MARGIN CALL RISK MANAGEMENT WITH FUTURES AND OPTIONS

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ABSTRACT

MARGIN CALL RISK MANAGEMENT WITH FUTURES AND OPTIONS

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This study examines dynamic hedge policy of a company in a multi-period framework. The company begins to operate a project for a customer and it also has a subcontractor which completes an important part of the project by using an economic commodity. The customer will pay a fixed price to the company at the end of the project. Meanwhile, the company needs to pay the debt to the subcontractor and the amount of the debt depends on the spot price of the commodity at that time. The company is allowed to hedge for the commodity price fluctuations via future and option contracts. Since the company has a limited cash reserve as well as previously planned payments, it may face financial distress when the net cash balance decreases below zero. Consequently, the company maximizes the expected value of itself by minimizing the expected financial distress cost.

Keywords: Risk Management, Price Risk, Margin Call, Futures, Options

ÖZ

VADELİ İŞLEMLER VE OPSİYONLAR İLE TEMİNAT TAMAMLAMA ÇAĞRI RİSK YÖNETİMİ

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Bu çalışma bir şirketin dinamik olarak riske karşı korunma politikasını çok periyotlu bir çerçevede incelemektedir. Şirketin müşterisi için bir proje yapım sorumluluğu bulunmakta ve belirli bir emtia kullanarak projenin önemli bir bölümünü gerçekleştirecek bir alt yükleniciye bu projede ihtiyaç duymaktadır. Müşteri bu proje başlangıcında belirlenen bir fiyatı proje sonunda şirkete ödeyecektir. Yine proje sonunda, alt yükleniciye borcunu ödeyecek ve borcun miktarı emtianın o andaki piyasa fiyatıyla doğrudan bağlantılı olacaktır. Bu senaryo kapsamında şirket, emtiaya bağlı vadeli türev işlemleri ve opsiyonlar aracılığıyla riskten korunabilir. Şirketin nakit hesap dengesinin sınırlılığı ve daha önceden planlanmış ödemelerinin varlığı sebebiyle nakit hesabının sıfırın altına indiği durumlarda şirkette mali sıkıntılar yaşanabilir. Bu sebeple şirket, gelecekteki beklenen değerini en üst düzeye çıkarmaya çalışırken nakit yetmezliğinden kaynaklanan mali sorunların beklenen maliyetini en düşük düzeye indirmelidir.

Anahtar Kelimeler: Risk Yönetimi, Fiyat Riski, Teminat Tamamlama Çağrısı, Vadeli İşlemler, Opsiyonlar

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CHAPTER 1

INTRODUCTION AND MOTIVATION

In today's economy, commodity prices in markets are important for companies that are utilizing them in their projects. When the prices are not predicted right and companies do not secure their financial positions against risks by hedging, changes in prices may lead to decrease in revenue. Such unexpected decreases may lead distress in the financial situation of companies and financial distress costs may incur when their cash balance becomes negative. The breakpoint values of the prices of commodities that may lead financial distress are approximately calculated by considering the cash exchange of the company during different time periods of the project. In the time that companies reach an agreement with their customers for the project that requires a specific delivery date in the future, companies already have planned deliverables or receivables to be realized during the project. Considering all those planned deliverables or receivables as well as the initial cash balance, they are generating a financial plan during the project. However, if companies face a financial distress, a loan may be received from the bank with an interest rate higher than the minimum rate of return in the market. Financial distress cost may be accumulated by both simple and compound interest models. Accumulated distress cost may be paid in the beginning of the next period, or it may be delayed for the following periods. In order to prevent the emergence of financial distress costs, hedging may be applied on the main commodities used during the project. However, the magnitude of the hedging decision and the derivatives used for it is a critical decision for companies.

In this paper, risk management decision of a firm which has an agreement to deliver a product mainly manufactured by using a common commodity will be considered. The firm will receive a revenue at the delivery of the products and will also pay the related debt to the subcontractor at that moment. The revenue to be received from the customer is known at the beginning of the contract. However, the debt of the firm to the subcontractor is a variable that is affected by the current price of the commodity and the profit margin of the subcontractor. Due to the price risk of the commodity, the firm may prefer to use financial instruments; especially over-the-counter derivatives that are derived by the price of the commodity used by the firm. Indeed, buying or selling forward contracts, futures contracts and call options maturing at the delivery time of the firm are probably the most common choice for such companies.

In both of the interest rate models, financial distress cost is accumulated by multiplying the cash need of the firm with bank loan interest rate. However, both models behave differently when time horizon is more than a single period. If simple interest is used, the interest is calculated on original principal only. However, compound interest takes original principal and all interest accumulated during past periods into account. Since the companies may be exposed to financial distress in some periods, some of the companies prefer to have overdraft arrangements with the banks. By using these agreements, the companies may borrow money from the bank and it will be exposed to a predefined interest rate for its loan up to a predefined limit. Although both compound and simple interest rates may be used for overdraft arrangements, simple interest rates are quite popular especially in case of long term commitments between the bank and the companies. Simple interest rates may be higher in some cases, however it is more secure in case of an unexpected long term scarce of cash.

CHAPTER 2

INTRODUCTION TO FINANCIAL DERIVATIVES

Derivatives are financial instruments whose values are derived from one or more underlying variables. These variables may have a wide variety, such that commodities (steel, silver, oil, gas, etc.), financial assets (bonds, stocks, etc.), another derivative (e.g. options on futures), index and interest rate. Forwards, futures and options will be mentioned in this paper.

Forwards are contracts that are held between two parties to buy or sell an asset at a predetermined price and a predetermined delivery date in future. The contracts are traded in over-the-counter markets and are non-standardized, as a result the traders can exchange all kind of assets without limitation. Thus, it is a popular derivative in the market and mainly does not have initial payment. The positions of the parties in the contract are classified as long and short positions. Having a long position means agreeing to buy the asset. On the other hand, the party agreeing to sell an asset in the forward contract is taking a short position. Suppose that S_i is the spot price of the asset at time i and $F_{0,i}$ is the agreed price by two parties at time 0 for the delivery time i. The payoff of the contract per unit is $(S_i - F_{0,i})$ for the party with long position and it is $(F_{0,i} - S_i)$ for the party with short position at the end of the contract, at time i. It means that taking long position lead to a profit in case that realized price of the commodity is higher than the future contract price, and it is vice versa for the short position. At the end, one party has a profit and this profit emerges as a cost to the other party.

Price risk may be completely eliminated by hedging through forwards. For instance, suppose that there is a company which needs to buy 100 tons of a certain commodity at time 2 and tries to avoid the price risk by hedging for the whole amount at time 1. The forward price of the commodity for time 2 is $ 10 per ton at time 1 and there is no initial payment for the forward contract. In all cases, the firm will pay $ 1000 totally due to the hedging policy. Suppose that the realized spot price of the commodity becomes $ 11 per ton. If the firm did not hedge at all at time 1, it would pay $ 1100 for the commodity purchase at time 2 in this case. Therefore the firm has $ 1000 of profit due to hedging. On the other hand, the firm would have a financial loss of $ 1000 if the spot price of the commodity would be $ 9 per ton, which is lower than the forward price. All financial transactions related to this scenario is shown at Table 2.1 and Table 2.2.

Strategies	Cash flows at time 1	Cash flows at time 2
Long a forward contract for 100 ton of commodity at time 1 for time 2	0.5	(11-10)*100 = $ 100
Pay for steel delivery at time 1	0	-11*100 = $-1100
TOTAL	**0**	**$ -1000**

Table 2.1: Cash flows when realized price is $ 11 in case of full hedge policy via forwards

Strategies	Cash flows at time 1	Cash flows at time 2
Long a forward contract for 100 ton of commodity at time 1 for time 2	0	(9-10)*100 = $ -100
Pay for steel delivery at time 1	0	-9*100 = $-900
TOTAL	**0**	**$ -1000**

Table 2.2: Cash flows when realized price is $ 9 in case of full hedge policy via forwards

Futures are standardized contracts that are held between two parties to buy or sell specified asset with standardized quantity and quality at a predetermined price and a predetermined delivery date in future, which are traded in over-the-counter markets. The party agreeing to buy the asset is said to be long positioned, the party agreed to sell the asset is said to be short positioned. Both parties have to post a margin when futures contract is agreed. Suppose that S_i is the spot price of the asset at time i, $F_{0,i}$ is the agreed price by two parties at time 0 for the delivery time t and κ_0 is the initial margin. Lastly, $F_{j,i}$ denotes the futures price of the commodity at time j for time i. When the change in the price of the future exceeds the margin, the party that has a negative payoff should post margin call. It means that the buyer needs to deposit a margin call κ_j when the futures price at time j decreased more than prepaid margin with respect to futures price at time 0, i.e. $F_{0,i} - F_{j,i} \geqq \kappa_0$. In the opposite case, i.e. $F_{j,i} - F_{0,i} \geqq \kappa_0$, the seller needs to deposit the margin. In this study, it will be assumed that there is no initial payment to enter into futures contract. However, the margin account of the firm needs to be updated between the time periods.

Suppose that a firm buys 10 units of futures of a commodity at 1st of February for 1st at a price of $ 100 per unit. It is also assumed that the initial margin for the contract is $ 50 and maintenance margin is $ 35; therefore the firm needs to pay $ 50 at the beginning of the contract and needs to compensate the margin account when it becomes less than $ 35. Daily margin account updates of the first three days are shown in Table 2.3. In the case denoted in this table, the firm has a loss of $ 10 at 2nd of February and balance of the margin account of the firm becomes $ 40 which is still more than maintenance margin. Additionally, margin account becomess less than the maintenance margin at 3rd of February and the firm needs to pay $ 40 to make its margin account $ 50 again. At 4th of February, the unit price of the contract increases to $ 97 and the daily gain of the firm becomes $ 10.

Day	Futures Price ($)	Daily Gain (Loss) ($)	Cumulative Gain (Loss) ($)	Margin Account Balance ($)	Margin call ($)
	100			50	
2-Feb	99	(10)	(10)	40	0
3-Feb	96	(30)	(40)	10	40
4-Feb	97	10	(30)	60	0

Table 2.3: Marking to market example for three days of operation

Options are contracts that are held between two parties which gives to buyer (also called as option holder) the right, but not the obligation, to buy or sell a specified asset at a predetermined quantity and a specified strike price on or before a specified date. The seller, in other words "option writer", has to fulfil the transaction when the buyer chooses to exercise the option. The buyer pays a premium to the seller for this right at the beginning of the contract. The agreement is called "call" option when the agreement is based on buying an asset, and it is called "put" option when the agreement is based on selling. The position of buyer is called long position, while the position of seller is described as short position.

Suppose that there is a company which needs to buy 100 tons of a certain commodity at time 2 and tries to avoid the price risk by hedging via call options on futures for the whole amount at time 1. The strike price of the call option for time 2 is $ 10 per ton at time 1 and the company needs to pay a risk premium of $ 0.5 per ton. Now, suppose that the realized spot price of the commodity becomes $ 11 per ton. If the firm did not hedge at all at time 1, it would pay $ 1100 for the commodity purchase at time 2 in this case. However, the firm totally pays an amount of $ 50 at time 1 and $ 1000 at time 2 due to hedging via call options, as shown in Table 2.4. On the other hand, the firm would totally pay $ 50 at time 1 and $ 900 at time 2 if the spot price of the commodity becomes $ 9 per ton of commodity as shown in Table 2.5.

Strategies	Cash flows at time 1	Cash flows at time 2
Long a call option for 100 ton of commodity at time 1 for time 2	-0.5*100 = $ -50	(11-10)*100 = $ 100
Pay for steel delivery at time 1	0	-11*100 = $ -1100
TOTAL	**$ -50**	**$ -1000**

Table 2.4: Cash flows when realized price is $ 11 in case of full hedge policy via call options

Strategies	Cash flows at time 1	Cash flows at time 2
Long a call option for 100 ton of commodity at time 1 for time 2	-0.5*100 = $ -50	0
Pay for steel delivery at time 1	0	-9*100 = $ -900
TOTAL	**$ -50**	**$ -900**

Table 2.5: Cash flows when realized price is $ 9 in case of full hedge policy via call options

CHAPTER 3

LITERATURE REVIEW

Companies try to secure their financial situation and avoid risks by hedging, especially in financially critical situations and low liquidity. They may use different kind of derivatives for hedging and most common examples are forward, future and option contracts. Moreover, using financial derivatives is a growing practice in different industries nowadays. Therefore, there is a vast collection of literature on using of derivatives and these studies are mainly divided mainly into three groups as utility maximization, value maximization and variance minimization problems.

The utility maximization models focus on utility function by considering risk approach of the decisionmakers. Indeed, most of the studies on this area focus on hedging of risk averse decisionmakers. The study of Heifner (1972) on finding optimal hedge policies for cattle feeding is an early example of utility maximization models. Another example is Rolfo (1980) which focuses on optimal hedge policy for cocoa producers with a risk averse approach. The paper considers both price and quantity risks with logarithmic utility functions. On the other hand, variance minimization is another useful model used in risk management problems. Minimizing variance reduces the unexpected consequences for the decisionmaker. The traditional method of static minimum-variance hedges are discussed by Stulz (2003) and McDonald (2006). One of the latest studies, Basak and Chabakauri (2012), provides a simple analytical solution for hedge decisions in a dynamic programming framework and discrete time settings for incomplete markets. Besides, Xing and Pietola (2005) figure out optimal hedge ratios of a wheat producer under quantity and price risk using both mean-variance and expected utility frameworks. They suggest that the correlation between price and quantity is important in determination of the optimal hedge ratio. Their results suggest that price and quantity are negatively correlated which creates a natural hedge. They show that when there is natural hedge, the optimal hedge is always less than the expected quantity. Moreover, they suggest that hedging effectiveness decreases as quantity uncertainty increases. These two methodologies are compared in a perspective of decisionmakers in some studies. For example, Cecchetti, Cumby, Figlewski (1988) suggest utility maximization approach rather than variance minimization on 20 year treasury bonds that would be held for one month periods.

Utility function maximization and variance minimization are properly applicable frameworks for individual decisionmakers, risk averse and little sized companies, and may be more useful than other approaches. On the other hand, in the risky business environments of today's economy, the companies need to manage the possible risks and seek for opportunities to grow. Therefore, variance minimization may not be an appropriate decision framework for many companies. Besides, utility maximization may not be applicable to the decision process of large corporations because there are a lot of stakeholders which have different types of risk approaches. Therefore, setting a common objective to be accepted by all of the components is necessary.

Value maximization problems maximize the discounted values of the expected financial balance of the company; as a result it is more prone to be used as the main decision criteria for large corporations. The study of Modigliani and Miller (1958) proved that hedging does not add any value to the value of the company in perfect markets. On the other hand, Smith and Stulz (1985) demonstrated that hedging may increase the expected value of the company in case of market imperfections such as managerial risk aversion, financial distress costs, bankruptcy and taxes. Indeed, hedging is an important practice to prevent financial distress. A more recent study completed by data collecting from 400 UK companies, Judge (2006), shows that one of the most common reasons of hedging is avoiding financial distress and firms with higher cash balance are less likely to hedge.

In our study, value maximization approach is used to decrease the expected financial distress cost. The company may face financial distress due to the price risk of the commodity used in its processes. There is a large literature about the effect of price risk on hedging decision; yet, a few of them use value maximization approach. One of them is the study of Maes (2011) which uses value maximization approach in order to analyze the hedging decisions of a wheat miller that is confronted with price, quantity and blend risks. It shows that when uncertainty increases, a lower optimal hedge ratio appears. Goel and Gutierrez (2009) and Goel and Gutierrez (2011) also use value maximization framework.

Goel and Gutierrez (2009) show the effects of changing futures and spot prices on inventory management decision of a company. Likewise, Goel and Gutierrez (2011) examine the additional value due to procurement by using commodity markets. Additionally, Goel and Tanrisever (2011) analyzes the hedge behavior of a company using a commodity in procurement, its processes and distribution. They analyze the optimum policy to balance procurement in spot market and forward/options market in a multi-period framework while the transportation cost paid for the spot market is higher than forward/options market.

Tanrisever and Gutierrez (2011) examine the motives for hedge actions of a flour miller using value maximization under financial distress risk. Their study considers price risk of the commodity used and it shows that hedging increase the value of the firm. They showed that price risk can be eliminated completely by hedging of the related commodities in make-to-order production and production level may be increased.

Margin call risk management is also another important characteristic of our study. Since margin calls may create an unexpected cost to the company, literature partly focuses on avoiding margin calls due to future and option contracts. For example, Garner (2010) focuses on different hedge strategies in order to reduce margin requirements of hedging via futures and options. Another characteristic approach is provided by the study of Dau and Groch (2005) which suggests a decision framework for buying contracts with margin call risk by using historical data to examine buying power of the company in future for an automated risk management back office. One of the two studies that is largely benefited in this paper is Tanrisever and Levij (2011) which examine the hedge policy of an electricity trading company facing margin calls due to price risk in a value maximization perspective in a two-period framework. Another beneficial study is Tanrisever, Duran and Sumer (2011) which focuses on hedge behavior of an offshore wind-farm construction company which mainly uses steel for construction, under price and quantity risk in one and two-period framework. This study uses value maximization approach with compound interest rates and underhedging is proved to be optimal for two-period model under price risk.

Unlike the previous studies, we analyze the hedge policy of a company facing price risk by using value maximization approach. One, two and multi-period analysis is completed by using futures and forward contracts, and the decision of buying call options in one-period of time is also analyzed in order to benchmark with the usage of futures and forwards. Although simple interest rates are used by assuming that there is an overdraft agreement between the company and a bank, an approximation for the model with compound interest rates in two-period perspective is also examined.

CHAPTER 4

MATHEMATICAL MODELS

In this study, the hedging decision of a firm to manage its financial distress risk is examined. The firm undertakes an agreement with a customer to deliver a product at the agreed date. Customer will pay a predefined price p per unit for a predefined quantity ξ at the agreement date. The firm outsources a considerable amount of work to a subcontractor which produces the necessary portion of the project. The debt to the subcontractor will also be paid at the end of the agreement when the firm gets paid by the customer. However, the main part of the cost paid to the subcontractor depends on the price of the underlying commodity of the delivered asset at the end of the project. The remaining part is a fixed profit margin of the subcontractor. If the project is ending at time i, the total cost paid to the subcontractor is $(S_i + \lambda)\,\xi$ for ξ units of assets where spot price of the asset at the end of the project is denoted as S_i and the profit margin of the subcontractor is denoted as λ.

Cash reserve of the company at time i after the transactions are realized (except any possible financial distress) cost is denoted as y_i. During the fulfillment process of the agreement, the firm will also receive or pay its payments that are irrelevant to the project. These payments are already known at the beginning of the project. A payment to be completed at time i is denoted as n_i. The value of n_i may also be negative, in this case the company has receivables for time i. Additionally, the firm has an overdraft agreement with a bank using simple interest rates. Therefore, in case of a negative balance in the cash balance of the firm, it may borrow money at a predefined simple interest rate. In this study, the firm's account of loans from the bank will be totally evaluated after the project ends. For instance, the financial distress costs until time i (including financial distress cost at time i) will be taken into the account. The interest rate of bank loan is the same in all periods and denoted as r, as a result financial distress cost due to scarce of money at the beginning of period i is denoted as $r\,[y_i]^-$.

The aim of the company is to maximize the expected total change in the value of the company during the project. In the model, the firm is assumed to deal only with one project during the periods that are examined, or the others are isolated from the decision. In order to maximize the total value change in positive direction, the firm is able to long or short futures or call options. In case of futures are obtained for this aim, a higher hedging quantity from the beginning to the end of the project may decrease the financial distress risk at the delivery date of the project assets, yet it surely increases the financial distress risk in the periods before the delivery date due to margin call of futures and it is vice versa for lower hedging quantity. On the other hand, options do not have margin calls and the risk premium is paid at the beginning of the contract. Therefore, the firm pays the expected cost of the risk that will be faced and does not have margin call updates during the contract is active. Although this approach may prevent the financial distress due to unexpected price fluctuations, it decreases profitability due to risk premiums paid to purchase the contract and will be paid after the project ends (or the debt will be considered until the end of the project).

In this study, pricing of the futures contract of an asset is modeled as a stochastic process. Indeed $F_{j+1,i}$ has a normal distribution with mean $F_{j,i}$ and standard deviation σ. Finally the spot price of the underlying commodity $\tilde{S}_i$ has a mean of $F_{i-1,i}$ and standard deviation σ. Therefore the following expression is valid for a decisionmaker at time j.

$$F_{j,i} = E^{\phi}\left(\tilde{F}_{j+1,i}\right) = E^{\phi}\left(\tilde{F}_{j+2,i}\right) = ... = E^{\phi}\left(\tilde{F}_{i-1,i}\right) = E^{\phi}\left(\tilde{S}_i\right)$$

However, for the same decisionmaker, the expected price of $\tilde{F}_{j+2,i}$ may not be equal to $F_{j,i}$ because the realized price of $\tilde{F}_{j+1,i}$ may be different than $F_{j,i}$. Consequently, the following may be true for the decisionmaker in time $j+1$.

$$F_{j,i} \neq F_{j+1,i} = E^{\phi}\left(\tilde{F}_{j+2,i}\right) = ... = E^{\phi}\left(\tilde{F}_{i-1,i}\right) = E^{\phi}\left(\tilde{S}_i\right)$$

In this study, the payoff of the call option will be examined with strike price of the future price of the commodity maturing at the end of the delivery date of assets, i.e. $F_{j,i}$. Thus, the risk premium of the option will be calculated as $h = \int_{F_{1,2}}^{\infty}\left(\tilde{S}_2 - F_{1,2}\right)\phi_{S_2}^{Q}\left(S_2\right) dS_2$ and this amount is needed to be paid by the buyer when the call option is bought.

Finally, a minimum required rate of return is used to evaluate expected future transactions in the objective function. The minimum required rate of return is the same for all periods and is denoted as r_d. Besides β is defined in order to derive the value of a transaction of a discrete time to the previous one and it is formulized as $1/\left(1+r_d\right)$.

Notation used in this table is provided briefly in Appendix.

4.1 One-Period Model

This case is assumed to be a scenario where there are no financial transactions between sign of the contract and delivery of asset. The firm has a debt at the amount of n_2 that is planned to be paid at time 2. It will receive the revenue from customer at an amount of $p\xi$ and will have to pay the cost of the assets to the subcontractor at an amount of $\left(\tilde{S}_2 + \lambda\right)\xi$ at time 2. The cash balance of the firm at the beginning of the project is y_1.

In the beginning of the period, the firm considers to long $x_{1,2}$ units of commodity futures contracts at time 1 maturing at time 2 for the futures price $F_{1,2}$ while the spot price of the commodity is S_1 at this time. By this action, the firm aims to decrease the risk of having financial distress cost at time 2 due to unstable price of the commodity. In this part of the study, it is assumed that the account of the firm is only updated at time 2, similar to a forward contract there is no margin update during the period. Since initial margin that is to be modeled is not considered in the model, it is the same model with the futures contract for one-period when margin call is not deposited during the project. The amount of cash transaction at time 2 led by hedging is denoted by $\left(\tilde{S}_2 - F_{1,2}\right)x_{1,2}$. It is also assumed that the price of the commodity at time 2, which is denoted as $\tilde{S}_2$, has a normal distribution with mean $F_{1,2}$ and standard deviation σ. Another assumption is that the price of the futures contract maturing at a certain time is equal to the expected spot price of the underlying commodity at that time. Thus the expected value of the underlying commodity at time 2 is equal to the price of the futures contract maturing at time 2, i.e. $F_{1,2} = E^{\phi}\left(\tilde{S}_2\right)$. The decision variable for the firm is the amount of futures contract to buy which is denoted as $x_{1,2}$.

All financial transactions of the described problem are briefly shown at Figure 4.1.

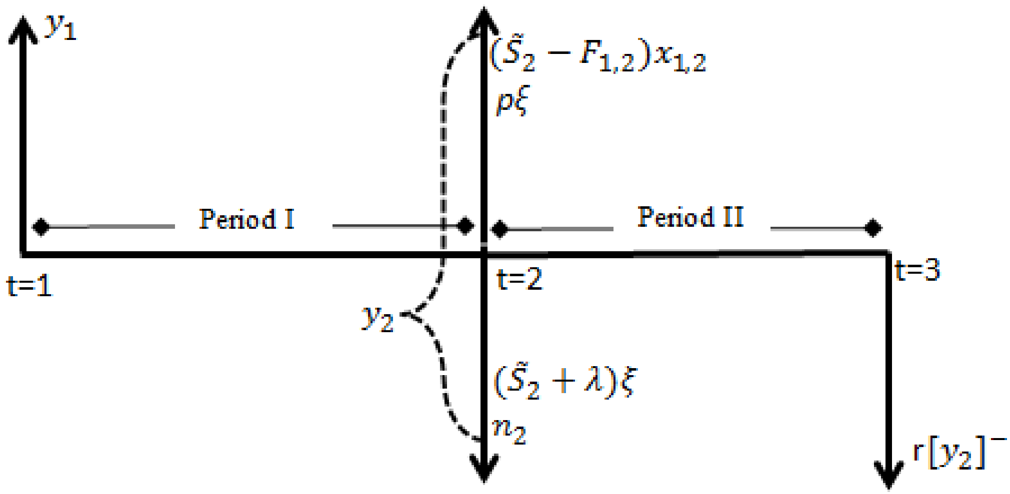

Figure 4.1: Summary of financial transactions for one-period model

To sum up, the firm has only one available decision, it is setting of hedging quantity at the at time 1. Besides it has to borrow money with interest rate r if the cash balance becomes negative. All financial transactions, decisions and actions of the company are shown together at the summary of events table below, at Table 4.1.

I Period Model		
t=1	**t=2**	**t=3**
Observe: $S_1, F_{1,2}$ **Decision:** ➢ Long $x_{1,2}$ units of futures contract	**Observe:** $\tilde{S}_2$ **Cash Flows:** ➢ Transactions included in y_2: • Receive of revenue from developer: $p\xi$ • Cost paid to manufacturer: $(\tilde{S}_2 + \lambda)\xi$ • Transaction due to futures position: $(\tilde{S}_2 - F_{1,2})x_{1,2}$ • Other payments of the firm: n_2 **Conditional Action:** ➢ Loan from bank is taken at the amount of $[y_2]^-$ if needed	**Cash Flows:** ➢ Accumulated FDC paid/accounted: $r[y_2]^-$

Table 4.1: Summary of events for one-period model

In this study, it is assumed that the company aims to maximize the expected total value of the company at the end of the contract. Therefore $V_j(y_i)$ is used as a function decision variables which maximizes the expected change in the value of the company at time j with an initial cash y_i. Besides another function $\triangle V_j(y_i)$ is defined to describe the change in the expected value of the company at the end of period j with an initial cash y_i. Finally, the objective function $V_j(y_i)$ can be defined as a function referring the maximum value of the function $\triangle V_j(y_i)$, which can be translated into a minimization problem by minimizing $\left(-\triangle V_j(y_i)\right)$. Another assumption is that the decisionmaker is risk neutral. Since the current problem is solved in one period framework and the only decision variable is $x_{1,2}$, the problem of the firm can be formulated as follows:

Objective Function:

$$\begin{aligned} V_2(y_1) &= \max_{x_{1,2}} \triangle V_2(y_1) = -\min_{x_{1,2}} \left(-\triangle V_2(y_1)\right) \\ &= \min_{x_{1,2}} E^{\phi}_{s_2}\left[n_2 - \left(p - \tilde{S}_2 - \lambda\right)\xi - \left(\tilde{S}_2 - F_{1,2}\right)x_{1,2} + \beta r\left[y_2\right]^{-}\right] \end{aligned}$$

Subject to:

$$y_2 = y_1 - n_2 + \left(p - \tilde{S}_2 - \lambda\right)\xi + \left(\tilde{S}_2 - F_{1,2}\right)x_{1,2}$$

The terms of the objective function may be explained referring to Table 4.1. The term n_2 is the planned payments of the company, the term
$\left(p - \tilde{S}_2 - \lambda\right)\xi$ is the profit of the company due to the project and
$\left(\tilde{S}_2 - F_{1,2}\right)x_{1,2}$ is the transaction due to the futures contract. Finally, $r\left[y_2\right]^{-}$ is the transaction due to financial distress cost incurred at time 3. Therefore, it is multiplied by β in order to derive the value of it to time 2.

It can easily be observed that the firm will have a positive outcome from the futures contract if the price of the derivative realizes higher than the price of futures contract, i.e. $\tilde{S}_2 > F_{1,2}$. On the other hand, the firm will have a negative outcome if the opposite scenario comes true and this negativity may cause financial distress. Indeed, in order to examine these situations, it is already known that the problem can be shown as the following by multiplying the whole expression by the risk neutral probability density function of random variable $\tilde{S}_2$:

$$V_2(y_1) = \min_{x_{1,2}} \int_0^{\infty} \left[-\triangle V_2(y_1)\right]\phi^{Q}_{S_2}(S_2)\,dS_2$$

As observed from the equation above that the only term in the equation above that is effected by decision $x_{1,2}$ is the financial distress cost premium. The decision variable has no effect on the magnitude of the term $y_1 - n_2 + \left(p - \tilde{S}_2 - \lambda\right)\xi$ and the expected value of the term $\left(\tilde{S}_2 - F_{1,2}\right)x_{1,2}$ is equal to zero. Thus, the first derivative of the objective function will be equal to the derivative of the financial distress cost at $t = 2$ with respect to $x_{1,2}$. Indeed, by minimizing the financial distress cost at $t = 2$, the decisionmaker maximizes the expected value of the company.

Theorem 1: *The optimum hedging decision of the company at time* 1 *for one-period model is* $x^*_{1,2} = \xi$.

In this model, the only risk of facing financial distress is because of the price risk of the commodity. The price risk is completely eliminated by applying the full-hedging decision, i.e. the company should hedge at the same quantity with amount used for the assets delivered at the end of the project. The result approves the deductions in the study of Tanrisever, Duran and Sumer (2011). The details of the proof can be found in Appendix.

The solution found in this part may also be applied to forward contracts as well as futures (when margin call updates are neglected). On the other hand, full-hedging decision is especially suggested to companies running their business with little amount of cash because the expected financial distress cost would be relatively high in such cases.

4.2 Two-Period Model

In this scenario where the firm signs a contract at time 1 for a delivery at time 3; and the company receives the revenue $p\xi$ from the contract and pays to the subcontractor at an amount of $\left(\tilde{S}_3 + \lambda\right)\xi$ at time 3. Similar to the previous model, the firm has a debt of n_i that is planned to be paid at time $t = i$ where $i = 2, 3$. Besides, the company has an initial cash reserve y_1 and the cash balance of the firm at time i after the transactions are realized is again denoted as y_i where $i = 2, 3$. It can also face a financial distress costs of $r\left[y_i\right]^-$ at time 2 and 3, if the net cash balance of the firm decreases to a value lower than zero. It is assumed that simple interest is used for calculation of the loans due to financial distress. Additionally, the firm is assumed to have an overdraft contract with a bank. In this model, financial distress costs during the project are considered and they are collected (or at least accounted) together at time 4.

The firm considers to long $x_{1,3}$ units of futures contracts at time 1 maturing at time 3 for the futures price $F_{1,3}$, and $x_{2,3}$ units of futures contracts at time 2 maturing at time 3 for the futures price $F_{2,3}$. By these actions, the firm aims to decrease the risk of having financial distress cost at time 2 and time 3 due to unstable price of the underlying commodity. In this part of the study, it is assumed that the margin account of the firm is updated at time 2. The amount of cash transactions at time 2 led by hedging are denoted by $\left(\tilde{F}_{2,3} - F_{1,3}\right)x_{1,3}$. Besides, it is assumed that the price of the futures contract maturing at a certain time is equal to the expected spot price of the underlying derivative at that time. Therefore at the end of the futures contract, the account of the firm is updated by addition of $\left(\tilde{S}_3 - F_{2,3}\right)x_{2,3}$. Figure 4.2 shows all cash transactions of the company in two-period problem.

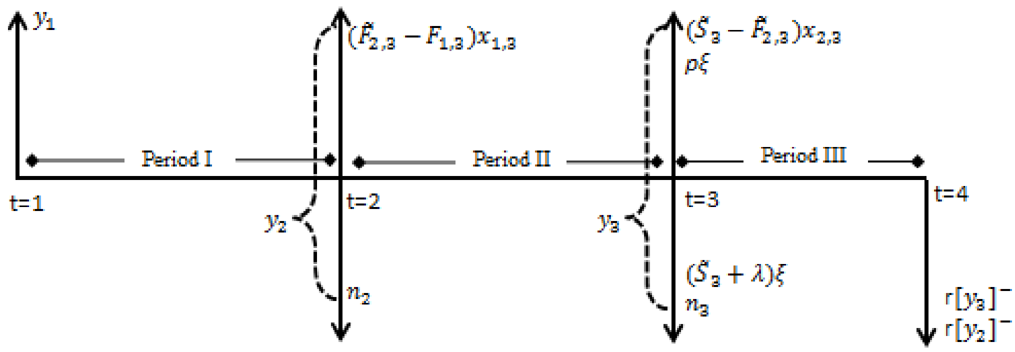

Figure 4.2: Summary of financial transactions for two-period model

To sum up, the firm has only two available decisions, it is the setting of hedging quantity at the at time 1 and 2. The company may change the amount of the futures contract on hand without any additional transactions because the prices are evaluated at the end of every period and initial margin payment is neglected in the model. In addition, it has to borrow money with interest rate r if the cash balance becomes negative. All financial transactions, decisions and actions of the company are shown together at Table 4.2.

