SALİH ZEKİ'S *DARÜLFÜNUN KONFERANSLARI*
AND
HIS TREATMENT OF THE DISCOVERY OF NON-EUCLIDEAN GEOMETRIES

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ABSTRACT

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This thesis examines *Darülfünun Konferansları* which consists of a series of lectures that were delivered by Salih Zeki in 1914 – 1915 in Ottoman State. These lectures were on geometry, its history and especially on the discovery of non-Euclidean geometries. And the purpose of this thesis is to propose the sufficiency and the legitimacy of these lectures as an account on the history of geometry. As a preliminary to analyzing Salih Zeki's lectures, different views on geometry's history and progress will be analyzed and compared. The results of this comparison will be the guide by means of which *Darülfünun Konferansları* will be examined. This thesis also serves as a source that makes Salih Zeki's ideas accessible, by presenting an English summary of his lectures which were originally published in Ottoman Turkish.

Keywords: Salih Zeki, Darülfünun Konferansları, History of Geometry, Discovery of non-Euclidean Geometries

ÖZ

SALİH ZEKİ'NİN *DARÜLFÜNUN KONFERANSLARI* VE GAYRİ ÖKLİDYEN GEOMETRİLERİN KEŞFİNE BAKIŞI

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Bu tez Salih Zeki'nin 1914 – 1915 yıllarında Osmanlı Devleti'nde vermiş olduğu bir grup dersten ibaret olan *Darülfünun Konferansları*'nı incelemektedir. Bu dersler geometri ve tarihi, özellikle de gayri Öklidyen geometrilerin keşfi üzerinedir. Tezin amacı bu derslerin geometri tarihini açıklamakta ne derece yeterli ve geçerli olduklarını ortaya koymaktır. Salih Zeki'nin derslerini incelemeye bir hazırlık olarak geometrinin tarihine ve gelişimine dair farklı görüşler değerlendirilecek ve karşılaştırılacaktır. Bu karşılaştırmanın sonuçları *Darülfünun Konferansları*'nın incelenmesinde bir rehber niteliği taşır. Ayrıca, bu tez aslen Osmanlı Türkçesi olarak yayınlanmış olan derslerin İngilizce bir özetini sunmak yoluyla, Salih Zeki'nin fikirlerini ulaşılabilir ve anlaşılabilir kılan bir kaynaktır.

Anahtar Kelimeler: Salih Zeki, Darülfünun Konferansları, Geometri Tarihi, Gayri Öklidyen Geometrilerin Keşfi.

To My Grandmother

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CHAPTER 1

INTRODUCTION

This thesis examines different perspectives on the development of the non-Euclidean Geometries, for the sake of determining the criteria that should be granted while constructing the history of geometry. I aim to propose a brief history of geometry and also to compare two different approaches on writing this history. The two different approaches to be compared may be titled as the "standard account" (Gray, 2003, p. 168) and the "historical account". Among these two views I should support the historical account. The summary of the history of geometry which will be provided in the second chapter of my thesis aims to be in agreement with the historical account rather than the standard account. The example that is provided to be a standard account of geometry's progress is Roberto Bonola's 1912 work.[1] And it will be criticized because of evaluating the history of geometry as a mere logical chain. When its history is regarded to be a logical chain, geometry turns out to be in a linear progress. All the theories in geometry's history flow from one another, no matter how they differ in their evaluation of space or what branch of mathematics they got their clues for dealing with their subject matter. The opposing example to Bonola's work is Jeremy Gray's historical account.[2] Gray's views on the history of geometry are going to be supported because of their success in proposing meaningful explanations on geometry's history, by means of leaving aside the logical or axiomatic treatment of its progress.

This comparison of the views on the history of geometry is a basis for evaluating a 20th century work on the discovery of non-Euclidean geometries. By determining the accurate view on the history of geometry, I aim to examine the 1914 –1915 (or 1330 – 1331) work of Salih Zeki (1864 – 1921). Salih Zeki presented a series of lectures on geometry and mathematics in Darülfünun in Istanbul between the mentioned

[1] *Non-Euclidean Geometry* (reprinted in 1955).

[2] Ideas of Space (2003).

years. Those lectures were published in two volumes and were entitled as "*Darülfünun Konferansları*".[3] Of these two volumes, this thesis is interested in the first one which deals with non-Euclidean geometries. The second volume of Salih Zeki's lectures presents mathematical concepts such as imaginary numbers and complex numbers. The volume on geometry consists of fourteen lectures (The last lecture is not published completely). The first five of the lectures on geometry discuss the discovery of non-Euclidean geometries, that is, they are on the history of geometry, while the rest of the lectures concentrate on geometrical systems in a more detailed sense. In my thesis I am going to concentrate on the history lectures provided by Salih Zeki, that is, the first five lectures of Darülfünun Konferansları.

With the justification of a legitimate account on geometry's development and by proposing the pitfalls of the standard logical view, I will analyze whether Salih Zeki's conferences were sufficient in proposing a proper examination of discovery of non-Euclidean geometries.

As I will point out in my summary of the history of geometry, a change in the mathematical methods that were applied in geometrical studies can be observed throughout geometry's history. This is a significant fact in explaining the change and development in geometry opening the way through non-Euclidean or the new geometries. The sufficiency of Salih Zeki's conferences in proposing the change in those mathematical methods is another point that will be taken into consideration in this thesis.

In order to carry out the above mentioned inquiries, that is, for analyzing Salih Zeki's views on the history of geometry and to figure out whether he emphasizes the change in mathematical methods, I will provide the study he carries out in his five lectures. The third chapter of my thesis consists of a summary of Salih Zeki's account on the discovery of non-Euclidean geometries. In providing this summary, I tried to be careful not to miss out any important detail that may give an idea about the writer's views on non-Euclidean geometries or his geometrical capacity. I think chapter three would make Salih Zeki's lectures accessible for those who cannot deal with Ottoman Turkish. The third chapter is merely a summary without any further explanations on Salih Zeki's ideas; I did not add any ideas that do not belong to

[3] Darülfünun Conferences.

Salih Zeki or correct any of his ideas. Also, in the second chapter of my thesis while presenting the significant studies that cannot be omitted in accounting for geometry's progress, I will provide explanations on the works of all the geometers that took place in Salih Zeki's lectures.

As I have mentioned in the previous paragraph, the examination of Darülfünun conferences will also reveal the extent of Salih Zeki's qualifications on the subject matter of his lectures. His capabilities may give us some clues about the mathematical capacity of the specific era he was dwelling in. This era was the beginning of the 20th century in Darülfünun, in my case it was the mathematical community of the late Ottoman State – considering that the audience of those lectures included "instructors of mathematics" as well as some amateur mathematicians as the cover of Darülfünun conferences informs us.[4] In other words, if Salih Zeki was instructing an audience which consisted of mathematicians then he was the authority among those scientists and I assume that the capabilities of his students on this very specific topic would not surpass his knowledge. My assumption also rests upon the fact that in his preface Salih Zeki says that the conferences aimed to fulfill a request of the audience for hearing about the new geometries which again implies his superiority on the examined subject.

Before I go through with the above mentioned promises, I should provide a short biography of Salih Zeki:

Salih Zeki was a mathematician who also studied physics, discussed philosophical topics in various articles and whose interest and efforts on the history of geometry cannot be underestimated. His interest in mathematics persisted through his education, first in Ottoman State then in Paris. After studying in Paris, he returned to Ottoman State in which he spent the rest of his life.

Salih Zeki was assigned to various governmental duties after his return from Paris, such as working in a ministry or the Observatory. He ended up as an instructor of mathematics and physics in several institutes among which was Darülfünun. During his years in Darülfünun he offered conferences on various subjects, five of which are examined in this thesis.

[4] "...riyaziyat muallim ve muhibblerine verilen konferanslar..."

Salih Zeki has written books on mathematics, geometry, algebra, astronomy, trigonometry and physics (Saraç, 2001, pp. 13-14). His translations of Poincaré's major works are among his known and accessible works. Salih Zeki translated *Science and Method*, *Science and Hypothesis*, and *The Value of Science* in to Ottoman Turkish. *Kamus-u Riyaziyat* should also be mentioned among his works which was an encyclopedia of mathematics.

One of his significant works, namely Asar-ı Bakiye implies that he spent quite a long time studying the works of oriental mathematicians of the middle ages. The preface of Asar-ı Bakiye tells us that an intention to analyze the works of the oriental scientists on geometry, astronomy, astrology and arithmetic led Salih Zeki to visit the libraries in Istanbul. After spending a short time on the available sources in those libraries he decided that in order to appreciate the works of the Eastern scientists he should study the Greek works on these sciences which preceded the Eastern contributions. Salih Zeki studied the Greek astronomy and geometry through the work of Paul Tannery, which Salih Zeki regards as a source to reach Ptolemy's Almagest and Euclid's geometry. Another step in Salih Zeki's preparation to study the Eastern mathematics was to read the translations of some old mathematics books in Sanskrit language. Afterwards he returned back to studying the sources in Istanbul libraries and also analyzed some works of European scholars on his main interest – an attempt which he says to be spread to three years.

In brief, Salih Zeki performed a study on mathematics and geometry which covers Ancient Greece and the Eastern World in the middle ages. *Asar-ı Bakiye* was published in 1913. Considering that he presented his conferences on non-Euclidean geometry between 1914 and 1915, this may be a glance on what he had studied preceding these conferences.

