omega_i H$'s) of this equation to construct the base functions (φ_i's), then we put these base functions in the scalar products in Eq. (2.17). This procedure will provide us with ixj coupled equations. We solved these coupled equations using Mathematica (instead of Gauss-Seidel iteration method), and we found a_i's, hence θ's (recall Eq. (2.10) and that $T^* = \Theta T_0$). The results are summarized in the Table 2.2.

Table 2.2: Comparison between the Bubnov-Galerkin solution and the exact solution for the new base function ($Bi = 10$, $\Psi = 1$, $\tau = +\infty$)

x/H	Bubnov-Galerkin Solution						Exact Solution
	$N = 1$	$N = 2$	$N = 4$	$N = 6$	$N = 8$	$N = 10$	
0.0	0.1700	0.2818	0.3788	0.4286	0.4591	0.4799	0.6000
0.2	0.1910	0.3251	0.4471	0.4994	0.5178	0.5244	0.5800
0.4	0.2374	0.3860	0.4464	0.4581	0.4791	0.4832	0.5200
0.6	0.2670	0.3538	0.3621	0.3839	0.3920	0.3954	0.4199
0.8	0.2302	0.2239	0.2553	0.2610	0.2626	0.2653	0.2800
1.0	0.0941	0.0740	0.0857	0.0905	0.0930	0.0945	0.1000

It is seen that the results are worse than we found before using $\varphi_i(x) = cos(\omega_i x)$ as the base function. The reason for this bad result might be explained like this; Eq. (2.6) looks like an inhomogenous heat equation when its last two terms are assumed constant. Heat equation is solved [7] using separation of variables method in which we split the assumed solution into *space* and *time* part. Space dependent part has a solution which has a *cosine* and *sine* . Here, we eliminated the *sine* part of the assumed solution because it does not obey the boundary conditions at hand. Most appropriate base function to be offered in Bubnov-Galerkin solution for this system seems to be $cos(\omega_i x)$ since now. In the following section we will try another base function.

2.1.3 Base function $\varphi_i(x) = P_2\left((\cos \omega_i x)\right)$

The transcendental equation for the Legendre Polynomial

$$\varphi_i(x) = P_2\left((\cos \omega_i x)\right) = \frac{1}{2}\left(3\cos^2(\omega_i x) - 1\right) \tag{2.21}$$

is of the form

$$(\omega_i H)\sin(2\omega_i H) = \frac{1}{3}Bi\left(3\cos^2(\omega_i H) - 1\right) \tag{2.22}$$

Roots of this nonlinear equation are $\omega_1 H = 2,0008$ and $\omega_2 H = 0,8669$. Utilizing a Mathematica code, we used these roots for the solution of equation (2.17) and found Bubnov-Galerkin solutions for θ at various x/H values for large time (τ) values. When these solutions are compared with the exact solutions, a fast convergence to exact results are observed.

The solutions for the other Legendre Polynomials $\varphi_i(x) = P_n\left((\cos \omega_i x)\right)$, $n = 2, 3, 4, ..., 8$ for large time are calculated as well. As an example solutions of P_2 and P_8 are given in tabular form in Table 2.3 and Table 2.4. As seen clearly, the convergence to exact solutions for various x/H values are still good with higher order Legendre Polynomials.

Table 2.3: Comparison between the Bubnov-Galerkin solution for $P_2\left((\cos \omega_i x)\right)$ and the exact solution ($Bi = 10$, $\Psi = 1$, $\tau = +\infty$)

x/H	Bubnov-Galerkin Solution for $P_2\left((\cos \omega_i x)\right)$						Exact Solution
	$N = 1$	$N = 2$	$N = 4$	$N = 6$	$N = 8$	$N = 10$	
0.0	0.6220	0.5953	0.5994	0.5999	0.6000	0.6000	0.6000
0.2	0.5943	0.5783	0.5803	0.5801	0.5800	0.5800	0.5800
0.4	0.5143	0.5234	0.5199	0.5199	0.5200	0.5200	0.5200
0.6	0.3916	0.4233	0.4197	0.4200	0.4200	0.4200	0.4199
0.8	0.2408	0.2763	0.2807	0.2801	0.2799	0.2800	0.2800
1.0	0.0798	0.0950	0.0989	0.0996	0.0998	0.0999	0.1000

Table 2.4: Comparison between the Bubnov-Galerkin solution for $P_8\left((\cos \omega_i x)\right)$ and the exact solution ($Bi = 10$, $\Psi = 1$, $\tau = +\infty$)

x/H	Bubnov-Galerkin Solution for $P_8\left((\cos \omega_i x)\right)$						Exact Solution
	$N = 1$	$N = 2$	$N = 4$	$N = 6$	$N = 8$	$N = 10$	
0.0	0.6225	0.5931	0.5993	0.5998	0.6000	0.6000	0.6000
0.2	0.5934	0.5777	0.5805	0.5801	0.5800	0.5800	0.5800
0.4	0.5101	0.5250	0.5198	0.5199	0.5201	0.5200	0.5200
0.6	0.3842	0.4234	0.4196	0.4200	0.4200	0.4200	0.4199
0.8	0.2328	0.2719	0.2809	0.2802	0.2799	0.2799	0.2800
1.0	0.0762	0.0915	0.0982	0.0994	0.0998	0.0999	0.1000

2.1.4 Results

As seen clearly; instead of a Cosine Function, a Legendre Polynomial (depending on cosine function) of any order n can be chosen as a basis function for implementation of Bubnov-Galerkin Method.

We may conclude that the most appropriate base functions to be offered in Bubnov-Galerkin solution for this system is $cos(\omega_i x)$ or $P_n\,((\cos\omega_i x))$.

In the following chapter, as a complementary work, we will examine the implementation of Bubnov Galerkin Method on a cylindrical reactor by choosing an appropriate basis function.

CHAPTER 3

HEAT AND MASS TRANSFER IN CYLINDRICAL ADSORPTION POROUS MEDIUM

3.1 Introduction

In recent years considerable attention has been paid to solid adsorption refrigeration systems. They are environmentally friendly alternatives to conventional vapor compression refrigeration systems, and they have been studied extensively [6]. They can use refrigerants that do not contribute to ozone layer depletion and global warming. When compared to mechanical vapor compression systems, benefits of the adsorption systems are; simpler control, no vibration and lower operation costs. On the other hand, in comparison with the absorption systems; they do not need a solution pump or rectifier for the refrigerant, do not present corrosion problems due to the working pairs normally used, they are less sensitive to shocks [2]. They could be operated with no-moving parts [3]. In spite of their advantages, the adsorption refrigeration systems have some disadvantages such as low coefficient of performance (COP), low specific cooling power (SCP), high weight and high cost. So as to overcome these inconveniences, scientists undertook various tasks such as improvement of heat and mass transfer in adsorbent beds, enhancement of the adsorption properties of the working pairs, design and study of different kind of cycles and improvement of regenerative heat and mass transfer between beds. However, the widespread use of the adsorption refrigeration systems is still limited by the technical and economic constraints. Works in this field are increasing to overcome these problems [4].

A more detailed schematic diagram of the solar powered continuous adsorption refrigeration system is presented by El Fadar et al [4]. The adsorption cooling machines consist essentially of an evaporator, a condenser and a reactor containing a porous medium (which is our task in this study). Several solar adsorption refrigeration units were tested with different combinations of adsorbents and adsorbates. The most studied pairs are activated carbon/ammonia, activated carbon/methanol, zeolite/water and silica gel/water. In this paper we study the reactor in which activated carbon is selected as adsorbent, and ammonia as refrigerant.

Assumptions in our theoretical model are;

◦ The adsorbent bed is considered as a continuous medium and the conduction heat transfer in the medium is characterized by an equivalent thermal conductivity, λ_e.

◦ The porous medium properties have a cylindrical symmetry.

◦ The heat transfer is radial and the convection heat transfer due to the radial mass transfer is neglected.

◦ The HTF temperature is uniform.

This chapter, through out the following sections, is organized as follows:

Heat & mass transfer equations and appropriate boundary conditions for a cylindrical reactor are developed in the light of cubic reactor equations. To simplify these equations a change of variable procedure is applied to the temperature. After proposing our basis function, using Bubnov-Galerkin method we end up with a system of N differential equations strongly nonlinear which is Eq. (3.18). In order to simplify this system of differential equations, some dimensionless quantities are substituted, and a simpler form is provided. This system of equations can be solved using Gauss-Seidel method. We used numerical techniques, instead. Finally a convergence study is presented with the help of tabulated values.

3.2 Cylindrical reactor

The adsorption refrigeration system we analyze has two cylindrical reactors (adsorbers) that contain the activated carbon-ammonia inside. An adsorber looks like the one shown in Fig. 3.1.

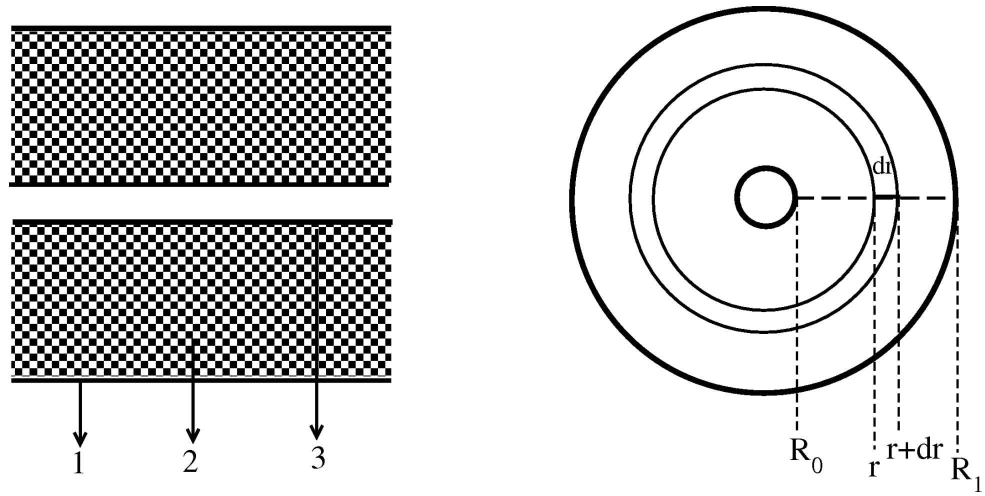

Figure 3.1: Sketch of the cylindrical solar reactor studied; 1. shell (insulating material), 2. adsorbent, 3. metal of heat exchange [4]

3.2.1 System Equations

The equation of heat and mass transfer in the porous medium is obtained by application of energy and mass conservation laws to a layer with radial coordinate r and thickness dr. [8]. Energy balance equation is

$$\frac{d}{dt}\Big[2\pi r dr L_r\Big((1-\varepsilon)\rho_s u_s + (\varepsilon-\theta)\rho_g u_g + \theta\rho_a u_a\Big)\Big] + q(r+dr,t)H_g(T\,(r+dr)\,,p) - q(r,t)H_g(T\,(r)\,,p) = 2\pi r\lambda_e dr L_r\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right] \tag{3.1}$$

where $H_g(T) = H_a(T) + \Delta H_{ads}$. $H_g(T)$ and $H_a(T)$ are specific enthalpies of ammonia at gaseous phase and adsorbed phase respectively. Mass conservation equation is

$$\frac{d}{dt}\Big[2\pi r dr L_r((\varepsilon-\theta)\rho_g + \theta\rho_a)\Big] = q(r,t) - q(r+dr,t) = -\frac{\partial q}{\partial r}dr \tag{3.2}$$

Heat and mass transfer equation in the porous medium is found by combination of equations Eq. (3.1) and Eq. (3.2).

$$\rho\, c_m(T)\,\frac{\partial T}{\partial t} - Q = \lambda_e \nabla_r^2 T \tag{3.3}$$

where

$$\rho c_m(T) = (1-\varepsilon)\rho_s C_{ps} + (\varepsilon - \theta)\rho_g C_{p_g} + \theta \rho_a C_{pa} \quad (3.4)$$

$$Q = \frac{\partial}{\partial t}((\varepsilon - \theta)\rho_g)\frac{p}{\rho_g} + \frac{1}{2\pi r dr L_r}\left(\frac{p}{\rho_a} + \Delta H_{ads}\right)\left(\frac{\partial m_a}{\partial t}\right) \quad (3.5)$$

$$\nabla_r^2 = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right] \quad (3.6)$$

∇_r^2 is the Laplacian. [4]

3.2.1.1 Temperature Boundary and Initial Conditions

The initial and boundary conditions depending on pressure and temperature are reported in detail for an adsorption refrigeration system which has two adsorbers in the work of El Fadar et al. [4]. Since this study deals with one of these adsorbers, we will only provide with temperature boundary conditions of one adsorber. Pressure boundary & initial conditions does not interest us for the time being.

Assuming that the metal of heat exchanger is thin, we take its inner and outer radius as R_0 for simplicity. Then, continuity of heat flux between adsorbent & metallic wall interface is written as

$$h_{gl}[T_{HTF} - T(R_0)] = -\lambda_e\left(\frac{\partial T}{\partial r}\right)_{r=R_0} \quad (3.7)$$

where h_{gl} is the heat transfer coefficient between the heat transfer fluid (HTF) and the adsorbent bed. Adiabatic condition at the rim of the cylinder shaped adsorber is

$$\left.\frac{\partial T(r,t)}{\partial r}\right|_{r=R_1} = 0 \quad (3.8)$$

The porous medium (or "adsorbent bed" as it is sometimes called) is assumed to be at the uniform temperature at $t = 0$

$$T(r,0) = T_0 \quad (3.9)$$

where T_0 is the morning ambient temperature for a cubic reactor [1], and T_{min} or T_{max} for a refrigeration system having two cylindrical adsorbers [4].

3.2.1.2 Bubnov-Galerkin Method

To eliminate the non-homogeneity of equations Eq. (3.7) and Eq. (3.9), let's make the following change of variable

$$T^* = T - T_{HTF} \tag{3.10}$$

Hence, Eqs. (3.3,3.7-3.9) become:

$$\rho\, c_m \frac{\partial T^*}{\partial t} - \lambda_e \nabla_r^2 T^* - Q + \rho\, c_m \frac{\partial T_{HTF}}{\partial t} = 0 \tag{3.11}$$

$$\left.\frac{\partial T^*}{\partial r}\right|_{r=R_o} = \frac{h_{gl}}{\lambda_e} T^*(R_o) \tag{3.12}$$

$$\left.\frac{\partial T^*}{\partial r}\right|_{r=R_1} = 0 \tag{3.13}$$

$$T^*(r,0) = 0 \tag{3.14}$$

For the resolution of the equations describing the coupling between heat and mass transfer, "Bubnov-Galerkin" method is adapted. In this method [10], the assumption is that the solution can be represented by an appropriate combination of analytical functions over the whole solution region without passing inevitably (in some conditions) by the discretization of the domain [11] as in the case of finite element method. By the Bubnov-Galerkin method, the solution is assumed by the form

$$T^*(r,t) = \sum_{i=1}^{N} a_i(t)\varphi_i(r) \tag{3.15}$$

The main problem at this stage is to choose an appropriate function $\varphi_i(r)$. In cartesian geometry [1], $\varphi_i(x)$ may be chosen as $cos(\omega_i x)$. Since our study is carried out on cylindrical coordinate, we should choose a basis function $\varphi_i(r)$ which might form a complete set of basis functions. The most appropriate one appears to be the First Kind, Zeroth Order Bessel function, $J_0(\omega_i r)$. [9, 12].