II Period Model			
t=1	**t=2**	**t=3**	**t=4**
Observe: $S_1, F_{1,3}$ **Decision:** ➢ Long $x_{1,3}$ units of futures contract	**Observe:** $\tilde{F}_{2,3}$ **Cash Flows:** ➢Transactions included in y_2: • Transaction due to futures position: $(\tilde{F}_{2,3} - F_{1,3})x_{1,3}$ • Other payments of the firm: n_2 **Conditional Action:** ➢Loan from bank is taken at the amount of $[y_2]^-$ **Decision:** ➢Long $x_{2,3}$ units of futures contract	**Observe:** $\tilde{S}_3$ **Cash Flows:** ➢ Transactions included in y_3: • Receive of revenue from developer: $p\xi$ • Cost paid to manufacturer: $(\tilde{S}_3 + \lambda)\xi$ • Transaction due to futures position: $(\tilde{S}_3 - \tilde{F}_{2,3})x_{2,3}$ • Other payments of the firm: n_3 **Conditional Action:** ➢ Loan from bank is taken at the amount of $[y_3]^-$	**Cash Flows:** ➢ Accumulated FDC paid/accounted: $r[y_2]^- + r[y_3]^-$

Table 4.2: Summary of the events for two-period model

As observed from the table above, the decisionmaker should optimize hedging decisions at time 1 and 2 by using the function $V_2(y_1)$. Since $V_2(y_1)$ includes the changes in the expected value of the company in the future, $V_3(y_2)$ should also be included in this function after multiplied by β. After considering all these facts and assumptions, the optimum hedging decision problem of the firm can be summarized as the following:

Objective Function at $t = 1$:

$$\begin{aligned} V_2(y_1) &= \max_{x_{1,3}} \triangle V_2(y_1) = \min_{x_{1,3}} (-\triangle V_2(y_1)) \\ &= \min_{x_{1,3}} E^{\phi}_{F_{2,3}} \left[n_2 + \left(F_{1,3} - \tilde{F}_{2,3}\right) x_{1,3} - \beta V_3(y_2) \right] \end{aligned}$$

subject to:

$$y_2 = y_1 - n_2 + \left(\tilde{F}_{2,3} - F_{1,3}\right) x_{1,3}$$

where:

$$\begin{aligned} V_3(y_2) &= \max_{x_{2,3}} \triangle V_3(y_2) = -\min_{x_{2,3}} (-\triangle V_3(y_2)) \\ &= \min_{x_{2,3}} E^{\phi}_{s_3} \left[n_3 - \left(p - \tilde{S}_3 - \lambda\right)\xi + \left(F_{2,3} - \tilde{S}_3\right) x_{2,3} + \beta r \left([y_2]^- + [y_3]^-\right) \right] \end{aligned}$$

subject to:

$$y_3 = y_2 - n_3 + \left(p - \tilde{S}_3 - \lambda\right)\xi + \left(\tilde{S}_3 - F_{2,3}\right) x_{2,3}$$

It is already mentioned that the decisionmaker aims to maximize the expected total value of the company at the end of the contract, $t = 3$. Therefore a dynamic programming approach is an appropriate way to deal with the problem. Firstly, the problem at time 2 should be solved. According to the previous section which examines the decision for one-period model, the optimum hedging decision of the decisionmaker at time 2 is $x^*_{2,3} = \xi$. In order to solve the optimization problem at $t = 1$, the solution of the first period should be placed into function $V_2(y_1)$, i.e. the decision variable $x_{2,3}$ in function $V_2(y_1)$ should be replaced by ξ. Moreover, it is already known that the objective function should be multiplied by the risk neutral probability density functions of the variables $F_{2,3}$ and S_3 to examine the expected value of it.

Lemma 1: *The optimum hedging decision of the company at $t = 2$ for two-period model is full hedging, i.e. $x^*_{2,3} = \xi$.*

By using Theorem 1, this result is obvious for the company. Unless the magnitude of the ξ changes, the optimum hedging quantity at time 2 stays at the same level.

Theorem 2: *The optimum hedging decision of the company at $t = 1$ for the two-period model is $0 \leqq x^*_{1,3} \leqq \xi$.*

The details of the solution can be found in the Appendix. This result proves that decisionmakers may under-hedge (in some special cases full hedge or no hedge) independent of any parameters if there is an opportunity to update the hedging amount before the project ends. This outcome is logical because company needs to balance two factors that can create financial distress while it can still update its decision in further periods.

- Risk Factor I: The first risk is facing financial distress in the project delivery time due to changing price of the commodity. In order to minimize this risk, the firm needs to hedge at an amount of ξ. If the amount becomes less or more than ξ, the risk of financial distress appears.
- Risk Factor II : The second factor is emergence of financial distress cost before the project delivery time, which is caused by margin call updates. In case of a decrease in the price of the future contract and the company has long position, the company needs to deposit margin call updates; and the same situation occurs in short position when price increased. However, if the hedging amount is too high, the company may face a shortage of cash to pay deposits. Therefore, in order to minimize this risk, the firm should not hedge at any amount.

In conclusion, one risk factor is minimized at full hedging decision and the other is minimized at no hedging. Thus, it is obvious that the optimum hedging amount will be between 0 and ξ but the exact optimum solution depends on the parameters in the model. The details of the proof can be found in Appendix.

For ease of notation, two terms are defined. $\triangle$ denotes the future cash reserve of the company at time 2 when the company does not hedge at all and it is equal to $y_1 - n_2$. Besides, δ is used to describe the expected profit margin of the company per unit and it is calculated as $p - \lambda - F_{1,3}$.

Theorem 3: *The optimum hedging decision at $t = 1$ is $x^*_{1,3} = \xi\triangle / (2\triangle - n_3 + \delta\xi)$ when $\triangle(\triangle - n_3 + \delta\xi) \geqq 0$, and $x^*_{1,3} = -\xi\triangle / (-n_3 + \delta\xi)$ when $\triangle(\triangle - n_3 + \delta\xi) \leqq 0$ satisfying that $0 \leqq x^*_{1,3} \leqq \xi$.*

The result is found by using the features of normal distribution, and true for any kind of central weighted symmetrical distributions. As well as theoretical findings, it may also be used as a beneficial approach in real life problems in case of roughly symmetrical price distribution assumptions. It also provides an opportunity for sensitivity analysis with respect to different values of the variables in the model. The details of the proof can be found in Appendix. The final solution can be summarized as the following.

$$x^*_{1,3} = \begin{cases} \xi\triangle / (2\triangle - n_3 + \delta\xi) & if\ \triangle(\triangle - n_3 + \delta\xi) \geqq 0 \\ -\xi\triangle / (-n_3 + \delta\xi) & if\ \triangle(\triangle - n_3 + \delta\xi) \leqq 0 \end{cases}$$

Theorem 4: *The optimum hedging decision of the company at $t = 1$ when $\triangle = 0$ and $\triangle \neq n_3 - \delta\xi$ for two-period model is not hedging any unit of* the underlying commodity*, i.e.* $x^*_{1,3} = 0$.

If the company will have a zero cash balance at $t = 2$ without hedging, then the cash balance of the company may decrease below zero and hedging could cause a financial distress cost (at %50 per cent probability for symmetrical distributions). In order to prevent a financial distress cost at the end of first period, the company should not hedge at all. The details of the proof can be found in Appendix.

Theorem 5: *In case of $\triangle = n_3 - \delta\xi$ and $\triangle \neq 0$, the optimum hedging decision at $t = 1$ is $x^*_{1,3} = \xi$.*

Expected cash balance of the company at the end of the project without hedging payment is 0 in these conditions. As a result, the cash balance of the company may decrease below zero and hedging could cause a financial distress cost (at %50 per cent probability for symmetrical distributions). In order to prevent a financial distress cost at the end of the project, the company should hedge at amount of ξ. In conclusion the optimum decision is $x^*_{1,3} = \xi$. The details of the proof can be found in Appendix.

Theorem 6: *In case of $\triangle = n_3 - \delta\xi = 0$ and $\triangle = 0$, the hedging decision becomes insignificant while $0 \leqq x^*_{1,3} \leqq \xi$, in other words it does not effect to the expected financial distress cost and all decisions satisfying $0 \leqq x^*_{1,3} \leqq \xi$ are optimum.*

In this case, the company will have a zero cash balance at $t = 2$ without hedging, then the cash balance of the company may decrease below zero and hedging could cause a financial distress cost. In order to prevent a financial distress cost at the end of first period, the company should not hedge at all. On the other hand, expected cash balance of the company at the end of the project without hedging transactions is 0 in these conditions. Therefore, the optimum decision to minimize this risk is full hedging.

In conclusion, both of the factors that affects to the hedging decision of the company are at their highest influence points. Besides, both have the same effect in magnitude but one tries to push the decision to the level 0 while the other pulls it to to ξ. Therefore, all hedging decisions in the interval $[0, \xi]$ are mathematically optimum in these conditions. The details of the proof can be found in Appendix. I

Theorem 7: *In case of $\triangle$ approaching plus or minus infinity, i.e. $\triangle \to \pm\infty$, the optimum hedging quantity line approaches to $\xi/2$. However, hedging lose its significance after some point and becomes a value-neutral action due to the probability density function of the normal variable $\tilde{F}_{1,2}$.*

In case that the firm has a lot of cash, it does not need to hedge its money because there is no risk of having financial distress cost in this condition. Similarly, if its cash reserve is hugely negative, it cannot prevent the financial distress cost by hedging; as a result it would not be necessary to hedge at any amount. The details of the proof can be found in Appendix.

Theorem 8: *In case of n_3 approaching positive or minus infinity, i.e. $n_3 \to \pm\infty$, the optimum hedging quantity line approaches to* 0*. However, multi-optimality occurs due to the probability density function of the normal variable $\tilde{F}_{1,2}$.*

In case that the firm will have a lot of cash at $t = 3$, it does not need to hedge its money because there is no risk of having financial distress cost in this condition. Similarly, if its cash reserve is hugely negative, it cannot prevent the financial distress cost by hedging, as a result it would not be necessary to hedge at any amount. The details of the proof can be found in Appendix.

Summary of theorems at two-period model:

After all the examinations completed in this section, hedging quantity is mainly dependent on the value of $\triangle$ and expression $n_3 + \delta\xi$. By analyzing the values of these variables, an appropriate decision can be made by the decisionmaker. As well as the outcomes in the Table 4.3, It is also true that hedging becomes a value neutral action while $\triangle$ or n_3 is getting closer to $\pm\infty$. The summary of theorems in this section are shown in Table 4.3.

		n_3		
		$\Delta < n_3 - \delta\xi$	$\Delta = n_3 - \delta\xi$	$\Delta > n_3 - \delta\xi$
Δ	$\Delta<0$	Under-hedge: $\xi\Delta/(2\Delta - n_3 + \delta\xi)$	Full-hedge: ξ	Under-hedge: $-\xi\Delta/(-n_3 + \delta\xi)$
	$\Delta=0$	No Hedge: 0	Under-hedge (Multioptimal): $(0, \xi)$	No Hedge: 0
	$\Delta>0$	Under-hedge: $-\xi\Delta/(-n_3 + \delta\xi)$	Full-hedge: ξ	Under-hedge: $\xi\Delta/(2\Delta - n_3 + \delta\xi)$

Table 4.3: Summary of theorems for two-period model

4.3 Three-Period Model

The company signs a project contract at $t = 1$. It receives a revenue of $p\xi$ from the customers when the project is completed at $t = 4$. On the other hand, it pays the cost of assets at an amount of $\left(\tilde{S}_4 + \lambda\right)\xi$ to the subcontractor at the end of the project, again at $t = 4$. Additionally, the firm has debts of n_i that is planned to be paid at $t = i$ where $i = 2, 3, 4$. The cash balance of the firm at time i after the transactions are realized is denoted as y_i where $i = 1, 2, 3, 4$. Obviously, the company may face financial distress costs when the net cash balance of the firm decreases to a value lower than zero at time i, where $i = 2, 3, 4$. In this model, accumulated financial distress costs during the project are paid or accounted at $t = 5$ and they are desired to be minimized.

The firm considers to long $x_{i,4}$ units of futures contracts at time i maturing at $t = 4$ for the futures price $F_{i,4}$ where $i = 1, 2, 3$. By hedging, the firm aims to decrease the risk of having financial distress costs, $r\left[y_i\right]^-$, at $i = 2, 3, 4$ due to the unstable price of the underlying commodity in its project. In this part of the study, it is assumed that the margin account of the firm is updated at $t = 2$ and $t = 3$. The amount of cash transactions at time i led by hedging are denoted by $\left(\tilde{F}_{i,4} - F_{i-1,4}\right)x_{i-1,4}$ where $i = 2, 3$. Due to the previously mentioned price assumptions of the futures contracts and spot prices of the commodity, it is assumed that $F_{1,4} = E^{\phi}\left(\tilde{F}_{2,4}\right) = E^{\phi}\left(\tilde{F}_{3,4}\right) = E^{\phi}\left(\tilde{S}_4\right)$ for a decisionmaker at $t = 1$. The decision variable at $t = i$ is the amount of the units of futures contract undertaken at $t = i$ for maturity date $t = 4$ which is denoted as $x_{i,4}$ where $i = 1, 2, 3$. All financial transactions of the described problem are briefly shown at Figure 4.3.

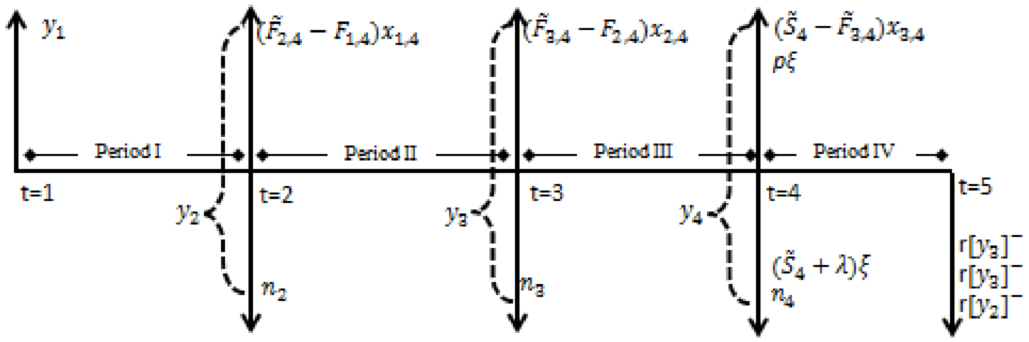

Figure 4.3: Summary of financial transactions for three-period model

Briefly, there are four time periods, as a result five different time epochs for events. The firm has to borrow money from the bank if cash balance goes to negative. Therefore, the company tries to minimize its expected financial distress cost by using three different decision variables. A clear and brief view of the information on the problem is shown in Table 4.4.

III Period Model				
t=1	t=2	t=3	t=4	t=5
Observe: $S_1, F_{1,4}$ **Decision:** ➢ Long $x_{1,4}$ units of futures contract	**Observe:** $\tilde{F}_{2,4}$ **Cash Flows:** ➢Transactions included in y_2: • Transaction due to futures position: $(\tilde{F}_{2,4} - F_{1,4})x_{1,4}$ • Other payments of the firm: n_2 **Conditional Action:** ➢Loan from bank is taken at the amount of $[y_2]^-$ **Decision:** ➢ Long $x_{2,4}$ units of futures contract	**Observe:** $\tilde{F}_{3,4}$ **Cash Flows:** ➢Transactions included in y_3: • Transaction due to futures position: $(\tilde{F}_{3,4} - F_{2,4})x_{2,4}$ • Other payments of the firm: n_3 **Conditional Action:** ➢Loan from bank is taken at the amount of $[y_3]^-$ **Decision:** ➢Long $x_{3,4}$ units of futures contract	**Observe:** $\tilde{S}_4$ **Cash Flows:** ➢ Transactions included in y_4: • Receive of revenue from developer: $p\xi$ • Cost paid to manufacturer: $(\tilde{S}_4 + \lambda)\xi$ • Transaction due to futures position: $(\tilde{S}_4 - \tilde{F}_{3,4})x_{3,4}$ • Other payments of the firm: n_4 **Conditional Action:** ➢ Loan from bank is taken at the amount of $[y_4]^-$	**Cash Flows:** ➢ Accumulated FDC paid/accounted: $r[y_2]^- + r[y_3]^- + r[y_4]^-$

Table 4.4: Summary of the events for three-period model

As observed from the table above, the decisionmaker should optimize hedging decisions at three different time epochs, time $1, 2$ and 3. Similar to the previous models, in order to achieve this aim, the company use the function $V_2(y_1)$, the maximum value of the expected the future cash change of the company at $t = 1$ which is discounted to the respective value of $t = 2$. After considering all these facts and assumptions, the optimum hedging decision problem of the company at $t = 1$ can be summarized as:

$$\begin{aligned} V_2(y_1) &= \max_{x_{1,3}} \triangle V_2(y_1) = \min_{x_{1,3}} (-\triangle V_2(y_1)) \\ &= \min_{x_{1,4}} E^{\phi}_{F_{2,4}} \left[n_2 + \left(F_{1,4} - \tilde{F}_{2,4} \right) x_{1,4} - \beta V_3(y_2) \right] \end{aligned}$$

subject to:

$$y_2 = y_1 - n_2 + \left(\tilde{F}_{2,4} - F_{1,4}\right) x_{1,4}$$

where:

$$V_3(y_2) \quad = \quad \max_{x_{2,4}} E^{\phi}_{F_{2,4}} \left[-n_2 + \left(\tilde{F}_{3,4} - F_{2,4}\right) x_{1,4} + \beta V_4(y_3)\right]$$

subject to:

$$y_3 = y_2 - n_3 + \left(\tilde{F}_{3,4} - F_{2,4}\right) x_{1,4}$$

and:

$$V_4(y_3) \quad = \quad \max_{x_{3,4}} E^{\phi}_{s_4}\left[-n_4 + \left(p - \tilde{S}_4 - \lambda\right)\xi + \left(\tilde{S}_4 - F_{3,4}\right) x_{3,4} - \beta r\left([y_2]^- + [y_3]^- + [y_4]^-\right)\right]$$

subject to:

$$y_4 = y_3 - n_4 + \left(p - \tilde{S}_4 - \lambda\right)\xi + \left(\tilde{S}_4 - F_{3,4}\right) x_{3,4}$$

Lemma 2: *The optimum hedging decision of the company at $t = 3$ for three-period model is full-hedging, i.e. $x^*_{3,4} = \xi$.*

This result follows from Theorem 1. Unless the magnitude of the ξ does not change, the optimum hedging quantity at $t = 3$ stays at the same level.

Lemma 3: *Suppose that the terms $\triangle'$ and δ are defined such as $\triangle' = y_2 - n_3$ and $\delta = p - \lambda - \tilde{F}_{2,4}$. The optimum hedging decision at $t = 1$ is $x^*_{2,4} = \xi\triangle' / \left(2\triangle' - n_4 + \delta\xi\right)$ when $\triangle'\left(\triangle' - n_4 + \delta\xi\right) \geqq 0$, and $x^*_{2,4} = -\xi\triangle' / (-n_3 + \delta\xi)$ when $\triangle'\left(\triangle' - n_4 + \delta\xi\right) \leqq 0$ satisfying that $x^*_{2,4} \leqq \xi$.*

The result directly follows from Theorem 2. On the other hand, both y_2 and δ has stochastic term $\tilde{F}_{2,4}$ inside and it means that the decision at $x^*_{2,4}$ is dependent on the realized value of the futures contract.

Theorem 9: *The optimum hedging decision of the company at $t = 1$ for three-period model is $0 \leqq x^*_{1,4} \leqq \overline{x}^*_{2,4}$.*

After examinations of all possible cases with respect to the decision variables, the optimum policy is found as underhedging and details of the solution may be found in Appendix.

The interpretation of Theorem 9 is that the decisionmaker should increase its hedging amount while getting closer to the delivery date in an optimum policy, such that $0 \leqq x^*_{1,4} \leqq \overline{x}^*_{2,4} \leqq x^*_{3,4} = \xi$. However, it should be noted that this problem is a dynamic problem and after the futures prices in a period is realized, it has to be solved again for the next period.

Moreover, Theorem 9 can be generalized for the cases where the decisionmaker has opportunities to update its hedge amount more than three times, in Theorem 10.

Theorem 10: *The optimum hedging decision of the decisionmaker at $t = 1$ for n period model is always underhedging and is less than or equal to the expected hedging amount of the next period, i.e. $0 \leqq x^*_{1,n} \leqq \overline{x}^*_{2,n} \leqq ... \leqq \overline{x}^*_{n-2,n} \leqq x^*_{n-1,n} = \xi$.*

After examinations of all possible cases with respect to the decision variables, the optimum policy is found as underhedging and details of the proof may be found in Appendix.

The interpretation of this theorem is the following. The decisionmaker should increase its hedging amount while getting closer to the delivery date in an optimum policy, such that $0 \leqq \overline{x}^*_{1,n} \leqq \overline{x}^*_{2,n} \leqq \ldots \leqq \overline{x}^*_{n-2,n} \leqq x^*_{n-1,n} = \xi$; since the decisionmaker has more flexibility to update the hedge amount further from the delivery date. As mentioned earlier, it is an important point that this problem is a dynamic problem and after the random variables in a period are realized, it has to be solved again for the next period. In case of a false decision that is made during a period, the decisionmaker may justify the optimum hedging quantities in the beginning of the periods again and should be able to increase or decrease the hedging quantity. In such a case, the inequality stated in Theorem 10 may not be satisfied; however the reason of it is the false hedging decisions during the previous periods.

Theorem 11: *Similar to two-period problem, hedging is becoming a value neutral action when $\triangle$ goes to $\pm\infty$ in multiperiod model.*

When the company has a lot of cash, it is not necessary to hedge at time 1. As long as facing financial distress is out of possibility, hedging at reasonable amounts does not affect its expected cash balance in the future. Consequently, hedging becomes a value neutral action as long as the company hedge at reasonable amounts. This amount is affected by the magnitude of $\triangle$ and standard deviation σ.

4.4 Approximation of Two-Period Model with Compound Interest

As mentioned earlier, this paper mainly aims to examine how to avoid financial distress cost under simple interest. However, compound interest is also charged by financial institutions in real life and the results found in two-period simple interest model may be used as an approximation to solution of the same problem with compound interest. By the simple interest approximation, we can provide a solid decision methodology for real life decisionmakers using compound interest rates.

The mathematical model is almost the same with simple interest except that the financial distress costs are directly accumulated at the period it is incurred. Therefore, interest of financial distress cost is also accounted in the model. After considering all these facts and assumptions, the optimum hedging decision problem with compound interest rates for a two-period problem may be summarized as the following:

$$
\begin{aligned}
V_2(y_1) &= \max_{x_{1,3}} \triangle V_2(y_1) = \min_{x_{1,3}} \left(-\triangle V_2(y_1)\right) \\
&= \min_{x_{1,3}} E^{\phi}_{F_{2,3}} \left[n_2 + \left(F_{1,3} - \tilde{F}_{2,3}\right) x_{1,3} + r\left[y_2\right]^- - \beta V_3(y_2) \right]
\end{aligned}
$$

subject to:

$$
y_2 = y_1 - n_2 + \left(\tilde{F}_{2,3} - F_{1,3}\right) x_{1,3}
$$

where:

$$
\begin{aligned}
V_3(y_2) &= \max_{x_{2,3}} \triangle V_3(y_2) = -\min_{x_{2,3}} \left(-\triangle V_3(y_2)\right) \\
&= \min_{x_{2,3}} E^{\phi}_{s_3} \left[n_3 - \left(p - \tilde{S}_3 - \lambda\right)\xi + \left(F_{2,3} - \tilde{S}_3\right) x_{2,3} + \beta r\left[y_3\right]^- \right]
\end{aligned}
$$

subject to:

$$y_3 = y_2 - n_3 + \left(p - \tilde{S}_3 - \lambda\right)\xi + \left(\tilde{S}_3 - F_{2,3}\right)x_{2,3} - r\left[y_2\right]^- .$$

The company tries to minimize the total of expected financial distress costs discounted to time 2 which is formulated as $r\left(\left[y_2\right]^- + \beta\left[y_3\right]^-\right)$. However, the most critical observation to make in this formulation is the fact that the term $\left[y_3\right]^-$ includes $\left[y_2\right]^-$ inside. This makes the problem harder to solve by classical mathematical approaches.

Lemma 4: *The optimum hedging decision of the company at $t = 2$ for two-period model is full hedging, i.e. $x^*_{2,3} = \xi$.*

In one-period problem, there is no difference in the optimum hedging quantity because there is no difference between compound and simple interest rates formulation for one-period. In addition, the company is again trying to minimize the expected value of the only possible financial distress cost in the future. Therefore, interest of interest is not considered and the optimum hedging decision at time 2 is equal to the quantity of commodity used for the project. When this result is used in the model, further calculations are possible.

Theorem 12: *The optimum hedging decision of the company at $t = 1$ for two-period model is $0 \leqq x^*_{1,3} \leqq \xi$.*

Similar to the model with simple interest, the optimum hedging decision at time 1 is also between 0 and ξ. The details of the proof can be found in Appendix.

Theorem 13: *The optimum hedging decision of the company at $t = 1$ when $\triangle = y_1 - n_2 = 0$ for two-period model with compound interest is not hedging any units at all, i.e. $x^*_{1,3} = 0$.*

Similar to the model with simple interest, the cash balance of the company may decrease below zero and hedging could cause a financial distress cost (at %50 per cent probability for symmetrical distributions) if the company will have a zero cash balance at $t = 2$ without hedging. In order to prevent a financial distress cost at the end of first period, the company should not hedge at all. The details of the proof can be found in Appendix.

Theorem 14: *The optimum hedging decision of the company at $t = 1$ cannot be higher than $\xi/(1+r)$ when $\triangle < 0$. The optimum hedging quantity will be in the interval of $0 \leqq x^*_{1,3} \leqq \xi/(1+r)$.*

If the net cash reserve of the company will be less than zero at time 2 except the transactions due to hedging, then the maximum optimum hedging amount of the decisionmaker is $\xi/(1+r)$. Otherwise the firm can face a higher expected financial distress cost due to hedging and the expected net cash reserve of the company will decrease at $t = 1$. The details of the proof can be found in Appendix.

Although the model with compound interest behaves similar to the model with simple interest for two-period, finding mathematical results for three or more periods is not as easy as the model with simple interest. Therefore, our idea is to use simple interest model as an approximation to this model and use numerical data analysis to judge the goodness of this approximation.

4.5 One-Period Model with Call Option

This case is assumed to be a scenario where there are no financial transactions between sign of the contract and delivery of project. The firm has a debt at the amount of n_2 that is planned to be paid at time 2. It will receive the revenue from customer at an amount of $p\xi$ and will have to pay the cost of the assets to the subcontractor at an amount of $(S_2 + \lambda)\,\xi$ also at time 2. The cash balance of the firm at the beginning of the project is y_1.

In the beginning of the period, the firm considers to long $x_{1,2}$ units of call options on the commodity futures at time 1 maturing at time 2 for the strike price $F_{1,2}$, which is the expected spot price of the commodity at time 2. By this action, the firm aims to decrease the risk of having financial distress cost at time 2 due to unstable price of the commodity. Unlike the using of future contract to hedge, this time the firm needs to pay risk premium to buy the contract at time 1 and the unit price of the option contract is equal to h where, $h = \int_{F_{1,2}}^{\infty} \left(\tilde{S}_2 - F_{1,2}\right) \phi^{Q}_{S_2}(S_2)\, dS_2$. By this action, the firm aims to decrease the risk of having financial distress cost at time 2 due to unstable price of commodity. It is assumed that the account of the firm is only updated at time 2. The amount of cash transaction at time 2 led by hedging is denoted by $\left(\tilde{S}_2 - F_{1,2}\right)^{+} x_{1,2}$. It means that the firm will get paid by $\left(\tilde{S}_2 - F_{1,2}\right) x_{1,2}$ when realized spot price of the commodity at time 2 is higher than the strike price of the option contract, which is $F_{1,2}$ for our model. Otherwise, there will be no financial transactions at time 2 due to the option contract. The only decision variable for the firm is the amount of call options to buy at time 1 for maturity date time 2, which is denoted as $x_{1,2}$.

Unlike the other models considered in this study, the firm is assumed to pay a risk premium to buy the financial derivative. Therefore, $y_1^{'}$ is defined the cash balance of the firm after buying the option contracts and it is formulated as $y_1^{'} = y_1 - hx_{1,2}$.

All financial transactions of the described problem are briefly shown at the Figure 4.4.

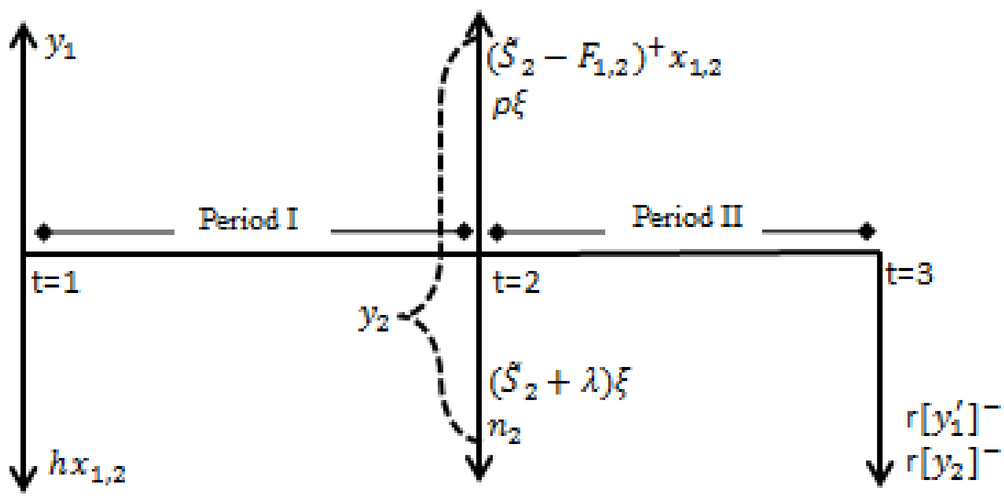

Figure 4.4: Summary of financial transactions for one-period model with call option

To sum up, the company has only one available decision, the hedging quantity at the at time 1. Besides it has to borrow money with interest rate r if the cash balance becomes negative. All financial transactions, decisions and actions of the company are shown together at the summary of events table below, at Table 4.5.

I Period Model with Call Option		
t=1	**t=2**	**t=3**
Observe: $S_1, F_{1,2}$ **Cash Flows:** ➢ Payment due to buying call options: $hx_{1,2}$ **Decision:** ➢ Long $x_{1,2}$ units of options contract	**Observe:** $\tilde{S}_2$ **Cash Flows:** ➢ Transactions included in y_2: • Receive of revenue from developer: $p\xi$ • Cost paid to manufacturer: $(\tilde{S}_2+\lambda)\xi$ • Transaction due to call option: $(\tilde{S}_2-F_{1,2})^+x_{1,2}$ • Other payments of the firm: n_2 **Conditional Action:** ➢ Loan from bank is taken at the amount of $[y_2]^-$ if needed	**Cash Flows:** ➢ Accumulated FDC paid/accounted: $r[y_1']^- + r[y_2]^-$

Table 4.5: Summary of events for one-period model with call option

Again the company aims to maximize the expected total value of the company at the end of the project. Therefore, $V_j(y_i)$ is used as the objective function which maximizes the expected change in the value of the company at $t = j$ with an initial cash y_i. Besides another function $\triangle V_{i+1}(y_i)$ is defined to describe the change in the expected value of the company in period i with an initial cash y_i. Thus, the objective function $V_j(y_i)$ can be defined as a function referring the maximum value of the function $\triangle V_{i+1}(y_i)$, which can be translated into a minimization problem by minimizing $-\triangle V_{i+1}(y_i)$. Since the only decision variable is $x_{1,2}$, the decision problem faced by the company can be summarized as follows:

$$
\begin{aligned}
V_2(y_1) &= \max_{x_{1,2}} \triangle V_2(y_1) \\
&= \min_{x_{1,2}} \; -\triangle V_2(y_1) \\
&= \min_{x_{1,2}} E^{\phi}_{s_2}\left[n_2 - \left(p - \tilde{S}_2 - \lambda\right)\xi + hx_{1,2} - \left[\tilde{S}_2 - F_{1,2}\right]^+ x_{1,2} \; - r\left[y_1'\right]^- + \beta r\left[y_2\right]^-\right]
\end{aligned}
$$

Subject to:

$$y_2 = y_1 - n_2 + \left(p - \tilde{S}_2 - \lambda\right)\xi - hx_{1,2} + \left[\tilde{S}_2 - F_{1,2}\right]^+ x_{1,2}$$

$$y_1' = y_1 - hx_{1,2}$$

where

$$h = \int_{F_{1,2}}^{\infty} \left(\tilde{S}_2 - F_{1,2}\right)\phi^{Q}_{S_2}(S_2)\, dS_2$$

The terms of the objective function may be explained referring to Figure 4.5. The term $hx_{1,2}$ is the cost of buying or selling call options, n_2 is the planned payments of the company, the term $\left(p - \tilde{S}_2 - \lambda\right)\xi$ is the profit of the company due to the project and $\left(\tilde{S}_2 - F_{1,2}\right)^+ x_{1,2}$ is the possible gain from call options. Finally, $r\left[y_2\right]^-$ is financial distress cost incurred at time 3. Therefore, $r\left[y_2\right]^-$ is multiplied by β in order to discount it to time 2.

The only term in the objective function affected by the decision of $x_{1,2}$ is the financial distress cost premium. The decision variable $x_{1,2}$ has no effect on the magnitude of the term $y_1 - n_2 + \left(p - \tilde{S}_2 - \lambda\right)\xi$ and the expected value of the term $\left(\tilde{S}_2 - F_{1,2}\right)x_{1,2}$ is equal to zero. Thus, the first derivative of the objective function will be equal to the derivative of the financial distress cost at $t = 2$ with respect to $x_{1,2}$. Indeed, by minimizing the financial distress cost at time 2, the decisionmaker maximizes the expected value of the company.

Theorem 15: *The optimum amount of call options to buy at time* 1 *with the strike price* $F_{1,2}$ *and maturity date* 2 *is always less than or equal to* $\frac{y_1}{h}$.

The company buys some call options to prevent financial distress costs at $t = 2$ by paying the expected payoff. However, if the firm pays expected payoff more than its budget at $t = 1$, it will immediately face financial distress cost instead of avoiding it. As a result, it should not buy more call options than its budget at $t = 1$. The details of the proof can be found in the Appendix.