Among Salih Zeki's notes, a mathematician was especially mentioned for his interest on the non-Euclidean geometries and their history, and for sharing his knowledge with Salih Zeki, namely Vidinli Tevfik Paşa (1832 – 1901) (Saraç, 2001, p. 52). Vidinli Tevfik Paşa appears to be the reason why Salih Zeki got acquainted with the non-Euclidean geometries. On the other hand, not all of the actual sources that guided *Darülfünun Konferansları* are cited explicitly in Salih Zeki's account. To illustrate, in his first lecture Salih Zeki provides the reader with a footnote in which he states that he quoted Gauss's letters from *Gauss, les deux Bolyai* which was translated into French by Laugel (S.Zeki, Darülfünun Konferansları, 1331, p. 9). And he writes down the original title of Cayley's article that is followed in the fourth lecture, namely A Sixth Memoir upon Quantics (S.Zeki, Darülfünun Konferansları, 1331, p. 47). However, when he gives an account of Riemann and Helmholtz's contributions Salih Zeki does not provide the titles of these works in any of the languages they appeared in, but gives Ottoman Turkish translations of them.

The fourth chapter of my thesis consists of an evaluation of Salih Zeki's lectures on geometry, in which one of the concerns will be the sources that were followed by him throughout his lectures. The main concern of my fourth chapter is to evaluate Salih Zeki's views on geometry and its history, to point out his mathematical knowledge and to figure out the erroneous parts of his lectures.

To sum up, in the following chapter I will provide a summary of the discovery of non-Euclidean geometries and also discuss how their history should be written by comparing the standard and historical accounts on geometry's progress. In the third chapter I will provide Salih Zeki's account on this discovery and the fourth chapter will consist of evaluating Salih Zeki's lectures.

CHAPTER 2

THE DISCOVERY OF NON-EUCLIDEAN GEOMETRIES AND DIFFERENT PERSPECTIVES

2.1 The Problem of Parallel Lines and the New Geometries

The non-Euclidean geometries, like any other innovations in the course of science, did not merely pop up at a definite time back in history. The discovery of the new geometries, if the desire is not to distort actual history, should be examined as spread to centuries. This discovery or the road which led the geometers to discover the new geometries can be traced back to Ancient Greece when the legitimacy of the Fifth Postulate in the first book of Euclid's *Elements* was doubted.

The first book of Euclid's *Elements* provides a "definition" of parallel lines: "Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction." (Euclid, 2002, p. 2). And the application of such lines in geometrical constructions was supported by a "postulate". The Fifth Postulate proposes "[t]hat, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles." (Euclid, 2002, p. 2).

Greeks regarded the Fifth Postulate to be problematic, since although it was titled to be a postulate, it required to be proven instead of being evident in the geometrical system of Euclid. The problem about the Fifth Postulate is that lines that approach but do not meet are conceivable (Proclus, 1970, p. 151). Actually, that is the case for asymptotic lines which were known by Greeks. If it is possible for some lines to approach without intersecting each other, this could be possible for straight lines too. In this sense, the claim that straight lines necessarily meet if they converge each other has to be proved. Also, it can be observed that the parallel postulate in its very statement, determines the “indefinite” in terms of the “definite”. It determines which cannot be demonstrated geometrically and is more likely to be in agreement with the experience of the external world.

One of its fundamental parts being dubious, Euclid’s geometry could be regarded to be in danger. However, geometers - just like any scientist would do about the theory she was used to understand the world with - did not stop relying on and working with this geometry for centuries. There has been quite an enormous effort in favor of the Fifth Postulate from the ancient times until the 19th century. And it is not the case that geometers after all their endeavors had to accept that the Fifth Postulate was not capable of having a proof and so left aside Euclid’s geometry and started to work with another one. On the contrary, all the attempts which aimed to support Euclid’s geometry were steps through the new geometries.

At the 2nd century AD Ptolemy believed that he provided a proof for the Fifth Postulate, regarding it as a theorem (Proclus, 1970, p. 285). Ptolemy started from parallel lines and believed that he had demonstrated that the sum of the interior angles on one side when such two lines are cut by a third line can only be equal to two right angles. Proclus in his *A Commentary on the First Book of Euclid’s Elements* thought that he had proved that if an arbitrary line cuts a second line then it has to cut a third line which is parallel to the second line. In doing so Proclus assumed that parallel lines are “equidistant”. This idea of equidistance can be observed in later geometrical works in history and is a determining concept in several geometrical studies on the parallel postulate.

Ptolemy and Proclus are examples of geometers which worked on the Fifth Postulate in the old times, none of which being successful to reach their goals. Their failure was mainly because of working with assumptions which were corresponding to what was desired to be proved. They were working within the limits of classical geometry. Long after their time, the geometers were able to come up with the idea that the solution to the problem of parallels would not come by means of the Euclidean practice.

Considering that I have limited my account to the Western contributions, I should continue my account with the 16th and 17th century investigations on the problem of parallels when it was being considered again in the Western world. It was about a millennium after Proclus's time when the geometers were considering this crucial problem with Euclid's geometry again. In the 16th and 17th centuries the studies on the problem of parallels got their clues from the idea of equidistance. The geometers that studied by means of the idea of equidistance of parallel lines are listed by Bonola to be Commandino (1509 – 1575), Clavio (1537 – 1612), Cataldi (… - 1626), Borelli (1608 – 1679) and Vitale (1633 – 1711).

Among these geometers Cataldi had a new idea. He put forward a hypothesis which said that "…straight lines which are not equidistant converge in one direction and diverge in the other" (Gray, 2003, p. 57). And Vitale is special for the simplicity of his formulation. Vitale worked with a claim that the locus of the points with equal distance to a straight line is itself a straight line. This assumption ended with the requirement of proving the existence of a single point on this locus (Bonola, 1955, p. 15).

The abandonment of the idea of equidistance enters the scene with the studies of Wallis (1616 – 1703). Wallis's constructions are on the idea of "similarity". He accepts that "…to every figure there exists a similar figure of arbitrary magnitude." (Bonola, 1955, p. 15). From its preliminary assumption Wallis's work has a deficiency since form which is independent of size is not self-evident (Gray, 2003, p. 58). When it is proposed that there is a similar figure of arbitrary magnitude to every figure, the parallel postulate is already assumed.

When we come from Wallis to the contributions of Saccheri (1667 – 1733), the formulation of the problem enters a quite interesting way. The method applied by Saccheri is not concerned with a direct proof of the parallel postulate anymore. Instead, his approach is constructed by means of "reductio ad absurdum". The way which was opened by Saccheri is an innovation that affected the studies of following geometers. The significance of Saccheri's work is that while applying reductio ad absurdum he assumed that some triangles may have an angle sum which is different than two right angles, hoping to prove their impossibility.

Saccheri set up three hypotheses: the hypothesis of the right angle (HRA), the hypothesis of the acute angle (HAA) and the hypothesis of the obtuse angle (HOA). These hypotheses were respectively dealing with triangles with an angle sum of two right angles, triangles with an angle sum which is less than two right angles and lastly triangles with a sum greater than two right angles. At the end he was able to prove HRA and reject that triangles may have an angle sum greater than two right angles. However, he was unsuccessful in rejecting the possibility of triangles with an angle sum less than two right angles.

Saccheri's work may look like a failure, but it has a definite significance in the course of the progress of geometry. This geometer worked with triangles and carried the problem to the realm of trigonometric studies from classical geometry. His followers such as Lambert (1728 – 1777) and Legendre (1752 – 1833) were occupied with the three hypotheses that were set may Saccheri.

Lambert and Legendre established some conclusions which may be characterized as non-Euclidean although any non-Euclidean geometry had not appeared in history yet. Lambert concluded that the difference of the angle sum of a polygon from the expected Euclidean sum and the area of this polygon would be proportional. And he was aware of "…the connection between spherical geometry and the geometry based on the HOA, and he also suggested that the AA'd [acute angled] geometry would be that on the imaginary sphere." (Gray, 2003, p. 102).[5] Legendre asserted as a theorem "…*that the sum of the angles of any triangle is either less than [Hypothesis of the Acute Angle] or equal to [Hypothesis of the Right Angle] two right angles."* (Bonola, 1955, p. 55).

Saccheri's, Lambert's and Legendre's studies were the last efforts in favor of the parallel postulate. From the time of Proclus until the 18th century the studies on the problem evolved from searching a direct proof to an indirect way of investigation. The application of classical geometry started to leave the scene for dealing with the problem of parallels by means of triangles. The last geometers of the Euclidean tradition could not exhaust the possibility of a triangle with an angle sum which is less than two right angles. And this triangle found its place in the geometries of two

[5] An imaginary sphere is a sphere with a radius which is an imaginary number. (eg. $\sqrt{-1}$)

later coming geometers: Janos Bolyai (1802 – 1860) and Nikolai Lobachevski (1793 – 1856).

As the history of geometry progresses through the first non-Euclidean geometries, the roles of Wolfgang Bolyai (1775 – 1856), Wachter (1792 – 1817), Gauss (1777 – 1855), Schweikart (1780 – 1859) and Taurinus (1794 – 1874) should also be mentioned.

Wolfgang Bolyai's contribution to the world of geometry is to propose that through three points not on a straight line, a circle can always be drawn (Bonola, 1955, p. 61). His aim was to prove the Fifth Postulate departing from such an assertion. Wachter followed Wolfgang Bolyai and based his argument upon four points and a sphere passing through them. Wachter could not provide an appropriate definition of the surface he takes into consideration. However, in a letter to Gauss, Wachter wrote about a "…surface to which a sphere tends as its radius approaches infinity, a surface on the Euclidean hypothesis identical with a plane." (Bonola, 1955, pp. 62-63).

Gauss, in his studies on parallel lines provides a definition of such lines. His definition is not a reformulation of the Euclidean one. On the contrary, it is a formulation of the negation of the Euclidean parallelism. Euclidean parallelism restricts the number of parallel lines to another line through a point which is not on this line to "one". When it is negated, the result is a plurality of such parallel lines. Gauss believed in the possibility of a non-Euclidean geometry and put forward a notion of parallelism which is no longer determined by Euclidean concepts. This new notion is parallelism in one direction.