There are two boundary conditions that $\varphi_i(r)$ should satisfy in these four equations; Eq. (3.12) and Eq. (3.13). When we try to satisfy the boundary condition Eq. (3.13), we face with different R_1 values for each index value "i" that we use. To get rid of this trouble, we use $J_0(\omega_i(R_1 - r))$ instead of $J_0(\omega_i r)$. Thus, the boundary condition is satisfied trivially without any discretization. Hence, Eq. (3.15) takes the form;

$$T^*(r,t) = \sum_{i=1}^{N} a_i(t) J_0(\omega_i(R_1 - r)) \tag{3.16}$$

By inserting Eq. (3.16) into the second boundary condition Eq. (3.12), we obtain

$$\frac{d}{d\bar{r}}\Big[J_0(X_i(1-\frac{R_0}{R_1}\bar{r}))\Big]\Big|_{\bar{r}=1} = Bi\,Jo(X_i(1-\frac{R_0}{R_1})) \tag{3.17}$$

where Bi is the Biot number; $Bi = \frac{h_{gl}R_0}{\lambda_e}$. The pulsations $X_i = \omega_i R_1$ $(i = 1, ..N)$ are solutions of this transcendent equation. R_0 and R_1 are known. Some numerical values are given in Table 3.1.

Table 3.1: Roots of Eq. (3.17) for $Bi = 10$

i	1	2	3	4	$\cdots$	N
$\omega_i R_1$	2.9709	6.8205	10.6952	14.5786	$\cdots$	$\cdots$

Next step is to find the coefficients $a_i(t)$ such that Eq. (3.16) satisfies Eqs. (3.11-3.14). The coefficients $a_i(t)$ are found by imposing that the scalar product of Eq. (3.11) and the weighting function $\varphi_j(r)$, $(j = 1, ...N)$ over the solution region is zero;

$$\sum_{i=1}^{N}\frac{da_i}{dt} < \varphi_i(r), \varphi_j(r) > -\sum_{i=1}^{N} a_i(t) < \frac{\lambda_e}{\rho\, c_m}\nabla_r^2\varphi_i, \varphi_j > + < \frac{\partial T_{HTF}}{\partial t} - \frac{Q}{\rho\, c_m}, \varphi_j >= 0 \tag{3.18}$$

where

$$< \varphi_i(r), \varphi_j(r) >= \int_{R_0}^{R_1} \varphi_i(r)\varphi_j(r) 2\pi r dr \tag{3.19}$$

In matrix notation this differential equation is equivalent to

$$[A]\frac{d}{dt}[a] + [B][a] = [C] \tag{3.20}$$

This represents a system of N differential equations strongly nonlinear. The resolution is made by implicit discretization. The coefficients in the discretization equations will themselves depend on T. Such situations may be handled by Gauss-Seidel iterations combined with the following expression of relaxation:

$$[a]^m = \Gamma\,[a]^{m^*} + (1-\Gamma)\,[a]^{m-1} \tag{3.21}$$

where $[a]^m$ is the approximation at the current iteration m, $[a]^{m-1}$ is the approximation at the previous iteration, m^* is the solution by Gauss-Seidel method at the current iteration m, Γ is the relaxation factor varying between 0 and 2. Utilizing a Fortran code, the required solution can be achieved.

3.2.2 Convergence Study and Calculations

The quantities ρ, c_m and Q depend strongly on the temperature and the pressure, so the problem is not linear. For a clear convergence analysis, the quantities ρ, c_m and Q can be considered as constants.

In order to simplify this system of differential equations, let's use the following dimensionless quantities;

$$\theta = \frac{T - T_{HTF}}{T_0}, \quad \bar{r} = \frac{r}{R_0}, \quad \tau = \frac{t\lambda_e}{\rho c_m R_0^2}, \quad Bi = \frac{h_{gl} R_0}{\lambda_e}, \quad \psi = \frac{R_0^2}{T_0 \lambda_e}\left(Q - \rho c_m \frac{dT_{HTF}}{dt}\right)$$

Eqs. (3.3,3.7-3.9) become

$$\frac{\partial \theta}{\partial \tau} - \nabla_{\bar{r}}^2 \theta - \psi = 0 \tag{3.22}$$

$$\left.\frac{\partial \theta(\bar{r}, \tau)}{\partial \bar{r}}\right|_{\bar{r}=\frac{R_1}{R_0}} = 0 \tag{3.23}$$

$$\left.\frac{\partial \theta(\bar{r}, \tau)}{\partial \bar{r}}\right|_{\bar{r}=1} = Bi\, \theta(1, \tau) \tag{3.24}$$

$$\theta(\bar{r}, 0) = 0 \tag{3.25}$$

For the sake of simplicity, the boundary condition T_{HTF} is considered as a linear function of time ($T_{HTF} = \varsigma t + T_0$); so the dimensionless source term ψ becomes $\psi = \frac{R_0^2}{T_0 \lambda_e}(Q - \rho c_m \varsigma)$. Remembering that $T^* = T - T_{HTF}$ we get

$$\theta(r, t) = \frac{T^*}{T_0} = \sum_{i=1}^{N} \frac{a_i(t) \varphi_i(r)}{T_0} \tag{3.26}$$

Putting this term into Eq. (3.22), then taking scalar product with φ_j gives

$$\sum_{i=1}^{N} \frac{da_i(t)}{d\tau} < \varphi_i, \varphi_j > - \sum_{i=1}^{N} a_i < \nabla_{\bar{r}}^2 \varphi_i, \varphi_j > - < \Psi, \varphi_j >= 0 \tag{3.27}$$

where

$$\Psi = \frac{R_0^2}{\lambda_e}(Q - \rho c_m \varsigma) \tag{3.28}$$

Eq. (3.27) may also be obtained by imposing new parameters $\bar{r}$ and τ on Eq. (3.18).

As we consider the parameter τ as a variable for the convergence analysis, the comparison between the Bubnov-Galerkin solution and the exact solution can be carried out in the stationary regime ($\tau \to \infty$) by writing $\frac{\partial \theta}{\partial \tau} = 0$ in Eq. (3.22). See Table 3.3. As $\tau \to \infty$ exact solution is

$$\theta(\bar{r}, \infty) = -\frac{\bar{r}^2}{4} + \frac{R_1^2}{2R_0^2} \ln(\bar{r}) + \frac{R_1^2}{20R_0^2} + \frac{1}{5} \tag{3.29}$$

Note also that as $\tau \to \infty$ first term of Eq. (3.27) vanishes, and equation becomes solvable analytically for small N integers.

3.3 Results

We have studied cylindrical reactor analytically by using Bubnov-Galerkin method. The problem is reduced solution of a system of N coupled differential equations strongly nonlinear. We have applied First kind, Zeroth order Bessel function which has been readily used for the solution of homogenous, cylindrical heat and mass transfer equation [9, 12]. The results shown in Tables 3.2 and 3.3 are coherent with the expectations. The percentage error in θ changes with the Biot number. However, the value of Biot number does not seem to affect the convergence pattern. See Fig. 3.2.

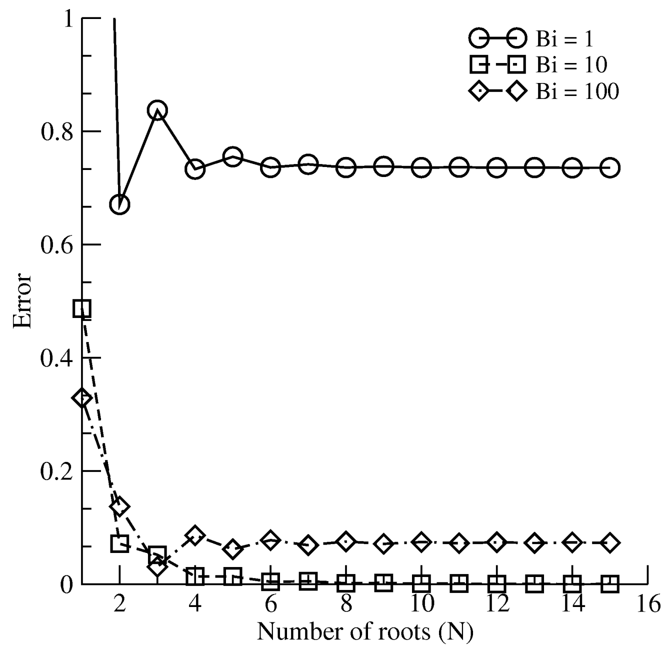

Figure 3.2: Relative deviation of error in θ with different Biot numbers. error=$\left|\frac{\theta_N(\bar{r})-\theta(\bar{r})}{\theta(\bar{r})}\right|$ $\theta_N(\bar{r})$ are the Bubnov-Galerkin solutions, $\theta(\bar{r})$ is the exact solution. $\bar{r}$ is chosen as 1.

Table 3.2: Comparison among different solutions of θ using First kind, Zeroth order Bessel function (J_0) for a cylindrical reactor (Bi = 10, Ψ = 1, τ = 1, R_0 = 53 cm, R_1 = 250 cm)

Radius(cm)	Bubnov-Galerkin			Numerical
	N = 5	N = 10	N = 15	
53.00	0.1159	0.1313	0.1351	0.1373
81.14	0.6404	0.6309	0.6290	0.6295
109.29	0.8553	0.8487	0.8460	0.8461
137.43	0.9276	0.9392	0.9401	0.9400
165.57	0.9890	0.9774	0.9786	0.9784
193.71	0.9915	0.9939	0.9931	0.9929
211.86	0.9928	0.9979	0.9980	0.9972
250.00	1.0103	0.9974	0.9992	0.9988

Table 3.3: Comparison among different solutions of θ using First kind, Zeroth order Bessel function (J_0) for a cylindrical reactor (Bi = 10, Ψ = 1, τ = $+\infty$ (chosen as 1000 in the numerical solution, 100 in Bubnov-Galerkin Solution), R_0 = 53 cm, R_1 = 250 cm)

Radius(cm)	Bubnov-Galerkin			Numerical	Exact
	N = 5	N = 10	N = 15		
53.00	0.9550	1.0294	1.0498	1.0617	1.0625
81.14	5.5728	5.4706	5.4624	5.4641	5.4648
109.29	8.4237	8.3121	8.3010	8.2997	8.3004
137.43	10.2528	10.2243	10.2335	10.2309	10.2316
165.57	11.7036	11.5359	11.5478	11.5445	11.5452
193.71	12.4921	12.3927	12.3948	12.3910	12.3917
211.86	12.9448	12.8559	12.8634	12.8592	12.8599
250.00	13.1844	12.9938	13.0104	13.0060	13.0067

CHAPTER 4

EXACT SOLUTION OF TRANSIENT HEAT TRANSFER EQUATION IN CYLINDRICAL MULTILAYERED ANNULUS

4.1 Introduction

An exact solution for time dependent heat transfer in cylindrical multilayered annulus is presented. Separation of variables method has been used to investigate the temperature behavior. An analytical double series relation is presented as a solution for the temperature distribution, and Fourier coefficients in each layer are obtained by solving some set of equations related to thermal boundary conditions at inside and outside of the cylinder. Thermal continuity and heat flux continuity between each layer is considered, as well. The method of Gauss-Jordan elimination will be used to solve some set of equations.

4.2 Formulation

Let us consider an n-layer annulus as shown schematically in Fig. 4.1 ($r_0 \leqslant r \leqslant r_n$). All the layers are assumed to be isotropic in thermal properties and are in perfect thermal contact. Let k_i and α_i be the temperature independent thermal conductivity and thermal diffusivity of the i^{th} layer, respectively. At the initial time $t = 0$, each i^{th} layer is at a specified temperature $f^{(i)}(r, \theta)$, and time-independent heat sources $g^{(i)}(r, \theta)$ are switched on at $t = 0$. The inner surface ($r = r_0, i = 1$), as well as the outer surface ($r = r_n, i = n$) of the annulus may be subjected to any combination of temperature and heat-flux boundary conditions. Since perfect thermal contact between the adjacent layers is not frequently observed in real materials, dealing with imperfect contact would require explicit modeling of the thermal resistance at the layer inter-

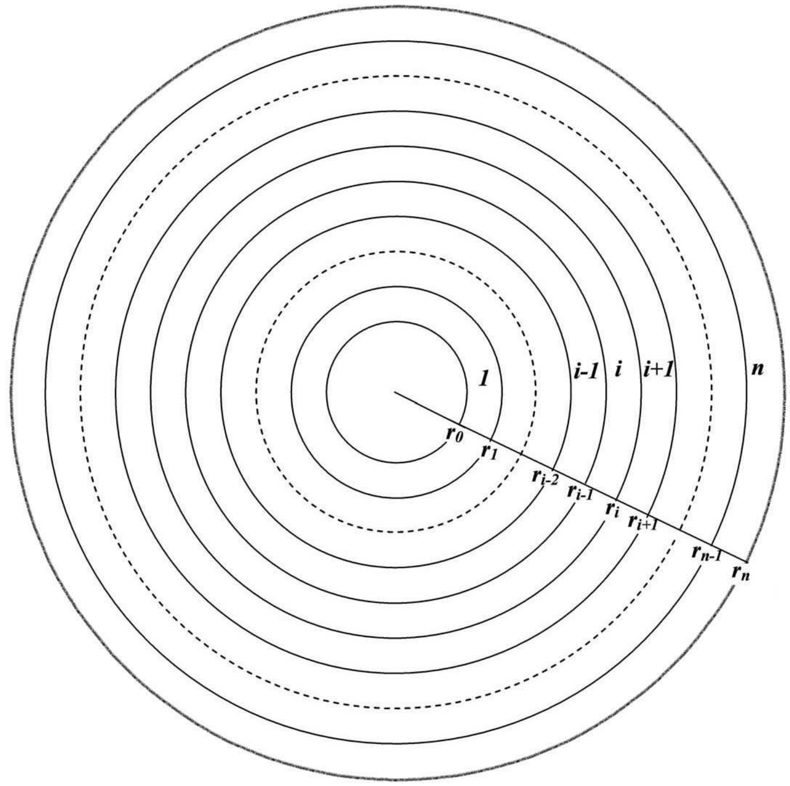

Figure 4.1: An n-layer annulus, i^{th} layer of which has an inner and outer radii equal to r_{i-1} and r_i, respectively.

faces [13–15]. Under such circumstances, the temperature at the contact interfaces will not be continuous.