When the result of Theorem 15 is utilized, the overall model can be rewritten:

$$
\begin{aligned}
V_2(y_1) &= \max_{x_{1,2}} \triangle V_2(y_1) \\
&= \min_{x_{1,2}} \; -\triangle V_2(y_1) \\
&= \min_{x_{1,2}} E^{\phi}_{s_2}\left[n_2 - \left(p - \tilde{S}_2 - \lambda\right)\xi + hx_{1,2} - \left[\tilde{S}_2 - F_{1,2}\right]^+ x_{1,2} + \beta r\left[y_2\right]^-\right]
\end{aligned}
$$

Subject to:

$$
y_2 = y_1 - n_2 + \left(p - \tilde{S}_2 - \lambda\right)\xi - hx_{1,2} + \left[\tilde{S}_2 - F_{1,2}\right]^+ x_{1,2}
$$

$$
x_{1,2} < \frac{y_1}{h}
$$

where

$$
h = \int_{F_{1,2}}^{\infty} \left(\tilde{S}_2 - F_{1,2}\right)\phi^{Q}_{S_2}(S_2)\, dS_2
$$

Theorem 16: $\frac{y_1}{h} + \left(\frac{(p-\lambda-F_{1,2})\xi - n_2}{h}\right)^+ \geqq x^*_{1,2} \geqq \xi$ *as long as* $\frac{y_1}{h} + \left(\frac{(p-\lambda-F_{1,2})\xi - n_2}{h}\right)^-$ *is more than* ξ.

The details of the proof may be found in Appendix. Supporting numerical results may also be found in Chapter 5, Numerical Analysis.

When $\frac{y_1}{h} + \left(\frac{(p-\lambda-F_{1,2})\xi-n_2}{h}\right)^- < \xi$, an optimal as suggested by Theorem 16 is not possible. In such situations, the company may intuitively try to equalize its hedging level to $\frac{y_1}{h} + \left(\frac{(p-\lambda-F_{1,2})\xi-n_2}{h}\right)^-$ in order to get closer to the optimality due to convexity. Although we do not prove this claim mathematically in this study, it will also be examined in numerical analysis part of this study.

CHAPTER 5

NUMERICAL ANALYSES

Numerical analysis is performed to observe the effects of mathematical results and gain further insight that could not be acquired by formal mathematical analysis. Simulation models are utilized in numerical analysis, and random variables are created by suitable random number generators for the related distributions. For all decision settings, the same random price streams are used and financial distress costs are calculated accordingly.

In mathematical models used throughout this study, futures contract prices are assumed to be stochastic. Indeed $F_{j+1,i}$ is assumed to follow a normal distribution with mean $F_{j,i}$ and standard deviation σ and the spot price of the underlying commodity $\tilde{S}_i$ has a mean of $F_{i-1,i}$ and standard deviation σ. Therefore, the realized value of the futures contract at a period is used as the mean of the futures contract price at the next period maturing at the same date in numerical analysis.

In order to eliminate the bias of random number generator, a specific approach is used. It is already stated that $F_{1,i}$ is the mean and σ is the standard deviation for the price of futures contract $F_{2,i}$ at the beginning of the project. These variables are utilized in the normal distribution random number generator and necessary random numbers for futures prices in the model are created. Suppose that m random variables are generated for period 2 (in other words, random $\tilde{F}_{2,i}$ values are created for m times) by using the generator, and j^{th} number is referred as $F_{2,i}^{(j,+)}$. After we create $F_{2,i}^{(j,+)}$ using the random number generator with the mean of $F_{1,i}$, another number $F_{2,i}^{(1,j,-)}$ is calculated such that $F_{2,i}^{(j,+)} + F_{2,i}^{(j,-)} = 2F_{1,i}$. Thus, totally $2m$ numbers are created denoted as $F_{2,i}^{(j,\Omega_1)}$ and where Ω_1 may be either $(+)$ or $(-)$ and $1 \leqq j \leqq m$.

For the next period, this time the numbers $F_{2,i}^{(j,\Omega_1)}$ are placed into the random number generators as mean of the price at the next period, and $2m$ numbers are generated denoted as $F_{3,i}^{(j,\Omega_1,+)}$. By using these numbers, the numbers $F_{3,i}^{(j,\Omega_1,-)}$ are also created such that $F_{3,i}^{(j,\Omega_1,+)} + F_{3,i}^{(j,\Omega_1,-)} = 2F_{2,i}^{(j,\Omega_1)}$ where Ω_1 may either be $+$ or $-$ and $1 \leqq j \leqq m$. As a result, $4m$ numbers are obtained as $F_{3,i}^{(j,\Omega_1,\Omega_2)}$ for the future price of the second period where Ω_1 and Ω_2 may either be $(+)$ or $(-)$ and $1 \leqq j \leqq m$. The random number generation procedure is illustrated below for the first two-periods.

$$F_{1,i} \begin{cases} F_{2,i} = F_{2,i}^{(j,+)} & \begin{cases} F_{3,i} = F_{3,i}^{(j,+,+)} \\ F_{3,i} = F_{3,i}^{(j,+,-)} \end{cases} \\ F_{2,i} = F_{2,i}^{(j,-)} & \begin{cases} F_{3,i} = F_{3,i}^{(j,-,+)} \\ F_{3,i} = F_{3,i}^{(j,-,-)} \end{cases} \end{cases}$$

By this procedure, $4m$ different price sets are composed for the first two-period, and the overall means of both of the random stages are equal to $F_{1,i}$. The price sets of futures contracts for $t = 1, 2, 3$ are as the following where j is an integer such that $1 \leqq j \leqq m$.

$$\begin{gathered}\left(F_{1,i}, F_{2,i}^{(j,+)}, F_{3,i}^{(j,+,+)}\right)\\ \left(F_{1,i}, F_{2,i}^{(j,+)}, F_{3,i}^{(j,+,-)}\right)\\ \left(F_{1,i}, F_{2,i}^{(j,-)}, F_{3,i}^{(j,-,+)}\right)\\ \left(F_{1,i}, F_{2,i}^{(j,-)}, F_{3,i}^{(j,-,-)}\right)\end{gathered}$$

As stated before, all sets should satisfy $F_{2,i}^{(j,+)} + F_{2,i}^{(j,-)} = 2F_{1,i}$ and $F_{3,i}^{(j,\Omega_1,+)} + F_{3,i}^{(j,\Omega_1,-)} = 2F_{2,i}^{(j,\Omega_1)}$for $\Omega_1 = \{+, -\}$. All numerical analyzes in this study are completed by using this procedure with 100,000 sample data size.

During numerical analysis, multi-optimality is observed as Theorem 7 and 8 suggested. The multi-optimal decisions exits due to the normally distributed future prices and significance limit of the decisionmaker for the expected financial distress cost. According to the observations, multi-optimality exists while the terms b and c are becoming far from the mean $F_{1,3}$ with respect to the standard deviation σ. In order to analyze this phenomenon, let us define a magnitude ζ that can be neglected by the decisionmaker in the objective function. In addition, a function $D(x_{1,3}, \alpha)$ is defined as the following:

$$D(x_{1,3}, \alpha) = \left| \triangle V_2\left(y_1 | x_{1,3} = x_{1,3}^*\right) - \triangle V_2\left(y_1 | x_{1,3} = x_{1,3}^* + \alpha\right) \right|$$

Suppose that an optimum interval of $x_{1,3}^*$ is identified such that $\left[x_{1,3}^* + \alpha^-, x_{1,3}^* + \alpha^+\right]$ where α^+ and α^- are integers as $\alpha^+ > 0 > \alpha^-$. It means that α^+ is defined as the positive deviation from the optimum decision $x_{1,3}^*$ and α^- is the negative deviation. Then the deviations α^+ and α^- are calculated by solving $D(x_{1,3}, \alpha^+) \cong D(x_{1,3}, \alpha^-) \geqq \zeta$. However, a linear differential approximation may be applied by assuming that new line $D(x_{1,3}, \alpha^+) \cong \left| \frac{\alpha^+ (1/2) d\left(\triangle V_2\left(y_1 | x_{1,3} = x_{1,3}^* + \alpha^+\right) + \triangle V_2\left(y_1 | x_{1,3} = x_{1,3}^*\right)\right)}{dx_{1,3}} \right|$. Since $\frac{d\left(\triangle V_2\left(y_1 | x_{1,3} = x_{1,3}^*\right)\right)}{dx_{1,3}}$ is equal to 0, the function $D(x_{1,3}, \alpha^+)$ becomes the following:

$$D(x_{1,3}, \alpha^+) \cong \left| \frac{\alpha^+ (1/2)\, d\left(\triangle V_2\left(y_1 | x_{1,3} = x_{1,3}^* + \alpha^+\right)\right)}{dx_{1,3}} \right| \cong \zeta$$

Then the value of α^+ is approximately calculated as the following:

$$\alpha^+ \cong \frac{2\zeta}{\left| \frac{d\left(\triangle V_2\left(y_1 | x_{1,3} = x_{1,3}^* + \alpha^+\right)\right)}{dx_{1,3}} \right|}$$

5.1 Analysis of Two-Period Model

The variables $\triangle$ (cash balance of the company at $t = 2$ without hedging) and n_3 (payment at $t = 3$) are the most important variables. Therefore, the behavior model will be examined as these two main parameters are changing.

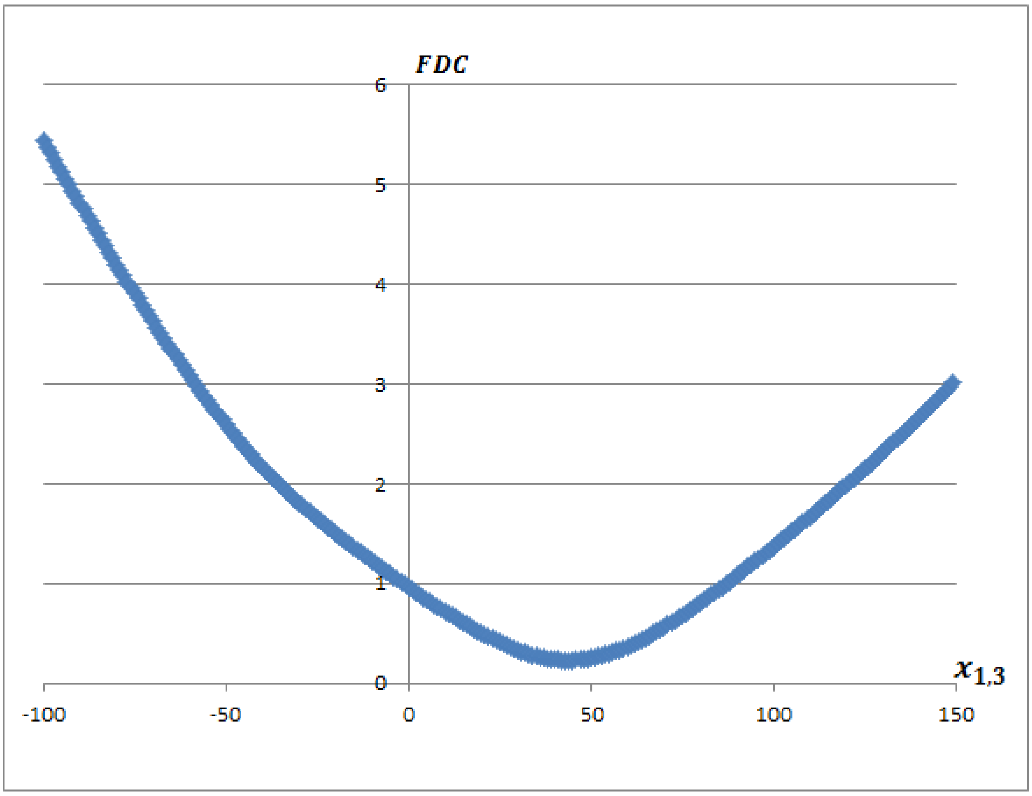

Figure 5.1: Expected total financial distress costs for $x_{1,3}$ values for two-period model with base parameter setting

A set of values of the parameters will be taken into account as base settings for numerical analysis. In this base setting, the values of the variables are chosen as $y_1 = 100$, $n_2 = 30$, $n_3 = 30$, $\xi = 100$, $p = 11$, $\lambda = 0.5$, $F_{1,3} = 10$, $\sigma = 1$, $r = 0.1$, $\beta = 0.95$. Besides, amounts less then $\zeta = 0.0001$ units of expected financial distress cost is supposed to be insignificant for the company, as a result multi-optimality exist for the company when difference of financial distress costs with respect to different decisions are less than $\zeta = 0.0001$. Expected total financial distress costs for the base case with respect to changing values of $x_{1,3}$ are shown below. For each $x_{1,3}$ decision in Figure 5.1, FDC represents the average total financial distress cost for $100,000$ different realizations of the future and commodity price realization. As observed from Figure 5.1, the function is convex and the optimum decision for the model is 45 which is less than $\xi = 100$ as Theorem 2 suggests.

5.1.1 Analysis of Two-Period Model With Changing Values of $\triangle$

$\triangle = y_1 - n_2$ is an effective parameter to the hedging decision and it is also important for real life decisionmakers. Moreover, while two-period model is examined mathematically in previous sections of this study, a summary of theorems table is already provided at Table 4.3 and it is primarily based on the values of $\triangle$. Therefore, the first parameter to consider should be $\triangle$. Typical behavior of the optimal decision for changing $\triangle$ values is obtained in Figure 5.2 when the model is solved with base setting parameters.

In Figure 5.2, X axis represents different $\triangle$ values while Y axis shows the optimum hedging deci-

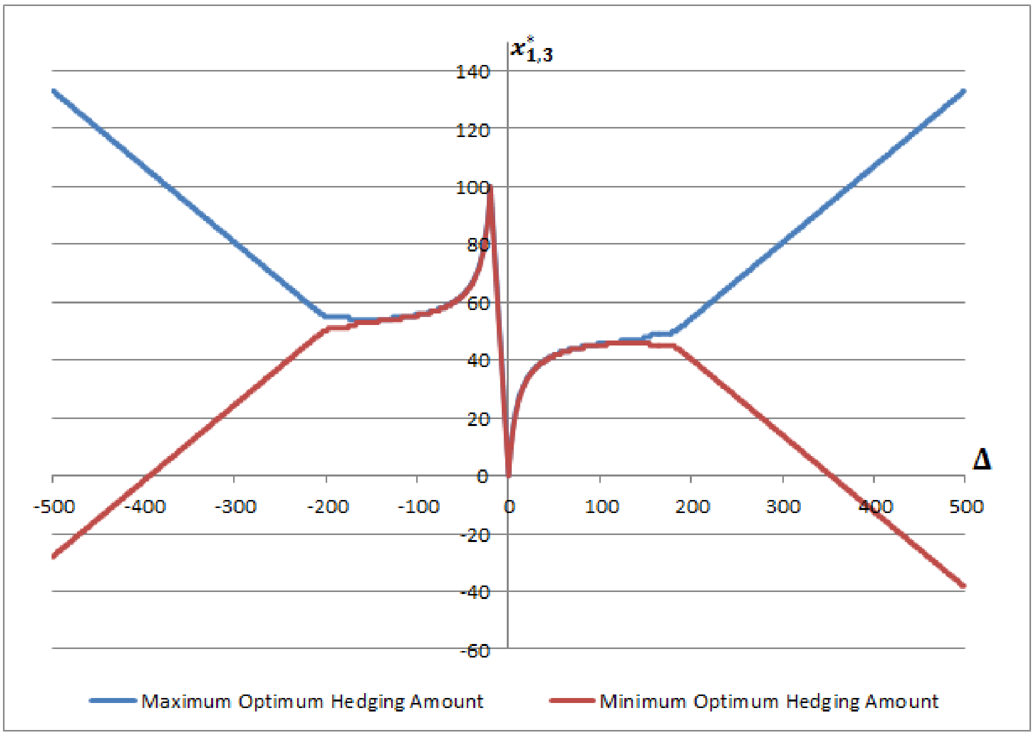

Figure 5.2: Simulated optimum hedging decisions corresponding to changing values of $\triangle$ for two-period model with base parameters

sions for corresponding $\triangle$ values. In both extreme sides of X axis, multiple optimality is observed as suggested in Theorem 7 and the points where multi-optimality started to be observed will also be examined in this section. On the other hand, firstly, the behavior of the optimal hedging quantity function should be inspected by using Theorem 3. Theorem 3 suggests the following result for solution of the model:

$$x^*_{1,3} = \begin{cases} \xi\triangle/(2\triangle - n_3 + \delta\xi) & if\, \triangle(\triangle - n_3 + \delta\xi) \geqq 0 \\ -\xi\triangle/(-n_3 + \delta\xi) & if\, \triangle(\triangle - n_3 + \delta\xi) \leqq 0 \end{cases}$$

In order to check the conformity to the theoretical results obtained in previous settings, the expressions $\xi\triangle/(2\triangle - n_3 + \delta\xi)$ and $-\xi\triangle/(-n_3 + \delta\xi)$ should also be drawn while $\triangle$ is changing and they will be called as Line 1 and Line 2, respectively in Figure 5.3. Figure 5.3 shows the optimum hedging values suggested by Theorem 3 with corresponding $\triangle$ values. It is observed that the solution found by using Theorem 3 is an exact fit to the solution found by simulation in Figure 5.3 except multi-optimum solutions. Consequently, the multi-optimality conditions should also be examined.

As stated in Theorem 4, the optimum hedging amount $x^*_{1,3}$ is equal to 0 when $\triangle = 0$. On the other hand, the optimum hedging amount is equal to ξ when $\triangle = n_3 - \delta\xi$. Between these two points, the optimum line is flat with a constant slope of $\xi/(n_3 - \delta\xi)$ which is shown as Line 2 at Figure 5.3. Outside this region, the optimum line has a curvy shape since the optimum hedging amount is equal to $\xi\triangle/(2\triangle - n_3 + \delta\xi)$ and it is represented as Line 1 in Figure 5.3. Besides, the line is converging to $\xi/2$ at both sides of X axis.

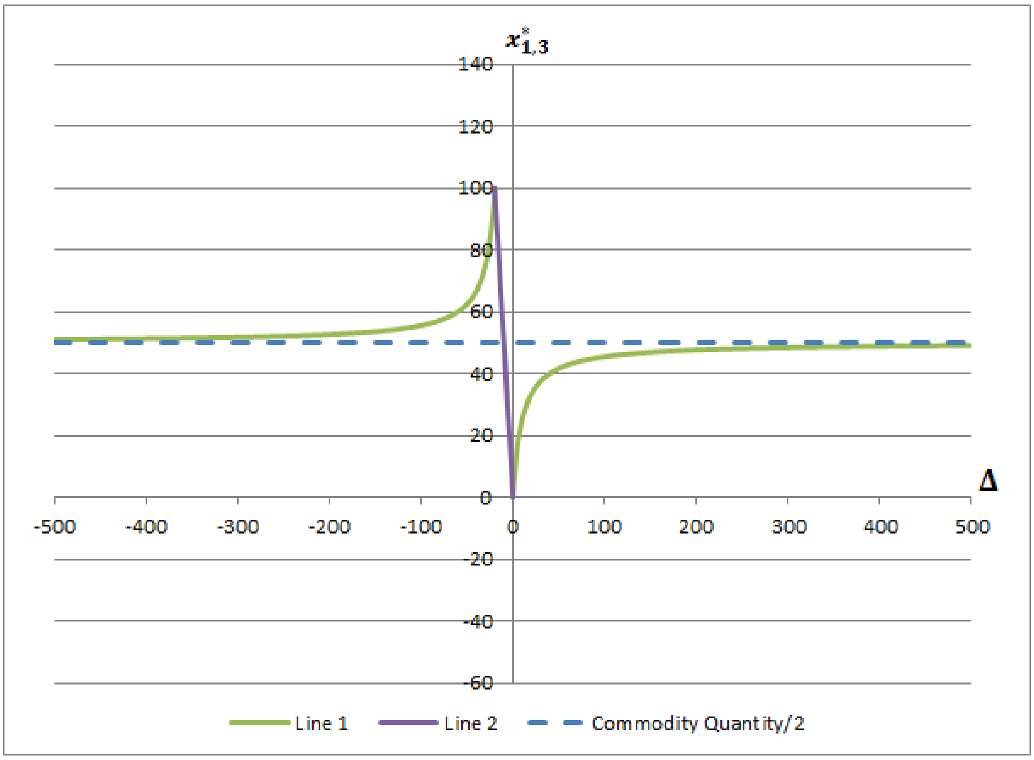

Figure 5.3: Theoretical optimum hedging decisions corresponding for different values of $\triangle$ for two-period model with base parameters

In order to analyze where multi-optimality starts while $\triangle \to \pm\infty$, b and c should carefully be analyzed. It is already known that Line 1 is the optimum line and $b = c = F_{1,3} - (2\triangle - n_3 + \delta\xi)/\xi$ while $\triangle \to \pm\infty$. As a result, the point where multi-optimality begins (such as the decision $x^*_{1,3}$ and $x^*_{1,3} + 1$ will be indifferent for the decisionmaker) is the point that satisfy $D(x_{1,3}, 1) \geqq \zeta$. Although, the first derivative of the expected total financial distress cost with respect to $x_{1,3}$ is hard to calculate, we may use the linear differential approximation mentioned in the beginning of this section in order to understand the variables affecting to the multioptimality.

Suppose that, $\triangle^+$ is the value of $\triangle$ where multi-optimality starts while $\triangle \to +\infty$. It is known that the optimum Line 1 (which represents $x^*_{1,3} = \xi\triangle/(2\triangle - n_3 + \delta\xi)$) is symmetrical with respect to the point $\left(\triangle = (-n_3 + \delta\xi)/2,\ x^*_{1,3} = \xi/2\right)$, and this situation is also observed at Figure 5.3. Therefore, the starting point of multi-optimality while $\triangle \to -\infty$ may be defined as $\triangle^-$such that $\triangle^+ + \triangle^- = n_3 - \delta\xi$. On the other hand, it is numerically observed that the one of the most important factor on where multi-optimality begins is standard deviation σ of futures prices of the commodity due to characteristic behavior of normal distribution. However, standard deviation σ does not affect magnitude of the optimum decision value, as observed in Theorem 3.

Now analysis of the total expected financial distress cost will be continued by considering other discrete parameters while $\triangle$ continuously changes. In these examinations, multi-optimality region will be ignored in the figures in order to improve the presentation.

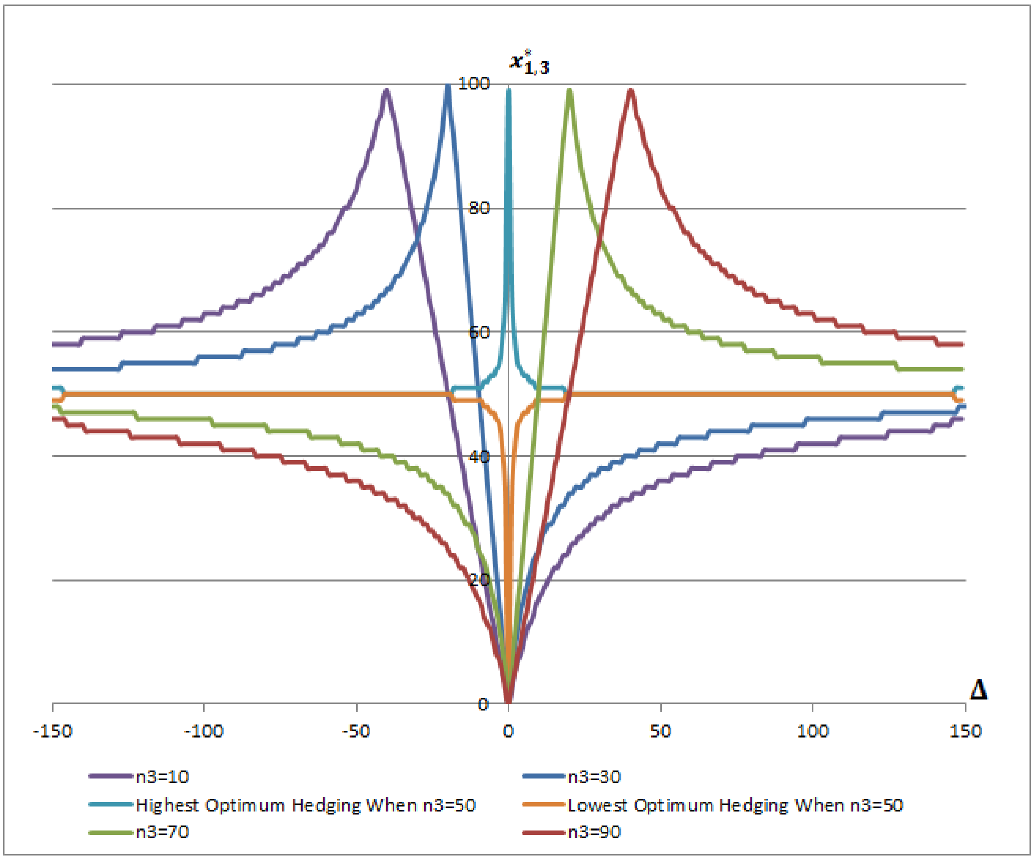

Figure 5.4: Optimum hedging decisions corresponding to changing values of $\triangle$ for two-period model with base parameters and different n_3 levels

5.1.1.1 Effect of n_3 while $\triangle$ values change:

In order to analyze the effect of n_3, again especially Theorem 3 should be considered. Therefore, the point where $\triangle = n_3 + \delta\xi = 0$ is important and this is satisfied by $n_3 = 50$ when predefined base values of the parameters are used. As stated in Theorem 8, there is multi-optimality while we get closer to the origin, $\triangle = 0$ when $n_3 = 50$.

Five different values of n_3 are utilized such as $10, 30, 50, 70, 90$ while the other parameters are set to their predefined base values. Figure 5.4 shows the respective optimum hedging decisions while $\triangle$ changes on a continuous interval and n_3 has these values. In addition, the optimum hedging line when $n_3 = 50$ is shown by two different lines to show the multi-optimal area. Moreover, in case of $i + j = 2 * 50$, the graphical representations of the optimum hedging lines with $n_3 = i$ and $n_3 = j$ will be symmetrical to each other with respect to Y axis. For example, the optimum value lines where the parameter value n_3 is 30 and 70 are symmetrical with respect to Y axis. As obtained from Theorem 5 and observed from Figure 5.4, the point where the optimum hedging quantity reaches to ξ shifts with direct proportion to value of n_3. Consequently, n_3 is a vital parameter for setting the hedging amount and decisionmakers should use approximations to set their hedging position on the optimum line in Figure 5.4.

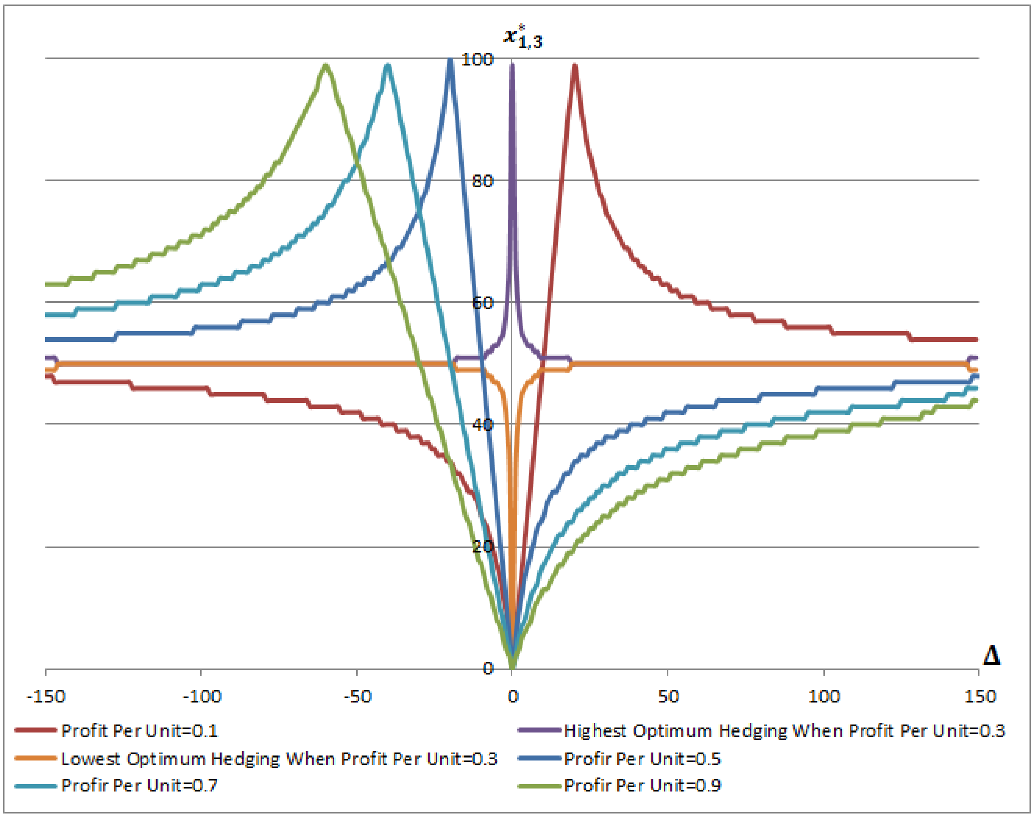

Figure 5.5: Optimum hedging decisions corresponding to changing values of $\triangle$ for two-period model with base parameters and different δ levels

5.1.1.2 Effect of δ while $\triangle$ values change

The value of the expected profit margin $\delta = p - \lambda - F_{1,3}$ is another important factor for the company while setting the hedging level. The value of δ should be positive, to get the company involved in the project. Similar to the analysis of the effect of n_3, the point where $\triangle = n_3 + \delta\xi = 0$ is important and this is satisfied by $\delta = 0.3$ when predefined base values of the parameters are used. Furthermore, again multi-optimality is observed as getting closer to origin, $\triangle = 0$ when $\delta = 0.3$.

Five different values of δ are exercised such as $0.1, 0.3, 0.5, 0.7, 0.9$ while the other parameters are set to their predefined base values. Figure 5.5 shows the respective optimum hedging decisions while $\triangle$ changes and δ has these five value levels. Besides, the optimum hedging line when $\delta = 0.3$ is shown by two different lines to show the multi-optimal area. The graphical representation of optimum hedging lines when value of the parameter δ is equal to i and j will be symmetric to each other with respect to Y axis as long as $i + j = 2 * 0.3$. For example, the optimum value lines where the parameter value δ is 0.1 and 0.5 are symmetric with respect to Y axis. As obtained from Theorem 5 and observed from Figure 5.5, the point where the optimum hedging quantity reaches to ξ shifts with direct proportion to value of δ. Consequently, δ is another vital parameter for setting the hedging amount and decisionmakers should use approximations to set their hedging position on the optimum curve in Figure 5.5.

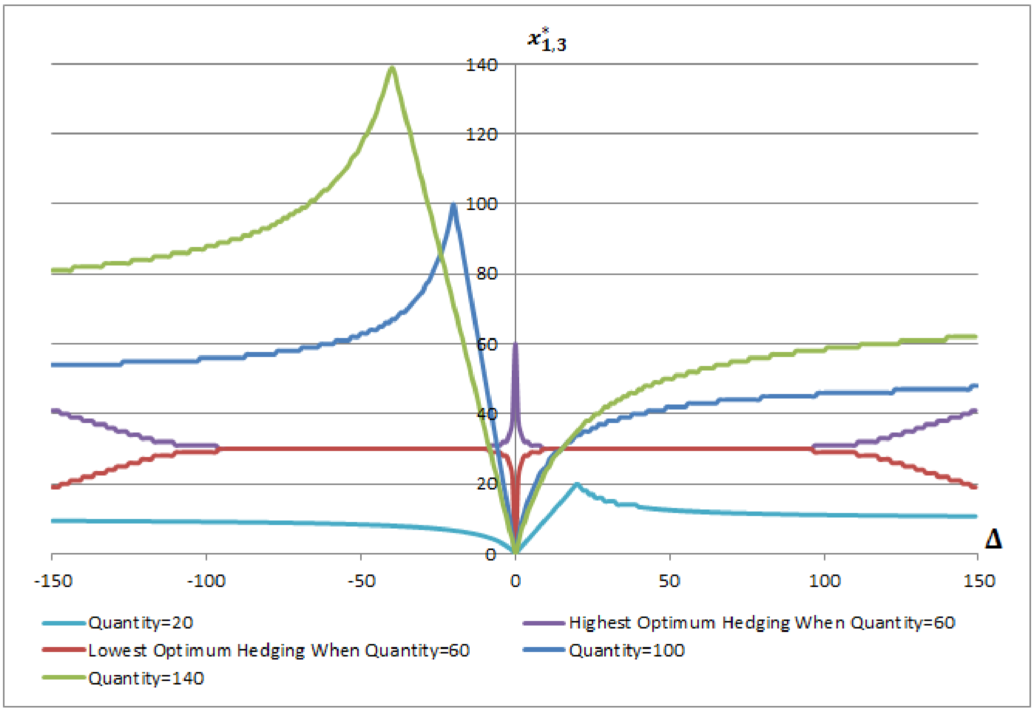

Figure 5.6: Optimum hedging decisions corresponding to changing values of $\triangle$ for two-period model with base parameters and different ξ levels

5.1.1.3 Effect of ξ while $\triangle$ values change

The value of the quantity of the assets required for the project ξ is a vital factor for the company to set its hedging position. Referring to Theorem 2, it is seen that the value of ξ determines the boundaries of the optimum region. Besides, Theorem 3 suggests that it also affects the slope of the lines of optimum solution while $\triangle$ changes. Similar to the analysis of the effect of n_3 and δ, the point where $\triangle = n_3 + \delta\xi = 0$ is important and this is satisfied by $\xi = 60$ when predefined base values of the parameters are used. Therefore, multi-optimality emerges while getting closer to origin, $\triangle = 0$ when $\xi = 60$.

Four different value levels of ξ are exercised; $20, 60, 100, 140$, while the other parameters are set to their predefined base values. Figure 5.6 shows the respective optimum hedging decisions while $\triangle$ changes and δ has these 4 value levels. Moreover, the optimum hedging line when $\xi = 60$ is shown by two different lines to show the multi-optimality region.

