BORN-INFELD GRAVITY THEORIES IN D-DIMENSIONS

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ABSTRACT

BORN-INFELD GRAVITY THEORIES IN D-DIMENSIONS

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Born-Infeld gravity proposed by Deser and Gibbons takes its origin from two ideas: Born-Infeld electrodynamics and Eddington's gravitational action. The theory is defined with a determinantal action involving the Ricci tensor as in the Eddington's theory; however, in contrast, the independent variable is the metric as in Einstein's gravity and the action is constructed in analogy with the action of the Born-Infeld electrodynamics. Main challenge in defining a Born-Infeld type gravity is obtaining a unitary theory around–at least–flat and maximally symmetric constant curvature backgrounds. In this thesis, a framework for analyzing the tree-level unitarity of a generic D-dimensional Born-Infeld type gravity is developed. Besides, in three dimensions, a Born-Infeld gravity theory which is unitary to all orders in the curvature is studied in detail. This theory was introduced as an extension of a specific quadratic curvature gravity theory dubbed as "new massive gravity" which is unitary with a massive spin-2 excitation in its spectrum. Besides having a unitary massive spin-2 excitation, the Born-Infeld gravity in three dimensions has a holographic c-function which is the same as Einstein's gravity. In addition, the theory has constant curvature Type-N and Type-D solutions which are the same as the cosmological topologically massive gravity.

Keywords: Born-Infeld gravity, new massive gravity, topologically massive gravity, Type-N solutions, Type-D solutions

ÖZ

D-BOYUTTA BORN-INFELD KÜTLEÇEKIM TEORILERI

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Deser ve Gibbons tarafından ortaya atılmış olan Born-Infeld kütleçekim, kaynağını Born-Infeld elektrodinamiği ve Eddington'un kütleçekim eylemi fikirlerinden almaktadır. Teori, Eddington'un teorisinde olduğu gibi Ricci tensörünün determinantını içeren bir eylemle tanımlanır, fakat bu teoriden farklı olarak Einstein teorisinde olduğu gibi bağımsız değişken metriktir ve eylem Born-Infeld elektrodinamiğinin eylemiyle analoji üzerinden kurulur. Born-Infeld tipi kütleçekim tanımlanırken karşılaşılan temel zorluk (en azından) düz ve maksimal olarak simetrik sabit eğrilikli arkaplanlar etrafında üniter bir teori elde edilmesi gereğidir. Bu tezde, D-boyutlu genel Born-Infeld tipi kütleçekimin ağaç mertebesi üniterliğini analiz etmek için gerekli çerçeve geliştirilecektir. Buna ek olarak, üç boyutta eğrilik terimlerinin bütün mertebeleri için üniter olan Born-Infeld kütleçekim teorisi detaylı bir şekilde incelenecektir. Bu teori, "yeni kütleli gravitasyon" olarak adlandırılan, üniter, spekturumunda kütleli spin-2 tedirgemeler olan özel bir ikinci dereceden eğrilikli kütleçekim teorisinin genişletilmesi olarak ortaya atılmıştır. Üç boyuttaki Born-Infeld kütleçekim teorisi, üniter kütleli spin-2 tedirgemeler içermesinin yanında Einstein kütleçekim ile aynı holografik c-fonksiyonuna sahiptir. Ek olarak, bu teori kozmolojik sabitli topolojik kütleli gravitasyon ile aynı Tip-N ve Tip-D çözümlerine sahiptir.

Anahtar Kelimeler: Born-Infeld kütleçekim, yeni kütleli kütleçekim, topolojik kütleli kütleçekim, Tip-N çözümler, Tip-D çözümler

To Kiraz, my dear wife

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INTRODUCTION

General relativity represents our current understanding of gravitation and at solar system scales, it is a well-tested theory. However, in larger scales, various observations imply a possible modification of the theory. For example, the accelerated expansion of the Universe can be explained by augmenting Einstein's gravity with a cosmological constant. Furthermore, the observations suggesting the existence of dark matter may also be an indication of a modification in the gravitational laws. In addition to this observational motivations, there are also theoretical considerations implying a modification of Einstein's gravity. Reconciling general relativity with quantum mechanics is a major theoretical problem and the nonrenormalizability of Einstein's gravity is an important issue in this respect. In various quantum gravity scenarios, such as string theory, asymptotic safety, Einstein's gravity appears as a low energy effective field theory which should be augmented by higher derivative terms to cure the nonrenormalizability behavior. Indeed, the extension of the theory with quadratic curvature terms was shown to be renormalizable [1]; however, the corresponding quantum theory is not unitary [1, 2]. With the effective field theory perspective, the generic form of the gravitational action is

$$I = \int d^4x \left\{ \frac{1}{\kappa} \left(R - 2\Lambda_0 \right) + \sum_{n=2}^{\infty} a_n \left(\text{Riem, Ric, R, } \nabla\text{Riem, } \dots \right)^n \right\}, \tag{1.1}$$

and the question is that which higher derivative and higher curvature terms appear in this action with which couplings. One may try to deduce the possible higher order terms and their couplings from a proposal of a UV-complete fundamental theory describing the quantum gravity. On the other hand, instead of this top-down approach, it is possible to consider various theoretical consistency requirements such as unitarity and try to figure out constraints on the possible terms and their couplings. In this regard, extending Einstein's gravity with higher curvature terms and studying viability of these theories can help to get general constraints on (1.1).

In this thesis, we focus on Born-Infeld gravity which is a specific infinite order higher curvature modification of Einstein's gravity. The bulk of the material presented here is based on the original research work whose results were published in the papers:

- I. Gullu, T. C. Sisman, B. Tekin, *"Born-Infeld extension of new massive gravity"* [3],
- I. Gullu, T. C. Sisman and B. Tekin, *"c-functions in the Born-Infeld extended New Massive Gravity"* [4],
- I. Gullu, T. C. Sisman and B. Tekin, *"Unitarity analysis of general Born-Infeld gravity theories"* [5],
- I. Gullu, T. C. Sisman, B. Tekin, *"All Bulk and Boundary Unitary Cubic Curvature Theories in Three Dimensions"* [6],
- M. Gurses, T. C. Sisman and B. Tekin, "*Some exact solutions of all $f\left(R_{\mu\nu}\right)$ theories in three dimensions*" [7].

Our main aim is to study the theoretical consistency of D-dimensional Born-Infeld gravity theories and to discuss a particularly successful three-dimensional Born-Infeld gravity theory.

The layout of this thesis is as follows: In the remaining sections of this chapter, first, we give a brief introduction on Born-Infeld gravity theories, and then, we discuss the tree-level unitarity of higher curvature gravity theories. In the second chapter, we analyze the unitarity of D-dimensional Born-Infeld gravity theories. Third chapter is devoted to the three-dimensional Born-Infeld gravity theory and its properties. A conclusion chapter is followed by two appendices on the metric perturbation expansions of various tensorial structures and the perturbative analysis of the quadratic curvature gravity action.

Now, let us give our conventions. The signature of the metric is taken to be mostly positive. The Riemann tensor is defined as $R^{\mu}{}_{\nu\rho\sigma} \equiv \partial_{\rho}\Gamma^{\mu}_{\sigma\nu} + \Gamma^{\mu}_{\rho\lambda}\Gamma^{\lambda}_{\sigma\nu} - \rho \leftrightarrow \sigma$, while the Ricci tensor is $R_{\nu\sigma} \equiv R^{\mu}{}_{\nu\mu\sigma}$. The determinant of the metric $g_{\mu\nu}$ is denoted as g. To avoid a possible confusion, the determinants of rank-$(0,2)$ and rank-$(1,1)$ tensors are shown explicitly; for example, as $\det(g_{\mu\nu} + A_{\mu\nu})$ and $\det(\delta^{\mu}_{\nu} + A^{\mu}_{\nu})$.

1.1 Born-Infeld gravity

The idea of the Born-Infeld (BI) type modifications of Einstein's gravity was proposed by Deser and Gibbons [8]. As the name suggests, the theory has common features with Born-Infeld electrodynamics [9]. In addition, Eddington's gravitational action [10] is the other source of inspiration for the proposal in [8].

BI electrodynamics is based on the principle of finiteness [9]. Born and Infeld considered the divergences in Maxwell electrodynamics as the failure of the theory and they defined their theory with the action having a determinantal form as

$$I = -b^2 \int d^4x \sqrt{-\det\left(g_{\mu\nu} + \frac{1}{b}F_{\mu\nu}\right)}, \tag{1.2}$$

which originated from the relativistic point particle action $I = -m \int dt \sqrt{1 - v^2}$. In the BI electrodynamics, there is an upper bound for the field strength whose scale is determined by the dimensionful parameter b. The finiteness of the field strength is due to the nonlinear nature of the theory and for small field strengths, the BI action generates the Maxwell action. Furthermore, the excitations of the theory, namely photons, are not ghost.

Determinantal actions also appeared as a modification of Einstein's gravity. The early proposal of Eddington [10], which even predates the BI electrodynamics, was motivated by the idea of writing a gravitational theory which has the connection as the fundamental geometric quantity rather than the metric. Then, Eddington preferred to write a generalized invariant volume element in the form $\int d^4x \sqrt{\det\left[R_{(\mu\nu)}(\Gamma)\right]}$ and proposed it as the action of the gravitational theory.

Following the ideas of Born and Infeld, and Eddington, the gravitational action

$$I = \int d^4x \sqrt{-\det(a g_{\mu\nu} + b R_{\mu\nu} + c X_{\mu\nu})}, \tag{1.3}$$

was proposed in [8]. Here, a, b, and c are the parameters of the theory. The action is a generalized invariant volume element which is constructed in a similar form to

(1.2). As opposed to Eddington's action, the fundamental geometric quantity in (1.3) is the metric. In addition, the tensor $X_{\mu\nu}$ is unknown and should be determined in such a way that the theory has a consistent spectrum which is free from ghosts and tachyons. There could of course be more constraints on the theory, such as being supersymmetrizable, but here we shall be only interested in the unitarity about flat and (anti)-de Sitter [(A)dS] backgrounds.

In three dimensions, there is a particularly successful example of BI-type gravity theories whose action has a rather elegant form as [3]

$$I_{\text{BINMG}} = -\frac{4m^2}{\kappa^2}\int d^3x \left[\sqrt{-\det\left(g_{\mu\nu} - \frac{1}{m^2}G_{\mu\nu}\right)} - \left(1 - \frac{\lambda_0}{2}\right)\sqrt{-g}\right], \qquad (1.4)$$

where $G_{\mu\nu}$ is the Einstein tensor, $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. When this action is expanded in curvature, at the quadratic curvature order, it reproduces new massive gravity (NMG) theory defined with the action [11, 12]

$$I_{\text{NMG}} = \frac{1}{\kappa^2}\int d^3x\sqrt{-g}\left[-R - 2\lambda_0 m^2 + \frac{1}{m^2}\left(R^\mu_\nu R^\nu_\mu - \frac{3}{8}R^2\right)\right], \qquad (1.5)$$

which is the unique quadratic curvature theory that is unitary around both flat and (A)dS backgrounds with a massive spin-2 excitation [12, 13, 14, 15, 16].[1] This feature of (1.4) was the original motivation for the construction of (1.4), so the theory was named as the Born-Infeld extension of NMG (BINMG) [3]. Just like NMG, BINMG is also unitary around both flat and (A)dS backgrounds with a massive spin-2 excitation [6], and in fact, BINMG is the first example of a unitary BI-type gravity around the (A)dS background. Both NMG and BINMG represent nonlinear generalizations of the Fierz-Pauli massive spin-2 theory which are free from the Boulware-Deser ghost [18] as shown in [19] for NMG and in [20] for BINMG. As in the case of BI electrodynamics for which there is a bound on the maximum attainable field strength, (1.4) puts a constraint on the curvature as $R \geq -6m^2$ for (A)dS spacetimes [21]. In addition, at cubic and quartic orders, the curvature expansion of (1.4) matches the cubic and the quartic curvature extensions of NMG which were constructed by AdS/CFT considerations in [22]. Certain aspects of BINMG such as its central charge [23, 4], c-functions [4], classical solutions [23, 24, 25, 7, 27], and Weyl invariant extension [28]

[1] Initially, NMG was thought to be renormalizable since the four-dimensional quadratic curvature theory is renormalizable. However, a careful study reveals that this is not the case for NMG due to the specific relation satisfied by the couplings of the quadratic curvature terms [17].

have been studied. Remarkably, BINMG has a holographic c-function which matches the c-function of Einstein's gravity. Furthermore, the gravitational charges for the BTZ black hole of BINMG were studied in [23, 29]. In addition, as a curious side remark let us note that BINMG appears as a cut-off independent counterterm to the four dimensional anti-de Sitter space [30].

1.2 Tree-level unitarity

A gravity theory has a physically consistent spectrum at the tree level if the spectrum is free from ghosts and tachyons. Ghosts are negative kinetic energy modes. A physically consistent mode is described with an action $I = \int dt\, L = \int dt\, (K - U)$, where K and U represent kinetic and potential energies, respectively. Then, the corresponding Hamiltonian, which represents the energy of the mode, is given as $H = K + U$, and it is positive definite for a positive-definite potential. However, for a ghost mode, one has $I = \int dt\, (-K - U)$ yielding $H = -K + U$, so the energy of the ghost mode is not positive definite. For such a case, the vacuum is not stable and the proliferation of negative energy modes is entropically favorable. An important feature of ghost instability is that it is relevant at all energy scales. In addition, the ghost modes correspond to the negative norm states in the Hilbert space of the corresponding quantum theory as $\langle\psi| H |\psi\rangle = -E \langle\psi|\psi\rangle \Rightarrow \langle\psi|\psi\rangle = -1$. Therefore, the unitarity of the theory is spoiled by the ghost modes as probabilities are not positive definite in the presence of negative norm states. On the other hand, tachyons have negative squared masses.[2] To understand the instability caused by tachyons, let us consider the example of a scalar field ϕ with the action $I[\phi] = \int dt\, [K(\partial_\mu \phi) - U(\phi)]$. If one considers the field fluctuation $\varphi \equiv \phi - \bar{\phi}$ around the vacuum $\bar{\phi}$, that is $\left[\frac{dU}{d\phi}\right]_{\bar{\phi}} = 0$, then one gets the action

$$I[\varphi] = \int dt \left(K(\partial_\mu \varphi) - U(\bar{\varphi}) - \frac{1}{2}\left[\frac{d^2U}{d\phi^2}\right]_{\bar{\phi}} \varphi^2 + \dots \right),$$

where, as usual, $\left[\frac{d^2U}{d\phi^2}\right]_{\bar{\phi}}$ corresponds to m^2. Thus, a tachyon, which is a negative m^2 mode, corresponds to a unstable vacuum as $\left[\frac{d^2U}{d\phi^2}\right]_{\bar{\phi}} < 0$. Note that the tachyon

[2] Note that in AdS if certain bounds are satisfied for various spins, a negative squared mass mode is allowed.

instability becomes relevant at energy scales close to m. To sum up, to have a physically consistent spectrum, excitations of a field theory should have correct signs at the action level. For example, for a scalar mode, the correct signs of the kinetic term and the mass term are given with the action $I = -\frac{1}{2}\int d^4x \left(\partial_\mu\varphi\partial^\mu\varphi + m^2\varphi^2\right)$ for the mostly plus convention of the metric. It is also possible to read these signs at the propagator level where one should have $\frac{1}{p^2+m^2}$ for a physically consistent scalar mode.

In this thesis, we study the unitarity of BI gravity theories which are infinite order higher curvature gravity theories with fixed couplings. To analyze the spectrum and its consistency for a higher curvature gravity theory, one needs to determine the free theory of metric fluctuations around a background spacetime (or, in other words, a vacuum) which solves the field equations of the higher curvature gravity theory. The free theory is described by the second order action in metric fluctuations around the background and in this study, the background is taken to be either the flat or the (A)dS spacetime.

In calculating the second order action in metric fluctuations, it is important to observe that for the flat background higher curvature terms beyond the quadratic curvature order do not yield any contribution, while for the (A)dS background all the terms in a higher curvature gravity action contribute in principle. Due to this fact the unitarity analysis of a higher curvature theory around the (A)dS background is a nontrivial task. To understand this better, let us consider the example of the cubic curvature term R^3. If the scalar curvature has the expansion in metric fluctuations as

$$R = \bar{R} + R_{(1)} + R_{(2)} + \ldots, \tag{1.6}$$

where $R_{(1)}$ and $R_{(2)}$ represent first and second orders of R in metric fluctuations and $\bar{R}$ is the background value of R, then the second order contributions that come from R^3 have the forms $\bar{R}R_{(1)}^2$ and $\bar{R}^2R_{(2)}$. Thus, for the flat background these terms become zero, while for the (A)dS background there are contributions to the second order action coming from the term R^3.

Another important observation in calculating the second order action in metric fluctuations around the (A)dS background is that the contributions coming from the terms beyond the quadratic curvature order have the same structure as the contributions coming from the quadratic curvature terms. To understand this point, let us resort to the R^3 example again. As discussed above the second order contributions coming from R^3 have the forms $\bar{R}R^2_{(1)}$ and $\bar{R}^2 R_{(2)}$, while the second order contributions coming from R^2 are $R^2_{(1)}$ and $\bar{R}R_{(2)}$. Thus, the term R^3 yields the same forms except the overall $\bar{R}$ multiplicity. In the same manner, the independent quadratic curvature scalars R^2, $R^\mu_\nu R^\nu_\mu$, and $R^{\mu\nu}_{\rho\sigma} R^{\rho\sigma}_{\mu\nu}$ determine the structure of the second order action and a higher curvature term of order n yields second order contributions which are the same as the ones that can come from these three quadratic curvature scalars except an overall factor proportional to $\bar{R}^{n-2}$. Instead of $R^{\mu\nu}_{\rho\sigma} R^{\rho\sigma}_{\mu\nu}$, the Gauss-Bonnet (GB) combination can also be considered as one of the independent scalars. Note that the GB combination is a boundary term in four dimensions and identically zero in three dimensions, so the GB combination (or the term $R^{\mu\nu}_{\rho\sigma} R^{\rho\sigma}_{\mu\nu}$) is relevant in dimensions greater than four.

Due to these two observations, the unitarity analysis of the most general quadratic curvature theory defined with the action

$$I = \int d^D x \sqrt{-g} \Bigg[\frac{1}{\kappa} (R - 2\Lambda_0) + \alpha R^2 + \beta R^\mu_\nu R^\nu_\mu + \gamma \left(R^{\mu\nu}_{\rho\sigma} R^{\rho\sigma}_{\mu\nu} - 4 R^\mu_\nu R^\nu_\mu + R^2 \right) \Bigg], \tag{1.7}$$

lay down the ground rules for the unitarity analysis of higher curvature theories. A higher curvature theory is unitary if and only if its propagator reduces to the propagator of one of the known unitary theories at the linear or the quadratic curvature order; therefore, it is essential to discuss unitary theories of quadratic curvature. For the flat background, the unitarity of the most general quadratic curvature theory in four dimensions was discussed in [2] and it was shown that the quadratic curvature theory is not unitary in the presence of the term $R^\mu_\nu R^\nu_\mu$. This result can be generalized to higher dimensions; however, in three dimensions, a subtlety occurs and one has NMG [11] for specific values of the parameters. For the (A)dS background, the unitarity of (1.7) is studied in [15] where the analysis is in D dimensions. Let us summarize the results of [2, 11, 15]:

- For $D = 4$, Einstein's gravity is unitary around both flat and (A)dS backgrounds with a massless spin-2 mode. When R^2 term is augmented to the Einstein's gravity, the theory is unitary with an additional massive spin-0 degree of freedom.

The original motivation for augmenting Einstein's gravity with the quadratic curvature terms is to endow it with renormalizability; however, its R^2 extension is not renormalizable, too. To have a renormalizable quadratic curvature theory, the term $R^\mu_\nu R^\nu_\mu$ is required, but this term spoils unitarity by introducing a massive spin-2 ghost mode to the spectrum.

- For $D = 3$, Einstein's gravity does not have any propagating degrees of freedom. Extending Einstein's gravity with R^2 term yields a unitary theory with a massive spin-0 excitation. However, for a generic quadratic curvature extension with $\alpha R^2 + \beta R^\mu_\nu R^\nu_\mu$, the spectrum consists of massive spin-0 and massive spin-2 excitations whose unitarity behaviors are in conflict. Remarkably, this conflict is resolved when the couplings satisfy $8\alpha + 3\beta = 0$, which is the NMG case [11]. With such couplings the massive spin-0 mode drops out the spectrum leaving the massless spin-2 mode which is unitary around both flat and (A)dS backgrounds.

- For $D > 4$, the GB combination becomes relevant and both Einstein's gravity and its extension with the GB combination have a unitary massless spin-2 mode around flat and (A)dS backgrounds. Augmenting R^2 to either Einstein's gravity or its GB extension does not effect the unitarity nature and extends the spectrum with a massive spin-0 mode. As in four dimensions, the presence of $R^\mu_\nu R^\nu_\mu$ in the action implies the existence of a massive spin-2 mode which is a ghost.

One can figure out all these results by analyzing the tree-level scattering amplitude for quadratic curvature gravity [15]. In the next section, we recapitulate this analysis revealing the propagator structure of quadratic curvature gravity.

1.2.1 Propagator Structure of Quadratic Curvature Gravity

The unitarity of the generic quadratic curvature theory (1.7) can be analyzed through the tree-level scattering amplitude between two background covariantly conserved sources, that is $\bar{\nabla}_\mu T^{\mu\nu} = 0$ where $\bar{\nabla}_\mu$ is the covariant derivative corresponding to the background metric $\bar{g}_{\mu\nu}$. The amplitude is described by the Feynman diagram given in Fig. 1.1 and to find the amplitude, one needs to calculate

$$A = \int d^D x \, \sqrt{-\bar{g}} T'_{\mu\nu}(x) \, h^{\mu\nu}(x) \,, \tag{1.8}$$

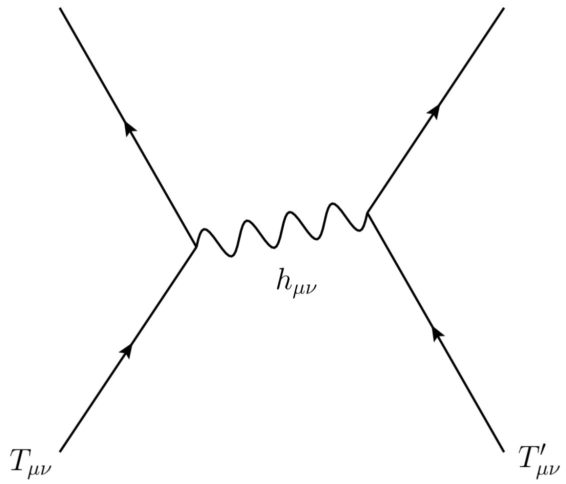

Figure 1.1: Tree-level scattering amplitude between two background covariantly conserved sources via the exchange of a graviton

where $h_{\mu\nu}$ is the metric fluctuation which is defined as $h_{\mu\nu} \equiv g_{\mu\nu} - \bar{g}_{\mu\nu}$ and created by the source $T_{\mu\nu}$. Here, the normalization of the amplitude is fixed such that the Newtonian potential can be reproduced for $\kappa = 8\pi G_N$ in four dimensions.

For quadratic curvature gravity, there are generically two (A)dS backgrounds. The (A)dS spacetime is maximally symmetric, so the form of the Riemann tensor is

$$\bar{R}_{\mu\rho\nu\sigma} = \frac{2\Lambda}{(D-1)(D-2)} \left(\bar{g}_{\mu\nu}\bar{g}_{\rho\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\rho\nu}\right), \tag{1.9}$$

and employing this form in the field equations of the quadratic curvature gravity theory yields

$$\frac{\Lambda - \Lambda_0}{2\kappa} + f\Lambda^2 = 0, \qquad f \equiv (D\alpha + \beta)\frac{(D-4)}{(D-2)^2} + \gamma\frac{(D-3)(D-4)}{(D-1)(D-2)}. \tag{1.10}$$

This result explicitly reveals that the GB combination does not have an effect on the effective cosmological constant in three and four dimensions.

The metric fluctuation $h_{\mu\nu}$ is determined through the linearized field equations which were found in [31, 32] as

$$\begin{aligned} T_{\mu\nu}(h) =& \frac{1}{\kappa_e}\mathcal{G}^L_{\mu\nu} + (2\alpha+\beta)\left(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu + \frac{2\Lambda}{D-2}\bar{g}_{\mu\nu}\right)R^L \\ &+ \beta\left(\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{D-1}\bar{g}_{\mu\nu}R^L\right), \end{aligned} \tag{1.11}$$

where the linearized Einstein and Ricci tensors and the linearized scalar curvature are given as

$$\begin{aligned}
\mathcal{G}^L_{\mu\nu} &= R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^L - \frac{2\Lambda}{D-2}h_{\mu\nu}, \\
R^L_{\mu\nu} &= \frac{1}{2}\left(\bar{\nabla}^\rho\bar{\nabla}_\mu h_{\nu\rho} + \bar{\nabla}^\rho\bar{\nabla}_\nu h_{\mu\rho} - \bar{\square}h_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu h\right), \\
R^L &= -\bar{\square}h + \bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu} - \frac{2\Lambda}{D-2}h.
\end{aligned} \tag{1.12}$$

Here, $\bar{\square}$ is the d'Alembertian defined as $\bar{\square} \equiv \bar{\nabla}^\mu\bar{\nabla}_\mu$. Furthermore, κ_e is the effective Newton's constant of the form

$$\frac{1}{\kappa_e} \equiv \frac{1}{\kappa} + \frac{4\Lambda D}{D-2}\alpha + \frac{4\Lambda}{D-1}\beta + \frac{4\Lambda\left(D-3\right)\left(D-4\right)}{\left(D-1\right)\left(D-2\right)}\gamma, \tag{1.13}$$

and $T_{\mu\nu}\left(h\right)$ involves all the higher order terms in $h_{\mu\nu}$ beyond the linear order in addition to the matter source.

To discuss the unitarity of the quadratic curvature theory, we need to put the amplitude (1.8) in a form where the propagator structure is explicit. However, this is not a trivial task since the differential operator $\mathcal{O}_{\mu\nu}{}^{\rho\sigma}$ which represents (1.11) as $\mathcal{O}_{\mu\nu}{}^{\rho\sigma}h_{\rho\sigma} = T_{\mu\nu}$ has a complicated form involving fourth and second order derivatives in addition to a constant term. Although we search for a symbolic form for the tensor Green's function of $\mathcal{O}_{\mu\nu}{}^{\rho\sigma}$, it is not possible to directly invert this operator. In [15], the desired form for the amplitude was found by first decomposing $h_{\mu\nu}$ as

$$h_{\mu\nu} \equiv h^{TT}_{\mu\nu} + \bar{\nabla}_{(\mu}V_{\nu)} + \bar{\nabla}_\mu\bar{\nabla}_\nu\phi + \bar{g}_{\mu\nu}\psi, \tag{1.14}$$

where $h^{TT}_{\mu\nu}$ is transverse-traceless part of $h_{\mu\nu}$ and V_ν is divergence free, and then by choosing an appropriate gauge[3] which makes possible to determine the physical parts of $h_{\mu\nu}$ and their relation to the sources. Finally, the amplitude becomes

$$\begin{aligned}
\frac{A}{\kappa_e} =& 2T'_{\mu\nu}\left[\left(\kappa_e\beta\bar{\square}+1\right)\left(\triangle^{(2)}_L - \frac{4\Lambda}{D-2}\right)\right]^{-1}T^{\mu\nu} \\
&+ \frac{2}{D-2}T'\left[\left(\kappa_e\beta\bar{\square}+1\right)\left(\bar{\square} + \frac{4\Lambda}{D-2}\right)\right]^{-1}T \\
&- \frac{2\left(\beta+c\right)}{c\left(D-1\right)\left(D-2\right)}T'\left[\left(\kappa_e\beta\bar{\square}+1\right)\left(\bar{\square}-m_s^2\right)\right]^{-1}T \\
&+ \frac{8\Lambda D\beta}{c\left(D-1\right)^2\left(D-2\right)^2} \\
&\times T'\left[\left(\kappa_e\beta\bar{\square}+1\right)\left(\bar{\square}-m_s^2\right)\left(\bar{\square}+\frac{2\Lambda D}{\left(D-1\right)\left(D-2\right)}\right)\right]^{-1}T,
\end{aligned} \tag{1.15}$$

[3] In the analysis of [15], the Fierz-Pauli mass term is augmented to the quadratic curvature action and the presence of this term helps to fix the gauge.

which is the reorganized form [33] of the result found in [15]. Here, $\triangle_L^{(2)}$ is the Lichnerowicz operator which acts on a symmetric rank-2 tensor, c is defined as $c \equiv \frac{4(D-1)\alpha+D\beta}{D-2}$, and m_s is the mass of the scalar excitation which has the form

$$m_s^2 = \frac{1}{c\kappa_e} - \frac{2\Lambda D}{(D-1)(D-2)}\left(1-\frac{\beta}{c}\right). \tag{1.16}$$

Note that in (1.15), integral signs and measures are omitted and the Green's functions are represented as inverse operators to simplify the expression.

Although (1.15) looks complicated, the important point is that each pole is multiplied with another pole. This feature of the amplitude (1.15) implies the presence of the ghosts since separating the poles yields a wrong sign propagator. After setting $\alpha = \beta = \gamma = 0$, one gets the part of (1.15) which represents the amplitude of Einstein's gravity as

$$A = 2\kappa\left[T'_{\mu\nu}\left(\triangle_L^{(2)} - \frac{4\Lambda}{D-2}\right)^{-1}T^{\mu\nu} + \frac{1}{D-2}T'\left(\bar{\Box} + \frac{4\Lambda}{D-2}\right)^{-1}T\right], \tag{1.17}$$

and we know that this amplitude represents the unitary interaction of the covariantly conserved sources with a massless spin-2 graviton for $\kappa > 0$ (except in three dimensions). On the other hand, the pole $\left(\kappa_e\beta\bar{\Box}+1\right)^{-1}$ represents the massive spin-2 mode as it couples to the tensorial sources, while the pole $\left(\bar{\Box}-m_s^2\right)^{-1}$ represents the massive spin-0 mode as it only couples to the trace of the sources. The multiplicative structure in (1.15) reveals that the unitarity of the massive spin-2 mode is in conflict with the unitarity of the massless spin-2 mode and the massive spin-0 mode. On the other hand, in the absence of the massive spin-2 mode, i.e. taking $\beta = 0$, the unitarity of the massless spin-0 mode is in accord with the unitarity of the Einstein mode as the amplitude takes the form

$$\begin{aligned} A = 2\kappa_e\Bigg[&T'_{\mu\nu}\left(\triangle_L^{(2)} - \frac{4\Lambda}{D-2}\right)^{-1}T^{\mu\nu} + \frac{1}{D-2}T'\left(\bar{\Box} + \frac{4\Lambda}{D-2}\right)^{-1}T \\ &- \frac{1}{(D-1)(D-2)}T'\left(\bar{\Box}-m_s^2\right)^{-1}T\Bigg], \end{aligned} \tag{1.18}$$

where m_s^2 reduces to

$$m_s^2 = \frac{D-2}{4(D-1)\alpha\kappa_e} - \frac{2\Lambda D}{(D-1)(D-2)}. \tag{1.19}$$

In (1.18), all the poles are separated and comparing with the amplitude of Einstein's gravity (1.17), one can figure out that it represents a unitary interaction if $\kappa_e > 0$. In addition, for the dS background, $m_s^2 > 0$ should hold, while for the AdS background, m_s^2 should satisfy the Breitenlohner-Freedman (BF) bound [34]

$$m_s^2 \geq \frac{D-1}{2(D-2)}\Lambda, \tag{1.20}$$

allowing for negative mass squared values. These conditions reduce to $\kappa > 0$ and $m_s^2 > 0$ for the flat background. Since there are two constraints on three theory parameters, namely, κ, α, γ, certainly there are parameter regions for which these unitarity constraints are satisfied.[4] Therefore, to have a unitary quadratic curvature theory, one needs to set $\beta = 0$ for $D > 3$. On the other hand, in three dimensions, a subtlety occurs since Einstein's gravity does not have a propagating degree of freedom. In the absence of the Einstein mode, the unitarity conflict between massive spin-2 and spin-0 modes can be resolved by choosing specific parameter values satisfying $8\alpha + 3\beta = 0$. For these values of the parameters, the massive spin-0 mode is removed from the spectrum [11] and the remaining massive spin-2 mode can be made unitary both around flat and (A)dS backgrounds by having negative κ .

From (1.15), the amplitude for the Einstein-Gauss-Bonnet theory can be obtained by setting $\alpha = 0$ and the amplitude has the same form as the amplitude for Einstein's gravity (1.17) except the appearance of the effective Newton's constant κ_e for the (A)dS background. Therefore, the Einstein-Gauss-Bonnet theory is unitary if $\kappa_e > 0$ for the (A)dS background and this constraint reduces to $\kappa > 0$ for the flat background. On the other hand, the amplitude for the R^2 extension of Einstein's gravity can be obtained by setting $\gamma = 0$ in (1.15) and as γ is implicit in κ_e, the amplitude has the same form as (1.15). Therefore, the unitarity constraints on the R^2 extension of the Einstein's gravity are the same as the ones for (1.15).

After elaborating on the unitary quadratic curvature theories by using the tree-level amplitude, let us discuss the relevance of the linearized field equations with the second order action in metric fluctuations. The second order action for the quadratic curvature theory simply has the from $I_{O(h^2)} = -\frac{1}{2}\int d^D x \sqrt{-\bar{g}}\, T_{\mu\nu} h^{\mu\nu}$ where $T_{\mu\nu}$ should be replaced with the corresponding expression coming from the linearized field equations (1.11). The structures involved in this action are important since they represent the generic building blocks for the second order action of any higher curvature theory.

[4] Please see [33], for explicit parameter regions where the theory is unitary.

For example, if the quadratic curvature theory is augmented with higher curvature terms of order n such as ηR^n, $\eta \left(R^\mu_\nu R^\nu_\mu \right)^{n/2}$, etc., then the second order contributions coming from these higher curvature terms of order n only introduce additional terms to $1/\kappa_e$ in the form $\eta \Lambda^{n-1}$ and to the coefficients $(2\alpha + \beta)$ and β in the form $\eta \Lambda^{n-2}$. Therefore, higher curvature terms do not generate new degrees of freedom other than the ones that are present in the quadratic curvature theory, but they may change their unitarity behavior depending on the couplings and the magnitude of curvature.