Under these assumptions; the governing heat transfer equation can be written as ($0 \leqslant \theta \leqslant 2\pi$, $r_{i-1} \leqslant r \leqslant r_i$,and $t > 0$, where $i = 1, 2, ..., n$);

$$\frac{1}{\alpha_i}\frac{\partial T^{(i)}}{\partial t}(r,\theta,t) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T^{(i)}}{\partial r}(r,\theta,t)\right) + \frac{1}{r^2}\frac{\partial^2 T^{(i)}}{\partial \theta^2}(r,\theta,t) + \frac{g^{(i)}(r,\theta)}{k_i} \tag{4.1}$$

Boundary conditions are;

∗ Inner surface of the first layer ($0 \leqslant \theta \leqslant 2\pi$ and $t > 0$)

$$A_{in}\frac{\partial T^{(1)}}{\partial r}(r_0,\theta,t) + B_{in}T^{(1)}(r_0,\theta,t) = C_{in}(\theta) \tag{4.2}$$

∗ Outer surface of the n^{th} layer ($0 \leqslant \theta \leqslant 2\pi$ and $t > 0$)

$$A_{out}\frac{\partial T^{(n)}}{\partial r}(r_n,\theta,t) + B_{out}T^{(n)}(r_n,\theta,t) = C_{out}(\theta) \tag{4.3}$$

$*$ Interface of the $(i-1)^{st}$ and the i^{th} layer ($0 \leqslant \theta \leqslant 2\pi$ and $t > 0$, where $i = 2, ..., n$);

$$T^{(i)}(r_{i-1}, \theta, t) = T^{(i-1)}(r_{i-1}, \theta, t) \tag{4.4}$$

$$k_i \frac{\partial T^{(i)}}{\partial r}(r_{i-1}, \theta, t) = k_{i-1} \frac{\partial T^{(i-1)}}{\partial r}(r_{i-1}, \theta, t) \tag{4.5}$$

$*$ Periodic boundary conditions ($r_{i-1} \leqslant r \leqslant r_i$, $t > 0$, where $i = 1, 2, ..., n$)

$$T^{(i)}(r, \theta = 0, t) = T^{(i)}(r, \theta = 2\pi, t) \tag{4.6}$$

$$\frac{\partial T^{(i)}}{\partial \theta}(r, \theta = 0, t) = \frac{\partial T^{(i)}}{\partial \theta}(r, \theta = 2\pi, t) \tag{4.7}$$

$*$ Initial condition ($r_{i-1} \leqslant r \leqslant r_i$, $0 \leqslant \theta \leqslant 2\pi$, where $i = 1, 2, ..., n$);

$$T^{(i)}(r, \theta, t = 0) = f^{(i)}(r, \theta) \tag{4.8}$$

Boundary conditions may be imposed at $r = r_0$ and $r = r_n$ by choosing the appropriate coefficients in Eq. (4.2) and Eq. (4.3). In addition, multiple layers with zero inner radius ($r_0 = 0$) can be simulated by assigning zero values to constants $B_{(in)}$ and $C_{(in)}$ in Eq. (4.2). Due to the limitation of the separation of variables method, the formulation presented in this paper only applies to time independent boundary conditions and/or source terms; the solution methodology presented here can not be extended to include the time dependence in boundary conditions and/or sources. Such problems can be solved analytically using the finite integral transform technique [16, 17].

4.3 Solution Method

In order to apply the separation of variables method, which is only applicable to homogenous problems, the nonhomogeneous problem has to be split into homogenous transient part and nonhomogeneous steady-state part. This is done by splitting transient temperature in governing equations Eqs. (4.1-4.8) as

$$T^{(i)}(r, \theta, t) = T_{ss}^{(i)}(r, \theta) + \hat{T}^{(i)}(r, \theta, t) \tag{4.9}$$

where $T_{ss}^{(i)}(r, \theta)$ is the steady-state part and $\hat{T}^{(i)}(r, \theta, t)$ is the ”complementary” transient part of the solution.

4.3.1 Inhomogeneous Steady State Part

Inhomogeneous steady-state equations corresponding to Eqs. (4.1-4.8) are as follows:

∗ Governing equation ($r_{i-1} \leqslant r \leqslant r_i$, $0 \leqslant \theta \leqslant 2\pi$, where $i = 1, 2, ..., n$);

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_{ss}^{(i)}}{\partial r}(r,\theta)\right) + \frac{1}{r^2}\frac{\partial^2 T_{ss}^{(i)}}{\partial \theta^2}(r,\theta) + \frac{g^{(i)}(r,\theta)}{k_i} \tag{4.10}$$

Boundary conditions are;

∗ Inner surface of the first layer ($0 \leqslant \theta \leqslant 2\pi$)

$$A_{in}\frac{\partial T_{ss}^{(1)}}{\partial r}(r_0,\theta) + B_{in}T_{ss}^{(1)}(r_0,\theta) = C_{in}(\theta) \tag{4.11}$$

∗ Outer surface of the n^{th} layer ($0 \leqslant \theta \leqslant 2\pi$)

$$A_{out}\frac{\partial T_{ss}^{(n)}}{\partial r}(r_n,\theta) + B_{out}T_{ss}^{(n)}(r_n,\theta) = C_{out}(\theta) \tag{4.12}$$

∗ Periodic boundary conditions ($r_{i-1} \leqslant r \leqslant r_i$, where $i = 1, 2, ..., n$);

$$T_{ss}^{(i)}(r,\theta = 0) = T_{ss}^{(i)}(r,\theta = 2\pi) \tag{4.13}$$

$$\frac{\partial T_{ss}^{(i)}}{\partial \theta}(r,\theta = 0) = \frac{\partial T_{ss}^{(i)}}{\partial \theta}(r,\theta = 2\pi) \tag{4.14}$$

∗ Interface of the $(i-1)^{st}$ and the i^{th} layer ($0 \leqslant \theta \leqslant 2\pi$, where $i = 1, 2, ..., n$)

$$T_{ss}^{(i)}(r_{i-1},\theta) = T_{ss}^{(i-1)}(r_{i-1},\theta) \tag{4.15}$$

$$k_i\frac{\partial T_{ss}^{(i)}}{\partial r}(r_{i-1},\theta) = k_{i-1}\frac{\partial T_{ss}^{(i-1)}}{\partial r}(r_{i-1},\theta) \tag{4.16}$$

4.3.2 Homogenous (Complementary) Transient Part

Homogeneous complementary transient equations corresponding to Eqs. (4.1-4.8) are as follows:

∗ Governing equation ($r_{i-1} \leqslant r \leqslant r_i$, $0 \leqslant \theta \leqslant 2\pi$ and $t > 0$, where $i = 1, 2, ..., n$);

$$\frac{1}{\alpha_i}\frac{\partial \hat{T}^{(i)}}{\partial t}(r,\theta,t) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{T}^{(i)}}{\partial r}(r,\theta,t)\right) + \frac{1}{r^2}\frac{\partial^2 \hat{T}^{(i)}}{\partial \theta^2}(r,\theta,t) \tag{4.17}$$

Boundary conditions are;

∗ Inner surface of the first layer ($0 \leqslant \theta \leqslant 2\pi$ and $t > 0$)

$$A_{in}\frac{\partial \hat{T}^{(1)}}{\partial r}(r_0, \theta, t) + B_{in}\hat{T}^{(1)}(r_0, \theta, t) = 0 \tag{4.18}$$

∗ Outer surface of the n^{th} layer ($0 \leqslant \theta \leqslant 2\pi$ and $t > 0$)

$$A_{out}\frac{\partial \hat{T}^{(n)}}{\partial r}(r_n, \theta, t) + B_{out}\hat{T}^{(n)}(r_n, \theta, t) = 0 \tag{4.19}$$

∗ Periodic boundary conditions ($r_{i-1} \leqslant r \leqslant r_i$, and $t > 0$), where $i = 1, 2, ..., n$)

$$\hat{T}^{(i)}(r, \theta = 0, t) = \hat{T}^{(i)}(r, \theta = 2\pi, t) \tag{4.20}$$

$$\frac{\partial \hat{T}^{(i)}}{\partial \theta}(r, \theta = 0, t) = \frac{\partial \hat{T}^{(i)}}{\partial \theta}(r, \theta = 2\pi, t) \tag{4.21}$$

∗ Interface of the $(i-1)^{st}$ and the i^{th} layer ($0 \leqslant \theta \leqslant 2\pi$, and $t > 0$, where $i = 1, 2, ..., n$)

$$\hat{T}^{(i)}(r_{i-1}, \theta, t) = \hat{T}^{(i-1)}(r_{i-1}, \theta, t) \tag{4.22}$$

$$k_i\frac{\partial \hat{T}^{(i)}}{\partial r}(r_{i-1}, \theta, t) = k_{i-1}\frac{\partial \hat{T}^{(i-1)}}{\partial r}(r_{i-1}, \theta, t) \tag{4.23}$$

∗ Initial condition ($r_{i-1} \leqslant r \leqslant r_i$, $0 \leqslant \theta \leqslant 2\pi$, where $i = 1, 2, ..., n$)

$$\hat{T}^{(i)}(r, \theta, t = 0) = f^{(i)}(r, \theta) - T_{ss}^{(i)}(r, \theta) \tag{4.24}$$

4.4 Solution to the Inhomogeneous Steady-State Part

Eigenfunction expansion method is used to solve the inhomogeneous steady-state problem [5]. The steady-state temperature distribution governed by Eq. (4.10) can be written as a generalized Fourier series in terms its angular eigenfunctions [22];

$$T_{ss}^{(i)}(r, \theta) = T_0^{(i)}(r) + \sum_{m=1}^{\infty} T_{c,m}^{(i)}(r)\cos(m\theta) + \sum_{m=1}^{\infty} T_{s,m}^{(i)}(r)\sin(m\theta) \tag{4.25}$$

where $r_{i-1} \leqslant r \leqslant r_i$ and $1 \leqslant i \leqslant n$.

The source term $g^{(i)}(r)$ of Eq. (4.10) can also be expanded in a generalized Fourier series as

$$g^{(i)}(r, \theta) = g_0^{(i)}(r) + \sum_{m=1}^{\infty} g_{c,m}^{(i)}(r)\cos(m\theta) + \sum_{m=1}^{\infty} g_{s,m}^{(i)}(r)\sin(m\theta) \tag{4.26}$$

where

$$g_0^{(i)}(r) = \frac{1}{2\pi}\int_0^{2\pi} g^{(i)}(r, \theta)d\theta \tag{4.27}$$

$$g_{c,m}^{(i)}(r) = \frac{1}{\pi}\int_0^{2\pi} g^{(i)}(r,\theta)\cos(m\theta)d\theta \tag{4.28}$$

$$g_{s,m}^{(i)}(r) = \frac{1}{\pi}\int_0^{2\pi} g^{(i)}(r,\theta)\sin(m\theta)d\theta \tag{4.29}$$

Substituting Eq. (4.25) and Eq. (4.26) into Eq. (4.10) gives

$$\frac{1}{r}\frac{d}{dr}(r\frac{dT_0^{(i)}(r)}{dr}) + \frac{g_0^{(i)}(r)}{k_i} = 0 \tag{4.30}$$

$$\frac{1}{r}\frac{d}{dr}(r\frac{dT_{c,m}^{(i)}(r)}{dr}) - \frac{m^2}{r^2}T_{c,m}^{(i)}(r) + \frac{g_{c,m}^{(i)}(r)}{k_i} = 0 \tag{4.31}$$

$$\frac{1}{r}\frac{d}{dr}(r\frac{dT_{s,m}^{(i)}(r)}{dr}) - \frac{m^2}{r^2}T_{s,m}^{(i)}(r) + \frac{g_{s,m}^{(i)}(r)}{k_i} = 0 \tag{4.32}$$

$C_{in}(\theta)$ and $C_{out}(\theta)$ in Eq. (4.11) and Eq. (4.12) can similarly be expanded in generalized Fourier series to give us boundary conditions for ordinary differential equations in Eqs. (4.30-4.32).

Eq. (4.31) and Eq. (4.32) are Euler equations, the solutions of which can be written as

$$T_{c,m}^{(i)}(r) = A_{c,m}^{(i)}r^m + B_{c,m}^{(i)}r^{-m} + f_{cp}(r) \tag{4.33}$$

$$T_{s,m}^{(i)}(r) = A_{s,m}^{(i)}r^m + B_{s,m}^{(i)}r^{-m} + f_{sp}(r) \tag{4.34}$$

where $f_{cp}(r)$ and $f_{sp}(r)$ are particular integrals which can be evaluated by applying the method of undetermined coefficients, or the method of variation of parameters. The constants $A_{c,m}^{(i)}$, $B_{c,m}^{(i)}$, $A_{s,m}^{(i)}$, $B_{s,m}^{(i)}$ can be evaluated using boundary and interface conditions for $T_{c,m}^{(i)}(r)$ and $T_{s,m}^{(i)}(r)$. The solutions for $T_0^{(i)}(r)$ are easily found once $g_0^{(i)}(r)$ are determined.

4.5 Solution to Homogeneous Transient Part

Transient part is solved using the separation of variables method. Transient temperature can be written as;

$$\hat{T}^{(i)}(r,\theta,t) = R^{(i)}(r)\,\Theta(\theta)\,\Gamma^{(i)}(t) \tag{4.35}$$

Applying the separation of variables:

For radial directions

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial R^{(i)}(r)}{\partial r}) + (\lambda_i^2 - \frac{m^2}{r^2})R^{(i)}(r) = 0 \tag{4.36}$$

in the range $r_{i-1} \leqslant r \leqslant r_i$, and where $i = 1, 2, ..., n$.

On the other hand, in between $0 \leqslant \theta \leqslant 2\pi$

$$\frac{\partial^2\Theta}{\partial\theta^2} + m^2\Theta(\theta) = 0 \tag{4.37}$$

Finally

$$\frac{1}{\alpha_i}\frac{\partial\Gamma^{(i)}(t)}{\partial t} + \lambda_i^2\Gamma^{(i)}(t) = 0 \tag{4.38}$$

where λ_i^2 are constants of separation.

In the view of these three ODE's given above, we are lead to a general solution for Eq. (4.17) as

$$\begin{aligned}\hat{T}^{(i)}(r,\theta,t) = \sum_{p=1}^{\infty} C_{0p} e^{-\alpha_i\lambda_{i0p}^2 t} R_{0p}^{(i)}(\lambda_{i0p} r) \\ + \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} D_{mp} e^{-\alpha_i\lambda_{imp}^2 t} R_{mp}^{(i)}(\lambda_{imp} r)\cos(m\theta) \\ + \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} E_{mp} e^{-\alpha_i\lambda_{imp}^2 t} R_{mp}^{(i)}(\lambda_{imp} r)\sin(m\theta)\end{aligned} \tag{4.39}$$

where continuity of the heat-flux at the layer interfaces is expressed by the following expression [16, 18–20].

$$\lambda_{imp} = \lambda_{1mp}\sqrt{\frac{\alpha_1}{\alpha_i}} \tag{4.40}$$

where $i = 1, 2, ..., n$.

The radial (transverse) eigenfunction $R_{mp}^{(i)}$ in Eq. (4.39) is in the form

$$R_{mp}^{(i)}(\lambda_{imp} r) = a_{imp} J_m(\lambda_{imp} r) + b_{imp} N_m(\lambda_{imp} r) \tag{4.41}$$

where $J_m(\lambda_{imp} r)$ and $N_m(\lambda_{imp} r)$ are Bessel functions of the first and second kind of order m, respectively. The corresponding orthogonality condition for R_{mp} is (see Appendix A.2)

$$\sum_{i=1}^{n}\frac{k_i}{\alpha_i}\int_{r_{i-1}}^{r_i} r R_{mp}^{(i)}(\lambda_{imp} r) R_{mq}^{(i)}(\lambda_{imq} r) dr = \begin{cases} N_{rmp} & p = q \\ 0 & p \neq q \end{cases} \tag{4.42}$$

where N_{rmp} is the normalization integral in the radial direction [21]. Standard orthogonality condition is valid for the angular eigenfunctions $\Theta_m(\theta)$. [16]

Coefficients C_{op}, D_{mp}, and E_{mp} in Eq. (4.39) are evaluated by applying the initial condition Eq. (4.24), and then making use of the orthogonality conditions in the radial and angular directions. (see Appendix A.1). Hence we get;

$$C_{0p} = \frac{1}{2\pi N_{r0p}} \sum_{i=1}^{n} \frac{k_i}{\alpha_i} \int_0^{2\pi} \int_{r_{i-1}}^{r_i} r R_{0p}^{(i)}(\lambda_{i0p} r) \hat{T}^{(i)}(r, \theta, t = 0) dr d\theta \tag{4.43}$$

$$D_{mp} = \frac{1}{\pi N_{rmp}} \sum_{i=1}^{n} \frac{k_i}{\alpha_i} \int_0^{2\pi} \int_{r_{i-1}}^{r_i} r R_{mp}^{(i)}(\lambda_{imp} r) \hat{T}^{(i)}(r, \theta, t = 0) \cos(m\theta) dr d\theta \tag{4.44}$$

$$E_{mp} = \frac{1}{\pi N_{rmp}} \sum_{i=1}^{n} \frac{k_i}{\alpha_i} \int_0^{2\pi} \int_{r_{i-1}}^{r_i} r R_{mp}^{(i)}(\lambda_{imp} r) \hat{T}^{(i)}(r, \theta, t = 0) \sin(m\theta) dr d\theta \tag{4.45}$$

Using the i^{th} interface condition, coefficients a_{imp} and b_{imp} in Eq. (4.41) are evaluated from the recurrence relationship (see Appendix A.3 for proof)

$$\begin{bmatrix} a_{i+1,mp} \\ b_{i+1,mp} \end{bmatrix} = \begin{bmatrix} J_m(\lambda_{i+1,mp} r_i) & N_m(\lambda_{i+1,mp} r_i) \\ k_{i+1} J'_m(\lambda_{i+1,mp} r_i) & k_{i+1} N'_m(\lambda_{i+1,mp} r_i) \end{bmatrix}^{-1} \begin{bmatrix} J_m(\lambda_{imp} r_i) & N_m(\lambda_{imp} r_i) \\ k_i J'_m(\lambda_{imp} r_i) & k_i N'_m(\lambda_{imp} r_i) \end{bmatrix} \begin{bmatrix} a_{imp} \\ b_{imp} \end{bmatrix} \tag{4.46}$$

where i=1,2,...,n-1, and also

$$b_{1mp} = -\frac{C_{1in}}{C_{2in}} a_{1mp} \tag{4.47}$$

valid for arbitrary a_{1mp}.