Since the magnitude of ξ has a direct proportion with the risk of financial distress due to price risk, it is an effective parameter. As mentioned above, the upper limits of hedge quantity is also determined by the value of ξ as observed from the curve of optimum hedge decision in Figure 5.6.

The following facts can be summarized as important for of real life decisionmakers:

- The optimum hedging quantity always follows the line $\xi\triangle/(n_3 - \delta\xi)$ starting from origin point until reaching to ξ where $\triangle = n_3 - \delta\xi$.
- When the expression $n_3 - \delta\xi$ is negative and multi-optimal region is ignored, the optimum hedging decision at $t = 1$ is always higher than $\xi/2$ as long as $\triangle < n_3 - \delta\xi$. On the other hand, it is

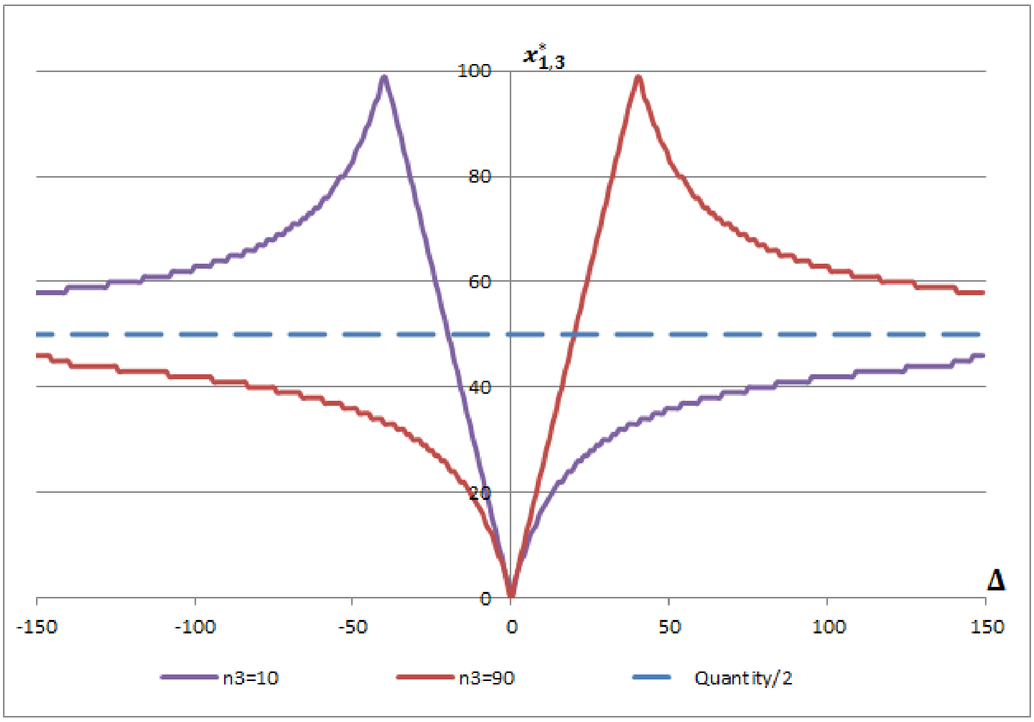

Figure 5.7: Optimum hedging decisions corresponding to changing values of $\triangle$ for two-period model with base parameters and different n_3 levels

always lower than $\xi/2$ as long as $\triangle > 0$ for this case.
The value of $n_3 - \delta\xi$ with base parameter values except $n_3 = 10$ is equal to 40. Therefore, the line representing the optimum hedging decisions when $n_3 = 10$ in Figure 5.7 is an example for the case described above. The optimum hedging line follows the line $\xi\triangle/(n_3 - \delta\xi)$ in the interval $[n_3 - \delta\xi, 0]$; additionally it is always higher than $\xi/2$ in the interval $[-\infty, n_3 - \delta\xi]$ and always lower than $\xi/2$ in the interval $[0, +\infty]$.

- When the expression $n_3 - \delta\xi$ is positive and multi-optimal region is ignored, optimum hedging decision at $t = 1$ is always higher than $\xi/2$ when $\triangle > n_3 - \delta\xi$. On the other hand, it is always lower than $\xi/2$ as long as $\triangle < 0$ for this case.
  The value of $n_3 - \delta\xi$ with base parameter values except $n_3 = 90$ is equal to -40. Therefore, the line representing the optimum hedging decisions when $n_3 = 90$ in Figure 5.7 is an example for the case described above. The optimum hedging line follows the line $\xi\triangle/(n_3 - \delta\xi)$ in the interval $[0, n_3 - \delta\xi]$; additionally it is always higher than $\xi/2$ in the interval $[n_3 - \delta\xi, +\infty]$ and always lower than $\xi/2$ in the interval $[-\infty, 0]$.
- Finally, the optimum hedging quantity line converges to $\xi/2$ while $\triangle \to \pm\infty$; yet multi-optimality emerges simultaneously.

5.1.2 Analysis of Two-Period Model With Changing Values of n_3

Another important parameter about the financial situation of the company is n_3 in this model. It is an important parameter and may also contain uncertainty in real life cases. Thus, an analysis on the effect of n_3 by simulation is necessary. A typical behavior of the optimum hedging amount function by changing n_3 values is given at Figure 5.8 when the model is used with the base parameters.

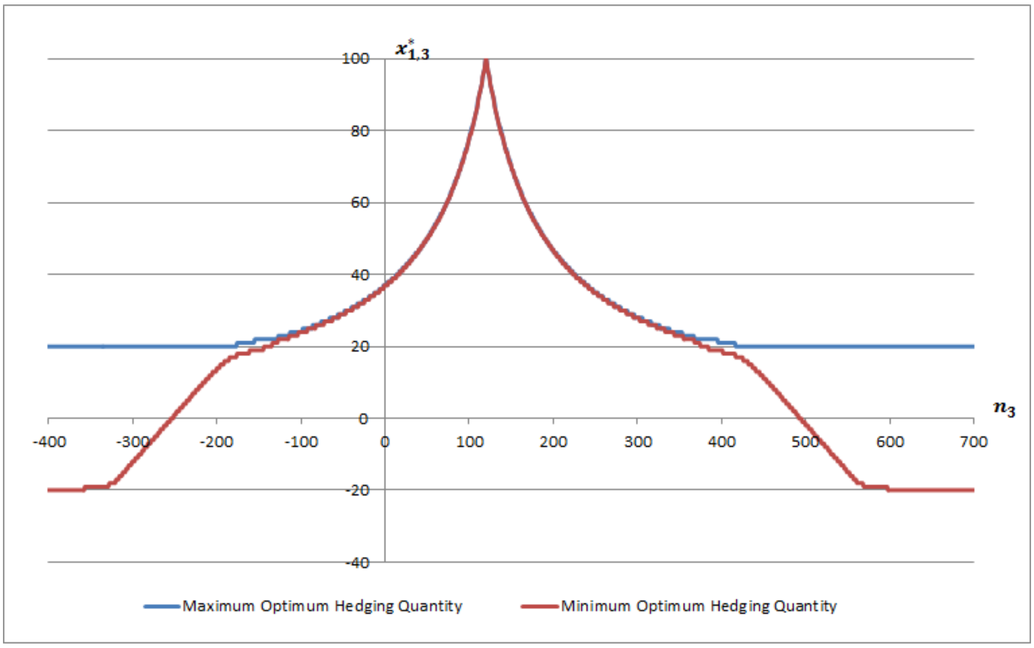

Figure 5.8: Simulated optimum hedging decisions corresponding to changing values of n_3 for two-period model with base parameters

In Figure 5.8, X axis shows continuously changing n_3 values while Y axis shows the optimum hedging decisions for corresponding n_3 values. In both extreme sides of X axis, multi-optimality is observed as suggested in Theorem 7 and the points where multi-optimality started to be observed will also be examined in this section. On the other hand, the behavior of the optimum hedging amount function should be inspected by using Theorem 3 which states:

$$x^*_{1,3} = \begin{cases} \xi\triangle/\left(2\triangle - n_3 + \delta\xi\right) & if\, \triangle\left(\triangle - n_3 + \delta\xi\right) \geqq 0 \\ -\xi\triangle/\left(-n_3 + \delta\xi\right) & if\, \triangle\left(\triangle - n_3 + \delta\xi\right) \leqq 0 \end{cases}$$

The expressions $\xi\triangle/\left(2\triangle - n_3 + \delta\xi\right)$ and $-\xi\triangle/\left(-n_3 + \delta\xi\right)$ will be drawn while n_3 is changing and they will be labeled as Line 1 and Line 2, respectively at Figure 5.9. Figure 5.9 shows the optimum values suggested by Theorem 3 with corresponding n_3 values and it is observed that these lines indicate the exact solution when two parts of lines 1 and 2 are considered together where $0 \leqq x^*_{1,3} \leqq \xi$. However, multi-optimality occurs while n_3 goes to $\pm\infty$ as Theorem 8 suggests.

The optimum hedging amount $x^*_{1,3}$ is equal to ξ when $\triangle = n_3 - \delta\xi$ as stated in Theorem 5. It is satisfied by $n_3 = 120$ when predefined base values of the parameters are used. As a result, the optimum line peaks at $n_3 = 120$ and it decreases while getting away from that point. By using Theorem 3, it is observed that the curves are symmetric with respect to the line $n_3 = 120$. Additionally, the optimum decision curve is converging to X axis while n_3 goes to $\pm\infty$.

The behavior of the optimum value function while n_3 is changing is similar to δ. However, the magnitude of the slopes and the point where it peaks is dependent to ξ. Therefore, analysis of the optimum hedging decision with respect to n_3 represents the analysis with respect to the corresponding value δ as long as the magnitude of ξ stays the same.

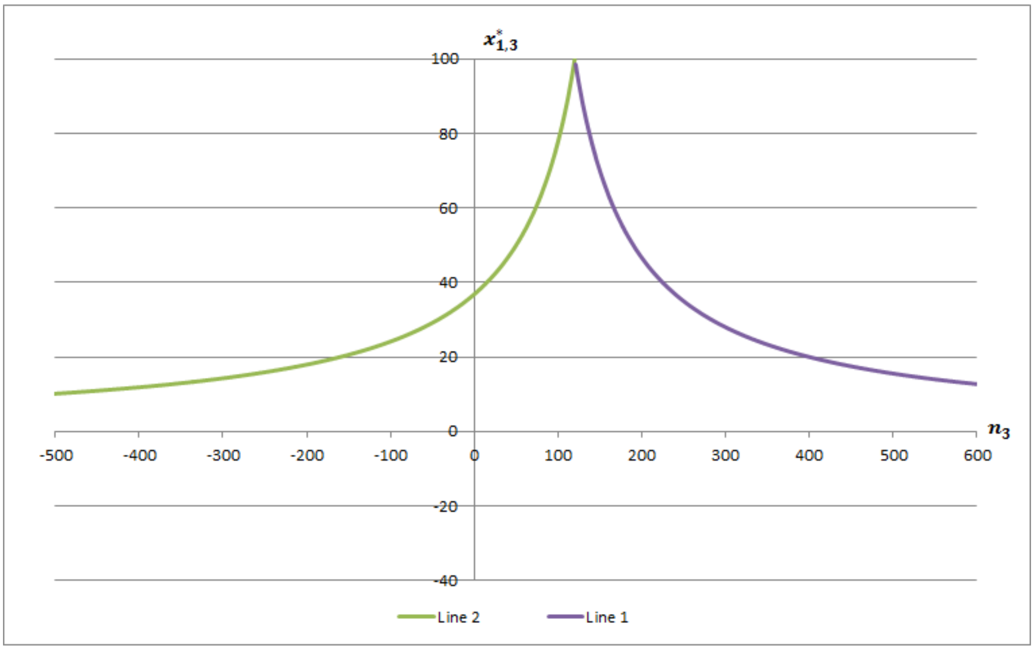

Figure 5.9: Theoretical optimum hedging decisions corresponding to changing values of $\triangle$ for two-period model with base parameters

5.2 Analysis of Three-Period Model

Similar to two-period model, the variables $\triangle$ (cash balance of the company at $t = 2$ without hedging), n_3 and n_4 are the most important variables.

A set of values of the parameters will be taken into account as a base for numerical analysis. In this base parameter value set, the values of the variables are $\triangle = y_1 - n_2 = 100$, $n_3 = 30$, $n_4 = 30$, $\xi = 100$, $p = 11$, $\lambda = 0.5$, $F_{1,4} = 10$, $\sigma = 1$, $r = 0.1$, $\beta = 0.95$. Besides, amounts less then $\zeta = 0.001$ units of expected financial distress cost is supposed to be insignificant for the company. Similar to two-period problem, multi-optimal decisions exist for the company when difference of financial distress costs are less than ζ. Expected total financial distress costs with base parameter values and with respect to changing values of $x_{1,3}$ are shown at Figure 5.10. Similar to the numeric analysis for two-period models, for each $x_{1,4}$ decision, financial distress cost is calculated as the average financial distress cost for 100,000 different realizations of the futures and commodity prices.

X axis denotes different $x_{1,4}$values while Y axis shows the corresponding expected total financial distress costs in Figure 5.10. As observed from Figure 5.10, the financial distress cost function is convex and the optimum decision for the model with base parameter value set is 44 which is less than ξ. Moreover, at time 1, it is observed that hedging amount gets smaller in expected value as time to go increases, i.e. $0 \leqq x^*_{1,4} \leqq x^*_{2,4} \leqq x^*_{3,4} = \xi$.

5.2.1 Analysis of Three-Period Model With Changing Values of $\triangle$

The first term to be analyzed is again $\triangle$. Typical behavior of the optimal hedging amount function by changing $\triangle$ values and with base parameter set is obtained, and shown at Figure 5.11.

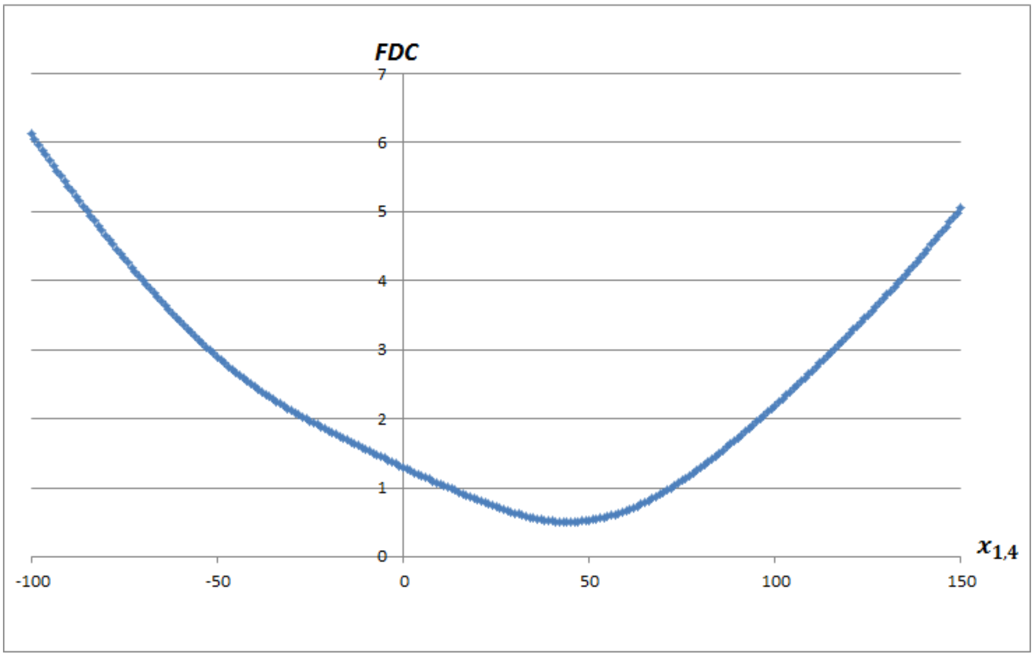

Figure 5.10: Expected total financial distress costs for three-period model with base parameters for different $x_{1,4}$ values

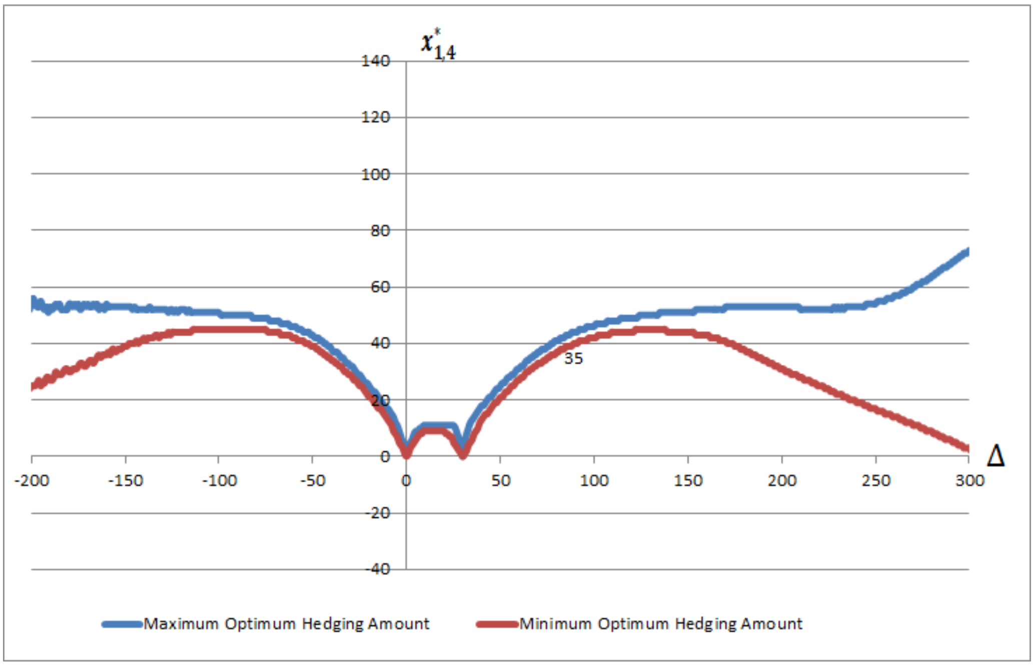

Figure 5.11: Optimum hedging amounts at $t = 1$ corresponding to changing values of $\triangle$ for three-period model with base parameters

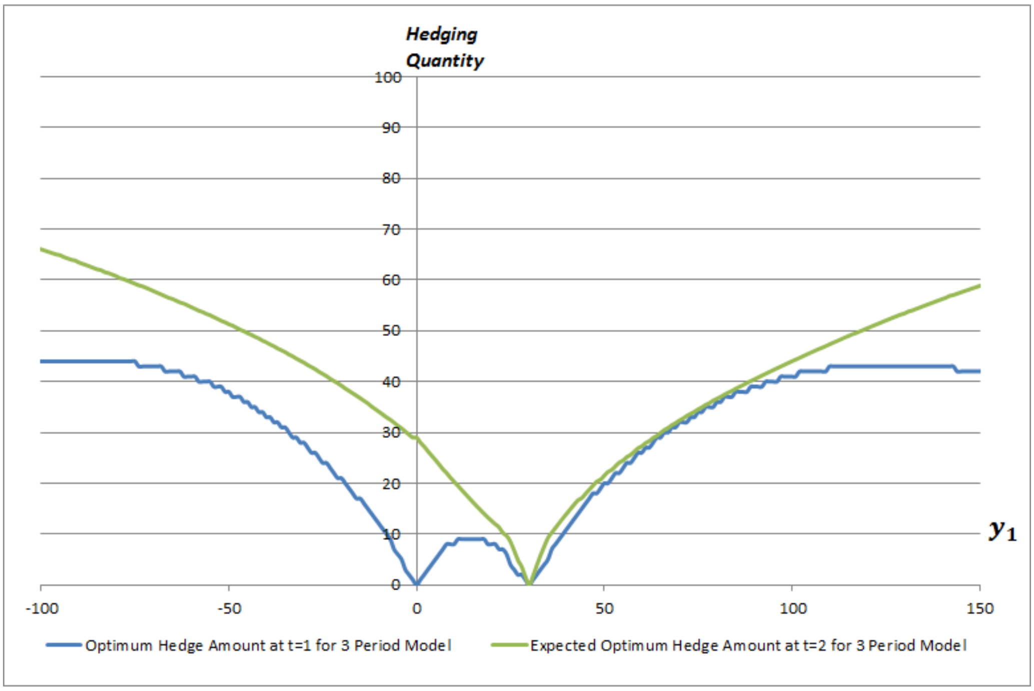

Figure 5.12: Optimum hedging decisions for $t = 1$ and expected optimum hedging decisions for $t = 2$ for three-period model at $t = 1$ with changing values of y_1 and base parameters

Just as Theorem 11 suggested, hedging is becoming a value neutral action while $\triangle$ is going through $\pm\infty$ in Figure 5.11. Therefore, as long as hedging quantity stays in the reasonable limits for corresponding $\triangle$ values, multi-optimality exists in both sides of Figure 5.11. It is also observed that the optimum hedging decision is equal to 0 when $\triangle = 0$ and $\triangle = n_3 = 30$.

As obtained from Theorem 9, the hedging policy will be as $0 \leqq x^*_{1,4} \leqq x^*_{2,4} \leqq x^*_{3,4} = \xi$. Therefore, the optimum hedging decision at time 1 is smaller than or equal to expected optimum hedging decision at time 2 and it is smaller than ξ when the company considers hedging at $t = 1$. However, the expected value of $x^*_{2,4}$ at time 1 may be very different at time 2, after the futures prices and cash flows related to future contracts are realized. Figure 5.12 shows the optimal hedging decision at $t = 1$ and the expected optimal hedging decision at $t = 2$ for three-period model with base parameter setting and with$n_2 = 0$, as well as the hedge quantity at $t = 1$ for two-period model with base parameter set. As observed from the figure, optimum hedging quantity at $t = 1$ is always lower than or equal to the optimum expected hedging quantity at $t = 2$.

Similar to two-period model, $\triangle = y_1 - n_2$ is an effective parameter to the hedging decision and it is also important for real life decisionmakers. As obtained from Theorem 9, the hedging policy will be as $0 \leqq x^*_{1,4} \leqq x^*_{2,4} \leqq x^*_{3,4} = \xi$ where expected values are considered for future decisions when the company is at time 1.

5.2.1.1 Effect of n_3 while $\triangle$ values change

It is known that n_3 is an important variable and it is one of the most effective factors on hedging quantity for three-period model. Figure 5.13 shows the respective optimum hedging decisions at time 1 while $\triangle$ changes and n_3 has three different values as $0, 15, 30$ while the other parameters are set to

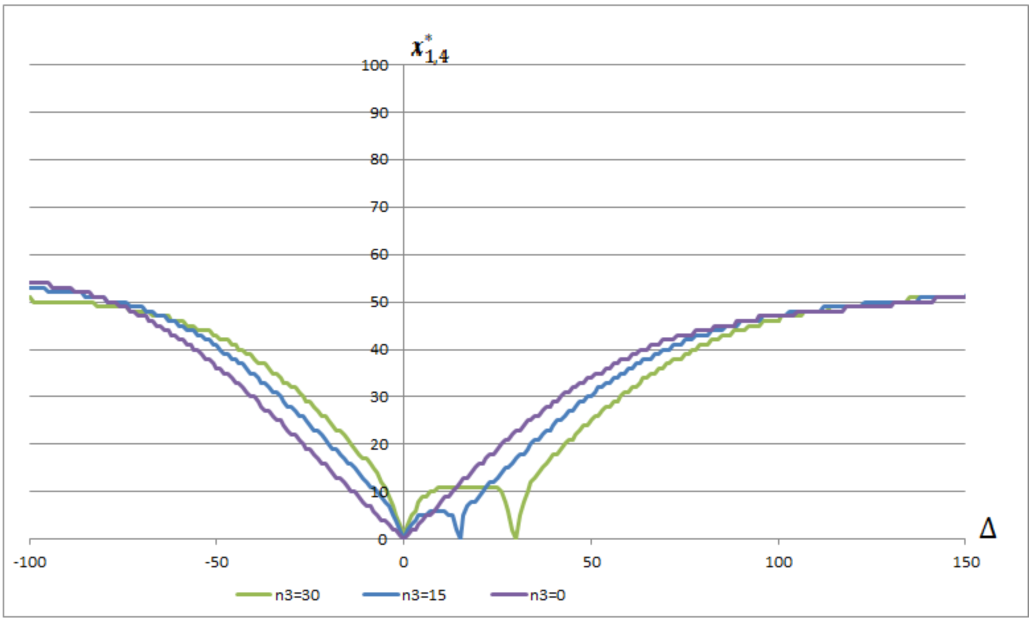

Figure 5.13: Optimum hedging decisions corresponding to changing values of $\triangle$ and n_3 for three-period model with base parameters

their predefined base value levels.

The first thing to notice in Figure 5.13 is that the company hedges nearly to amount of zero when the cash reserve of the company at t=1 except hedging is zero, i.e. $\triangle = y_1 - n_2 = 0$. This is a similar result to the two-period model. The reason is that the company will have financial distress at time 2 cost at 50% probability if it hedges.

Another important thing to notice is that the optimum hedging quantity becomes zero when the cash reserve of the company at t=3 except hedging is zero, i.e. $\triangle - n_3 = 0$. Thus, the company does not hedge at the points where $\triangle = n_3$ in Figure 5.13. Another observed point is that the optimum decision lines are converging to $\xi/2$ while $\triangle \to \pm\infty$ and this behavior is similar to the optimum solution at time 1 for two-period model.

5.3 Analysis of the Approximation of the Model with Compound Interest

In this section, numeric comparison of compound and simple interest model will be performed for two-period model. The base parameter set is the same as the one with simple interest. Typical behavior of the optimum decision function as $\triangle$ values are changing is obtained when the simulation model with base parameters is used, and shown at Figure 5.14. Similar to the model with simple interest, hedging is a value neutral action while $\triangle \to \pm\infty$. Likewise, the optimum decision lines are also converging to $\xi/2$ while $\triangle \to \pm\infty$.

As mentioned in the previous parts, the model with simple interest rates is aimed to be used in estimation of the optimum hedging decision at t=1 for the model with compound interest. The discount factor β is chosen to be 1 to equate the weight of both of financial distress costs in the model with compound interest, just as simple interest model. The comparison of these two models with the base

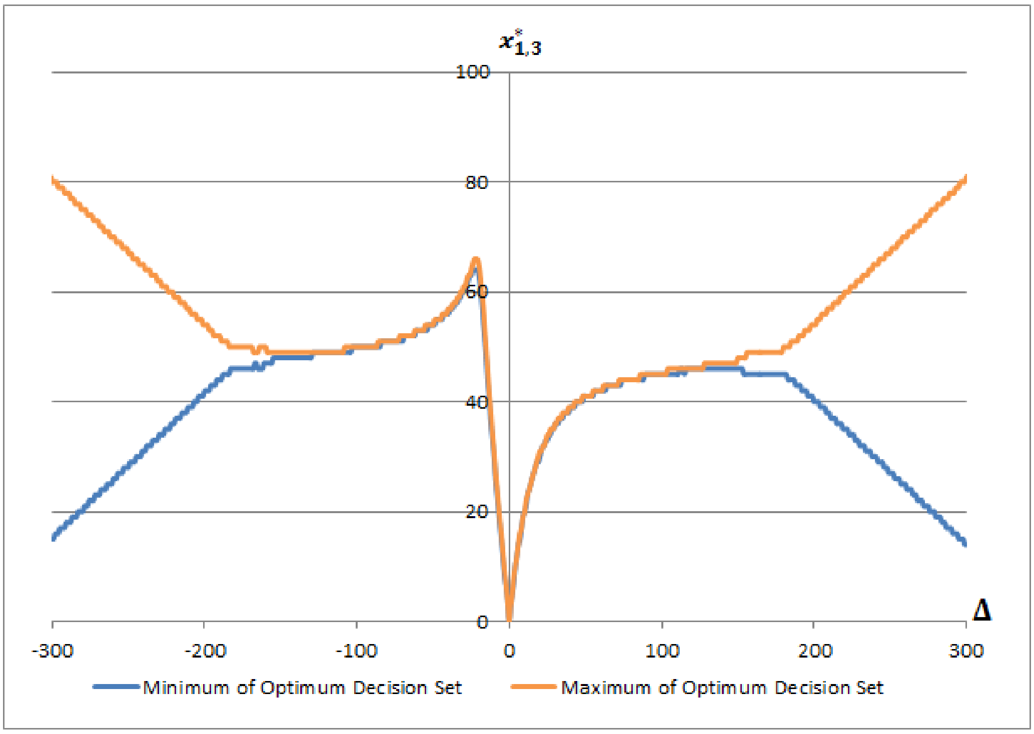

Figure 5.14: Optimum hedging decisions corresponding to changing values of $\triangle$ for two-period model with compound interest and base parameters

parameters (except $\beta = 1$) is shown in Figure 5.15.

Similar to the model with simple interest, there is a flat line between $\triangle = (n_3 - \delta\xi) / (1 + r)$ and $\triangle = 0$ when $n_3 - \delta\xi < 0$ and $\beta = 1$. It is known that the expression $(n_3 - \delta\xi)$ is equal to -20 for the base parameters, which is lower than zero. As an illustration to this case, Figure 5.15 shows the optimum hedging amounts corresponding to changing values of $\triangle$ with base parameters. It is observed that, the optimum decision of the model with compound interest is less than the other one as long as $\triangle$ is between $-\infty$ and $-n_3 + \delta\xi$. Another fact is that, all possible optimum decisions when $\triangle < 0$ are less than $\xi / (1 + r)$.

In case of $n_3 - \delta\xi > 0$, again a flat line starting from the origin takes place in the optimum hedging decision figures; yet this line may reach to the level $x^*_{1,3} = \xi$ in extreme positive values of $n_3 - \delta\xi$. Figure 5.16 shows that two models almost behave the same in the interval $(0, \infty)$ of $\triangle$ when $\beta = 1$ and $n_3 - \delta\xi = 20$, which is more than zero. It is seen that, the optimum hedging quantity of the model with compound interest is lower than the one with simple interest in the interval $(-\infty, 0)$ of $\triangle$.

When the expression $n_3 - \delta\xi\beta \cong 0$, multi-optimality exists around $\triangle = 0$; and this is also a similar behavior of the model with simple interest. Besides the optimum hedging quantity becomes close to $\xi/2$ while $\triangle$ goes to $\pm\infty$. Figure 5.17 shows the behavior of the optimum hedging decisions of two-period model with compound interest rates when $n_3 - \delta\xi\beta = 0$. Minimum and maximum values of optimum decision sets are shown as separate lines because there is multi-optimality. It may be concluded that the optimum hedging quantity line for model with compound interest and $\beta = 1$ has a similar shape to the one with simple interest when Figure 5.17 and Figure 5.5 are compared.