CHAPTER 2

UNITARITY ANALYSIS OF BORN-INFELD GRAVITY THEORIES

Born-Infeld (BI) gravity is an appealing modification of Einstein's gravity. When considering such a modification, the immediate questions are how the spectrum of Einstein's gravity is changed and whether the modes in the spectrum are theoretically consistent. In a theoretically consistent modification, the theory should be free from ghosts and tachyons that are negative kinetic energy and negative square mass modes, respectively; and the ghosts of the classical level imply that the quantum theory described by the theory is not unitary. This chapter is based on [5] and devoted to analyze the spectrum and its consistency for a generic D-dimensional BI gravity theory around its maximally symmetric vacuum that is (anti)-de Sitter [(A)dS] spacetime. For a generic BI gravity theory, we developed a formulation from which the second order action in the metric perturbation, $h_{\mu\nu} \equiv g_{\mu\nu} - \bar{g}_{\mu\nu}$, around (A)dS vacua, $\bar{g}_{\mu\nu}$, can be obtained. The $O\left(h^2\right)$ action represents the free theory of the BI gravity theory and naturally shares the same structure with the free theory of the quadratic curvature gravity. Furthermore, we presented procedures to obtain equivalent actions whose free theory and vacua are equal to specific BI gravity theories.

To analyze the spectrum of a generic BI gravity theory around its (A)dS vacua is a nontrivial task compared to the flat background analysis. For a BI gravity theory, the Maclaurin series expansion in curvature represents an infinite series in higher curvature terms. Around the flat background, the free theory of BI gravity only depends on the terms up to the second order in the curvature expansion. For example, it is rather simple to show the gravity theory described by the BI-type Lagrangian

$$\mathcal{L}=\sqrt{-\det\left(g_{\mu\nu}+\alpha R_{\mu\nu}\right)}-\sqrt{-g}, \tag{2.1}$$

is not unitary. To demonstrate this, one needs to expand (2.1) up to the second order in curvature which yields

$$\mathcal{L}_{O(R^2)}=\frac{\alpha}{2}R-\frac{\alpha^2}{4}\left(R_{\mu\nu}R^{\mu\nu}-\frac{1}{2}R^2\right), \tag{2.2}$$

Based on the results of [2] where the $O\left(h^2\right)$ action of a generic quadratic curvature gravity was analyzed, one can decide on the spectrum of the theory (2.1) around the flat background. The spectrum consists of massive spin-0 and massive spin-2 modes in addition to the massless spin-2 Einstein mode, and the massive spin-2 mode is a ghost. To have a unitary theory around flat backgrounds with the same spectrum as Einstein's gravity, the quadratic curvature terms in (2.2) should be eliminated and one way to achieve this is to introduce the quadratic curvature combination $\frac{\alpha^2}{2}\left(R_{\mu\rho}R^{\rho}_{\nu}-\frac{1}{2}RR_{\mu\nu}\right)$ into the BI action [8] as

$$\mathcal{L}=\sqrt{-\det\left[g_{\mu\nu}+\alpha R_{\mu\nu}+\frac{\alpha^2}{2}\left(R_{\mu\rho}R^{\rho}_{\nu}-\frac{1}{2}RR_{\mu\nu}\right)\right]}-\sqrt{-g}. \tag{2.3}$$

On the other hand, around (A)dS backgrounds, all the higher curvature terms in the Maclaurin series expansion of a generic BI gravity theory contribute to the free theory in principle, and these contributions are in the same form as the contribution coming from quadratic curvature terms. For example, the contribution coming from the cubic curvature term R^3 is the same as the contribution coming from the quadratic curvature term R^2 except for the overall factor $\bar{R}$, which is the scalar curvature of the background (A)dS spacetime. Therefore, to analyze the spectrum of a BI gravity theory around (A)dS backgrounds, one should directly find the second order expansion in the metric perturbation for the action of the theory and the unitarity of the free theory can be analyzed by following [15]. To find the $O\left(h^2\right)$ action is a rather cumbersome task; however, the determinantal form of the BI action and maximally symmetric nature of the background yield compact expressions.

The techniques we developed in this chapter are crucial in the unitarity analysis of BINMG around (A)dS backgrounds. Furthermore, to construct unitary theories in higher dimensions, one should rely on the results that we obtained.

2.1 BI-Type Actions at $O\left(h^2\right)$

2.1.1 General analysis

In this section, we obtain the $O\left(h^2\right)$ action for generic BI gravity with the assumptions that cosmological Einstein's gravity is the leading order in the small curvature expansion of BI gravity and the BI action does not involve covariant derivatives of the curvature tensors. The action of BI gravity is taken as

$$I = \frac{2}{\kappa\alpha}\int d^D x \left[\sqrt{-\det\left(g_{\mu\nu}+A_{\mu\nu}\right)} - \left(\alpha\Lambda_0+1\right)\sqrt{-g}\right]. \tag{2.4}$$

Here, to reproduce the Einstein-Hilbert action as the first order of the small curvature expansion, $A_{\mu\nu}$ should have the form $A_{\mu\nu} = \alpha\left(R_{\mu\nu}+\beta S_{\mu\nu}\right) + O\left(R^2\right)$, where $S_{\mu\nu}$ is the traceless-Ricci tensor, $S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{D}g_{\mu\nu}R$, and $O\left(R^2\right)$ represents any quadratic curvature rank $(0,2)$ tensor that can be constructed from the contractions of the Riemann tensor. In addition, the zeroth order of the small curvature expansion yields a fixed cosmological constant and to make it undetermined, one needs to augment BI-type Lagrangian with the D-dimensional invariant volume term with the factor $(-\alpha\Lambda_0 - 1)$. Furthermore, we introduced the dimensionful parameter α with the $(\text{mass})^{-2}$ dimension in addition to the (bare) cosmological constant Λ_0 and the gravitational coupling κ of Einstein's gravity. One does not need to introduce an additional dimensionful parameter and may prefer to use a combination of the parameters of cosmological Einstein's gravity with the dimension $(\text{mass})^{-2}$. In contrast to the gravitational setting, in the Born-Infeld electrodynamics, one has to introduce a dimensionful parameter as the Maxwell electrodynamics is a scale invariant theory.

Instead of focusing on a specific BI gravity theory for a given $A_{\mu\nu}$, we perform a general analysis and obtain the $O\left(h^2\right)$ action of the generic BI gravity (2.4) in terms of $A^{(1)}_{\mu\nu}$ and $A^{(2)}_{\mu\nu}$ which are the first and the second order terms in the metric perturbation expansion of $A_{\mu\nu}$:

$$A_{\mu\nu} \equiv \bar{A}_{\mu\nu} + \tau A^{(1)}_{\mu\nu} + \tau^2 A^{(2)}_{\mu\nu} + O\left(\tau^3\right). \tag{2.5}$$

Here, $\bar{A}_{\mu\nu}$ represents the evaluation of $A_{\mu\nu}$ for the background spacetime $\bar{g}_{\mu\nu}$ and the dimensionless parameter τ is introduced to keep track of orders in $h_{\mu\nu}$ as $\tau h_{\mu\nu} \equiv g_{\mu\nu} - \bar{g}_{\mu\nu}$. Note that calculating $A_{\mu\nu}$ for an (A)dS background yields a value proportional to $\bar{g}_{\mu\nu}$ as $\bar{A}_{\mu\nu} \equiv a\bar{g}_{\mu\nu}$, where a is a dimensionless parameter that is a function of the effective cosmological constant Λ and the parameters of the theory such as α. To obtain $O(h^2)$ expansion, we used the Maclaurin series expansion of $\sqrt{\det(I+M)}$ which can be obtained from the identity $\det N = \exp(\mathrm{Tr}(\ln N))$ as

$$\sqrt{\det(I+M)} = I + \frac{1}{2}\mathrm{Tr}M + \frac{1}{8}(\mathrm{Tr}M)^2 - \frac{1}{4}\mathrm{Tr}\left(M^2\right) + O\left(M^3\right), \tag{2.6}$$

where I is the identity matrix. To employ (2.6), first one needs to rewrite (2.4) as

$$I = \frac{2}{\kappa\alpha}\int d^Dx\,\sqrt{-g}\left[\sqrt{-\det\left(\delta^\rho_\nu + A^\rho_\nu\right)} - (\alpha\Lambda_0 + 1)\right]. \tag{2.7}$$

At this level, although expanding $\sqrt{-g}$ by using (2.6) yields a perturbative expansion in h as

$$\begin{aligned}\sqrt{-g} &= \sqrt{-\det\left(\bar{g}_{\mu\nu} + \tau h_{\mu\nu}\right)} \\ &= \sqrt{-\bar{g}}\left[1 + \frac{\tau}{2}h + \frac{1}{8}\tau^2\left(h^2 - 2h^2_{\mu\nu}\right) + O\left(\tau^3\right)\right],\end{aligned} \tag{2.8}$$

for the term $\sqrt{-\det\left(\delta^\rho_\nu + A^\rho_\nu\right)}$, to get an expansion in h, zeroth order of A^ρ_ν, that is $a\delta^\rho_\nu$, should be separated from the first and second orders in the h expansion of A^ρ_ν which are the relevant orders in obtaining the $O(h^2)$ action. Up to second order, A^ρ_ν can be expanded in h as

$$\begin{aligned}A^\rho_\nu &\equiv a\delta^\rho_\nu + \tau B^\rho_\nu \\ &= a\delta^\rho_\nu + \tau\left(\bar{g}^{\rho\mu}A^{(1)}_{\mu\nu} - ah^\rho_\nu\right) + \tau^2\left(\bar{g}^{\rho\mu}A^{(2)}_{\mu\nu} - h^{\rho\mu}A^{(1)}_{\mu\nu} + ah^{\rho\sigma}h_{\sigma\nu}\right),\end{aligned} \tag{2.9}$$

where B^ρ_ν is defined as a bookkeeping device and the second line follows from (2.5) and the $O(h^2)$ expansion of the inverse metric

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \tau h^{\mu\nu} + \tau^2 h^{\mu\rho}h^\nu_\rho + O\left(\tau^3\right). \tag{2.10}$$

Use of (2.9) in (2.7) yields

$$\begin{aligned}I &= \frac{2}{\kappa\alpha}\int d^Dx\,\sqrt{-g}\left\{\sqrt{-\det\left[(1+a)\,\delta^\rho_\nu + \tau B^\rho_\nu\right]} - (\alpha\Lambda_0 + 1)\right\} \\ &= \frac{2}{\kappa\alpha}(1+a)^{\frac{D-4}{2}}\int d^Dx\,\sqrt{-g}\Bigg\{(1+a)^2\sqrt{-\det\left[\delta^\rho_\nu + \frac{\tau}{(1+a)}B^\rho_\nu\right]} \\ &\qquad - (1+a)^{\frac{4-D}{2}}(\alpha\Lambda_0 + 1)\Bigg\}.\end{aligned} \tag{2.11}$$

With this result, we have achieved to put the generic BI action (2.4) in a form which is convenient to obtain a perturbative expansion in h. As apparent in the first line of

(2.11), $a=-1$ value eliminates the leading order and one cannot have a well-defined expansion in h. However, we assume $a\neq -1$ because for a BI gravity theory for which $a=-1$, the couplings in $A_{\mu\nu}$ are proportional to inverse powers of Λ, for example $A_{\mu\nu}=-\frac{(D-2)}{2\Lambda}R_{\mu\nu}$, so they diverge in the flat spacetime limit and higher curvature terms dominate over the Einstein-Hilbert term for small curvature backgrounds. In addition, we keep the factor $(1+a)^2$ in front of the determinantal form on purpose. In this way, inverse powers of $(1+a)$ resulting from the expansion of the determinant are canceled and the $O\left(h^2\right)$ action takes a form where one can trace the origin of the contributions.

Now let us expand (2.11) by using (2.6). One of the determinantal forms appearing in (2.11) is $\sqrt{-g}$ and its $O\left(h^2\right)$ expansion is already given in (2.8). Using this result and expanding the other determinantal form via (2.6) yields

$$
\begin{aligned}
I=&\frac{2}{\kappa\alpha}(1+a)^{\frac{D-4}{2}}\int d^Dx\,\sqrt{-\bar{g}}\Bigg\{\left[(1+a)^2-(1+a)^{\frac{4-D}{2}}(\alpha\Lambda_0+1)\right]\\
&+\frac{\tau}{2}\left[(1+a)\,B^{\rho}_{\rho}+\left[(1+a)^2-(1+a)^{\frac{4-D}{2}}(\alpha\Lambda_0+1)\right]h\right]\\
&+\frac{\tau^2}{8}\Bigg[\left(B^{\rho}_{\rho}\right)^2-2B^{\rho}_{\nu}B^{\nu}_{\rho}+2\,(1+a)\,hB^{\rho}_{\rho}\\
&\qquad+\left[(1+a)^2-(1+a)^{\frac{4-D}{2}}(\alpha\Lambda_0+1)\right]\left(h^2-2h^2_{\mu\nu}\right)\Bigg]\Bigg\}.
\end{aligned}
\tag{2.12}
$$

This action incorporates all the terms up to $O\left(h^2\right)$ in the metric perturbation expansion of the generic BI gravity (2.4). However, since B^{ρ}_{ν} involves an $O\left(\tau\right)$ term [see (2.9)], the $O\left(h\right)$ and $O\left(h^2\right)$ terms are not explicit. In addition, there are some $O\left(h^3\right)$ terms in (2.12), but they do not represent all the terms appearing at the $O\left(h^3\right)$ of (2.4). The $O\left(h^0\right)$ terms in (2.12) give the value of (2.4) for (A)dS backgrounds and this value is irrelevant for our purposes. On the other hand, the $O\left(h\right)$ terms determine the vacuum of the generic BI theory around which we analyze the spectrum for and check the consistency of the theory through the use of $O\left(h^2\right)$ action. Furthermore, note that for odd dimensions to have a real-valued action, the parameter a should satisfy $a>-1$ which puts a constraint on the effective cosmological constant Λ.

The $O\left(h\right)$ action of the generic BI gravity (2.4) provides an easy way to find the (A)dS vacua of a specific BI gravity once $A^{(1)}_{\mu\nu}$ is calculated. The canonical way to find the vacua of a gravity theory is to first derive the field equations by taking the variation of the action

$$\delta I=\int d^Dx\,\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}}\delta g_{\mu\nu}, \tag{2.13}$$

which often becomes cumbersome for higher curvature theories, then put the maximally symmetric Riemann tensor

$$\bar{R}_{\mu\alpha\nu\beta}=\frac{2\Lambda}{(D-1)(D-2)}\left(\bar{g}_{\mu\nu}\bar{g}_{\alpha\beta}-\bar{g}_{\mu\beta}\bar{g}_{\alpha\nu}\right), \tag{2.14}$$

in the field equations. This yields the field equation for the effective cosmological constant Λ as

$$\left.\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}}\right|_{\bar{g}_{\mu\nu}}=0. \tag{2.15}$$

On the other hand, by finding the $O(h)$ action for (2.4) around the (A)dS background, symbolically we have found

$$I_{O(h)}=\int d^Dx\left[\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}}\right]_{\bar{g}_{\mu\nu}}\delta g_{\mu\nu}, \tag{2.16}$$

where $\delta g_{\mu\nu}=h_{\mu\nu}$ and the explicit form of $\delta\mathcal{L}/\delta g_{\mu\nu}$ depends on $A_{\mu\nu}$. Therefore, being in line with the spirit of variational principle, if one requires (2.16) to be zero for an arbitrary $h_{\mu\nu}$, then one gets the field equation for Λ which is (2.15).[1]

Now, to obtain the $O(h)$ action for the generic BI gravity (2.4) from (2.12), one just needs the leading order of B_ρ^ρ which is simply

$$B_\rho^\rho=\bar{g}^{\rho\mu}A_{\mu\rho}^{(1)}-ah+O(\tau). \tag{2.17}$$

Then, the $O(h)$ action for (2.4) becomes

$$I_{O(h)}=\frac{1}{\kappa\alpha}\int d^Dx\sqrt{-\bar{g}} \\ \left[(1+a)^{\frac{D-2}{2}}\left(\bar{g}^{\rho\mu}A_{\mu\rho}^{(1)}\right)+\left((1+a)^{\frac{D-2}{2}}-1-\alpha\Lambda_0\right)h\right]. \tag{2.18}$$

Therefore, for a specific BI gravity defined with $A_{\mu\nu}$, one needs to calculate the value of $A_{\mu\nu}$ for the (A)dS background, $\bar{A}_{\mu\nu}=a\bar{g}_{\mu\nu}$, and linearize $A_{\mu\nu}$ in h, $A_{\mu\nu}^{(1)}$. After finding a and $A_{\mu\nu}^{(1)}$, to find the vacuum of a specific BI gravity in a rather economical way, one just needs to remove the (possible) boundary terms and solve $I_{O(h)}=0$ for arbitrary $h_{\mu\nu}$.

The $O(h^2)$ action of the generic BI gravity (2.4) can be extracted from (2.12) by calculating the leading order contributions coming from the terms $\left(B_\rho^\rho\right)^2-2B_\nu^\rho B_\rho^\nu+$

[1] Note that $\sqrt{-g}$ factors are treated as usual.

$2\left(1+a\right)hB^{\rho}_{\rho}$ and the next to leading order contribution coming from the term $\tau\left(1+a\right)B^{\rho}_{\rho}$. Using the definition of B^{ρ}_{ν} given in (2.9), these contributions can be found as

$$\begin{aligned}\left(B^{\rho}_{\rho}\right)^2-2B^{\rho}_{\nu}B^{\nu}_{\rho}+2\left(1+a\right)hB^{\rho}_{\rho}=&\left(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu}\right)^2-2A^{(1)}_{\mu\nu}A^{\mu\nu}_{(1)}\\&+2h^{\mu\nu}\left(2aA^{(1)}_{\mu\nu}+\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}A^{(1)}_{\rho\sigma}\right)\\&-ah^{\mu\nu}\left(2ah_{\mu\nu}+\left(2+a\right)\bar{g}_{\mu\nu}h\right),\end{aligned}\tag{2.19}$$

and

$$B^{\rho}_{\rho}=O\left(\tau^0\right)+\tau\left[\bar{g}^{\mu\nu}A^{(2)}_{\mu\nu}-h^{\mu\nu}\left(A^{(1)}_{\mu\nu}-ah_{\mu\nu}\right)\right].\tag{2.20}$$

Once again we did not explicitly put the $O\left(\tau^0\right)$ term which is not relevant for our purposes. Employing these results in (2.12) yields the $O\left(h^2\right)$ action of (2.4) as

$$\begin{aligned}I_{O(h^2)}=-\frac{\left(1+a\right)^{\frac{D-4}{2}}}{\kappa\alpha}\int d^Dx\,\sqrt{-\bar{g}}&\\\Bigg\{&\frac{1}{2}\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}A^{(1)}_{\mu\nu}A^{(1)}_{\alpha\beta}-\frac{1}{4}\left(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu}\right)^2-\left(1+a\right)\bar{g}^{\mu\nu}A^{(2)}_{\mu\nu}\\&+h^{\mu\nu}\left(A^{(1)}_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}A^{(1)}_{\rho\sigma}\right)\\&-\frac{1}{4}\left[1-\left(1+a\right)^{\frac{4-D}{2}}\left(\alpha\Lambda_0+1\right)\right]\left(h^2-2h^2_{\mu\nu}\right)\Bigg\}.\end{aligned}\tag{2.21}$$

Note that (2.21) is valid for any value of the background curvature. In the course of obtaining (2.21), we have not done a small curvature expansion. In fact, all the perturbative expansions are in terms of the metric perturbation $h_{\mu\nu}$. Since no assumption on the magnitude of the background curvature is made, an infinite amount of terms in the curvature expansion of the generic BI gravity (2.4) contribute to the $O\left(h^2\right)$ action and all of these contributions are incorporated in (2.21).

The $O\left(h^2\right)$ action (2.21) is one of the most important results in this work. It provides a compact formulation applicable to *any* BI gravity theory. To obtain the $O\left(h^2\right)$ action of a specific BI gravity, one needs to expand the given $A_{\mu\nu}$ tensor as in the symbolic expansion (2.5) by using the metric perturbation expansion of the curvature tensors given in Appendix A. The resulting action is going to be in a complicated form which should be rearranged by following the examples in Appendix B which are about calculating the $O\left(h^2\right)$ action for Einstein's gravity and quadratic curvature gravity.

Upon contemplating at the $O\left(h\right)$ and the $O\left(h^2\right)$ actions of the generic BI gravity given in (2.18) and (2.21), respectively, one can make an intriguing observation: for *even* dimensions, only finite number of higher curvature terms contribute to these actions. As we discuss below, this observation relies on the fact that for even dimensions the parameter a, which implies the prior existence of an $A_{\mu\nu}$ tensor that subsequently takes a nonvanishing background value, has finite powers in the $O\left(h\right)$ and the $O\left(h^2\right)$ actions given in (2.18) and (2.21), respectively. The most striking example of this observation is in $D=4$, which we discuss now. The $O\left(h\right)$ and the $O\left(h^2\right)$ actions of the generic four-dimensional BI gravity

$$I=\frac{2}{\kappa\alpha}\int d^4x\left[\sqrt{-\det\left(g_{\mu\nu}+A_{\mu\nu}\right)}-\left(\alpha\Lambda_0+1\right)\sqrt{-g}\right], \tag{2.22}$$

are exactly the same as the $O\left(h\right)$ and the $O\left(h^2\right)$ actions, respectively, of the higher curvature theory

$$I_{O(A^2)}=\frac{1}{\kappa\alpha}\int d^4x\sqrt{-g}\left(A^\mu_\mu-2\alpha\Lambda_0+\frac{1}{4}A^\mu_\mu A^\nu_\nu-\frac{1}{2}A^\nu_\mu A^\mu_\nu\right). \tag{2.23}$$

So in other words, these actions represent gravity theories which remarkably have the same spectrum and the same vacua. The action (2.23) is nothing but the up to $O\left(A^2\right)$ expansion of (2.22) via (2.6). Let us elaborate on this point. The generic BI gravity action (2.4) has a Maclaurin series expansion in A which symbolically has the form

$$\begin{aligned}I&=\frac{2}{\kappa\alpha}\int d^Dx\left[\sqrt{-\det\left(g_{\mu\nu}+A_{\mu\nu}\right)}-\left(\alpha\Lambda_0+1\right)\sqrt{-g}\right]\\&\sim\frac{2}{\kappa\alpha}\int d^Dx\sqrt{-g}\left[\sum_{n=0}^{\infty}c_nA^n-\left(\alpha\Lambda_0+1\right)\right]\\&\sim\int d^Dx\sqrt{-g}\left[\frac{1}{\kappa}\left(R-2\Lambda_0\right)+\frac{2}{\kappa\alpha}\sum_{n=2}^{\infty}c_nA^n\right],\end{aligned} \tag{2.24}$$

where the last line follows from the assumption that $A_{\mu\nu}$ has the form $A_{\mu\nu}=\alpha\left(R_{\mu\nu}+\beta S_{\mu\nu}\right)+O\left(R^2\right)$ such that the leading order in a curvature expansion of the generic BI gravity should produce the cosmological Einstein's gravity theory. Here, the term A^n represents all possible contractions that can be obtained with n number of $A_{\mu\nu}$ tensors such as $\left(A^\mu_\mu\right)^n$, $\left(A^\mu_\mu\right)^{n-2}A^\mu_\nu A^\nu_\mu$, $\left(A^\mu_\mu\right)^{n-3}A^\mu_\rho A^\rho_\nu A^\nu_\mu$, *etc.* Note that if there are $O\left(R^2\right)$ terms in $A_{\mu\nu}$, the expansion of the generic BI gravity (2.4) at a given order in A does not match the expansion in curvature, that is in αR, at the corresponding order. If one wants to compute the $O\left(h\right)$ and $O\left(h^2\right)$ of the generic BI gravity (2.4), all orders in A will contribute in principle. However, in $D=4$ curiously all the contributions to the $O\left(h\right)$ and $O\left(h^2\right)$ of (2.22) coming from the terms beyond $O\left(A^2\right)$ vanish identically. These remarkable cancellations are due to the form of the BI action as a square root of a determinant and due to the maximally symmetric nature of the background. For generic higher curvature theories such a cancellation does not work.

Now let us discuss how to figure out which orders in the A expansion of the generic BI gravity (2.4) contribute to $O(h)$ and $O(h^2)$ of (2.4) by just counting the powers of the parameter a. To obtain the $O(h)$ and $O(h^2)$ contributions coming from the terms in the A expansion, we need the expansion of A^μ_μ up to $O(h^2)$ in addition to up-to-$O(h^2)$ expansion of A^ρ_ν given in (2.9). By using (2.9), this expansion simply becomes

$$A^\mu_\mu = aD + \tau\left(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu} - ah\right) + \tau^2\left(\bar{g}^{\mu\nu}A^{(2)}_{\mu\nu} - h^{\mu\nu}A^{(1)}_{\mu\nu} + ah^\mu_\nu h^\nu_\mu\right). \tag{2.25}$$

First, we analyze the form of the $O(h)$ contribution coming from each order in the A expansion (2.24). The zeroth order action in (2.24) is proportional to the invariant spacetime volume as $I^{(0)} = -\frac{2\Lambda_0}{\kappa}\int d^Dx\sqrt{-g}$ whose $O(h)$ expansion is

$$I^{(0)}_{O(h)} = -\frac{1}{\kappa}\int d^Dx\sqrt{-\bar{g}}\Lambda_0 h, \tag{2.26}$$

which follows from the h expansion of $\sqrt{-g}$ in (2.8). Then, the first order in the A expansion of (2.4) is $\frac{1}{2}A^\mu_\mu$ via (2.5). Using up to $O(h)$ expansions of A^μ_μ and $\sqrt{-g}$ from (2.25) and (2.8), respectively, the $O(h)$ of the first order action in the A expansion of (2.4), $I^{(1)} = \frac{1}{\kappa\alpha}\int d^Dx\sqrt{-g}A^\mu_\mu$, takes the form

$$I^{(1)}_{O(h)} = \frac{1}{\kappa\alpha}\int d^Dx\sqrt{-\bar{g}}\left[\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu} + \frac{(D-2)}{2}ah\right]. \tag{2.27}$$

Then, let us consider the $O(h)$ contribution coming from the second order in A expansion of (2.4) which is $\left(\frac{1}{8}A^\mu_\mu A^\nu_\nu - \frac{1}{4}A^\nu_\mu A^\mu_\nu\right)$ via (2.5). The background values of the terms $A^\mu_\mu A^\nu_\nu$ and $A^\nu_\mu A^\mu_\nu$ are proportional to a^2. On the other hand, in calculating the $O(h)$ part of $A^\mu_\mu A^\nu_\nu$ and $A^\nu_\mu A^\mu_\nu$, one of the A tensors takes the background value while the other yields the $O(h)$ contribution in the form $\left(A^\mu_\mu\right)_{(1)}$. Up to $O(h)$, the second order terms in the A expansion of (2.4) have the h expansion

$$\frac{1}{8}A^\mu_\mu A^\nu_\nu - \frac{1}{4}A^\nu_\mu A^\mu_\nu = a\left[\frac{D(D-2)}{8}a + \tau\frac{(D-2)}{4}\left(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu} - ah\right)\right] + O\left(\tau^2\right). \tag{2.28}$$

This result has the same structure as the up to $O(h)$ expansion of A^μ_μ, so the $O(h)$ of the second order action in the A expansion of (2.4) that is

$$I^{(2)} = \frac{1}{\kappa\alpha}\int d^Dx\sqrt{-g}\left(\frac{1}{4}A^\mu_\mu A^\nu_\nu - \frac{1}{2}A^\nu_\mu A^\mu_\nu\right), \tag{2.29}$$

has the same structure as $I^{(1)}_{O(h)}$ given in (2.27). The explicit calculation of the $O(h)$ of (2.29) yields

$$I^{(2)}_{O(h)} = \frac{1}{\kappa\alpha}\int d^D x\,\sqrt{-\bar{g}}\,a\left[\frac{(D-2)}{2}\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu} + \frac{(D-2)(D-4)}{8}ah\right], \tag{2.30}$$

where, as it is clear by the overall factor of a, the powers of a in the coefficients of $\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu}$ and h increase by one when compared to (2.27). Here, note that choice of dimension can cancel some of the terms in (2.27) and (2.30). To consider the $O(h)$ contribution coming from the terms at $O(A^n)$, one can follow a similar logic. First, one needs the h expansion of the $O(A^n)$ terms up to $O(h)$. The background value of the $O(A^n)$ terms is proportional to a^n and in calculating the $O(h)$ part of the $O(A^n)$ terms, $(n-1)$ number of A tensors take the background value while the remaining one yields the $O(h)$ contribution in the form $\left(A^\mu_\mu\right)_{(1)}$. Therefore, up to $O(h)$, the $O(A^n)$ terms have the h expansion in the form

$$c_n A^n = a^{n-1}\left[b_{n1}a + \tau b_{n2}\left(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu} - ah\right)\right] + O\left(\tau^2\right), \tag{2.31}$$

where b_{n1} and b_{n2} are just numbers whose values depend on c_n; however, their specific values are not important for our discussion unless they happen to be zero. With (2.31), the $O(h)$ of the n^{th} order action in the A expansion of (2.4), $I^{(n)} = \frac{2}{\kappa\alpha}\int d^D x\,\sqrt{-g}\,c_n A^n$, has the form

$$I^{(n)}_{O(h)} = \frac{2}{\kappa\alpha}\int d^4 x\,\sqrt{-\bar{g}}\,a^{n-1}\left[d_{n1}\left(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu}\right) + d_{n2}ah\right], \tag{2.32}$$

where d_{n1} and d_{n2} are again just numbers depending on c_n. The important point to notice in (2.32) is that the structure of the action is the same as the $O(A)$ and $O(A^2)$ cases given in (2.27) and (2.30), respectively, and there is the overall factor of a^{n-1} showing from which order in the A expansion the contribution is coming. Thus, for $n \geq 1$, unless d_{n1} and d_{n2} are zero, each order n in the A expansion (2.24) has a similar contribution to the $O(h)$ action of the generic BI gravity (2.4) differing by just an overall factor of a^{n-1}.

Now, we can determine which order in the A expansion (2.24) of the generic BI gravity (2.4) contributes to the $O(h)$ action of (2.4) given in (2.18). We just need to consider the coefficients of the terms $\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu}$ and h, and from (2.18) they are $(1+a)^{\frac{D-2}{2}}$ and $\left((1+a)^{\frac{D-2}{2}} - 1 - \alpha\Lambda_0\right)$, respectively. For odd dimensions, these coefficients are infinite series in a, so all the order in the A expansion give a contribution to the $O(h)$

action. On the other hand, for even dimensions, from $O(A)$ to $O\left(A^{\frac{D}{2}}\right)$ in the A expansion of (2.4) give an $O(h)$ contribution in the form $\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu}$, while from $O(A^0)$ to $O\left(A^{\frac{D}{2}-1}\right)$ give an $O(h)$ contribution in the form h. Beyond $O\left(A^{\frac{D}{2}}\right)$, the d_{n1} and d_{n2} coefficients are zero due to the specific values of c_n coefficients in the A expansion of (2.4).

For even D dimensions, remarkably the generic BI gravity (2.4) and its up to $O\left(A^{\frac{D}{2}}\right)$ expansion have the same vacua. For example, in $D=4$, the four-dimensional BI gravity (2.22) and its up to $O(A^2)$ expansion (2.23) have the same vacua as mentioned above. To provide concrete verification of this result, let us find the $O(h)$ of (2.23). Collecting the $O(h)$ of the zeroth, first, and second orders in the A expansion of the generic BI gravity (2.4) given in (2.26), (2.27), and (2.30), respectively, yields the $O(h)$ of up to $O(A^2)$ expansion of (2.4) as

$$I^{O(A^2)}_{O(h)} = \frac{1}{\kappa\alpha}\int d^Dx\,\sqrt{-\bar{g}}\Bigg[\left(1+\frac{a(D-2)}{2}\right)\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu} + \frac{a(D-2)}{2}\left(1+\frac{a(D-4)}{4}\right)h - \alpha\Lambda_0 h\Bigg]. \tag{2.33}$$

For $D=4$, both (2.18) and (2.33) reduce to the same action as

$$I_{O(h)} = \frac{1}{\kappa\alpha}\int d^4x\,\sqrt{-\bar{g}}\left[(1+a)\,\bar{g}^{\rho\mu}A^{(1)}_{\mu\rho} + (a-\alpha\Lambda_0)\,h\right]. \tag{2.34}$$

This equivalence in $D=4$ occurs in a nontrivial way as the coefficients of the corresponding terms in (2.18) and (2.33) have totally different structures.