4.6 Illustrative example: three-layer system

Now, we will find the transient temperature profile of a three layer annulus, with $r_0 \leqslant r \leqslant r_3$, $0 \leqslant \theta \leqslant 2\pi$ and $1 \leqslant i \leqslant 3$. See Fig. 4.2; each layer has a different thermal diffusivity (α_i) and thermal conductivity (k_i). The lower-half of the annulus where $\pi \leqslant \theta \leqslant 2\pi$ is kept insulated, while the upper-half ($0 \leqslant \theta \leqslant \pi$) is subjected to a θ-dependent heat-flux. The system is initially at a uniform zero temperature at $t = 0$, which means $f^{(i)}(r, \theta) = 0$ in Eq. (4.8) and Eq. (4.24). For $t > 0$ heat flux is given as

$$q''(r = r_3, \theta) = \begin{cases} q_0 \sin(\theta) & \text{for} \quad 0 \leqslant \theta \leqslant \pi \\ 0 & \text{for} \quad \pi \leqslant \theta \leqslant 2\pi \end{cases} \tag{4.48}$$

is applied towards the outer surface ($r = r_3$), while the inner surface ($r = r_0$) is constantly kept at zero temperature. These conditions lead us to define the relevant coefficients as $A_{in} = 0$, $B_{in} = 1$, $C_{in}(\theta) = 0$, $A_{out} = k_3$, $B_{out} = 0$, and $C_{out}(\theta) = q''(r = r_3, \theta)$ in the boundary condition equations Eq. (4.11) and Eq. (4.12). We assume that there is no heat generation in any of the layers, which means $g^{(i)}(r,\theta) = 0$.

We arbitrarily choose the parameter values in this example as $k_2/k_1 = 2$, $k_3/k_1 = 4$, $\alpha_2/\alpha_1 = 4$, $\alpha_3/\alpha_1 = 9$, $r_1/r_0 = 2$, $r_2/r_0 = 4$, and $r_3/r_0 = 6$.

In the results that follow; r, t, and $T^{(i)}(r,\theta,t)$ are expressed in terms of r_0, r_0^2/α_1, and, $q_0 r_0/k_1$, respectively.

For this particular example, the infinite series solution for the complementary transient temperature $\hat{T}^{(i)}(r,\theta,t)$ is truncated at $m = 10$ and $p = 10$.

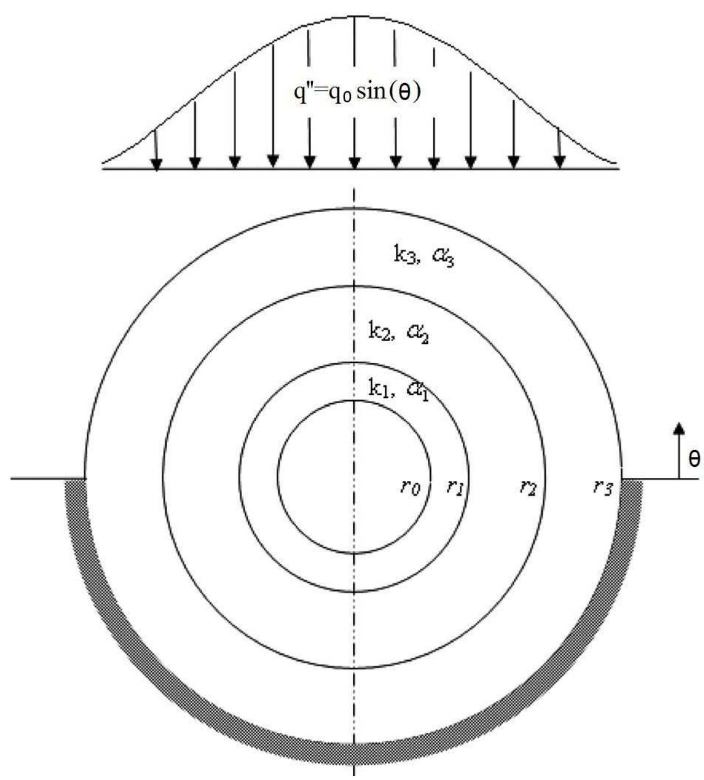

Figure 4.2: Three layer annulus.

4.6.1 Solution to steady-state part for three-layer system

Since the heat generation term $g^{(i)}(r,\theta)$ is assumed to be zero, the term $g_0^{(i)}(r)$ in Eq. (4.30) is zero, as well. Then, the solution to this equation is straightforward, which is

$$T_0^{(i)}(r) = C_{i0}\ln(r) + C_i \tag{4.49}$$

The heat generation terms in Euler equations Eq. (4.31) and Eq. (4.32) are also zero. Solutions to these homogeneous equations are written as

$$T_{c,m}^{(i)}(r) = A_{c,m}^{(i)} r^m + B_{c,m}^{(i)} r^{-m} \tag{4.50}$$

$$T_{s,m}^{(i)}(r) = A_{s,m}^{(i)} r^m + B_{s,m}^{(i)} r^{-m} \tag{4.51}$$

Using three results above in the eigenfunction expansion, the steady-state temperature distribution in Eq. (4.25) is written as

$$T_{ss}^{(i)}(r,\theta) = C_{i0}\ln(r) + C_i + \sum_{m=1}^{\infty}\left[A_{c,m}^{(i)} r^m + B_{c,m}^{(i)} r^{-m}\right]\cos(m\theta) + \sum_{m=1}^{\infty}\left[A_{s,m}^{(i)} r^m + B_{s,m}^{(i)} r^{-m}\right]\sin(m\theta) \tag{4.52}$$

We may define a new function as another solution to Eq. (4.10); Instead of $T_{ss}^{(i)}(r,\theta)$,

$$\Gamma_{ss}^{(i)}(r,\theta) = T_{ss}^{(i)}(r,\theta) - T_{ss}^{(1)}(r_0,\theta) \tag{4.53}$$

is also a solution to Eq. (4.10), as $T_{ss}^{(1)}(r_0,\theta) = 0$ in Eq. (4.11). Alternatively we may write

$$\Gamma_{ss}^{(i)}(r,\theta) = C_{i0}\ln(r) + C_i + \sum_{m=1}^{\infty}\left[A_{c,m}^{(i)} r^m + B_{c,m}^{(i)} r^{-m}\right]\cos(m\theta) + \sum_{m=1}^{\infty}\left[A_{s,m}^{(i)} r^m + B_{s,m}^{(i)} r^{-m}\right]\sin(m\theta) \tag{4.54}$$

Introducing unitless temperature written in terms of $(q_0 r_0/k_1)$, and writing r in terms or r_0

$$\tilde{\Gamma}_{ss}^{(i)}(\frac{r}{r_0},\theta) = \tilde{C}_{i0}\ln\left(\frac{r}{r_0}\right) + \tilde{C}_i + \sum_{m=1}^{\infty}\left[\tilde{A}_{c,m}^{(i)}\left(\frac{r}{r_0}\right)^m + \tilde{B}_{c,m}^{(i)}\left(\frac{r}{r_0}\right)^{-m}\right]\cos(m\theta) + \sum_{m=1}^{\infty}\left[\tilde{A}_{s,m}^{(i)}\left(\frac{r}{r_0}\right)^m + \tilde{B}_{s,m}^{(i)}\left(\frac{r}{r_0}\right)^{-m}\right]\sin(m\theta) \tag{4.55}$$

where $\tilde{\Gamma}_{ss}^{(i)}(\frac{r}{r_0},\theta) = \Gamma_{ss}^{(i)}(\frac{r}{r_0},\theta)/(q_0 r_0/k_1)$ ...and etc.

Since $T_{ss}^{(1)}(r_0, \theta) = 0$, $C_{in}(\theta) = 0$ in Eq. (4.11). This means $\Gamma_{ss}^{(1)}(1, \theta) = 0$. Then

$$\tilde{C}_{10} = \text{unknown for now} \quad (4.56)$$

$$\tilde{C}_1 = 0 \quad (4.57)$$

$$\left[A_{c,m}^{(1)} r^m + B_{c,m}^{(1)} r^{-m}\right] = 0 \quad (4.58)$$

$$\left[A_{s,m}^{(1)} r^m + B_{s,m}^{(1)} r^{-m}\right] = 0 \quad (4.59)$$

Noting that $\tilde{T}_{ss}^{(i)} = \tilde{\Gamma}_{ss}^{(i)}(r, \theta) + \tilde{T}_{ss}^{(1)}(r_0, \theta)$ and inserting this into Eq. (4.15) (interface condition) for $i = 2, 3$;

$$\tilde{C}_{i0} \ln\left(\frac{r_{i-1}}{r_0}\right) + \tilde{C}_i = \tilde{C}_{i-1,0} \ln\left(\frac{r_{i-1}}{r_0}\right) + \tilde{C}_{i-1} \quad (4.60)$$

$$\tilde{A}_{c,m}^{(i)} \left(\frac{r_{i-1}}{r_0}\right)^m + \tilde{B}_{c,m}^{(i)} \left(\frac{r_{i-1}}{r_0}\right)^{-m} - \tilde{A}_{c,m}^{(i-1)} \left(\frac{r_{i-1}}{r_0}\right)^m - \tilde{B}_{c,m}^{(i-1)} \left(\frac{r_{i-1}}{r_0}\right)^{-m} = 0 \quad (4.61)$$

$$\tilde{A}_{s,m}^{(i)} \left(\frac{r_{i-1}}{r_0}\right)^m + \tilde{B}_{s,m}^{(i)} \left(\frac{r_{i-1}}{r_0}\right)^{-m} - \tilde{A}_{s,m}^{(i-1)} \left(\frac{r_{i-1}}{r_0}\right)^m - \tilde{B}_{s,m}^{(i-1)} \left(\frac{r_{i-1}}{r_0}\right)^{-m} = 0 \quad (4.62)$$

Inserting $\tilde{T}_{ss}^{(i)}$ this time into Eq. (4.16) (interface condition) for $i = 2, 3$;

$$\tilde{k}_i \tilde{C}_{i0} = \tilde{k}_{i-1} \tilde{C}_{i-1,0} \quad (4.63)$$

$$\tilde{k}_i \tilde{A}_{c,m}^{(i)} \left(\frac{r_{i-1}}{r_0}\right)^{m-1} - \tilde{k}_i \tilde{B}_{c,m}^{(i)} \left(\frac{r_{i-1}}{r_0}\right)^{-m-1} - \tilde{k}_{i-1} \tilde{A}_{c,m}^{(i-1)} \left(\frac{r_{i-1}}{r_0}\right)^{m-1} + \tilde{k}_{i-1} \tilde{B}_{c,m}^{(i-1)} \left(\frac{r_{i-1}}{r_0}\right)^{-m-1} = 0 \quad (4.64)$$

$$\tilde{k}_i \tilde{A}_{s,m}^{(i)} \left(\frac{r_{i-1}}{r_0}\right)^{m-1} - \tilde{k}_i \tilde{B}_{s,m}^{(i)} \left(\frac{r_{i-1}}{r_0}\right)^{-m-1} - \tilde{k}_{i-1} \tilde{A}_{s,m}^{(i-1)} \left(\frac{r_{i-1}}{r_0}\right)^{m-1} + \tilde{k}_{i-1} \tilde{B}_{s,m}^{(i-1)} \left(\frac{r_{i-1}}{r_0}\right)^{-m-1} = 0 \quad (4.65)$$

On the other hand, Eq. (4.12) turns into

$$\left(\frac{k_3}{k_1}\right) \frac{\partial \tilde{\Gamma}_{ss}^{(3)}(\frac{r_3}{r_0}, \theta)}{\partial(\frac{r}{r_0})} = \sin(\theta) \quad (4.66)$$

Integrating both sides of this equation with $\int_0^{2\Pi} cos(n\theta) d\theta$, $\int_0^{2\Pi} sin(n\theta) d\theta$ and $\int_0^{2\Pi} d\theta$, we get

$$\tilde{A}_{c,m}^{(3)} \left(\frac{r_3}{r_0}\right)^{m-1} - \tilde{B}_{c,m}^{(3)} \left(\frac{r_3}{r_0}\right)^{-m-1} = \frac{(-1)^{m+1} - 1}{\pi \frac{k_3}{k_1} m(m^2 - 1)} \quad (4.67)$$

where m=1,2,3. Also we get

$$\tilde{A}_{s,m}^{(3)} \left(\frac{r_3}{r_0}\right)^{m-1} - \tilde{B}_{s,m}^{(3)} \left(\frac{r_3}{r_0}\right)^{-m-1} = \begin{cases} \frac{1}{2\frac{k_3}{k_1}} & \text{for} \quad m = 1 \\ 0 & \text{for} \quad m > 1 \end{cases} \quad (4.68)$$

and thirdly

$$\tilde{C}_{30} = 2 \tag{4.69}$$

If we organize everything we found through Eqs. (4.56-4.69), we obtain;

$$\begin{aligned}
\tilde{C}_1 &= 0 && (4.70)\\
\tilde{C}_{30} &= 2 && (4.71)\\
\tilde{C}_{10} &= \tilde{k}_2\tilde{C}_{20} && (4.72)\\
\tilde{k}_2\tilde{C}_{20} &= \tilde{k}_3\tilde{C}_{30} && (4.73)\\
\tilde{C}_{20}\ln\left(\frac{r_1}{r_0}\right) + \tilde{C}_2 &= \tilde{C}_{10}\ln\left(\frac{r_1}{r_0}\right) && (4.74)\\
\tilde{C}_{30}\ln\left(\frac{r_2}{r_0}\right) + \tilde{C}_3 &= \tilde{C}_{20}\ln\left(\frac{r_2}{r_0}\right) + \tilde{C}_2 && (4.75)
\end{aligned}$$

There are six equations with six unknowns above. Hence, this system of equations is solvable.