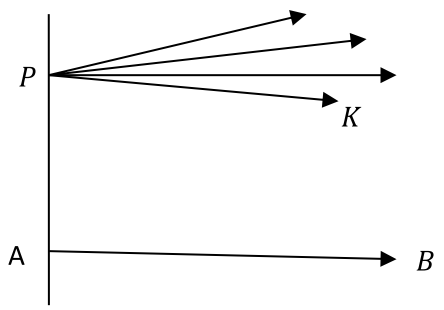

Figure 1

In the figure PK and AB are said to be parallel to the right. There is a plurality of lines through P that do not cut AB. And if Euclid's parallelism was considered they would all be parallel to AB. Gauss reduces the number of parallel lines through P to AB, by defining the first one to be the parallel line to AB (Bonola, 1955, p. 68).

In a non-Euclidean geometry which agrees with the hypothesis of the acute angle, that is, in which the angle sum of a triangle is less than two right angles, there is a concept which is called a "horocycle". "The horocycle is not a circle and indeed has the remarkable property that no three points on it can be joined by a circle…" (Gray, 2003, p. 91). A horocycle passes through the corresponding points on a pencil of parallel lines. The corresponding points would be two points A and B on two parallel lines a and b, such that AB makes equal angles with a and b.[6]

Gauss introduced a new way of treating parallels and he also put forward the notion of corresponding points. However, he did not work on the horocycle any further (Gray, 2003, p. 91).

Schweikart was a contemporary of Gauss whose investigation was entitled as "Astral Geometry". In Schweikart's geometry the angle sum of a triangle was less

[6] In Euclidean geometry there are two types of pencil of lines: A pencil of lines through a point and a pencil of lines with a common perpendicular. The lines with a common perpendicular are parallel lines in Euclid's geometry. However, for the non-Euclidean geometry in this case, a pencil of parallel lines is a third type.

than two right angles and he wrote that "…the sum becomes ever less, the greater the area of the triangle…" (Bonola, 1955, p. 76). Another point about Schweikart which should not be overlooked is that he proposed that the altitude of an isosceles right angled triangle cannot get larger than a certain length which he calls a "constant".

Taurinus, just like Lambert, recognized the relation between the hypothesis of the obtuse angle and the spherical trigonometry (Bonola, 1955, p. 82). He believed in the possibility of an inverse spherical geometry (Gray, 2003, p. 101). And his intention was to derive the formulae of a geometry which corresponds to the hypothesis of the acute angle from the formulae of spherical trigonometry.

It can be concluded that with Saccheri's approach a new era started in the realm of geometrical enterprise. The negations of the Euclidean postulate were studied. A new definition of parallelism was proposed. Geometers noticed the relation between spherical trigonometry and the hypothesis of the obtuse angle. Ideas about surfaces in which the hypothesis of the acute angle would be true were revealed. And some properties of a non-Euclidean geometry, such as a "constant" and the "corresponding points" came out. Still, no geometer was able to construct a non-Euclidean geometry as a system.

The 19th century finally brings the first non-Euclidean geometrical systems by means of the works of Nikolai Lobachevski and Janos Bolyai. Lobachevski thought that the difficulties faced by the geometers concerning the problem of parallels throughout history, were because the proof of the parallel postulate could not be in terms of the available data (Bonola, 1955, p. 92). In other words, the proof would not come from Euclid's geometry itself. Lobachevski's system agrees with one of the negations of the Fifth Postulate of Euclid; the hypothesis of the acute angle. Lobachevski did not start with Euclidean thinking, but constructed a brand new geometry in which triangles have an angle sum less than two right angles, the loci of the corresponding points of parallel lines are horocycles and length is absolute. Lobachevski's geometry allowed more than one parallel line through a point to another line. And the title of this geometry was "Pangeometry".

Bolyai's geometry has the same properties as Pangeometry. Bolyai named his geometry as "Absolute Geometry", a name which implies his main intention in constructing his geometry. Lobachevski set up a new universe in which Euclid's parallels would not hold anymore. Though, Bolyai was not interested in negating the Fifth Postulate. He was looking for the absolute theorems in Euclid's geometry which are the theorems that are true independently of the parallel postulate (Gray, 2003, p. 108). And the theorems that are absolute in this sense are in agreement with the hypothesis of the acute angle.

Lobachevski's and Bolyai's geometries were independent studies. Considering that while Lobachevski was studying with the negation of the parallel postulate, Bolyai was investigating the independent parts of the ordinary geometry from the parallel postulate, it can be concluded that Euclid's geometry and Pangeometry have the Absolute Geometry in common (Bonola, 1955, p. 102).

With the studies of Lobachevski and Bolyai, Euclid's geometry was not the only geometry anymore and the Euclidean plane was not the only one to work with. However, those two geometries had not brought the all-embracing outlook required by geometry yet. When the problem of parallels was totally worked out, the Euclidean plane and the surfaces that Lobachevski's and Bolyai's geometries were true for, turned out to be examples of many possible spaces. It was the hypotheses of Riemann (1826 – 1866) which supplied geometry with the way of constructing Euclidean and non-Euclidean geometries from one point of view.

Riemann, in his hypotheses[7] did not use the name "non-Euclidean geometry", but he opened the way to a more general geometrical thinking than Euclid's (Gray, 2003, p. 141). Riemann regards the problems that were faced by geometers to result from the fact that multiply extended magnitudes were not studied. In his work he constructs the n-fold magnitudes or the notion of the "manifolds". A manifold may be of one, two, three or more folds and has an intrinsic curvature. The experienced space is an example of a three-fold manifold. And a three-fold manifold with zero curvature would be the Euclidean space; a two-fold one with the same curvature would be the Euclidean plane. Curvature has to be constant, not necessarily equal to zero, if figures are to move without being subject to any change. Zero curvature means that a manifold is flat. If curvature is different than zero, either positive or negative, a manifold would be curved. In this sense, a spherical surface can have its

[7] On the Hypotheses which lie at the Bases of Geometry (1854)

own geometry independent of any Euclidean concept. Curvature would determine the angle sum of a triangle and when it is constant this sum would be true for all triangles in a manifold.

Riemann’s approach brought the subject to a general ground for geometry. His work did not present a particular type of geometry that was different from the long Euclidean tradition or the following non-Euclidean geometries. Riemann’s investigation was on “space”; the subject matter of geometry.

The idea of treating all geometry from one point of view and regarding the traditional geometry not “the” but “a” geometry was not a simple change in the realm of geometrical practice. However, the studies of some following geometers strengthened its stance. Beltrami (1835 – 1899) constructed a model of non-Euclidean geometry. Beltrami’s model mapped a Euclidean plane and a non-Euclidean surface. He made “…a region of the Euclidean plane… exhibit the descriptive features of a new geometry…” (Gray, 2003, p. 150). This could be understood by imagining light passing through a gridded plane and projecting grids on to a sphere (Gray, 2003, p. 147). In this sense, distances and angles on the gridded plane would be altered on the sphere but the two different surfaces would be mapped. Geometers had always regarded Euclidean geometry as reliable grounds. And since non-Euclidean geometries could be mapped with the reliable geometry, it was shown that if Euclidean geometry was worth working with then so were the non-Euclidean geometries. Any non-Euclidean figure could be interpreted as a Euclidean one and “…every statement about it made in the course of any proof is likewise interpretable in strictly Euclidean terms” (Gray, 2003, p. 149).

Helmholtz (1821 – 1894) was another scientist who independently came up with the similar formulations as Riemann. Helmholtz described a manifold of n dimensions, a more general notion than space for the possibility of the origin of the concept of space. He set forth the necessary grounds for the free mobility of figures in a manifold, so that geometrical constructions would be possible.

Although Riemann and Helmholtz’s studies are concerned with the same concepts such as manifolds and the necessity of constant curvature for the possibility of free mobility, they differ in the paths they follow through the course of their declarations. Riemann starts form the notion of extended magnitudes and then considers the measure-relations in a manifold of n-dimensions. Measurements in a manifold are possible on the condition that lengths are independent of their positions by means of which Riemann describes the linear element. For the application of his ideas to space Riemann considers the idea of flat manifolds which is a special case of manifolds with constant curvature that allow figures to move without being subject to any stretching or contraction. In this sense, free mobility is the end result of Riemann's investigation. On the other hand, Helmholtz's course of study starts from searching for the necessary grounds that supply figures in space with free mobility.

Helmholtz's interest in free mobility stems from his idea that "…primary measurement of space is entirely based upon the observation of congruence…" (Helmholtz, 1977, p. 41). And the determination of congruence requires the motion of bodies in space without changing their forms. As a result, Helmholtz describes the notion of an n-dimensional manifold with constant curvature.

While Riemann was assuming the expression for the line element from the beginning, Helmholtz showed that this expression was the only possible one for surfaces of constant curvature, if free mobility of figures was assumed in the first place. Sophus Lie (1842 – 1899) generalized the difference between the approaches of the two geometers by means of his work on the continuous groups of transformations. Sophus Lie's formulation of the views of Riemann and Helmholtz was "*To determine all the continuous groups in space which, in a bounded region, have the property of displacements*".

Another geometrical study that put forward a relation between Euclid's geometry and non-Euclidean geometries is Felix Klein's (1849 – 1925) work in terms of projective geometry. It was Cayley (1821 – 1895) who provided a projective definition of distance by means of a conic which he termed the "Absolute" but he did not consider the relation of his theory to non-Euclidean geometries (Bonola, 1955, p. 148). Klein put forward that "Cayley's theory of projective measurement leads… directly to the three possible cases of non-Euclidean geometry: hyperbolic, parabolic, and elliptic, according as the measure of curvature k is < 0, $= 0$, or > 0 (Klein, 1894, pp. 85-86).