Now let us analyze the form of the $O(h^2)$ contribution coming from each order in the A expansion (2.24). For the $O(h^2)$ case, the contribution coming from each order in the A expansion has the same form as the $O(h^2)$ contribution coming from the $O(A^2)$ terms. As in the $O(h)$ case, the only difference is the introduction of an overall a factor for each order beyond $O(A^2)$. Before moving to the $O(h^2)$ of the $O(A^2)$ terms, let us obtain the $O(h^2)$ contributions of the zeroth order and the first order actions in the A expansion of the generic BI gravity (2.4). By using the $O(h^2)$ expansion of $\sqrt{-g}$ in (2.8), the $O(h^2)$ of the zeroth order action, $I^{(0)} = -\frac{2\Lambda_0}{\kappa}\int d^Dx\,\sqrt{-g}$, becomes

$$I^{(0)}_{O(h)} = -\frac{1}{\kappa}\int d^Dx\,\sqrt{-\bar{g}}\frac{\Lambda_0}{4}\left(h^2 - 2h^\mu_\nu h^\nu_\mu\right). \tag{2.35}$$

On the other hand, to obtain $O(h^2)$ of the first order action, $I^{(1)} = \frac{1}{\kappa\alpha}\int d^Dx\,\sqrt{-g}A^\mu_\mu$, one needs again (2.8) and the $O(h^2)$ expansion of A^μ_μ given in (2.25) and the resulting action is

$$I^{(1)}_{O(h^2)} = \frac{1}{\kappa\alpha}\int d^Dx\,\sqrt{-\bar{g}}\left\{\bar{g}^{\mu\nu}A^{(2)}_{\mu\nu} - h^{\mu\nu}\left(A^{(1)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}A^{(1)}_{\rho\sigma}\right)\right.$$
$$\left.+\frac{a\,(D-4)}{8}\left(h^2 - 2h^\mu_\nu h^\nu_\mu\right)\right\}. \qquad (2.36)$$

Now let us move on to the $O\left(h^2\right)$ contribution coming from the second order in the A expansion of (2.22). To calculate this contribution, one needs the $O\left(h^2\right)$ expansion of the $O\left(A^2\right)$ terms, and up to $O\left(h\right)$ part of this expansion is given in (2.28). To yield an $O\left(h^2\right)$ part, either both of the A tensors of the $O\left(A^2\right)$ terms should be first order in h as $A_{(1)}A_{(1)}$ or one of them should be second order in h while the other takes background value as $aA_{(2)}$. Therefore, in terms $(A^\mu_\nu)_{(1)}$ and $(A^\mu_\nu)_{(2)}$, which are the first and second orders of A^μ_ν in h , the $O\left(h^2\right)$ expansion of the $O\left(A^2\right)$ terms has the form

$$\begin{aligned}\left(\frac{1}{8}A^\mu_\mu A^\nu_\nu - \frac{1}{4}A^\nu_\mu A^\mu_\nu\right) =& \frac{a^2D\,(D-2)}{8} + \tau\frac{a\,(D-2)}{4}\left(A^\mu_\mu\right)_{(1)}\\ &+\tau^2\left[\frac{1}{8}\left(A^\mu_\mu\right)_{(1)}(A^\nu_\nu)_{(1)} - \frac{1}{4}\left(A^\nu_\mu\right)_{(1)}(A^\mu_\nu)_{(1)}\right.\\ &\left.+\frac{a\,(D-2)}{4}\left(A^\mu_\mu\right)_{(2)}\right] + O\left(\tau^3\right).\end{aligned} \qquad (2.37)$$

We want to express this expansion in terms of $A^{(1)}_{\mu\nu}$ and $A^{(2)}_{\mu\nu}$ and up to $O\left(h\right)$ part in the first line is already expressed in this way in (2.28), and the remaining $O\left(h^2\right)$ part can be written in this way by using (2.9) and (2.25) as

$$\begin{aligned}\left(\frac{1}{8}A^\mu_\mu A^\nu_\nu - \frac{1}{4}A^\nu_\mu A^\mu_\nu\right)_{(2)} =& \frac{1}{8}\left(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu}\right)^2 - \frac{1}{4}\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}A^{(1)}_{\mu\nu}A^{(1)}_{\alpha\beta}\\ &-2ah^{\mu\nu}\left(\frac{1}{8}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}A^{(1)}_{\rho\sigma} - \frac{1}{4}A^{(1)}_{\mu\nu}\right) + a^2\left(\frac{1}{8}h^2 - \frac{1}{4}h^\mu_\nu h^\nu_\mu\right)\\ &+\frac{a\,(D-2)}{4}\left(\bar{g}^{\mu\nu}A^{(2)}_{\mu\nu} - h^{\mu\nu}A^{(1)}_{\mu\nu} + ah^\mu_\nu h^\nu_\mu\right).\end{aligned} \qquad (2.38)$$

This result involves all the possible seven forms that can appear in the $O\left(h^2\right)$ of any $O\left(A^n\right)$ term which are $\left(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu}\right)^2$, $\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}A^{(1)}_{\mu\nu}A^{(1)}_{\alpha\beta}$, $\bar{g}^{\mu\nu}A^{(2)}_{\mu\nu}$, $h\bar{g}^{\rho\sigma}A^{(1)}_{\rho\sigma}$, $h^{\mu\nu}A^{(1)}_{\mu\nu}$, $h^\mu_\nu h^\nu_\mu$, and h^2. Using (2.28), (2.38), and (2.8), one can calculate the $O\left(h^2\right)$ contribution coming from the second order action in the A expansion of (2.4) given in (2.29) as

$$I^{(2)}_{O(h^2)} = -\frac{1}{\kappa\alpha}\int d^Dx\,\sqrt{-\bar{g}}$$
$$\times\left\{\frac{1}{2}\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}A^{(1)}_{\mu\nu}A^{(1)}_{\alpha\beta}-\frac{1}{4}\left(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu}\right)^2-\frac{a\,(D-2)}{2}\bar{g}^{\mu\nu}A^{(2)}_{\mu\nu}\right. \tag{2.39}$$
$$+\frac{a\,(D-4)}{2}h^{\mu\nu}\left(A^{(1)}_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}A^{(1)}_{\rho\sigma}\right)$$
$$\left.-\frac{a^2\,(D-4)\,(D-6)}{32}\left(h^2-2h^\mu_\nu h^\nu_\mu\right)\right\}, \tag{2.40}$$

where all the possible seven forms that can appear in $O\left(h^2\right)$ are present. The structure of (2.40) represents the generic structure of the $O\left(h^2\right)$ contributions coming from any order $O\left(A^n\right)$. To understand this, first one needs to consider the $O\left(h^2\right)$ expansion of the $O\left(A^n\right)$ terms and up to $O\left(h\right)$ this expansion is given in (2.31). Obtaining the $O\left(h^2\right)$ part of the $O\left(A^n\right)$ terms is similar to the $O\left(A^2\right)$ case: either two of the A tensors are first order in h and the others take the background value as $a^{n-2}A_{(1)}A_{(1)}$, or one of them is second order in h while the others take background value as $a^{n-1}A_{(2)}$. Therefore, the $O\left(h^2\right)$ expansion of the $O\left(A^n\right)$ terms has the form

$$c_nA^n = a^{n-2}\Big\{b_{n1}\,a^2+\tau\,b_{n2}\,a\left(A^\mu_\mu\right)_{(1)}$$
$$+\tau^2\Big[b_{n3}\left(A^\mu_\mu\right)_{(1)}\left(A^\nu_\nu\right)_{(1)}+b_{n4}\left(A^\nu_\mu\right)_{(1)}\left(A^\mu_\nu\right)_{(1)}$$
$$+b_{n2}\,a\left(A^\mu_\mu\right)_{(2)}\Big]\Big\}+O\left(\tau^3\right), \tag{2.41}$$

where b_{ni} coefficients are just numbers whose specific values depend on the c_n coefficients; however, again their specific values are not important unless they are zero. Apart from the b_{ni} coefficients and the important overall a^{n-2} factor, (2.41) has the same structure as the $O\left(h^2\right)$ expansion of $O\left(A^2\right)$ given in (2.37). Therefore, the $O\left(h^2\right)$ contribution coming from the n^{th} order action in the A expansion of (2.4), $I^{(n)}=\frac{2}{\kappa\alpha}\int d^Dx\,\sqrt{-g}c_nA^n$, should have the same structure as the $O\left(h^2\right)$ contribution of the $O\left(A^2\right)$ action given (2.40) and the $O\left(h^2\right)$ contribution of the $O\left(A^n\right)$ action has the form

$$I^{(n)}_{O(h^2)} = \frac{2}{\kappa\alpha}\int d^Dx\,\sqrt{-\bar{g}}$$
$$\times\,a^{n-2}\left\{d_{n1}\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}A^{(1)}_{\mu\nu}A^{(1)}_{\alpha\beta}+d_{n2}\left(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu}\right)^2+d_{n3}a\bar{g}^{\mu\nu}A^{(2)}_{\mu\nu}\right.$$
$$+ah^{\mu\nu}\left(d_{n4}A^{(1)}_{\mu\nu}+d_{n5}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}A^{(1)}_{\rho\sigma}\right)$$
$$\left.+a^2\left(d_{n6}h^2+d_{n7}h^\mu_\nu h^\nu_\mu\right)\right\}, \tag{2.42}$$

where the d_{ni} coefficients are again just numbers depending on c_n. For $n \geq 1$, unless the d_{ni} coefficients are zero, each order n in the A expansion (2.24) has a similar contribution to the $O\left(h^2\right)$ action of the generic BI gravity (2.4) differing by just an overall factor of a^{n-2}.

Now, let us determine which order in the A expansion (2.24) of the generic BI gravity (2.4) contributes to the $O\left(h^2\right)$ action of (2.4) given in (2.21). For odd dimensions and for $D = 2$, the overall factor $(1+a)^{\frac{D}{2}-2}$ in (2.21) is an infinite series in a, while it is a polynomial of degree $D/2-2$ for even dimensions (higher than two). Therefore, for odd dimensions and for $D = 2$, all the terms in the A expansion of (2.4) contributes to the $O\left(h^2\right)$ action, while for even dimensions only finite number of terms contributes to the $O\left(h^2\right)$ action and the number of contributing terms depends on the dimension of the spacetime. Let us analyze which orders contribute to each of the seven terms in the $O\left(h^2\right)$ action for even dimensions. For the first two terms $\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}A^{(1)}_{\mu\nu}A^{(1)}_{\alpha\beta}$ and $\left(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu}\right)^2$, the overall factor $(1+a)^{\frac{D}{2}-2}$ shows that from $O\left(A^2\right)$ to $O\left(A^{\frac{D}{2}}\right)$ in the A expansion of (2.4) yield these terms. The coefficient of the term $\bar{g}^{\mu\nu}A^{(2)}_{\mu\nu}$ is $(1+a)^{\frac{D}{2}-1}$, so from $O\left(A\right)$ to $O\left(A^{\frac{D}{2}}\right)$ give an $O\left(h^2\right)$ contribution in this form. For the terms $h^{\mu\nu}A^{(1)}_{\mu\nu}$ and $h\bar{g}^{\rho\sigma}A^{(1)}_{\rho\sigma}$, again the overall factor $(1+a)^{\frac{D}{2}-2}$ is the coefficient, so from $O\left(A\right)$ to $O\left(A^{\frac{D}{2}-1}\right)$ give an $O\left(h^2\right)$ contribution in these forms. The terms h^2 and $h^2_{\mu\nu}$ share the same factor of $\left[(1+a)^{\frac{D}{2}-2} - (\alpha\Lambda_0 + 1)\right]$ which shows that from $O\left(A^0\right)$ to $O\left(A^{\frac{D}{2}-2}\right)$ yield these terms. Just like the $O\left(h\right)$ case, beyond $O\left(A^{\frac{D}{2}}\right)$ the d_{ni} coefficients are zero due to the specific values of c_n coefficients in the A expansion of (2.4).

For even D dimensions, both $O\left(h\right)$ and $O\left(h^2\right)$ analyses yield the remarkable conclusion that the generic BI gravity (2.4) and its up to $O\left(A^{\frac{D}{2}}\right)$ expansion has the same spectrum around the same vacua. For example, in $D = 4$, the four-dimensional BI gravity (2.22) and its up to $O\left(A^2\right)$ expansion (2.23) are equivalent with respect to their spectra and vacua. As we explicitly verified the equivalence in the $O\left(h\right)$ case for $D = 4$, let us also show the $O\left(h^2\right)$ equivalence explicitly by finding the $O\left(h^2\right)$ of (2.23). Collecting the $O\left(h^2\right)$ contributions coming from of the zeroth, first, and second orders in the A expansion of the generic BI gravity (2.4) given in (2.35), (2.36), and (2.38), respectively, yields the $O\left(h^2\right)$ of up to $O\left(A^2\right)$ expansion of (2.4) as

$$I_{O(h^2)}^{O(A^2)} = -\frac{1}{\kappa\alpha}\int d^D x\,\sqrt{-\bar{g}}$$
$$\times\left\{\frac{1}{2}\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}A^{(1)}_{\mu\nu}A^{(1)}_{\alpha\beta} - \frac{1}{4}\left(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu}\right)^2 - \left(1+\frac{aD}{2}-a\right)\bar{g}^{\mu\nu}A^{(2)}_{\mu\nu}\right.$$
$$+\left(1+\frac{aD}{2}-2a\right)h^{\mu\nu}\left(A^{(1)}_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}A^{(1)}_{\rho\sigma}\right) \tag{2.43}$$
$$\left.-\frac{(D-4)}{8}\left[a+\frac{(D-6)\,a^2}{4}\right]\left(h^2-2h^\mu_\nu h^\nu_\mu\right)+\frac{\alpha\Lambda_0}{4}\left(h^2-2h^\mu_\nu h^\nu_\mu\right)\right\}.$$

Although the dependence of the coefficients on D in the $O\left(h^2\right)$ actions (2.21) and (2.43) are totally different, for $D=4$ both of them reduce to the same action as

$$I_{O(h^2)} = -\frac{1}{\kappa\alpha}\int d^4x\,\sqrt{-\bar{g}}\left\{\frac{1}{2}\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}A^{(1)}_{\mu\nu}A^{(1)}_{\alpha\beta}-\frac{1}{4}\left(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu}\right)^2-(1+a)\,\bar{g}^{\mu\nu}A^{(2)}_{\mu\nu}\right.$$
$$\left.+h^{\mu\nu}\left(A^{(1)}_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}A^{(1)}_{\rho\sigma}\right)+\frac{1}{4}\alpha\Lambda_0\left(h^2-2h^\mu_\nu h^\nu_\mu\right)\right\}. \tag{2.44}$$

With this last verification, we have established the totally nontrivial equivalence with respect to the spectrum and vacua level between the four-dimensional BI gravity (2.22), which is an infinite order in curvature theory, and the higher curvature gravity (2.23) which is finite order in curvature and can be obtained by expanding (2.22) in A up to second order. The $D=4$ case is the most striking case of the equivalence between the even D dimensional generic BI gravity theory and the higher curvature theory which is obtained by the $O\left(A^{D/2}\right)$ expansion of BI gravity. It is worth noting that this equivalence is exact, that is, we have not assumed smallness of the scalar curvature or smallness of the A tensor at any step in obtaining the equivalence.

Let us lay out the procedure for the canonical analysis of the spectrum and the consistency of a given BI gravity. First, one needs to find $\bar{A}_{\mu\nu}$, $A^{(1)}_{\mu\nu}$, and $A^{(2)}_{\mu\nu}$. Then, the vacua of the theory should be found by using (2.18). Finally, the free theory described by the $O\left(h^2\right)$ action can be found by using (2.21). Once the vacua and the free theory of the given BI gravity is determined, then one can use the standard techniques discussed in [15] to find whether or not the theory is free from ghosts and tachyons. Note that if the $A_{\mu\nu}$ tensor defining the BI gravity has a complicated structure, then using (2.21) to find the free theory will become a demanding job.

Another way to analyze the unitarity of the BI gravity is to obtain an equivalent quadratic curvature gravity action. In [35], the procedure to obtain the equivalent quadratic curvature gravity is given for a generic higher curvature gravity which is constructed by the contractions of the Riemann tensor (but not its derivatives). The equivalent quadratic curvature gravity for a generic higher curvature gravity represents the vacua and the free theory of the generic higher curvature gravity theory, but differs at the interaction level. Once the equivalent quadratic curvature gravity is obtained, then the unitarity analysis is relatively straightforward by using the already known results of the quadratic curvature gravity. One can apply the method of finding equivalent quadratic curvature action either directly to the BI gravity action or for even dimensional BI gravity theories, to the equivalent $O\left(A^{D/2}\right)$ actions.

In the following two subsections, we provide two examples. The equivalences we observe rely on two facts: the form of the BI gravity action as the square root of the determinant and the maximal symmetry of the background. To make this point more explicit, in the first example we give the explicit calculations in the simplest setting, that is the linear order equivalence in two dimensions. In the second example, we study the unitarity of the simplest BI gravity defined by $A_{\mu\nu} \equiv \alpha R_{\mu\nu}$ in four dimensions. Although we know that this theory is not unitary even around the flat background, it provides the simplest setting in which we can analyze the unitarity of the theory by both using the $O\left(h\right)$ and $O\left(h^2\right)$ actions, and using the equivalence between the four-dimensional BI gravity and its $O\left(A^2\right)$ expansion.

2.1.2 $O\left(h\right)$ equivalence in two dimensions

We observe that in $D = 2$ the generic BI gravity and its $O\left(A\right)$ expansion are equivalent at the linear level in h. This case is the simplest one of the equivalences and studying this example explicitly with matrix forms clarifies the key roles of the functional form of the BI gravity and the maximal symmetry of the background. The functional form of the two-dimensional "BI gravity" can be represented as

$$f\left(\tau,\gamma\right) = \sqrt{\det\left[\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right) + \gamma\left(\begin{array}{cc} a\left(\tau\right) & b\left(\tau\right) \\ c\left(\tau\right) & d\left(\tau\right) \end{array}\right)\right]}, \tag{2.45}$$

where the parameters τ and γ are introduced to represent the h and A dependence of the BI gravity. Hence, expanding $f\left(\tau,\gamma\right)$ in τ mimics the h expansion of BI gravity, while the τ expansion of $f\left(\tau,\gamma\right)$ corresponds to the h expansion. The background spacetime is maximally symmetric; therefore, any $\left(1,1\right)$ rank tensor that can be constructed with the contractions of the curvature tensors should be proportional to the identity matrix as for the case of $A^{\rho}_{\nu} = a\delta^{\rho}_{\nu}$. In analogy, we assume that

$$\left[\begin{pmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{pmatrix}\right]_{\tau=0} = \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix}. \tag{2.46}$$

The $O(h)$ level equivalence between the BI gravity and its $O(A)$ expansion implies that $f(\tau,\gamma)$ and its $O(\gamma)$ expansion should have the same $O(\tau)$ expansion. We name the $O(\gamma)$ expansion of $f(\tau,\gamma)$ as $g(\tau,\gamma)$ and it can be found by using (2.6) as

$$g(\tau,\gamma) \equiv 1 + \frac{1}{2}\gamma\left[a(\tau) + d(\tau)\right]. \tag{2.47}$$

Before showing that the $O(\tau)$ expansions of $f(\tau,\gamma)$ and $g(\tau,\gamma)$ are indeed the same, let us discuss the case of a generic two parameter function $\phi(\tau,\gamma)$. In the γ expansion of $\phi(\tau,\gamma)$, the coefficient of each order depends on τ and the expansion has the form $\phi(\tau,\gamma) = \sum_{i=0}^{\infty}\psi_i(\tau)\gamma^i$, where

$$\psi_i(\tau) = \frac{1}{i!}\left(\frac{\partial^i\phi}{\partial\gamma^i}\right)_{\gamma=0}. \tag{2.48}$$

If one considers the $O(\tau)$ expansion of $\phi(\tau,\gamma)$ having the form

$$\phi(\tau,\gamma) = \phi(\tau=0,\gamma) + \left(\frac{\partial\phi}{\partial\tau}\right)_{\tau=0}\tau + O\left(\tau^2\right), \tag{2.49}$$

then each order in the γ expansion should yield a contribution to the orders $O(\tau^0)$ and $O(\tau)$. Expanding the ψ_i coefficients up to $O(\tau)$ yields these contributions as

$$\phi(\tau,\gamma) = \sum_{i=0}^{\infty}\psi_i(\tau=0)\gamma^i + \left[\sum_{i=0}^{\infty}\left(\frac{\partial\psi_i}{\partial\tau}\right)_{\tau=0}\gamma^i\right]\tau + O\left(\tau^2\right). \tag{2.50}$$

Thus, unless $\psi_i(\tau=0)$ and $\left(\frac{\partial\psi_i}{\partial\tau}\right)_{\tau=0}$ are zero, each order in the γ expansion yields a contribution to the $O(\tau)$ expansion of $\phi(\tau,\gamma)$. However, for $f(\tau,\gamma)$, the proposal is that $\psi_i(\tau=0)$ and $\left(\frac{\partial\psi_i}{\partial\tau}\right)_{\tau=0}$ are zero for $i \geq 2$. For example, from (2.6), the $O(\gamma^2)$ of $f(\tau,\gamma)$ can be obtained as

$$\begin{aligned}\left(\sqrt{\det\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \gamma\begin{pmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{pmatrix}\right]}\right)_{(2)} &= \frac{1}{8}(\mathrm{Tr}M)^2 - \frac{1}{4}\mathrm{Tr}\left(M^2\right) \\ &= \frac{1}{8}\gamma^2\left[a(\tau) + d(\tau)\right]^2 - \frac{1}{4}\gamma^2\left[a^2(\tau) + 2b(\tau)c(\tau) + d^2(\tau)\right]. \end{aligned} \tag{2.51}$$

To calculate $\psi_2(\tau=0)$ and $\left(\frac{\partial\psi_2}{\partial\tau}\right)_{\tau=0}$, let us write

$$\begin{pmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{pmatrix} = \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix} + \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\tau + O\left(\tau^2\right), \tag{2.52}$$

where the zeroth order has the maximally symmetric structure (2.46). Then, $\psi_2(\tau=0)$ and $\left(\frac{\partial\psi_2}{\partial\tau}\right)_{\tau=0}$ become

$$\psi_2(\tau=0) = \frac{1}{8}\gamma^2(a_0+a_0)^2 - \frac{1}{4}\gamma^2\left(a_0^2+a_0^2\right) = 0, \tag{2.53}$$

and

$$\begin{aligned}\left(\frac{\partial\psi_2}{\partial\tau}\right)_{\tau=0} =& \frac{1}{4}\gamma^2(a_0+a_0)(a_1+d_1) \\ &-\frac{1}{2}\gamma^2(a_0a_1+b_1c_0+b_0c_1+a_0d_1) = 0,\end{aligned} \tag{2.54}$$

where we have given the details of the calculations to show the effect of the maximally symmetric background for which the diagonal terms have the same value and the off-diagonal ones are zero. In addition, the special form of $O(\gamma^2)$ term of $f(\tau,\gamma)$ is also crucial to get these results. Note that for the flat background, these forms are also zero [actually this is true for any functional form $\phi(\tau,\gamma)$], since the zeroth order in (2.52) is a matrix whose entries are just zero. In addition, around the flat background for any $\phi(\tau,\gamma)$, all the $\psi_i(\tau=0)$ and $\left(\frac{\partial\psi_i}{\partial\tau}\right)_{\tau=0}$ are zero for $i\geq 2$. Therefore, as we show below, around the maximally symmetric background, the specific functional form of the square root of the determinant yields the same behavior.

Now to verify the equivalence, let us obtain the $O(\tau)$ expansions of $f(\tau,\gamma)$ and $g(\tau,\gamma)$. By using (2.52), the $O(\tau)$ expansion of $g(\tau,\gamma)$ can be found as

$$g(\tau,\gamma) = (1+\gamma a_0) + \frac{1}{2}\gamma\tau(a_1+d_1). \tag{2.55}$$

Then, employing (2.52) in $f(\tau,\gamma)$ yields

$$\begin{aligned}&\sqrt{\det\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \gamma\begin{pmatrix} a_0+\tau a_1 & \tau b_1 \\ \tau c_1 & a_0+\tau d_1 \end{pmatrix} + O(\tau^2)\right]} \\ &= (1+\gamma a_0)\sqrt{\det\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\gamma\tau}{(1+\gamma a_0)}\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + O(\tau^2)\right]},\end{aligned} \tag{2.56}$$

where we assumed $(1+\gamma a_0) > 0$ and by using (2.6), the $O(\tau)$ expansion of $f(\tau,\gamma)$ becomes

$$f(\tau,\gamma) = (1+\gamma a_0)\left[1+\frac{1}{2}\frac{\gamma\tau}{(1+\gamma a_0)}(a_1+d_1)+O\left(\tau^2\right)\right]$$
$$= (1+\gamma a_0)+\frac{1}{2}\gamma\tau(a_1+d_1)+O\left(\tau^2\right), \tag{2.57}$$

which is the same as the $O(\tau)$ expansion of $g(\tau,\gamma)$ in (2.55). Therefore, in the γ expansion of $f(\tau,\gamma)$, the terms beyond $O(\gamma)$ do not contribute to the terms in the $O(\tau)$ expansion of $f(\tau,\gamma)$.

To understand why the τ expansion of $f(\tau,\gamma)$ is in accord with the γ expansion of $f(\tau,\gamma)$, let us take a closer look at (2.56). For the maximally symmetric zeroth order term in (2.52), due to the determinantal form, one can extract the overall factor $(1+\gamma a_0)$ and, in D dimensions, this factor becomes $(1+\gamma a_0)^{D/2}$. Then, in the τ expansion of the square root of the determinant, the n^{th} order in τ has the factor $\gamma^n(1+\gamma a_0)^{-n}$ and this is the case in any dimensions. Therefore, together with the overall factor, the $O(\tau^n)$ terms have the factor of $\gamma^n(1+\gamma a_0)^{D/2-n}$ and once the power $(D/2-n)$ is zero or a positive integer, only the terms up to $O\left(\gamma^{D/2}\right)$ in the γ expansion yield contributions to the terms in the τ expansion up to $O(\tau^n)$.

This analysis has implications on the interacting level of BI gravities. For $D=4$ although the BI gravity and its $O(A^2)$ expansion have the same spectrum around the same maximally symmetric vacua, they are different at the interacting level [$O(h^3)$ and beyond]. On the other hand, for $D=6$, the BI gravity and its $O(A^3)$ expansion not only have the same spectrum around the same maximally symmetric vacua but also the same cubic interactions, and the higher even dimensional interaction level equivalences follow similarly.

2.1.3 Unitarity analysis of BI gravity $A_{\mu\nu} = \alpha R_{\mu\nu}$ in $D=4$

In this subsection, the techniques we developed for analyzing the unitarity of the BI gravity theories are applied to a specific theory which is the four-dimensional BI gravity with $A_{\mu\nu} = \alpha R_{\mu\nu}$, whose action reads

$$I = \frac{2}{\kappa\alpha}\int d^4x\left[\sqrt{-\det(g_{\mu\nu}+\alpha R_{\mu\nu})}-(\alpha\Lambda_0+1)\sqrt{-g}\right], \tag{2.58}$$

and the theory is nonunitary even around the flat background. This theory is the simplest BI gravity, so it provides a suitable example to demonstrate the canonical way to analyze the unitarity of a specific BI gravity around the (A)dS background by first finding the vacuum through the $O(h)$ action and then finding the $O(h^2)$ action describing the free theory. In addition, the spacetime dimension is chosen as four to employ the $O(h)$ and $O(h^2)$ level equivalence between the BI gravity and its $O(A^2)$ expansion. Since $A_{\mu\nu} = \alpha R_{\mu\nu}$, the A expansion is nothing but the curvature expansion in αR. By using (2.23), the $O\left[(\alpha R)^2\right]$ expansion of (2.58) becomes

$$I_{O(R^2)} = \frac{1}{\kappa}\int d^4x\sqrt{-g}\left[R - 2\Lambda_0 - \frac{\alpha}{2}\left(R^\mu_\nu R^\nu_\mu - \frac{1}{2}R^2\right)\right], \tag{2.59}$$

which has the same spectrum around the same (A)dS background as the four-dimensional BI gravity (2.58). The field equation for the (A)dS background can be found as $\Lambda = \Lambda_0$ by using the field equation for the (A)dS spacetime for the generic quadratic curvature gravity theories given in [15].

One can decide on the unitarity of the four-dimensional BI gravity (2.58) around the (A)dS background by analyzing the unitarity of (2.59) around the same background. For generic quadratic curvature theories, the spectrum and its consistency are well-studied for the flat background [2] and for the (A)dS background [15]. From the flat spacetime limit [2] and from the pole structure in the tree level scattering amplitude around the (A)dS background [15], one knows that the existence of the terms $R^\mu_\nu R^\nu_\mu$ and R^2 in the action (2.59) indicate the presence of the massive spin-2 and massive spin-0 modes, respectively, in addition to the massless spin-2 Einstein mode.[2] The unitarity of the massless spin-2 and the massive spin-2 modes are always in conflict, since the modes have kinetic energy terms with opposite signs. Thus, the quadratic curvature gravity (2.59) is nonunitary due to massive spin-2 ghost mode. Since (2.58) and (2.59) have the same spectrum around the same (A)dS background, the BI gravity theory is also nonunitary around its (A)dS vacua as expected.

The unitarity analysis of the four-dimensional BI gravity (2.58) around its (A)dS vacua is significantly simplified by use of the $O(h)$ and $O(h^2)$ level equivalence between (2.58) and (2.59). However, if the $O(R^2)$ terms exist in the A tensor, then the A expansion differs from the curvature expansion and the $O(A^2)$ expansion of (2.58)

[2] If the couplings of the terms $\beta R^\mu_\nu R^\nu_\mu$ and αR^2 are related as $4(D-1)\alpha + D\beta = 0$, then the massive spin-0 mode can be eliminated from the spectrum [15].

does not yield a quadratic curvature gravity; instead, it also involves the cubic and quartic curvature terms. Hence, to analyze the unitarity for the cases in which cubic and quartic curvature terms appear in the $O\left(A^2\right)$ expansion, one needs to use other techniques such as calculating the $O\left(h\right)$ and $O\left(h^2\right)$ actions explicitly for the cubic and quartic curvature terms or the method of Hindawi *et al* [35] discussed in the next section.

Now, we analyze the unitarity of the four-dimensional BI gravity theory (2.58) by finding the $O\left(h\right)$ and $O\left(h^2\right)$ actions via (2.18) and (2.21), respectively. Following the procedure, first we need to find $\bar{A}_{\mu\nu}$, $A^{(1)}_{\mu\nu}$, and $A^{(2)}_{\mu\nu}$ which are simply $\bar{A}_{\mu\nu} = \alpha\Lambda\bar{g}_{\mu\nu}$ implying $a = \alpha\Lambda$, $A^{(1)}_{\mu\nu} = \alpha R^L_{\mu\nu}$, and $A^{(2)}_{\mu\nu} = \alpha R^{(2)}_{\mu\nu}$, and the explicit forms of $R^L_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$ are given in the Appendix A. Then, to find the (A)dS vacua, one needs to calculate the $O\left(h\right)$ action (2.18), and using $a = \alpha\Lambda$ and the explicit form of $R^L_{\mu\nu}$, (2.18) becomes

$$I_{O(h)} = \frac{1}{\kappa\alpha}\int d^4x\,\sqrt{-\bar{g}}\left[\alpha\left(1+\alpha\Lambda\right)\bar{\nabla}_\mu\left(\bar{\nabla}_\nu h^{\mu\nu} - \bar{\nabla}^\mu h\right) + \alpha\left(\Lambda-\Lambda_0\right)h\right]. \qquad (2.60)$$

After dropping the boundary term, the $O\left(h\right)$ action is zero for arbitrary h if and only if $\Lambda = \Lambda_0$ which is the field equation for the (A)dS background. Now, turning to the $O\left(h^2\right)$ action (2.21), for the four-dimensional BI gravity theory (2.58), the $O\left(h^2\right)$ action takes the form

$$\begin{aligned}I_{O(h^2)} = -\frac{1}{\kappa\alpha}\int d^4x\,\sqrt{-\bar{g}}\bigg\{&\frac{\alpha^2}{2}\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}R^L_{\mu\nu}R^L_{\alpha\beta} - \frac{\alpha^2}{4}\left(R_L+\Lambda h\right)^2\\ &-\left(1+\alpha\Lambda\right)\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu} + \alpha h^{\mu\nu}\left[R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\left(R_L+\Lambda h\right)\right]\\ &+\frac{\alpha}{4}\Lambda_0\left(h^2 - 2h^\mu_\nu h^\nu_\mu\right)\bigg\}, \end{aligned} \qquad (2.61)$$

where the integrals of the terms $\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}R^L_{\mu\nu}R^L_{\alpha\beta}$ and $\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}$ are calculated in the Appendix B as

$$\begin{aligned}\int d^4x\,\sqrt{-\bar{g}}\,\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}R^L_{\mu\nu}R^L_{\alpha\beta} = -\frac{1}{2}\int d^4x\,\sqrt{-\bar{g}}h^{\mu\nu}&\\ \times\bigg[\left(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu + \Lambda\bar{g}_{\mu\nu}\right)R_L&\\ +\left(\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{3}\bar{g}_{\mu\nu}R_L\right)&\\ -\frac{14\Lambda}{3}R^L_{\mu\nu} + \frac{\Lambda}{3}\bar{g}_{\mu\nu}R_L + \frac{8\Lambda^2}{3}h_{\mu\nu}\bigg]&, \end{aligned} \qquad (2.62)$$

and

$$\int d^4x\,\sqrt{-\bar{g}}\,\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu} = -\frac{1}{2}\int d^4x\,\sqrt{-\bar{g}}h^{\mu\nu}\left(\frac{1}{2}R^L_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}R_L - \frac{\Lambda}{4}\bar{g}_{\mu\nu}h\right). \tag{2.63}$$

Here, $\mathcal{G}^L_{\mu\nu}$ and R_L represent the linear orders in h for the cosmological Einstein tensor, $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}$, and the scalar curvature, respectively. With these results, the $O\left(h^2\right)$ action becomes

$$\begin{aligned} I_{O(h^2)} = -\frac{1}{\alpha\kappa}\int d^4x\,\sqrt{-\bar{g}}\Bigg\{ h^{\mu\nu}\left[\left(\frac{\alpha}{2} + \frac{2\alpha^2\Lambda}{3}\right)\mathcal{G}^L_{\mu\nu} - \frac{\alpha^2}{4}\left(\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{3}\bar{g}_{\mu\nu}R_L\right)\right] \\ -\frac{\alpha}{4}\left(\Lambda - \Lambda_0\right)\left(h^2 - 2h^\mu_\nu h^\nu_\mu\right)\Bigg\}, \end{aligned} \tag{2.64}$$

where the first line is in a background gauge-invariant form (upon use of $\bar{\nabla}_\mu\mathcal{G}^{\mu\nu}_L = 0$), while the second line is not. However, once the terms in the $O\left(h^2\right)$ action other than the term $\left(h^2 - 2h^\mu_\nu h^\nu_\mu\right)$ are put in a background gauge invariant form, the coefficient of the term $\left(h^2 - 2h^\mu_\nu h^\nu_\mu\right)$ takes the form of the field equations for the (A)dS background which is $\Lambda = \Lambda_0$ in our example. Then, upon use of $\Lambda = \Lambda_0$, the $O\left(h^2\right)$ action takes its final form which is background gauge invariant as

$$\begin{aligned} I_{O(h^2)} = -\frac{1}{\alpha\kappa}\int d^4x\,\sqrt{-\bar{g}} \\ \times h^{\mu\nu}\left[\left(\frac{\alpha}{2} + \frac{2\alpha^2\Lambda_0}{3}\right)\mathcal{G}^L_{\mu\nu} - \frac{\alpha^2}{4}\left(\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda_0}{3}\bar{g}_{\mu\nu}R_L\right)\right], \end{aligned} \tag{2.65}$$

from which one can find the linearized field equations by taking variation with respect to $h_{\mu\nu}$. Since the operators in $\mathcal{G}^L_{\mu\nu}$ and R_L are self-adjoint operators, the linearized field equations simply become

$$\left(1 + \frac{4\alpha\Lambda_0}{3}\right)\mathcal{G}^L_{\mu\nu} - \frac{\alpha}{2}\left(\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda_0}{3}\bar{g}_{\mu\nu}R_L\right) = 0, \tag{2.66}$$

and these equations are the same as the linearized field equations of the $O\left[(\alpha R)^2\right]$ action (2.59), which can be found by using the linearized field equations of the generic quadratic curvature gravity given in [31, 32, 15]. By using the results of [15], one can figure out that the linearized field equations represent the same spectrum with the massive spin-2 ghost mode as depicted above. The linearized cosmological Einstein tensor $\mathcal{G}^L_{\mu\nu}$ indicates[3] the existence of the massless spin-2 mode, while the presence of the form $\left(\bar{\Box} - m^2\right)\mathcal{G}^L_{\mu\nu} = 0$ in (2.66) implies the existence of the massive spin-2

[3] Note that to actually work out the spectrum, one must find the wave type equations of the microscopic fields (such as the physical parts of $h_{\mu\nu}$) not on derived fields such as $\mathcal{G}^L_{\mu\nu}$.

mode whose propagator comes with the opposite sign compared to the Einstein mode, that is a ghost. In addition, the trace of (2.66) has the form $\left(\bar{\square} - m^2\right) R_L = 0$ which implies the existence of the massive spin-0 mode.