Although the models with compound and simple interest rates are pretty much similar when $\beta = 1$,

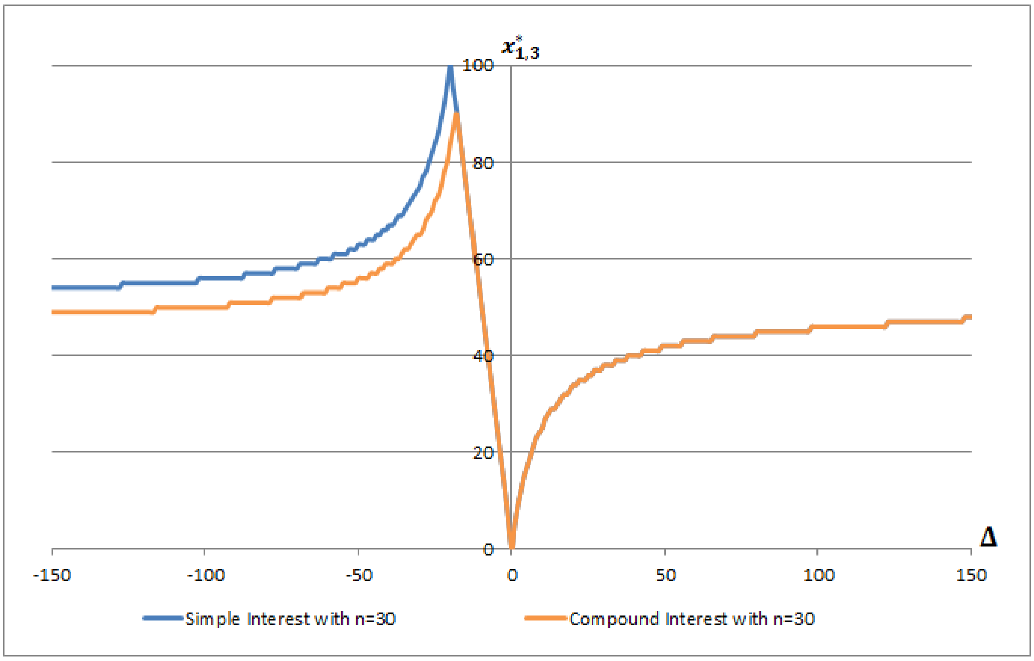

Figure 5.15: Optimum hedging decisions of two-period models with simple and compound interest corresponding to changing values of △ by using base parameters (except $\beta = 1$)

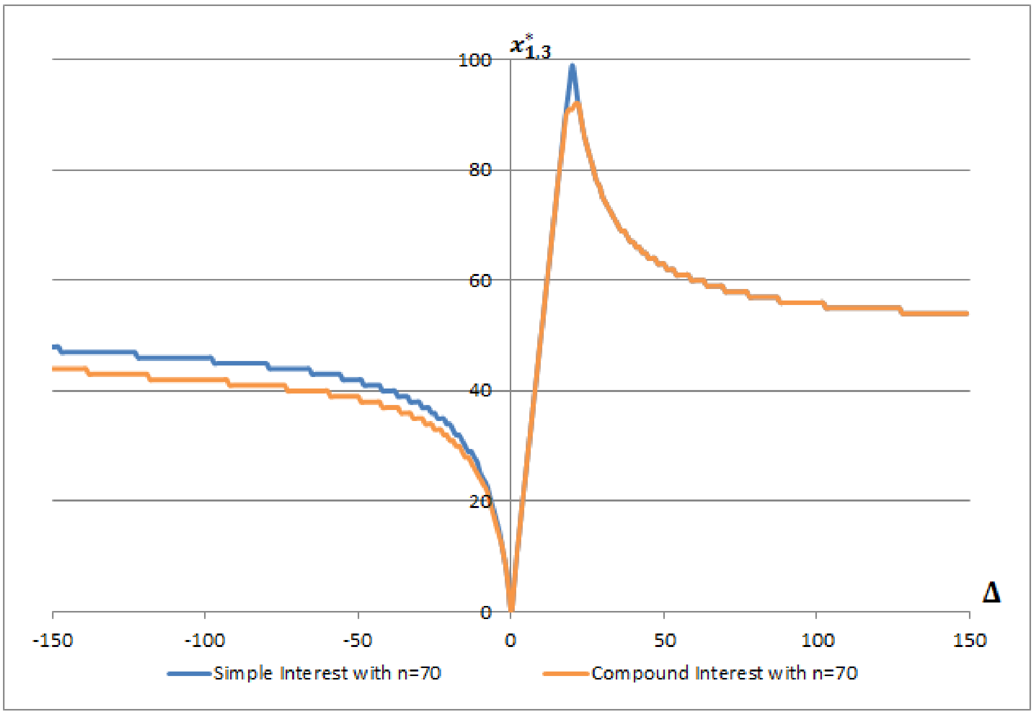

Figure 5.16: Optimum hedging decisions of two-period models with simple and compound interest corresponding to changing values of △ by using base parameters (except $\beta = 1$ and $n_3 = 70$)

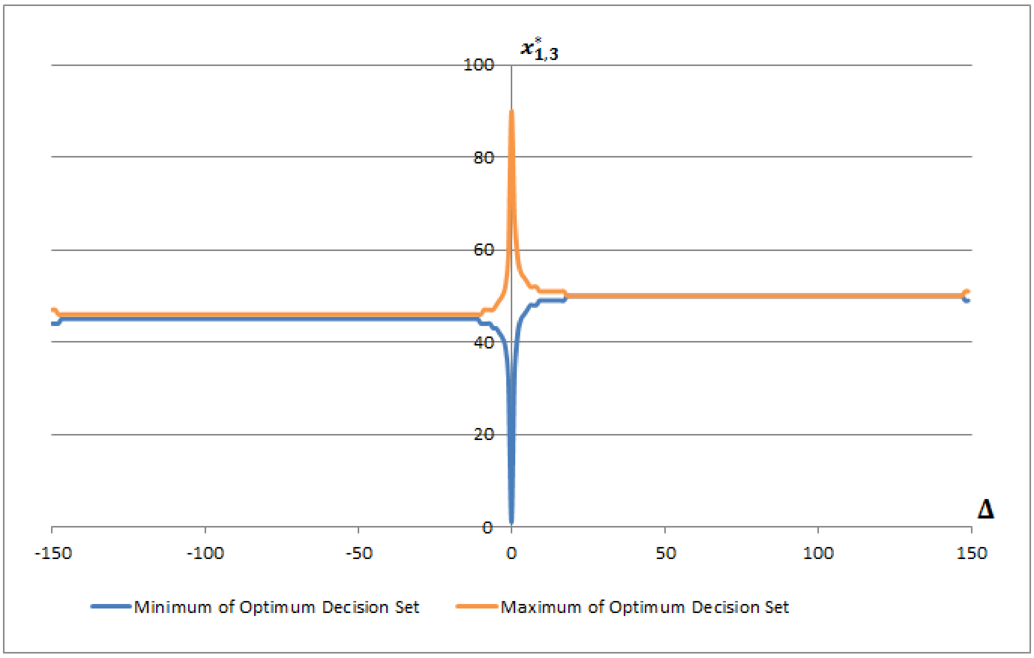

Figure 5.17: Optimum hedging decisions of two-period model with compound interest corresponding to changing values of $\triangle$ by using base parameters (except $\beta = 1$ and $n_3 = 50$)

differences become more apparent while β decreases. Figure 5.18 shows the optimum hedging decision of the company at time 1 with compound interest rates and for different β and $\triangle$ values. It is observed that, the optimum hedging amount decreases while β decreases. The decisionmakers should also consider β as an important factor for the hedging decision.

In conclusion, the model with simple interest may be helpful in estimating the optimum hedging amount of the model with compound interest. Two models behaves very similar between $max\,(0, n_3 - \delta\xi)$ and $+\infty$ when $\beta = 1$. On the other hand, the optimum hedging decision at time 1 for the model with compound interest decreases in some regions while β is getting smaller. Briefly, the optimum hedging amount with compound interest at time 1 is always less than or equal to the optimum hedging quantity with simple interest.

5.4 Analysis of One-Period Model With Call Option

As stated in the mathematical model, buying call option decision should be examined and profitability of this decision with respect to the model with futures contract should be compared. The same base parameter set is used in these comparisons. In this base setting, the values of the variables are chosen as $y_1 = 100$, $n_2 = 30$, $\xi = 100$, $p = 11.5$, $\lambda = 0.5$, $F_{1,2} = 10$, $\sigma = 1$, $r = 0.1$, $\beta = 0.95$ and $\zeta = 0.0001$. By using the value of $F_{1,2}$ and σ, the value of h is found as approximately 0.4.

The first step for testifying claims of Theorem 15 and 16 is to prove that expected total financial distress cost is convex with respect to $x_{1,2}$. Possible hedging decisions and corresponding expected financial distress costs in the model with base settings are shown in Figure 5.19. For each $x_{1,2}$ decision in Figure 5.19, FDC represents the average total financial distress cost for 100,000 different realizations of the future and commodity price realization. As observed, the function is convex with respect to the

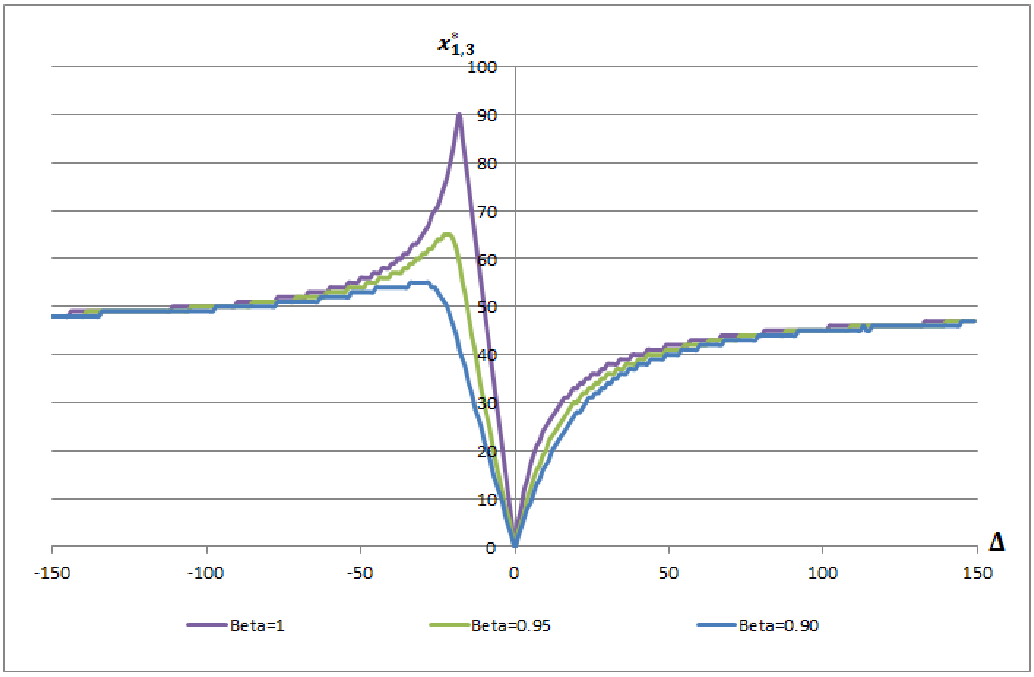

Figure 5.18: Optimum hedging decisions of two-period model with compound interest corresponding to changing values of $\triangle$ and β by using base parameters

decision variable $x_{1,2}$.

The next step is to examine a typical behavior of the function. Figure 5.20 shows the optimum decision sets (by denoting highest and lowest optimum values) corresponding to changing values of y_1. As suggested in Theorem 16, $\frac{y_1}{h} + \left(\frac{(p-\lambda-F_{1,2})\xi-n_2}{h}\right)^-$ is always an optimum decision for the company where $\frac{y_1}{h} + \left(\frac{(p-\lambda-F_{1,2})\xi-n_2}{h}\right)^- > \xi$. It is also observed that the solution $\frac{y_1}{h} + \left(\frac{(p-\lambda-F_{1,2})\xi-n_2}{h}\right)^-$ is always an optimum solution as suggested in our intuition.

Finally a numerical comparison between using futures and call options should be made in order to understand which policy is more profitable for the company. It is already known that the optimum hedging decision is always equal to ξ regardless of any other parameters and $\xi = 100$ in the base parameter set. On the other hand, the optimum amount of call options are shown at Figure 5.20 for base parameter set and changing values of y_1. Corresponding financial distress costs at optimum policy of two different models for base parameter settings and different y_1 values are shown in Figure 5.21.

As observed, buying future contract (or forwards) is a better choice in the case of one-period model since the firm sets the price of the commodity to a fixed price without paying a considerable amount of money. However, the firm pays the risk premium while buying call options and this causes a decrease in its cash balance. Therefore, expected financial distress cost becomes more when option contract is used.

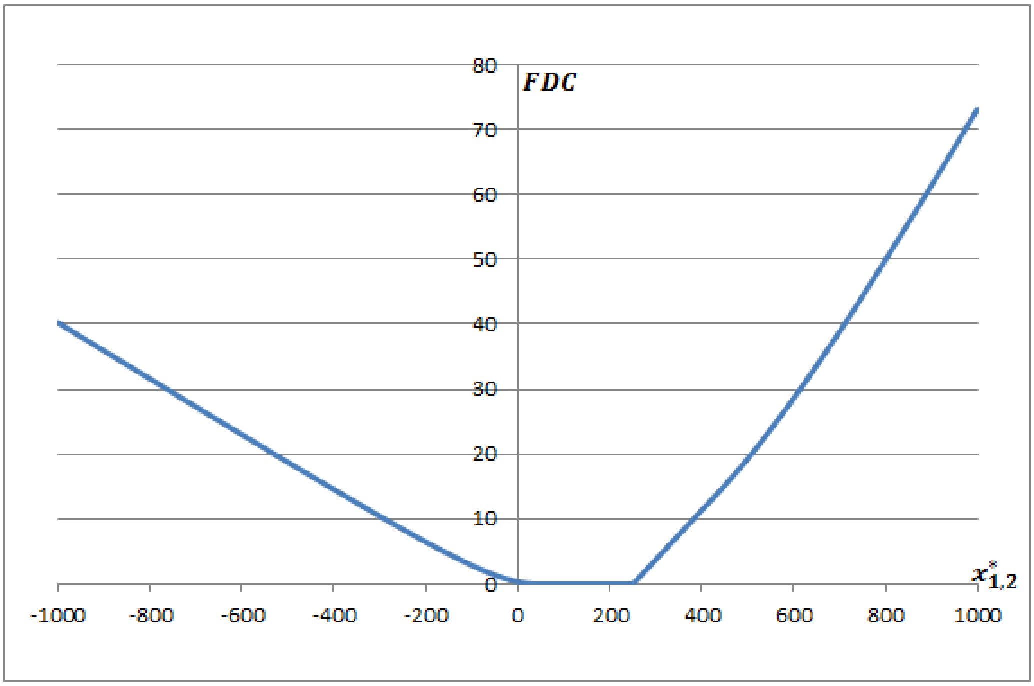

Figure 5.19: Expected total financial distress costs corresponding to different $x_{1,2}$ values for one-period model with call options and with base parameters

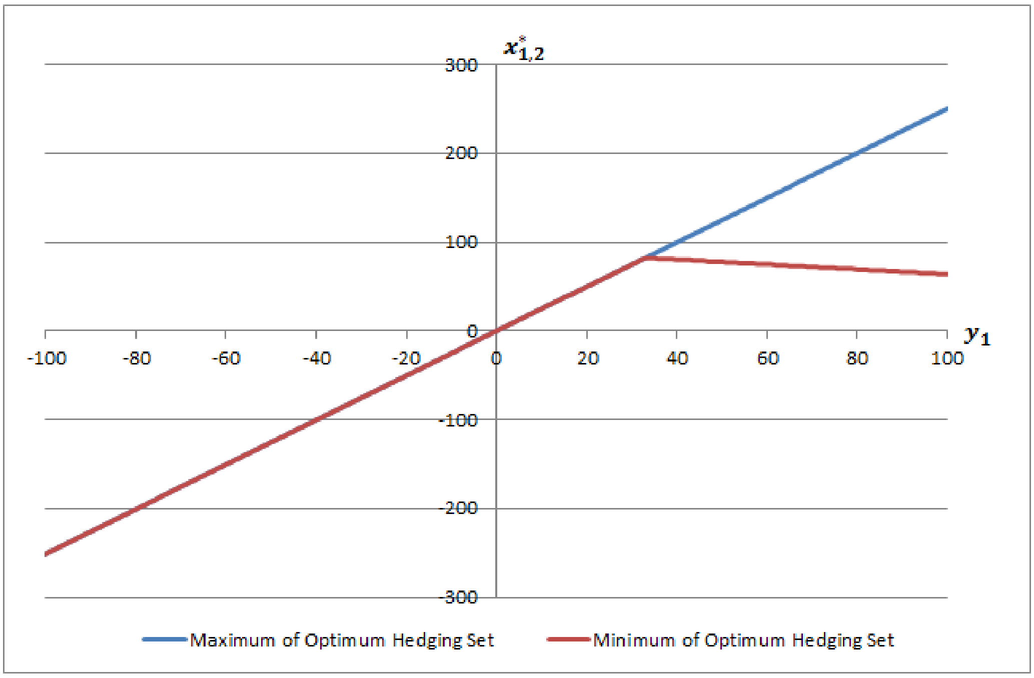

Figure 5.20: Optimum hedging decisions at $t = 1$ corresponding to changing values of $\triangle$ for one-period model with call options and with base parameters

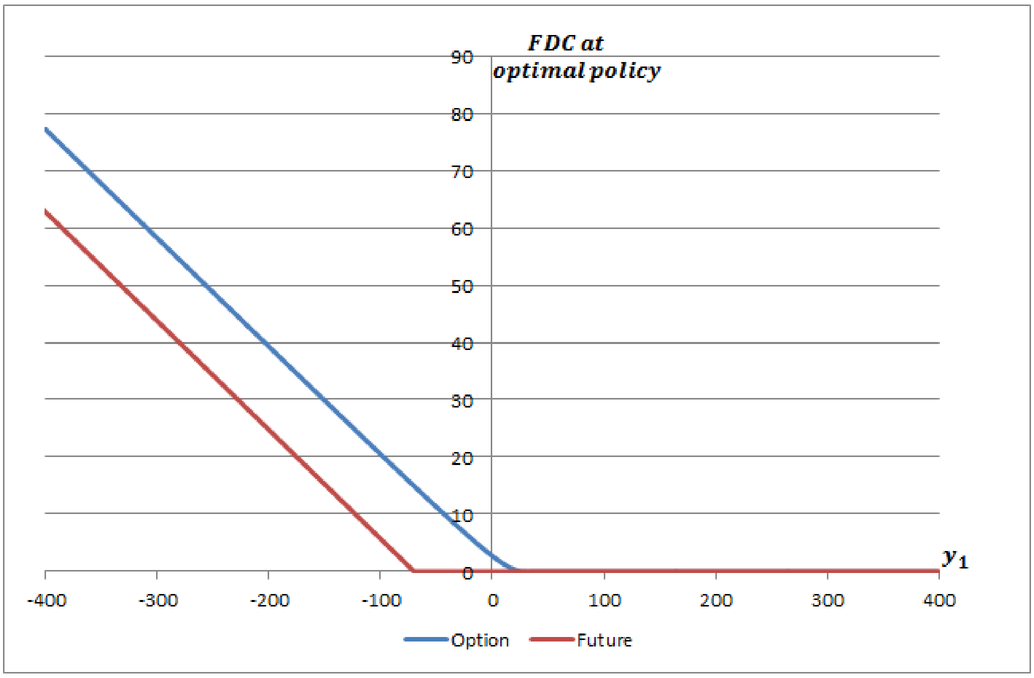

Figure 5.21: Expected financial distress costs of one-period models with future and option contracts corresponding to different y_1 values at optimum policy

CHAPTER 6

CONCLUSION AND FUTURE WORK

6.1 Conclusion

This study focuses on hedge decision of a company mainly using a specific commodity for a project with a predetermined delivery date and amount. Besides it benefits from a subcontractor to produce the necessary output for the project which consists of the commodity. The revenue from the customer of the company is at a fixed amount and is received at the end of the project; however the cost paid to subcontractor is changing with respect to the spot price of the commodity at the payment date and again is to be paid at the delivery date when revenue is received. Therefore, the company is open to the price risk and it may use a proper hedge policy to avoid facing financial distress. As well as financial transactions due to the project, the company has also previously planned payments to be paid during the project. The decision is examined both under simple and compound interest rates.

The outcomes of mathematical models and computational results suggests that the firm needs to hedge at the same amount of commodity needed for the project when using forward contracts or future contracts in one-period (it means that margin call requirements of future contracts are completely ignored). By having such an action, the risk of financial distress at the end of the project due to change in price of the commodity is fully eliminated. On the other hand, the firm needs to decrease hedge amounts while the number of periods is increasing because of another risk; the risk of financial distress during the project due to margin calls of future contracts. This risk is completely eliminated when hedge amount is zero. Since the risk of financial distress at the end of the project is eliminated at a perfect hedge policy, the firm needs to have a balance between these two risks; as a result the firm needs to hedge at an amount between zero and the amount of commodity necessary for the project. Besides, the optimum hedge amount decreases in average while the number of periods where margin calls are adjusted are increasing, because the risk of financial distress due to margin call requirements is increasing.

Decision of buying call options, with a strike price of the expected commodity price at the end of the project, is also examined in one-period framework in order to benchmark the profitability with the hedge decision via futures. The optimum hedging policy via call options is mainly analyzed by using numerical results. When using call options, it is found out that the firm may both under-hedge or over-hedge in expected value maximization perspective with respect to the values of the parameters in the model. In addition, according to the numerical results of our models, hedging via futures (or forward contracts) is more profitable than hedging via call options in one-period.

6.2 Future Research

In this study, decision of hedging via futures contract of a company under price risk is elaborately analyzed in different number of periods of time and through expected value maximization framework. However, initial margins of future contracts are not taken into account in these studies. Therefore, this study can be improved by considering different initial margin models.

Call option decision with a strike price of the expected price of the commodity in the end of the project (which is equal to the future contract price of the commodity maturing at the end of the project) is also examined in this study. However, further analyzes are needed in order to fully analyze the decision of hedging via option contracts. The decision should be analyzed in different time frames and especially with different strike price levels.

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APPENDIX A

LIST OF NOTATIONS

In this study, discrete points in time are used and time i happens before time $i+1$. The random variables are shown with the sign "∼" on the top of the variable whereas the realization of the corresponding random variables are denoted as standard uppercase letters.

S_i: Unit price of the commodity at the time i

p : Unit contract price fixed at time 1 to be paid by the customer to the company

$(S_i + \lambda)$: Unit price paid by the company to the subcontractor for asset delivery at time i.

ξ : Units of commodity required for the project

$F_{i,j:}$Futures contract price of commodity starting at time i, maturing at time j

$x_{i,j}$: Amount of futures or call options longed at time i for the maturity time j

n_i : Cash outflow of the company at time i

y_i : Cash reserve of the company at time i after the transactions are realized except FDC premium

r_d : Minimum required rate of return which is used to discount future cash flows and is assumed to be constant for all periods

β : The multiplier used to discount a cash flow to the previous time point, formulized as $1/(1 + r_d)$

r : Cost of financial distress per unit which is assumed to be constant at all time points

$\phi_J^Q(.)$: Risk neutral probability density function of random variable J

$E_J^Q(.)$: The expected value function which is found by using risk neutral probability function of random variable J

$\triangle V_{i+1}(y_i)$: Expected future cash change of the company discounted to respective value of time $i + 1$ when the company is at time i

$V_{i+1}(y_i)$: Maximum value of the expected future cash change of the company discounted to respective value of time $i + 1$ when the company is at time i , i.e. $V_{i+1}(y_i) = max \triangle V_{i+1}(y_i)$

$\triangle$: Net cash balance of the company after the preplanned cash transactions are completed at time 2, i.e. $\triangle = y_1 - n_2$

δ : Expected unit profit margin of the company, calculated as $p - \lambda - S_i$ for a project starting at time 1 and ending at time i

h : Unit risk premium price of option contract

APPENDIX B

PROOFS

B.1 Theorem 1

The first derivative of the term $(-\triangle V_2(y_1))$ with respect to $x_{1,2}$ is shown as the following:

$$\begin{aligned}\frac{d\left(-\triangle V_2(y_1)\right)}{dx_{1,2}} &= \frac{d\int_0^\infty \left[n_2 - \left(p - \tilde{S}_2 - \lambda\right)\xi - \left(\tilde{S}_2 - F_{1,2}\right)x_{1,2} + \beta r\left[y_2\right]^-\right]\phi_{S_2}^Q(S_2)\,dS_2}{dx_{1,2}} \\ &= \frac{d\int_0^\infty \left[\left(F_{1,2} - \tilde{S}_2\right)x_{1,2} + \beta r\left[y_2\right]^-\right]\phi_{S_2}^Q(S_2)\,dS_2}{dx_{1,2}} \\ &= \frac{d\int_0^\infty \beta r\left[y_2\right]^- \phi_{S_2}^Q(S_2)\,dS_2}{dx_{1,2}}\end{aligned}$$

Since $x_{1,2}$ is included only in the terms $\left(F_{1,2} - \tilde{S}_2\right)x_{1,2}$ and $-r\left[y_2\right]^-$, the derivatives of these terms are taken. However, it is already known that $\int_0^\infty \tilde{S}_2\phi_{S_2}^Q dS_2 = F_{1,2}$. Consequently, the only term left over is the term referring financial distress cost, i.e. $r\left[y_2\right]^-$. As a result, the expression can be transformed into the following by writing the term $r\left[y_2\right]^-$ explicitly:

$$\frac{d\left(-\triangle V_2(y_1)\right)}{dx_{1,2}} = \frac{d\int_0^\infty \beta r\left[y_1 + \left(\tilde{S}_2 - F_{1,2}\right)x_{1,2} + \left(p - \tilde{S}_2 - \lambda\right)\xi - n_2\right]^- \phi_{S_2}^Q(S_2)\,dS_2}{dx_{1,2}}$$

For y_2 to be negative, there are two cases to be considered. The cases are shown as the following where $\frac{-y_1 + F_{1,2}x_{1,2} - (p-\lambda)\xi + n_2}{x_{1,2} - \xi}$ is denoted as a:

$$y_2 < 0 \Leftrightarrow \begin{cases}\tilde{S}_2 < a & if\, x_{1,2} > \xi \\ \tilde{S}_2 > a & if\, x_{1,2} < \xi\end{cases}$$

By using the only the terms having $x_{1,2}$ as a multiplier and Leibniz Rule on taking derivative of y_2, the first derivative of the function $(-\triangle V_2(y_1))$ can be referred as the following:

$$\frac{d\left(-\triangle V_2(y_1)\right)}{dx_{1,2}} = \begin{cases}\beta r\int_0^a \left(F_{1,2} - \tilde{S}_2\right)\phi_{S_2}^Q(S_2)\,dS_2 & for\, x_{1,2} > \xi \\ \beta r\int_a^\infty \left(F_{1,2} - \tilde{S}_2\right)\phi_{S_2}^Q(S_2)\,dS_2 & for\, x_{1,2} < \xi\end{cases}$$

The point where the first derivative of the function is zero will be an extreme point. The main term in both of the cases are partial moments of normal distribution of order 1. Therefore, for the case of

$x_{1,2} > \xi$, the first derivative of $V_2(y_1)$ is positive and it is vice versa for the case of $x_{1,2} < \xi$. By referring to the paper "the Determination of Partial Moments" written by Winkler, Roodman and Britney (1972), the first derivative of the function can only be equal to zero when $x_{1,2} = \xi$.

At the next step, convexity of the term $(-\triangle V_2(y_1))$ should be examined. In the term $(-\triangle V_2(y_1))$, the expression $n_2 - \left(p - \tilde{S}_2 - \lambda\right)\xi - \left(\tilde{S}_2 - F_{1,2}\right)x_{1,2}$ is linear in $x_{1,2}$. Therefore the term $\beta r\,[y_2]^-$ should be examined. Obviously the term is a piecewise linear function in $x_{1,2}$ too. It is known that if a function is convex, its expected value function is also convex. Finally the term $(-\triangle V_2(y_1))$ is concluded to be convex, because all parts of it are convex. Due to convexity, the extreme point found is the universal minimum point of the function. Therefore, the decision $x^*_{1,2} = \xi$ is proved to be the optimum hedge quantity.

B.2 Theorem 2

In order to prove the theorem, firstly the term of $(-\triangle V_2(y_1))$ should be minimized similar to the proof of Theorem 1. The term $(-\triangle V_2(y_1))$ at $t = 1$ can be described mathematically as the following after replacing $x_{2,3}$ by ξ due to Lemma 1:

$$\begin{aligned}(-\triangle V_2(y_1)) = \int_0^\infty \Bigg[& n_2 - \left(\tilde{F}_{2,3} - F_{1,3}\right)x_{1,3} \\ & + \beta \int_0^\infty \Bigg[n_3 - \left(p - \tilde{S}_3 - \lambda\right)\xi - \left(\tilde{S}_3 - \tilde{F}_{2,3}\right)\xi \\ & + \beta\left(r\,[y_2]^- + r\,[y_3]^-\right)\Bigg]\phi^Q_{S_3}(S_3)\,dS_3\Bigg]\phi^Q_{F_{2,3}}(F_{2,3})\,dF_{2,3}\end{aligned}$$

where

$$y_2 = y_1 - n_2 + \left(\tilde{F}_{2,3} - F_{1,3}\right)x_{1,3}$$

$$y_3 = y_2 - n_3 + \left(p - \tilde{F}_{2,3} - \lambda\right)\xi$$

In order to find the optimum hedging quantity at time 1, it is necessary to analyze the first and the second derivatives of the term $(-\triangle V_2(y_1))$ with respect to the decision variable at $t = 1$, i.e. $x_{1,3}$. Since $F_{1,3} = \int_0^\infty \left[\left(\tilde{F}_{2,3}\right)\right]\phi^Q_{F_{2,3}}dF_{2,3}$ and the rest of the expression do not contain $x_{1,3}$ except financial distress cost premiums, the first derivative of the term $(-\triangle V_2(y_1))$ is the following.

$$\frac{d\,(-\triangle V_2(y_1))}{dx_{1,3}} = \frac{\beta\, d\left(\int_0^\infty \left[r\,[y_2]^- + \int_0^\infty r\,[y_3]^-\,\phi^Q_{S_3}dS_3\right]\phi^Q_{F_{2,3}}(F_{2,3})\,dF_{2,3}\right)}{dx_{1,3}}$$

In order to examine the expression above, the breakpoints where y_2 and y_3 becomes zero should be found. The condition that makes the expression $y_2 = y_1 - n_2 + \left(\tilde{F}_{2,3} - F_{1,3}\right)x_{1,3}$ becomes less than zero is the following where $b = \frac{-y_1 + F_{1,3}x_{1,3} + n_2}{x_{1,3}}$.

$$y_2 < 0 \Leftrightarrow \begin{cases} \tilde{F}_{2,3} < b & if\, x_{1,3} > 0 \\ \tilde{F}_{2,3} > b & if\, x_{1,3} < 0 \end{cases}$$

Since the expected financial distress cost at the end of the second period is also aimed to be found, the conditions that satisfy $y_3 < 0$ should be examined. Thus y_3 will be equal to $y_1 + \left(\tilde{F}_{2,3} - F_{1,3}\right) x_{1,3} + \left(p - \tilde{F}_{2,3} - \lambda\right) \xi - n_3 - n_2$. The condition for $y_3 < 0$ where $c = \frac{-y_1 + F_{1,3} x_{1,3} - (p-\lambda)\xi + n_2 + n_3}{x_{1,3} - \xi}$ is the following.

$$y_3 < 0 \Leftrightarrow \begin{cases} F_{2,3} < c & if\, x_{1,3} > \xi \\ F_{2,3} > c & if\, x_{1,3} < \xi \end{cases}$$

We will prove that the firm should underhedge in order to decrease the total of the expected financial distress costs. By examining the negativity conditions of y_2 and y_3, and corresponding first derivative of the objective function in these 2 different intervals:

Case I: $x_{1,3} > \xi$

As a result, the expression will be as the following.

$$\begin{aligned} \frac{d\left(-\triangle V_2\left(y_1\right)\right)}{dx_{1,3}} \quad = \quad & \frac{d\left[\beta r \int_0^b \left(-y_1 + \left(F_{1,3} - \tilde{F}_{2,3}\right) x_{1,3} + n_2\right) \phi^Q_{F_{2,3}}\left(F_{2,3}\right) dF_{2,3}\right]}{dx_{1,3}} \\ & + \frac{d\left[\beta r \int_0^c \left[-y_1 + \left(F_{1,3} - \tilde{F}_{2,3}\right) x_{1,3} + n_2 - \left(p - \tilde{F}_{2,3} - \lambda\right)\xi + n_3\right] \phi^Q_{F_{2,3}}\left(F_{2,3}\right) dF_{2,3}\right]}{dx_{1,3}} \end{aligned}$$

This expression can be transformed into the following by only considering the terms including $x_{1,3}$ and use of Leibniz rule:

$$\begin{aligned} \frac{d\left(-\triangle V_2\left(y_1\right)\right)}{dx_{1,3}} \quad = \quad & \beta r \int_0^b \left(F_{1,3} - \tilde{F}_{2,3}\right) \phi^Q_{F_{2,3}}\left(F_{2,3}\right) dF_{2,3} \\ & + \beta r \int_0^c \left(F_{1,3} - \tilde{F}_{2,3}\right) \phi^Q_{F_{2,3}}\left(F_{2,3}\right) dF_{2,3} \end{aligned}$$

Both of the terms $\beta r \int_0^b \left(F_{1,3} - \tilde{F}_{2,3}\right) \phi^Q_{F_{2,3}}\left(F_{2,3}\right) dF_{2,3}$ and $\beta r \int_0^c \left(F_{1,3} - \tilde{F}_{2,3}\right) \phi^Q_{F_{2,3}}\left(F_{2,3}\right) dF_{2,3}$ are positive. Consequently, there cannot be an extreme point when $x_{1,3} > \xi$.

Case II: $0 \leqq x_{1,3} \leqq \xi$

In this case we have:

$$\frac{d\left(-\triangle V_2\left(y_1\right)\right)}{dx_{1,3}} = \frac{d\left[\beta r\int_0^b \left(-y_1+\left(F_{1,3}-\tilde{F}_{2,3}\right)x_{1,3}+n_2\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}\right]}{dx_{1,3}} + \frac{d\beta r\int_c^\infty \left[-y_1+\left(F_{1,3}-\tilde{F}_{2,3}\right)x_{1,3}+n_2-\left(p-\tilde{F}_{2,3}-\lambda\right)\xi+n_3\right]\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}}{dx_{1,3}}$$

This expression can be transformed into the following to use Leibniz rule easier.

$$\frac{d\left(-\triangle V_2\left(y_1\right)\right)}{dx_{1,3}} = \beta r\int_0^b \left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3} + \beta r\int_c^\infty \left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}$$

The term $\beta r\int_0^b \left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}$ is positive and the term $\beta r\int_0^c \left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}$ is negative. Consequently, an extreme point may exist when $0 \leqq x_{1,3} \leqq \xi$.

Case III: $x_{1,3} < 0$

In this case the company is selling futures contracts. As a result, the expression will be as the following.

$$\frac{d\left(-\triangle V_2\left(y_1\right)\right)}{dx_{1,3}} = \frac{d\left[\beta r\int_b^\infty \left(-y_1+\left(F_{1,3}-\tilde{F}_{2,3}\right)x_{1,3}+n_2\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}\right]}{dx_{1,3}} + \frac{d\beta r\int_c^\infty \left[-y_1+\left(F_{1,3}-\tilde{F}_{2,3}\right)x_{1,3}+n_2-\left(p-\tilde{F}_{2,3}-\lambda\right)\xi+n_3\right]\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}}{dx_{1,3}}$$

Ignoring terms not including $x_{1,3}$ and use of Leibniz rule, we have:

$$\frac{d\left(-\triangle V_2\left(y_1\right)\right)}{dx_{1,3}} = \beta r\int_b^\infty \left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3} + \beta r\int_c^\infty \left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}$$

Both of the terms are negative. Consequently, an extreme point may not exist when $x_{1,3} < 0$.

Similar to Theorem 1, the all terms except $\beta\left(r\left[y_2\right]^- + r\left[y_3\right]^-\right)$ are linear in $x_{1,3}$ in $-\triangle V_2\left(y_1\right)$. Besides, it is known that both of the financial distress costs at time 2 and 3 are piecewise linear functions in $x_{1,3}$. It is known that if a function is convex, its expected value function is also convex. Finally the term

$-\triangle V_2(y_1)$ is concluded to be convex, because all parts of it are convex. Due to convexity, the extreme points found are universal minimum points of the function. In other words, they are the solutions that generate the minimum value of the term. Consequently, there cannot be an optimum solution when $x_{1,3} > \xi$ or $x_{1,3} < 0$, optimum solution can take place when $0 \leqq x^*_{1,3} \leqq \xi$.