2.2 The Different Perspectives on the Discovery of non-Euclidean Geometries

In the previous section I tried to propose a short history of geometry whose focus was the discovery of non-Euclidean geometries. The story that appeared in the previous section was limited to the studies in the western world. The reason for this limitation is to keep a parallel study with Salih Zeki's account on the discovery of the new geometries. Salih Zeki's account starts from the problem of parallelism and presents Legendre's study to be the last one that was held in favor of Euclidean geometry. After mentioning Legendre, his account introduces the non-Euclidean geometries of Lobachevski, Bolyai and Riemann; and Salih Zeki continues his lectures with the interpretations on the new geometries in term of projective geometry. Apparently, Salih Zeki's lectures were organized as an account of the major innovations in the west.

An account on the progress of geometry may provide a story that lasts for centuries, or it may be limited to a specific era. Either way, it is possible to put forward various accounts on the progress of geometry that interpret the facts in terms of different perspectives. In the first section of this chapter I preferred to emphasize the intentions of geometers and the mathematical methods they applied. Also, I tried to point out the new notions that were introduced in geometrical studies and their innovative conclusions.

The progress of geometry is treated from several points of view. As a result, the history of geometry is written in various manners. The treatment of geometry's progress affects the descriptions of the aspects of its history. To illustrate, while one account describes Beltrami's work to be the required grounds for the consistency of the new geometries, another account may propose that this work showed that it is possible to practice with non-Euclidean geometries. Saccheri's study can be designated to give the idea that constructing a geometry in terms of the hypothesis of the acute angle is logically possible. On the other hand, the emphasis may be on Saccheri's treatment of the subject in terms of triangles. Moreover, Lobachevski's intention may be characterized as to construct a geometry on the negation of the fifth postulate of Euclid's geometry; while also it can be asserted that he worked with the idea that a non-Euclidean geometry is possible.

Geometry's progress can be regarded to be axiomatic or it can be examined as a historical process. When the emphasis is on the axiomatic structure of geometry's progress, the result would be an account that is mostly interested in the logical properties and the logical relations of the innovations. Such an account would be what Jeremy Gray terms as the "standard account" of the history of geometry (Gray, 2003, p. 168).

Treating the progress of geometry was a result of the axiomatization of geometry at the beginning of the 20th century. A name that can be pointed out to have an important role in this fashion is David Hilbert (1862 – 1943), with his *Foundations of Geometry* (first appeared in 1899). "Hilbert believed that the proper way to develop any scientific subject rigorously required an axiomatic approach" (Zach, 2009). Hilbert put forward a treatment of Euclidean geometry by means of a set of twenty axioms. These twenty axioms were presented in five groups: 1. Axioms of connection, 2. Axioms of order, 3. Axiom of parallels, 4. Axioms of congruence, 5. Axiom of continuity (Hilbert, 1950, p. 2). The five groups of axioms are not contradictory to one another; in other words, "…it is not possible to deduce from these axioms, by any logical process of reasoning, a proposition which is contradictory to any of them (Hilbert, 1950, p. 17). The twenty axioms of Hilbert are also mutually independent, where no axiom is the logical result of the other.

When Hilbert considers the third group of his axioms, that is, the parallel postulate of Euclid he restates the postulate into two assertions. The first assertion is that when a straight line is considered there is always another straight line that does not intersect it through a point that is not on the first straight line. The second assertion is on the uniqueness of the second straight line. "The first statement of the axiom of parallels can be demonstrated by aid of the axiom groups I, II, and IV" (Hilbert, 1950, p. 19). However, the second assertion is independent of all the other axioms.

In his book Hilbert provides a geometry where all the axioms of congruence hold except the sixth one. The sixth axiom of congruence is on triangles. It states that if in two triangles ABC and $A'B'C'$, the side AB is equal to $A'B'$, and AC is equal to $A'C'$ and also the angles at A and A' are equal then the angles at B and C would be equal to B^{ι} andC^{ι} respectively. Hilbert proposes that this axiom is not always confirmed and that it is possible to find two triangles in which the sixth axiom of congruence is not valid.

The fifth group of Hilbert's axioms consists of the Postulate of Archimedes. Archimedes' postulate is based on the infinite extendibility of the straight line and assumes a notion of continuity. Hilbert's axiom of continuity is again an independent axiom which he proves by producing a geometry in which all axioms of his system are confirmed except the axiom of continuity.

Hilbert based Euclid's geometry on axiomatic foundations. The postulate of parallels and the Archimedean postulate were the independent parts of this formulation. In this respect, geometrical practice would still be possible if these axioms were excluded from a theory. Their independence suggested the logical possibility of non-Archimedean and non-Euclidean geometries.

The axiomatization of geometry influenced the philosophy of mathematics. When geometry is regarded to be an axiomatic discipline, the historian of geometry provides a history of axioms. As a result, the history of geometry is examining the equivalency of axioms, independence claims and the consistency of geometrical systems.

The influence of the axiomatization of geometry in writing its history can be observed in Roberto Bonola's *Non-Euclidean Geometries* (first English edition was published in 1912). Bonola's work provides a significant analysis of the geometrical works in the western world including original works of some geometers. However, the effect of the axiomatic ideas on geometry persists throughout this work. As I have mentioned before Jeremy Gray terms Bonola's account to be an example of the "standard account" on the progress of geometry. The standard account examines the axioms of geometrical systems, together with the demonstrations of geometers and the results of geometrical studies (Gray, 1998, p. 58). Bonola provides alternative proofs for the theorems of various geometers in order to exclude the assumption of the notion of continuity or the application of the Archimedean postulate in demonstrations. *Non-Euclidean Geometries* ends with a section on the impossibility of proving the parallel postulate (Bonola, 1955, p. 177). Bonola states that that the long lasting efforts to prove parallel postulate did not bring out any success, suggests the impossibility of demonstrating this postulate. He regards the works of Gauss, Lobachevski and Bolyai to be the results of working without the parallel postulate, and to be systems that are free from any contradictions. Consequently, according to Bonola, the parallel postulate cannot be demonstrated since it is independent from the first principles of geometry. Therefore, geometrical systems can be constructed on the negation of the parallel postulate, while keeping the rest of Euclid's postulates.

If geometrical systems are axiomatic disciplines then the notion of consistency is a required aspect for them. In this sense, the idea of consistency of geometrical systems is constantly visited in Bonola's account:

> Still, though it failed in its aim, Saccheri's work is of great importance. …the fact that he did not succeed in discovering any contradictions among the consequences of the *Hypothesis of the Acute Angle*, could not help suggesting the question, whether a consistent logical geometrical system could not be built upon this hypothesis… (Bonola, 1955, p. 43).

In terms of an axiomatic outlook their consistency should be provided for the possibility of the application of the new geometries. As a consequence, the significance of Beltrami's work is characterized to show "…from the properties of surfaces of constant curvature, that the chain of deductions from the three hypotheses regarding the sum of the angles of a triangle must lead to logically consistent systems of geometry" (Bonola, 1955, p. 139).

Since Bonola regards geometry as a logical chain, his account evolves around logical aspects. Bonola provides facts from the history of geometry without making sufficient interpretations on them. The standard account on the history of geometry does not seem to be interested in assigning any meaning to the works of geometers or provide them a reasonable place in geometry's progress. Bonola's work is more like a geometry book organized in a chronological order. The rare interpretations on geometrical studies appear by means of the desire to provide theories with consistency – which can be observed from my quotations from Bonola's book, related to Saccheri and Beltrami. Once the progress of geometry is regarded to be axiomatic, the interpretations distort the meaning of geometrical studies or assign them meanings that were not present in the first place. Standard account's interpretation of Beltrami's work is in terms of the relative consistency of geometries.

An axiomatic history emphasizes that since Beltrami put forward a map between the old and the new geometries, his work yielded that Euclidean and non-Euclidean geometries are relatively consistent. Since whatever is done in Euclidean geometry can also be performed in non-Euclidean geometries by means of Beltrami's model, the inconsistency of the new geometries would imply that Euclid's geometry is also inconsistent. Therefore, Beltrami's model supplies the new geometries with consistency which is required in axiomatic systems.

Klein's work is also important for a standard account of the history of geometry from the same point of view. The projective interpretation of non-Euclidean geometries owes its significance to the relations it sets forth between the hyperbolic, elliptic, parabolic and Euclidean geometries – which again implies their relative consistency.

The required consistency of the new geometries for the axiomatic approach could not be ascertained with the exclusion of the parallel postulate, since the new geometries were still open to future contradictions. However, the notion of relative consistency was able to solve the problem.

If geometry is an axiomatic discipline and its progress can be understood by examining axioms, then history of geometry is a linear continuum and it can be written as a bunch of results. If this is the case, then non-Euclidean geometries must have appeared as a mere step in this continuum. On the other hand, historical facts confirm that there are gaps between major geometrical works and they do not flow from one another in an axiomatic sense. An account on the progress of geometry should provide reasons for geometrical changes. A comprehensive account should leave aside examining axioms and concentrate on the different approaches of geometers. In other words, it is more likely to provide a meaningful account when the different intentions of geometers, the various types of stating problems and the application of different mathematical methods were taken into consideration. It was suggested by Gray to focus on "…the mathematical methods and intentions of the actors in the historical drama" (Gray, 2003, p. 170).

Gray states that the history of geometry was not an investigation of axioms, since the application of the parallel postulate and the idea of extending straight lines indefinitely were not left aside because of their logical independence (Gray, 1998, p.

58). In this respect, treating the progress of geometry as if it was axiomatic is not supported by history itself.