We used two techniques of analyzing unitarity in the example of the four-dimensional BI gravity (2.58). First, the $O(h)$ and $O(h^2)$ level equivalence between the four-dimensional BI gravity and its $O(A^2)$ expansion is used and due to the absence of the $O(R^2)$ terms in the A tensor, the $O(A^2)$ expansion yields a quadratic curvature gravity theory from which we can decide on the unitarity of the theory. Secondly, the unitarity of the four-dimensional BI gravity is discussed by finding the $O(h)$ and $O(h^2)$ actions. Although the first technique is specific to the rather restricted class of the four-dimensional BI gravity theories defined with an A tensor that is linear in curvature, the second technique can be applied to any BI gravity in any dimensions by following the same procedure.

Now, let us discuss the cancellation of the $O(h^2)$ contribution coming from the $O\left[(\alpha R)^3\right]$ of the four-dimensional BI gravity (2.58), although the results we obtained so far is sufficient to prove it. The main reason for this discussion is that the procedure that we follow to explicitly verify this cancellation is applicable to cases when the $O(A^2)$ expansion of a four-dimensional BI gravity involves cubic and quartic curvature terms. In addition, when the higher even D-dimensional equivalences are used, again higher curvature terms appear in the $O\left(A^{D/2}\right)$ expansion, and these terms can be analyzed in the same way.

To find the $O\left[(\alpha R)^3\right]$ expansion of the four-dimensional BI gravity (2.58), we need the cubic order in the expansion of $\sqrt{\det(I+M)}$ which is

$$\left(\sqrt{\det(I+M)}\right)_{(3)} = \frac{1}{6}\mathrm{Tr}\left(M^3\right) - \frac{1}{8}\mathrm{Tr}\left(M^2\right)\mathrm{Tr}M + \frac{1}{48}\left(\mathrm{Tr}M\right)^3, \tag{2.67}$$

and using this result one can find the $O\left[(\alpha R)^3\right]$ expansion of (2.58) as

$$I_{O(R^3)} = \frac{1}{\kappa}\int d^4x\sqrt{-g}\left[(R-2\Lambda_0) - \frac{\alpha}{2}K + \frac{\alpha^2}{24}S\right], \tag{2.68}$$

where K and S are defined as

$$K \equiv R^{\mu}_{\nu}R^{\nu}_{\mu} - \frac{1}{2}R^2, \qquad S \equiv 8R^{\mu}_{\nu}R^{\alpha}_{\mu}R^{\nu}_{\alpha} - 6RR_{\mu\nu}R^{\mu\nu} + R^3. \tag{2.69}$$

We obtain the $O\left(h^2\right)$ action for the $O\left[(\alpha R)^3\right]$ action (2.68) to show the cancellation of the $O\left(h^2\right)$ contribution coming from the $O\left[(\alpha R)^3\right]$ terms. In calculating the $O\left(h^2\right)$ action for the generic BI gravity, we expanded the action in h; however, following this path for (2.68) is rather cumbersome. Instead, we find the field equations for (2.68), then linearize them in h. Once the linearized field equations are found, we use the self-adjointness of the operators appearing in the linearized field equations to obtain the $O\left(h^2\right)$ action for (2.68).

The field equations for the $O\left[(\alpha R)^3\right]$ action (2.68) can be found as

$$\begin{aligned}
0 = &-\frac{1}{2}\left[(R-2\Lambda_0) - \frac{\alpha}{2}K + \frac{\alpha^2}{24}S\right] g_{\mu\nu} + R_{\mu\nu} \\
&+ \frac{\alpha}{2}\left[RR_{\mu\nu} - 2R_{\lambda\nu\alpha\mu}R^{\lambda\alpha} - \Box\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\right] \\
&+ \frac{\alpha^2}{2}\left(2R^\rho_\mu R_{\rho\alpha}R^\alpha_\nu + \left[g_{\mu\nu}\nabla_\alpha\nabla_\beta\left(R^{\beta\rho}R^\alpha_\rho\right) + \Box\left(R^\rho_\nu R_{\mu\rho}\right) - 2\nabla_\alpha\nabla_\mu\left(R^\rho_\nu R^\alpha_\rho\right)\right]\right) \\
&+ \frac{\alpha^2}{4}\left(\left[2\nabla_\alpha\nabla_\mu\left(RR^\alpha_\nu\right) - g_{\mu\nu}\nabla_\alpha\nabla_\beta\left(RR^{\alpha\beta}\right) - \Box\left(RR_{\mu\nu}\right)\right] - 2RR^\rho_\nu R_{\mu\rho}\right) \\
&- \frac{\alpha^2}{4}\left[\left(g_{\mu\nu}\Box - \nabla_\nu\nabla_\mu\right) + R_{\mu\nu}\right]\left(R^2_{\alpha\beta} - \frac{1}{2}R^2\right).
\end{aligned} \tag{2.70}$$

If the Ricci tensor for the (A)dS background, that is, $\bar{R}_{\mu\nu} = \Lambda\bar{g}_{\mu\nu}$ is put in (2.70), one can rederive $\Lambda = \Lambda_0$. By using the variations given in Appendix A.2, the linearized field equations can be calculated as

$$0 = \left(1 + \frac{4\alpha\Lambda_0}{3}\right)\mathcal{G}^L_{\mu\nu} - \frac{\alpha}{2}\left(\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda_0}{3}\bar{g}_{\mu\nu}R_L\right), \tag{2.71}$$

which is the same as (2.66). Here, the linearization of the terms originated from the cubic curvature terms in (2.68) remarkably cancel each other. Since $\mathcal{G}^L_{\mu\nu}$ and R_L are self-adjoint operators, one can obtain the Lagrangian density of the $O\left(h^2\right)$ action by simply multiplying the linearized field equations with $h_{\mu\nu}$. The overall factor for the $O\left(h^2\right)$ action can be fixed by considering that the $O\left(h^2\right)$ action of the Einstein-Hilbert action $\int d^4x\sqrt{-g}\,(R-2\Lambda_0)$ is $-\frac{1}{2}\int d^4x\,\sqrt{-\bar{g}}h^{\mu\nu}\mathcal{G}^L_{\mu\nu}$, and one finally obtains the $O\left(h^2\right)$ action for the $O\left[(\alpha R)^3\right]$ action (2.68) which is the same as (2.65). By following the same procedure, the higher curvature terms appearing in the $O\left(A^{D/2}\right)$ expansion of any even D-dimensional BI gravity can be analyzed. However, with increasing order of curvature the analysis becomes more involved.

2.2 Equivalent Quadratic Curvature Actions

In this section, we present another method to analyze the unitarity of the BI gravity theories. This method is developed by Hindawi *et al* [35] and it is not restricted to the BI gravity theories only. Any higher curvature gravity theory based on the contractions of the Riemann tensor, but not its derivatives, can be analyzed by this method.

The method is based on the construction of an equivalent quadratic curvature action which has the same vacua and the same spectrum as the higher curvature gravity theory whose spectrum and its viability are under investigation. Let us consider a generic gravity theory having the Lagrangian density $\mathcal{L} \equiv \sqrt{-g}\, f\left(R^{\mu\nu}_{\rho\sigma}\right)$ and construct the equivalent quadratic curvature action for this theory. We specifically choose the Lagrangian density to depend on the Riemann tensor with two up and two down indices because with this form of the Riemann tensor, any higher curvature scalar can be constructed without the need for metric or its inverse. Furthermore, $R^{\mu\nu}_{\rho\sigma}$ has the form $R^{\mu\nu}_{\rho\sigma} \sim \delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}$ for the (A)dS background and the absence of the background metric $\bar{g}_{\mu\nu}$ and its inverse in this form simplifies the calculations of the equivalent quadratic curvature action.

The spectrum of the $f\left(R^{\mu\nu}_{\rho\sigma}\right)$ theory around the (A)dS background is analyzed by expanding the action $\int d^D x\, \mathcal{L}$ in the metric perturbation $h_{\mu\nu}$. The $O\left(h\right)$ term in this expansion determines the (A)dS vacua, while the $O\left(h^2\right)$ term represents the free theory of excitations in the spectrum. In calculating $O\left(h\right)$ and $O\left(h^2\right)$ actions, only the up-to-$O\left(h^2\right)$ expansion of $f\left(R^{\mu\nu}_{\rho\sigma}\right)$ is required. Therefore, two gravity theories having Lagrangian densities $\mathcal{L}_1 \equiv \sqrt{-g}\, f_1\left(R^{\mu\nu}_{\rho\sigma}\right)$ and $\mathcal{L}_1 \equiv \sqrt{-g}\, f_2\left(R^{\mu\nu}_{\rho\sigma}\right)$ can have the same spectrum around the same background if the expansions of f_1 and f_2 in h are the same up to $O\left(h^2\right)$.

Now, let us consider the Taylor series expansion of $f\left(R^{\mu\nu}_{\rho\sigma}\right)$ in curvature around the (A)dS background as

$$f\left(R^{\mu\nu}_{\rho\sigma}\right) = \sum_{i=0}^{\infty} \frac{1}{i!} \left[\frac{\partial^i f}{\partial \left(R^{\mu\nu}_{\rho\sigma}\right)^i}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} \left(R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma}\right)^i . \tag{2.72}$$

Note that as the expansion is around the (A)dS background, the tensorial structures of $\bar{R}^{\mu\nu}_{\rho\sigma}$ and the derivatives calculated at the background consist of only Kronecker deltas.

At this point, it is simple but important to observe that expanding $\left(R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma}\right)$ in h yields a linear term in h at the leading order as

$$R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma} = \tau \left(R^{\mu\nu}_{\rho\sigma}\right)_{(1)} + \tau^2 \left(R^{\mu\nu}_{\rho\sigma}\right)_{(2)} + O\left(\tau^3\right), \tag{2.73}$$

so if one expands the i^{th} order of (2.72), then the leading order in h is $O\left(h^i\right)$ as

$$\left(R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma}\right)^i = \tau^i \left[\left(R^{\mu\nu}_{\rho\sigma}\right)_{(1)}\right]^i + O\left(\tau^{i+1}\right). \tag{2.74}$$

This observation implies that the up-to-$O\left(h^2\right)$ expansion of $f\left(R^{\mu\nu}_{\rho\sigma}\right)$ involves contributions coming from the first three terms of (2.72); therefore, by truncating (2.72) at $i = 2$, one can define a new function which has the same $O\left(h^2\right)$ expansion as f. This truncation yields a quadratic curvature theory for an (A)dS background with the action $\int d^D x\, \sqrt{-g}\, f_{\text{quad-equal}}\left(R^{\mu\nu}_{\rho\sigma}\right)$ where

$$f_{\text{quad-equal}}\left(R^{\mu\nu}_{\rho\sigma}\right) \equiv \sum_{i=0}^{2} \frac{1}{i!} \left[\frac{\partial^i f}{\partial \left(R^{\mu\nu}_{\rho\sigma}\right)^i}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} \left(R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma}\right)^i, \tag{2.75}$$

and by definition, this theory has the same (A)dS vacua and the same spectrum as the original theory with the action $\int d^D x\, \sqrt{-g}\, f\left(R^{\mu\nu}_{\rho\sigma}\right)$. Once (2.75) is calculated for a given f, in the final form one obtains

$$\begin{aligned} f_{\text{quad-equal}}\left(R^{\mu\nu}_{\rho\sigma}\right) =& \frac{1}{\tilde{\kappa}}\left(R - 2\tilde{\Lambda}_0\right) \\ &+ \tilde{\alpha} R^2 + \tilde{\beta} R^{\mu}_{\nu} R^{\nu}_{\mu} + \tilde{\gamma}\left(R^{\mu\nu}_{\rho\sigma} R^{\rho\sigma}_{\mu\nu} - 4 R^{\mu}_{\nu} R^{\nu}_{\mu} + R^2\right), \end{aligned} \tag{2.76}$$

where the couplings depend on the parameters of the original theory defined by f and the effective cosmological constant Λ. Furthermore, the function f may depend on the Ricci tensor R^{μ}_{ν} and in such a case, although it is still valid to use (2.75), it is more convenient to use

$$f_{\text{quad-equal}}\left(R^{\mu}_{\nu}\right) \equiv \sum_{i=0}^{2} \frac{1}{i!} \left[\frac{\partial^i f}{\partial \left(R^{\mu}_{\nu}\right)^i}\right]_{\bar{R}^{\mu}_{\nu}} \left(R^{\mu}_{\nu} - \bar{R}^{\mu}_{\nu}\right)^i, \tag{2.77}$$

which follows from the same observations and is equivalent to (2.75). Note that for this case, $f_{\text{quad-equal}}\left(R^{\mu}_{\nu}\right)$ does not involve the Gauss-Bonnet combination in the final form.

After finding the equivalent quadratic curvature action $\int d^D x\, \sqrt{-g}\, f_{\text{quad-equal}}\left(R^{\mu\nu}_{\rho\sigma}\right)$ from (2.75), the (A)dS vacuum can be found and the spectrum around this background can be analyzed by using the results of [15]. To have a unitary theory in $D > 3$, the unique criterion is the absence of the term $R^{\mu}_{\nu}R^{\nu}_{\mu}$ in the equivalent quadratic curvature action. To sum up, to analyze the unitarity of a higher curvature gravity theory, the method of finding the equivalent quadratic curvature action is quite efficient because one just needs to use (2.75) without any need to do an expansion in h and to find the field equations.

To find the equivalent quadratic curvature action for a BI gravity theory, one may follow different ways. The immediate option is to work with the BI action in the original determinantal form as we do in this work. On the other hand, one may rewrite the determinant in terms of traces by using the exact trace expansion of the determinant which are specific to each dimension. For example, in [6] the unitarity of BINMG and in [5] the equivalent quadratic curvature action for the four-dimensional BI gravity defined with $A_{\mu\nu} = \alpha R_{\mu\nu}$ are studied by use of the exact trace expansions. In addition, if the spacetime is even D-dimensional, one can use the $O(h)$ and $O(h^2)$ level equivalences between BI gravity and its $O\left(A^{D/2}\right)$ expansion. Then, the higher curvature gravity theory resulting from the $O\left(A^{D/2}\right)$ expansion can be used to find the equivalent quadratic curvature action for the BI gravity theory. In [5], for example, the unitarity of the four-dimensional BI gravity theory proposed by Deser and Gibbons is analyzed via the $O(A^2)$ expansion of the theory. Note that, with the increase in the number of dimensions, the number of terms in the exact trace expansion and in the $O\left(A^{D/2}\right)$ expansion increase, and if in addition the A tensor has a complex form, then the use of the exact trace expansion and the $O\left(A^{D/2}\right)$ expansion become elaborate compared to the use of the original determinantal form.

For a generic BI gravity theory, let us provide general formulas that are useful in calculating $f_{\text{quad-equal}}$ via (2.75) or (2.77). To find $f_{\text{quad-equal}}$, one needs to calculate the (A)dS background values for the matrix function $\sqrt{\det\left(\delta^{\rho}_{\nu} + A^{\rho}_{\nu}\right)}$ and its first and second derivatives. First, the background value of $\sqrt{\det\left(\delta^{\rho}_{\nu} + A^{\rho}_{\nu}\right)}$ is

$$\sqrt{\det\left(\delta^{\rho}_{\nu} + \bar{A}^{\rho}_{\nu}\right)} = (1+a)^{\frac{D}{2}}\,, \tag{2.78}$$

where a is defined by $\bar{A}^{\rho}_{\nu} = a\delta^{\rho}_{\nu}$ as before. Then, by using $\det N = \exp\left(\text{Tr}\left(\ln N\right)\right)$, the first and second order differentials of $\sqrt{\det\left(\delta^{\rho}_{\nu} + A^{\rho}_{\nu}\right)}$ have the form

$$\partial\left(\sqrt{\det\left(\delta^{\rho}_{\nu} + A^{\rho}_{\nu}\right)}\right) = \frac{1}{2}\sqrt{\det\left(\delta^{\rho}_{\nu} + A^{\rho}_{\nu}\right)}C^{\lambda}_{\gamma}\partial A^{\gamma}_{\lambda}, \tag{2.79}$$

and

$$\partial^2\left(\sqrt{\det\left(\delta^\rho_\nu + A^\rho_\nu\right)}\right) = \frac{1}{2}\sqrt{\det\left(\delta^\rho_\nu + A^\rho_\nu\right)}\left[C^\lambda_\gamma \partial^2 A^\gamma_\lambda - C^\lambda_\theta C^\tau_\gamma\left(\partial A^\theta_\tau\right)\partial A^\gamma_\lambda + \frac{1}{2}\left(C^\lambda_\gamma \partial A^\gamma_\lambda\right)^2\right], \quad (2.80)$$

where C^λ_γ represents the inverse of $\left(\delta^\lambda_\gamma + A^\lambda_\gamma\right)$ and for the differential of C we use $\partial C = -C\left(\partial A\right)C$. Note that one may not find the explicit form of the C tensor for a given A tensor, and in fact, even for $A_{\mu\nu} = \alpha R_{\mu\nu}$ it is not possible to find an explicit form. However, just the (A)dS background value of the C tensor is required to calculate the background values for the first and second derivative of $\sqrt{\det\left(\delta^\rho_\nu + A^\rho_\nu\right)}$, and one can calculate it as

$$\bar{C}^\lambda_\gamma = \left(1+a\right)^{-1}\delta^\lambda_\gamma. \quad (2.81)$$

Note that the matrix $(I + A)$ becomes singular for $a = 1$, and we have already assumed that $a \neq -1$. In the absence of the specific definition for the A tensor, there is no need to further study the background values of (2.79) and (2.80) by employing (2.78) and (2.81).

To find $f_{\text{quad-equal}}$ for a specific BI gravity theory, one needs to find a and needs to calculate the first and second derivatives of the A tensor with respect to the Riemann tensor, $R^{\mu\nu}_{\rho\sigma}$, or the Ricci tensor, R^μ_ν, depending on the form of A. Then, the formulas (2.78)–(2.81) are enough to work out $f_{\text{quad-equal}}$ for the BI gravity theory.

In the following two subsections, we apply the method of equivalent quadratic curvature action in the example of the four-dimensional BI gravity defined by $A_{\mu\nu} = \alpha R_{\mu\nu}$. We first find the equivalent quadratic curvature action for the $O\left[(\alpha R)^3\right]$ expansion of the theory, then we consider the theory to all orders in curvature.

2.2.1 Analysis of cubic order of BI gravity $A_{\mu\nu} = \alpha R_{\mu\nu}$

For even D dimensions, the $O\left(h\right)$ and $O\left(h^2\right)$ level equivalence between BI gravity and its $O\left(A^{D/2}\right)$ expansion may yield higher curvature actions. To develop techniques for analyzing these cases, in Sec. 2.1.3, we consider the example of the $O\left[(\alpha R)^3\right]$ expansion of the four-dimensional BI gravity with the A tensor $A_{\mu\nu} = \alpha R_{\mu\nu}$. In this subsection, we analyze the same example by finding the equivalent quadratic curvature action. As we demonstrate, analyzing the unitarity via the equivalent quadratic curvature action is more efficient than the technique applied in Sec. 2.1.3 where we find the vacua and calculate the $O\left(h^2\right)$ action by use of the field equations.

Now, let us calculate the equivalent quadratic curvature action for the $O\left[(\alpha R)^3\right]$ action (2.68). To calculate $f_{\text{quad-equal}}\left(R^\mu_\nu\right)$, it is better to use (2.77) whose expanded form is

$$\begin{aligned} f_{\text{quad-equal}}\left(R^\mu_\nu\right) =& f\left(\bar{R}^\mu_\nu\right) + \left[\frac{\partial f}{\partial R^\alpha_\beta}\right]_{\bar{R}^\mu_\nu}\left(R^\alpha_\beta - \bar{R}^\alpha_\beta\right) \\ &+ \frac{1}{2}\left[\frac{\partial^2 f}{\partial R^\rho_\sigma \partial R^\alpha_\beta}\right]_{\bar{R}^\mu_\nu}\left(R^\alpha_\beta - \bar{R}^\alpha_\beta\right)\left(R^\rho_\sigma - \bar{R}^\rho_\sigma\right). \end{aligned} \tag{2.82}$$

The function $f\left(R^\mu_\nu\right)$ for the $O\left[(\alpha R)^3\right]$ action (2.68) is

$$\begin{aligned} f\left(R^\mu_\nu\right) \equiv& R - 2\Lambda_0 - \frac{\alpha}{2}\left(R^\mu_\nu R^\nu_\mu - \frac{1}{2}R^2\right) \\ &+ \frac{\alpha^2}{24}\left(8R^\mu_\rho R^\rho_\nu R^\nu_\mu - 6R^\mu_\nu R^\nu_\mu R + R^3\right). \end{aligned} \tag{2.83}$$

One needs to calculate the background values for f and for its first and second order derivatives which turns out to be

$$\begin{aligned} f\left(\bar{R}^\mu_\nu\right) &= 4\Lambda - 2\Lambda_0 + 2\alpha\Lambda^2, \\ \left[\frac{\partial f}{\partial R^\alpha_\beta}\right]_{\bar{R}^\mu_\nu} &= \left(1 + \alpha\Lambda\right)\delta^\beta_\alpha, \\ \left[\frac{\partial^2 f}{\partial R^\rho_\sigma \partial R^\alpha_\beta}\right]_{\bar{R}^\mu_\nu} &= \frac{\alpha}{2}\left(\delta^\sigma_\rho\delta^\beta_\alpha - 2\delta^\beta_\rho\delta^\sigma_\alpha\right). \end{aligned} \tag{2.84}$$

Using these results in (2.82) yields $f_{\text{quad-equal}}$ as

$$f_{\text{quad-equal}}\left(R^\mu_\nu\right) = R - 2\Lambda_0 - \frac{\alpha}{2}\left(R^\mu_\nu R^\nu_\mu - \frac{1}{2}R^2\right), \tag{2.85}$$

which is the same as $f\left(R^\mu_\nu\right)$ up to the quadratic curvature order. This is an expected result, since we know that the $O\left[(\alpha R)^3\right]$ terms of the four-dimensional BI gravity defined by $A_{\mu\nu} = \alpha R_{\mu\nu}$ do not yield $O(h)$ and $O\left(h^2\right)$ contributions.

The spectrum and the vacua of the $O\left[(\alpha R)^3\right]$ action (2.68) are determined by the equivalent quadratic curvature action, $\int d^D x\sqrt{-g}\, f_{\text{quad-equal}}\left(R^\mu_\nu\right)$. The equivalent quadratic curvature action is the same as the $O\left[(\alpha R)^2\right]$ expansion of the four-dimensional BI gravity defined by $A_{\mu\nu} = \alpha R_{\mu\nu}$ given in (2.59) whose spectrum and vacuum were analyzed in Sec. 2.1.3.

As revealed by this example, using the method of equivalent quadratic curvature action for the higher curvature gravity theories originated from the $O\left(A^{D/2}\right)$ expansion of the even-dimensional BI gravity theories is straightforward and less demanding compared to explicit computation of the $O\left(h^2\right)$ action of the $O\left(A^{D/2}\right)$ expansion.

2.2.2 Analysis of BI gravity $A_{\mu\nu} = \alpha R_{\mu\nu}$

Now, using the formulation given in (2.78)–(2.81), we analyze the unitarity of the four-dimensional BI gravity theory defined by $A_{\mu\nu} = \alpha R_{\mu\nu}$. From the action of the theory (2.22), $f\left(R^\mu_\nu\right)$ can be defined as

$$f\left(R^\mu_\nu\right) \equiv \frac{2}{\alpha}\left[\sqrt{\det\left(\delta^\rho_\nu + \alpha R^\rho_\nu\right)} - \left(\alpha\Lambda_0 + 1\right)\right]. \tag{2.86}$$

To calculate $f_{\text{quad-equal}}\left(R^\mu_\nu\right)$, first one needs to find the (A)dS background values of the A tensor, and its first and second derivatives. For $A_{\mu\nu} = \alpha R_{\mu\nu}$, the background values of the A tensor and its first derivative are simply $\bar{A}^\rho_\nu = \alpha\Lambda\delta^\rho_\nu$ and $\partial A^\nu_\rho/\partial R^\alpha_\beta = \delta^\nu_\alpha\delta^\beta_\rho$, while the second derivative is zero. Employing these results in (2.78)–(2.81), one can calculate the background values of $f\left(R^\mu_\nu\right)$, (2.86), and its first and second order derivatives, and they turn out to be the same as (2.84) which are calculated for the $O\left[(\alpha R)^3\right]$ expansion of (2.86) given in (2.83). Thus, $f_{\text{quad-equal}}\left(R^\mu_\nu\right)$ of (2.86) is the same as (2.85).

The equivalent quadratic curvature action, $\int d^D x\,\sqrt{-g}\,f_{\text{quad-equal}}\left(R^\mu_\nu\right)$, for the four-dimensional BI gravity defined by $A_{\mu\nu} = \alpha R_{\mu\nu}$ is the same as the $O\left(A^2\right)$ expansion of the BI gravity given in (2.59), since the A tensor is simply linear in curvature. As discussed in Sec. 2.1.3, the appearance of the term $R^\mu_\nu R^\nu_\mu$ in the equivalent quadratic curvature action (2.59) implies that the four-dimensional BI gravity defined by $A_{\mu\nu} = \alpha R_{\mu\nu}$ is nonunitary due to the massive spin-2 ghost in the spectrum.

So far, the unitarity of the four-dimensional BI gravity defined by $A_{\mu\nu} = \alpha R_{\mu\nu}$ is analyzed in three ways: using the equivalence between the BI gravity and its $O\left(A^2\right)$ expansion, using the $O\left(h\right)$ and $O\left(h^2\right)$ actions of generic BI gravity, and finding the equivalent quadratic curvature action. The first way is the simplest way; however, the simplicity of this way is due to the linearity of A in curvature and the fact that we worked in four dimensions. In generic even D dimensions, for an A tensor involving higher orders in curvature, one needs to analyze the $O\left(A^{D/2}\right)$ expansion of a specific even-dimensional BI gravity by finding either the equivalent quadratic curvature action or the $O\left(h^2\right)$ expansion for the $O\left(A^{D/2}\right)$ action. On the other hand, the other two methods involve straightforward calculations; however, finding the equivalent quadratic curvature action is less involved compared to the use of the $O\left(h\right)$ and $O\left(h^2\right)$ actions, where one needs to find the $O\left(h^2\right)$ expansion of the A tensor and needs to rewrite the $O\left(h^2\right)$ action in a manifestly background gauge invariant form.

2.3 Unitarity analysis of Born-Infeld Gravity Proposed by Deser and Gibbons

So far, we have worked on the basic example of the four-dimensional BI gravity defined by $A_{\mu\nu} = \alpha R_{\mu\nu}$, which is not unitary around flat and (A)dS backgrounds simply due to the appearance of the $R^{\mu}_{\nu} R^{\nu}_{\mu}$ in the $O\left[(\alpha R)^2\right]$ expansion (2.59). To cure the unitarity around the flat background, one needs to add specific $O\left(R^2\right)$ terms to the A tensor such that the quadratic curvature expansion of the BI gravity takes the form $\frac{1}{\kappa}R + \alpha R^2 + \gamma\left(R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu} - 4R^{\mu}_{\nu}R^{\nu}_{\mu} + R^2\right)$, where either α or γ or both can be zero. One of the alternatives is taking the A tensor in the form $A_{\mu\nu} = \alpha R_{\mu\nu} + \frac{\alpha^2}{2}\left(R_{\mu\rho}R^{\rho}_{\nu} - \frac{1}{2}RR_{\mu\nu}\right)$ as proposed by Deser and Gibbons [8]. With this choice of the A tensor, the BI gravity theory

$$I = \frac{2}{\kappa\alpha}\int d^4x \left\{\sqrt{-\det\left[g_{\mu\nu} + \alpha R_{\mu\nu} + \frac{\alpha^2}{2}\left(R_{\mu\rho}R^{\rho}_{\nu} - \frac{1}{2}RR_{\mu\nu}\right)\right]} - \sqrt{-g}\right\}, \quad (2.87)$$

has the quadratic curvature expansion in the form $\frac{1}{\kappa}R$; therefore, it is a unitary theory around the flat background with a massless spin-2 mode. Note that to have a flat background Λ_0 is taken to be zero.

In this section, we analyze the unitarity of this theory around the (A)dS background and it turns out to be nonunitary. Any one of the techniques we have developed can be used to analyze the unitarity of the theory. In [5], the $O\left(A^2\right)$ expansion of (2.87) is found which is known to have the same vacua and the same spectrum as (2.87). Then, the $O\left(A^2\right)$ action of (2.87), which involves cubic and quartic curvature terms, is analyzed by finding the equivalent quadratic curvature action. One can also use the $O\left(h\right)$ and $O\left(h^2\right)$ actions of generic BI gravity to determine the vacua and the free-theory of (2.87). Here, we prefer to find the equivalent quadratic curvature action directly from (2.87) by using the formulation given in (2.78)–(2.81), so from (2.87) $f\left(R^{\mu}_{\nu}\right)$ is

$$f\left(R^{\mu}_{\nu}\right) \equiv \frac{2}{\alpha}\left[\sqrt{\det\left[\delta^{\rho}_{\nu}+\alpha R^{\rho}_{\nu}+\frac{\alpha^2}{2}\left(R^{\rho}_{\mu}R^{\mu}_{\nu}-\frac{1}{2}RR^{\rho}_{\nu}\right)\right]}-1\right]. \tag{2.88}$$

To find $f_{\text{quad-equal}}\left(R^{\mu}_{\nu}\right)$, first the background values of the A tensor and its first and second order derivatives should be calculated, and they become

$$\begin{aligned}
\bar{A}^{\gamma}_{\lambda} &= \left(\alpha\Lambda-\frac{1}{2}\alpha^2\Lambda^2\right)\delta^{\gamma}_{\lambda} \\
\left[\frac{\partial A^{\gamma}_{\lambda}}{\partial R^{\alpha}_{\beta}}\right]_{\bar{R}^{\mu}_{\nu}} &= \alpha\left(\delta^{\gamma}_{\alpha}\delta^{\beta}_{\lambda}-\frac{1}{4}\alpha\Lambda\delta^{\beta}_{\alpha}\delta^{\gamma}_{\lambda}\right), \\
\left[\frac{\partial^2 A^{\gamma}_{\lambda}}{\partial R^{\rho}_{\sigma}\partial R^{\alpha}_{\beta}}\right]_{\bar{R}^{\mu}_{\nu}} &= \frac{\alpha^2}{2}\left(\delta^{\gamma}_{\alpha}\delta^{\beta}_{\rho}\delta^{\sigma}_{\lambda}+\delta^{\gamma}_{\rho}\delta^{\sigma}_{\alpha}\delta^{\beta}_{\lambda}-\frac{1}{2}\delta^{\beta}_{\alpha}\delta^{\gamma}_{\rho}\delta^{\sigma}_{\lambda}-\frac{1}{2}\delta^{\sigma}_{\rho}\delta^{\gamma}_{\alpha}\delta^{\beta}_{\lambda}\right),
\end{aligned} \tag{2.89}$$

and from $\bar{A}^{\gamma}_{\lambda}$ the value of a is $a=\alpha\Lambda-\frac{1}{2}\alpha^2\Lambda^2$. Then, employing these results in (2.78)–(2.81), one can find the background values of $f\left(R^{\mu}_{\nu}\right)$ and its first and second derivatives as

$$\begin{aligned}
f\left(\bar{R}^{\mu}_{\nu}\right) &= 4\Lambda-2\alpha^2\Lambda^3+\frac{1}{2}\alpha^3\Lambda^4, \\
\left[\frac{\partial f}{\partial R^{\alpha}_{\beta}}\right]_{\bar{R}^{\mu}_{\nu}} &= \left(1-\frac{3}{2}\alpha^2\Lambda^2+\frac{1}{2}\alpha^3\Lambda^3\right)\delta^{\beta}_{\alpha}, \\
\left[\frac{\partial^2 f}{\partial R^{\rho}_{\sigma}\partial R^{\alpha}_{\beta}}\right]_{\bar{R}^{\mu}_{\nu}} &= \alpha^2\Lambda\left(1-\frac{1}{2}\alpha\Lambda\right)\left(\delta^{\beta}_{\rho}\delta^{\sigma}_{\alpha}-\delta^{\beta}_{\alpha}\delta^{\sigma}_{\rho}\right).
\end{aligned} \tag{2.90}$$

With these results, one can find $f_{\text{quad-equal}}\left(R^{\mu}_{\nu}\right)$ via (2.82) and the equivalent quadratic curvature action, $\int d^Dx\,\sqrt{-g}\,f_{\text{quad-equal}}\left(R^{\mu}_{\nu}\right)$, for (2.87) takes the form

$$I_{\text{quad-equal}} = \int d^4x\sqrt{-g}\left[\frac{1}{\tilde{\kappa}}\left(R-2\tilde{\Lambda}_0\right)+\tilde{\alpha}R^2+\tilde{\beta}R^{\mu}_{\nu}R^{\nu}_{\mu}\right], \tag{2.91}$$

where

$$\begin{aligned}
\frac{1}{\tilde{\kappa}} &= \frac{1}{\kappa}\left(1+\frac{3}{2}\alpha^2\Lambda^2-\alpha^3\Lambda^3\right), \qquad \tilde{\Lambda}_0=\frac{\tilde{\kappa}}{\kappa}\left(\alpha^2\Lambda^3-\frac{3}{4}\alpha^3\Lambda^4\right), \\
\tilde{\beta} &= \frac{\alpha^2\Lambda}{2\kappa}\left(1-\frac{1}{2}\alpha\Lambda\right), \qquad \tilde{\alpha}=-\tilde{\beta}.
\end{aligned} \tag{2.92}$$

Now, let us determine the vacua and the spectrum of the BI gravity proposed by Deser and Gibbons (2.87) via the equivalent quadratic curvature action (2.91). In four dimensions, the vacua of (2.91) satisfies $\Lambda=\tilde{\Lambda}_0$ [15], so one gets

$$\Lambda+\frac{\alpha^2\Lambda^3}{2}-\frac{\alpha^3\Lambda^4}{4}=0, \tag{2.93}$$

where clearly the flat background $\Lambda = 0$ is a solution. There is another real solution for (2.93) whose explicit form is not particularly illuminating, but it has the approximate value $\Lambda \approx 2.59/\alpha$. For $\Lambda = 0$, (2.91) reduces to $\frac{1}{\kappa}R$ by construction, while for the (A)dS background it takes the form

$$I_{\text{quad-equal}} = \frac{1}{\kappa}\int d^4x\sqrt{-g}\left[-\left(3+\frac{1}{2}\alpha^2\Lambda^2\right)(R-2\Lambda) - \frac{1}{\Lambda}\left(R^\mu_\nu R^\nu_\mu - R^2\right)\right], \quad (2.94)$$

by use of (2.93). Since the coefficient of R in (2.94) is negative, κ should also be negative to have a unitary massless spin-2 mode. However, due to presence of the term $R^\mu_\nu R^\nu_\mu$ for any value of Λ, we know that there is a massive spin-2 mode in the spectrum whose unitarity is in conflict with the massless spin-2 mode. Therefore, the BI gravity theory (2.87) is nonunitary around the (A)dS background, although it is unitary around the flat background.