B.3 Theorem 3

We can rewrite b and c as $\left(F_{1,3} - \frac{\triangle}{x_{1,3}}\right)$ and $\left(F_{1,3} + \frac{-\triangle+n_3-\delta\xi}{x_{1,3}-\xi}\right)$ respectively. Setting $\frac{d(-\triangle V_2(y_1))}{dx_{1,3}} = 0$ at case II (which suggests $0 \leqq x_{1,3} \leqq \xi$) at Theorem 2, we get $\int_0^{F_{1,3}-\frac{\triangle}{x_{1,3}}} \left(F_{1,3} - \tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}(F_{2,3})\,dF_{2,3} = \int_0^{F_{1,3}+\frac{-\triangle+n_3-\delta\xi}{x_{1,3}-\xi}} \left(F_{1,3} - \tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}(F_{2,3})\,dF_{2,3}$ since the random variable $\tilde{F}_{2,3}$ has a normal distribution with mean $F_{1,3}$. Indeed, in order to satisfy $\frac{d}{dx_{1,3}}(-V_2(y_1)) = 0$, the following equation should also be satisfied:

$$\left|\frac{\triangle}{x_{1,3}}\right| = \left|\frac{\triangle - n_3 + \delta\xi}{\xi - x_{1,3}}\right|$$

Case I: $\triangle(\triangle - n_3 + \delta\xi) > 0$

Suppose that $\triangle > 0$ and $\triangle - n_3 + \delta\xi > 0$, or $\triangle < 0$ and $\triangle - n_3 + \delta\xi < 0$.

$$\xi\triangle - x_{1,3}\triangle = x_{1,3}\triangle - n_3x_{1,3} + \delta\xi x_{1,3}$$

Finally $x^*_{1,3} = \xi\triangle/(2\triangle - n_3 + \delta\xi)$ satisfying $x^*_{1,3} < \xi$ and $\triangle(\triangle - n_3 + \delta\xi) > 0$.

Case II: $\triangle(\triangle - n_3 + \delta\xi) < 0$

Suppose that $\triangle < 0$ and $\triangle - n_3 + \delta\xi > 0$, or $\triangle > 0$ and $\triangle - n_3 + \delta\xi < 0$.

$$-\xi\triangle + x_{1,3}\triangle = x_{1,3}\triangle - n_3x_{1,3} + \delta\xi x_{1,3}$$

Finally $x^*_{1,3} = -\xi\triangle/(-n_3 + \delta\xi)$ satisfying $x^*_{1,3} < \xi$ and $\triangle(\triangle - n_3 + \delta\xi) > 0$.

B.4 Theorem 4

When $\triangle = y_1 - n_2 = 0$, we have $b = F_{1,3}$ and $c = F_{1,3} + \frac{\delta\xi-n_3}{\xi-x_{1,3}}$. Thus, the first derivative of the term $(-\triangle V_2(y_1))$ in the region where optimal hedging amount is found $0 \leqq x_{1,3} \leqq \xi$ is:

$$\begin{aligned}\frac{d(-\triangle V_2(y_1))}{dx_{1,3}} &= \beta r\int_0^{F_{1,3}}\left(F_{1,3} - \tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}(F_{2,3})\,dF_{2,3} \\ &\quad +\beta r\int_{F_{1,3}+\frac{\delta\xi-n_3}{\xi-x_{1,3}}}^{\infty}\left(F_{1,3} - \tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}(F_{2,3})\,dF_{2,3}\end{aligned}$$

It is already known that $\int_0^{F_{1,3}} \left(F_{1,3} - \tilde{F}_{2,3}\right) \phi^Q_{F_{2,3}} (F_{2,3})\, dF_{2,3}$ $+ \int_{F_{1,3}+\frac{\delta\xi - n_3}{x_{1,3}-\xi}}^{\infty} \left(F_{1,3} - \tilde{F}_{2,3}\right) \phi^Q_{F_{2,3}} (F_{2,3})\, dF_{2,3} > 0$, as a result $\frac{d}{dx_{1,3}} (-\triangle V_2(y_1))$ is positive. Indeed, increasing $x_{1,3}$ will increase the expected financial distress cost and we already know that the optimum decision should satisfy $0 \leqq x_{1,3} \leqq \xi$. Consequently, $x^*_{1,3}$ is equal to 0 when $\triangle = 0$.

B.5 Theorem 5

It is already known that $b = F_{1,3} - \frac{\triangle}{x_{1,3}}$ and $c = F_{1,3} + \frac{-\triangle + n_3 - \delta\xi}{x_{1,3}-\xi}$. When $\triangle = n_3 - \delta\xi$, we have $c = F_{1,3}$. Finally the first derivative of $(-\triangle V_2(y_1))$ is the following for $x_{1,3} < \xi$:

$$\begin{aligned} \frac{d(-\triangle V_2(y_1))}{dx_{1,3}} &= \beta r \int_0^{F_{1,3}-\frac{\triangle}{x_{1,3}}} \left(F_{1,3} - \tilde{F}_{2,3}\right) \phi^Q_{F_{2,3}} (F_{2,3})\, dF_{2,3} \\ &\quad + \beta r \int_{F_{1,3}}^{\infty} \left(F_{1,3} - \tilde{F}_{2,3}\right) \phi^Q_{F_{2,3}} (F_{2,3})\, dF_{2,3} \end{aligned}$$

Since $\int_0^{F_{1,3}-\frac{\triangle}{x_{1,3}}} \left(F_{1,3} - \tilde{F}_{2,3}\right) \phi^Q_{F_{2,3}} dF_{2,3} + \int_{F_{1,3}}^{\infty} \left(F_{1,3} - \tilde{F}_{2,3}\right) \phi^Q_{F_{2,3}} (F_{2,3})\, dF_{2,3} < 0$, the expression above is negative and the decisionmaker should increase its hedging quantity until it reaches to ξ. The optimum hedging decision is full-hedging, i.e. $x^*_{1,3} = \xi$

B.6 Theorem 6

When $\triangle = y_1 - n_2 = n_3 - \delta\xi = 0$, we have $b = c = F_{1,3}$, and the first derivative of the objective function is for $0 \leqq x_{1,3} \leqq \xi$:

$$\begin{aligned} \frac{d(-\triangle V_2(y_1))}{dx_{1,3}} &= \beta r \int_0^{F_{1,3}} \left(F_{1,3} - \tilde{F}_{2,3}\right) \phi^Q_{F_{2,3}} (F_{2,3})\, dF_{2,3} \\ &\quad + \beta r \int_{F_{1,3}}^{\infty} \left(F_{1,3} - \tilde{F}_{2,3}\right) \phi^Q_{F_{2,3}} (F_{2,3})\, dF_{2,3} \\ &= 0 \end{aligned}$$

In this case, hedging quantity is not important in terms of financial distress cost minimization.

Figure B.1 shows numerical simulation results of a model with 100,000 runs and values $n_3 = 50$, $\xi = 100$, $p = 11$, $\lambda = 0.5$, $F_{1,3} = 10$, $\sigma = 1$, $r = 0.1$, $\beta = 0.95$. Besides, amounts less then $\zeta = 0.0001$ units of expected financial distress cost is supposed to be neglected for the company.

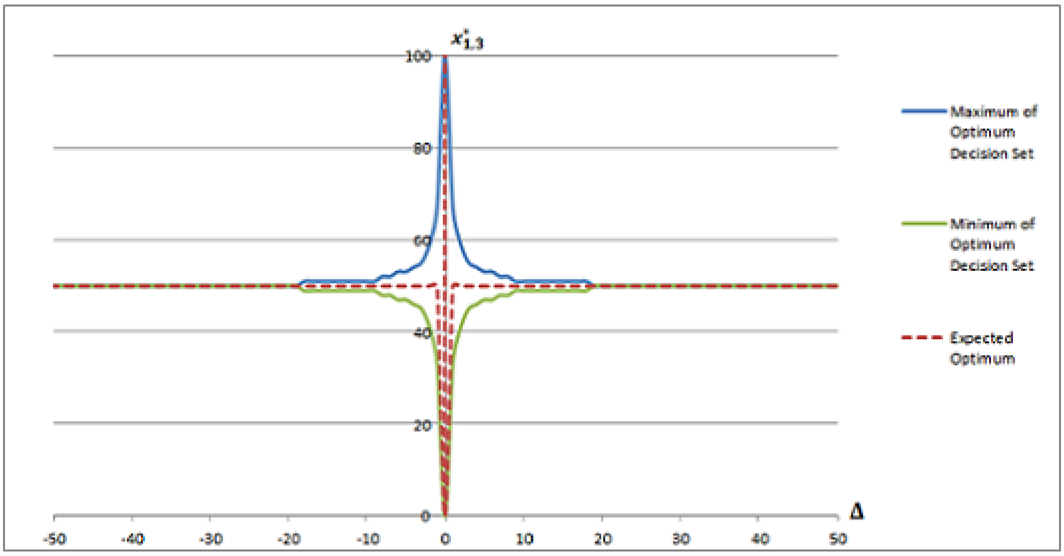

Figure B.1: Expected total financial distress costs by changing values of $x_{1,3}$ for two-period model with base parameters

In Figure B.1, two different lines show the maximum and minimum hedging amounts for the decision-maker at optimality. Moreover, the dashed line shows the numerically calculated optimum points. As observed from Figure B.1, multi-optimal decisions exist for the company when getting closer to the point where $\triangle$ and $n_3 - \delta\xi$ equal to 0.

B.7 Theorem 7

As stated above, the optimum hedging policy follows the function below.

$$x^*_{1,3} = \begin{cases} \xi\triangle/\left(2\triangle - n_3 + \delta\xi\right) & if\, \triangle\left(\triangle - n_3 + \delta\xi\right) \geqq 0 \\ -\xi\triangle/\left(-n_3 + \delta\xi\right) & if\, \triangle\left(\triangle - n_3 + \delta\xi\right) \leqq 0 \end{cases}$$

The optimum hedging decision line will follow $-\xi\triangle/\left(-n_3 + \delta\xi\right)$ when $min\left(0, n_3 - \delta\xi\right) < \triangle <$ $max\left(0, n_3 - \delta\xi\right)$. On the other hand, at the both edges while $\triangle$ goes to $\pm\infty$, the optimum hedging decision will be equal to $\xi\triangle/\left(2\triangle - n_3 + \delta\xi\right)$. If the limits of the lines are calculated, it can already be seen that:

$$\lim_{\triangle\to\pm\infty}\left(\xi\triangle/\left(2\triangle - n_3 + \delta\xi\right)\right) = \xi/2$$

It means that the optimum hedging quantity is $\xi/2$ while $\triangle$ goes to $\pm\infty$. However it should also be noted that the normal distribution is used in the model. Thus, the total expected financial distress cost does not change in significant amounts in both infinite ends and the behavior of the function should be examined in the region where we have the optimal hedging decisions $0 \leqq x^*_{1,3} \leqq \xi$, and $\triangle \to \pm\infty$. As $\triangle \to \infty$; $b \to -\infty$ and $c \to \infty$; and as $\triangle \to -\infty$; $b \to \infty$ and $c \to -\infty$. Finally, the first derivative of $\left(-\triangle V_2\left(y_1\right)\right)$ is:

$$\begin{aligned}\frac{d\left(-\triangle V_2\left(y_1\right)\right)}{dx_{1,3}} &= \beta r\int_0^b\left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^{Q}_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}\\ &\quad +\beta r\int_c^{\infty}\left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^{Q}_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}\\ &= 0\end{aligned}$$

The expressions $\int_0^{\infty}\left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^{Q}_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}$ and $\int_{\infty}^{\infty}\left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^{Q}_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}$ are known to be zero. As a result, there is no effect of hedging in these conditions as shown above.

Figure B.2 shows numerical simulation results of a model with $100,000$ runs and values $n_3 = 30$, $\xi = 100$, $p = 11$, $\lambda = 0.5$, $F_{1,3} = 10$, $\sigma = 1$, $r = 0.1$, $\beta = 0.95$. Besides, amounts less then $\zeta = 0.0001$ units of expected financial distress cost is supposed to be neglected for the company.

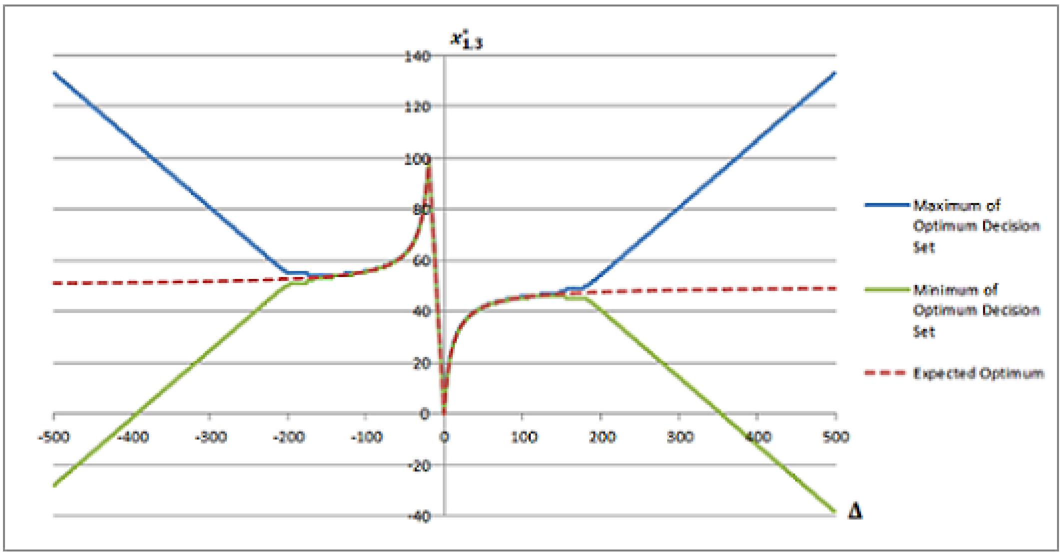

Figure B.2: Expected total financial distress costs by changing values of $x_{1,3}$ for two-period model with base parameters

In Figure B.2, two different lines show the maximum and minimum hedging amounts for the decision-maker at optimality. Moreover, the dashed line shows the numerically calculated optimum points. As observed from Figure B.2, multi-optimal decisions exist for the company when $\triangle \to \pm\infty$. However, the calculated optimum quantity line converges to $\xi/2$.

B.8 Theorem 8

As stated above, the optimum hedging policy follows the function below.

$$x^*_{1,3} = \begin{cases}\xi\triangle/\left(2\triangle - n_3 + \delta\xi\right) & if\ \triangle\left(\triangle - n_3 + \delta\xi\right) \geqq 0\\ -\xi\triangle/\left(-n_3 + \delta\xi\right) & if\ \triangle\left(\triangle - n_3 + \delta\xi\right) \leqq 0\end{cases}$$

The optimum hedging decision line may follow both of the scenarios with respect to the sign of $\triangle$ and n_3 while n_3 goes to $\pm\infty$. When $\triangle > 0$ and n_3 goes to $+\infty$ or $\triangle < 0$ and n_3 goes to $-\infty$, the optimum hedging decision at $t = 1$ will follow the line $-\xi\triangle/(-n_3 + \delta\xi)$. On the contrast, $x^*_{1,3}$ will follow the line $\xi\triangle/(2\triangle - n_3 + \delta\xi)$ when $\triangle < 0$ and n_3 goes to $+\infty$ or $\triangle > 0$ and n_3 goes to $-\infty$. Thus the optimum decision will converge to 0 in both ends as shown below.

$$\underset{\triangle\rightarrow\pm\infty}{lim} (\xi\triangle/(2\triangle - n_3 + \delta\xi)) = 0$$

$$\underset{\triangle\rightarrow\pm\infty}{lim} (-\xi\triangle/(-n_3 + \delta\xi)) = 0$$

It means that the optimum hedging quantity is 0 while n_3 goes to $\pm\infty$. However, again it should be noted that the normal distribution is used in the model. Thus, the total expected financial distress cost does not change in significant amounts in both ends and the behavior of the function should be examined in the region where we have the optimal hedging decisions $0 \leqq x^*_{1,3} \leqq \xi$, and $n_3 \rightarrow \pm\infty$. As $n_3 \rightarrow +\infty$ and $\triangle > 0$; $x^*_{1,3} = -\xi\triangle/(-n_3 + \delta\xi)$, $b = F_{1,2} - \frac{n_3+\delta\xi}{\xi}$ and $c = F_{1,2} + \frac{n_3+\delta\xi}{\xi}$; $b \rightarrow -\infty$ and $c \rightarrow +\infty$. As $n_3 \rightarrow -\infty$ and $\triangle < 0$; $x^*_{1,3} = -\xi\triangle/(-n_3 + \delta\xi)$, $b = F_{1,2} - \frac{n_3+\delta\xi}{\xi}$ and $c = F_{1,2} + \frac{n_3+\delta\xi}{\xi}$; $b \rightarrow +\infty$ and $c \rightarrow -\infty$. As $n_3 \rightarrow +\infty$ and $\triangle < 0$; $x^*_{1,3} = \xi\triangle/(2\triangle - n_3 + \delta\xi)$, $b = F_{1,2} - \frac{2\triangle-n_3+\delta\xi}{\xi}$ and $c = F_{1,2} + \frac{2\triangle-n_3+\delta\xi}{\xi}$; $b \rightarrow +\infty$ and $c \rightarrow -\infty$. As $n_3 \rightarrow -\infty$ and $\triangle > 0$; $x^*_{1,3} = \xi\triangle/(2\triangle - n_3 + \delta\xi)$, $b = F_{1,2} - \frac{2\triangle-n_3+\delta\xi}{\xi}$ and $c = F_{1,2} + \frac{2\triangle-n_3+\delta\xi}{\xi}$; $b \rightarrow -\infty$ and $c \rightarrow +\infty$. Consequently, the first derivative of $(-\triangle V_2(y_1))$ is known to be the following expression:

$$\begin{aligned}
\frac{d}{dx_{1,3}}(-\triangle V_2(y_1)) &= \beta r \int_0^b \left(F_{1,3} - \tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}(F_{2,3})\,dF_{2,3} \\
&\quad + \beta r \int_c^\infty \left(F_{1,3} - \tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}(F_{2,3})\,dF_{2,3} \\
&= 0
\end{aligned}$$

Since, the expressions $\int_0^0 \left(F_{1,3} - \tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}(F_{2,3})\,dF_{2,3}$,
$\int_\infty^\infty \left(F_{1,3} - \tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}(F_{2,3})\,dF_{2,3}$ and $\int_0^\infty \left(F_{1,3} - \tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}(F_{2,3})\,dF_{2,3}$ are all known to be equal to zero. Therefore, in the region $0 \leqq x^*_{1,3} \leqq \xi$ that we considered, there is multioptimality due to characteristics of normal distribution.

Figure B.3 shows numerical simulation results of a model with 100,000 runs and values $y_1 = 100$, $n_2 = 30$, $n_3 = 30$, $\xi = 100$, $p = 11$, $\lambda = 0.5$, $F_{1,3} = 10$, $\sigma = 1$, $r = 0.1$, $\beta = 0.95$. Besides, amounts less then $\zeta = 0.0001$ units of expected financial distress cost is supposed to be neglected for the company.

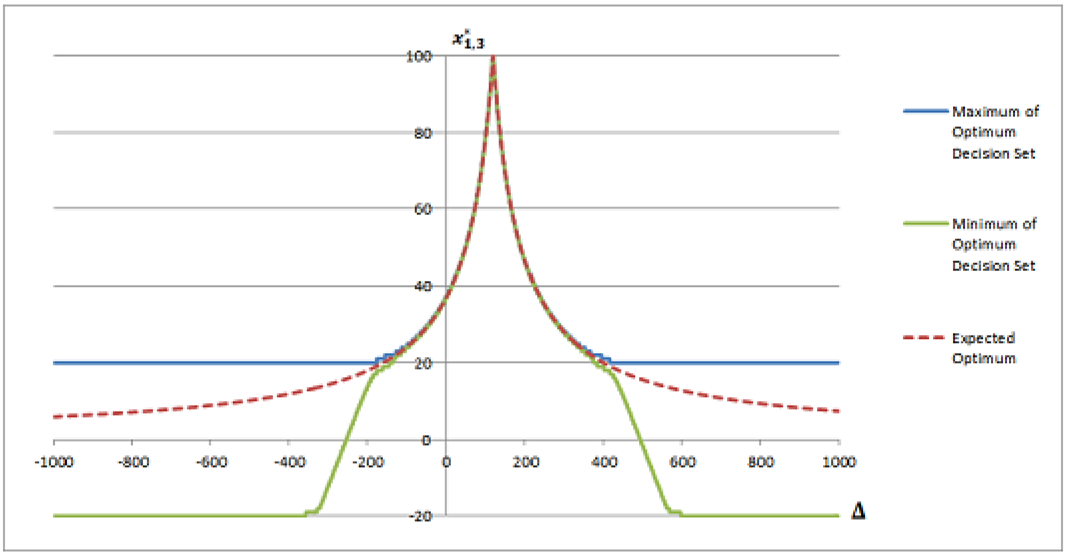

Figure B.3: Expected total financial distress costs by changing values of $x_{1,3}$ for two-period model with base parameters

In Figure B.3, two different lines show the maximum and minimum hedging amounts for the decision-maker at optimality. Moreover, the dashed line shows the numerically calculated optimum points. As observed from Figure B.3, multi-optimal decisions exist for the company when $n_3 \to \pm\infty$. However, the calculated optimum quantity line converges to 0.

B.9 Theorem 9

The first derivative is to be examined in order to observe the behavior of the function.

$$\frac{d\left(-\triangle V_2\left(y_1\right)\right)}{dx_{1,4}} = \frac{d\beta^3 r\left[E_F^{\phi}\left([y_2]^- + [y_3]^- + r\,[y_4]^-\right)\right]}{dx_{1,4}}$$

The expression above should better be analyzed in three different components. First of all, the break-points of financial distress costs in all those components should be found.

It is already known that $y_2 = y_1 + \left(\tilde{F}_{2,4} - F_{1,4}\right)x_{1,4} - n_2$. It behaves different in two different cases with respect to the value of $x_{1,4}$. Suppose that $e = \frac{-y_1+F_{1,4}x_{1,4}+n_2}{x_{1,4}}$. Then the negativity condition of y_2 is $\tilde{F}_{2,4} < e$ when $x_{1,4} > 0$; and the condition in case of $x_{1,4} < 0$ is $\tilde{F}_{2,4} > e$.

- Suppose that $x_{1,4} > 0$:

Then the expected value of $r\,[y_2]^-$ will be as the following:

$$E_F^{\phi}\left([y_2]^-\right) = \int_0^e \left(-y_1 + \left(F_{1,4} - \tilde{F}_{2,4}\right)x_{1,4} + n_2\right)\phi_{F_{2,4}}^{Q}\left(F_{2,4}\right)dF_{2,4}$$

Finally, first derivative of expected value of $r\left[y_2\right]^-$ is:

$$\frac{d\left(E_F^\phi\left(\left[y_2\right]^-\right)\right)}{dx_{1,4}} = \int\limits_0^e \left(F_{1,4} - \tilde{F}_{2,4}\right)\phi_{F_{2,4}}^Q\left(F_{2,4}\right)dF_{2,4}$$

- Suppose that $x_{1,4} < 0$:

Then the expected value of $r\left[y_2\right]^-$ will be as the following:

$$E_F^\phi\left(\left[y_2\right]^-\right) = \int\limits_e^{+\infty}\left(-y_1 + \left(F_{1,4} - \tilde{F}_{2,4}\right)x_{1,4} + n_2\right)\phi_{F_{2,4}}^Q\left(F_{2,4}\right)dF_{2,4}$$

First derivative of expected value of $r\left[y_2\right]^-$when $x_{1,4} > 0$:

$$\frac{d\left(E_F^\phi\left(\left[y_2\right]^-\right)\right)}{dx_{1,4}} = \int\limits_e^{+\infty}\left(F_{1,4} - \tilde{F}_{2,4}\right)\phi_{F_{2,4}}^Q\left(F_{2,4}\right)dF_{2,4}$$

We have $y_3 = y_1 + \left(\tilde{F}_{2,4} - F_{1,4}\right)x_{1,4} - n_2 + \left(\tilde{F}_{3,4} - \tilde{F}_{2,4}\right)x_{2,4} - n_3$. Negativity condition of y_3 should be found by assuming that $y_3 < 0$, then the condition becomes the following:

$$\tilde{F}_{2,4}\left(x_{1,4} - x_{2,4}\right) < -y_1 + F_{1,4}x_{1,4} + n_2 + n_3 - \tilde{F}_{3,4}x_{2,4}$$

For the sake of notation ease, suppose that $g = -y_1 + F_{1,4}x_{1,4} + n_2 + n_3$ and $f = x_{1,4} - x_{2,4}$. There are two cases with respect to value of $x_{1,4}$ as the following. In case of $x_{1,4} > x_{2,4}$, the condition of y_3 is $\tilde{F}_{2,4} < \frac{g}{f}$. On the other hand, the condition evolves into $\tilde{F}_{2,4} > \frac{g}{f}$ in case of $x_{1,4} < x_{2,4}$.

Note that, the decision variable is $x_{1,4}$ at $t = 1$. Therefore, the first derivative of $\left[y_3\right]^-$ with respect to $x_{1,4}$ is to be analyzed. Moreover, $F_{1,4}$ is known, while $\tilde{F}_{2,4}$ and $\tilde{F}_{3,4}$ are random stochastic variables at $t = 1$; as a result recursive integral will be used to calculate the expected financial distress cost.

- Suppose that $x_{1,4} > x_{2,4}$:

First derivative of $r\left[y_3\right]^-$ when $x_{1,4} > x_{2,4}$:

$$\begin{aligned}\frac{d\left(E_F^\phi\left(\left[y_3\right]^-\right)\right)}{dx_{1,4}} &= \frac{d\int_{F_{3,4}=0}^{+\infty}\int_{F_{2,4}=0}^{\frac{g-F_{3,4}x_{2,4}}{f}}\left(-y_3\right)\phi_{F_{2,4}}^Q\left(F_{2,4}\right)dF_{2,4}\phi_{F_{3,4}}^Q\left(F_{3,4}\right)dF_{3,4}}{dx_{1,4}}\\ &= \int\limits_{F_{3,4}=0}^{+\infty}\int\limits_{F_{2,4}=0}^{\frac{g-F_{3,4}x_{2,4}}{f}}\left(F_{1,4} - \tilde{F}_{2,4}\right)\phi_{F_{2,4}}^Q\left(F_{2,4}\right)dF_{2,4}\phi_{F_{3,4}}^Q\left(F_{3,4}\right)dF_{3,4}\end{aligned}$$

- Suppose that $x_{1,4} < x_{2,4}$:

First derivative of $r\left[y_3\right]^-$ when $x_{1,4} < x_{2,4}$:

$$\frac{d\left(E_F^{\phi}\left([y_3]^-\right)\right)}{dx_{1,4}} = \frac{d\int_{F_{3,4}=0}^{+\infty}\int_{F_{2,4}=\frac{g-F_{3,4}x_{2,4}}{f}}^{+\infty}(-y_3)\,\phi_{F_{2,4}}^{Q}(F_{2,4})\,dF_{2,4}\phi_{F_{3,4}}^{Q}(F_{3,4})\,dF_{3,4}}{dx_{1,4}}$$

$$= \int_{F_{3,4}=0}^{+\infty}\int_{F_{2,4}=\frac{g-F_{3,4}x_{2,4}}{f}}^{+\infty}\left(F_{1,4}-\tilde{F}_{2,4}\right)\phi_{F_{2,4}}^{Q}(F_{2,4})\,dF_{2,4}\phi_{F_{3,4}}^{Q}(F_{3,4})\,dF_{3,4}$$

The term y_4 is equal to $y_1 + \left(\tilde{F}_{2,4} - F_{1,4}\right)x_{1,4} - n_2 + \left(\tilde{F}_{3,4} - \tilde{F}_{2,4}\right)x_{2,4} - n_3 + \left(\tilde{S}_4 - \tilde{F}_{3,4}\right)x_{3,4}$ $+ \left(p - \tilde{S}_4 - \lambda\right)\xi - n_4$. Negativity condition of y_4 should be found by assuming that $y_4 < 0$. By using Lemma 2, it is already known that: $x^*_{3,4} = \xi$. Furthermore, Theorem 2 suggests that $(x_{2,4} - \xi) \leqq 0$ regardless of the value of the other variables. Therefore the expression becomes the following:

$$F_{2,4}\left(x_{1,4} - x_{2,4}\right) < -y_1 + F_{1,4}x_{1,4} + n_2 + n_3 + n_4 - (p-\lambda)\,\xi - \tilde{F}_{3,4}\left(x_{2,4} - \xi\right)$$

There are two different conditions for the different cases with respect to the value of $x^*_{1,4}$. For the sake of notation ease, suppose that $f = x_{1,4} - x_{2,4}$, $k = -y_1 + F_{1,4}x_{1,4} + n_2 + n_3 + n_4 - (p-\lambda)\,\xi$ and $l = x_{2,4} - \xi$. The negativity condition of y_4 for $x_{1,4} > x_{2,4}$ is $\tilde{F}_{2,4} < \frac{k-\tilde{F}_{3,4}l}{f}$; while the negativity condition for $x_{1,4} < x_{2,4}$ is $\tilde{F}_{2,4} > \frac{k-\tilde{F}_{3,4}l}{f}$.

- Suppose that $x_{1,4} > x_{2,4}$:

The first derivative of $[y_4]^-$ with respect to $x_{1,4}$ is shown below.

$$\frac{d\left(E_F^{\phi}\left([y_4]^-\right)\right)}{dx_{1,4}} = \frac{d\left(\int_{F_{3,4}=0}^{+\infty}\int_{F_{2,4}=0}^{\frac{k-F_{3,4}l_1}{f}}(-y_4)\,\phi_{F_{2,4}}^{Q}(F_{2,4})\,dF_{2,4}\phi_{F_{3,4}}^{Q}(F_{3,4})\,dF_{3,4}\right)}{dx_{1,4}}$$

$$= \int_{F_{3,4}=0}^{+\infty}\int_{F_{2,4}=0}^{\frac{k-F_{3,4}l}{f}}\left(F_{1,4}-\tilde{F}_{2,4}\right)\phi_{F_{2,4}}^{Q}(F_{2,4})\,dF_{2,4}\phi_{F_{3,4}}^{Q}(F_{3,4})\,dF_{3,4}$$

- Suppose that $x_{1,4} < x_{2,4}$:

The first derivative of $[y_4]^-$ with respect to $x_{1,4}$ is shown below.

$$\frac{d\left(E_F^{\phi}\left([y_4]^-\right)\right)}{dx_{1,4}} = \frac{d\left(\int_{F_{3,4}=0}^{+\infty}\int_{F_{2,4}=\frac{k-F_{3,4}l}{f}}^{+\infty}(-y_4)\,\phi_{F_{2,4}}^{Q}(F_{2,4})\,dF_{2,4}\phi_{F_{3,4}}^{Q}(F_{3,4})\,dF_{3,4}\right)}{dx_{1,4}}$$

$$= \int_{F_{3,4}=0}^{+\infty}\int_{F_{2,4}=\frac{k-F_{3,4}l}{f}}^{+\infty}\left(F_{1,4}-\tilde{F}_{2,4}\right)\phi_{F_{2,4}}^{Q}(F_{2,4})\,dF_{2,4}\phi_{F_{3,4}}^{Q}(F_{3,4})\,dF_{3,4}$$

The convexity of the function may be proved by using the same methodology used in Theorem 2. Similar to the previous models, it is observed that the expressions $[y_2]^-$,$[y_3]^-$and $[y_4]^-$are piecewise convex functions and other variables in $-\triangle V_2(y_1)$ are all linear, and if a function is convex, its expected value function is also convex. Consequently, it is obvious that the term $-\triangle V_2(y_1)$ is convex in $x_{1,4}$.

From the point of decisionmaker, there are two different intervals of $x_{1,4}$.

Case I: $x_{1,4} > x_{2,4}$ and $x_{1,4} > 0$

$$\begin{aligned}\frac{d\left(-\triangle V_2(y_1)\right)}{dx_{1,4}} &= \int\limits_{F_{2,4}=0}^{e}\left(F_{1,4}-\tilde{F}_{2,4}\right)\phi^{Q}_{F_{2,4}}(F_{2,4})\,dF_{2,4} \\ &+ \int\limits_{F_{3,4}=0}^{+\infty}\int\limits_{F_{2,4}=0}^{\frac{g-F_{3,4}x_{2,4}}{f}}\left(F_{1,4}-\tilde{F}_{2,4}\right)\phi^{Q}_{F_{2,4}}(F_{2,4})\,dF_{2,4}\phi^{Q}_{F_{3,4}}(F_{3,4})\,dF_{3,4} \\ &+ \int\limits_{F_{3,4}=0}^{+\infty}\int\limits_{F_{2,4}=0}^{\frac{k-F_{3,4}l}{f}}\left(F_{1,4}-\tilde{F}_{2,4}\right)\phi^{Q}_{F_{2,4}}(F_{2,4})\,dF_{2,4}\phi^{Q}_{F_{3,4}}(F_{3,4})\,dF_{3,4}\end{aligned}$$

This expression is positive, it means that the expected total financial distress cost will increase while $x_{1,4}$ is increasing. Indeed, an extreme point cannot exist here; as a result it is not the optimum interval.

Case II: $x_{1,4} < x_{2,4}$ and $x_{1,4} > 0$

$$\begin{aligned}\frac{d\left(-\triangle V_2(y_1)\right)}{dx_{1,4}} &= \int\limits_{F_{2,4}=0}^{e}\left(F_{1,4}-\tilde{F}_{2,4}\right)\phi^{Q}_{F_{2,4}}(F_{2,4})\,dF_{2,4} \\ &+ \int\limits_{F_{3,4}=0}^{+\infty}\int\limits_{F_{2,4}=\frac{g-F_{3,4}x_{2,4}}{f}}^{+\infty}\left(F_{1,4}-\tilde{F}_{2,4}\right)\phi^{Q}_{F_{2,4}}(F_{2,4})\,dF_{2,4}\phi^{Q}_{F_{3,4}}(F_{3,4})\,dF_{3,4} \\ &+ \int\limits_{F_{3,4}=0}^{+\infty}\int\limits_{F_{2,4}=\frac{k-F_{3,4}l}{f}}^{+\infty}\left(F_{1,4}-\tilde{F}_{2,4}\right)\phi^{Q}_{F_{2,4}}(F_{2,4})\,dF_{2,4}\phi^{Q}_{F_{3,4}}(F_{3,4})\,dF_{3,4}\end{aligned}$$

This expression can either be positive or negative or zero. Therefore, the optimum decision may exist in this interval.