We also obtain the following matrix relation

$$\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0\\
(1+\frac{k_2}{k_1})(\frac{r_1}{r_0})^m & (\frac{k_2}{k_1}-1)(\frac{r_1}{r_0})^{-m} & -2\frac{k_2}{k_1}(\frac{r_1}{r_0})^m & 0 & 0 & 0\\
0 & 2(\frac{r_1}{r_0})^{-m} & (\frac{k_2}{k_1}-1)(\frac{r_1}{r_0})^m & -(1+\frac{k_2}{k_1})(\frac{r_1}{r_0})^{-m} & 0 & 0\\
0 & 0 & (1+\frac{k_3}{k_2})(\frac{r_2}{r_0})^m & (\frac{k_3}{k_2}-1)(\frac{r_2}{r_0})^{-m} & -2\frac{k_3}{k_2}(\frac{r_2}{r_0})^m & 0\\
0 & 0 & 0 & 2(\frac{r_2}{r_0})^{-m} & (\frac{k_3}{k_2}-1)(\frac{r_2}{r_0})^m & -(1+\frac{k_3}{k_2})(\frac{r_2}{r_0})^{-m}\\
0 & 0 & 0 & 0 & (\frac{r_3}{r_0})^{m-1} & -(\frac{r_3}{r_0})^{-m-1}
\end{bmatrix}$$

$$\text{x}\begin{bmatrix}\tilde{A}^{(1)}_{c,m}\\ \tilde{B}^{(1)}_{c,m}\\ \tilde{A}^{(2)}_{c,m}\\ \tilde{B}^{(2)}_{c,m}\\ \tilde{A}^{(3)}_{c,m}\\ \tilde{B}^{(3)}_{c,m}\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\\ \Omega\end{bmatrix} \tag{4.76}$$

where $\Omega = \frac{(-1)^{m+1}-1}{\pi\frac{k_3}{k_1}m(m^2-1)}$ and $i = 1, 2, 3$.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ (1+\frac{k_2}{k_1})(\frac{r_1}{r_0})^m & (\frac{k_2}{k_1}-1)(\frac{r_1}{r_0})^{-m} & -2\frac{k_2}{k_1}(\frac{r_1}{r_0})^m & 0 & 0 & 0 \\ 0 & 2(\frac{r_1}{r_0})^{-m} & (\frac{k_2}{k_1}-1)(\frac{r_1}{r_0})^m & -(1+\frac{k_2}{k_1})(\frac{r_1}{r_0})^{-m} & 0 & 0 \\ 0 & 0 & (1+\frac{k_3}{k_2})(\frac{r_2}{r_0})^m & (\frac{k_3}{k_2}-1)(\frac{r_2}{r_0})^{-m} & -2\frac{k_3}{k_2}(\frac{r_2}{r_0})^m & 0 \\ 0 & 0 & 0 & 2(\frac{r_2}{r_0})^{-m} & (\frac{k_3}{k_2}-1)(\frac{r_2}{r_0})^m & -(1+\frac{k_3}{k_2})(\frac{r_2}{r_0})^{-m} \\ 0 & 0 & 0 & 0 & (\frac{r_3}{r_0})^{m-1} & -(\frac{r_3}{r_0})^{-m-1} \end{bmatrix}$$

$$\text{x} \begin{bmatrix} \tilde{A}^{(1)}_{s,m} \\ \tilde{B}^{(1)}_{s,m} \\ \tilde{A}^{(2)}_{s,m} \\ \tilde{B}^{(2)}_{s,m} \\ \tilde{A}^{(3)}_{s,m} \\ \tilde{B}^{(3)}_{s,m} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \Delta \end{bmatrix} \tag{4.77}$$

where

$$\Delta = \begin{cases} \frac{1}{2\frac{k_3}{k_1}} & \text{if} \quad m = 1 \\ 0 & \text{if} \quad m > 1 \end{cases} \tag{4.78}$$

and $i = 1, 2, 3$.

Eqs. (4.70-4.77) allow us to find all coefficients of Eq. (4.55).

By using the method of Gauss-Jordon elimination (simply row reduction), or LU factorization, these solutions can be obtained for matrices Eq. (4.76) and Eq. (4.77). Solving these systems by using the method of Gauss-Jordon elimination (in general for $1 \le i \le n_L$), we obtain;

$$\begin{bmatrix} \tau_1 & \nu_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \phi_1 & \tau_2 & \nu_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \phi_2 & \tau_3 & \nu_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \phi_3 & \tau_4 & \nu_4 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \phi_{2n_L-3} & \tau_{2n_L-2} & \nu_{2n_L-2} & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \phi_{2n_L-2} & \tau_{2n_L-1} & \nu_{2n_L-1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \phi_{2n_L-1} & \tau_{2n_L} \end{bmatrix} \begin{bmatrix} \tilde{A}^{(1)}_{c,m} \\ \tilde{B}^{(1)}_{c,m} \\ \tilde{A}^{(2)}_{c,m} \\ \tilde{B}^{(2)}_{c,m} \\ \vdots \\ \tilde{B}^{(n_L-1)}_{c,m} \\ \tilde{A}^{(n_L)}_{c,m} \\ \tilde{B}^{(n_L)}_{c,m} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \Omega \end{bmatrix} \tag{4.79}$$

$$\begin{bmatrix} \tau_1 & \nu_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \phi_1 & \tau_2 & \nu_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \phi_2 & \tau_3 & \nu_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \phi_3 & \tau_4 & \nu_4 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \phi_{2n_L-3} & \tau_{2n_L-2} & \nu_{2n_L-2} & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \phi_{2n_L-2} & \tau_{2n_L-1} & \nu_{2n_L-1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \phi_{2n_L-1} & \tau_{2n_L} \end{bmatrix} \begin{bmatrix} \tilde{A}_{s,m}^{(1)} \\ \tilde{B}_{s,m}^{(1)} \\ \tilde{A}_{s,m}^{(2)} \\ \tilde{B}_{s,m}^{(2)} \\ \vdots \\ \tilde{B}_{s,m}^{(n_L-1)} \\ \tilde{A}_{s,m}^{(n_L)} \\ \tilde{B}_{s,m}^{(n_L)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \Delta \end{bmatrix} \tag{4.80}$$

where the coefficients are found as

$$\tilde{A}_{c,m}^{(i)} = \frac{(-1)^{m+2}+1}{\pi\frac{k_{n_L}}{k_1}m(m^2-1)} \frac{\prod\limits_{\ell=2i-1}^{2n_L-1}\nu_\ell}{\prod\limits_{\ell=2i-1}^{2n_L}\psi_\ell} = \frac{(-1)^{m+2}+1}{\pi\frac{k_{n_L}}{k_1}m(m^2-1)} \frac{1}{\psi_{2n_L}} \prod_{\ell=2i-1}^{2n_L-1}\frac{\nu_\ell}{\psi_\ell} \tag{4.81}$$

$$\tilde{B}_{c,m}^{(i)} = \frac{(-1)^{m+1}-1}{\pi\frac{k_{n_L}}{k_1}m(m^2-1)} \frac{\prod\limits_{\ell=2i}^{2n_L-1}\nu_\ell}{\prod\limits_{\ell=2i}^{2n_L}\psi_\ell} = \frac{(-1)^{m+1}-1}{\pi\frac{k_{n_L}}{k_1}m(m^2-1)} \frac{1}{\psi_{2n_L}} \prod_{\ell=2i}^{2n_L-1}\frac{\nu_\ell}{\psi_\ell} \tag{4.82}$$

$$\tilde{A}_{s,1}^{(i)} = -\frac{1}{2\frac{k_{n_L}}{k_1}} \frac{\prod\limits_{\ell=2i-1}^{2n_L-1}\nu_\ell}{\prod\limits_{\ell=2i-1}^{2n_L}\psi_\ell} = -\frac{1}{2\frac{k_{n_L}}{k_1}} \frac{1}{\psi_{2n_L}} \prod_{\ell=2i-1}^{2n_L-1}\frac{\nu_\ell}{\psi_\ell} \tag{4.83}$$

$$\tilde{B}_{s,1}^{(i)} = \frac{1}{2\frac{k_{n_L}}{k_1}} \frac{\prod\limits_{\ell=2i}^{2n_L-1}\nu_\ell}{\prod\limits_{\ell=2i}^{2n_L}\psi_\ell} = \frac{1}{2\frac{k_{n_L}}{k_1}} \frac{1}{\psi_{2n_L}} \prod_{\ell=2i}^{2n_L-1}\frac{\nu_\ell}{\psi_\ell} \tag{4.84}$$

$$\tilde{A}_{s,m}^{(i)} = 0 \quad \text{and} \quad \tilde{B}_{s,m}^{(i)} = 0 \quad \text{for } m > 1 \tag{4.85}$$

for $1 \leq j \leq n_L - 1$, where

$$
\begin{aligned}
\nu_1 &= 1 && (4.86)\\
\nu_{2j} &= -2\frac{k_{j+1}}{k_j}\left(\frac{r_j}{r_0}\right)^m && (4.87)\\
\nu_{2j+1} &= -\left(1+\frac{k_{j+1}}{k_j}\right)\left(\frac{r_j}{r_0}\right)^{-m} && (4.88)\\
\psi_1 &= 1 && (4.89)\\
\psi_{j+1} &= \tau_{j+1} - \frac{\phi_j \nu_j}{\psi_j} && (4.90)\\
\tau_1 &= 1 && (4.91)\\
\tau_{2j} &= \left(\frac{k_{j+1}}{k_j}-1\right)\left(\frac{r_j}{r_0}\right)^{-m} && (4.92)\\
\tau_{2j+1} &= \left(\frac{k_{j+1}}{k_j}-1\right)\left(\frac{r_j}{r_0}\right)^{m} && (4.93)\\
\tau_{2n_L} &= -\left(\frac{r_{n_L}}{r_0}\right)^{-m-1} && (4.94)\\
\phi_{2j-1} &= \left(1+\frac{k_{j+1}}{k_j}\right)\left(\frac{r_j}{r_0}\right)^{m} && (4.95)\\
\phi_{2j} &= 2\left(\frac{r_j}{r_0}\right)^{-m} && (4.96)\\
\phi_{2n_L-1} &= \left(\frac{r_{n_L}}{r_0}\right)^{m-1} && (4.97)\\
\Omega &= \frac{(-1)^{m+1}-1}{\pi\frac{k_{n_L}}{k_1}m(m^2-1)} && (4.98)\\
\Delta &= \begin{cases} \dfrac{1}{2\frac{k_{n_L}}{k_1}} & m = 1 \\ 0 & m > 1 \end{cases} && (4.99)
\end{aligned}
$$

Using these relations, we can find all the coefficients in Eq. (4.55), hence we can calculate steady-state temperature for each layer.

4.6.2 Solution to transient part for three-layer system

Now that we found the steady-state solution, we are ready to continue with the transient solution and finally, we should add the two results in order to get the resultant temperature. Since $f^{(i)}(r,\theta) = 0$ in Eq. (4.24), what we found is actually the unitless transient temperature at $t = 0$, cause Eq. (4.24) now becomes

$$\hat{T}^{(i)}(r,\theta,t=0) = -T_{ss}^{(i)}(r,\theta) \tag{4.100}$$

Initial transient temperature is used to find the coefficients C_{0p}, D_{mp} and E_{mp} in Eqs. (4.43-4.45), which will then be used in Eq. (4.39). In order to obtain (4.39) as whole, we should also find λ_{1mp} and write it in terms of λ_{imp} by using Eq. (4.40). We should also find radial transverse eigenfunctions $R^{(i)}_{mp}(\lambda_{imp})$ by using Eq. (4.41). Coefficients of radial transverse eigenfunctions a_{imp} and b_{imp} are found using Eq. (4.46) in Section 4.6.2.2 .

4.6.2.1 Finding the normalization integral N_{rmp}

Normalization integral N_{rmp} that appears in Eq. (4.42) is found as (see Appendix A.4);

$$N_{rmp} = \left(\frac{k_1 r_0^2}{\alpha_1}\right)\sum_{i=1}^{3}\tilde{N}^{(i)}_{rmp} \tag{4.101}$$

where

$$\tilde{N}^{(i)}_{rmp} = \left(\frac{\frac{k_i}{k_1}}{\frac{\alpha_i}{\alpha_1}}\right)\int_{\tilde{r}_{i-1}}^{\tilde{r}_i} \tilde{r}\left(R^{(i)}_{mp}(\tilde{\lambda}_{imp}\tilde{r})\right)^2 d(\tilde{r}) \tag{4.102}$$

is a unitless integral to be found for each i^{th} layer, and where $\tilde{r} = r/r_0$ and $\tilde{\lambda}_{imp} = \lambda_{imp}r_0$ are unitless variables.

4.6.2.2 Finding a_{imp} and b_{imp}

There are 2 interfaces for 3 layers. For each interface, application of Eq. (4.18) for $i = 1, 2$ gives us a_{2mp}, b_{2mp}, a_{3mp} and b_{3mp}. Note that a_{1mp} is arbitrary; so b_{1mp} is found using the relation

$$b_{1mp} = -\frac{C_{1in}}{C_{2in}}a_{1mp} \tag{4.103}$$

where C_{1in} and C_{2in} can be found by application of $R^{(i)}_{mp}(\lambda_{imp})$ in Eq. (4.41), to Eq. (4.18); then

$$C_{1in} = A_{in}\frac{\partial}{\partial r}J_m(\lambda_{1mp}r_0) + B_{in}J_m(\lambda_{1mp}r_0) \tag{4.104}$$

$$C_{2in} = A_{in}\frac{\partial}{\partial r}N_m(\lambda_{1mp}r_0) + B_{in}N_m(\lambda_{1mp}r_0) \tag{4.105}$$

where $A_{in} = 0$, $B_{in} = 1$ for our system. Thus

$$C_{1in} = J_m(\lambda_{1mp}r_0), \qquad C_{2in} = N_m(\lambda_{1mp}r_0) \tag{4.106}$$

for three-layer system.