CHAPTER 3

BORN-INFELD EXTENSION OF NEW MASSIVE GRAVITY

Born-Infeld extension of new massive gravity (BINMG) is the most remarkable example of the Born-Infeld gravity theories alas in three dimensions. BINMG possesses interesting properties such as it is unitary around flat and (A)dS backgrounds [6], it has constant scalar curvature Type-O, Type-N, and Type-D solutions which are the same as the solutions of cosmological topologically massive gravity (TMG) [7], and it has two *simple* holographic c-functions one of which is the same as the holographic c-function of Einstein's gravity [4]. In this chapter, after discussing the construction of the theory, we study these properties of BINMG in detail.

3.1 Constructing the Born-Infeld extension of New Massive Gravity

In this section, we discuss the idea underlying BINMG and construct its action. As discussed in the previous section, the unitarity of a BI gravity theory around the flat background is determined by only the quadratic curvature expansion of the BI gravity theory. The appearance of the terms $\beta R^{\mu}_{\nu} R^{\nu}_{\mu}$ and αR^2 in the quadratic curvature expansion implies the existence of the massive spin-2 and massive spin-0 modes, respectively, except for the specific parameter values satisfying the relation $4\,(D-1)\,\alpha + D\beta = 0$ and once this relation is satisfied, the spin-0 mode is eliminated from the spectrum [15]. The unitarity of the massive spin-2 mode is in conflict with the Einstein mode, while the unitarity of the spin-0 mode is in accord with the Einstein mode. Therefore, to have a unitary BI gravity theory, the quadratic curvature expansion of BI gravity should be in the form $\frac{1}{\kappa}R+\alpha R^2+\gamma\left(R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}-4R^{\mu}_{\nu}R^{\nu}_{\mu}+R^2\right)$.[1] However, in three dimensions an interesting opportunity appears. The three-dimensional Einstein-Hilbert action does not have a propagating degree of freedom; therefore, it is possible to resolve the unitarity conflict by eliminating the spin-0 mode via adjusting the relative coefficients of the $\beta R^{\mu}_{\nu}R^{\nu}_{\mu}$ and αR^2 terms as $8\alpha+3\beta=0$, and by introducing an overall minus sign to the quadratic curvature gravity action to make massive spin-2 mode nonghost. The resulting theory is new massive gravity (NMG) [11, 12] with the action

$$I_{\text{NMG}}=\frac{1}{\kappa^2}\int d^3x\,\sqrt{-g}\left[-R+\frac{1}{m^2}\left(R^2_{\mu\nu}-\frac{3}{8}R^2\right)\right],\tag{3.1}$$

where the Einstein-Hilbert term has the *wrong* sign as compared to the four-dimensional Einstein's gravity and the overall sign of the quadratic curvature terms is fixed such that the massive spin-2 mode is nontachyonic. Therefore, in three dimensions, there is an additional option that to have a unitary BI gravity theory around the flat background, the quadratic curvature expansion of the BI gravity theory can have the NMG form. Note that in higher dimensions the desired form of the quadratic curvature expansion, that is $\frac{1}{\kappa}R+\alpha R^2+\gamma\left(R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}-4R^{\mu}_{\nu}R^{\nu}_{\mu}+R^2\right)$, can only be obtained by having specific $O\left(R^2\right)$ terms in the A tensor (cf. Sec. 2.3) which are chosen such that they cancel out the $R^{\mu}_{\nu}R^{\nu}_{\mu}$ term originating from the leading order terms in A, that is $A_{\mu\nu}=\alpha\left(R_{\mu\nu}+\beta S_{\mu\nu}\right)+O\left(R^2\right)$ where $S_{\mu\nu}$ is the traceless-Ricci tensor. However, in three dimensions due to NMG form involving the $R^{\mu}_{\nu}R^{\nu}_{\mu}$ term, there is not any need to have the $O\left(R^2\right)$ terms in the A tensor, and the A tensor can take the simple linear-in-curvature form of BINMG.

Now, let us construct the action of BINMG. Using the $O\left(A^2\right)$ expansion[2] given in (2.23) for the three-dimensional generic BI gravity

$$I=\frac{2}{\kappa\alpha}\int d^3x\left[\sqrt{-\det\left(g_{\mu\nu}+A_{\mu\nu}\right)}-\sqrt{-g}\right],\tag{3.2}$$

one can get the quadratic curvature expansion for $A_{\mu\nu}=\alpha\left(R_{\mu\nu}+\beta S_{\mu\nu}\right)$ as

$$I_{O(R^2)}=\frac{1}{\kappa}\int d^3x\,\sqrt{-g}\left\{R-\frac{1}{2}\alpha\left(1+\beta\right)^2\left[R^{\alpha}_{\beta}R^{\beta}_{\alpha}-\frac{2\beta^2+4\beta+3}{6\left(1+\beta\right)^2}R^2\right]\right\}.\tag{3.3}$$

To obtain (3.1) from (3.3), first there is a need for an overall minus sign, then the dimensionful α parameter should take the value $\alpha=1/2m^2$, while the dimensionless

[1] The Gauss-Bonnet combination is identically zero in three dimensions.
[2] Note that the form of the A expansion for BI gravity does not change with dimension.

β parameter can either be -3 or 1. For $\beta = -3$, one gets the BINMG action

$$I_{\mathrm{BINMG}} = -\frac{4m^2}{\kappa^2} \int d^3x \left[\sqrt{-\det\left(g_{\mu\nu} - \frac{1}{m^2} G_{\mu\nu}\right)} - \sqrt{-g} \right], \tag{3.4}$$

where $G_{\mu\nu}$ is the Einstein tensor, that is $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. For $\beta = 1$, the BI action takes the form

$$I = -\frac{4m^2}{\kappa^2} \int d^3x \left\{ \sqrt{-\det\left[g_{\mu\nu} + \frac{1}{m^2}\left(R_{\mu\nu} - \frac{1}{6} g_{\mu\nu} R\right)\right]} - \sqrt{-g} \right\}. \tag{3.5}$$

Although both (3.4) and (3.5) reduce to NMG at the quadratic curvature order, BINMG has a more appealing form with the appearance of the Einstein tensor in the action. In addition, more important than the elegancy of (3.4), the curvature expansion of BINMG reproduces the cubic and the quartic curvature extensions of NMG [22] that are originated from ideas based on AdS/CFT while (3.5) does not yield a relevant higher curvature expansion.

As the quadratic curvature expansions of both (3.4) and (3.5) yield NMG, the theories are unitary around the flat background by construction. We study the unitarity of the two actions around the (A)dS background in the next section, but to consider unitarity around (A)dS, first one needs to introduce the bare cosmological constant simply as

$$I_{\mathrm{BINMG}} = -\frac{4m^2}{\kappa^2} \int d^3x \left[\sqrt{-\det\left(g_{\mu\nu} - \frac{1}{m^2} G_{\mu\nu}\right)} - \left(1 - \frac{\lambda_0}{2}\right)\sqrt{-g} \right]. \tag{3.6}$$

This action obviously reproduces the cosmological NMG at the quadratic curvature order and one can introduce higher curvature extensions of NMG by studying the curvature expansion of (3.6). For example, the quartic curvature expansion of this action can be found by using

$$\begin{aligned}\sqrt{\det(I+M)} =& I + \frac{1}{2}\mathrm{Tr}M + \frac{1}{8}(\mathrm{Tr}M)^2 - \frac{1}{4}\mathrm{Tr}\left(M^2\right) \\ &+ \frac{1}{6}\mathrm{Tr}\left(M^3\right) - \frac{1}{8}\mathrm{Tr}\left(M^2\right)\mathrm{Tr}M + \frac{1}{48}(\mathrm{Tr}M)^3 \\ &- \frac{1}{8}\mathrm{Tr}\left(M^4\right) + \frac{1}{32}\left[\mathrm{Tr}\left(M^2\right)\right]^2 + \frac{1}{12}\mathrm{Tr}\left(M^3\right)\mathrm{Tr}M \\ &- \frac{1}{32}(\mathrm{Tr}M)^2\,\mathrm{Tr}\left(M^2\right) + \frac{1}{384}(\mathrm{Tr}M)^4 + O\left(M^5\right).\end{aligned} \tag{3.7}$$

The following trace identities for the Einstein tensor are required for the calculation

$$
\begin{aligned}
G^{\alpha}_{\alpha} = -\frac{R}{2}, \qquad G^{\nu}_{\mu}G^{\mu}_{\nu} = R^{\nu}_{\mu}R^{\mu}_{\nu} - \frac{1}{4}R^2, \\
G^{\mu}_{\nu}G^{\nu}_{\alpha}G^{\alpha}_{\mu} = R^{\mu}_{\nu}R^{\nu}_{\alpha}R^{\alpha}_{\mu} - \frac{3}{2}R^{\nu}_{\mu}R^{\mu}_{\nu}R + \frac{3}{8}R^3, \\
G^{\mu}_{\nu}G^{\nu}_{\alpha}G^{\alpha}_{\beta}G^{\beta}_{\mu} = R^{\mu}_{\nu}R^{\nu}_{\alpha}R^{\alpha}_{\beta}R^{\beta}_{\mu} - 2R^{\mu}_{\nu}R^{\nu}_{\alpha}R^{\alpha}_{\mu}R + \frac{3}{2}R^{\nu}_{\mu}R^{\mu}_{\nu}R^2 - \frac{5}{16}R^4.
\end{aligned}
\tag{3.8}
$$

Using these identities, the quartic curvature expansion of BINMG becomes

$$
\begin{aligned}
I^{O(R^4)}_{\text{BINMG}} = \frac{1}{\kappa^2}\int d^3x\,\sqrt{-g}\Big\{ & -R - 2\lambda_0 m^2 + \frac{1}{m^2}\Big(R^{\nu}_{\mu}R^{\mu}_{\nu} - \frac{3}{8}R^2\Big) \\
& + \frac{2}{3m^4}\Big(R^{\mu}_{\nu}R^{\nu}_{\alpha}R^{\alpha}_{\mu} - \frac{9}{8}R^{\nu}_{\mu}R^{\mu}_{\nu}R + \frac{17}{64}R^3\Big) \\
& - \frac{1}{8m^8}\Big[R^{\mu}_{\nu}R^{\nu}_{\alpha}R^{\alpha}_{\beta}R^{\beta}_{\mu} - \frac{5}{3}R^{\mu}_{\nu}R^{\nu}_{\alpha}R^{\alpha}_{\mu}R \\
& \qquad + \frac{19}{16}R^{\nu}_{\mu}R^{\mu}_{\nu}R^2 - \frac{1}{4}\Big(R^{\nu}_{\mu}R^{\mu}_{\nu}\Big)^2 - \frac{169}{768}R^4\Big]\Big\},
\end{aligned}
\tag{3.9}
$$

which is the quartic curvature extension of NMG based on BINMG.

Now, let us compare (3.9) with the cubic and the quartic curvature extensions of NMG [22] based on AdS/CFT considerations. In [22], it is shown that among the three-dimensional quadratic curvature theories, NMG is singled out when one requires the existence of a *simple* (in some sense simply integrable) (Zamolodchikov's) c-function, and following the same idea the cubic and the quartic curvature extensions of NMG was introduced. Remarkably, the cubic curvature extension of [22] has the same relative coefficients of $(1, -9/8, 17/64)$ as the cubic curvature order of BINMG in (3.9). At the quartic curvature order, the set of relative coefficients of BINMG satisfies the one-parameter family of conditions defining the quartic curvature extension of [22]. This remarkable match between the curvature expansion of BINMG and the extensions of NMG that are based on the existence of the holographic c-theorem motivates the search for a simple c-function for BINMG and it turns out that BINMG indeed has a simple c-function which matches the one for Einstein's gravity [4]. We discuss this issue in the last section of this chapter.

BINMG (3.4) and its cousin (3.5) are the only unitary theories around the flat background for which the A tensor is linear in curvature. However, if one allows for the $O(R^2)$ terms in the A tensor, then it is possible to construct other three-dimensional BI gravity theories which reduce to NMG at the quadratic curvature order and, in turn, which are unitary around the flat background with a massive spin-2 excitation. For example, insisting on a A tensor which is formed solely by the Ricci tensor but not the metric and the scalar curvature, that is $A_{\mu\nu} = \alpha R_{\mu\nu} + \beta R_{\mu\rho} R_{\nu}^{\rho}$, yields a nonminimal BI gravity theory

$$I = -\frac{4m^2}{3\kappa^2} \int d^3x \left\{ \sqrt{-\det \left[g_{\mu\nu} + \frac{3}{2m^2} \left(R_{\mu\nu} - \frac{1}{4m^2} R_{\mu\rho} R_{\nu}^{\rho} \right) \right]} - \left(1 - \frac{3}{2}\lambda_0 \right) \sqrt{-g} \right\}, \tag{3.10}$$

which reduces to NMG when it is expanded up to quadratic curvature. Once the metric and the scalar curvature are allowed to appear in the $O\left(R^2\right)$ terms of the A tensor, in addition to $R_{\mu\rho} R_{\nu}^{\rho}$, the terms $g_{\mu\nu} R_{\beta}^{\alpha} R_{\alpha}^{\beta}$, $R_{\mu\nu} R$, and $g_{\mu\nu} R^2$ are the other possible terms that can be considered as a $O\left(R^2\right)$ term. In the most general case, A tensor consists of six terms, that are two linear and four quadratic in curvature terms, and one may construct four-parameter family of BI gravity theories which reproduce NMG at the quadratic curvature order.

3.2 Unitarity around (A)dS and central charge

In this section, we study the unitarity around the (A)dS background for BINMG (3.4) and its cousin (3.5), and the BI gravity theory (3.10) which is a nonminimal BI type extension of NMG in the sense that it involves quadratic curvature terms in the A tensor. In [6], it was shown that BINMG and its cousin are unitary around the (A)dS background both by calculating the $O\left(h^2\right)$ action directly and by finding the equivalent quadratic curvature action by use of the explicit trace expansion of the determinantal action. Here, we prefer to find the equivalent quadratic curvature action from the determinantal form of the BI action directly via the formulation developed in Sec. 2.2. Instead of finding the equivalent quadratic curvature action for BINMG and its cousin separately, we find equivalent quadratic curvature action for the three-dimensional BI gravity theory defined by the A tensor that is linear in curvature as $A_{\mu\nu} = \alpha \left(R_{\mu\nu} + \beta S_{\mu\nu} \right)$. In this way, it is shown that BINMG and its cousin are the only unitary BI gravity theories of this form around the (A)dS background similar to the case of the flat background. On the other hand, the nonminimal extension turns out to be nonunitary around the (A)dS background. Therefore, the unitarity around the flat background does not suggest the unitarity around the (A)dS background (see also Sec. 2.3 for the four-dimensional case), and the unitarity of BINMG and its cousin around both the flat and the (A)dS backgrounds are totally nontrivial.

Before proceeding to the analysis on the BI gravity theories, let us first recapitulate the unitarity of NMG around the (A)dS background, because to have a unitary BI gravity theory around the (A)dS background with the massive spin-2 excitation, the equivalent quadratic curvature action for the BI gravity theory should have the NMG form as in the flat background case. The cosmological NMG theory has the action

$$I = \frac{1}{\kappa^2}\int d^3x\sqrt{-g}\left[\sigma R - 2\lambda_0 m^2 + \frac{\omega}{m^2}\left(R^\mu_\nu R^\nu_\mu - \frac{3}{8}R^2\right)\right], \tag{3.11}$$

where in addition to the bare cosmological constant λ_0, $\sigma = \pm 1$ and $\omega = \pm 1$ parameters are introduced because around the (A)dS background there are unitary regions in the parameter space for both signs of the Einstein-Hilbert term and the parameter m^2. The field equation for the (A)dS vacua, $\bar{R}_{\mu\nu} = 2\lambda m^2 \bar{g}_{\mu\nu}$, of NMG is

$$\omega\lambda^2 + 4\sigma\lambda - 4\lambda_0 = 0, \tag{3.12}$$

and NMG is unitary around this (A)dS vacua when the two conditions

$$\omega\lambda - 2\sigma > 0, \tag{3.13}$$

and

$$\frac{2\sigma}{\omega} + \lambda \leq 0, \tag{3.14}$$

are satisfied (see [12] for the detailed analysis of the unitary regions in the parameter space). The former condition is required for the absence of ghosts, while the latter is required to have a nontachyonic mode and it comes from the Breitenlohner-Freedman (BF) bound [34] for AdS and the Higuchi bound [36] for dS. On the other hand, to have a unitary CFT on the boundary, the central charge of the AdS cosmological NMG, which is [12]

$$c = \frac{3\ell}{2G_3}\left(\sigma - \frac{\omega\lambda}{2}\right), \tag{3.15}$$

should be positive. Here, ℓ is the AdS length defined by $\lambda m^2 \equiv -1/\ell^2$, and G_3 is the three-dimensional Newton's constant, $\kappa^2 \equiv 16\pi G_3$. When (3.13) and (3.15) are compared, it is obvious that the bulk and the boundary unitarity for NMG is in conflict for any choice of the parameters. In fact, this conflict is one of the motivations for the extensions of NMG [22, 3]; however, this conflict is not resolved for the extensions [22, 6]. We explicitly show this for the BINMG case below.

BINMG, its cousin, and the nonminimal extension of NMG are constructed with the requirement that the quadratic curvature expansion of these theories match NMG. In fact, the quadratic curvature expansion that is found by use of (3.7) (up to the quadratic order) is nothing but the equivalent quadratic curvature action for the background $\bar{R}_{\mu\nu} = 0$. Both the quadratic curvature expansion and the equivalent quadratic curvature action are the Taylor series expansion in curvature for the BI action around the flat and the (A)dS backgrounds, respectively. Thus, vanishing effective cosmological constant limit[3] of the equivalent quadratic curvature action reduces to the quadratic curvature expansion. To have such a smooth limit for a *small* (absolute) value of the effective cosmological constant $|\lambda|$ for the (A)dS background, the equivalent quadratic curvature action for BINMG, its cousin, and the nonminimal extension should have the generic form $\frac{1}{m^2}\Big((1+c\lambda)\, R^{\nu}_{\mu}R^{\mu}_{\nu} - \frac{3}{8}(1+d\lambda)\, R^2\Big)$ at the quadratic curvature order. Here, c and d are numbers that depend on the BI theory. A BI gravity theory which has the quadratic curvature expansion in the NMG form is unitary around the (A)dS background if and only if $c = d$ holds, which yields $\frac{1}{m^2}(1+c\lambda)\Big(R^{\nu}_{\mu}R^{\mu}_{\nu} - \frac{3}{8}R^2\Big)$, so the equivalent quadratic curvature action has the desired NMG form for the (A)dS unitarity. For arbitrary values of λ, the idea is the same with the obvious adjustment that $(1+c\lambda)$ and $(1+d\lambda)$ are replaced with the functional forms $c(\lambda)$ and $d(\lambda)$ having the $\lambda \to 0$ limits $\lim_{\lambda\to 0} c(\lambda) = 1$ and $\lim_{\lambda\to 0} d(\lambda) = 1$.

Now, let us move on to the calculation of the equivalent quadratic curvature action for the three-dimensional BI gravity theory defined by the A tensor $A_{\mu\nu} = \alpha(R_{\mu\nu} + \beta S_{\mu\nu})$, so $f(R^{\mu}_{\nu})$ has the form

$$f(R^{\mu}_{\nu}) = \frac{2\sigma}{\alpha}\left[\sqrt{\det\left[\delta^{\rho}_{\nu} + \alpha\left(R^{\rho}_{\nu} + \beta S^{\rho}_{\nu}\right)\right]} - \Big(\sigma\alpha\lambda_0 m^2 + 1\Big)\right], \tag{3.16}$$

where the σ factors are introduced such that the form $(\sigma R - 2\lambda_0 m^2)$ is obtained as in the NMG action (3.11) when (3.16) is expanded in curvature around the flat background. To calculate $f_{\text{quad-equal}}(R^{\mu}_{\nu})$, one needs to find the (A)dS background values of $f(R^{\mu}_{\nu})$ and its first and second order derivatives, and they can be calculated

[3] This limit requires that the bare cosmological constant should also vanish.

through the use of the formulation in (2.78)–(2.81) as

$$
\begin{aligned}
f\left(\bar{R}^{\mu}_{\nu}\right) &= \frac{2\sigma}{\alpha}\left[\left(1+2\alpha\lambda m^{2}\right)^{3/2}-\left(\sigma\alpha\lambda_{0}m^{2}+1\right)\right],\\
\left[\frac{\partial f}{\partial R^{\alpha}_{\beta}}\right]_{\bar{R}^{\mu}_{\nu}} &= \sigma\left(1+2\alpha\lambda m^{2}\right)^{1/2}\delta^{\beta}_{\alpha},\\
\left[\frac{\partial^{2} f}{\partial R^{\rho}_{\sigma}\partial R^{\alpha}_{\beta}}\right]_{\bar{R}^{\mu}_{\nu}} &= -\sigma\alpha\left(1+2\alpha\lambda m^{2}\right)^{-1/2}\\
&\quad\times\left[(1+\beta)^{2}\delta^{\beta}_{\rho}\delta^{\sigma}_{\alpha}-\frac{1}{6}\left(2\beta^{2}+4\beta+3\right)\delta^{\beta}_{\alpha}\delta^{\sigma}_{\rho}\right].
\end{aligned}
\tag{3.17}
$$

Employing these results in (2.82) yields $f_{\text{quad-equal}}\left(R^{\mu}_{\nu}\right)$ for (3.16) as

$$
\begin{aligned}
f_{\text{quad-equal}}\left(R^{\mu}_{\nu}\right) = &-\frac{2\sigma}{\alpha}\left[\sigma\alpha\lambda_{0}m^{2}+1-\left(1+2\alpha\lambda m^{2}\right)^{-1/2}\right.\\
&\qquad\left.\times\left(1+\alpha\lambda m^{2}-\frac{1}{2}\alpha^{2}\lambda^{2}m^{4}\right)\right]\\
&+\left(1+2\alpha\lambda m^{2}\right)^{-1/2}\left(\sigma+\sigma\alpha\lambda m^{2}\right)R\\
&-\frac{1}{2}\left(1+2\alpha\lambda m^{2}\right)^{-1/2}\\
&\times\sigma\alpha\left(1+\beta\right)^{2}\left[R^{\mu}_{\nu}R^{\nu}_{\mu}-\frac{2\beta^{2}+4\beta+3}{6\left(1+\beta\right)^{2}}R^{2}\right],
\end{aligned}
\tag{3.18}
$$

whose $\lambda\to 0$ limit (with $\sigma=1$) matches the quadratic curvature expansion (3.3). To have the quadratic curvature terms in the NMG form as $R^{\mu}_{\nu}R^{\nu}_{\mu}-\frac{3}{8}R^{2}$, the parameter β can only take the values -3 or 1, which are the values for BINMG and its cousin, respectively. Therefore, BINMG and its cousin are the only (A)dS unitary BI gravity theories in the form $A_{\mu\nu}=\alpha\left(R_{\mu\nu}+\beta S_{\mu\nu}\right)$, if they satisfy the unitarity constraints corresponding to (3.13) and (3.14). Then, after putting $\alpha=-\frac{\sigma}{2m^{2}}$ value and either one of the β values in (3.18), the equivalent quadratic curvature action, $\frac{1}{\kappa^{2}}\int d^{3}x\sqrt{-g}\,f_{\text{quad-equal}}\left(R^{\mu}_{\nu}\right)$, for BINMG and its cousin can be found as

$$
I_{\text{quad-equal}}=\frac{1}{\kappa^{2}}\int d^{3}x\sqrt{-g}\left[\tilde{\sigma}R-2m^{2}\tilde{\lambda}_{0}+\frac{\tilde{\omega}}{m^{2}}\left(R^{2}_{\mu\nu}-\frac{3}{8}R^{2}\right)\right],
\tag{3.19}
$$

where, for $\sigma\lambda<1$,

$$
\begin{aligned}
\tilde{\sigma} &= \frac{\left(\sigma-\frac{1}{2}\lambda\right)}{\sqrt{1-\sigma\lambda}}, \qquad \tilde{\lambda}_{0}=\lambda_{0}-2+\frac{1}{\sqrt{1-\sigma\lambda}}\left(2-\sigma\lambda-\frac{\lambda^{2}}{4}\right),\\
\tilde{\omega} &= \frac{1}{\sqrt{1-\sigma\lambda}}.
\end{aligned}
\tag{3.20}
$$

Thus, the equivalent quadratic curvature action for BINMG and its cousin is in the cosmological NMG form (3.11) with redefined parameters. As BINMG and its cousin have the same equivalent quadratic curvature action, curiously they have the same vacua and the same spectrum which can be determined via (3.19). The vacuum equation, the unitarity constraints, and the central charge for (3.19) are the same as the ones for the cosmological NMG, but in terms of the tilded variables.

To find the (A)dS vacua of BINMG, one needs to solve

$$\tilde{\omega}\lambda^2 + 4\tilde{\sigma}\lambda - 4\tilde{\lambda}_0 = 0, \tag{3.21}$$

and using (3.20) it becomes

$$\sqrt{1-\sigma\lambda} = 1 - \frac{1}{2}\lambda_0, \tag{3.22}$$

which requires $\lambda_0 < 2$, then taking the square of the equation yields the effective cosmological constant for BINMG and its cousin as

$$\lambda = \sigma\lambda_0\left(1 - \frac{1}{4}\lambda_0\right), \qquad \lambda_0 < 2. \tag{3.23}$$

This result is found through the field equations in [4, 23] and from the equivalent linear curvature action, which is the equivalent theory of BINMG with respect to vacua, in [6]. Note that BINMG has a *unique* vacuum as opposed to double vacua of NMG, and for $\sigma = -1$ ($\sigma = 1$) there is a minimum (maximum) of $\lambda\,(\lambda_0)$ at $\lambda_0 = 2$ with the value $\lambda_{\text{min}} = -1$ ($\lambda_{\text{max}} = 1$).

The equivalent quadratic curvature action (3.19) and in turn BINMG represent a unitary theory around (A)dS background, if the conditions $\tilde{\omega}\lambda - 2\tilde{\sigma} > 0$ and $\frac{2\tilde{\sigma}}{\tilde{\omega}} + \lambda \leq 0$ are satisfied, and using (3.20) these conditions become $\lambda > \sigma$ and $\sigma \leq 0$, respectively. Then, one needs to have $\sigma = -1$, and $\lambda > \sigma$ is automatically satisfied as $\lambda_{\text{min}} = -1$ for $\sigma = -1$. Thus, if $\sigma = -1$, BINMG and its cousin are unitary both around dS and AdS for all values of λ allowed by (3.23).

Similar to NMG, the bulk and the boundary unitarity is in conflict for BINMG. The central charge for BINMG has been found in [23, 4] and it can also be calculated from (3.19) via (3.15) as

$$c = \frac{3\ell}{2G_3}\left(\tilde{\sigma} - \frac{\tilde{\omega}\lambda}{2}\right) = \frac{3\sigma\ell}{4G_3}\left(2 - \lambda_0\right), \tag{3.24}$$

which is always negative for $\sigma = -1$ which is required for bulk unitarity and for $\lambda_0 < 2$. In fact, for any theory which has an equivalent quadratic curvature action in the form of NMG, the bulk and the boundary unitarity conflict cannot be resolved because the conflict is inherited through the apparent contradiction between the no-ghost condition and the positivity of the central charge. Furthermore, for NMG, $c = 0$ value of the central charge represents a special point in the parameter space of NMG and at this point NMG has logaritmic solutions; however, for BINMG, the central charge cannot attain the value $c = 0$ due to the constraint $\lambda_0 < 2$ and this is an indication for the absence of logarithmic solutions of BINMG which was shown in [24]. In addition, let us discuss why the equivalent quadratic curvature action of BINMG (3.19) has the same central charge as BINMG. As shown in [37, 38, 39], the central charge of a higher curvature gravity theory in three dimensions can be calculated via

$$c = \frac{\ell}{2G_3} g_{\mu\nu} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu}}, \tag{3.25}$$

which was used to calculate the central charge of BINMG in [4]. As (3.25) implies, by definition the equivalent quadratic curvature action has the same central charge as BINMG.

Now, let us discuss the unitarity of the nonminimal BI extension of NMG. The function $f\left(R^{\mu}_{\nu}\right)$ can be defined from (3.10) as

$$\begin{aligned} f\left(R^{\mu}_{\nu}\right) = -\frac{4m^2}{3} \Bigg\{ & \sqrt{\det\left[\delta^{\rho}_{\nu} + \frac{3}{2m^2}\left(R^{\rho}_{\nu} - \frac{1}{4m^2} R^{\rho}_{\mu} R^{\mu}_{\nu}\right)\right]} \\ & - \left(1 - \frac{3}{2}\lambda_0\right) \Bigg\}. \end{aligned} \tag{3.26}$$

The (A)dS background values of $f\left(R^{\mu}_{\nu}\right)$ and its first and second order derivatives need to be calculated to find $f_{\text{quad-equal}}\left(R^{\mu}_{\nu}\right)$, and using (2.78)–(2.81) they can be found as

$$\begin{aligned} f\left(\bar{R}^{\mu}_{\nu}\right) &= \frac{4m^2}{3}\left[1 - \frac{3}{2}\lambda_0 - \left(1 + 3\lambda - \frac{3}{2}\lambda^2\right)^{3/2}\right], \\ \left[\frac{\partial f}{\partial R^{\alpha}_{\beta}}\right]_{\bar{R}^{\mu}_{\nu}} &= -\left(1 + 3\lambda - \frac{3}{2}\lambda^2\right)^{1/2} (1 - \lambda)\, \delta^{\beta}_{\alpha}, \\ \left[\frac{\partial^2 f}{\partial R^{\rho}_{\sigma} \partial R^{\alpha}_{\beta}}\right]_{\bar{R}^{\mu}_{\nu}} &= \frac{2}{m^2}\left(1 + 3\lambda - \frac{3}{2}\lambda^2\right)^{-1/2} \\ & \quad \times \left[\left(1 - \frac{3}{4}\lambda + \frac{3}{8}\lambda^2\right) \delta^{\sigma}_{\alpha} \delta^{\beta}_{\rho} - \frac{3}{8}(1 - \lambda)^2 \delta^{\beta}_{\alpha} \delta^{\sigma}_{\rho}\right]. \end{aligned} \tag{3.27}$$

With these results, $f_{\text{quad-equal}}\left(R^{\mu}_{\nu}\right)$ can be calculated from (2.82) as

$$f_{\text{quad-equal}}\left(R^{\mu}_{\nu}\right) = -2m^2\Big[\lambda_0 - \frac{2}{3} + \left(1+3\lambda-\frac{3}{2}\lambda^2\right)^{-1/2} \times\left(\frac{2}{3}+\lambda-\frac{5}{4}\lambda^2-\frac{3}{2}\lambda^3+\frac{3}{2}\lambda^4\right)\Big] - \left(1+3\lambda-\frac{3}{2}\lambda^2\right)^{-1/2}\left(1+\frac{3}{2}\lambda+\frac{3}{2}\lambda^2-\frac{3}{2}\lambda^3\right)R + \frac{1}{m^2}\left(1+3\lambda-\frac{3}{2}\lambda^2\right)^{-1/2} \times\left(1-\frac{3}{4}\lambda+\frac{3}{8}\lambda^2\right)\left[R^{\alpha}_{\beta}R^{\beta}_{\alpha}-\frac{3}{8}\frac{(1-\lambda)^2}{\left(1-\frac{3}{4}\lambda+\frac{3}{8}\lambda^2\right)}R^2\right]. \tag{3.28}$$

Here, the quadratic curvature terms have a form that differs from the NMG form $R^{\mu}_{\nu}R^{\nu}_{\mu}-\frac{3}{8}R^2$; therefore, (A)dS unitarity fails for the nonminimal BI extension of NMG (3.10).

To conclude, in this section we showed that BINMG (3.4) and its cousin (3.5) are unitary around the (A)dS background in addition to the flat background, and they are the only unitary BI theories whose A tensor is linear in curvature. However, for these two theories, in the region where the bulk theory is unitary around AdS, the boundary CFT becomes nonunitary.[4] Furthermore, it is shown that for the AdS spacetime solution of BINMG, the absolute value of curvature scalar has an upper bound. It is intriguing that BINMG and its cousin only differ at the interaction levels. Also, the nonminimal BI extension of NMG (3.10) is shown to be nonunitary around (A)dS. Therefore, this example demonstrates that the flat space unitarity for a BI gravity theory is necessary but not sufficient for the (A)dS unitarity.

3.3 Exact solutions

In this section, we study the solutions of BINMG which are constant scalar invariant (CSI) spacetimes of Type N and Type D. These solutions were obtained in [7] where the same type of solutions for all three-dimensional higher curvature theories were studied. Type-N and Type-D spacetimes are classified according to the particular

[4] One might be tempted to consider the vanishing central charge limit for $\lambda_0 = 2$, but then AdS is not the vacuum of the theory.

form that the traceless-Ricci tensor takes.[5] For the constant scalar invariant (CSI) spacetimes of Type N and Type D, the field equations of BINMG are greatly simplified and reduce to the field equations of TMG (in the quadratic form) and NMG. Then, the Type-N and the Type-D solutions of TMG and NMG can be used to obtain the solutions of BINMG. For TMG, these types of solutions are compiled in [40] and for NMG, various solutions of these types are found in [41, 42, 43, 44].