Although the standard account ignores the intentions and different reasoning of geometers, it does catch up with the various mathematical methods that appeared through the course of geometry. Bonola's account divides the history of non-Euclidean geometries into three periods. The first period is entitled as "The Forerunners of Non-Euclidean Geometry" and includes the geometrical works in the 18th century, starting from explaining Saccheri's contributions and bringing its subject until Gauss' studies. The following chapter in Bonola's account is "The Founders of Non-Euclidean Geometry" and it includes the studies of Gauss, Lobachevski and Bolyai. Finally, the third period includes Riemann, Helmholtz and the studies by means of projective geometry whose title is "The Later Development of Non-Euclidean Geometries". Gray puts forward that such a classification is in agreement with both the evident chronological divisions and the changes in mathematical methods. Gray explains this fact by proposing that "…in the eighteenth century Saccheri and Lambert used classical geometry; in the early nineteenth century Bolyai and Lobachevskii used analysis; in the mid-nineteenth century Riemann and Beltrami turned to the techniques of differential geometry" (Gray, 2003, p. 168). However, being in agreement with the various mathematical methods that were applied in geometry is required yet not sufficient for providing meaningful explanations on the progress of geometry.

The comparison of the expressions on the problem of parallels in Bonola's and Gray's accounts may give an idea about the role of "meaningful explanations" in understanding the progress of geometry. It is a popular fact that the discovery of non-Euclidean geometries is rooted in the ancient times when geometers were troubled with the parallel postulate. By means of Gray's account we can learn why this problem was highly disturbing for the world of geometry: "Congruence" was a basic concept in Greek geometry and two figures are said to be congruent if they can coincide (Gray, 2003, p. 26). The demonstration of the congruence of two figures would be by moving one until it coincides the other. The motions that were performed in Greek geometry were translations, rotations and reflections which are possible by another basic concept of this tradition, that is, parallelism. By means of parallel lines angles could be transported and similarity could be obtained. In a space, which is apparently assumed to be homogeneous, most of the geometrical constructions were relying on the idea of parallelism. Euclid's geometry provided a safe basis for geometrical constructions by placing the idea of parallelism among the guides of construction of this geometry. It can be observed that, the problem of parallel lines was of great significance since it was the concern of not all but many of the performed constructions. And naturally, geometers were driven to save the basis for the application of the idea of parallelism.

While Gray's account is concerned with what was the actual problem about the parallel postulate, Bonola merely puts forward that the parallel postulate was not evident and needed a proof even for the earliest commentators on Euclid's text (Bonola, 1955, p. 2). Afterwards, he directly introduces the various reformulations of the parallel postulate and the proofs that are based on them. Bonola does not need to bring forward any explanations of what was problematic about the mentioned postulate, since at the end he is going to point out the logical independency of it as causing problems. In this sense, in an axiomatic approach what is considered regarding the parallel postulate is that it is not capable of any logical proof; instead of its not being evident or necessarily true or whether it could be applied to the external world.

The axiomatic account, by concentrating on the history of axioms, misses some useful aspects which can be examined for the sake of comprehensive explanations on geometry's evolution. A proper account on the history of geometry should consider the mathematical methods together with why the path of a geometrical study is constructed as the way it is. In this sense, an account would successfully explain how geometry proceeded through the non-Euclidean geometries, without distorting the actual history. Gray's endeavor to emphasize the mentioned aspects of the history of geometry can be illustrated by the following explanations:

Saccheri and Lambert were studying in terms of classical geometry. The appearance of the first non-Euclidean geometries had to wait until the hyperbolic trigonometric functions were applied. Janos Bolyai and Lobachevski concluded that a geometry which is not Euclidean was possible for space, in terms of the trigonometric formulae. The two geometers were working directly with three dimensional non-Euclidean space and this was the end of embedding a non-

Euclidean plane in a three dimensional Euclidean space. Gauss considered geometry intrinsic to a surface and there were studies on surfaces of constant curvature. However, these works were carried out by the hyperbolic formulae without making any connections to non-Euclidean geometries. It was Riemann who could appreciate the intrinsic nature of non-Euclidean geometries, by means of differential geometry. Riemann and Beltrami formulated geometry in local terms, and based the earlier studies to solid grounds (Gray, 2003, pp. 170-171).

The previous paragraph is an example of how the history of geometry can be written instead of setting forth a sequence of axioms and results. The axiomatic account can be criticized from various respects, such as ignoring the requirement of providing reasons for some obvious problems in the history of geometry. In conclusion, the axiomatic progress of geometry is not in agreement with historical facts and is unable to assign proper meanings to the aspects of geometry's progress, as well as proposing misleading accounts that fulfill its major concerns.

CHAPTER 3

DARÜLFÜNUN KONFERANSLARI

The present chapter consists of a summary of Salih Zeki's lectures on the discovery of non-Euclidean geometries. In the preface of *Darülfünun Konferansları*, Salih Zeki informs the reader that the mathematical developments in the nineteenth century can be evaluated not only by means of their mathematical value but also philosophically. The lectures in Darülfünun are constructed in accordance with such a view. Salih Zeki presents a significant amount of geometrical knowledge and also considers philosophical debates concerning geometry. In the first five lectures most of the effort is devoted to examine the geometries of Lobachevski and Riemann, Cayley's studies in terms of projective geometry and Klein's interpretation of the relation between Euclidean and Non-Euclidean geometries. As an overall description of these lectures, it can be said that Salih Zeki's endeavor is to clarify two philosophical problems regarding the progress of geometry which are the problem of space and the problem of the principles of geometry. He provides the required grounds for the discussion of these two problems in his lectures and ends his account on the history of geometry by Poincaré's (1854 – 1912) views on the two mentioned problems.

I prefer to focus on Salih Zeki's account in this chapter and leave the evaluation of his ideas on the history of geometry to the following chapter.

3.1 Lecture 1 [27 November 1914]

Salih Zeki starts his first lecture by mentioning the significance of Euclidean geometry and putting forward that it was applied to the external world for centuries without any hesitation. However, this geometry was not as strong as it was regarded to be and had a weak point. The problematic point about Euclid's geometry was the parallel postulate [muvazat mevzu'esi]. This postulate proclaimed that "…from a point on a plane only one straight line can be drawn that does not intersect another straight line on the same plane in both directions" (S.Zeki, Darülfünun Konferansları, 1331, p. 4). Salih Zeki explains that such a postulate would mean that "…from a point on a plane only one straight line can be drawn parallel to another straight line on the same plane…" (S.Zeki, Darülfünun Konferansları, 1331, p. 4). The parallel postulate can be objected firstly because of not being related to the rest of the postulates, and secondly since the non-intersection of lines does not necessarily mean that they are parallel lines. The first four postulates of Euclid's geometry were:

1. A straight line can always be drawn between two given points.
2. A straight line can be prolonged indefinitely[8] in both directions.
3. Two straight lines with two common points necessarily coincide between these two points.
4. Given its origin and radius a circle can always be drawn (S.Zeki, Darülfünun Konferansları, 1331, p. 5).

Salih Zeki characterizes the parallel postulate to be independent of these four postulates. In addition, it is possible to consider straight lines that converge but do not intersect each other when prolonged in both directions, which means that non-intersection is not a distinguishing property of parallel straight lines.

According to Salih Zeki, many geometers worked in favor of Euclid's geometry, that is, their purpose was to strengthen the weakness that stems from the parallel postulate throughout history; yet none could achieve anything. Among these geometers Legendre is worth to be mentioned since Salih Zeki regards his work to be the last step before the non-Euclidean geometries were discovered. Legendre should be appreciated since he set off to prove a direct consequence of the parallel postulate without applying this postulate. Legendre wanted to prove that the sum of the interior angles of a rectilinear triangle was equal to two right angles. His attempt

[8] The word that is used by Salih Zeki is "ila-gayri'n-nihaye" and I prefer to translate it as "indefinite". It can be observed from one of Salih Zeki's articles which was titled as "Namütenahi" (Saraç, 2001, pp. 38-45) and throughout *Darülfünun Konferansları* that the Ottoman phrase he uses for "infinite" is "namütenahi".

ended with proving that the angle sum of a triangle cannot exceed two right angles. However, he was not able to prove that this sum could not be less than two right angles. Legendre achieved to show that if this sum was equal to two right angles in one triangle then it would be so for all triangles. Still, he could not put forward a triangle with the angle sum of two right angles.

Salih Zeki affirms that it was after Legendre's failure that the world of geometry was convinced that the Euclidean postulate required to be proved. He regards the parallel postulate not to be a geometrical proposition at all. A geometrical proposition would be provable but the parallel postulate was rather approximately true and seems to be borrowed from experience. Salih Zeki asserts that the angles of any triangle can be measured to be equal to two right angles, whereas the geometrical proof of that would necessitate the application of Euclid's parallel postulate.

In Salih Zeki's account, Legendre's failure is set as a reason for the more ambitious attacks against Euclid's geometry. And finally, Gauss was the geometer to defeat the Euclidean tradition. However, Lobachevski and Bolyai were the ones that appeared to be the first successful geometers in this field in the history and that was the time when the new geometries were brought out.

The lecture in Darülfünun provides the following expression and the claim is that Gauss, Lobachevski and Bolyai stated it independently.

> If the Euclidean postulate can be logically derived from the other postulates and axioms, in a geometry that is constructed upon the negation of this postulate and the affirmation of the rest, it would be impossible not to face any contradictions. (S.Zeki, Darülfünun Konferansları, 1331, p. 6)

These three geometers departed from this idea and constructed geometries on the negation of the parallel postulate and faced no contradictions.