4.6.2.3 Finding λ_{imp} terms

Application of the boundary condition equations Eqs. (4.18-4.19) and interface condition equations Eqs. (4.22-4.23) to the transverse eigenfunction $R_{mp}^{(i)}$ yields a $2n_L x 2n_L$ matrix for each integer value of m. Transverse eigencondition is obtained by setting the determinant of this matrix equal to zero, roots of which, in turn, yield the infinite number of eigenvalues λ_{1mp} corresponding to the first layer for each integer value of m. For three layers, the matrix is found as

$$\begin{bmatrix} C_{1in} & C_{2in} & 0 & 0 & 0 & 0 \\ -J_m(\lambda_{1mp}r_1) & -N_m(\lambda_{1mp}r_1) & J_m(\lambda_{2mp}r_1) & N_m(\lambda_{2mp}r_1) & 0 & 0 \\ -k_1 J'_m(\lambda_{1mp}r_1) & -k_1 N'_m(\lambda_{1mp}r_1) & k_2 J'_m(\lambda_{2mp}r_1) & k_2 N'_m(\lambda_{2mp}r_1) & 0 & 0 \\ 0 & 0 & -J_m(\lambda_{2mp}r_2) & -N_m(\lambda_{2mp}r_2) & J_m(\lambda_{3mp}r_2) & N_m(\lambda_{3mp}r_2) \\ 0 & 0 & -k_2 J'_m(\lambda_{2mp}r_2) & -k_2 N'_m(\lambda_{2mp}r_2) & k_3 J'_m(\lambda_{3mp}r_2) & k_3 N'_m(\lambda_{3mp}r_2) \\ 0 & 0 & 0 & 0 & C_{1out} & C_{2out} \end{bmatrix}$$

$$\text{x} \begin{bmatrix} a_{1mp} \\ b_{1mp} \\ a_{2mp} \\ b_{2mp} \\ a_{3mp} \\ b_{3mp} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.107}$$

where C_{1out} and C_{2out} can be found by application of $R_{mp}^{(i)}(\lambda_{imp})$ in Eq. (4.41), to Eq. (4.19); so

$$C_{1out} = A_{out}\frac{\partial}{\partial r}J_m(\lambda_{n_L,mp}r_{n_L}) + B_{out}J_m(\lambda_{n_L,mp}r_{n_L}) \tag{4.108}$$

$$C_{2out} = A_{out}\frac{\partial}{\partial r}N_m(\lambda_{n_L,mp}r_{n_L}) + B_{in}N_m(\lambda_{n_L,mp}r_{n_L}) \tag{4.109}$$

where $A_{out} = k_3$, $B_{out} = 0$ and $n_L = 3$ for the system we chose. Thus

$$C_{1out} = k_3\frac{\partial}{\partial r}J_m(\lambda_{3mp}r_3), \qquad C_{2out} = k_3\frac{\partial}{\partial r}N_m(\lambda_{3mp}r_3) \tag{4.110}$$

Inserting C_{1in} and C_{2in} from Eq. (4.106), C_{1out} and C_{2out} from Eq. (4.110), recalling $\tilde{r} = \frac{r}{r_0}$, $\frac{d}{dr} = \frac{1}{r_0}\frac{d}{d\tilde{r}}$, $\tilde{\lambda} = \lambda r_0$, and remembering that we should write k_i in terms of k_1, we can write the determinant of this matrix equation as

$$\begin{bmatrix} J_m(\tilde{\lambda}_{1mp}) & N_m(\tilde{\lambda}_{1mp}) & 0 & 0 & 0 & 0 \\ -J_m(\tilde{\lambda}_{1mp}\tilde{r}_1) & -N_m(\tilde{\lambda}_{1mp}\tilde{r}_1) & J_m(\tilde{\lambda}_{2mp}\tilde{r}_1) & N_m(\tilde{\lambda}_{2mp}\tilde{r}_1) & 0 & 0 \\ -\frac{d}{d\tilde{r}}J_m(\tilde{\lambda}_{1mp}\tilde{r}_1) & -\frac{d}{d\tilde{r}}N_m(\tilde{\lambda}_{1mp}\tilde{r}_1) & \frac{k_2}{k_1}\frac{d}{d\tilde{r}}J_m(\tilde{\lambda}_{2mp}\tilde{r}_1) & \frac{k_2}{k_1}\frac{d}{d\tilde{r}}N_m(\tilde{\lambda}_{2mp}\tilde{r}_1) & 0 & 0 \\ 0 & 0 & -J_m(\tilde{\lambda}_{2mp}\tilde{r}_2) & -N_m(\tilde{\lambda}_{2mp}\tilde{r}_2) & J_m(\tilde{\lambda}_{3mp}\tilde{r}_2) & N_m(\tilde{\lambda}_{3mp}\tilde{r}_2) \\ 0 & 0 & -\frac{k_2}{k_1}\frac{d}{d\tilde{r}}J_m(\tilde{\lambda}_{2mp}\tilde{r}_2) & -\frac{k_2}{k_1}\frac{d}{d\tilde{r}}N_m(\tilde{\lambda}_{2mp}\tilde{r}_2) & \frac{k_3}{k_1}\frac{d}{d\tilde{r}}J_m(\tilde{\lambda}_{3mp}\tilde{r}_2) & \frac{k_3}{k_1}\frac{d}{d\tilde{r}}N_m(\tilde{\lambda}_{3mp}\tilde{r}_2) \\ 0 & 0 & 0 & 0 & \frac{k_3}{k_1}\frac{d}{d\tilde{r}}J_m(\tilde{\lambda}_{3mp}\tilde{r}_3) & \frac{k_3}{k_1}\frac{d}{d\tilde{r}}N_m(\tilde{\lambda}_{3mp}\tilde{r}_3) \end{bmatrix}$$

In order to find the values of λ_{1mp}, we should write every $\tilde{\lambda}_{imp}$ in terms of $\tilde{\lambda}_{1mp}$ in this matrix. Hence we obtain

$$\begin{bmatrix} J_m(\tilde{\lambda}_{1mp}) & N_m(\tilde{\lambda}_{1mp}) & 0 & 0 & 0 & 0 \\ -J_m(\tilde{\lambda}_{1mp}\tilde{r}_1) & -N_m(\tilde{\lambda}_{1mp}\tilde{r}_1) & J_m(\tilde{\lambda}_{1mp}\sqrt{\frac{\alpha_1}{\alpha_2}}\tilde{r}_1) & N_m(\tilde{\lambda}_{1mp}\sqrt{\frac{\alpha_1}{\alpha_2}}\tilde{r}_1) & 0 & 0 \\ -\frac{d}{d\tilde{r}}J_m(\tilde{\lambda}_{1mp}\tilde{r}_1) & -\frac{d}{d\tilde{r}}N_m(\tilde{\lambda}_{1mp}\tilde{r}_1) & \frac{k_2}{k_1}\frac{d}{d\tilde{r}}J_m(\tilde{\lambda}_{1mp}\sqrt{\frac{\alpha_1}{\alpha_2}}\tilde{r}_1) & \frac{k_2}{k_1}\frac{d}{d\tilde{r}}N_m(\tilde{\lambda}_{1mp}\sqrt{\frac{\alpha_1}{\alpha_2}}\tilde{r}_1) & 0 & 0 \\ 0 & 0 & -J_m(\tilde{\lambda}_{1mp}\sqrt{\frac{\alpha_1}{\alpha_2}}\tilde{r}_2) & -N_m(\tilde{\lambda}_{1mp}\sqrt{\frac{\alpha_1}{\alpha_2}}\tilde{r}_2) & J_m(\tilde{\lambda}_{1mp}\sqrt{\frac{\alpha_1}{\alpha_3}}\tilde{r}_2) & N_m(\tilde{\lambda}_{1mp}\sqrt{\frac{\alpha_1}{\alpha_3}}\tilde{r}_2) \\ 0 & 0 & -\frac{k_2}{k_1}\frac{d}{d\tilde{r}}J_m(\tilde{\lambda}_{1mp}\sqrt{\frac{\alpha_1}{\alpha_2}}\tilde{r}_2) & -\frac{k_2}{k_1}\frac{d}{d\tilde{r}}N_m(\tilde{\lambda}_{1mp}\sqrt{\frac{\alpha_1}{\alpha_2}}\tilde{r}_2) & \frac{k_3}{k_1}\frac{d}{d\tilde{r}}J_m(\tilde{\lambda}_{1mp}\sqrt{\frac{\alpha_1}{\alpha_3}}\tilde{r}_2) & \frac{k_3}{k_1}\frac{d}{d\tilde{r}}N_m(\tilde{\lambda}_{1mp}\sqrt{\frac{\alpha_1}{\alpha_3}}\tilde{r}_2) \\ 0 & 0 & 0 & 0 & \frac{k_3}{k_1}\frac{d}{d\tilde{r}}J_m(\tilde{\lambda}_{1mp}\sqrt{\frac{\alpha_1}{\alpha_3}}\tilde{r}_3) & \frac{k_3}{k_1}\frac{d}{d\tilde{r}}N_m(\tilde{\lambda}_{1mp}\sqrt{\frac{\alpha_1}{\alpha_3}}\tilde{r}_3) \end{bmatrix}$$

Making the determinant of this matrix equal to zero gives us infinitely many λ_{1mp} roots for each m and p.

4.7 Calculations and Results

We are ready to find temperature distribution for our system. Isotherms in three-layer annulus are shown in Fig. 4.3. Maximum temperatures on the outer part of the cylinder ($r = r_3$) are located around $\theta = \pi/2$, and minimum temperatures around $\theta = 3\pi/2$, as expected. Discontinuities on the temperature slopes on the interfaces indicate that each layer has different thermal properties.

Radial temperature variations at some different angular positions ($\theta = 0$, $\theta = \pi/2$, and $\theta = 3\pi/2$) are given in Fig. 4.4. At any radius and time (r, t), minimum temperatures are observed at $\theta = 3\pi/2$), and maximum temperatures at $\theta = \pi/2$). This is an expected result, since the heat-flux applied above the system has its maximum at $\theta = \pi/2$).

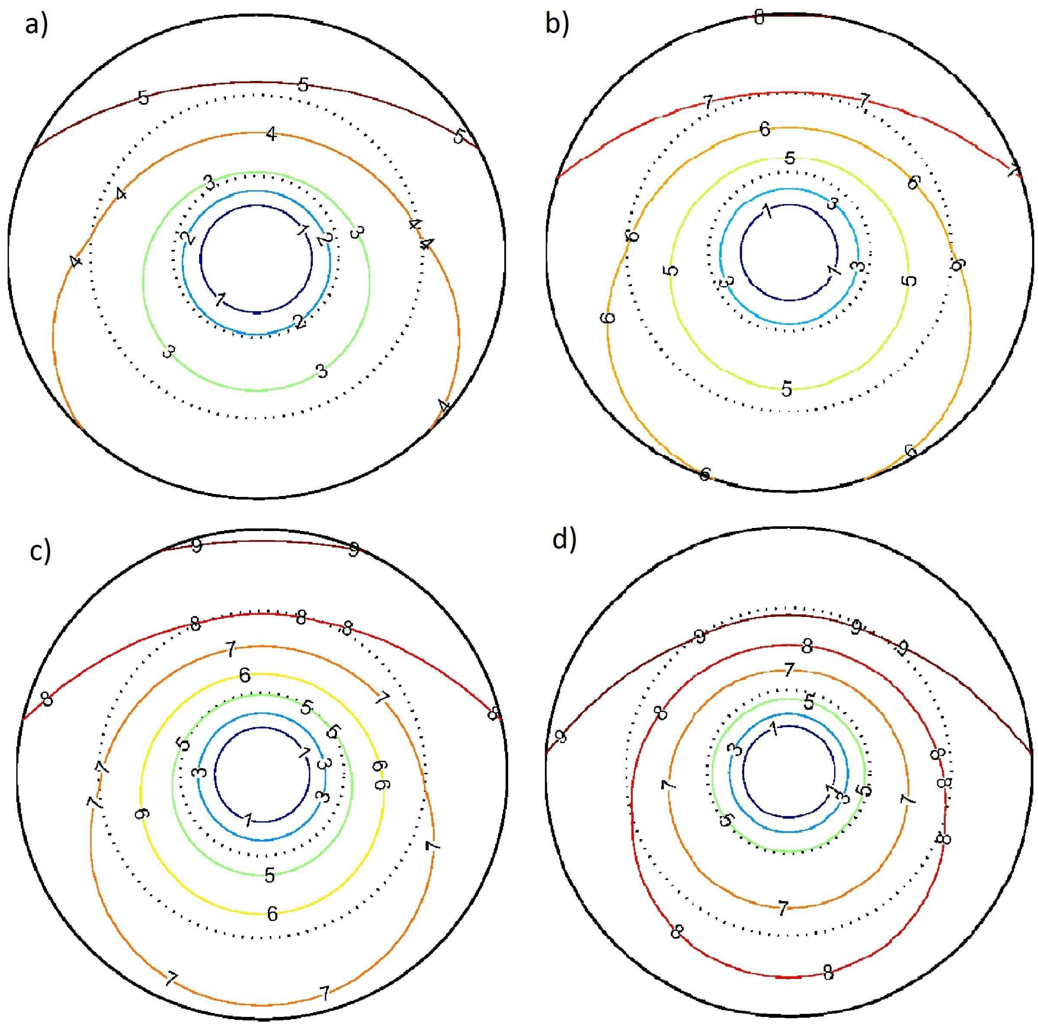

Figure 4.3: Transient isotherms in three-layer annulus: a) t=5, b) t=10, c) t=15, d) steady-state

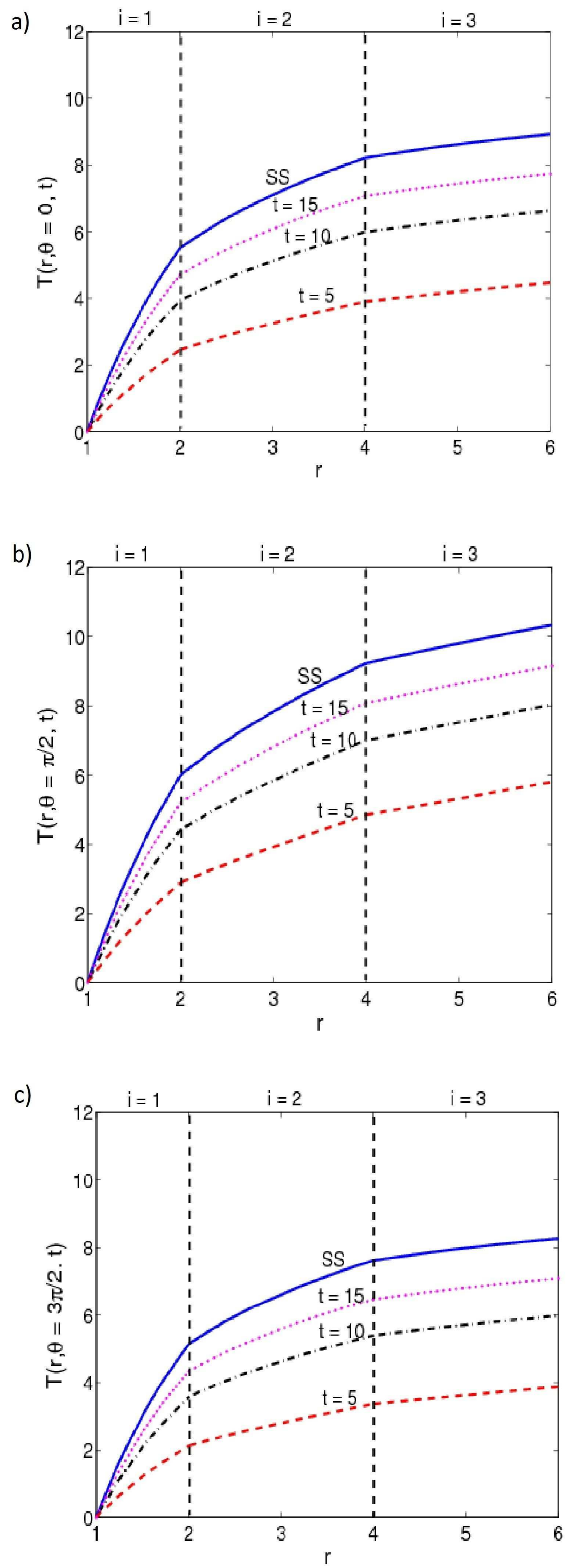

Figure 4.4: Transient temperature variation in the radial direction at a) $\theta = 0$, b) $\theta = \pi/2$, c) $\theta = 3\pi/2$.

CHAPTER 5

CONCLUSIONS

Heat equation is solved [7] using separation of variables method in which we split the assumed solution into *space* and *time* parts. Space dependent part has a solution which has *cosine* and *sine* trigonometric functions. Here, we eliminated the *sine* part of the assumed solution because it does not obey the boundary conditions at hand. So we may conclude that one of the most appropriate base functions to be offered in Bubnov-Galerkin solution for our system in this chapter is $cos(\omega_i x)$. We also see that $P_n((\cos \omega_i x))$ is another suitable one.

A cylindrical reactor has been studied analytically using Bubnov-Galerkin method. The problem is reduced solution of a system of N coupled differential equations strongly nonlinear. We have applied First kind, Zeroth order Bessel function which has been readily used for the solution of homogenous, cylindrical heat and mass transfer equation [9, 12]. The results shown in Table 3.2 and Table 3.3 are coherent with the expectations. The percentage error in θ changes with the Biot number. However, the value of Biot number does not seem to affect the convergence pattern. Fig. 3.2 makes this point clear.