To study the solutions of BINMG, we first describe the Type-N and Type-D spacetimes, then we give the field equations of TMG and NMG for these spacetimes. After laying out this background, we derive the field equations of BINMG, and finally discuss the Type-N and Type-D solutions for the theory.

3.3.1 Classification of three-dimensional spacetimes

In this section, we summarize the algebraic classification of curvature in three dimensions discussed in [40]. Three-dimensional spacetimes can be classified in analogy with the Petrov and the Segre classifications of four-dimensional spacetimes. In four dimensions, the Petrov classification is based on the algebraic classification of the Weyl tensor. In three dimensions the Weyl tensor is identically zero, so the Cotton tensor can be used instead and one can classify the three-dimensional spacetimes according to the eigenvalue equation of the Cotton tensor (C^{μ}_{ν}) [45]. On the other hand, the Segre classification of four dimensions is based on the analysis of the eigenvalue equation for the traceless-Ricci tensor (S^{μ}_{ν}) and the same idea follows in three dimensions.

For the solutions of a generic three-dimensional gravity theory, the Petrov and the Segre classifications are distinct; however, for the case of TMG they coincide because the field equations of TMG relate the Cotton and the traceless-Ricci tensors. Let us first observe this point. The cosmological TMG has the action [46, 47]

$$I = -\frac{1}{\kappa}\int d^3x\sqrt{-g}\left[R - 2\Lambda + \frac{1}{2\mu}\eta^{\alpha\beta\gamma}\Gamma^{\mu}_{\alpha\nu}\left(\partial_\beta\Gamma^{\nu}_{\gamma\mu} + \frac{2}{3}\Gamma^{\nu}_{\beta\rho}\Gamma^{\rho}_{\gamma\mu}\right)\right], \qquad (3.29)$$

where $\eta_{\mu\sigma\rho}$ is the Levi-Civita tensor, $\eta_{\mu\sigma\rho} = \sqrt{-g}\varepsilon_{\mu\sigma\rho}$, with the convention $\varepsilon_{012} = +1$.

[5] The terminology of the Petrov classification which is based on the analysis of the Cotton tensor is used here, although the form of the traceless-Ricci tensor is used to classify spacetimes which is in fact the Segre classification. However, as we shall discuss below, the two classifications coincide for TMG and we stick to this choice of terminology for the cases of NMG and BINMG for which this coincidence fails in general.

The field equations for this action can be found as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0, \tag{3.30}$$

where $C_{\mu\nu}$ is the Cotton tensor defined as

$$C_{\mu\nu} \equiv \eta_{\mu\alpha\beta} \nabla^{\alpha} \left(R^{\beta}_{\nu} - \frac{1}{4} \delta^{\beta}_{\nu} R \right), \tag{3.31}$$

and it is a symmetric, traceless and covariantly conserved tensor. Taking the trace of (3.30) yields $R = 6\Lambda$; therefore, the scalar curvature is constant for cosmological TMG and with this result (3.30) can be put in the form

$$\mu S_{\mu\nu} = -C_{\mu\nu}, \tag{3.32}$$

where the traceless-Ricci tensor defined as $S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{3} g_{\mu\nu} R$. Hence, the Petrov and the Segre classifications for the solutions of cosmological TMG are identical.

To classify the solutions of TMG, one needs to determine the eigenvalues and their algebraic multiplicities for S^{μ}_{ν} and they can be determined through the use of two scalar invariants,

$$A \equiv S^{\mu}_{\nu} S^{\nu}_{\mu}, \qquad B \equiv S^{\mu}_{\nu} S^{\nu}_{\sigma} S^{\sigma}_{\mu}. \tag{3.33}$$

For the Petrov-Segre Types O, N, and III, A and B should be zero; while for the Types D and II, they are related as $A^3 = 6B^2 \neq 0$. Finally, for the Types $\mathrm{I}_{\mathbb{R}}$ and $\mathrm{I}_{\mathbb{C}}$, one has $A^3 > 6B^2$ and $A^3 < 6B^2$, respectively [40].

Here, we focus on the Type-N and the Type-D solutions of BINMG, since for Type-N and Type-D spacetimes the field equations of BINMG simplify significantly. These spacetimes are characterized by the canonical form of the traceless-Ricci tensor. For the Type-N spacetimes, the canonical form of $S_{\mu\nu}$ is

$$S_{\mu\nu} = \rho\, \xi_{\mu} \xi_{\nu}, \tag{3.34}$$

where ρ is a scalar function and ξ^{μ} is a null Killing vector [41]. For Type-D spacetimes, $S_{\mu\nu}$ has the canonical form

$$S_{\mu\nu} = p \left(g_{\mu\nu} - \frac{3}{\sigma} \xi_{\mu} \xi_{\nu} \right), \tag{3.35}$$

where p is a scalar function and ξ^{μ} is a vector of unit norm as $\xi^{\mu} \xi_{\mu} \equiv \sigma = \pm 1$. In addition, to achieve the desired simplification in the field equations of BINMG, we consider the Type-N and the Type-D solutions of BINMG which are also constant scalar invariant (CSI) spacetimes. To have a CSI spacetime of Type N, the scalar curvature should be constant. On the other hand, to have a CSI spacetime of Type D, both the scalar curvature and the scalar function p need to be a constant.[6]

3.3.2 Field equations of TMG and NMG for Type-N and Type-D spacetimes

We find Type-N and Type-D solutions of BINMG by using the corresponding solutions of cosmological TMG and NMG. The approach we used is based on the fact that the field equations of cosmological TMG[7], NMG, and BINMG reduce to the same form for the CSI spacetimes of Type N and Type D.[8] For these types of spacetimes, the trace field equations of the theories determine the value of scalar curvature in terms of the theory parameters, while the traceless field equations of the theories have the form

$$(\Box - c)\, S_{\mu\nu} = 0, \tag{3.36}$$

where c is a function of parameters of the theories. In this section, for Type-N and Type-D spacetimes, we discuss the field equations of cosmological TMG in the quadratic form and the field equations of NMG.

To discuss the field equations of TMG for Type-N and Type-D spacetimes, first we need to put the field equations of TMG in a second order wavelike equation form for the Ricci tensor. Using $R = 6\Lambda$, (3.30) can be written as

$$R_{\mu\nu} - \frac{1}{3} g_{\mu\nu} R + \frac{1}{\mu} \eta_{\mu\alpha\beta} \nabla^{\alpha} R^{\beta}_{\nu} = 0. \tag{3.37}$$

Then, multiplying this equation with $\eta^{\mu}{}_{\sigma\rho} \nabla^{\sigma}$ yields the quadratic field equations for cosmological TMG as

$$\Box R_{\mu\nu} = \mu^2 \left(R_{\mu\nu} - 2\Lambda g_{\mu\nu}\right) + 3 R_{\mu\lambda} R^{\lambda}_{\nu} - g_{\mu\nu} R^{\rho}_{\sigma} R^{\sigma}_{\rho} - \frac{3}{2} R R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R^2, \tag{3.38}$$

[6] In three dimensions, all the scalar invariants that can be constructed from the Riemann tensor can be written in terms of the scalar curvature and the two scalar invariants A and B [48]. For Type-N spacetimes, A and B are zero, while for Type-D spacetimes they become $A = 6p^2$ and $B = -6p^3$. Therefore, p must also be constant in addition to the scalar curvature to have a CSI spacetime of Type D.

[7] We mean the field equations of TMG in the quadratic form.

[8] In fact, the field equations of all the higher curvature theories in three dimensions has the same form for the CSI spacetimes of Type N and Type D [7].

upon use of the usual identity of the Levi-Civita tensor;

$$\eta^{\mu\alpha\beta}\eta_{\mu\sigma\rho} = -\left(\delta^{\alpha}_{\sigma}\delta^{\beta}_{\rho} - \delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma}\right). \tag{3.39}$$

The $\Lambda = 0$ version of (3.38) was given in [46, 47], where an operator $\mathcal{O}_{\mu\nu}{}^{\lambda\sigma}(\mu)$ was defined by writing the $\Lambda = 0$ form of (3.30) as $\mathcal{O}_{\mu\nu}{}^{\lambda\sigma}(\mu)\, R_{\lambda\sigma} = 0$ and the quadratic field equations were obtained by calculating

$$\mu^2\mathcal{O}_{\alpha\beta}{}^{\mu\nu}(-\mu)\,\mathcal{O}_{\mu\nu}{}^{\lambda\sigma}(\mu)\, R_{\lambda\sigma} = 0. \tag{3.40}$$

One may follow the same route and can define

$$\mathcal{O}_{\mu\nu}{}^{\lambda\sigma}(\mu) \equiv \delta^{\lambda}_{\mu}\delta^{\sigma}_{\nu} - \frac{1}{2}g_{\mu\nu}g^{\lambda\sigma}\left(1 - \frac{2\Lambda}{R}\right) + \frac{1}{\mu}\eta_{\mu}{}^{\alpha\beta}\left(\delta^{\lambda}_{\beta}\delta^{\sigma}_{\nu} - \frac{1}{4}g^{\lambda\sigma}g_{\nu\beta}\right)\nabla_{\alpha}, \tag{3.41}$$

for (3.30), then (3.38) can be obtained by calculating $\mu^2\mathcal{O}_{\alpha\beta}{}^{\mu\nu}(-\mu)\,\mathcal{O}_{\mu\nu}{}^{\lambda\sigma}(\mu)\, R_{\lambda\sigma} = 0$.

Now, let us express the content of the quadratic field equations of TMG as the trace and the traceless field equations which are

$$R = 6\Lambda, \tag{3.42}$$

$$\left(\Box - \mu^2 - 3\Lambda\right) S_{\mu\nu} = 3S_{\mu\rho}S^{\rho}_{\nu} - g_{\mu\nu}S_{\sigma\rho}S^{\sigma\rho}, \tag{3.43}$$

respectively, where we converted the Ricci tensor to the traceless-Ricci tensor. Note that each solution of cosmological TMG, that is the spacetime solving (3.30), also solves the quadratic field equations of TMG; however, the solutions of the quadratic field equations need not to solve (3.30).

For Type-N spacetimes, the traceless field equations (3.43) take the form

$$\Box S_{\mu\nu} = \left(\mu^2 + 3\Lambda\right) S_{\mu\nu}, \tag{3.44}$$

after using the canonical form of $S_{\mu\nu}$ given in (3.34). On the other hand, for Type-D spacetimes (3.43) becomes

$$\Box S_{\mu\nu} = \left(\mu^2 + 3\Lambda - 3p\right) S_{\mu\nu}, \tag{3.45}$$

after using the canonical form of $S_{\mu\nu}$ given in (3.35). To have a Type-D solution of TMG, the vector ξ^{μ} appearing in (3.35) should be a Killing vector and satisfy [49, 50]

$$\nabla_{\mu}\xi_{\nu} = \frac{\mu}{3}\eta_{\mu\nu\rho}\xi^{\rho}, \tag{3.46}$$

which implies that the function p needs to be constant and should satisfy

$$p = \frac{\mu^2}{9} + \Lambda. \tag{3.47}$$

One may use this relation to eliminate μ^2 in (3.45) which then becomes

$$\left[\Box + 6\left(\Lambda - p\right)\right] S_{\mu\nu} = 0. \tag{3.48}$$

For cosmological TMG, the trace field equation implies the constancy of the scalar curvature, so the Type-N solutions of TMG are CSI spacetimes. Since p should also be a constant to have the Type-D solutions of TMG, then the Type-D solutions of TMG are also CSI spacetimes.

Now, let us obtain the field equations of NMG for Type-N and Type-D spacetimes. The action for NMG is

$$I_{\text{NMG}} = -\frac{1}{16\pi G}\int d^3x\sqrt{-g}\left[R - 2\lambda_0 - \frac{1}{m^2}\left(R^\mu_\nu R^\nu_\mu - \frac{3}{8}R^2\right)\right], \tag{3.49}$$

from which the field equations can be found as [11]

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \lambda_0 g_{\mu\nu} - \frac{1}{m^2}\Box R_{\mu\nu} + \frac{1}{4m^2}\left(\nabla_\mu\nabla_\nu R + g_{\mu\nu}\Box R\right) \\ + \frac{4}{m^2}R^\alpha_\mu R_{\nu\alpha} - \frac{9}{4m^2}RR_{\mu\nu} - \frac{1}{2m^2}g_{\mu\nu}\left(3R_{\alpha\beta}R^{\alpha\beta} - \frac{13}{8}R^2\right) = 0. \end{aligned} \tag{3.50}$$

These equations can be separated into the trace and the traceless parts and rewritten in terms of the traceless-Ricci tensor as

$$S_{\mu\nu}S^{\mu\nu} + m^2 R - \frac{1}{24}R^2 = 6m^2\lambda_0, \tag{3.51}$$

and

$$\left(\Box - m^2 - \frac{5}{12}R\right)S_{\mu\nu} = 4\left(S_{\mu\rho}S^\rho_\nu - \frac{1}{3}g_{\mu\nu}S_{\sigma\rho}S^{\sigma\rho}\right) + \frac{1}{4}\left(\nabla_\mu\nabla_\nu - \frac{1}{3}g_{\mu\nu}\Box\right)R. \tag{3.52}$$

In [43, 44], the traceless field equations of NMG were given as

$$\left(\not{D}^2 - m^2\right)S_{\mu\nu} = S_{\mu\rho}S^\rho_\nu - \frac{1}{3}g_{\mu\nu}S_{\sigma\rho}S^{\sigma\rho} - \frac{R}{12}S_{\mu\nu}, \tag{3.53}$$

where the operator $\not{D}$ is defined by its action on a symmetric rank-2 tensor $\Phi_{\mu\nu}$ as

$$\not{D}\Phi_{\mu\nu} \equiv \frac{1}{2}\left(\eta_\mu{}^{\alpha\beta}\nabla_\beta\Phi_{\nu\alpha} + \eta_\nu{}^{\alpha\beta}\nabla_\beta\Phi_{\mu\alpha}\right), \tag{3.54}$$

and these two forms of the traceless field equations are totally equivalent.

For Type-N spacetimes characterized by the traceless-Ricci tensor having the canonical form (3.34), the trace field equation (3.51) takes the form

$$m^2 R - \frac{1}{24} R^2 = 6 m^2 \lambda_0, \tag{3.55}$$

which implies the constancy of the scalar curvature. Hence, the Type-N solutions of NMG are CSI spacetimes. On the other hand, the traceless field equations (3.52) become

$$\Box S_{\mu\nu} = \left(m^2 + \frac{5}{12} R \right) S_{\mu\nu}, \tag{3.56}$$

after using (3.34) and the constancy of the scalar curvature. Note that (3.44) and (3.56) are the same equation with different parametrizations which are related by

$$\mu^2 = m^2 - \frac{R}{12}. \tag{3.57}$$

In [42, 43], this fact was used to obtain the Type-N solutions of NMG inherited from the Type-N solutions of TMG. In addition, there are other Type-N solutions of NMG which solve (3.44) [or equivalently (3.56)] but not (3.32), and these solutions were also found in [42, 43].

For Type-D spacetimes characterized by the traceless-Ricci tensor having the canonical form (3.35), the trace and the traceless field equations of NMG given in (3.51) and (3.52), respectively, reduce to the forms

$$6p^2 + m^2 R - \frac{1}{24} R^2 = 6 m^2 \lambda_0, \tag{3.58}$$

and

$$\left(\Box - m^2 - \frac{5}{12} R + 4p \right) S_{\mu\nu} = \frac{1}{4} \left(\nabla_\mu \nabla_\nu - \frac{1}{3} g_{\mu\nu} \Box \right) R. \tag{3.59}$$

If one considers constant-scalar-curvature Type-D solutions, then the trace field equation will imply that the function p should also be constant, so the constant-scalar-curvature Type-D solutions of NMG are necessarily be CSI spacetimes. With the constant scalar curvature assumption, (3.59) becomes

$$\Box S_{\mu\nu} = \left(m^2 + \frac{5}{12} R - 4p \right) S_{\mu\nu}, \tag{3.60}$$

which is the same equation as (3.45) with different parametrizations related by

$$\mu^2 = m^2 - \frac{R}{12} - p. \tag{3.61}$$

Again, this fact was used to obtain the Type-D solutions of NMG that are inherited from the Type-D solutions of TMG in [41, 42]. For these solutions, the parameters m^2, p, and R should satisfy the relation

$$p = \frac{m^2}{10} + \frac{17}{120}R, \tag{3.62}$$

which can be obtained from (3.47) and (3.61). In addition to the TMG-based solutions, there are other constant-scalar-curvature Type-D solutions of NMG which solve (3.45) [or equivalently (3.60)] but not (3.32) [42, 44]. To have these solutions, the vector ξ^μ appeared in (3.35) should be either a hyper-surface orthogonal Killing vector or a covariantly divergence-free vector but not a Killing vector. Like the TMG-based Type-D solutions, the parameters m^2, p and R should satisfy specific relations to have these solutions. In the case for which ξ^μ is a hyper-surface orthogonal Killing vector, this relation is

$$p = \frac{R}{6} = \frac{2}{3}m^2, \tag{3.63}$$

which implies $\lambda_0 = m^2/5$, while in the case for which ξ^μ is a divergence-free vector, the relation is

$$p = -\frac{R}{3} = -\frac{4}{15}m^2, \tag{3.64}$$

which implies $\lambda_0 = m^2$. Note that in these relations, the two theory parameters appearing in the action, that are m^2 and λ_0, are related and these Type-D solutions of NMG found in [44] are parametrized by only m^2.

3.3.3 Field Equations of BINMG

To investigate the Type-N and the Type-D solutions of a three-dimensional higher curvature theory, it is better to write the field equations of the theory in terms of the traceless-Ricci tensor, simply because the canonical forms of the traceless-Ricci tensor given in (3.34) and (3.35) can easily be employed as an ansatz to the field equations (see [7]). Thus, we derive the field equations of BINMG in such a form.

We take the action of BINMG as

$$I_{\text{BINMG}} = -\frac{4\tilde{m}^2}{\kappa^2}\int d^3x \left[\sqrt{-\det\left(g_{\mu\nu} - \frac{1}{\tilde{m}^2}G_{\mu\nu}\right)} - \left(1 - \frac{\tilde{\lambda}_0}{2}\right)\sqrt{-g}\right], \tag{3.65}$$

where $G_{\mu\nu}$ is the Einstein tensor and we introduced tildes to the mass and the bare cosmological constant parameters to avoid confusion with the corresponding parameters of NMG. As shown in [7], the action for any three-dimensional higher curvature gravity theory can be put in the form $\int d^3x\sqrt{-g}F\left(R,A,B\right)$, where A and B are defined by $A\equiv S^\mu_\nu S^\nu_\mu$ and $B\equiv S^\mu_\rho S^\nu_\mu S^\rho_\nu$ as before. To write BINMG action in this form, one can use

$$\det A=\frac{1}{6}\left[(\mathrm{Tr}A)^3-3\mathrm{Tr}A\mathrm{Tr}\left(A^2\right)+2\mathrm{Tr}\left(A^3\right)\right], \tag{3.66}$$

which is an exact expression for 3×3 matrices and (3.65) takes the form [21]

$$I_{\mathrm{BINMG}}=-\frac{4\tilde{m}^2}{\kappa^2}\int d^3x\sqrt{-g}F\left(R,A,B\right), \tag{3.67}$$

where $F\left(R,A,B\right)$ is

$$F\left(R,A,B\right)\equiv\sqrt{\left(1+\frac{R}{6\tilde{m}^2}\right)^3-\frac{A}{2\tilde{m}^4}\left(1+\frac{R}{6\tilde{m}^2}\right)-\frac{B}{3\tilde{m}^6}}-\left(1-\frac{\tilde{\lambda}_0}{2}\right). \tag{3.68}$$

To find the field equations, taking the variation of the action yields

$$\delta I=\int d^3x\sqrt{-g}\left(F_R\delta R+F_A\delta A+F_B\delta B-\frac{1}{2}g_{\mu\nu}F\delta g^{\mu\nu}\right), \tag{3.69}$$

where F_R, F_A, and F_B for BINMG have the form

$$\begin{aligned}
F_R&\equiv\frac{\partial F}{\partial R}=\frac{1}{4\tilde{m}^2}\left(F+1-\frac{\tilde{\lambda}_0}{2}\right)^{-1}\left[\left(1+\frac{R}{6\tilde{m}^2}\right)^2-\frac{A}{6\tilde{m}^4}\right],\\
F_A&\equiv\frac{\partial F}{\partial A}=-\frac{1}{4\tilde{m}^4}\left(F+1-\frac{\tilde{\lambda}_0}{2}\right)^{-1}\left(1+\frac{R}{6\tilde{m}^2}\right),\\
F_B&\equiv\frac{\partial F}{\partial B}=-\frac{1}{6\tilde{m}^6}\left(F+1-\frac{\tilde{\lambda}_0}{2}\right)^{-1}.
\end{aligned} \tag{3.70}$$

Then, the field equations of BINMG can be found as

$$\begin{aligned}
&-\frac{1}{2}g_{\mu\nu}F+2F_AS^\rho_\mu S_{\rho\nu}+3F_BS^\rho_\mu S_{\rho\sigma}S^\sigma_\nu+\left(\Box+\frac{2}{3}R\right)\left(F_AS_{\mu\nu}+\frac{3}{2}F_BS^\rho_\mu S_{\rho\nu}\right)\\
&\qquad+\left(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu+S_{\mu\nu}+\frac{1}{3}g_{\mu\nu}R\right)\left(F_R-F_BS^\rho_\sigma S^\sigma_\rho\right)\\
&-2\nabla_\alpha\nabla_{(\mu}\left(S^\alpha_{\nu)}F_A+\frac{3}{2}S^\rho_{\nu)}S^\alpha_\rho F_B\right)+g_{\mu\nu}\nabla_\alpha\nabla_\beta\left(F_AS^{\alpha\beta}+\frac{3}{2}F_BS^{\alpha\rho}S^\beta_\rho\right)=0,
\end{aligned} \tag{3.71}$$

by using the δR, δA and δB results given in Appendix A.3. After obtaining the field equations, first one can easily verify the (A)dS solution of BINMG given in (3.23). For (A)dS spacetime $S_{\mu\nu}=0$ and then (3.71) reduces to

$$\frac{3}{2}F-RF_R=0, \tag{3.72}$$

where $R = 6\tilde{m}^2\lambda$,

$$F = (1+\lambda)^{3/2} - \left(1 - \frac{\tilde{\lambda}_0}{2}\right), \qquad F_R = \frac{1}{4\tilde{m}^2}(1+\lambda)^{1/2}, \tag{3.73}$$

and after some algebra, one gets (3.23). Furthermore, note that for Type-N spacetimes F, F_R, F_A, F_B are functions of R only since A and B are zero for Type-N spacetimes. On the other hand, for Type-D spacetimes they are functions of R and p since $A \sim p^2$ and $B \sim p^3$ for Type-D spacetimes.

Now, let us study the Type-N and the Type-D solutions of BINMG which are also solutions of the cosmological TMG or NMG. In finding these solutions, we will assume that the spacetime is CSI. This assumption implies that the scalar curvature is constant in addition to the constancy of p for Type-D spacetimes.

3.3.4 Type-N solutions

In this section, we find the field equations of BINMG for Type-N spacetimes and discuss their solutions. Note that the vector ξ^μ appearing in the canonical form of $S_{\mu\nu}$ for Type-N spacetimes given in (3.34) is a null vector; therefore, contractions of two and more traceless-Ricci tensors vanish and the field equations of BINMG given in (3.71) reduce to

$$-\frac{1}{2}g_{\mu\nu}F + \left(\Box + \frac{2}{3}R\right)(F_A S_{\mu\nu}) + \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + S_{\mu\nu} + \frac{1}{3}g_{\mu\nu}R\right)F_R \\ - 2\nabla_\alpha\nabla_{(\mu}\left(S^\alpha_{\nu)}F_A\right) + g_{\mu\nu}\nabla_\alpha\nabla_\beta\left(F_A S^{\alpha\beta}\right) = 0. \tag{3.74}$$

In addition, we are interested in CSI solutions, so the scalar curvature and, in turn, F, F_R, F_A, F_B are all constant. Furthermore, for constant scalar curvature, the Bianchi identity becomes $\nabla_\mu S^\mu_\nu = \frac{1}{6}\nabla_\nu R = 0$. Employing all of them in (3.74) yields

$$\left(\frac{1}{3}RF_R - \frac{1}{2}F\right)g_{\mu\nu} + \left(F_A\Box - \frac{1}{3}RF_A + F_R\right)S_{\mu\nu} = 0, \tag{3.75}$$

which can be separated into its trace and traceless parts as[9]

$$\frac{3}{2}F - RF_R = 0, \tag{3.76}$$

[9] The trace and the traceless field equations, (3.76) and (3.77), are also the field equations of a generic three-dimensional higher curvature gravity theory for the Type-N spacetimes with constant curvature [7].

$$\left(F_A\Box - \frac{1}{3}RF_A + F_R\right)S_{\mu\nu} = 0. \tag{3.77}$$

Here, the functions F, F_R, and F_A of BINMG, which are given in (3.68) and (3.70), become

$$F = \left(1+\frac{R}{6\tilde{m}^2}\right)^{\frac{3}{2}} - \left(1-\frac{\tilde{\lambda}_0}{2}\right), \qquad F_R = \frac{1}{4\tilde{m}^2}\left(1+\frac{R}{6\tilde{m}^2}\right)^{\frac{1}{2}},$$
$$F_A = -\frac{1}{4\tilde{m}^4}\left(1+\frac{R}{6\tilde{m}^2}\right)^{-\frac{1}{2}}, \tag{3.78}$$

by using the fact that A and B are zero for Type-N spacetimes. Note that $R > -6\tilde{m}^2$, which sets a lower bound on the scalar curvature, should be satisfied to make (3.78) consistent.

The trace field equation (3.76) determines the value of the scalar curvature in terms of the theory parameters. Using (3.78) in (3.76) and solving the resulting equation yield

$$R = -6\tilde{m}^2\tilde{\lambda}_0\left(1-\frac{\tilde{\lambda}_0}{4}\right), \qquad \tilde{\lambda}_0 < 2, \tag{3.79}$$

which is the same as the (A)dS result given in (3.23). Note that R cannot attain the value $R = -6\tilde{m}^2$ which will be important in discussing the solutions of the traceless field equations below.

On the other hand, after using (3.78) in (3.77) the traceless field equations of BINMG for Type-N spacetimes become

$$\left(\Box - \tilde{m}^2 - \frac{R}{2}\right)S_{\mu\nu} = 0. \tag{3.80}$$

After simply setting $\tilde{m}^2 = \mu^2$ and $R \equiv 6\Lambda$, these equations are the same as the traceless field equations of TMG in the quadratic form for Type-N spacetimes given in (3.44). Therefore, the Type-N solutions of TMG which solve $\eta_{\mu\alpha\beta}\nabla^\alpha S^\beta_\nu + \tilde{m}S_{\mu\nu} = 0$ are also solutions of BINMG for the constant curvature value given in (3.79). In [43], for a negative constant curvature $R \equiv -6n^2$, the metric solving (3.80) was found as

$$ds^2 = d\rho^2 + \frac{2}{n^2-\beta^2}dudv + \left(Z\left(u,\rho\right) - \frac{v^2}{n^2-\beta^2}\right)du^2, \tag{3.81}$$

where β is either $\beta = n\tanh\left(n\rho\right)$ or $\beta = n\coth\left(n\rho\right)$, and the function $Z\left(u,\rho\right)$ has the form

$$Z(u,\rho)=\frac{1}{\sqrt{n^2-\beta^2}}\Big(\cosh(\tilde{m}\rho)F_1(u)+\sinh(\tilde{m}\rho)F_2(u)$$
$$+\cosh(n\rho)f_1(u)+\sinh(n\rho)f_2(u)\Big). \quad (3.82)$$

Here, $F_1(u)$, $F_2(u)$, $f_1(u)$, and $f_2(u)$ are arbitrary functions of u. The Type-N solutions of BINMG inherited from the Type-N solutions of TMG are also involved in (3.81), and they can be obtained by setting $F_1(u)=\pm F_2(u)$. In addition, the AdS-wave solution of BINMG found in [24] can also be obtained from (3.81) by having the limit $\beta^2\to n^2$ for which $\xi_\mu=\partial_v$ becomes a null-Killing vector (for the details, see [43]), and then the metric for the AdS-wave solution has the form

$$ds^2=d\rho^2+2e^{2n\rho}dudv+e^{n\rho}\Big(\cosh(\tilde{m}\rho)F_1(u)+\sinh(\tilde{m}\rho)F_2(u)$$
$$+e^{n\rho}f_1(u)+e^{-n\rho}f_2(u)\Big)du^2. \quad (3.83)$$

Furthermore, for $\tilde{m}^2=0$ and $n^2=\tilde{m}^2$, (3.80) has special solutions that require different $Z(u,\rho)$ functions; however, these parameter values are not possible for the case of BINMG since for $\tilde{m}^2=0$ BINMG cannot be defined and the value $n^2=\tilde{m}^2$ cannot be attained for BINMG. Since $n^2=\tilde{m}^2$ is not possible, the logarithmic solutions of BINMG are absent which was also demonstrated in [24].

3.3.5 Type-D solutions

To derive the field equations of BINMG for Type-D spacetimes, first note that the rank $(0,2)$ tensors $S_\mu^\rho S_{\rho\nu}$ and $S_\mu^\rho S_{\rho\sigma}S_\nu^\sigma$ appearing in (3.71) can be written as

$$S_\mu^\rho S_{\rho\nu}=p(2pg_{\mu\nu}-S_{\mu\nu}), \qquad S_\mu^\rho S_{\rho\sigma}S_\nu^\sigma=p^2(3S_{\mu\nu}-2pg_{\mu\nu}), \quad (3.84)$$

by using the canonical form of $S_{\mu\nu}$ given in (3.35). Therefore, the only rank $(0,2)$ tensors that can appear in the field equations of BINMG for Type-D spacetimes are the metric and the traceless-Ricci tensor. Using (3.84), the field equations of BINMG given in (3.71) take the form

$$\begin{aligned}\left(-\frac{1}{2}F+4p^2F_A-6p^3F_B\right)g_{\mu\nu}+\left(-2pF_A+9p^2F_B\right)S_{\mu\nu}\\ +\left(\Box+\frac{2}{3}R\right)\left[3p^2F_Bg_{\mu\nu}+\left(F_A-\frac{3}{2}pF_B\right)S_{\mu\nu}\right]\\ +\left(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu+S_{\mu\nu}+\frac{1}{3}g_{\mu\nu}R\right)\left(F_R-6p^2F_B\right)\\ -2\nabla_\alpha\nabla_{(\mu}\left[\delta^\alpha_{\nu)}3p^2F_B+S^\alpha_{\nu)}\left(F_A-\frac{3}{2}pF_B\right)\right]\\ +g_{\mu\nu}\nabla_\alpha\nabla_\beta\left[3p^2F_Bg^{\alpha\beta}+\left(F_A-\frac{3}{2}pF_B\right)S^{\alpha\beta}\right]=0.\end{aligned}\tag{3.85}$$

Furthermore, we consider the solutions of BINMG which are CSI spacetimes; therefore, the scalar curvature and the functions p, F, F_R , F_A, and F_B are all constant. Using the constancy of these terms and the Bianchi identity, that is $\nabla_\mu S^\mu_\nu=\frac{1}{6}\nabla_\nu R=0$, (3.85) reduces to

$$\begin{aligned}0=\left[-\frac{1}{2}F+\frac{1}{3}RF_R+4p^2\left(F_A-\frac{3}{2}pF_B\right)\right]g_{\mu\nu}\\ +\left[F_R+\left(F_A-\frac{3}{2}pF_B\right)\left(\Box-\frac{1}{3}R+4p\right)\right]S_{\mu\nu},\end{aligned}\tag{3.86}$$

which can be separated into its trace and traceless parts as[10]

$$\frac{3}{2}F-RF_R-6p^2\left(2F_A-3pF_B\right)=0,\tag{3.87}$$

$$\left[F_R+\left(F_A-\frac{3}{2}pF_B\right)\left(\Box-\frac{1}{3}R+4p\right)\right]S_{\mu\nu}=0.\tag{3.88}$$

Here, the functions F, F_R, F_A, and F_B of BINMG, which are given in (3.68) and (3.70), become

$$\begin{aligned}F=\sqrt{\left(1+\frac{R}{6\tilde{m}^2}+\frac{2}{\tilde{m}^2}p\right)\left(1+\frac{R}{6\tilde{m}^2}-\frac{p}{\tilde{m}^2}\right)^2}-\left(1-\frac{\tilde{\lambda}_0}{2}\right),\\ F_R=\frac{1}{4\tilde{m}^2}\left(F+1-\frac{\tilde{\lambda}_0}{2}\right)^{-1}\left[\left(1+\frac{R}{6\tilde{m}^2}\right)^2-\frac{p^2}{\tilde{m}^4}\right],\\ F_A=-\frac{1}{4\tilde{m}^4}\left(F+1-\frac{\tilde{\lambda}_0}{2}\right)^{-1}\left(1+\frac{R}{6\tilde{m}^2}\right),\qquad F_B=-\frac{1}{6\tilde{m}^6}\left(F+1-\frac{\tilde{\lambda}_0}{2}\right)^{-1},\end{aligned}\tag{3.89}$$

by using $A=6p^2$ and $B=-6p^3$, and the consistency of (3.89) requires $R\neq 6\left(p-\tilde{m}^2\right)$ and $R>-6\left(\tilde{m}^2+2p\right)$. Employing (3.89) in (3.87) and (3.88) yields the final forms of the field equations of BINMG for CSI Type-D spacetimes as

$$\left(F+1-\frac{\tilde{\lambda}_0}{2}\right)^{-1}\left[\left(1+\frac{R}{6\tilde{m}^2}\right)^2-\frac{p^2}{\tilde{m}^4}\right]-\left(1-\frac{\tilde{\lambda}_0}{2}\right)=0,\tag{3.90}$$

[10] The trace and the traceless field equations, (3.87) and (3.88), are also the field equations of a generic three-dimensional higher curvature gravity theory for the Type-D spacetimes with constant R and p [7].

$$\left(\Box - \tilde{m}^2 - \frac{R}{2} + 3p\right) S_{\mu\nu} = 0. \tag{3.91}$$

Here, the traceless field equations (3.91) are the same equations as the traceless field equations of TMG in the quadratic form for Type-D spacetimes given in (3.45) after setting $\tilde{m}^2 = \mu^2$. Therefore, the Type-D solutions of TMG compiled in [40] are also solutions of BINMG with the constant scalar curvature satisfying (3.90). For a negative constant scalar curvature value $R \equiv -6n^2$, the metrics solving (3.91) are

$$ds^2 = -\left(dt + \frac{6\tilde{m}}{\tilde{m}^2 + 27n^2} \cosh\theta \, d\phi\right)^2 + \frac{9}{\tilde{m}^2 + 27n^2} \left(d\theta^2 + \sinh^2\theta \, d\phi^2\right), \tag{3.92}$$

which has a timelike Killing vector, and

$$ds^2 = \frac{9}{\tilde{m}^2 + 27n^2} \left(-\cosh^2\rho \, d\tau^2 + d\rho^2\right) + \left(dy + \frac{6\tilde{m}}{\tilde{m}^2 + 27n^2} \sinh\rho \, d\tau\right)^2, \tag{3.93}$$

which has a spacelike Killing vector. For both of these metrics, $\tilde{m}^2$, p, and R should satisfy

$$p = \frac{\tilde{m}^2}{9} + \frac{R}{6}, \tag{3.94}$$

which can be obtained from (3.47) by setting $\mu^2 = \tilde{m}^2$.