Lobachevski examined the theorems that are independent of the parallel postulate in Euclidean geometry and provided a definition of parallelism which stated that: Straight lines through a given point on a plane are of two groups due to a given straight line. While one group intersects the given straight line, the other group does not. The limiting line of the two groups is the parallel straight line through the given point to the given straight line (S.Zeki, Darülfünun Konferansları, 1331, p. 8). Later on, Lobachevski put forward a system which was entitled as *Pangéométrie* [Hendese-i Cami'a] and which was a surface geometry.

Salih Zeki provides the properties of Lobachevski's geometry in order to illustrate how this new geometry differs from Euclid's geometry. He informs the audience about the angle sum of a triangle in this new geometry and how it changes with the length of the sides. The notion of the equidistance of parallel lines does not exist in this geometry and it is not the case that a circle passes through every three points that are not on a straight line. Also, there is no concept of similarity in Lobachevski's geometry unless two figures are congruent.

In *Darülfünun* Konferansları, explaining the geometry of Janos Bolyai is regarded to be unnecessary since Salih Zeki considers this geometry to include nothing different than Lobachevski's geometry.

It is Gauss who put forward the fundamentals of a non-Euclidean geometry in1792, long before Lobachevski and Bolyai. The letters between Gauss and Schumacher (1780 – 1850) are taken to be the evidence for Gauss' success. In a letter to Schumacher in 1821 Gauss introduced some non-Euclidean ideas. This letter is quoted by Salih Zeki, in which Gauss affirms that: Non-Euclidean geometry includes no contradictions. Its conclusions may seem weird at a first glance but this is only because Euclidean geometry has long been accepted to be absolute. Non-Euclidean geometry requires equivalence for any similarity of figures. An angle of an equilateral triangle differs from 60 degrees and angles are dependent on the length of the sides of a triangle. While Euclidean geometry includes nothing absolute, this is a distinguishing property of non-Euclidean geometry.

In his letters to Schumacher and Wolfgang Bolyai, Gauss declares that he agrees with Lobachevski's and Janos Bolyai's ideas. However, he regards the new geometries to include nothing new for him, since he has been thinking in the same grounds for a long time before the works of Lobachevski and Janos Bolyai were published.

Salih Zeki's conclusion on the first non-Euclidean geometries is that those geometries were constructions on the negation of the parallel postulate, and they emerged from the aim of proving that the parallel postulate is logically independent from the other axioms and postulates of Euclid's geometry. When Lobachevski and Bolyai were working with their geometries they checked out whether there were any contradictions among the consequences. There were no contradictive results but the possibility that contradictions may occur among the results of non-Euclidean geometries in the future persisted. Later on this possibility was going to be exhausted by Beltrami.

Beltrami is introduced in Salih Zeki's first lecture but he does not explain this geometer's studies yet and pays attention to the contributions of Riemann. Riemann's work is accounted for while a parallel evaluation of Helmholtz's studies is provided.

Riemann and Helmholtz had been occupied with the same subject independently. The aim of the two geometers was not only to deal with the parallel postulate but to analyze all geometrical postulates. They considered space as a manifold [zu-kesirü'l-enva'] or as an aggregate of magnitudes [mecma'-i mekadir] with a curvature. Their common purpose was to show that measure relations were possible for manifolds and that the real space or the geometrical space was a simple form of a manifold.

Following Klein's categorization of the studies in the course of non-Euclidean geometries, Salih Zeki proposes that the era that started with Riemann differed from the previous non-Euclidean studies. Gauss, Lobachevski and Janos Bolyai constitute the first era of non-Euclidean geometries and the purpose which prevailed in this era was already explained by Salih Zeki. The second era that starts with Riemann was not dealing with the theory of parallels or the non-Euclidean geometries directly. Its subject matter was "space". The studies of Riemann and Helmholtz are of philosophical significance as well as mathematical, which derives Salih Zeki to examine the studies of the two in detail.

An examination of the hypotheses put forward by Riemann constitutes the rest of the first lecture in Darülfünun and it continues in the second lecture too. Salih Zeki quotes and explains Riemann's expressions and tries to clarify the notion of curvature by means of Gauss' studies. Riemann's aim is described to provide a logical definition of what is generally called space and which is the subject matter of Euclid's geometry. In this sense, Riemann was looking for a more general notion than space. He regarded space to be a special case of an aggregate of magnitudes. And he handled the subject by assigning values to the elements and points of such an aggregate. The notion of magnitude was not limited to the capabilities of increasing and decreasing, but could be measured and determined in various ways. The whole of this quantitative determination constituted the aggregate of magnitudes which Riemann called a “manifold” (S.Zeki, Darülfünun Konferansları, 1331, p. 14). Such a general notion or manifold was necessary for the comprehension of magnitudes. If it was possible to pass from one type of determination to the other continuously the manifold would be continuous; if not the manifold would be discrete.

Salih Zeki’s presentation of the properties of a manifold continues with the notion of n-dimensional manifolds . Measure-relations in a manifold of n dimensions require that quantities or length of lines are independent of their positions. It is stated by Salih Zeki that the idea of quantities independent of their positions, brings the next notion which was specified by Riemann: curvature.

Gauss applied curvature to surfaces. However, Riemann was applying it to manifolds of n dimensions. The curvature of a surface and the curvature of a manifold were different notions, and in the case of a manifold curvature should be regarded as its “constant”. Salih Zeki explains how the radius of a circle and its curvature are related: Curvature of a circle is proportional to the inverse of its radius. While the curvature of a circle is constant, it may vary on a curve from point to point. The determination of the curvature at one point of a curve would be by the same method for both curves on a plane and three dimensional curves. Curvature at this point would proportional to the radius of a circle passing through it and two infinitesimal points, namely osculating circle [daire-i mukterine] (S.Zeki, Darülfünun Konferansları, 1331, p. 16).

When a surface is considered, in order to determine its curvature at point, a tangent plane through this point is required. The number of the circles through this point and its infinitesimal points would not be limited to one as in the case of a curve. In other words, there would be an infinite number of such circles since this point is on a surface. And each circle is on a planar section which is perpendicular to the tangent plane, namely a normal section [makta’-ı nazımi]. However, the radii of these circles would be bounded by a minimum (r_1) and a maximum (r_2) value which are the principal radii of curvature [nısf-ı kutr-u inhina-i asli] (S.Zeki, Darülfünun Konferansları, 1331, p. 17). The two circles with the radii r_1 and r_2 would be on two normal sections that are perpendicular to each other. The Gaussian curvature at this point of the surface would be equal to $\frac{1}{r_1} \times \frac{1}{r_2}$. Salih Zeki states that the product of the inverse of the principal radii would determine more than the curvature of the surface. If the two radii were equal the surface would be a sphere. If one of the two radii was infinite, curvature would be equal to zero which is the case for a cylinder or cone. As well as being positive or zero, curvature can also be negative. The rotation of a hyperbola around its directrix would generate a surface with a negative curvature.

Gauss proved that for two surfaces to be applicable to each other they should have equal curvatures at corresponding points. In this sense, a section of one surface can be generated on the other without any stretching or contraction.

Through the end of his first lecture Salih Zeki introduces the notion of free mobility of surfaces, sections of surfaces or figures which necessitates constant curvature. All the surfaces with constant positive curvature would be applicable to a sphere. If the curvature was negative and constant, the surface would be applicable to a pseudo-sphere. And the surfaces with constant zero curvature are the developable surfaces, that is, applicable to a plane.

Salih Zeki had presented the curvature of curves and surfaces and with these last remarks the first lecture at Darülfünun is finished. The promise for the next lecture is to examine the application of curvature to a manifold.

3.2 Lecture 2 [11 December 1914]

For the application of curvature to a manifold of n dimensions, Riemann regarded the normal section curves [makta'-ı nazımi münhanileri] that are generated for the determination of curvature at a point on a surface to be the properties of this surface

(S.Zeki, Darülfünun Konferansları, 1331, p. 20). A surface was taken to be a two dimensional space, independent of the third dimension of the ambient space. The infinitesimal curves on the normal sections were determined by means of two coordinate axes on this surface. The infinitesimal curve d_s on this surface with the coordinates x and y was described by Gauss as the following:

$$d_s{}^2 = g_1(d_x)^2 + g_2(d_x)(d_y) + g_3(d_y)^2$$

If the curvature of an infinitesimal [asgar-ı namütenahi] part of a geodesic [hatt-ı aksar] was determined by this formula then curvature would also be an intrinsic property of the surface. In this sense, the application of a third dimension would not be required. In a manifold of n dimensions each element or point was described by n continuous variables. And since the distance between two elements or points was a quantity, every distance could be measured by another length. Moreover, the infinitesimal distance d_l between two points was represented by the square root of a quadratic function of the coordinates of these points:

$$d_l = \sqrt{\sum (d_x)^2}$$

This expression constitutes the line element formula [unsur-u hatti düsturu] and Riemann preferred it because of its simplicity, that is, he regarded this formula to be simplest form of a line element (S.Zeki, Darülfünun Konferansları, 1331, p. 21). After introducing the line element, the next step for Salih Zeki is to define the curvature of a manifold.

If it was assumed that a geodesic is determined by a point on it and its direction at this point, or by two points on it with infinitesimal distance in between, then a manifold must have a curvature. Curvature can be directly determined by Gauss' general formula or by Riemann's line element formula.

An intelligible description of the curvature at a point and a given surface-direction through this point would be by accepting the fact that: The geodesics proceeding from a point are determined if their initial direction is given. Therefore, if the given geodesics proceeding form a point are prolonged in all directions the result would be a determinate surface whose curvature at this given point would also be the curvature of the n-dimensional manifold and of the surface element at this point.