Isotherms in three-layer annulus are shown in Fig. 4.3. Maximum temperatures on the outer part of the cylinder ($r = r_3$) are located around $\theta = \pi/2$, and minimum temperatures around $\theta = 3\pi/2$, as expected. Discontinuities on the temperature slopes on the interfaces indicate that each layer has different thermal properties. Radial temperature variations at some different angular positions ($\theta = 0$, $\theta = \pi/2$, and $\theta = 3\pi/2$) are given in Fig. 4.4. At any radius and time (r, t), minimum temperatures are observed at $\theta = 3\pi/2$), and maximum temperatures at $\theta = \pi/2$). This is an expected result, since the heat-flux applied above the system has its maximum at $\theta = \pi/2$).

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APPENDIX A

Derivations

A.1 Determination of coefficients C_{0p}, D_{mp} and E_{mp} in Eq. (4.39)

Eq. (4.39) is in the form

$$\hat{T}^{(i)}(r,\theta,t) = \sum_{p=1}^{\infty} C_{0p} e^{-\alpha_i \lambda_{i0p}^2 t} R_{0p}^{(i)}(\lambda_{i0p} r) + \sum_{m=1}^{\infty}\sum_{p=1}^{\infty}(\ldots)\cos(m\theta) + \sum_{m=1}^{\infty}\sum_{p=1}^{\infty}(\ldots)\sin(m\theta) \quad \text{(A.1)}$$

Integrating both sides of this equation with $\int_0^{2\pi} d\theta$ gives

$$\int_0^{2\pi} \hat{T}^{(i)}(r,\theta,t)d\theta = \int_0^{2\pi} \sum_{p=1}^{\infty} C_{0p} e^{-\alpha_i \lambda_{i0p}^2 t} R_{0p}^{(i)}(\lambda_{i0p} r) d\theta \quad \text{(A.2)}$$

$$= 2\pi \sum_{p=1}^{\infty} C_{0p} e^{-\alpha_i \lambda_{i0p}^2 t} R_{0p}^{(i)}(\lambda_{i0p} r) \quad \text{(A.3)}$$

Operating both sides with $\sum_{i=1}^{n} \frac{k_i}{\alpha_i} \int_{r_{i-1}}^{r_i} r R_{0q}^{(i)}(\lambda_{i0q} r) dr$

$$\sum_{i=1}^{n} \frac{k_i}{\alpha_i} \int_0^{2\pi}\int_{r_{i-1}}^{r_i} r R_{0q}^{(i)}(\lambda_{i0q} r)\hat{T}(r,\theta,t) dr d\theta = 2\pi \sum_{p=1}^{\infty} C_{0p} \sum_{i=1}^{n} e^{-\alpha_i \lambda_{i0p}^2 t} \frac{k_i}{\alpha_i} \int_{r_{i-1}}^{r_i} r R_{0p}^{(i)}(\lambda_{i0p} r) R_{0q}^{(i)}(\lambda_{i0q} r) dr$$

At initial time (namely, at $t = 0$) we obtain

$$\sum_{i=1}^{n} \frac{k_i}{\alpha_i} \int_0^{2\pi}\int_{r_{i-1}}^{r_i} r R_{0q}^{(i)}(\lambda_{i0q} r)\hat{T}(r,\theta,t=0) dr d\theta = 2\pi \sum_{p=1}^{\infty} C_{0p} \underbrace{\sum_{i=1}^{n} \frac{k_i}{\alpha_i} \int_{r_{i-1}}^{r_i} r R_{0p}^{(i)}(\lambda_{i0p} r) R_{0q}^{(i)}(\lambda_{i0q} r) dr}_{=\begin{cases} N_{r0p} & \text{for } p = q \\ 0 & \text{for } p \neq q \end{cases}}$$

$$= 2\pi C_{0p} N_{r0p}$$

Hence,

$$C_{0p} = \frac{1}{2\pi N_{r0p}} \sum_{i=1}^{n} \frac{k_i}{\alpha_i} \int_0^{2\pi}\int_{r_{i-1}}^{r_i} r R_{0p}^{(i)}(\lambda_{i0p} r)\hat{T}(r,\theta,t=0) dr d\theta \quad \text{(A.4)}$$

which is equal to Eq. 4.43.

In order to find D_{mp}, let's begin with integrating both sides of Eq. (4.39) with $\int_0^{2\pi} cos(l\theta)d\theta$

$$\begin{aligned}\int_0^{2\pi} \hat{T}^{(i)}(r,\theta,t)cos(l\theta)d\theta &= \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} D_{mp}e^{-\alpha_i\lambda_{imp}^2 t}R_{mp}^{(i)}(\lambda_{imp}r)\underbrace{\int_0^{2\pi} cos(l\theta)cos(m\theta)d\theta}_{=\begin{cases}\pi & \text{for } l=m\\ 0 & \text{for } l\neq m\end{cases}} \\ &= \pi\sum_{p=1}^{\infty} D_{mp}e^{-\alpha_i\lambda_{imp}^2 t}R_{mp}^{(i)}(\lambda_{imp}r)\end{aligned}$$

operating both sides with $\sum_{i=1}^{n}\frac{k_i}{\alpha_i}\int_{r_{i-1}}^{r_i} rR_{mq}^{(i)}(\lambda_{imq}r)dr$ gives

$$\sum_{i=1}^{n}\frac{k_i}{\alpha_i}\int_0^{2\pi}\int_{r_{i-1}}^{r_i} rR_{mq}^{(i)}(\lambda_{imq}r)\hat{T}^{(i)}(r,\theta,t)cos(m\theta)drd\theta$$
$$= \pi\sum_{p=1}^{\infty} D_{mp}\sum_{i=1}^{n} e^{-\alpha_i\lambda_{imp}^2 t}\frac{k_i}{\alpha_i}\int_{r_{i-1}}^{r_i} rR_{mq}^{(i)}(\lambda_{imq}r)R_{mp}^{(i)}(\lambda_{imp}r)dr \quad (A.5)$$

at $t = 0$, this relation changes into

$$\begin{aligned}&\sum_{i=1}^{n}\frac{k_i}{\alpha_i}\int_0^{2\pi}\int_{r_{i-1}}^{r_i} rR_{mq}^{(i)}(\lambda_{imq}r)\hat{T}^{(i)}(r,\theta,t=0)cos(l\theta)drd\theta \\ &= \pi\sum_{p=1}^{\infty} D_{mp}\underbrace{\sum_{i=1}^{n}\frac{k_i}{\alpha_i}\int_{r_{i-1}}^{r_i} rR_{mq}^{(i)}(\lambda_{imq}r)R_{mp}^{(i)}(\lambda_{imp}r)dr}_{=\begin{cases}N_{rmp} & \text{for } p=q\\ 0 & \text{for } p\neq q\end{cases}} \\ &= \pi\sum_{p=1}^{\infty} D_{mp}N_{rmp}\end{aligned}$$

Hence we obtain

$$D_{mp} = \frac{1}{\pi N_{rmp}}\sum_{i=1}^{n}\frac{k_i}{\alpha_i}\int_0^{2\pi}\int_{r_{i-1}}^{r_i} rR_{mp}^{(i)}(\lambda_{imp}r)\hat{T}^{(i)}(r,\theta,t=0)cos(m\theta)drd\theta \quad (A.6)$$

which is equal to Eq. (4.44).

In order to find E_{mp}, first, we integrate both sides of Eq. (4.39) with $\int_0^{2\pi} sin(n\theta)d\theta$), then operate both sides of the resultant equation with $\sum_{i=1}^{n}\frac{k_i}{\alpha_i}\int_{r_{i-1}}^{r_i} rR_{mq}^{(i)}(\lambda_{imq}r)dr$ and evaluate this equation at

$t = 0$ again. Similar steps lead us to Eq. (4.45), which is;

$$E_{mp} = \frac{1}{\pi N_{rmp}} \sum_{i=1}^{n} \frac{k_i}{\alpha_i} \int_0^{2\pi} \int_{r_{i-1}}^{r_i} r R_{mp}^{(i)}(\lambda_{imp} r) \hat{T}^{(i)}(r, \theta, t = 0) \sin(m\theta) dr d\theta$$

Since we want to find the transient temperature of three-layer system in terms of $q_0 r_0 / k_1$ (this term is in Kelvin unit), C_{0p}, D_{mp} and E_{mp} coefficients in Eqs. (4.43-4.45) should be modified according to this aim. Note that, these coefficients are in the unit of Kelvin. On the other hand, unit of k_i are W/mK, unit of α_i are m^2/s, $R_{imp}(\lambda_{imp} r)$ are unitless functions. Our aim is to write C_{0p}, D_{mp} and E_{mp} in terms $q_0 r_0 / k_1$, k_i in terms of k_1 and α_i in terms of α_1 and r in terms of r_0.

So, let us begin with defining $\tilde{r} = \frac{r}{r_0}$. Then $dr = r_0 d\tilde{r}$ and $dr = r_0 d\tilde{r}$. Let us also define $\tilde{\lambda}_{imp} = \lambda_{imp} r_0$. Thus, $\left(\lambda_{imp} r\right) = (\lambda_{imp} r_0 . \frac{r}{r_0}) = (\tilde{\lambda}_{imp} \tilde{r})$.

Inserting these variables into Eq.4.43

$$C_{0p} = \frac{1}{2\pi N_{r0p}} \left(\frac{k_1 r_0^2}{\alpha_1}\right) \sum_{i=1}^{3} \frac{\frac{k_i}{k_1}}{\frac{\alpha_i}{\alpha_1}} \int_0^{2\pi} \int_{\tilde{r}_{i-1}}^{\tilde{r}_i} \tilde{r} R_{0p}^{(i)}(\tilde{\lambda}_{i0p} \tilde{r}) \hat{T}(r_0 \tilde{r}, \theta, t = 0) d\tilde{r} d\theta \tag{A.7}$$

where $\tilde{N}_{r0p} = N_{r0p} / (\frac{k_1 r_0^2}{\alpha_1})$ (recall Eq. (A.48)) and $\hat{T}(r_0 \tilde{r}, \theta, t = 0) = T_{ss}^{(i)}(r_0 \tilde{r}, \theta)$ cause we take $f^{(i)} = 0$ in Eq. (4.24) for three-layer system. Hence can we write

$$C_{0p} = -\frac{1}{2\pi \tilde{N}_{r0p}} \sum_{i=1}^{3} \frac{\frac{k_i}{k_1}}{\frac{\alpha_i}{\alpha_1}} \int_0^{2\pi} \int_{\tilde{r}_{i-1}}^{\tilde{r}_i} \tilde{r} R_{0p}^{(i)}(\tilde{\lambda}_{i0p} \tilde{r}) T_{ss}^{(i)}(r_0 \tilde{r}, \theta) d\tilde{r} d\theta \tag{A.8}$$

dividing both sides of this relation with $q_0 r_0 / k_1$

$$\tilde{C}_{0p} = -\frac{1}{2\pi \tilde{N}_{r0p}} \sum_{i=1}^{3} \frac{\frac{k_i}{k_1}}{\frac{\alpha_i}{\alpha_1}} \int_0^{2\pi} \int_{\tilde{r}_{i-1}}^{\tilde{r}_i} \tilde{r} R_{0p}^{(i)}(\tilde{\lambda}_{i0p} \tilde{r}) \tilde{T}_{ss}^{(i)}(r_0 \tilde{r}, \theta) d\tilde{r} d\theta \tag{A.9}$$

Applying similar steps to Eqs. (4.44-4.45) gives

$$\tilde{D}_{mp} = -\frac{1}{\pi \tilde{N}_{rmp}} \sum_{i=1}^{3} \frac{\frac{k_i}{k_1}}{\frac{\alpha_i}{\alpha_1}} \int_0^{2\pi} \int_{\tilde{r}_{i-1}}^{\tilde{r}_i} \tilde{r} R_{mp}^{(i)}(\tilde{\lambda}_{imp} r) \tilde{T}_{ss}^{(i)}(r_0 \tilde{r}, \theta) cos(m\theta) d\tilde{r} d\theta \tag{A.10}$$

$$\tilde{E}_{mp} = -\frac{1}{\pi \tilde{N}_{rmp}} \sum_{i=1}^{3} \frac{\frac{k_i}{k_1}}{\frac{\alpha_i}{\alpha_1}} \int_0^{2\pi} \int_{\tilde{r}_{i-1}}^{\tilde{r}_i} \tilde{r} R_{mp}^{(i)}(\tilde{\lambda}_{imp} r) \tilde{T}_{ss}^{(i)}(r_0 \tilde{r}, \theta) cos(m\theta) d\tilde{r} d\theta \tag{A.11}$$

A.2 Orthogonality condition: Eq. (4.42)

Let $R_{mp}^{(i)}$ and $R_{mq}^{(i)}$ be transverse eigenfunctions satisfying Eq. (4.36). So

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR_{mp}^{(i)}}{dr}\right)+\left(\lambda_{imp}^2-\frac{m^2}{r^2}\right)R_{mp}^{(i)}=0 \tag{A.12}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR_{mq}^{(i)}}{dr}\right)+\left(\lambda_{imq}^2-\frac{m^2}{r^2}\right)R_{mq}^{(i)}=0 \tag{A.13}$$

Boundary and interface conditions for $\hat{T}^{(i)}(r,\theta,t)$ (Eqs. (4.18-4.23)) are also valid for transverse eigenfunctions.