Now, let us find the scalar curvature for the TMG-based Type-D solutions of BINMG. Putting (3.94) in the trace field equation (3.90) yields

$$\sqrt{\frac{11}{9} + \frac{R}{2\tilde{m}^2}} = \left(1 - \frac{\tilde{\lambda}_0}{2}\right)^{-1} \left(\frac{10}{9} + \frac{R}{3\tilde{m}^2}\right), \tag{3.95}$$

which requires $R > -\frac{22\tilde{m}^2}{9}$ and $\tilde{\lambda}_0 < 2$, and then solving this equation yields the scalar curvature as

$$R = \frac{9}{16}\tilde{m}^2 \left[\left(\tilde{\lambda}_0^2 - 4\tilde{\lambda}_0 - \frac{52}{27}\right) \pm \left(\tilde{\lambda}_0 - 2\right) \sqrt{\left(\tilde{\lambda}_0 - \frac{2}{9}\right)\left(\tilde{\lambda}_0 - \frac{34}{9}\right)}\right], \tag{3.96}$$

where $\tilde{\lambda}_0 \leq 2/9$. Thus, the metrics (3.92) and (3.93), where $n^2 = -R/6$ has the value (3.96), are the CSI Type-D solutions of BINMG which are inherited from the Type-D solutions of TMG.

Now, let us consider the CSI Type-D solutions of BINMG which are also constant curvature Type-D solutions of NMG but not TMG; i.e. these solutions solve (3.91) but not $\eta_{\mu\alpha\beta}\nabla^\alpha S^\beta_\nu + \tilde{m} S_{\mu\nu} = 0$. For these solutions, the parameters of NMG satisfy specific relations which are (3.63) and (3.64). Let us start with the latter relation which corresponds to the case in which ξ_μ appearing in (3.35) is a covariantly divergence-free vector which is not a Killing vector. Employing (3.64) in the traceless field equations of NMG (3.60) and BINMG (3.91) reduce these equations to $\Box S_{\mu\nu} = 3RS_{\mu\nu}$ and $\Box S_{\mu\nu} = \left(\tilde{m}^2 + 3R/2\right) S_{\mu\nu}$, respectively. If one requires that the Type-D solution of NMG also solves BINMG, then the value of scalar curvature for the BINMG solution immediately follows from these two equations as $R = \frac{2}{3}\tilde{m}^2$. In addition, the relation between the mass parameters of NMG and BINMG can be found as $m^2 = \frac{5}{6}\tilde{m}^2$ via (3.64). Furthermore, after putting $p = -\frac{R}{3} = -\frac{2}{9}\tilde{m}^2$ in the trace field equation (3.90), one can determine $\tilde{\lambda}_0$ as $\tilde{\lambda}_0 = 2 - \frac{8\sqrt{6}}{9}$. To write the metric that solves the BINMG field equations, one just needs $m^2 = \frac{5}{6}\tilde{m}^2$ as the solutions given in [44] are parametrized with m. Then, with the solutions in [44], the following two metrics are also solutions of BINMG:

$$ds^2 = -d\tau^2 + e^{\frac{2}{\sqrt{3}}\tilde{m}\tau} dx^2 + e^{-\frac{2}{\sqrt{3}}\tilde{m}\tau} dy^2, \tag{3.97}$$

$$ds^2 = \cos\left(\frac{2}{\sqrt{3}}\tilde{m}x\right)\left(-dt^2 + dy^2\right) + dx^2 + 2\sin\left(\frac{2}{\sqrt{3}}\tilde{m}x\right) dtdy. \tag{3.98}$$

On the other hand, if one considers the parameter relation (3.63) for which ξ_μ is a hyper-surface orthogonal Killing vector, then the traceless field equations of NMG (3.60) and BINMG (3.91) reduce to $\Box S_{\mu\nu} = 0$ and $\Box S_{\mu\nu} = \tilde{m}^2 S_{\mu\nu}$, respectively. Then, the Type-D solution of NMG also solves BINMG if and only if $\tilde{m}^2 = 0$; however, for this value of $\tilde{m}^2$, BINMG is not defined. Hence, BINMG does not have a constant scalar curvature Type-D solution with a hypersurface orthogonal Killing vector just as TMG [51].

3.4 Holographic c-theorem

In this section, we show that with the use of holography and the null-energy condition, that is for an arbitrary null vector ζ the energy-momentum tensor $T_{\mu\nu}$ satisfies $T_{\mu\nu}\zeta^\mu\zeta^\nu \geq 0$, one can define a c-function for BINMG which is equal to the c-function of Einstein's gravity. In addition, it is shown that at the fixed point of the renormalization group flow, the value of the c-function concurs with the central charge of the Virasoro algebra and the coefficient of the Weyl anomaly up to a constant. To allow for both signs of the Einstein-Hilbert term at the leading order of the curvature expansion, we take the BINMG action in the form

$$I_{\rm BINMG} = -\frac{4m^2}{\kappa^2}\int d^3x \left[\sqrt{-\det\left(g_{\mu\nu}+\frac{\sigma}{m^2}G_{\mu\nu}\right)} - \left(1-\frac{\lambda_0}{2}\right)\sqrt{-g}\right], \qquad (3.99)$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ and $\sigma = \pm 1$. The field equations of this action are the same as (3.71) except for one adjustment, that each $1/m^2$ factor in (3.70) should be multiplied with σ. Now, let us investigate the c-function of the BINMG theory, and compute the Weyl anomaly coefficient.

3.4.1 c-functions

The approach of using holography to define a c-theorem for the boundary field theory via the bulk gravity theory first used in [52, 53]. This approach was recently applied to NMG in [22] where the c-function for NMG was constructed. In addition, cubic and quartic curvature extensions of NMG were found by requiring the existence of a c-theorem [22]. As these extensions match the cubic and the quartic curvature expansions of BINMG, it is only natural to ask whether a c-theorem for BINMG exists or not. The answer is affirmative as shown in [4] whose analysis is given below.

Before moving to the construction of c-function for BINMG, let us discuss why the existence of a c-theorem is important. In [54], Zamolodchikov proved that for two-dimensional quantum field theories, there exists a positive function c which depends on coupling constants and decreases monotonically under the RG flow, and the c-function becomes stationary at any fixed point of the RG flow where it takes a value that coincides with the central charge of the conformal field theory that corresponds to the fixed point. The significance of the c-theorem is due to the fact that it is a proof of irreversibility for the RG flow. It is important to extend this result beyond two dimensions, and AdS/CFT correspondence provides a framework for a D-dimensional construction [52, 53].[11] Since the radial coordinate of AdS spacetime can be considered as a measure of energy in context of the AdS/CFT correspondence, the bulk solution interpolating from $r \to \infty$ to $r \to -\infty$ corresponds to a construction of RG flow for the boundary field theory from UV fixed point to IR fixed point. To construct the

[11] For a recent important achievement in extending c-theorem to higher dimensions, see [55] where the c-theorem in four dimensions was proved by using field theoretic considerations.

c-function, it is assumed that the bulk theory is well described by classical gravity and the null-energy condition, $T_{\mu\nu}\zeta^{\mu}\zeta^{\nu} \geq 0$, is satisfied.

Let us recapitulate the discussion of [52] where the bulk theory is Einstein's gravity. We work in three dimensions and to have the Poincaré symmetry for the boundary theory, we take the metric ansatz

$$ds^2 = e^{2A(r)}\left(-dt^2 + dx^2\right) + dr^2, \tag{3.100}$$

which is the most general metric satisfying this symmetry and becomes AdS spacetime for $A(r) = r/\ell$, where ℓ is the AdS length $\ell \equiv \sqrt{|\Lambda|}$. We consider a geometry which is AdS at both UV and IR, so $A(r)$ should be linear in r as $r \to \pm\infty$. Putting (3.100) in the field equations of Einstein's gravity, one can calculate that

$$\frac{2A''}{\kappa^2} = T^t_t - T^r_r \leq 0, \tag{3.101}$$

where the inequality follows from the null-energy condition. As $A'' \leq 0$, the function $c(r) = c_0/A'(r)$ is a monotonic function which decreases from UV $(r \to \infty)$ to IR $(r \to -\infty)$ for a positive constant c_0 which can be chosen such that it matches the central charge of the Virasoro algebra [56] at the boundary of AdS;

$$c(r) \equiv \frac{24\pi}{\kappa^2 A'(r)} = \frac{3}{2G_3 A'(r)}. \tag{3.102}$$

Therefore, one has a holographic realization of two-dimensional c-theorem as the c-function (3.102) is positive and monotonic, and as $r \to \infty$, it attains a value which matches the central charge of the boundary CFT.

Now, let us move to the BINMG case. Putting the metric ansatz (3.100) in the field equations of BINMG (3.71) and using the null-energy condition yield

$$\left(\frac{2m}{\kappa^2}\right)\frac{\left[A'' + (A')^2 + \sigma m^2\right]A''}{\sqrt{\left[m^2 + \sigma(A')^2\right]\left[A'' + (A')^2 + \sigma m^2\right]^2}} = T^t_t - T^r_r \leq 0, \tag{3.103}$$

where m is assumed to be positive.[12] The left-hand side expression is finite and real, if the constraints

$$\left[A'' + (A')^2 + \sigma m^2\right] \neq 0, \qquad m^2 + \sigma(A')^2 > 0, \tag{3.104}$$

[12] Note that here we consider a matter source satisfying null-energy condition, so on the right-hand side of (3.71), one should place the energy-momentum tensor which comes with the factor $-\frac{\kappa^2}{8m^2}$.

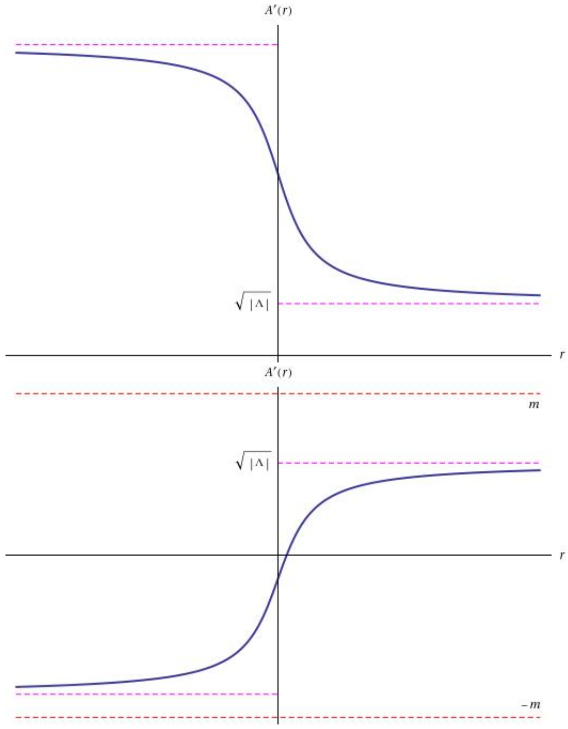

Figure 3.1: The first graph gives a possible behavior for $A'(r)$ in the $\sigma = +1$ case, while the second one shows a possible behavior for $A'(r)$ in the $\sigma = -1$ case.

are satisfied. The first constraint implies that the expression $\left[A'' + (A')^2 + \sigma m^2\right]$ cannot change sign for $r \in \mathbb{R}$; therefore, to satisfy the inequality in (3.103) A'' should also not change sign. Thus, A' has a monotonic behavior which is crucial in constructing the c-function. On the other hand, the implications of the second constraint depends on σ, and it is better to discuss each value of σ separately:

- $\sigma = +1$ **case:** The second constraint is automatically satisfied for this case. In addition, one can decide on the sign of A'' by observing that for $A'' \geq 0$ the term $\left[A'' + (A')^2 + m^2\right]$ is positive which is in conflict with (3.103), so $A'' \leq 0$ should hold with the constraint $A'' > -\left[(A')^2 + m^2\right]$. As we consider a spacetime which is AdS at both $r \to \infty$ (the UV region) and $r \to -\infty$ (the IR region), then $A(r)$ asymptotically has the linear behavior $A(r) \sim r\sqrt{|\Lambda|}$ as $r \to \pm\infty$ with $\sqrt{|\Lambda_{\text{UV}}|} < \sqrt{|\Lambda_{\text{IR}}|}$. Overall, for $\sigma = +1$, A' is a positive monotonically-decreasing function which has the behavior given in the first graph of Fig.3.1.

- $\sigma = -1$ **case:** The second constraint, that is $m^2 - (A')^2 > 0$, implies that A'

is a bounded function as $m > A' > -m$. Since A' is monotonic and bounded, A'' should become zero as $r \to \pm\infty$. In addition, to decide on the sign of A'', observe that for $A'' \leq 0$ the term $\left[A'' + (A')^2 - m^2\right]$ is negative due to the second constraint and this result is in conflict with (3.103), so $A'' \geq 0$ should hold with the bound $A'' < m^2 - (A')^2$. We consider a spacetime which is AdS as $r \to \pm\infty$ and due to the monotonically increasing nature of A', an interesting possibility appears: As $r \to +\infty$, A' may attain a positive constant value and as $r \to -\infty$, A' may attain a negative constant value, so it is possible to have a spacetime with two AdS "boundaries". To sum up, for $\sigma = -1$, A' is a monotonically increasing function in the interval $m > A' > -m$. A possible form for A', which yields a spacetime with two AdS boundaries, is given in the second graph of Fig.3.1.

As a result, the null-energy condition implies that $\sigma A'' \leq 0$; therefore, the c-function for BINMG can be defined as

$$c(r) \equiv \frac{3\sigma}{2G_3 A'(r)} \quad \Rightarrow \quad \frac{dc}{dr} \geq 0, \tag{3.105}$$

which is the same as the c-function for Einstein's gravity. As $r \to \infty$, the value of the c-function matches the central charge of the boundary CFT (3.24) and the Weyl anomaly coefficient, which is discussed in the next section, up to a factor.

In addition to (3.105), one can define another c-function for BINMG, as the analysis of (3.103) also implies the inequality

$$\frac{\sigma A''}{\sqrt{m^2 + \sigma (A')^2}} \leq 0. \tag{3.106}$$

The second c-function for BINMG can be defined as

$$c(r) \equiv -\sigma \arctan\left(\frac{A'(r)}{\sqrt{m^2 + \sigma (A')^2}}\right), \tag{3.107}$$

where the minus sign is introduced to make $c(r)$ monotonically increasing function. For $\sigma = -1$, the derivative of (3.107) directly gives the form in (3.106), while for $\sigma = +1$, there is a positive factor of $m^2/\left[m^2 + 2(A')^2\right]$ which does not effect the inequality.[13] To understand the relevance of this c-function, one needs to consider

[13] For $\sigma = +1$, one can define the c-function as $c(r) = \ln\left[A' + \sqrt{m^2 + (A')^2}\right]$ by directly integrating the form in (3.106). However, we prefer to give a single c-function for both values of σ.

the expansion of (3.106) in $(A'/m)^2$. At the desired order, this expansion gives the c-functions that were found in [22], but in the forms where the arbitrary coefficients in [22] are fixed by the curvature expansion of BINMG.

3.4.2 Weyl anomaly

Now, by using holographic techniques let us calculate the Weyl anomaly[14] for the boundary CFT theory that is assumed to be dual of BINMG. As shown in [58], to find the holographic Weyl anomaly, one needs to calculate the action of the bulk gravity theory for the Einstein metric which induces a conformal structure on the boundary and the logarithmic divergence that appears in this result yields the Weyl anomaly for the boundary CFT. Furthermore, note that for a two-dimensional CFT the Weyl anomaly coefficient a, which is defined by $g^{\mu\nu}\langle T_{\mu\nu}\rangle = a\,R$, and the central charge of the Virasoro algebra are related as $c = 24\pi a$. Therefore, one can obtain the c charge of the two-dimensional boundary CFT through a holographic calculation of the Weyl anomaly from the three-dimensional bulk gravity theory.

To find the Weyl anomaly for BINMG, we follow [59] and calculate the BINMG action for the Euclidean AdS_3 metric

$$ds^2 = \frac{dr^2}{1+\frac{r^2}{\ell^2}} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{3.108}$$

whose boundary is S_2. This metric solves the field equations of BINMG, if $\ell^2 \equiv -\frac{1}{\lambda m^2}$ satisfies (3.23). One can infer from (3.23) that AdS spacetime is possible if the bare cosmological constant λ_0 is in the interval $2 > \lambda_0 > 0$ for $\sigma = -1$ or $\lambda_0 < 0$ for $\sigma = +1$. Putting (3.108) in the BINMG action (3.99) yields

$$I_{\text{BINMG}} = -\frac{16\pi\ell m^2}{\kappa^2}\left[\left(1+\frac{\sigma}{\ell^2 m^2}\right)^{3/2} - \left(1-\frac{\lambda_0}{2}\right)\right]\int_0^r d\bar{r}\,\frac{\bar{r}^2}{\sqrt{\ell^2+\bar{r}^2}}, \tag{3.109}$$

and after calculating the integral and using (3.22), one gets

$$I_{\text{BINMG}} = \frac{16\pi\ell m^2\lambda_0}{\kappa^2}\left(1-\frac{\lambda_0}{4}\right)\left(1-\frac{\lambda_0}{2}\right)\frac{r^2}{2} \times\left[\sqrt{1+\left(\frac{\ell}{r}\right)^2} - \left(\frac{\ell}{r}\right)^2 \operatorname{arcsinh}\left(\frac{r}{\ell}\right)\right]. \tag{3.110}$$

[14] For a nice review on Weyl anomaly, see [57].

Note that this result is valid for both values of σ as $\text{sign}(\lambda_0) = -\sigma$. Here, we are interested in the logarithmic term that appears in the asymptotic expansion of (3.110) as $r \to \infty$ and it can be found as

$$\frac{4\pi\sigma\ell}{\kappa^2}(2-\lambda_0)\ln\left(\frac{2r}{\ell}\right), \tag{3.111}$$

after using (3.23) and the expansion

$$\sqrt{1+x^2} - x^2 \text{arcsinh}\left(x^{-1}\right) = 1 + \frac{1}{2}x^2\left[1 + 2\ln\left(\frac{x}{2}\right)\right] + O\left(x^3\right). \tag{3.112}$$

As discussed in [59], the coefficient of the logarithmic term is equal to $c/3$, so one arrives at the same central charge result given in (3.24) which is obtained via the method of equivalent quadratic curvature action.

CHAPTER 4

CONCLUSIONS

In this thesis, we developed the tools to analyze the unitarity of the D-dimensional generic Born-Infeld (BI) gravity theory around the (A)dS background. In addition, we studied a remarkable example of the Born-Infeld gravity theories which is the Born-Infeld extension of new massive gravity (BINMG) in detail.

To analyze the unitarity of BI gravity theories around the (A)dS background, we developed two general techniques. In constructing a modified gravity theory, the absence of ghosts and tachyons in the spectrum is an important consistency criteria. For BI gravity theories, checking this criteria around the flat background is rather simple compared to the (A)dS background because only up to quadratic curvature terms determine the flat space unitarity, while for the (A)dS space case all the terms in the curvature expansion of the BI gravity effect the unitarity in principle. Thus, constructing a unitary BI gravity theory around flat backgrounds is fairly easy and there are examples starting with the original paper of Deser and Gibbons [8]. On the other hand, without the techniques we developed such a construction is not possible for the (A)dS background. In [6], using these techniques, the three-dimensional BINMG theory is shown to be the first example of the unitary BI gravity theory around the (A)dS background. Hopefully, these techniques enable one to construct further higher-dimensional BI gravity theories around the (A)dS background.

The first of the techniques which can be applied to any BI gravity theory is based on finding the $O\left(h^2\right)$ action which represents the free-theory of the excitations in the theory. The $O\left(h\right)$ and $O\left(h^2\right)$ actions for the generic BI gravity given in (2.18) and (2.21), respectively, provide the required formulation to find the vacua and the spectrum for a specific BI gravity once the $O\left(h^2\right)$ expansion of the A tensor is found. On the other hand, the forms of $O\left(h\right)$ and $O\left(h^2\right)$ actions for the generic BI gravity reveal the remarkable fact that the even-dimensional BI gravity theory and its $O\left(A^{D/2}\right)$ expansion are equivalent with respect to vacua and spectrum. This equivalence may prove to be useful in constructing an even dimensional unitary BI gravity theory around the (A)dS background, since one needs to consider a finite order in curvature theory, the $O\left(A^{D/2}\right)$ expansion, instead of the original BI gravity which is infinite order in curvature.

The second general technique is based on the construction of an equivalent quadratic curvature action which have the same vacua and the same spectrum as the BI gravity. The construction of the equivalent quadratic curvature action just involves calculating the Taylor series expansion in curvature for the Lagrangian around the, yet to be found, (A)dS background. Differing form [5], the equivalent quadratic curvature action is calculated from the original determinantal form of the BI action using the formulation in (2.78–2.81) and in this way the technique becomes more streamlined.

When the two techniques are compared in application, the use of the $O\left(h\right)$ and $O\left(h^2\right)$ actions for the generic BI gravity is rather involved than finding the equivalent quadratic curvature action because the metric perturbation, h, expansions of the curvature tensors yield rather complicated expressions. In both of the techniques, the final expressions are analyzed by using the existing unitarity discussions of the generic quadratic curvature theories in [15].

To conclude, the techniques that we developed can be straightforwardly applied to analyze unitarity of a specific BI gravity around the (A)dS background. Furthermore, they are vital for a construction of a consistent BI gravity theory which is unitary around the (A)dS background.

In considering BINMG, we discussed the construction of the theory in detail. Then, using the techniques we developed, we demonstrated the unitarity of BINMG around the (A)dS backgrounds. In addition, the Type-N and the Type-D solutions of BINMG were found by using the Type-N and the Type-D solutions of TMG and NMG. Finally, the two c-functions of BINMG were calculated.

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APPENDIX A

EXPANSIONS IN METRIC PERTURBATION

A.1 Second Order Expansions of Curvature Tensors

One of the approaches we use to analyze the spectrum of a BI gravity theory is to calculate the second order action in metric perturbation. For such a calculation, the second order expansions of curvature tensors are required and we derive them here. The metric perturbation $h_{\mu\nu}$ is defined as

$$g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \tau h_{\mu\nu}, \tag{A.1}$$

where $\bar{g}_{\mu\nu}$ is the background metric and τ is introduced for bookkeeping purposes. In considering expansions in $h_{\mu\nu}$, one assumes that it is possible to find a coordinate frame where components of $h_{\mu\nu}$ are small (in absolute value) compared to the components of $\bar{g}_{\mu\nu}$. Surely, the (second order) expansions of curvature tensors in $h_{\mu\nu}$ have already been considered in the literature, but here we derive the expansions relevant for us and they are in the forms used in the subsequent calculations.

First, the second order expansion of the inverse metric can be found as

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \tau h^{\mu\nu} + \tau^2 h^{\mu\rho} h^{\nu}_{\rho} + O\left(\tau^3\right). \tag{A.2}$$

With this result, the Levi-Civita connection for $g_{\mu\nu}$ can be expanded up to second order in $h_{\mu\nu}$ as

$$\Gamma^{\rho}_{\mu\nu} = \bar{\Gamma}^{\rho}_{\mu\nu} + \tau \left(\Gamma^{\rho}_{\mu\nu}\right)_L - \tau^2 h^{\rho}_{\beta} \left(\Gamma^{\beta}_{\mu\nu}\right)_L + O\left(\tau^3\right), \tag{A.3}$$

where $\left(\Gamma^{\rho}_{\mu\nu}\right)_L$ is the linearized Christoffel connection defined as

$$\left(\Gamma^{\rho}_{\mu\nu}\right)_L \equiv \frac{1}{2} \bar{g}^{\rho\lambda} \left(\bar{\nabla}_\mu h_{\nu\lambda} + \bar{\nabla}_\nu h_{\mu\lambda} - \bar{\nabla}_\lambda h_{\mu\nu}\right), \tag{A.4}$$

and $\bar{\Gamma}^{\rho}_{\mu\nu}$ is the Levi-Civita connection for the background metric.

After finding the expansion for the Levi-Civita connection, we can derive our main result, namely, the second order expansion of the Riemann tensor from which we find the relevant expansions for the Ricci tensor and the scalar curvature. A variation in the connection, that is $\delta\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} - \bar{\Gamma}^{\rho}_{\mu\nu}$, yields a variation in the Riemann tensor $R^{\mu}{}_{\nu\rho\sigma} \equiv \partial_\rho \Gamma^{\mu}_{\sigma\nu} + \Gamma^{\mu}_{\rho\lambda}\Gamma^{\lambda}_{\sigma\nu} - \rho \leftrightarrow \sigma$ as

$$R^{\mu}{}_{\nu\rho\sigma} = \bar{R}^{\mu}{}_{\nu\rho\sigma} + \bar{\nabla}_\rho \left(\delta\Gamma^{\mu}_{\sigma\nu}\right) - \bar{\nabla}_\sigma \left(\delta\Gamma^{\mu}_{\rho\nu}\right) + \delta\Gamma^{\mu}_{\rho\lambda}\delta\Gamma^{\lambda}_{\sigma\nu} - \delta\Gamma^{\mu}_{\sigma\lambda}\delta\Gamma^{\lambda}_{\rho\nu}, \tag{A.5}$$

and for the second order expansion in $h_{\mu\nu}$, the relevant $\delta\Gamma^{\rho}_{\mu\nu}$ is $\delta\Gamma^{\rho}_{\mu\nu} = \tau\left(\Gamma^{\rho}_{\mu\nu}\right)_L - \tau^2 h^{\rho}_{\beta}\left(\Gamma^{\beta}_{\mu\nu}\right)_L$ which yields

$$\begin{aligned} R^{\mu}{}_{\nu\rho\sigma} =& \bar{R}^{\mu}{}_{\nu\rho\sigma} + \tau\left(R^{\mu}{}_{\nu\rho\sigma}\right)_L - \tau^2 h^{\mu}_{\beta}\left(R^{\beta}{}_{\nu\rho\sigma}\right)_L \\ &- \tau^2 \bar{g}^{\mu\alpha}\bar{g}_{\beta\gamma}\left[\left(\Gamma^{\gamma}_{\rho\alpha}\right)_L\left(\Gamma^{\beta}_{\sigma\nu}\right)_L - \left(\Gamma^{\gamma}_{\sigma\alpha}\right)_L\left(\Gamma^{\beta}_{\rho\nu}\right)_L\right] + O\left(\tau^3\right). \end{aligned} \tag{A.6}$$

Here, $\left(R^{\mu}{}_{\nu\rho\sigma}\right)_L$ is the linearized Riemann tensor with the definition

$$\begin{aligned} \left(R^{\mu}{}_{\nu\rho\sigma}\right)_L \equiv \frac{1}{2}\Big(& \bar{\nabla}_\rho\bar{\nabla}_\sigma h^{\mu}_{\nu} + \bar{\nabla}_\rho\bar{\nabla}_\nu h^{\mu}_{\sigma} - \bar{\nabla}_\rho\bar{\nabla}^{\mu} h_{\sigma\nu} \\ &- \bar{\nabla}_\sigma\bar{\nabla}_\rho h^{\mu}_{\nu} - \bar{\nabla}_\sigma\bar{\nabla}_\nu h^{\mu}_{\rho} + \bar{\nabla}_\sigma\bar{\nabla}^{\mu} h_{\rho\nu}\Big). \end{aligned} \tag{A.7}$$

Using (A.6), the second order expansions of the Ricci tensor and the scalar curvature can be found as

$$\begin{aligned} R_{\nu\sigma} =& \bar{R}_{\nu\sigma} + \tau\left(R_{\nu\sigma}\right)_L - \tau^2 h^{\mu}_{\beta}\left(R^{\beta}{}_{\nu\mu\sigma}\right)_L \\ &- \tau^2 \bar{g}^{\mu\alpha}\bar{g}_{\beta\gamma}\left[\left(\Gamma^{\gamma}_{\mu\alpha}\right)_L\left(\Gamma^{\beta}_{\sigma\nu}\right)_L - \left(\Gamma^{\gamma}_{\sigma\alpha}\right)_L\left(\Gamma^{\beta}_{\mu\nu}\right)_L\right] + O\left(\tau^3\right), \end{aligned} \tag{A.8}$$

$$\begin{aligned} R =& \bar{R} + \tau R_L + \tau^2\Big\{\bar{R}^{\rho\lambda} h_{\alpha\rho} h^{\alpha}_{\lambda} - h^{\nu\sigma}\left(R_{\nu\sigma}\right)_L - \bar{g}^{\nu\sigma} h^{\mu}_{\beta}\left(R^{\beta}{}_{\nu\mu\sigma}\right)_L \\ &- \bar{g}^{\nu\sigma}\bar{g}^{\mu\alpha}\bar{g}_{\beta\gamma}\left[\left(\Gamma^{\gamma}_{\mu\alpha}\right)_L\left(\Gamma^{\beta}_{\sigma\nu}\right)_L - \left(\Gamma^{\gamma}_{\sigma\alpha}\right)_L\left(\Gamma^{\beta}_{\mu\nu}\right)_L\right]\Big\} + O\left(\tau^3\right), \end{aligned} \tag{A.9}$$

where the linearized Ricci tensor $R^{L}_{\nu\sigma}$ and the linearized curvature scalar R_L are defined as

$$R^{L}_{\nu\sigma} \equiv \frac{1}{2}\left(\bar{\nabla}_\mu\bar{\nabla}_\sigma h^{\mu}_{\nu} + \bar{\nabla}_\mu\bar{\nabla}_\nu h^{\mu}_{\sigma} - \bar{\Box} h_{\sigma\nu} - \bar{\nabla}_\sigma\bar{\nabla}_\nu h\right), \tag{A.10}$$

$$R_L = \bar{g}^{\alpha\beta} R^{L}_{\alpha\beta} - \bar{R}^{\alpha\beta} h_{\alpha\beta}. \tag{A.11}$$

Here, h is the trace of $h_{\mu\nu}$ with the definition $h \equiv \bar{g}^{\mu\nu} h_{\mu\nu}$. Note that the results we obtained are valid for any background spacetime; however, in this thesis (A)dS spacetimes are considered as the background spacetime when the second order action for a BI gravity theory is studied.