Now Salih Zeki can introduce the notion of flat manifoldness since the curvature of a manifold was described. Flat manifoldness is a requirement for applying Riemann's ideas to space. An n-dimensional manifold is flat, if its curvature is zero at every point and all directions. Actually, such a surface would be a special case of manifolds with constant curvature. The common characteristic among manifolds with constant curvature is that they allow the motion of objects and figures without any stretching. And free motion would not be possible if curvature was varying from a point to another. Also, since the measure-relations of a manifold at a point are determined by the curvature at this point, these relations must also be constant. In such a manifold a figure can be constructed from any initial point or may have any position. Riemann reformulated the line element formula for a flat manifold as the following, in which b represents the constant curvature:

$$\sqrt{\sum \left(d_{x_k}\right)^2}\,\frac{1}{1+\frac{b}{4}\sum x_k^2}$$

In order to apply the provided ideas to space, it must be assumed that lines in space are independent of their position and line-elements are expressed by the given formula. These requirements can be fulfilled in several ways. One way is to assume curvature to be zero at every point and in all three dimensions of space. In this sense, if the angle sum of a triangle is everywhere equal to two right angles, the measure-relations of space can also be determined. Or it could be accepted that bodies as well as lines are independent of position in space, like Euclid did. This again would mean that curvature is constant and the sum of the interior angles of a triangle would be true for all triangles. Another option is to regard the length and direction of lines independent of position. All three assumptions would be true for a flat manifold.

Riemann describes space to be of three dimensions and it is an unbounded but a finite [gayr-i mahdud fakat mütenahi] manifold (S.Zeki, Darülfünun Konferansları, 1331, p. 23). Its being unbounded agrees with the external world and this is always confirmed by experience. However, the infinity of space does not flow from being unbounded. On the contrary, if figures were independent of position by means of a positive curvature then space is necessarily finite.

After this general account on Riemann's work and considering the basic notions that were introduced by the geometer, Salih Zeki provides the properties of Riemann's geometry shortly: Riemann constructed a surface geometry in which Euclid's parallel postulate was not the only one to be rejected. His geometry also rejected that two straight lines with two common points necessarily coincide between these two points. In Riemannian geometry straight lines were unbounded but finite and they could not have the property of being parallel. The sum of the angles of a triangle was greater than two right angles. Riemann's geometry was another non-Euclidean geometry and different than Lobachevski's system.

In this way Salih Zeki concludes his explanations about Riemann's contributions to geometry and concentrates on Helmholtz's work. Helmholtz was interested in the origin of space while he was working on the objects in the field of vision. His aim was to figure out which geometrical theorems were based on experience and which theorems were conventional [i'tibari] or consisted of definitions and consequences of definitions (S.Zeki, Darülfünun Konferansları, 1331, p. 24). Helmholtz entered a complex subject: Geometrical figures were not existing bodies and the objects of the external world could hardly represent those figures. Also, geometrical axioms and postulates were not sufficient to generate the notion of space. Helmholtz proposed that any relations other than the magnitude-relations that are possible in the external world could not be considered. Because of visual intuition empirical properties of the external world were regarded to be evident, even if they were not. As a consequence, Helmholtz appealed to analytic geometry which worked with the pure notion of quantity, considering that visual intuition cannot interfere in. And he searched for the analytic properties of space.

Helmholtz constructed a manifold independently of Riemann. Each point of this manifold is determined by n quantities or coordinates which are required for providing properties of space such as continuity and being n-dimensional. These coordinates are continuous variables [suret-i gayr-i münkati'ede mütehavvil] and independent from each other. According to Helmholtz a manifold is of n dimensions or extended in n directions. And if it is going to be regarded as space, then a line element in space in any direction should be comparable to any other line element.

If x, y, z represent the coordinates of the initial point of a line element, then the coordinates of another point with infinitesimal distance to the first one would be $x + dx$, $y + dy$, $z + dz$. The length of such a line element would be determined by a function of dx, dy, dz or the General Pythagorean Theorem [Fisagorat Da'va-i 'Umumisi] (S.Zeki, Darülfünun Konferansları, 1331, p. 26). This theorem did not require any kind of measurement and constituted a distinguishing property of our space from other manifolds.

Riemann regarded this theorem as the simplest form for a line element and added that objects were not subject to any change during their motions in space. For Helmholtz free mobility was a postulate and he concluded the General Pythagorean Theorem by means of analysis.

Both Riemann and Helmholtz based the possibility of measurement on congruence [muvakafat] (S.Zeki, Darülfünun Konferansları, 1331, p. 26). This requirement, that is, congruence was provided by Helmholtz's four postulates. The first postulate was on the continuity of manifolds, and as an instance the continuity of space. Continuity of manifolds was a definition in Riemann's study. The other three postulates were on the existence of a rigid body in motion, freedom of motion and transition and the independence of rotation of a rigid body. These three postulates were not explained in detail in Riemann's work, but they were generally accepted. Helmholtz's ideas show that he assumed that the constituting points of rigid bodies are independent of position, just like Riemann's expressions.

Helmholtz compared space to a system of colors. The medium for measurement is mixture for colors and any quantity-relations cannot be determined unless mixture plays its role. This tool for measurement provides the possibility of comparing one relation with another. In a color system relations are possible between three colors, one of which is the mixture of the other two. Such a relation corresponds to the quantitative-relations in a set of three points on a straight line. In this sense, the color system is more complex than a manifold, since there is a quantitative relation between any two points in a geometric space. The color system and the geometric space are both continuous manifolds, whereas the complexity of the color system makes it impossible to apply the General Pythagorean Theorem or Riemann's general formula.

At this point, Salih Zeki states that what he proposed was the ideas of Riemann and Helmholtz in general. Subsequently, he aims to answer the question of a possible future coming contradiction among Riemann's theorems and his answer is in the negative sense. Riemann's geometry was about figures on a surface of constant positive curvature and such a geometry would not differ from a spherical geometry. In Riemann's geometry Euclid's spherical surface was a plane. The shortest path between two points on a sphere was a section of a great circle, while the other lines between two points would be curves. By mapping Riemann's and Euclid's geometries, Salih Zeki proposes that the theorems and conclusions of Riemann's geometry were not open to any contradictions unless it was accepted that Euclidean spherical geometry includes contradictions.

In accordance with his aim of establishing the legitimacy of the new geometries Salih Zeki accounts for Lobachevski's geometry too. The possibility of any contradictions in the future for Lobachevski's geometry was exhausted when Riemann and Helmholtz put forward the necessity of curvature for surface geometries. By means of the studies of Riemann and Helmholtz, Beltrami interpreted Lobachevski's geometry to be a case of Euclid's geometry. Short after the publication of Riemann's hypotheses it was clear that Lobachevski's geometry belonged to the surfaces with constant negative curvature. Riemann's and Lobachevski's geometries were the application of Euclid's geometry to surfaces with constant positive curvature and constant negative curvature respectively. In other words, Euclid's plane geometry was a special case of both Riemann's and Lobachevski's geometries. If the radius of either the sphere or the pseudosphere was taken to be infinite, curvature would be zero and the surface would Euclid's plane.

The rest of the second lecture in Darülfünun is driven by Salih Zeki's intention of considering the mathematical progress in the 19^{th} century not only from a mathematical stance but also in terms of its philosophical implications. The new geometries were surface geometries and as a result they operated on two dimensional spaces which could be embedded in a three dimensional Euclidean space. However, when three dimensional figures were considered in these geometries there appeared a problem. Three dimensional figures of non-Euclidean geometries indicated the application to the fourth dimension in Euclidean sense; yet the fourth dimension was not comprehensible.

Salih Zeki states that the idea of a fourth dimension resulted in a philosophical debate. And by “philosophical debate” he refers to the empiricist-rationalist dispute concerning space. He provides the general ideas of the two groups on the nature of space rather than on the idea of a fourth dimension, such as objective reality of space according to the empiricists or the rationalist idea that whatever time is with respect to events, space is so with respect to objects.

The empiricist and rationalist expressions are the last remarks in Salih Zeki’s second lecture. At the end of his presentation Salih Zeki mentions the lectures of Helmholtz in the 1870s and sets the plan of his third lecture in Darülfünun to be evaluating Helmholtz’s lectures for the sake of clarifying Riemann’s and Helmholtz’s ideas.

3.3 Lecture 3 [25 December 1914]

In the third lecture Salih Zeki directly starts to present the illustrations of Helmholtz in which the types of surface geometries are compared. The first example is about a plane [sath-ı müstevi] and two dimensional intelligent beings dwelling on it (S.Zeki, Darülfünun Konferansları, 1331, p. 32). If these creatures wanted to measure their space, the resulting geometry would be a two dimensional or a surface geometry. The creatures would be familiar with shortest distances or straight lines and curves. They would draw only one parallel line to another straight line through a point that is not on it. They would come up with many geometrical theorems. This geometry would include triangles, quadrilaterals, polygons and circles. Even ellipses, parabolas, hyperbolas and similarity of figures would be discovered by the two dimensional creatures. However, they would never come up with the idea of a solid body. The similarity of two triangles would be comprehensible for these creatures if and only if the figures could be applied to each other by transition. The similarity of equal but symmetric figures would not be known since this would require picking up one of the figures and putting it on the other one which can only be by means of a third dimension.

There could be other surfaces existing along with the world of the two dimensional creatures, some of them being parallel to the first one. These creatures would not be able to know anything about the parallel surfaces. Even though another surface was intersecting theirs, they would still be ignorant. Passing to another surface which intersects their surface would be impossible for them, since it necessitates the appeal to the third dimension. It would be beyond the comprehension of the planar creatures to think by means of three dimensions.

The same example can be constructed about a spherical surface which would again be two dimensional according to its dwellers. The spherical creatures would compose a geometry of two dimensions, not planar but spherical. Shortest distances would be geodesics. The length of curves would be proportional to their radii and radii would be geodesics too. If two straight lines were prolonged enough they would intersect in both directions. There would be no room for the notion of parallelism in this geometry. The tools for measuring lengths and areas would be a geodesic and a spherical square respectively. Any similar figures could not exist on such a surface since the angle sum of triangle would be dependent on the length of its sides. The spherical creatures would not know anything about a third dimension and as a result they would not be aware of the radius of their surface. It would be natural to end up at the initial point of a path that is followed on a straight line, since it would be taught by experience.