Since $\alpha_i\lambda_{imp}^2=\alpha_1\lambda_{1mp}^2$ from Eq. (4.40), we can write

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR_{mp}^{(i)}}{dr}\right)+\left(\frac{\alpha_1\lambda_{1mp}^2}{\alpha_i}-\frac{m^2}{r^2}\right)R_{mp}^{(i)}=0 \tag{A.14}$$

and similarly

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR_{mq}^{(i)}}{dr}\right)+\left(\frac{\alpha_1\lambda_{1mq}^2}{\alpha_i}-\frac{m^2}{r^2}\right)R_{mq}^{(i)}=0 \tag{A.15}$$

Multiplying Eq. (A.14) by $R_{mq}^{(i)}$ and Eq. (A.15) by $R_{mp}^{(i)}$ and subtracting from each other, we obtain

$$R_{mq}^{(i)}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR_{mp}^{(i)}}{dr}\right)-R_{mp}^{(i)}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR_{mq}^{(i)}}{dr}\right)+\alpha_1\left(\frac{\lambda_{1mp}^2}{\alpha_i}-\frac{\lambda_{1mq}^2}{\alpha_i}\right)R_{mq}^{(i)}R_{mp}^{(i)}=0 \tag{A.16}$$

Integrating both sides with $\int\limits_{r_{i-1}}^{r_i} rdr$

$$\int\limits_{r_{i-1}}^{r_i}\left[R_{mq}^{(i)}\frac{d}{dr}\left(r\frac{dR_{mp}^{(i)}}{dr}\right)\right]dr-\int\limits_{r_{i-1}}^{r_i}\left[R_{mp}^{(i)}\frac{d}{dr}\left(r\frac{dR_{mq}^{(i)}}{dr}\right)\right]dr+\alpha_1\int\limits_{r_{i-1}}^{r_i}\left(\frac{\lambda_{1mp}^2}{\alpha_i}-\frac{\lambda_{1mq}^2}{\alpha_i}\right)rR_{mq}^{(i)}R_{mp}^{(i)}dr=0 \tag{A.17}$$

Applying integration by parts twice on the first integral in Eq. (A.17)

$$\int\limits_{r_{i-1}}^{r_i}\left[R_{mq}^{(i)}\frac{d}{dr}\left(r\frac{dR_{mp}^{(i)}}{dr}\right)\right]dr=\left[rR_{mq}^{(i)}\frac{dR_{mp}^{(i)}}{dr}-rR_{mp}^{(i)}\frac{dR_{mq}^{(i)}}{dr}\right]_{r=r_{i-1}}^{r=r_i}+\int\limits_{r_{i-1}}^{r_i}\left[R_{mp}^{(i)}\frac{d}{dr}\left(r\frac{dR_{mq}^{(i)}}{dr}\right)\right]dr \tag{A.18}$$

Substituting Eq. (A.18) in Eq. (A.17), second term in Eq. (A.17) cancels last term in Eq. (A.18) and we will get

$$\left[rR_{mq}^{(i)}\frac{dR_{mp}^{(i)}}{dr}-rR_{mp}^{(i)}\frac{dR_{mq}^{(i)}}{dr}\right]_{r=r_{i-1}}^{r=r_i}+\alpha_1\int\limits_{r_{i-1}}^{r_i}\left(\frac{\lambda_{1mp}^2}{\alpha_i}-\frac{\lambda_{1mq}^2}{\alpha_i}\right)rR_{mq}^{(i)}R_{mp}^{(i)}dr=0 \tag{A.19}$$

Multiplying Eq. (A.19) by k_i and then summing over all i, we obtain

$$\sum_{i=1}^{n}\left[k_i r R_{mq}^{(i)}\frac{dR_{mp}^{(i)}}{dr} - k_i r R_{mp}^{(i)}\frac{dR_{mq}^{(i)}}{dr}\right]_{r=r_{i-1}}^{r=r_i} + \sum_{i=1}^{n}\frac{\alpha_1 k_i}{\alpha_i}\int_{r_{i-1}}^{r_i}\left(\lambda_{1mp}^2 - \lambda_{1mq}^2\right) r R_{mq}^{(i)} R_{mp}^{(i)} dr = 0 \quad (A.20)$$

Applying interface conditions Eq. (4.22) and Eq. (4.23), we get

$$\left[k_n r R_{mq}^{(n)}\frac{dR_{mp}^{(n)}}{dr} - k_n r R_{mp}^{(n)}\frac{dR_{mq}^{(n)}}{dr}\right]_{r=r_n} - \left[k_1 r R_{mq}^{(1)}\frac{dR_{mp}^{(1)}}{dr} - k_1 r R_{mp}^{(1)}\frac{dR_{mq}^{(1)}}{dr}\right]_{r=r_0}$$
$$+ \sum_{i=1}^{n}\frac{\alpha_1 k_i}{\alpha_i}\int_{r_{i-1}}^{r_i}\left(\lambda_{1mp}^2 - \lambda_{1mq}^2\right) r R_{mq}^{(i)} R_{mp}^{(i)} dr = 0 \quad (A.21)$$

On the other hand, from the outer layer boundary condition Eq. (4.19), we have

$$\left[A_{out}\frac{dR_{mp}^{(n)}}{dr} + B_{out}R_{mp}^{(n)}\right]_{r=r_n} = 0 \quad (A.22)$$

$$\left[A_{out}\frac{dR_{mq}^{(n)}}{dr} + B_{out}R_{mq}^{(n)}\right]_{r=r_n} = 0 \quad (A.23)$$

Multiplying Eq. (A.22) by $k_n r_n R_{mq}^{(n)}(r = r_n)$ and Eq. (A.23) by $k_n r_n R_{mp}^{(n)}(r = r_n)$ and subtracting,

$$A_{out}\left[k_n r R_{mq}^{(n)}\frac{dR_{mp}^{(n)}}{dr} - k_n r R_{mp}^{(n)}\frac{dR_{mq}^{(n)}}{dr}\right]_{r=r_n} = 0 \quad (A.24)$$

Now, consider three different cases;

(a) $A_{out} \neq 0$ and $B_{out} \neq 0$

(b) $A_{out} \neq 0$ and $B_{out} = 0$

(c) $A_{out} = 0$ and $B_{out} \neq 0$

For cases (a) and (b), Eq. (A.24) reduces to

$$\left[k_n r R_{mq}^{(n)}\frac{dR_{mp}^{(n)}}{dr} - k_n r R_{mp}^{(n)}\frac{dR_{mq}^{(n)}}{dr}\right]_{r=r_n} = 0 \quad (A.25)$$

For case (c), Eq. (A.22) and Eq. (A.23) respectively imply that $R_{nmp}(r = r_n) = 0$ and $R_{nmq}(r = r_n) = 0$. Thus, we see that Eq. (A.25) is also true for case (c).

Similarly, we can show that

$$\left[k_1 r R_{mq}^{(1)}\frac{dR_{mp}^{(1)}}{dr} - k_1 r R_{mp}^{(1)}\frac{dR_{mq}^{(1)}}{dr}\right]_{r=r_0} = 0 \quad (A.26)$$

In the view of Eq. (A.25) and Eq. (A.26), Eq. (A.21) gives

$$\left(\lambda_{1mp}^2 - \lambda_{1mq}^2\right) \sum_{i=1}^{n} \frac{\alpha_1 k_i}{\alpha_i} \int_{r_{i-1}}^{r_i} r R_{mp}^{(i)} R_{mq}^{(i)} dr = 0 \tag{A.27}$$

For $p \neq q$, $\left(\lambda_{1mp}^2 - \lambda_{1mq}^2\right) \neq 0$. Therefore [21]

$$\sum_{i=1}^{n} \frac{k_i}{\alpha_i} \int_{r_{i-1}}^{r_i} r R_{mp}^{(i)} R_{mq}^{(i)} dr = 0, \qquad \text{for} \;\; p \neq q \tag{A.28}$$

Eq. (A.27) implies that, Eq. (A.28) is not necessarily equal to zero for $p = q$. This explains the existence of normalization integral N_{rmp} in Eq. (4.42).

A.3 Recurrence relationship: Eq. (4.46)

From Eqs. (4.4-4.5), for the interface of the $(i-1)^{st}$ and the i^{th} layer ($0 \leqslant \theta \leqslant 2\pi$ and $t > 0$, where $i = 2, ..., n$), we have;

$$\begin{aligned} T^{(i)}(r_{i-1}, \theta, t) &= T^{(i-1)}(r_{i-1}, \theta, t) && \text{(A.29)} \\ k_i \frac{\partial T^{(i)}}{\partial r}(r_{i-1}, \theta, t) &= k_{i-1} \frac{\partial T^{(i-1)}}{\partial r}(r_{i-1}, \theta, t) && \text{(A.30)} \end{aligned}$$

We can alternatively write Eqs. (A.29-A.30) as

$$\begin{aligned} T^{(i+1)}(r_i, \theta, t) &= T^{(i)}(r_i, \theta, t) && \text{(A.31)} \\ k_{i+1} \frac{\partial T^{(i+1)}}{\partial r}(r_i, \theta, t) &= k_i \frac{\partial T^{(i)}}{\partial r}(r_i, \theta, t) && \text{(A.32)} \end{aligned}$$

where $0 \leqslant \theta \leqslant 2\pi$, $t > 0$, and $i = 1, 2, 3, ..., (n-1)$. Inserting the radial function $R_{mp}^{(i)}(\lambda_{imp} r)$ in Eqs. (A.31-A.32)

$$\begin{aligned} R_{mp}^{(i+1)}(\lambda_{(i+1)mp} r_i) &= R_{mp}^{(i)}(\lambda_{imp} r_i) && \text{(A.33)} \\ k_{i+1} \frac{\partial R_{mp}^{(i+1)}}{\partial r}(\lambda_{(i+1)mp} r_i) &= k_i \frac{\partial R_{mp}^{(i)}}{\partial r}(\lambda_{imp} r_i) && \text{(A.34)} \end{aligned}$$

where

$$R_{mp}^{(i+1)}(\lambda_{(i+1)mp} r) = a_{(i+1)mp} J_m(\lambda_{(i+1)mp} r) + b_{(i+1)mp} N_m(\lambda_{(i+1)mp} r)$$

Hence we obtain

$$a_{(i+1)mp} J_m(\lambda_{(i+1)mp} r_i) + b_{(i+1)mp} N_m(\lambda_{(i+1)mp} r_i) = a_{imp} J_m(\lambda_{imp} r_i) + b_{imp} N_m(\lambda_{imp} r_i) \tag{A.35}$$

$$k_{i+1}a_{(i+1)mp}J'_m(\lambda_{(i+1)mp}r_i) + k_{i+1}b_{(i+1)mp}N'_m(\lambda_{(i+1)mp}r_i)$$
$$= k_i a_{imp}J'_m(\lambda_{imp}r_i) + k_i b_{imp}N'_m(\lambda_{imp}r_i) \quad (A.36)$$

where $J'_m(\lambda_{imp}r_i) \equiv \left(\frac{dJ_m}{dr}\right)_{r=r_i}$ and $N'_m(\lambda_{imp}r_i) \equiv \left(\frac{dN_m}{dr}\right)_{r=r_i}$.

Writing Eqs. (A.35-A.36) in matrix form gives

$$\begin{bmatrix} J_m(\lambda_{i+1,mp}r_i) & N_m(\lambda_{i+1,mp}r_i) \\ k_{i+1}J'_m(\lambda_{i+1,mp}r_i) & k_{i+1}N'_m(\lambda_{i+1,mp}r_i) \end{bmatrix} \begin{bmatrix} a_{i+1,mp} \\ b_{i+1,mp} \end{bmatrix} = \begin{bmatrix} J_m(\lambda_{imp}r_i) & N_m(\lambda_{imp}r_i) \\ k_iJ'_m(\lambda_{imp}r_i) & k_iN'_m(\lambda_{imp}r_i) \end{bmatrix} \begin{bmatrix} a_{imp} \\ b_{imp} \end{bmatrix} \quad (A.37)$$

where i=1,2,...,n-1. Eq. (A.37) is equal to Eq. (4.46)

A.4 Normalization integral: Eq. (4.101)

Using Eq. (4.42), for the first layer ($i = 1$), we can write

$$N^{(1)}_{rmp} = \frac{k_1}{\alpha_1}\int_{r_0}^{r_1} r\left(R^{(1)}_{mp}\left(\lambda_{imp}r\right)\right)^2 dr \quad (A.38)$$

where $R^{(1)}_{mp}$ is a unitless function. Let us define $\tilde{r} = \frac{r}{r_0}$, then $r = r_0\tilde{r}$, and $dr = r_0 d\tilde{r}$. Let us also define $\tilde{\lambda}_{imp} = \lambda_{imp}r_0$. Thus, $\left(\lambda_{imp}r\right) = (\lambda_{imp}r_0.\frac{r}{r_0}) = (\tilde{\lambda}_{imp}\tilde{r})$. Hence, we can write

$$N^{(1)}_{rmp} = \frac{k_1}{\alpha_1}\int_{1}^{r_1/r_0} r_0\tilde{r}\left(R^{(1)}_{mp}(\tilde{\lambda}_{1mp}\tilde{r})\right)^2 r_0 d\tilde{r} \quad (A.39)$$

$$= \left(\frac{k_1 r_0^2}{\alpha_1}\right)\underbrace{\int_{1}^{r_1/r_0} \tilde{r}\left(R^{(1)}_{mp}(\tilde{\lambda}_{1mp}\tilde{r})\right)^2 d\tilde{r}}_{\equiv \tilde{N}^{(1)}_{rmp}} \quad (A.40)$$

where $\tilde{N}^{(1)}_{rmp}$ is a unitless integral.

For $i = 2$, we get

$$N^{(2)}_{rmp} = \frac{k_2}{\alpha_2}\int_{r_1}^{r2} r\left(R^{(2)}_{mp}\left(\lambda_{2mp}r\right)\right)^2 dr \quad (A.41)$$

$$= \frac{k_2}{\alpha_2}\int_{r_1/r_0}^{r_2/r_0} r_0\tilde{r}\left(R^{(2)}_{mp}(\tilde{\lambda}_{2mp}\tilde{r})\right)^2 r_0 d\tilde{r} \quad (A.42)$$

$$= \frac{k_2 r_0^2}{\alpha_2}\int_{r_1/r_0}^{r_2/r_0} \tilde{r}\left(R^{(2)}_{mp}(\tilde{\lambda}_{2mp}\tilde{r})\right)^2 d\tilde{r} \quad (A.43)$$

Multiplying the last equation above with $(\frac{k_1}{k_1}\frac{\alpha_1}{\alpha_1})$ does not harm the equality

$$N^{(2)}_{rmp} = \frac{k_2 r_0^2}{\alpha_2}\left(\frac{k_1}{k_1}\frac{\alpha_1}{\alpha_1}\right)\int\limits_{r_1/r_0}^{r_2/r_0} {}_0\tilde{r}\left(R^{(2)}_{mp}(\tilde{\lambda}_{2mp}\tilde{r})\right)^2 d\tilde{r} \tag{A.44}$$

$$= \frac{k_1 r_0^2}{\alpha_1}\left(\frac{k_2}{k_1}\frac{\alpha_1}{\alpha_2}\right)\int\limits_{r_1/r_0}^{r_2/r_0} \tilde{r}\left(R^{(2)}_{mp}(\tilde{\lambda}_{2mp}\tilde{r})\right)^2 d\tilde{r} \tag{A.45}$$

$$= \left(\frac{k_1 r_0^2}{\alpha_1}\right)\underbrace{\left(\frac{\frac{k_2}{k_1}}{\frac{\alpha_2}{\alpha_1}}\right)\int\limits_{r_1/r_0}^{r_2/r_0} \tilde{r}\left(R^{(2)}_{mp}(\tilde{\lambda}_{2mp}\tilde{r})\right)^2 d\tilde{r}}_{\equiv \tilde{N}^{(2)}_{rmp}} \tag{A.46}$$

Similar calculation for i=3 gives

$$N^{(3)}_{rmp} = \left(\frac{k_1 r_0^2}{\alpha_1}\right)\underbrace{\left(\frac{\frac{k_3}{k_1}}{\frac{\alpha_3}{\alpha_1}}\right)\int\limits_{r_2/r_0}^{r_3/r_0} \tilde{r}\left(R^{(3)}_{mp}(\tilde{\lambda}_{3mp}\tilde{r})\right)^2 d\tilde{r}}_{\equiv \tilde{N}^{(3)}_{rmp}} \tag{A.47}$$

Comparing Eq. (A.40), Eq. (A.46) and Eq. (A.47), we see that we can use induction method to generalize the normalization integral for i^{th} layer as

$$N^{(i)}_{rmp} = \left(\frac{k_1 r_0^2}{\alpha_1}\right)\tilde{N}^{(i)}_{rmp} \tag{A.48}$$

where

$$\tilde{N}^{(i)}_{rmp} = \left(\frac{\frac{k_i}{k_1}}{\frac{\alpha_i}{\alpha_1}}\right)\int\limits_{r_{i-1}/r_0}^{r_i/r_0} \tilde{r}\left(R^{(i)}_{mp}(\tilde{\lambda}_{imp}\tilde{r})\right)^2 d\tilde{r} \tag{A.49}$$

Note that, Eq. (A.49) is eqaul to Eq. (4.102).

The normalization constant in radial direction for $i = n$ layers can be found by summing up the results of Eq. (A.48) found for each n layer. Then;

$$N_{rmp} = N^{(1)}_{rmp} + N^{(2)}_{rmp} + \ldots + N^{(n)}_{rmp} \tag{A.50}$$

$$= \left(\frac{k_1 r_0^2}{\alpha_1}\right)\sum_{i=1}^{n}\tilde{N}^{(i)}_{rmp} \tag{A.51}$$

Our illustrative example has three layers. Hence, we take $n = 3$ in the equation above, which gives us Eq. (4.101).

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