A.2 Some linearization results in four dimensions

To linearize the field equations of the $O\left[(\alpha R)^3\right]$ expansion of BI gravity

$$I = \frac{2}{\kappa\alpha}\int d^4x\left[\sqrt{-\det\left(g_{\mu\nu}+\alpha R_{\mu\nu}\right)}-\left(\alpha\Lambda_0+1\right)\sqrt{-g}\right], \tag{A.12}$$

one needs the following linearization results which are specific to four dimensions:

$$\begin{gathered}
\delta\left(R_{\lambda\nu\alpha\mu}R^{\lambda\alpha}\right)=\frac{2\Lambda}{3}R^L_{\mu\nu}+\frac{\Lambda}{3}\bar{g}_{\mu\nu}R_L+\frac{\Lambda^2}{3}h_{\mu\nu},\\
\delta K=-2\Lambda R_L, \qquad \delta S=0,\\
\delta\left(\Box R_{\mu\nu}\right)=\bar{\Box}R^L_{\mu\nu}-\Lambda\bar{\Box}h_{\mu\nu}, \qquad \delta\left(\nabla_\mu\nabla_\nu R\right)=\bar{\nabla}_\mu\bar{\nabla}_\nu R_L, \qquad \delta\left(\Box R\right)=\bar{\Box}R_L,\\
\delta\left(R^\rho_\mu R_{\rho\alpha}R^\alpha_\nu\right)=3\Lambda^2R^L_{\mu\nu}-2\Lambda^3h_{\mu\nu}, \qquad \delta\left(R_{\mu\nu}R^2_{\alpha\beta}\right)=4\Lambda^2R^L_{\mu\nu}+2\Lambda^2\bar{g}_{\mu\nu}R_L, \quad \text{(A.13)}\\
\delta\left(R_{\mu\nu}R^2\right)=16\Lambda^2R^L_{\mu\nu}+8\Lambda^2\bar{g}_{\mu\nu}R_L, \qquad \delta\left(RR^\rho_\nu R_{\mu\rho}\right)=8\Lambda^2R^L_{\mu\nu}+\Lambda^2\bar{g}_{\mu\nu}R_L-4\Lambda^3h_{\mu\nu},\\
\delta\left(\nabla_\alpha\nabla_\mu R^\alpha_\nu\right)=\frac{1}{2}\bar{\nabla}_\mu\bar{\nabla}_\nu R_L+\frac{4\Lambda}{3}R^L_{\mu\nu}-\frac{\Lambda}{3}\bar{g}_{\mu\nu}R_L-\frac{4\Lambda^2}{3}h_{\mu\nu},\\
\delta\left(\nabla_\mu\nabla_\nu R_{\alpha\beta}\right)=\bar{\nabla}_\mu\bar{\nabla}_\nu R_{\alpha\beta}-\Lambda\bar{\nabla}_\mu\bar{\nabla}_\nu h_{\alpha\beta}.
\end{gathered}$$

Some of these results were already given in [32], but they are reproduced here for completeness. Note that the last two equations are related through the linearized Bianchi identity:

$$\bar{\nabla}^\mu\mathcal{G}^L_{\mu\nu}=0, \qquad \mathcal{G}^L_{\mu\nu}\equiv R^L_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}R_L-\Lambda h_{\mu\nu}. \tag{A.14}$$

A.3 Variations of cubic curvature terms

Variations of the cubic curvature terms $R^\mu_\nu R^\rho_\mu R^\nu_\rho$, $RR^\mu_\nu R^\nu_\mu$ and R^3 can be found as

$$\delta\left(R^\mu_\nu R^\rho_\mu R^\nu_\rho\right)=3\left[R^\rho_\mu R_{\rho\alpha}R^\alpha_\nu+\frac{1}{2}\left(g_{\mu\nu}R^{\beta\rho}R^\alpha_\rho\nabla_\beta\nabla_\alpha+R^\rho_\nu R_{\mu\rho}\Box-2R^\rho_\nu R^\alpha_\rho\nabla_\mu\nabla_\alpha\right)\right]\delta g^{\mu\nu}, \tag{A.15}$$

$$\begin{aligned}
\delta\left(RR^\mu_\nu R^\nu_\mu\right)=&R\left[\left(g_{\mu\nu}R^{\alpha\beta}\nabla_\beta\nabla_\alpha+R_{\mu\nu}\Box-2R^\alpha_\nu\nabla_\mu\nabla_\alpha\right)+2R^\rho_\nu R_{\mu\rho}\right]\delta g^{\mu\nu}\\
&+R^\alpha_\beta R^\beta_\alpha\left[\left(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu\right)+R_{\mu\nu}\right]\delta g^{\mu\nu},
\end{aligned} \tag{A.16}$$

$$\delta\left(R^3\right) = 3R^2\left[\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right) + R_{\mu\nu}\right]\delta g^{\mu\nu}. \tag{A.17}$$

The variation of $S_{\alpha\beta}$ can be calculated by using

$$\delta R_{\alpha\beta} = \frac{1}{2}\left(g_{\mu\nu}\nabla_\alpha\nabla_\beta + g_{\mu\alpha}g_{\beta\nu}\Box - g_{\beta\nu}\nabla_\mu\nabla_\alpha - g_{\alpha\nu}\nabla_\mu\nabla_\beta\right)\delta g^{\mu\nu}, \tag{A.18}$$

$$\delta R = \left[R_{\mu\nu} + \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)\right]\delta g^{\mu\nu}, \tag{A.19}$$

as

$$\begin{aligned}\delta S_{\alpha\beta} =& \frac{1}{2}\left[\left(g_{\mu\nu}\nabla_\alpha\nabla_\beta + g_{\mu\alpha}g_{\beta\nu}\Box - g_{\beta\nu}\nabla_\mu\nabla_\alpha - g_{\alpha\nu}\nabla_\mu\nabla_\beta\right) - \frac{2}{3}g_{\alpha\beta}\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)\right]\delta g^{\mu\nu} \\ &+ \frac{1}{3}\left[\left(g_{\mu\alpha}g_{\nu\beta} - \frac{1}{3}g_{\alpha\beta}g_{\mu\nu}\right)R - g_{\alpha\beta}S_{\mu\nu}\right]\delta g^{\mu\nu}.\end{aligned} \tag{A.20}$$

Using this result, one can find $\delta A \equiv \delta\left(S^\alpha_\beta S^\beta_\alpha\right)$ and $\delta B \equiv \delta\left(S^\alpha_\rho S^\beta_\alpha S^\rho_\beta\right)$ as

$$\delta A = \left[2\left(S^\alpha_\mu S_{\alpha\nu} + \frac{1}{3}RS_{\mu\nu}\right) + \left(g_{\mu\nu}S^{\alpha\beta}\nabla_\alpha\nabla_\beta + S_{\mu\nu}\Box - 2S^\alpha_\nu\nabla_\mu\nabla_\alpha\right)\right]\delta g^{\mu\nu}, \tag{A.21}$$

$$\begin{aligned}\delta B =& \left[\frac{3}{2}\left(g_{\mu\nu}S^\alpha_\rho S^{\beta\rho}\nabla_\alpha\nabla_\beta + S_{\mu\rho}S^\rho_\nu\Box - 2S^\alpha_\rho S^\rho_\nu\nabla_\mu\nabla_\alpha\right) - S^\alpha_\rho S^\rho_\alpha\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)\right]\delta g^{\mu\nu} \\ &+ \left[3S^\rho_\mu S^\sigma_\rho S_{\nu\sigma} - S^\alpha_\rho S^\rho_\alpha S_{\mu\nu} + \left(S_{\mu\rho}S^\rho_\nu - \frac{1}{3}g_{\mu\nu}S^\alpha_\rho S^\rho_\alpha\right)R\right]\delta g^{\mu\nu}.\end{aligned} \tag{A.22}$$

APPENDIX B

ANALYZING EINSTEIN'S GRAVITY AND QUADRATIC CURVATURE GRAVITY WITH SECOND ORDER PERTURBATIONS

In calculating the $O\left(h^2\right)$ action for a generic BI gravity theory, one encounters with the $O\left(h^2\right)$ terms

$$\int d^4x\sqrt{-\bar{g}}\,R_{(2)},\qquad \int d^4x\sqrt{-\bar{g}}\,\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu},\qquad \int d^4x\sqrt{-\bar{g}}\,R^{\mu\nu}_{L}R^{L}_{\mu\nu},$$

$$\int d^4x\sqrt{-\bar{g}}\,\left(R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}\right)^{(2)},\qquad \int d^4x\sqrt{-\bar{g}}\,\bar{g}^{\sigma\nu}\bar{g}^{\lambda\gamma}\left(R^{\mu}{}_{\rho\sigma\lambda}\right)^{(1)}\left(R^{\rho}{}_{\mu\gamma\nu}\right)^{(1)},\qquad \text{(B.1)}$$

which should be written in terms of the building blocks that appear in Eqn. (25) of [32]. To obtain the desired forms for these terms, we study them in the well-known examples of Einstein's gravity and quadratic curvature gravity.

B.1 Analysis of Einstein's gravity

Expanding the cosmological Einstein-Hilbert action,

$$I=\frac{1}{\kappa}\int d^4x\sqrt{-g}\left(R-2\Lambda_0\right),\qquad \text{(B.2)}$$

up to second order in metric perturbation $h_{\mu\nu}$ yields

$$\begin{aligned}
I =& \frac{1}{\kappa}\int d^4x\sqrt{-\bar{g}}\left[1+\frac{\tau}{2}h+\frac{1}{8}\tau^2\left(h^2-2h_{\mu\nu}^2\right)+O\left(\tau^3\right)\right] \\
&\times\left[\left(\bar{R}-2\Lambda_0\right)+\tau R_L+\tau^2 R_{(2)}+O\left(\tau^3\right)\right] \\
=& \frac{1}{\kappa}\int d^4x\sqrt{-\bar{g}}\Bigg\{\left(\bar{R}-2\Lambda_0\right)+\tau\left[\frac{1}{2}h\left(\bar{R}-2\Lambda_0\right)+R_L\right] \\
&+\tau^2\left[\frac{1}{8}\left(\bar{R}-2\Lambda_0\right)\left(h^2-2h_{\mu\nu}^2\right)+\frac{1}{2}hR_L+R_{(2)}\right]+O\left(\tau^3\right)\Bigg\}.
\end{aligned} \tag{B.3}$$

The first order term in this result can be used to find the field equations for the (A)dS background. By using $\bar{R}=4\Lambda$ and R_L definition given in (A.11), the first order term becomes

$$I_{O(h)}=\frac{1}{\kappa}\int d^4x\sqrt{-\bar{g}}\,h\left(\Lambda-\Lambda_0\right), \tag{B.4}$$

where a boundary term is dropped out. Then, after taking variation with respect to $h_{\mu\nu}$, the field equations for the (A)dS background follow as $\left(\Lambda-\Lambda_0\right)\bar{g}_{\mu\nu}=0$.

Now, let us consider the second order action which is

$$I_{O(h^2)}=\frac{1}{\kappa}\int d^4x\sqrt{-\bar{g}}\left\{h^{\mu\nu}\left[\frac{1}{2}\left(\Lambda-\frac{1}{2}\Lambda_0\right)\left(\bar{g}_{\mu\nu}h-2h_{\mu\nu}\right)+\frac{1}{2}\bar{g}_{\mu\nu}R_L\right]+R_{(2)}\right\}, \tag{B.5}$$

where $R_{(2)}$ is

$$\begin{aligned}
R_{(2)}=&\bar{R}^{\rho\lambda}h_{\alpha\rho}h_{\lambda}^{\alpha}-h^{\mu\nu}R_{\mu\nu}^L-\bar{g}^{\nu\sigma}h_{\beta}^{\mu}\left(R^{\beta}{}_{\nu\mu\sigma}\right)_L \\
&-\bar{g}^{\nu\sigma}\bar{g}^{\mu\alpha}\bar{g}_{\beta\gamma}\left[\left(\Gamma_{\mu\alpha}^{\gamma}\right)_L\left(\Gamma_{\sigma\nu}^{\beta}\right)_L-\left(\Gamma_{\sigma\alpha}^{\gamma}\right)_L\left(\Gamma_{\mu\nu}^{\beta}\right)_L\right],
\end{aligned} \tag{B.6}$$

from (A.9). To write $\int d^4x\sqrt{-\bar{g}}\,R_{(2)}$ in terms of the building blocks of [32], one should work out the last three terms in $R_{(2)}$ by using the results in Appendix A.1 and the maximally symmetric Riemann tensor form

$$\bar{R}_{\mu\alpha\nu\beta}=\frac{2\Lambda}{\left(D-1\right)\left(D-2\right)}\left(\bar{g}_{\mu\nu}\bar{g}_{\alpha\beta}-\bar{g}_{\mu\beta}\bar{g}_{\alpha\nu}\right). \tag{B.7}$$

The term $\bar{g}^{\nu\sigma}h_{\beta}^{\mu}\left(R^{\beta}{}_{\nu\mu\sigma}\right)_L$ can be written as

$$\bar{g}^{\nu\sigma}h_{\beta}^{\mu}\left(R^{\beta}{}_{\nu\mu\sigma}\right)_L=h^{\mu\nu}\left(R_{\mu\nu}^L-\frac{4\Lambda}{3}h_{\mu\nu}+\frac{\Lambda}{3}\bar{g}_{\mu\nu}h\right), \tag{B.8}$$

while the other two terms take the forms

$$\int d^4x\sqrt{-\bar{g}}\,\bar{g}^{\nu\sigma}\bar{g}^{\mu\alpha}\bar{g}_{\beta\gamma}\left(\Gamma^{\gamma}_{\mu\alpha}\right)_L\left(\Gamma^{\beta}_{\sigma\nu}\right)_L = \\ \int d^4x\sqrt{-\bar{g}}\left[-\frac{1}{2}h^{\mu\nu}\left(\bar{\nabla}^{\sigma}\bar{\nabla}_{\mu}h_{\nu\sigma}+\bar{\nabla}^{\sigma}\bar{\nabla}_{\nu}h_{\mu\sigma}-\frac{3}{2}\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}h\right)\right. \\ \left.+h^{\mu\nu}\left(\frac{4\Lambda}{3}h_{\mu\nu}-\frac{\Lambda}{12}\bar{g}_{\mu\nu}h\right)+\frac{1}{4}h^{\mu\nu}\bar{g}_{\mu\nu}R_L\right], \quad \text{(B.9)}$$

$$\int d^4x\sqrt{-\bar{g}}\,\bar{g}^{\nu\sigma}\bar{g}^{\mu\alpha}\bar{g}_{\beta\gamma}\left(\Gamma^{\gamma}_{\sigma\alpha}\right)_L\left(\Gamma^{\beta}_{\mu\nu}\right)_L = \\ \int d^4x\sqrt{-\bar{g}}\left[-\frac{1}{4}h^{\mu\nu}\left(3\bar{\Box}h_{\mu\nu}-\bar{\nabla}^{\sigma}\bar{\nabla}_{\mu}h_{\sigma\nu}-\bar{\nabla}^{\sigma}\bar{\nabla}_{\nu}h_{\mu\sigma}\right)\right], \quad \text{(B.10)}$$

after several integration by parts. Employing these results in $\int d^4x\sqrt{-\bar{g}}\,R_{(2)}$, finally, one gets

$$\int d^4x\sqrt{-\bar{g}}\,R_{(2)} = \int d^4x\sqrt{-\bar{g}}\,h^{\mu\nu}\left(-\frac{1}{2}R^L_{\mu\nu}-\frac{1}{4}\bar{g}_{\mu\nu}R_L+\Lambda h_{\mu\nu}-\frac{\Lambda}{4}\bar{g}_{\mu\nu}h\right). \quad \text{(B.11)}$$

With this result, the $O\left(h^2\right)$ action for cosmological Einstein's gravity (B.5) becomes

$$I_{O(h^2)} = -\frac{1}{2\kappa}\int d^4x\sqrt{-\bar{g}}\,h^{\mu\nu}\left[\mathcal{G}^L_{\mu\nu}+\frac{1}{2}\left(\Lambda_0-\Lambda\right)\left(\bar{g}_{\mu\nu}h-2h_{\mu\nu}\right)\right], \quad \text{(B.12)}$$

and this result further reduces to the well-known form

$$I_{O(h^2)} = -\frac{1}{2\kappa}\int d^4x\sqrt{-\bar{g}}\,h^{\mu\nu}\mathcal{G}^L_{\mu\nu}, \quad \text{(B.13)}$$

after using the field equation for the (A)dS background, that is $\Lambda = \Lambda_0$.

B.2 Analysis of quadratic curvature gravity

Now, let us study the most general quadratic curvature gravity in four dimensions;

$$I = \int d^4x\sqrt{-g}\left[\frac{1}{\kappa}\left(R-2\Lambda_0\right)+\alpha R^2+\beta R^{\mu}_{\nu}R^{\nu}_{\mu}\right]. \quad \text{(B.14)}$$

One can expand this action up to second order in $h_{\mu\nu}$ as

$$
\begin{aligned}
I = \int d^4x\sqrt{-\bar{g}}\Big\{ & \left[\frac{1}{\kappa}\left(\bar{R}-2\Lambda_0\right)+\alpha\bar{R}^2+\beta\bar{R}^{\mu}_{\nu}\bar{R}^{\nu}_{\mu}\right] \\
& +\tau\left[\frac{1}{2}h\left(\frac{1}{\kappa}\left(\bar{R}-2\Lambda_0\right)+\alpha\bar{R}^2+\beta\bar{R}^{\mu}_{\nu}\bar{R}^{\nu}_{\mu}\right)\right. \\
& \left.+\left(\frac{1}{\kappa}R_L+2\alpha\bar{R}R_L+\beta\bar{R}^{\mu\nu}R^L_{\mu\nu}+\beta\left(R^{\mu\nu}\right)_{(1)}\bar{R}_{\mu\nu}\right)\right] \\
& +\tau^2\left[\frac{1}{8}\left(h^2-2h^2_{\mu\nu}\right)\left(\frac{1}{\kappa}\left(\bar{R}-2\Lambda_0\right)+\alpha\bar{R}^2+\beta\bar{R}^{\mu}_{\nu}\bar{R}^{\nu}_{\mu}\right)\right. \\
& +\frac{1}{2}h\left(\frac{1}{\kappa}R_L+2\alpha\bar{R}R_L+\beta\bar{R}^{\mu\nu}R^L_{\mu\nu}+\beta\left(R^{\mu\nu}\right)_{(1)}\bar{R}_{\mu\nu}\right) \\
& \left.+\left(\frac{1}{\kappa}R_{(2)}+2\alpha\bar{R}R_{(2)}+\alpha R_L^2+\beta\bar{R}^{\mu\nu}R^{(2)}_{\mu\nu}+\beta\left(R^{\mu\nu}\right)_{(1)}R^L_{\mu\nu}+\beta\left(R^{\mu\nu}\right)_{(2)}\bar{R}_{\mu\nu}\right)\right]\Big\},
\end{aligned}
\tag{B.15}
$$

where $\left(R^{\mu\nu}\right)_{(1)}$ and $\left(R^{\mu\nu}\right)_{(2)}$ represent the first and the second orders of $R^{\mu\nu}$ in metric perturbation, respectively. From the linear order of this expansion, one should first find the field equation for the (A)dS background. By using the definitions of $R^L_{\mu\nu}$ and R_L, the linear order terms take the form

$$
I_{O(h)} = \frac{1}{\kappa}\int d^4x\sqrt{-\bar{g}}\,h\left(\Lambda-\Lambda_0\right), \tag{B.16}
$$

after dropping out the boundary terms, and one can directly read the field equation as $\Lambda=\Lambda_0$.

Now, let us consider the $O\left(h^2\right)$ action which takes the form

$$
\begin{aligned}
I_{O(h^2)} = -\frac{1}{2}\int d^4x\sqrt{-\bar{g}}\Big\{ & \left(\frac{1}{\kappa}+8\alpha\Lambda+4\beta\Lambda\right)h^{\mu\nu}\mathcal{G}^L_{\mu\nu} \\
& -\frac{1}{2}h^2\left[\frac{1}{\kappa}\left(\Lambda-\Lambda_0\right)+2\beta\Lambda^2\right] \\
& +h^2_{\mu\nu}\left[\frac{1}{\kappa}\left(\Lambda-\Lambda_0\right)+6\beta\Lambda^2\right] \\
& +2\alpha h^{\mu\nu}\left(\bar{g}_{\mu\nu}\bar{\Box}-\bar{\nabla}_\mu\bar{\nabla}_\nu+\Lambda\bar{g}_{\mu\nu}\right)R_L \\
& -2\beta\left(\Lambda\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}+R^{\mu\nu}_L R^L_{\mu\nu}+R^{\mu\nu}_{(2)}\Lambda\bar{g}_{\mu\nu}\right)\Big\},
\end{aligned}
\tag{B.17}
$$

after using (B.11). We should rewrite the three terms in the last line in terms of the building blocks of [32]. First, the term $\int d^4x\sqrt{-\bar{g}}\,R^{\mu\nu}_L R^L_{\mu\nu}$ can be put in the desired form as

$$
\begin{aligned}
\int d^4x\sqrt{-\bar{g}}\,R^{\mu\nu}_L R^L_{\mu\nu} = -\frac{1}{2}\int d^4x\sqrt{-\bar{g}}\,h^{\mu\nu}\Big[& \left(\bar{g}_{\mu\nu}\bar{\Box}-\bar{\nabla}_\mu\bar{\nabla}_\nu+\Lambda\bar{g}_{\mu\nu}\right)R_L \\
& +\left(\bar{\Box}\mathcal{G}^L_{\mu\nu}-\frac{2\Lambda}{3}\bar{g}_{\mu\nu}R_L\right) \\
& -\frac{14\Lambda}{3}R^L_{\mu\nu}+\frac{\Lambda}{3}\bar{g}_{\mu\nu}R_L+\frac{8\Lambda^2}{3}h_{\mu\nu}\Big],
\end{aligned}
\tag{B.18}
$$

after several integration by parts and using the linearized Bianchi identity;

$$\bar{\nabla}^{\mu}\mathcal{G}^{L}_{\mu\nu}=0, \qquad \mathcal{G}^{L}_{\mu\nu}\equiv R^{L}_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}R_{L}-\Lambda h_{\mu\nu}, \tag{B.19}$$

and its variant;

$$\bar{\nabla}^{\mu}\bar{\nabla}^{\nu}R^{L}_{\mu\nu}=\frac{1}{2}\bar{\square}R_{L}+\Lambda\bar{\nabla}^{\mu}\bar{\nabla}^{\nu}h_{\mu\nu}. \tag{B.20}$$

Turning to the other two terms, note that the term $\bar{g}_{\mu\nu}\left(R^{\mu\nu}\right)_{(2)}$ is simply related to $\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}$ as

$$\bar{g}_{\mu\nu}\left(R^{\mu\nu}\right)_{(2)}=\bar{g}_{\mu\nu}\left(g^{\mu\alpha}g^{\nu\beta}R_{\alpha\beta}\right)^{(2)}=\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}-2h^{\mu\nu}R^{L}_{\mu\nu}+3\Lambda h^{2}_{\mu\nu}, \tag{B.21}$$

and the term $\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}$ can be written in terms of $R_{(2)}$ as

$$R_{(2)}=\left(g^{\mu\nu}R_{\mu\nu}\right)_{(2)}=\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}-h^{\mu\nu}R^{L}_{\mu\nu}+\Lambda h^{2}_{\mu\nu}. \tag{B.22}$$

By using this result and (B.11), $\int d^4x\sqrt{-\bar{g}}\,\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}$ takes the form

$$\int d^4x\sqrt{-\bar{g}}\,\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}=\int d^4x\sqrt{-\bar{g}}\,h^{\mu\nu}\left(\frac{1}{2}R^{L}_{\mu\nu}-\frac{1}{4}\bar{g}_{\mu\nu}R_{L}-\frac{\Lambda}{4}\bar{g}_{\mu\nu}h\right). \tag{B.23}$$

By using (B.18), (B.21), (B.23), and the field equation for the (A)dS background, the second order action for the most general quadratic curvature gravity in four dimensions becomes

$$\begin{aligned}I_{O(h^2)}=-\frac{1}{2}\int d^4x\sqrt{-\bar{g}}\,h^{\mu\nu}\Bigg[&\left(\frac{1}{\kappa}+8\alpha\Lambda+\frac{4}{3}\beta\Lambda\right)\mathcal{G}^{L}_{\mu\nu}\\&+(2\alpha+\beta)\left(\bar{g}_{\mu\nu}\bar{\square}-\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}+\Lambda\bar{g}_{\mu\nu}\right)R_{L}\\&+\beta\left(\bar{\square}\mathcal{G}^{L}_{\mu\nu}-\frac{2\Lambda}{3}\bar{g}_{\mu\nu}R_{L}\right)\Bigg],\end{aligned} \tag{B.24}$$

which matches the form given in [32].

Now, let us study the Einstein-Gauss-Bonnet theory,

$$I=\int d^4x\sqrt{-g}\left[\frac{1}{\kappa}\left(R-2\Lambda_0\right)+\gamma\left(R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}-4R^{\mu}_{\nu}R^{\nu}_{\mu}+R^{2}\right)\right], \tag{B.25}$$

to work out the terms in the second line of (B.1). Note that Gauss-Bonnet combination is a boundary term in four dimensions, so the field equation for the (A)dS background and the $O\left(h^2\right)$ action should the same as the corresponding results in the Einstein's gravity case.

The $O\left(h^2\right)$ action for the terms other than $R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}$ can be given as

$$I_{O(h^2)} = -\frac{1}{2}\int d^4x\sqrt{-\bar{g}}\,h^{\mu\nu}\Big[\left(\frac{1}{\kappa}+\frac{8}{3}\gamma\Lambda\right)\mathcal{G}^L_{\mu\nu} - 2\gamma\left(\bar{g}_{\mu\nu}\bar{\Box}-\bar{\nabla}_\mu\bar{\nabla}_\nu+\Lambda\bar{g}_{\mu\nu}\right)R_L \\ - 4\gamma\left(\bar{\Box}\mathcal{G}^L_{\mu\nu}-\frac{2\Lambda}{3}\bar{g}_{\mu\nu}R_L\right)\Big], \tag{B.26}$$

by using (B.24). On the other hand, to find the $O\left(h^2\right)$ action for $R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}$, let us expand it up to $O\left(h^2\right)$ as

$$I = \gamma\int d^4x\sqrt{-\bar{g}}\Big\{\bar{R}^{\mu\nu}_{\rho\sigma}\bar{R}^{\rho\sigma}_{\mu\nu}+\tau\left[\left(R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}\right)^{(1)}+\frac{1}{2}h\bar{R}^{\mu\nu}_{\rho\sigma}\bar{R}^{\rho\sigma}_{\mu\nu}\right] \\ +\tau^2\left[\left(R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}\right)^{(2)}+\frac{1}{2}h\left(R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}\right)^{(1)}+\frac{1}{8}\bar{R}^{\mu\nu}_{\rho\sigma}\bar{R}^{\rho\sigma}_{\mu\nu}\left(h^2-2h^2_{\mu\nu}\right)\right]\Big\}. \tag{B.27}$$

One expects that the linear order of this expansion should become a boundary term in order not to effect the field equation for the (A)dS background. One can calculate that

$$\bar{R}^{\mu\nu}_{\rho\sigma}\bar{R}^{\rho\sigma}_{\mu\nu} = \frac{8\Lambda^2}{3}, \qquad \left(R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}\right)^{(1)} = \frac{4\Lambda}{3}R_L, \tag{B.28}$$

then the linear order of (B.27) becomes

$$I_{O(h)} = \int d^4x\sqrt{-\bar{g}}\left[\left(\bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu}-\bar{\Box}h\right)\right], \tag{B.29}$$

which is a boundary term.

Now, let us study the $O\left(h^2\right)$ of (B.27) which becomes

$$I_{O(h^2)} = \gamma\int d^4x\,\sqrt{-\bar{g}}\left[\left(R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}\right)^{(2)}+\frac{2\Lambda}{3}hR_L+\frac{\Lambda^2}{3}\left(h^2-2h^2_{\mu\nu}\right)\right], \tag{B.30}$$

after using (B.28). Here, $\left(R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}\right)^{(2)}$ can be written as

$$\left(R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}\right)^{(2)} = \left(R^\mu{}_{\rho\sigma\lambda}R^\rho{}_{\mu\gamma\nu}g^{\sigma\nu}g^{\lambda\gamma}\right)^{(2)} \\ = 2\bar{R}^\rho{}_\mu{}^{\lambda\sigma}\left(R^\mu{}_{\rho\sigma\lambda}\right)^{(2)}+2\bar{R}^\mu{}_{\rho\sigma\lambda}\bar{R}^\rho{}_\mu{}^\lambda{}_\nu g^{\sigma\nu}_{(2)}+\bar{g}^{\sigma\nu}\bar{g}^{\lambda\gamma}\left(R^\mu{}_{\rho\sigma\lambda}\right)^{(1)}\left(R^\rho{}_{\mu\gamma\nu}\right)^{(1)} \\ +2\left[\bar{R}^\rho{}_\mu{}^\lambda{}_\nu\left(R^\mu{}_{\rho\sigma\lambda}\right)^{(1)}+\bar{R}^\mu{}_{\rho\sigma}{}^\gamma\left(R^\rho{}_{\mu\gamma\nu}\right)^{(1)}\right]g^{\sigma\nu}_{(1)}+\bar{R}^\mu{}_{\rho\sigma\lambda}\bar{R}^\rho{}_{\mu\gamma\nu}g^{\sigma\nu}_{(1)}g^{\lambda\gamma}_{(1)}, \tag{B.31}$$

and by employing (B.7), (B.8), and $R_{(2)} = \bar{g}^{\rho\sigma}R^{(2)}_{\rho\sigma}+g^{\rho\sigma}_{(1)}R^{(1)}_{\rho\sigma}+\bar{R}_{\rho\sigma}g^{\rho\sigma}_{(2)}$, it further reduces to

$$\left(R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}\right)^{(2)} = \frac{4\Lambda}{3}R_{(2)}-\bar{g}^{\sigma\nu}\bar{g}^{\lambda\gamma}\left(R^\mu{}_{\rho\lambda\sigma}\right)^{(1)}\left(R^\rho{}_{\mu\gamma\nu}\right)^{(1)} \\ -\frac{4\Lambda}{3}h^{\mu\nu}R^L_{\mu\nu}+\frac{14\Lambda^2}{9}h^2_{\mu\nu}-\frac{2\Lambda^2}{9}h^2. \tag{B.32}$$

Let us consider the integral of this form where the second term can be calculated as

$$\begin{aligned}
&\int d^4x\sqrt{-\bar{g}}\,\bar{g}^{\mu\nu}\bar{g}^{\rho\alpha}\left({R^{\lambda}}_{\sigma\rho\mu}\right)^{(1)}\left({R^{\sigma}}_{\lambda\alpha\nu}\right)^{(1)} \\
&\quad=\int d^4x\sqrt{-\bar{g}}\Bigg\{h^{\mu\nu}\left[2\left(\bar{\Box}\mathcal{G}^L_{\mu\nu}-\frac{2\Lambda}{3}\bar{g}_{\mu\nu}R_L\right)+\left(\bar{g}_{\mu\nu}\bar{\Box}-\bar{\nabla}_\mu\bar{\nabla}_\nu+\Lambda\bar{g}_{\mu\nu}\right)R_L\right] \\
&\qquad\qquad-\frac{\Lambda}{9}h^{\mu\nu}\left(30R^L_{\mu\nu}-9\bar{g}_{\mu\nu}R_L-32\Lambda h_{\mu\nu}+2\Lambda\bar{g}_{\mu\nu}h\right)\Bigg\}, \qquad \text{(B.33)}
\end{aligned}$$

after a rather lengthy but straightforward computation involving several integration by parts. Putting this result and (B.11) in $\int d^4x\,\sqrt{-\bar{g}}\,\left(R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}\right)^{(2)}$, one finally gets

$$\begin{aligned}
&\int d^4x\sqrt{-\bar{g}}\left(R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}\right)^{(2)}= \\
&\quad\int d^4x\sqrt{-\bar{g}}\,h^{\mu\nu}\Bigg\{-\left[2\left(\bar{\Box}\mathcal{G}^L_{\mu\nu}-\frac{2\Lambda}{3}\bar{g}_{\mu\nu}R_L\right)+\left(\bar{g}_{\mu\nu}\bar{\Box}-\bar{\nabla}_\mu\bar{\nabla}_\nu+\Lambda\bar{g}_{\mu\nu}\right)R_L\right] \\
&\qquad\qquad+\frac{\Lambda}{3}\left(8\mathcal{G}^L_{\mu\nu}-4R^L_{\mu\nu}+6\Lambda h_{\mu\nu}-\Lambda\bar{g}_{\mu\nu}h\right)\Bigg\}. \qquad \text{(B.34)}
\end{aligned}$$

Now, we are ready to write the $O\left(h^2\right)$ of $I=\gamma\int d^4x\sqrt{-g}\,R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}$ which becomes

$$\begin{aligned}
I_{O(h^2)}=&-\frac{1}{2}\gamma\int d^4x\sqrt{-\bar{g}}\,h^{\mu\nu} \\
&\left[-\frac{8\Lambda}{3}\mathcal{G}^L_{\mu\nu}+2\left(\bar{g}_{\mu\nu}\bar{\Box}-\bar{\nabla}_\mu\bar{\nabla}_\nu+\Lambda\bar{g}_{\mu\nu}\right)R_L+4\left(\bar{\Box}\mathcal{G}^L_{\mu\nu}-\frac{2\Lambda}{3}\bar{g}_{\mu\nu}R_L\right)\right].
\end{aligned} \qquad \text{(B.35)}$$

One can obtain the $O\left(h^2\right)$ of the Einstein-Gauss-Bonnet theory by adding this result and (B.26), and in the final form there is not any contribution coming from the Gauss-Bonnet term as expected.

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Presentations

- Title of the talk: *Critical gravity in D-dimensions*,
  - "Sixth Aegean Summer School - Quantum Gravity and Quantum Cosmology," Chora, Greece, September 16, 2011;
  - "International School of Subnuclear Physics," Erice, Italy, June 25, 2011;
  - "SIGRAV Graduate School - IX Edition Analogue Gravity," Como, Italy, May 21, 2011;
  - "Applied Mathematics Group: Thursday Seminars," Department of Mathematics, Bilkent University, Ankara, Turkey, January 27, 2011.
- Title of the talk: *Unitarity analysis of general Born-Infeld gravity theories*,
  - "10th Workshop on Quantization, Dualities and Integrable Systems," Magusa, Turkish Republic of Northern Cyprus, April 22, 2011;
  - "Applied Mathematics Group: Thursday Seminars," Department of Mathematics, Bilkent University, Ankara, Turkey, October 21, 2010.
- Title of the talk: *New Massive Gravity in de-Sitter Space and Chiral Gravity*,
  - "9th Workshop on Quantization, Dualities and Integrable Systems," Istanbul, Turkey, April 24, 2010.
- Title of the talk: *Quantum Corrections to the Instantons in the Two Dimensional Abelian Higgs Model: Functional Determinants and Green's Function Techniques*,
  - "8th Workshop on Quantization, Dualities and Integrable Systems," Ankara, Turkey, April 23, 2009.

Conferences, Workshops, Schools

- *Workshop on Infrared Modifications of Gravity*, The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, September 26–30, 2011.
- *Sixth Aegean Summer School - Quantum Gravity and Quantum Cosmology*, Chora, Naxos, Greece, September 12–17, 2011.

- *Istanbul 2011: Strings, Branes and Supergravity*, Koc University, Istanbul, Turkey, August 01–05, 2011.

- *International School of Subnuclear Physics (49th Course) - Searching for the Unexpected at LHC and Status of Our Knowledge*, Ettore Majorana Foundation and Centre for Scientific Culture, Erice, Italy, June 24 – July 03, 2011.

- *ICTP Summer School on Particle Physics*, The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, June 06–17, 2011.

- *SIGRAV Graduate School in Contemporary Relativity and Gravitational Physics - IX Edition Analogue Gravity*, Centro di Cultura Scientifica ”Alessandro Volta”, Villa Olmo, Como, Italy, May 16–21, 2011.

- *10th Workshop on Quantization, Dualities and Integrable Systems*, Eastern Mediterranean University, Magusa, Turkish Republic of Northern Cyprus, April 22–24, 2011.

- *9th Workshop on Quantization, Dualities and Integrable Systems*, Yeditepe University, Istanbul, Turkey, April 23–25, 2010.

- *Lectures on Neutrino Mass and Grand Unified Theories*, Feza Gursey Institute for Fundamental Sciences, Istanbul, Turkey, July 27 – August 01, 2009.

- *8th Workshop on Quantization, Dualities and Integrable Systems*, Ankara University, Ankara, Turkey, April 23–25, 2009.

- *A Short Course on Supergravity*, Istanbul Center for Mathematical Sciences, Turkey, February 05–06, 2009.

- *Lie Groups and Representation Theory*, Feza Gursey Institute for Fundamental Sciences, Istanbul, Turkey, July 28 – August 08, 2008.

- *Phase Transitions and Renormalization Group*, Feza Gursey Institute for Fundamental Sciences, Istanbul, Turkey, July 13–24, 2008.

- *Rigorous Results in Statistical Mechanics and Quantum Field Theory*, Feza Gursey Institute for Fundamental Sciences, Istanbul, Turkey, June 12–20, 2008.

- *7th Workshop on Quantization, Dualities and Integrable Systems*, Anadolu University, Eskisehir, Turkey, April 23–25, 2008.
- *High Energy Physics School*, Feza Gursey Institute for Fundamental Sciences, Istanbul, Turkey, June 25 – August 03, 2007.
- *6th Workshop on Quantization, Dualities and Integrable Systems*, Middle East Technical University, Ankara, Turkey, April 20–22, 2007.