Case III: $x_{1,4} > x_{2,4}$ and $x_{1,4} < 0$

This case suggests that $x_{2,4} < 0$. However, this assumption contradicts with our previously proved knowledge via Lemma 2; $x_{2,4} \geq 0$. As a result, there is no need to examine this situation.

Case IV: $x_{1,4} < x_{2,4}$ and $x_{1,4} < 0$

$$\frac{d\left(-\triangle V_2(y_1)\right)}{dx_{1,4}} = \int\limits_{F_{2,4}=e}^{\infty}\left(F_{1,4}-\tilde{F}_{2,4}\right)\phi^{Q}_{F_{2,4}}(F_{2,4})\,dF_{2,4}$$

$$+\int_{F_{3,4}=0}^{+\infty}\int_{F_{2,4}=\frac{g-F_{3,4}x_{2,4}}{f}}^{+\infty}\left(F_{1,4}-\tilde{F}_{2,4}\right)\phi^{Q}_{F_{2,4}}\left(F_{2,4}\right)dF_{2,4}\phi^{Q}_{F_{3,4}}\left(F_{3,4}\right)dF_{3,4}$$

$$+\int_{F_{3,4}=0}^{+\infty}\int_{F_{2,4}=\frac{k-F_{3,4}l}{f}}^{+\infty}\left(F_{1,4}-\tilde{F}_{2,4}\right)\phi^{Q}_{F_{2,4}}\left(F_{2,4}\right)dF_{2,4}\phi^{Q}_{F_{3,4}}\left(F_{3,4}\right)dF_{3,4}$$

This expression can be only negative and it means that the total expected financial distress cost always increases while $x_{1,4}$ is increasing in the interval $(-\infty, 0]$. Consequently, there cannot be an optimum decision in this interval and optimum $x^*_{1,4}$ should be equal to or higher than 0 because of the convexity.

After examining all the possible cases with respect to the values of the decision variables and convexity of $-\triangle V_2\left(y_1\right)$, that suggests the optimum decision policy is under-hedging, such that $x^*_{1,4} \leqq \bar{x}^*_{2,4} \leqq \xi$ where $x^*_{2,4}$ is considered as the mean of the expected optimum hedge amount at time 2.

B.10 Theorem 10

Initially, the mathematical model should be composed. The model of n period model is shown below.

$$V_2\left(y_1\right) = \min_{x_{1,n+1}} E^{\phi}_{F_{2,n+1}}\left[n_2 + \left(F_{1,n+1} - \tilde{F}_{2,n+1}\right)x_{1,n+1} - \beta V_3(y_2)\right]$$

Subject to:

$$y_2 = y_1 - n_2 + \left(\tilde{F}_{2,n+1} - F_{1,n+1}\right)x_{1,n+1}$$

$$V_3\left(y_2\right) = \min_{x_{2,n+1}} E^{\phi}_{F_{2,n}}\left[n_2 + \left(F_{2,n+1} - \tilde{F}_{3,n+1}\right)x_{1,n+1} - \beta V_4(y_3)\right]$$

Subject to:

$$y_3 = y_2 - n_3 + \left(\tilde{F}_{3,n+1} - F_{2,n+1}\right)x_{1,n+1}$$

....

$$V_n\left(y_{n-1}\right) = \min_{x_{n-1,n+1}} E^{\phi}_{F_{2,n+1}}\left[n_{n-1} + \left(F_{n-1,n+1} - \tilde{F}_{n,n+1}\right)x_{1,n+1} - \beta V_{n+1}(y_n)\right]$$

Subject to:

$$y_n = y_{n-1} - n_n + \left(\tilde{F}_{3,n} - F_{2,n}\right)x_{1,n}$$

$$V_{n+1}(y_n) = \min_{x_{n,n+1}} E^{\phi}_{s_{n+1}}\left[n_{n+1} - \left(p - \tilde{S}_{n+1} - \lambda\right)\xi + \left(F_{n,n+1} - \tilde{S}_{n+1}\right)x_{n,n+1} - \beta V_{n+2}(y_{n+1})\right]$$

Subject to:

$$y_{n+1} = y_n - n_{n+1} + \left(p - \tilde{S}_{n+1} - \lambda\right)\xi + \left(\tilde{S}_{n+1} - F_{n,n+1}\right)x_{n,n+1}$$

There is no decision variable at $t = n + 1$, as a result the function $V_{n+2}(y_{n+1})$ is as the following:

$$V_{n+2}(y_{n+1}) = -\sum_{i=2}^{n+1} r\,[y_i]^-$$

Our aim is to solve the problem at $t = 1$. Thus, the first derivative is to be examined in order to observe the behavior of the function.

$$\frac{d\left(-\triangle V_2\left(y_1\right)\right)}{dx_{1,n+1}} = \frac{d\left[\beta^n r \sum_{i=2}^{n+1}\left[y_i\right]^-\right]}{dx_{1,n+1}}$$

Similar to the proof of Theorem 9, the expression above should better to be analyzed in three different components. Initially, the breakpoints of financial distress costs in all those components should be found.

It is already known that $y_2 = y_1 + \left(\tilde{F}_{2,n+1} - F_{1,n+1}\right) x_{1,n+1} - n_2$. Negativity condition of y_2 should be found by assuming that $y_2 < 0$. It behaves different in two different cases with respect to the value of $x_{1,n+1}$. Suppose that $m_1 = \frac{-y_1 + F_{1,n+1} x_{1,n+1} + n_2}{x_{1,n+1}}$. Then the negativity condition is the following when $x_{1,4} > 0$:

$$\tilde{F}_{2,n+1} < m_1$$

The condition in case of $x_{1,n+1} < 0$ is the following:

$$\tilde{F}_{2,n+1} > m_1$$

- Suppose that $x_{1,n+1} > 0$:

Then the expected value of $r\,[y_2]^-$ will be as the following:

$$E_F^{\phi}\left(\left[y_2\right]^-\right) = \int_0^{m_1} \left(-y_1 + \left(F_{1,n+1} - \tilde{F}_{2,n+1}\right) x_{1,n+1} + n_2\right) \phi^{Q}_{F_{2,n+1}}\left(F_{2,n+1}\right) dF_{2,n+1}$$

Finally, first derivative of expected value of $r\,[y_2]^-$ is:

$$\frac{d\left(E_F^{\phi}\left(\left[y_2\right]^-\right)\right)}{dx_{1,n+1}} = \int_0^{m_1} \left(F_{1,n+1} - \tilde{F}_{2,n+1}\right) \phi^{Q}_{F_{2,n+1}}\left(F_{2,n+1}\right) dF_{2,n+1}$$

- Suppose that $x_{1,n+1} < 0$:

Then the expected value of $r\,[y_2]^-$ will be as the following:

$$E_F^{\phi}\left(\left[y_2\right]^-\right) = \int_{m_1}^{+\infty} \left(-y_1 + \left(F_{1,n+1} - \tilde{F}_{2,n+1}\right) x_{1,4} + n_2\right) \phi^{Q}_{F_{2,n+1}}\left(F_{2,n+1}\right) dF_{2,n+1}$$

First derivative of expected value of $r\,[y_2]^-$when $x_{1,4} > 0$:

$$\frac{d\left(E_F^{\phi}\left([y_2]^-\right)\right)}{dx_{1,n+1}} = \int_{m_1}^{+\infty} \left(F_{1,n+1} - \tilde{F}_{2,n+1}\right) \phi_{F_{2,n+1}}^{Q} \left(F_{2,n+1}\right) dF_{2,n+1}$$

It is already known that $y_3 = y_1 + \left(\tilde{F}_{2,n+1} - F_{1,n+1}\right) x_{1,n+1} - n_2$ $+ \left(\tilde{F}_{3,n+1} - \tilde{F}_{2,n+1}\right) x_{2,n+1} - n_3$. Negativity condition of y_3 should be found by assuming that $y_3 < 0$, then the condition becomes the following:

$$\tilde{F}_{2,n+1} \left(x_{1,n+1} - x_{2,n+1}\right) < -y_1 + F_{1,n+1} x_{1,n+1} + n_2 + n_3 - \tilde{F}_{3,n+1} x_{2,n+1}$$

For the sake of notation ease, suppose that $m_2 = \frac{-y_1 + F_{1,n+1} x_{1,n+1} + n_2 + n_3 - \tilde{F}_{3,n+1} x_{2,n+1}}{\left(x_{1,n+1} - x_{2,n+1}\right)}$. There are two cases with respect to value of $x_{1,4}$ as the following. In case of$x_{1,4} > x_{2,4}$, the condition is:

$$\tilde{F}_{2,n+1} < m_2$$

On the other hand, the condition evolves into the following expression in case of $x_{1,4} < x_{2,4}$.

$$\tilde{F}_{2,n+1} > m_2$$

Note that, the decision variable is $x_{1,4}$ at $t = 1$. Therefore the first derivative of $[y_3]^-$ with respect to $x_{1,n+1}$ is to be analyzed. Moreover, $F_{1,n+1}$is known, while $\tilde{F}_{2,n+1}$ and $\tilde{F}_{3,n+1}$ are random stochastic variables at $t = 1$; as a result recursive integral will be used to calculate the expected financial distress cost.

- Suppose that $x_{1,n+1} > x_{2,n+1}$:

First derivative of $r\,[y_3]^-$ when $x_{1,n+1} > x_{2,n+1}$:

$$\begin{aligned}
\frac{d\left(E_F^{\phi}\left([y_3]^-\right)\right)}{dx_{1,n+1}} &= \frac{d \int_{F_{3,n+1}=0}^{+\infty} \int_{F_{2,n+1}=0}^{m_2} \left(-y_3\right) \phi_{F_{2,n+1}}^{Q} \left(F_{2,n+1}\right) dF_{2,n+1} \phi_{F_{3,n+1}}^{Q} \left(F_{3,n+1}\right) dF_{3,n+1}}{dx_{1,n+1}} \\
&= \int_{F_{3,4}=0}^{+\infty} \int_{F_{2,4}=0}^{m_2} \left(F_{1,n+1} - \tilde{F}_{2,n+1}\right) \phi_{F_{2,n+1}}^{Q} \left(F_{2,n+1}\right) dF_{2,n+1} \phi_{F_{3,n+1}}^{Q} \left(F_{3,n+1}\right) dF_{3,n+1}
\end{aligned}$$

- Suppose that $x_{1,n+1} < x_{2,n+1}$:

First derivative of $r\,[y_3]^-$ when $x_{1,n+1} < x_{2,n+1}$:

$$\begin{aligned}
\frac{d\left(E_F^{\phi}\left([y_3]^-\right)\right)}{dx_{1,n+1}} &= \frac{d \int_{F_{3,n+1}=0}^{+\infty} \int_{F_{2,n+1}=m_2}^{+\infty} \left(-y_3\right) \phi_{F_{2,n+1}}^{Q} \left(F_{2,n+1}\right) dF_{2,n+1} \phi_{F_{3,n+1}}^{Q} \left(F_{3,n+1}\right) dF_{3,n+1}}{dx_{1,n+1}} \\
&= \int_{F_{3,n+1}=0}^{+\infty} \int_{F_{2,n+1}=m_2}^{+\infty} \left(F_{1,n+1} - \tilde{F}_{2,n+1}\right) \phi_{F_{2,n+1}}^{Q} \left(F_{2,n+1}\right) dF_{2,n+1} \phi_{F_{3,n+1}}^{Q} \left(F_{3,n+1}\right) dF_{3,n+1}
\end{aligned}$$

After all those observations, a generalized examination is necessary to produce general formulas. Suppose that $3 < i < n+1$. Then y_i will be defined as the following:

$$\begin{aligned} y_i &= y_3 + \sum_{j=4}^{i} \left(\left(\tilde{F}_{j,n+1} - \tilde{F}_{j-1,n+1} \right) x_{j-1,n+1} - n_j \right) \\ &= y_1 + \left(\tilde{F}_{2,n+1} - F_{1,n+1} \right) x_{1,n+1} + \left(\tilde{F}_{3,n+1} - \tilde{F}_{2,n+1} \right) x_{2,n+1} \\ &\quad -n_2 - n_3 + \sum_{j=4}^{i} \left(\left(\tilde{F}_{j,n+1} - \tilde{F}_{j-1,n+1} \right) x_{j-1,n+1} - n_j \right) \end{aligned}$$

For the sake of ease of notation Γ is defined such as $\Gamma = -y_1 + F_{1,n+1}x_{1,n+1} - \tilde{F}_{3,n+1}x_{2,n+1} + n_2 + n_3$. The breakpoint of negativity of y_i is also defined as the following:

$$m_{i-1} = \frac{\Gamma - \sum_{j=4}^{i} \left(\left(\tilde{F}_{j,n+1} - \tilde{F}_{j-1,n+1} \right) x_{j-1,n+1} - n_j \right)}{\left(x_{1,n+1} - x_{2,n+1} \right)}$$

There are two cases with respect to value of $x_{1,4}$ as the following. In case of$x_{1,n+1} > x_{2,n+1}$, the condition is:

$$\tilde{F}_{2,n+1} < m_{i-1}$$

On the other hand, the condition evolves into the following expression in case of $x_{1,n+1} < x_{2,n+1}$.

$$\tilde{F}_{2,n+1} > m_{i-1}$$

Note that, the decision variable is $x_{1,n+1}$ at $t = 1$. Therefore the first derivative of $[y_i]^-$ with respect to $x_{1,n+1}$ is to be analyzed. Moreover, $F_{1,n+1}$ is known, while $\tilde{F}_{j,n+1}$ is a random stochastic variable at $t = 1$ while $j = 2, 3, 4, ..., i-2, i$; as a result recursive integral will be used to calculate the expected financial distress cost.

- Suppose that $x_{1,n+1} > x_{2,n+1}$:

First derivative of $r\left[y_i\right]^-$ when $x_{1,n+1} > x_{2,n+1}$:

$$\begin{aligned} \frac{d\left(E_F^{\phi}\left([y_i]^-\right)\right)}{dx_{1,n+1}} &= \frac{d\int_{F_{i,n+1}=0}^{+\infty} \cdots \int_{F_{3,n+1}=0}^{+\infty} \int_{F_{2,n+1}=0}^{m_{i-1}} (-y_i)\, \phi_{F_{2,n+1}}^{Q}\left(F_{2,n+1}\right) dF_{2,n+1} \phi_{F_{3,n+1}}^{Q}\left(F_{3,n+1}\right) dF_{3,n+1} \ldots \phi_{F_{i,n+1}}^{Q}\left(F_{i,n+1}\right) dF_{i,n+1}}{dx_{1,n+1}} \\ &= \int\limits_{F_{i,n+1}=0}^{+\infty} \cdots \int\limits_{F_{3,n+1}=0}^{+\infty} \int\limits_{F_{2,n+1}=0}^{m_{i-1}} \left(F_{1,n+1} - \tilde{F}_{2,n+1}\right) \phi_{F_{2,n+1}}^{Q}\left(F_{2,n+1}\right) dF_{2,n+1} \phi_{F_{3,n+1}}^{Q}\left(F_{3,n+1}\right) dF_{3,n+1} \ldots \phi_{F_{i,n+1}}^{Q}\left(F_{i,n+1}\right) dF_{i,n+1} \end{aligned}$$

- Suppose that $x_{1,n+1} < x_{2,n+1}$:

First derivative of $r\left[y_i\right]^-$ when $x_{n+1} < x_{n+1}$:

$$\begin{aligned} \frac{d\left(E_F^{\phi}\left([y_i]^-\right)\right)}{dx_{1,n+1}} &= \frac{d\int_{F_{i,n+1}=0}^{+\infty} \cdots \int_{F_{3,n+1}=0}^{+\infty} \int_{F_{2,n+1}=m_{i-1}}^{+\infty} (-y_i)\, \phi_{F_{2,n+1}}^{Q}\left(F_{2,n+1}\right) dF_{2,n+1} \phi_{F_{3,n+1}}^{Q}\left(F_{3,n+1}\right) dF_{3,n+1} \ldots \phi_{F_{i,n+1}}^{Q}\left(F_{i,n+1}\right) dF_{i,n+1}}{dx_{1,n+1}} \\ &= \int\limits_{F_{i,n+1}=0}^{+\infty} \cdots \int\limits_{F_{3,n+1}=0}^{+\infty} \int\limits_{F_{2,n+1}=m_{i-1}}^{+\infty} \left(F_{1,n+1} - \tilde{F}_{2,n+1}\right) \phi_{F_{2,n+1}}^{Q}\left(F_{2,n+1}\right) dF_{2,n+1} \phi_{F_{3,n+1}}^{Q}\left(F_{3,n+1}\right) dF_{3,n+1} \ldots \phi_{F_{i,n+1}}^{Q}\left(F_{i,n+1}\right) dF_{i,n+1} \end{aligned}$$

Time $n+1$ is the time for delivery of the assets in the agreement of the firm. After all those observations, a generalized examination is necessary to produce a general formula for this instance. The value of the term y_{n+1} is defined as the following:

$$\begin{aligned} y_{n+1} &= y_n + \left(\tilde{S}_{n+1} - \tilde{F}_{n,n+1}\right) x_{3,n+1} + \left(p - \tilde{S}_{n+1} - \lambda\right)\xi - n_{n+1} \\ &= y_1 + \left(\tilde{F}_{2,n+1} - F_{1,n+1}\right) x_{1,4} + \left(F_{3,n+1} - F_{2,n+1}\right) x_{2,n+1} - n_2 - n_3 + \left(\tilde{S}_{n+1} - \tilde{F}_{n,n+1}\right) x_{n,n+1} \\ &\quad + \left(p - \tilde{S}_{n+1} - \lambda\right)\xi - n_{n+1} + \textstyle\sum_{j=4}^{n} \left(\left(\tilde{F}_{j,n+1} - \tilde{F}_{j-1,n+1}\right) x_{j-1,n+1} - n_j\right) \end{aligned}$$

Negativity condition of y_{n+1} should be found by assuming that $y_{n+1} < 0$. By using Lemma 2, it is already known that: $x^*_{3,4} = \xi$. Furthermore, Theorem 2 suggests that $(x_{2,4} - \xi) \leqq 0$ regardless of the value of the other variables.

There are two different conditions for the different cases with respect to the value of $x^*_{1,4}$. For the sake of ease of notation ω is defined such as
$\omega = -\sum_{j=4}^{n} \left(\left(\tilde{F}_{j,n+1} - \tilde{F}_{j-1,n+1}\right) x_{j-1,n+1} - n_j\right)$. Then, the variable breakpoint value m_n will be defined such that:

$$m_n = \frac{\Gamma + \omega - \sum_{j=4}^{n} \left(\left(\tilde{F}_{j,n+1} - \tilde{F}_{j-1,n+1}\right) x_{j-1,n+1} - n_j\right)}{\left(x_{1,n+1} - x_{2,n+1}\right)}$$

Finally, the negativity condition for $x_{1,n+1} > x_{2,n+1}$ is shown below.

$$\tilde{F}_{2,n+1} < m_n$$

The negativity condition for $x_{1,n+1} < x_{2,n+1}$ is shown below.

$$\tilde{F}_{2,n+1} > m_n$$

- Suppose that $x_{1,n+1} > x_{2,n+1}$:

The first derivative of $[y_4]^-$ with respect to $x_{1,4}$ is shown below.

$$\begin{aligned} \frac{d\left(E_F^{\phi}\left([y_{n+1}]^-\right)\right)}{dx_{1,n+1}} &= \frac{d \int_{F_{n,n+1}=0}^{+\infty} \cdots \int_{F_{3,n+1}=0}^{+\infty} \int_{F_{2,n+1}=0}^{m_n} (-y_{n+1})\, \phi^Q_{F_{2,n+1}}\left(F_{2,n+1}\right) dF_{2,n+1} \phi^Q_{F_{3,n+1}}\left(F_{3,n+1}\right) dF_{3,n+1} \ldots \phi^Q_{F_{i,n+1}}\left(F_{n,n+1}\right) dF_{i,n+1}}{dx_{1,n+1}} \\ &= \int\limits_{F_{n,n+1}=0}^{+\infty} \cdots \int\limits_{F_{3,n+1}=0}^{+\infty} \int\limits_{F_{2,n+1}=0}^{m_n} \left(F_{1,n+1} - \tilde{F}_{2,n+1}\right) \phi^Q_{F_{2,n+1}}\left(F_{2,n+1}\right) dF_{2,n+1} \phi^Q_{F_{3,n+1}}\left(F_{3,n+1}\right) dF_{3,n+1} \ldots \phi^Q_{F_{i,n+1}}\left(F_{n,n+1}\right) dF_{i,n+1} \end{aligned}$$

- Suppose that $x_{1,n+1} < x_{2,n+1}$:

The first derivative of $[y_4]^-$ with respect to $x_{1,4}$ is shown below.

$$\begin{aligned} \frac{d\left(E_F^{\phi}\left([y_{n+1}]^-\right)\right)}{dx_{1,n+1}} &= \frac{d \int_{F_{n,n+1}=0}^{+\infty} \cdots \int_{F_{3,n+1}=0}^{+\infty} \int_{F_{2,n+1}=m_n}^{+\infty} (-y_{n+1})\, \phi^Q_{F_{2,n+1}}\left(F_{2,n+1}\right) dF_{2,n+1} \phi^Q_{F_{3,n+1}}\left(F_{3,n+1}\right) dF_{3,n+1} \ldots \phi^Q_{F_{i,n+1}}\left(F_{n,n+1}\right) dF_{i,n+1}}{dx_{1,n+1}} \\ &= \int\limits_{F_{n,n+1}=0}^{+\infty} \cdots \int\limits_{F_{3,n+1}=0}^{+\infty} \int\limits_{F_{2,n+1}=m_n}^{+\infty} \left(F_{1,n+1} - \tilde{F}_{2,n+1}\right) \phi^Q_{F_{2,n+1}}\left(F_{2,n+1}\right) dF_{2,n+1} \phi^Q_{F_{3,n+1}}\left(F_{3,n+1}\right) dF_{3,n+1} \ldots \phi^Q_{F_{i,n+1}}\left(F_{n,n+1}\right) dF_{i,n+1} \end{aligned}$$

The convexity of the function may be proved by using the same methodology used in Theorem 2. Similar to the previous models, it is observed that the expressions $[y_2]^-$,$[y_3]^-$,..., $[y_n]^-$,and $[y_n]^-$are piecewise convex functions and other variables in $-\triangle V_2(y_1)$ are all linear. It is trivial that if a function is convex, its expected value function is also convex. Consequently, it is obvious that the term $-\triangle V_2(y_1)$ is convex in this model.

After a similar examination to the Theorem 9, it is observed that there are 4 different cases with respect to the decision variable.

- Case I: $x_{1,n+1} > x_{2,n+1}$ and $x_{1,n+1} > 0$: In this case, the value of $\frac{d(-\triangle V_2(y_1))}{dx_{1,n+1}}$ is observed to be positive. It means that the expected total financial distress cost will increase while $x_{1,n+1}$ is increasing. Indeed, an extreme point cannot exist here, as a result it is not the optimum interval.
- Case II: $x_{1,n+1} < x_{2,n+1}$ and $x_{1,n+1} > 0$: The expression $\frac{d}{dx_{1,n+1}}(-\triangle V_2(y_1))$ can either be positive or negative or zero. Therefore, the optimum decision may exist in this interval.
- Case III: $x_{1,n+1} > x_{2,n+1}$ and $x_{1,n+1} < 0$: This case suggests that $x_{2,n+1} < 0$. However, this assumption cannot be a valid one because it is already known that $x_{2,n+1} \geq 0$ by using Theorem 9. As a result, there is no need to examine this situation.
- Case IV: $x_{1,n+1} < x_{2,n+1}$ and $x_{1,n+1} < 0$: In this case, the value of $\frac{d(-\triangle V_2(y_1))}{dx_{1,n+1}}$ can only be negative and it means that the total expected financial distress cost always increases while $x_{1,n+1}$ is increasing in the interval $(-\infty, 0]$. Consequently, there cannot be an optimum decision in this interval and optimum $x^*_{1,n+1}$ should be equal to or higher than 0 because of the convexity.

After examining all the possible cases with respect to the expected values of the decision variables, it is observed that the optimum decision policy is underhedging, such that $0 \leqq x^*_{1,n} \leqq \bar{x}^*_{2,n} \leqq ... \leqq \bar{x}^*_{n-2,n} \leqq x^*_{n-1,n} = \xi$.

B.11 Theorem 11

It is already stated that Case 2 leads to the optimum solution. When $\triangle$ approaches to $\pm\infty$, upper and lower limits of the integrals goes to minus or positive infinity too. Therefore, the first derivative of $-\triangle V_2(y_1)$ with respect to $x_{1,4}$ is given by the equation below and it is equal to 0 because of the characteristics of normal distribution.

$$\begin{aligned}\frac{d}{dx_{1,4}}(-\triangle V_2(y_1)) \quad = \quad & \lim_{\triangle\to\pm\infty} \int_{F_{2,4}=0}^{e} \left(F_{1,4} - \tilde{F}_{2,4}\right) \phi^Q_{F_{2,4}}(F_{2,4})\, dF_{2,4} \\ & + \lim_{\triangle\to\pm\infty} \int_{F_{3,4}=0}^{+\infty} \int_{F_{2,4}=\frac{g-F_{3,4}x_{2,4}}{f}}^{+\infty} \left(F_{1,4} - \tilde{F}_{2,4}\right) \phi^Q_{F_{2,4}}(F_{2,4})\, dF_{2,4} \phi^Q_{F_{3,4}}(F_{3,4})\, dF_{3,4} \\ & + \lim_{\triangle\to\pm\infty} \int_{F_{3,4}=0}^{+\infty} \int_{F_{2,4}=\frac{k-F_{3,4}l}{f}}^{+\infty} \left(F_{1,4} - \tilde{F}_{2,4}\right) \phi^Q_{F_{2,4}}(F_{2,4})\, dF_{2,4} \phi^Q_{F_{3,4}}(F_{3,4})\, dF_{3,4}\end{aligned}$$

$$= 0$$

B.12 Theorem 12

In order to prove the theorem, firstly the behavior of the term of $-\triangle V_2(y_1)$ should be examined, similar to the proof of Theorem 2. The term $-\triangle V_2(y_1)$ at $t = 1$ can be described mathematically as the following after replacing the term $x_{2,3}$ by ξ due to Lemma 2 (the optimum decision quantity for one-period model):

$$\begin{aligned} -\triangle V_2(y_1) &= \int_0^\infty \Bigg[n_2 - \left(\tilde{F}_{2,3} - F_{1,3}\right) x_{1,3} + r\,[y_2]^- \\ &\quad y + \beta \int_0^\infty \left[n_3 - \left(p - \tilde{S}_3 - \lambda\right)\xi - \left(\tilde{S}_3 - \tilde{F}_{2,3}\right)\xi + r\,[y_3]^- \right] \phi^Q_{S_3}(S_3)\, dS_3 \Bigg] \phi^Q_{F_{2,3}}(F_{2,3})\, dF_{2,3} \end{aligned}$$

where

$$y_2 = y_1 - n_2 + \left(\tilde{F}_{2,3} - F_{1,3}\right) x_{1,3}$$

$$y_3 = y_2 - n_3 + \left(p - \tilde{S}_3 - \lambda\right)\xi + \left(\tilde{S}_3 - \tilde{F}_{2,3}\right)\xi - r\,[y_2]^-$$

In order to find the optimum hedging quantity, it is necessary to analyze the first and the second derivatives of the term $-\triangle V_2(y_1)$ with respect to the decision variable at $t = 1$, i.e. $x_{1,3}$. Since $F_{1,3} = \int_0^\infty \left[\left(\tilde{F}_{2,3}\right)\right] \phi^Q_{F_{2,3}} dF_{2,3}$ and the rest of the expression does not contain $x_{1,3}$ except financial distress cost premiums, the first derivative of the term $-\triangle V_2(y_1)$ is:

$$\frac{d\left(-\triangle V_2(y_1)\right)}{dx_{1,3}} = \frac{d\int_0^\infty \left[r\,[y_2]^- + \int_0^\infty \beta r\,[y_3]^- \phi^Q_{S_3}(S_3)\, dS_3 \right] \phi^Q_{F_{2,3}}(F_{2,3})\, dF_{2,3}}{dx_{1,3}}$$

In order to examine the expression above, the breakpoints where y_2 and y_3 become zero should be found. Same notations used in the simple interest case are used also for breakpoints in this case. The condition that makes the expression $y_2 = y_1 - n_2 + \left(\tilde{F}_{2,3} - F_{1,3}\right) x_{1,3}$ less than zero is the following where $b = \frac{-y_1 + F_{1,3}x_{1,3} + n_2}{x_{1,3}}$.

$$y_2 < 0 \Leftrightarrow \begin{cases} \tilde{F}_{2,3} < b & if\; x_{1,3} > 0 \\ \tilde{F}_{2,3} > b & if\; x_{1,3} < 0 \end{cases}$$

Since the expected financial distress cost at the end of the second period is also aimed to be found, the conditions that satisfy $y_3 < 0$ should be examined with respect to the negativity condition of y_2. In the case of $y_2 > 0$, y_3 will be equal to $y_1 + \left(\tilde{F}_{2,3} - F_{1,3}\right) x_{1,3} + \left(p - \tilde{F}_{2,3} - \lambda\right)\xi - n_3 - n_2$. The condition of $y_3 < 0$ where $y_2 > 0$ and $c = \frac{-y_1 + F_{1,3}x_{1,3} - (p-\lambda)\xi + n_2 + n_3}{x_{1,3} - \xi}$ is:.

$$y_3 < 0 \Leftrightarrow \begin{cases} \tilde{F}_{2,3} < c & if\; x_{1,3} > \xi\; and\; y_2 > 0 \\ \tilde{F}_{2,3} > c & if\; x_{1,3} < \xi\; and\; y_2 > 0 \end{cases}$$

Similarly, y_3 becomes equal to $\left(y_1 + \left(\tilde{F}_{2,3} - F_{1,3}\right) x_{1,3} - n_2\right)(1+r) + \left(\tilde{S}_3 - \tilde{F}_{2,3}\right)\xi$
$+ \left(p - \tilde{S}_3 - \lambda\right)\xi - n_3$ when $y_2 < 0$. Thus the condition of $y_3 < 0$ where $y_2 < 0$ and $d = \frac{(-y_1+F_{1,3}x_{1,3}+n_2)(1+r)-(p-\lambda)\xi+n_3}{x_{1,3}(1+r)-\xi}$ is:

$$y_3 < 0 \Leftrightarrow \begin{cases} \tilde{F}_{2,3} < d & if\, x_{1,3} > \xi/(1+r)\ and\, y_2 < 0 \\ \tilde{F}_{2,3} > d & if\, x_{1,3} < \xi/(1+r)\ and\, y_2 < 0 \end{cases}$$

We will prove that the firm should underhedge by contradiction, by supposing $x_{1,3} > \xi$. The value of d will be composed with respect to the values of b and c. Besides, the only intervals where variables b and c are larger than zero will be examined because hedging will become a value neutral action in other case (the fact denoted by Theorem 7 and 8 is also valid for this model).

Decision Interval I: $x_{1,3} > \xi$

Case I: $b > c$

In this case, we have $c < d < b$. As a result, the expression will be as the following.

$$\begin{aligned} \frac{d\left(-\triangle V_2(y_1)\right)}{dx_{1,3}} \quad = \quad & \frac{dr\int_0^b \left(-y_1 + \left(F_{1,3} - \tilde{F}_{2,3}\right)x_{1,3} + n_2\right)\phi^Q_{F_{2,3}}(F_{2,3})\,dF_{2,3}}{dx_{1,3}} \\ & + \frac{d\beta r\int_0^d \left[\left(-y_1 + \left(F_{1,3} - \tilde{F}_{2,3}\right)x_{1,3} + n_2\right)(1+r) - \left(p - \tilde{F}_{2,3} - \lambda\right)\xi + n_3\right]\phi^Q_{F_{2,3}}(F_{2,3})\,dF_{2,3}}{dx_{1,3}} \end{aligned}$$

By applying Leibniz Rule, the following expression is obtained.

$$\frac{d}{dx_{1,3}}\left(-\triangle V_2(y_1)\right) = r\int_0^b \left(F_{1,3} - \tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}(F_{2,3})\,dF_{2,3} + \beta r\int_0^d \left(F_{1,3} - \tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}(F_{2,3})\,dF_{2,3}$$

The expression above is definitely positive. As a result, decreasing hedging amount will also decrease expected financial distress cost.

Case II: $b < c$

In this case, we have $b < d < c$. As a result, the expression will be as the following.

$$\begin{aligned} \frac{d\left(-\triangle V_2(y_1)\right)}{dx_{1,3}} \quad = \quad & \frac{dr\int_0^b \left(-y_1 + \left(F_{1,3} - \tilde{F}_{2,3}\right)x_{1,3} + n_2\right)\phi^Q_{F_{2,3}}(F_{2,3})\,dF_{2,3}}{dx_{1,3}} \\ & + \frac{d\beta r\int_0^b \left[\left(-y_1 + \left(F_{1,3} - \tilde{F}_{2,3}\right)x_{1,3} + n_2\right)(1+r) - \left(p - \tilde{F}_{2,3} - \lambda\right)\xi + n_3\right]\phi^Q_{F_{2,3}}(F_{2,3})\,dF_{2,3}}{dx_{1,3}} \\ & + \frac{d\beta r\int_b^c \left[-y_1 + \left(F_{1,3} - \tilde{F}_{2,3}\right)x_{1,3} + n_2 - \left(p - \tilde{F}_{2,3} - \lambda\right)\xi + n_3\right]\phi^Q_{F_{2,3}}(F_{2,3})\,dF_{2,3}}{dx_{1,3}} \end{aligned}$$

This expression can be transformed into:

$$
\begin{aligned}
\frac{d\left(-\triangle V_2\left(y_1\right)\right)}{dx_{1,3}} &= \frac{dr\int_0^b \left(-y_1+\left(F_{1,3}-\tilde{F}_{2,3}\right)x_{1,3}+n_2\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}}{dx_{1,3}} \\
&+\frac{d\beta r\int_0^c \left[-y_1+\left(F_{1,3}-\tilde{F}_{2,3}\right)x_{1,3}+n_2-\left(p-\tilde{F}_{2,3}-\lambda\right)\xi+n_3\right]\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}}{dx_{1,3}} \\
&+\frac{d\beta r^2\int_0^b \left(-y_1+\left(F_{1,3}-\tilde{F}_{2,3}\right)x_{1,3}+n_2\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}}{dx_{1,3}}
\end{aligned}
$$

After applying the Leibniz Rule, we obtain:

$$
\begin{aligned}
\frac{d\left(-\triangle V_2\left(y_1\right)\right)}{dx_{1,3}} &= r\int_0^b \left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3} \\
&+\beta r\int_0^c \left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3} \\
&+\beta r^2\int_0^b \left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}
\end{aligned}
$$

The expression above is also positive. Consequently, all of the first derivatives are positive in both cases if b, c , d are larger than zero. Then, hedging amount should be decreased as long as $x_{1,3} > \xi$.

Decision Interval II: $x_{1,3} < 0$

This case means that the company gets short position. Again, there are 2 different cases with respect to values of b and c.

Case I: $b > c$

In this case, we have $c < d < b$. As a result, the expression will be as the following.

$$
\begin{aligned}
\frac{d\left(-\triangle V_2\left(y_1\right)\right)}{dx_{1,3}} &= \frac{dr\int_b^\infty \left(-y_1+\left(F_{1,3}-\tilde{F}_{2,3}\right)x_{1,3}+n_2\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}}{dx_{1,3}} \\
&+\frac{d\beta r\int_b^\infty \left[\left(-y_1+\left(F_{1,3}-\tilde{F}_{2,3}\right)x_{1,3}+n_2\right)(1+r)-\left(p-\tilde{F}_{2,3}-\lambda\right)\xi+n_3\right]\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}}{dx_{1,3}} \\
&+\frac{d\beta r\int_c^b \left[\left(-y_1+\left(F_{1,3}-\tilde{F}_{2,3}\right)x_{1,3}+n_2\right)-\left(p-\tilde{F}_{2,3}-\lambda\right)\xi+n_3\right]\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}}{dx_{1,3}}
\end{aligned}
$$

This expression can be transformed into:

$$
\begin{aligned}
\frac{d\left(-\triangle V_2\left(y_1\right)\right)}{dx_{1,3}} &= \frac{dr\int_b^\infty \left(-y_1+\left(F_{1,3}-\tilde{F}_{2,3}\right)x_{1,3}+n_2\right)\phi_{F_{2,3}}^{Q}\left(F_{2,3}\right)dF_{2,3}}{dx_{1,3}} \\
&+\frac{d\beta r^2\int_b^\infty \left(-y_1+\left(F_{1,3}-\tilde{F}_{2,3}\right)x_{1,3}+n_2\right)\phi_{F_{2,3}}^{Q}\left(F_{2,3}\right)dF_{2,3}}{dx_{1,3}} \\
&+\frac{d\beta r\int_c^\infty \left[\left(-y_1+\left(F_{1,3}-\tilde{F}_{2,3}\right)x_{1,3}+n_2\right)-\left(p-\tilde{F}_{2,3}-\lambda\right)\xi+n_3\right]\phi_{F_{2,3}}^{Q}\left(F_{2,3}\right)dF_{2,3}}{dx_{1,3}}
\end{aligned}
$$

By applying Leibniz Rule, we obtain:

$$
\begin{aligned}
\frac{d\left(-\triangle V_2\left(y_1\right)\right)}{dx_{1,3}} &= r\int_b^\infty \left(F_{1,3}-\tilde{F}_{2,3}\right)\phi_{F_{2,3}}^{Q}\left(F_{2,3}\right)dF_{2,3} \\
&+\beta r\int_c^\infty \left(F_{1,3}-\tilde{F}_{2,3}\right)\phi_{F_{2,3}}^{Q}\left(F_{2,3}\right)dF_{2,3} \\
&+\beta r^2\int_b^\infty \left(F_{1,3}-\tilde{F}_{2,3}\right)\phi_{F_{2,3}}^{Q}\left(F_{2,3}\right)dF_{2,3}
\end{aligned}
$$

The expression above is definitely negative, therefore the decision variable should be increased in this case.

Case II: $b < c$

In this case, we have got $b < d < c$. As a result, the expression will be as the following.

$$
\begin{aligned}
\frac{d\left(-\triangle V_2\left(y_1\right)\right)}{dx_{1,3}} &= \frac{dr\int_b^\infty \left(-y_1+\left(F_{1,3}-\tilde{F}_{2,3}\right)x_{1,3}+n_2\right)\phi_{F_{2,3}}^{Q}\left(F_{2,3}\right)dF_{2,3}}{dx_{1,3}} \\
&+\frac{d\beta r\int_d^\infty \left[\left(-y_1+\left(F_{1,3}-\tilde{F}_{2,3}\right)x_{1,3}+n_2\right)(1+r)-\left(p-\tilde{F}_{2,3}-\lambda\right)\xi+n_3\right]\phi_{F_{2,3}}^{Q}\left(F_{2,3}\right)dF_{2,3}}{dx_{1,3}}
\end{aligned}
$$

After applying the Leibniz Rule, we get:

$$
\frac{d\left(-\triangle V_2\left(y_1\right)\right)}{dx_{1,3}} = r\int_b^\infty \left(F_{1,3}-\tilde{F}_{2,3}\right)\phi_{F_{2,3}}^{Q}\left(F_{2,3}\right)dF_{2,3}+\beta r\int_d^\infty \left(F_{1,3}-\tilde{F}_{2,3}\right)\phi_{F_{2,3}}^{Q}\left(F_{2,3}\right)dF_{2,3}
$$

The expression above is also negative. Consequently, all of the first derivatives are negative in both cases if b, c , d are larger than zero and $x_{1,3} < 0$. Thus we should increase $x_{1,3}$ if $x_{1,3} < 0$

Now let us examine the convexity of $-\triangle V_2\left(y_1\right)$. Similar to the model with simple interest rates, all the terms in $-\triangle V_2\left(y_1\right)$ except financial distress cost are linear. It is already known that the terms related to financial distress costs are piecewise convex, it means that all terms in the function are convex. Since if a function is convex, its expected function is also convex; t $-\triangle V_2(y_1)$ is concluded to be convex in $x_{1,3}$.

In conclusion, we increase $x_{1,3}$ in the region $x_{1,3} < 0$ and we decrease $x_{1,3}$ when $x_{1,3} > \xi$; and the function is convex in $x_{1,3}$. Therefore, the optimal solution must be in the region $0 \leqq x_{1,3} \leqq \xi$.

B.13 Theorem 13

In the case that $\triangle = y_1 - n_2 = 0$, the variables are realized as $b = F_{1,3}$, $c = \frac{(p-\lambda)\xi - F_{1,3}x_{1,3} - n_3}{\xi - x_{1,3}}$ and $d = \frac{(p-\lambda)\xi - F_{1,3}x_{1,3}(1+r) - n_3}{\xi - x_{1,3}(1+r)}$. Now suppose that $x_{1,3} = 0$, then the variables are: $b = F_{1,3}$ and $c = d = \frac{(p-\lambda)\xi - n_3}{\xi}$. Consequently, the first derivative of the term $-\triangle V_2(y_1)$ is:

$$\frac{d}{dx_{1,3}}\left(-\triangle V_2(y_1)\right) = \beta r\left(\int_0^{F_{1,3}}\left(F_{1,3} - \tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3} + \int_{\frac{(p-\lambda)\xi - n_3}{\xi}}^{\infty}\left(F_{1,3} - \tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}\right)$$

Similar to Theorem 4, it is known that this term is positive. Thus, the financial distress costs increase while $x_{1,3}$ increases when $x_{1,3} = 0$ which proves that the company should not hedge at all, i.e. $x^*_{1,3} = 0$ when $\triangle = 0$.

B.14 Theorem 14

For this theorem, we will prove that there cannot be an optimal solution such that $\xi/(1+r) < x_{1,3} < \xi$ when $\triangle < 0$.

Decision Interval III: $\xi/(1+r) < x_{1,3} < \xi$

Case I: $b > c$

When $b > c$, both numerator and denominator of c are negative while it is vice versa for b. Therefore, it is also obvious that the denominator and numerator of d are positive, and $d > b$. As a result, the first derivative will be as the following.

$$\begin{aligned}
\frac{d}{dx_{1,3}}\left(-\triangle V_2(y_1)\right) \quad = \quad & \frac{dr\int_0^b\left(-y_1 + \left(F_{1,3} - \tilde{F}_{2,3}\right)x_{1,3} + n_2\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}}{dx_{1,3}} \\
& + \frac{d\beta r\int_0^b\left[\left(-y_1 + \left(F_{1,3} - \tilde{F}_{2,3}\right)x_{1,3} + n_2\right)(1+r) - \left(p - \tilde{F}_{2,3} - \lambda\right)\xi + n_3\right]\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}}{dx_{1,3}} \\
& + \frac{d\beta r\int_b^\infty\left[-y_1 + \left(F_{1,3} - \tilde{F}_{2,3}\right)x_{1,3} + n_2 - \left(p - \tilde{F}_{2,3} - \lambda\right)\xi + n_3\right]\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}}{dx_{1,3}}
\end{aligned}$$

Then the expression can be transformed into the following by Leibniz Rule:

$$\begin{aligned}\frac{d}{dx_{1,3}}\left(-\triangle V_2\left(y_1\right)\right) &= r\left(1+\beta r\right)\int\limits_0^b \left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}+\beta r\int\limits_0^\infty \left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}\\ &= r\left(1+\beta r\right)\int\limits_0^b \left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}\end{aligned}$$

Since the first derivative is always positive in the region, there cannot be an optimum solution here.

Case II: $b < c$

When $b < c$, both numerator and denominator of c are negative while it is vice versa for b. Therefore, the denominator of d is positive while the numerator may either be positive or negative. Thus, $d < b$.

As a result, the first derivative of $-\triangle V_2\left(y_1\right)$ is:

$$\begin{aligned}\frac{d}{dx_{1,3}}\left(-\triangle V_2\left(y_1\right)\right) &= \frac{dr\int_0^b\left(-y_1+\left(F_{1,3}-\tilde{F}_{2,3}\right)x_{1,3}+n_2\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}}{dx_{1,3}}\\ &+\frac{d\beta r\int_0^{max(0,d)}\left[\left(-y_1+\left(F_{1,3}-\tilde{F}_{2,3}\right)x_{1,3}+n_2\right)\left(1+r\right)-\left(p-\tilde{F}_{2,3}-\lambda\right)\xi+n_3\right]\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}}{dx_{1,3}}\\ &+\frac{d\beta r\int_c^\infty\left[-y_1+\left(F_{1,3}-\tilde{F}_{2,3}\right)x_{1,3}+n_2-\left(p-\tilde{F}_{2,3}-\lambda\right)\xi+n_3\right]\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}}{dx_{1,3}}\end{aligned}$$

The expression above can be simplified into the following by using Leibniz Rule:

$$\begin{aligned}\frac{d}{dx_{1,3}}\left(-\triangle V_2\left(y_1\right)\right) &= r\int\limits_0^b \left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}+\beta r\int\limits_c^\infty \left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}\\ &+\beta r\int\limits_0^{max(0,d)} \left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}\end{aligned}$$

b is equal to $F_{1,3} - \triangle/x_{1,3}$ which is larger than $F_{1,3}$, since $\triangle < 0$. Thus, $F_{1,3} < b < c$. Note that, the biggest value of the expression $\int_0^i\left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}$ realizes at $i = F_{1,3}$. While drifting away from $F_{1,3}$, the value of the expression decreases, and finally it reaches to zero in both ends. Due to $F_{1,3} < b < c$ and $\beta < 1$, the following the expression $r\int_0^b\left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}$ $+\beta r\int_c^\infty\left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}$ becomes larger than zero. In addition,
$\beta r\int_0^{max(0,d)}\left(F_{1,3}-\tilde{F}_{2,3}\right)\phi^Q_{F_{2,3}}\left(F_{2,3}\right)dF_{2,3}$ is also a nonnegative variable. Consequently, the whole expression is concluded to be positive when $\triangle < 0$ and there cannot be an optimal solution in the region of $\xi/\left(1+r\right) < x_{1,3} < \xi$. On the other hand, the expression may be equal to zero when $\triangle > 0$.

In conclusion, the first derivative never reaches to zero and there cannot exist an optimum decision such as $\xi/\left(1+r\right) < x^*_{1,3} < \xi$ when $\triangle < 0$. Yet such an optimal decision is possible when $\triangle > 0$.

B.15 Theorem 15

The objective function of the problem can be written more implicitly as:

$$\begin{aligned}
V_2(y_1) &= \min_{x_{1,2}} \int_0^{\infty} \left[-\triangle V_2(y_1)\right] \phi^{Q}_{S_2}(S_2)\, dS_2 \\
&= \min_{x_{1,2}} \Bigg[\int_{F_{1,2}}^{\infty} \left(n_2 - \left(p - \tilde{S}_2 - \lambda\right)\xi + h x_{1,2} - \left[\tilde{S}_2 - F_{1,2}\right]^{+} x_{1,2} + r\left[y_1^{'}\right]^{-} + \beta r \left[y_2\right]^{-} \right) \phi^{Q}_{S_2}(S_2)\, dS_2 \\
&\quad + \int_0^{F_{1,2}} \left(n_2 - \left(p - \tilde{S}_2 - \lambda\right)\xi + h x_{1,2} + r\left[y_1^{'}\right]^{-} + \beta r \left[y_2\right]^{-} \right) \phi^{Q}_{S_2}(S_2)\, dS_2 \Bigg]
\end{aligned}$$

Firstly, the conditions to face financial distress costs should be examined. We will prove the theorem by contradiction. The condition for incurring a financial distress cost due to the negativity of $y_1^{'}$ is:

$$\frac{y_1}{h} < x_{1,2}$$

Additionally, financial distress cost incurs due to y_2 when:

$$y_1 - n_2 + \left(p - \tilde{S}_2 - \lambda\right)\xi - h x_{1,2} + \left[\tilde{S}_2 - F_{1,2}\right]^{+} x_{1,2} < 0$$

However, two different cases occur with respect to the value of $\tilde{S}_2$. When the realized derivative price is higher than the strike price of the underlying derivative, i.e. $\tilde{S}_2 > F_{1,2}$, and $x_{1,2} > \xi$, the negativity condition of y_2 is the following:

$$\tilde{S}_2 < \frac{-y_1 + n_2 - (p - \lambda)\xi + h x_{1,2} + F_{1,2} x_{1,2}}{(x_{1,2} - \xi)}$$

On the other hand, the condition is the following for $\tilde{S}_2 > F_{1,2}$ and decision $x_{1,2} < \xi$:

$$\tilde{S}_2 > \frac{-y_1 + n_2 - (p - \lambda)\xi + h x_{1,2} + F_{1,2} x_{1,2}}{(x_{1,2} - \xi)}$$

The term $\frac{-y_1 + n_2 - (p-\lambda)\xi + h x_{1,2} + F_{1,2} x_{1,2}}{(x_{1,2} - \xi)}$ may be written as $F_{1,2} + \frac{-y_1 + n_2 - \left(p - \lambda - F_{1,2}\right)\xi + h x_{1,2}}{\left(x_{1,2} - \xi\right)}$ to benefit from the characteristics of normal distribution. Additionally, $\frac{-y_1 + n_2 - (p-\lambda)\xi + h x_{1,2} + F_{1,2} x_{1,2}}{(x_{1,2} - \xi)}$ will be denoted as a, and $\frac{-y_1 + n_2 - \left(p - \lambda - F_{1,2}\right)\xi + h x_{1,2}}{\left(x_{1,2} - \xi\right)}$ will be denoted as θ_1 for ease in illustration.

For $\tilde{S}_2 < F_{1,2}$, the financial distress cost occurs at the following condition.

$$\tilde{S}_2 > \frac{-y_1 + n_2 - (p - \lambda)\xi + h x_{1,2}}{-\xi}$$

Since $\frac{-y_1+n_2-(p-\lambda)\xi+hx_{1,2}}{-\xi}$ may be written as $F_{1,2}+\frac{-y_1+n_2-\left(p-\lambda-F_{1,2}\right)\xi+hx_{1,2}}{-\xi}$, $\frac{-y_1+n_2-(p-\lambda)\xi+hx_{1,2}}{-\xi}$ may be called as b and $\frac{-y_1+n_2-\left(p-\lambda-F_{1,2}\right)\xi+hx_{1,2}}{-\xi}$ may be called as θ_2.

In order to prove the theorem, $x_{1,2}$ is divided into two pieces. Suppose that $x_{1,2}=x^u_{1,2}+x^o_{1,2}$ such that $\frac{y_1}{h}\geqq x^u_{1,2}$; and $x^o_{1,2}\geqq 0$ when $\frac{y_1}{h}=x^u_{1,2}$. Here, $x^u_{1,2}$ denotes the magnitude of call options under the budget. On the other hand $x^o_{1,2}$ represents call options bought over the budget. As stated, the theorem will be proved by contradiction by supposing $x^o_{1,2}>0$. Thus, the first derivative of the objective function should be examined in two different intervals with respect to $x^o_{1,2}$. If the result is positive, the firm should not hedge more than $\frac{y_1}{h}$ units.

In case that $(x_{1,2}>\xi)$ and $\frac{y_1}{h}<x_{1,2}$, the value of $-\triangle V_2(y_1)$ is the following:

$$
\begin{aligned}
-\triangle V_2(y_1) \quad = \quad & \int\limits_{max\left(a,F_{1,2}\right)}^{\infty}\left(n_2-\left(p-\tilde{S}_2-\lambda\right)\xi+hx_{1,2}-\left(\tilde{S}_2-F_{1,2}\right)x_{1,2}-ry_1'\right)\phi^Q_{S_2}(S_2)\,dS_2\\
&+\int\limits_{F_{1,2}}^{max\left(a,F_{1,2}\right)}\left(n_2-\left(p-\tilde{S}_2-\lambda\right)\xi+hx_{1,2}-\left(\tilde{S}_2-F_{1,2}\right)x_{1,2}-ry_1'-\beta ry_2\right)\phi^Q_{S_2}(S_2)\,dS_2\\
&+\int\limits_{min\left(b,F_{1,2}\right)}^{F_{1,2}}\left(n_2-\left(p-\tilde{S}_2-\lambda\right)\xi+hx_{1,2}-ry_1'-\beta ry_2\right)\phi^Q_{S_2}dS_2\\
&+\int\limits_{0}^{min\left(b,F_{1,2}\right)}\left(n_2-\left(p-\tilde{S}_2-\lambda\right)\xi+hx_{1,2}-ry_1'\right)\phi^Q_{S_2}dS_2\\
= \quad & \left(n_2-\left(p-\tilde{S}_2-\lambda\right)\xi\right)-\beta ry_1'-\int\limits_{F_{1,2}}^{max\left(a,F_{1,2}\right)}\beta ry_2\phi^Q_{S_2}dS_2-\int\limits_{min\left(b,F_{1,2}\right)}^{F_{1,2}}\beta ry_2\phi^Q_{S_2}(S_2)\,dS_2
\end{aligned}
$$

Now, the first derivative of the expression above with respect to $x^o_{1,2}$ should be checked in order to understand whether hedging over the budget is profitable or not.

$$
\begin{aligned}
\frac{d\left(-\triangle V_2(y_1)\right)}{dx^o_{1,2}} \quad = \quad & \frac{d\left[\left(n_2-\left(p-\tilde{S}_2-\lambda\right)\xi\right)-\beta ry_1'\right]}{dx^o_{1,2}}\\
&-\left(\frac{\int_{F_{1,2}}^{max\left(a,F_{1,2}\right)}\beta ry_2\phi^Q_{S_2}dS_2+\int_{min\left(b,F_{1,2}\right)}^{F_{1,2}}\beta ry_2\phi^Q_{S_2}(S_2)\,dS_2}{dx^o_{1,2}}\right)\\
= \quad & \frac{-\beta r\,d\left[\int_0^{\infty}\left(-hx^o_{1,2}\right)\phi^Q_{S_2}(S_2)\,dS_2\right]}{dx^o_{1,2}}\\
&+\frac{-\beta r\,d\left[\int_{F_{1,2}}^{max\left(a,F_{1,2}\right)}\left(y_1-n_2+\left(p-\tilde{S}_2-\lambda\right)\xi-hx_{1,2}+\left(\tilde{S}_2-F_{1,2}\right)x_{1,2}\right)\phi^Q_{S_2}(S_2)\,dS_2\right]}{dx^o_{1,2}}\\
&+\frac{-\beta r\,d\left[\int_{min\left(b,F_{1,2}\right)}^{F_{1,2}}\left(y_1-n_2+\left(p-\tilde{S}_2-\lambda\right)\xi-hx_{1,2}\right)\phi^Q_{S_2}(S_2)\,dS_2\right]}{dx^o_{1,2}}\\
= \quad & \beta\left(rh+\int\limits_{F_{1,2}}^{max\left(a,F_{1,2}\right)}\left(h-\left(\tilde{S}_2-F_{1,2}\right)\right)\phi^Q_{S_2}dS_2+\int\limits_{min\left(b,F_{1,2}\right)}^{F_{1,2}}h\phi^Q_{S_2}dS_2\right)
\end{aligned}
$$

$$= \beta r\Bigg[\int_{F_{1,2}}^{\infty}\left(\tilde{S}_2 - F_{1,2}\right)\phi_{S_2}^{Q}(S_2)\,dS_2\left(\int_{F_{1,2}}^{max(a,F_{1,2})}\phi_{S_2}^{Q}dS_2 + \int_{min(b,F_{1,2})}^{F_{1,2}}\phi_{S_2}^{Q}(S_2)\,dS_2 + 1\right)$$

$$- \int_{max(a,F_{1,2})}^{\infty}\left(\tilde{S}_2 - F_{1,2}\right)\phi_{S_2}^{Q}(S_2)\,dS_2\Bigg]$$

Here, the result is always positive regardless of the values of a and b, due to the characteristics of normal distribution. As a result, the company should decrease the value of $x_{1,2}^{o}$ to its minimum value, zero when $x_{1,2} > \xi$.

Then the second decision, under-hedging, should be analyzed. The value of $-\triangle V_2(y_1)$ will be as the following when $x_{1,2} < \xi$.

$$\begin{aligned}
-\triangle V_2(y_1) &= \int_{F_{1,2}}^{max(a,F_{1,2})}\left(n_2 - \left(p - \tilde{S}_2 - \lambda\right)\xi + hx_{1,2} - \left[\tilde{S}_2 - F_{1,2}\right]^{+}x_{1,2} - ry_1'\right)\phi_{S_2}^{Q}(S_2)\,dS_2 \\
&+ \int_{max(a,F_{1,2})}^{\infty}\left(n_2 - \left(p - \tilde{S}_2 - \lambda\right)\xi + hx_{1,2} - \left[\tilde{S}_2 - F_{1,2}\right]^{+}x_{1,2} - ry_1' - \beta ry_2\right)\phi_{S_2}^{Q}(S_2)\,dS_2 \\
&+ \int_{min(b,F_{1,2})}^{F_{1,2}}\left(n_2 - \left(p - \tilde{S}_2 - \lambda\right)\xi + hx_{1,2} - ry_1' - \beta ry_2\right)\phi_{S_2}^{Q}(S_2)\,dS_2 \\
&+ \int_{0}^{min(b,F_{1,2})}\left(n_2 - \left(p - \tilde{S}_2 - \lambda\right)\xi + hx_{1,2} - ry_1'\right)\phi_{S_2}^{Q}(S_2)\,dS_2 \\
&= \left(n_2 - \left(p - \tilde{S}_2 - \lambda\right)\xi\right) - \int_{0}^{\infty}\beta ry_1'\phi_{S_2}^{Q}(S_2)\,dS_2 \\
&- \int_{max(a,F_{1,2})}^{\infty}\beta ry_2\phi_{S_2}^{Q}(S_2)\,dS_2 - \int_{min(b,F_{1,2})}^{F_{1,2}}\beta ry_2\phi_{S_2}^{Q}(S_2)\,dS_2
\end{aligned}$$

Let us take the first derivative of the expression with respect to $x_{1,2}^{o}$

$$\begin{aligned}
\frac{d\left(-\triangle V_2(y_1)\right)}{dx_{1,2}^{o}} &= \frac{d\left[\left(n_2 - \left(p - \tilde{S}_2 - \lambda\right)\xi\right) - \int_0^{\infty}\beta ry_1'\phi_{S_2}^{Q}dS_2\right]}{dx_{1,2}^{o}} \\
&- \left(\frac{\int_{max(a,F_{1,2})}^{\infty}\beta ry_2\phi_{S_2}^{Q}dS_2 + \int_{min(b,F_{1,2})}^{F_{1,2}}\beta ry_2\phi_{S_2}^{Q}(S_2)\,dS_2}{dx_{1,2}^{o}}\right) \\
&= \frac{-\beta r\,d\left[\int_0^{\infty}\left(-hx_{1,2}^{o}\right)\phi_{S_2}^{Q}(S_2)\,dS_2\right]}{dx_{1,2}^{o}} \\
&+ \frac{-\beta r\,d\left[\int_{max(a,F_{1,2})}^{\infty}\beta\left(y_1 - n_2 + \left(p - \tilde{S}_2 - \lambda\right)\xi - hx_{1,2} + \left(\tilde{S}_2 - F_{1,2}\right)x_{1,2}\right)\phi_{S_2}^{Q}(S_2)\,dS_2\right]}{dx_{1,2}^{o}}
\end{aligned}$$

$$+\frac{-\beta r\, d\left[\int_{min\left(b,F_{1,2}\right)}^{F_{1,2}}\left(y_1 - n_2 + \left(p - \tilde{S}_2 - \lambda\right)\xi - hx_{1,2}\right)\phi_{S_2}^{Q}\left(S_2\right)dS_2\right]}{dx_{1,2}^{o}}$$

$$= \beta r\int_{F_{1,2}}^{\infty}\left(\tilde{S}_2 - F_{1,2}\right)\phi_{S_2}^{Q}\left(S_2\right)dS_2\left(\int_{max\left(a,F_{1,2}\right)}^{\infty}\beta\phi_{S_2}^{Q}\left(S_2\right)dS_2 + \int_{min\left(b,F_{1,2}\right)}^{F_{1,2}}\phi_{S_2}^{Q}\left(S_2\right)dS_2 + 1\right)$$

$$-\int_{F_{1,2}}^{max\left(a,F_{1,2}\right)}\left(\tilde{S}_2 - F_{1,2}\right)\phi_{S_2}^{Q}\left(S_2\right)dS_2$$

Here, the result is always positive regardless of the values of a and b. Therefore, the company should decrease the value of $x_{1,2}^{o}$ to its minimum value, zero. Since both cases suggest the same result, it can be said that the company should not have a decision such as $\frac{y_1}{h} < x_{1,2}$.

B.16 Theorem 16

At the proof of the Theorem 15, the terms a, b, θ_1 and θ_2 are defined. It is already known that both of the terms θ_1 and θ_2 have the same expression $-y_1 + n_2 - (p - \lambda - F_{1,2})\xi + hx_{1,2}$ as numerator. As a result, both of the terms θ_1 and θ_2 have to be in the opposite sign when $x_{1,2} > \xi$. Thus there are three cases that can be valid when $x_{1,2} < \xi$ as the following.

- $a > F_{1,2}, b < F_{1,2}$
- $a < F_{1,2}, b > F_{1,2}$
- $a = F_{1,2}, b = F_{1,2}$

Similarly, the terms θ_1 and θ_2 have to be in the same sign when $x_{1,2} < \xi$. Thus there are three cases that can be valid when $x_{1,2} < \xi$.

- $a > F_{1,2}, b > F_{1,2}$
- $a < F_{1,2}, b < F_{1,2}$
- $a = F_{1,2}, b = F_{1,2}$

We also know that the term $-\triangle V_2\left(y_1\right)$ is convex with respect to $x_{1,2}$. The value of $-\triangle V_2\left(y_1\right)$ is shown below.

$$-\triangle V_2\left(y_1\right) = \left(n_2 - \left(p - \tilde{S}_2 - \lambda\right)\xi\right) + \int_0^{\infty}\left(hx_{1,2} - \left[\tilde{S}_2 - F_{1,2}\right]^{+}x_{1,2} + \beta r\left[y_2\right]^{-}\right)\phi_{S_2}^{Q}\left(S_2\right)dS_2$$

Since the term $\left(n_2 - \left(p - \tilde{S}_2 - \lambda\right)\xi\right)$ is linear, which means it is both convex and concave. When the probability distribution functions are ignored, it is observed that the expression $hx_{1,2} - \left[\tilde{S}_2 - F_{1,2}\right]^{+}x_{1,2} + \beta r\left[y_2\right]^{-}$ is also convex. It is known that if a function is convex, its expected function is also convex; as a result $-\triangle V_2\left(y_1\right)$ is concluded to be convex.

Since $-\triangle V_2\left(y_1\right)$ is convex and optimum solutions may exist in different intervals of $x_{1,2}$, we will provide just one optimum region that satisfy optimality conditions in some realistic cases.

After all these analysis, two different decision should be examined by getting the first derivative of the objective function with respect to $x_{1,2}$.The examination of the decision $x_{1,2} > \xi$

Decision I: $x_{1,2} \geqq \xi$

$$
\begin{aligned}
-\triangle V_2(y_1) &= \int_{max(a,F_{1,2})}^{\infty} \left(n_2 - \left(p - \tilde{S}_2 - \lambda\right)\xi + hx_{1,2} - \left(\tilde{S}_2 - F_{1,2}\right)x_{1,2}\right)\phi^{Q}_{S_2}(S_2)\,dS_2 \\
&+ \int_{F_{1,2}}^{max(a,F_{1,2})} \left(n_2 - \left(p - \tilde{S}_2 - \lambda\right)\xi + hx_{1,2} - \left(\tilde{S}_2 - F_{1,2}\right)x_{1,2} - \beta r y_2\right)\phi^{Q}_{S_2}(S_2)\,dS_2 \\
&+ \int_{min(b,F_{1,2})}^{F_{1,2}} \left(n_2 - \left(p - \tilde{S}_2 - \lambda\right)\xi + hx_{1,2} - \beta r y_2\right)\phi^{Q}_{S_2}(S_2)\,dS_2 \\
&+ \int_{0}^{min(b,F_{1,2})} \left(n_2 - \left(p - \tilde{S}_2 - \lambda\right)\xi + hx_{1,2}\right)\phi^{Q}_{S_2}(S_2)\,dS_2 \\
&= \left(n_2 - \left(p - \tilde{S}_2 - \lambda\right)\xi\right) - \int_{F_{1,2}}^{max(a,F_{1,2})} \beta r y_2 \phi^{Q}_{S_2}(S_2)\,dS_2 - \int_{min(b,F_{1,2})}^{F_{1,2}} \beta r y_2 \phi^{Q}_{S_2}(S_2)\,dS_2
\end{aligned}
$$

The value of the expression above changes with respect to the values of a and b. Therefore, subcases should be analyzed separately.

Decision I.II: $a < F_{1,2}$, $b > F_{1,2}$ and $x_{1,2} \geqq \xi$

First derivative:

$$
\begin{aligned}
\frac{d\left(-\triangle V_2(y_1)\right)}{dx_{1,2}} &= \frac{d\left[\left(n_2 - \left(p - \tilde{S}_2 - \lambda\right)\xi\right) - \int_{F_{1,2}}^{max(a,F_{1,2})} \beta r y_2 \phi^{Q}_{S_2}(S_2)\,dS_2\right]}{dx_{1,2}} \\
&\quad - \frac{d\left(\int_{min(b,F_{1,2})}^{F_{1,2}} \beta r y_2 \phi^{Q}_{S_2}(S_2)\,dS_2\right)}{dx_{1,2}} \\
&= \frac{-\beta r\, d\left[\int_{F_{1,2}}^{\infty} \left(y_1 - n_2 + (p - S_2 - \lambda)\xi\right)\phi^{Q}_{S_2}(S_2)\,dS_2\right]}{dx_{1,2}} \\
&= 0
\end{aligned}
$$

The cluster of the decisions that satisfies the condition above may cause multi-optimality because all are local extreme points. If the problem is proved to be convex, it means that this is the optimum decision.

Decision I.III: $a = F_{1,2}$, $b = F_{1,2}$ and $x_{1,2} \geqq \xi$

It is the same with case $a < F_{1,2}$, $b > F_{1,2}$. It is always zero, as a result there is an extreme point here.

Now, it is analyzed that both of the cases always provide an optimal solution. It is already known that the conditions satisfying $a \leqq F_{1,2}$, $b \geqq F_{1,2}$ should also satisfy $\theta_1 \leqq 0$, $\theta_2 \geqq 0$. Therefore, the expression $-y_1 + n_2 - (p - \lambda - F_{1,2})\xi + hx_{1,2}$ has to be less than zero because it is the numerator of both θ_1 and θ_2 terms. Then, another condition may be written for optimal hedging decision at time 1 such as $x_{1,2} < \frac{y_1}{h} + \frac{(p-\lambda-F_{1,2})\xi-n_2}{h}$.

It is observed that the decision of $x^*_{1,2} \geqq \xi$ is an optimal decision as long as $x^*_{1,2} < \frac{y_1}{h} + \frac{(p-\lambda-F_{1,2})\xi-n_2}{h}$; and also the budget constraint at time 1, which is $x^*_{1,2} < \frac{y_1}{h}$, are satisfied together. Two maximum limit conditions may be united as $x^*_{1,2} \leqq \frac{y_1}{h} + \left(\frac{(p-\lambda-F_{1,2})\xi-n_2}{h}\right)^-$. Therefore, the optimum decision is the decisions satisfying $\frac{y_1}{h} + \left(\frac{(p-\lambda-F_{1,2})\xi-n_2}{h}\right)^- \geqq x^*_{1,2} \geqq \xi$ as long as $\frac{y_1}{h} + \left(\frac{(p-\lambda-F_{1,2})\xi-n_2}{h}\right)^-$ is more than ξ.