Apart from their inability to think of a third dimension, the spherical creatures would not know that their geometry is not absolute or understand a planar geometry.

If a planar creature was somehow transported to another plane, it would not face any trouble and continue in its habits. On the other hand, when the same process was considered for a spherical creature, there would be two possibilities. If the new surface had the same curvature as the first one, nothing would be new for the spherical creature. However, if the radius of the second surface happened to be different than the radius of the first surface the creature would face serious problems because of being in a surrounding with which its curvature does not agree.

Each surface has a special property, that is, a constant. The creatures in a two dimensional space cannot determine how a quantity it is, but since we live in three dimensional space we that the constant for a sphere is its radius or its curvature.

Salih Zeki continues explaining two dimensional geometries by comparing spheres of different radii. A straight line on a sphere may become a curve on another sphere with a greater radius. A curve on a sphere may be a straight line for another sphere with a smaller radius. However, the transition between spheres of different radii would destroy triangles, squares, and the rest of geometrical figures. Every space has its own figures in agreement with its distinguishing property, that is, curvature.

One last remark before Salih Zeki considers the notion of a fourth dimension is to exemplify developable surfaces. When a piece of paper with a figure drawn on it is folded around a cylinder or cone, the figure would seem different. Though, neither the lengths of its sides nor its angles would be altered as a consequence of the constant zero curvature which is common to the plane, the cylinder and the cone. On the contrary, a planar figure cannot be transported upon an ellipsoid without any change.

When Salih Zeki considers the fourth dimension he sets an analogy between the reaction of a two dimensional creature to the idea of a third dimension, and the difficulty that a three dimensional creature would have in thinking of a fourth dimension. The fact that two dimensional creatures would not be able to think in terms of three dimensions may seem ridiculous to us, since we already live in three dimensions. In the same way, if there were creatures living in a four dimensional space, the difficulty in understanding Lobachevski's and Riemann's geometries would be absurd to them.

An inductive way of reasoning is suggested by Salih Zeki for understanding the position of point in a four dimensional space. A point on a straight line or curve is determined by its distance to a fixed point on this line. The position of a point that moves on a surface is due to its distance to two fixed lines on this surface. And in our space the position of a point is given by means of three fixed lines. This similarity suggests that the position of a point in a four dimensional space is determined by four three dimensional spaces.

Salih Zeki names the four dimensional space as Hyperspace [mekan-ı zaidi] (S.Zeki, Darülfünun Konferansları, 1331, p. 39). Afterwards, he describes Euclid's three dimensional space.

In Euclid's geometry a right triangle can be rotated in order to generate a cone. Furthermore, the generated cone can be cut by a plane and it is possible to perform geometrical measurements on the resulting surface section on this cone. Through this process it is assumed that the cone shows no resistance against our actions. The freedom of geometrical processes is by means of a property of Euclidean space; Euclid's space is homogeneous [mütecanis] (S.Zeki, Darülfünun Konferansları, 1331, p. 40). In Euclidean space a solid can move in any direction without any resistance. And any two points can be joined by a straight line. Together with the notion of homogeneity which is required for free motion, Salih Zeki considers the effect of a change in curvature on motion. The Euclidean straight line is not the shortest path anymore if it was transported to a non-Euclidean space. The straight lines on non-Euclidean surfaces are determined by curvature and such surfaces would resist a Euclidean straight line.

Straight lines in the spaces that were considered by Riemann and Helmholtz are geodesics on a sphere or pseudosphere. The planes in these spaces are spherical or pseudospherical surfaces. The variety of straight lines and planes is unbounded, and it is due to the variety of the spaces of constant curvature. The straight lines and planes in non-Euclidean geometries and for instance Riemann's geometry are curves and surfaces in Euclid's geometry. Salih Zeki provides some further illustrations to point out that there is a variety of straight lines and planes in geometry:

If space was merely a straight line, the only possible constructions would generate line sections. As a result, it would be impossible to rotate a line section since rotation requires two dimensions. When a straight line is half rotated around its mid-point, it coincides with its initial position. If the same procedure was applied on a curve the result would be a symmetrical curve through a tangent line to the initial curve at its mid-point. Still, there is a way for a curve to coincide its initial position after a half rotation. Rotating a curve around its diameter would provide the desired conclusion but this can only be possible in three dimensions and not on a plane. However, if a curve was on a sphere provided that the curve and the sphere have the same radius, then such a rotation would be possible in two dimensions. A geodesic on a sphere would coincide with its initial position by half a rotation around its mid-point. The relation between a straight line and a plane does not differ from the relation between a geodesic and a spherical surface.

The illustrations on different types of straight lines and planes are followed by Salih Zeki's explanation on the necessity of the notion of curvature for space. The case on the rotation of straight lines and curves shows that a two dimensional spherical surface corresponds to Euclid's three dimensional space. In other words, a two dimensional space with a curvature acts like three dimensional which is true for both the sphere and the pseudosphere. Salih Zeki considers rotation in three dimensions by means of a plane which is half rotated around a straight line on it. After the half rotation each half of the plane would be placed in the initial position of the other half. However, a figure on one of the halves would gain a symmetrical position due to its initial position. If the same procedure was to be held on a spherical surface in order to get symmetrical figures, the application to a fourth dimension would be required. If there were a fourth dimension a spherical surface could also be turned inside out like the plane. A figure on a spherical surface would gather a symmetrical position to its initial one if it could be half rotated around a straight line of this surface. Such a procedure is not comprehensible in the case of a closed sphere, since we live in a three dimensional space without a curvature. In the case of two cylinders which are symmetrical by means of a plane, it is possible to determine the equality of the two bodies, but they cannot be rotated to coincide. Salih Zeki states as a justification that Newcomb (1835 – 1909) affirmed that a fourth dimension would make it possible to turn a sphere inside out.

If Euclid's geometry has a unique surface, that is, a plane that can be rotated around a straight line on it, then the non-Euclidean geometries should have their unique surfaces with the same capability or their own planes. It is accepted that a plane can be rotated in the mentioned way and a plane is nothing but a surface with an infinite radius.

Salih Zeki concludes his third lecture by putting forward the distinguishing properties of the three geometries in terms of triangles. The angle sum of a spherical triangle is greater than two right angles, but when the radius of the surface is infinite this sum would be equal to two right angles and the curvature would be zero. For an angle on a pseudosphere this angle sum is less than two right angles and again in the case of an infinite radius or when curvature is zero, triangles behave like Euclidean ones. In this sense, there are three types of trigonometry: Spherical trigonometry, Pseudospherical trigonometry [müsellesat-ı küreviyye-ı kazibe] and plane trigonometry (S.Zeki, Darülfünun Konferansları, 1331, p. 45). The difference between them is in terms of their theorems for the angles of triangles: If x, y, z are the angles and x^{ι}, y^{ι}, z^{ι} are the opposite sides, then

$\frac{\sin x}{x^{\iota}} = \frac{\sin y}{y^{\iota}} = \frac{\sin z}{z^{\iota}}$ is true for a Euclidean triangle,

$\frac{\sin x}{\sin x^{\iota}} = \frac{\sin y}{\sin y^{\iota}} = \frac{\sin z}{\sin z^{\iota}}$ is the case for a triangle in Riemann's geometry, and

$\frac{\sin x}{\sinh x^{\iota}} = \frac{\sin y}{\sinh y^{\iota}} = \frac{\sin z}{\sinh z^{\iota}}$ is the theorem for a triangle in Lobachevski's geometry.

In a footnote Salih Zeki points out that $\sinh x^{\iota}$ means the hyperbolic sine of x^{ι} or the hyperbolic sine of the curve x^{ι} (S.Zeki, Darülfünun Konferansları, 1331, p. 46).

Lastly, as the topic of the following lecture Salih Zeki promises to examine the situation of the non-Euclidean geometry back at the beginning of the 20th century.

3.4 Lecture 4 [8 January 1915]

In his fourth lecture Salih Zeki concentrates on the non-Euclidean geometries, mainly in terms of the contributions of Cayley (1821 – 1895) and Sophus Lie (1842 – 1899). He examines Cayley's "A Sixth Memoir upon Quantics" [Kemiyyata Da'ir Altı Muhtıra (S.Zeki, Darülfünun Konferansları, 1331, p. 47)] and Lie's studies on the transformation groups [Zümre-i Mütemadiyeler Nazariyyesi (S.Zeki, Darülfünun Konferansları, 1331, p. 56)].

Salih Zeki characterizes Cayley's aim to base all considerations concerning space on purely projective principles. Cayley transformed geometrical terms such as distance and angle to projective grounds. Elementary geometry was dealing with magnitudes in terms of quantities and Cayley was looking for a more general system.

Cayley's geometry constituted the third era in Klein's classification of the history of the new geometries, which was introduced in the first lecture by Salih Zeki. Cayley showed that a purely projective interpretation of distance can be provided by means of circular points or a straight line at infinity. Afterwards, he described the distance between two points by the inverse sine or the inverse cosine of a function of the quantities which he called "projective coordinates" [kemiyyat-ı vaz'iyye-i irtisamiye]. In this sense, he changed the measurable properties of figures into projective properties with respect to a conic [mahrutiyye] that he called the "absolute" (S.Zeki, Darülfünun Konferansları, 1331, p. 47).

In analytic geometry, circular points at infinity constitute an imaginary conic. When Cayley generalized the notion of distance in terms of such a conic, he also proved that the two dimensional geometry which was obtained by projection based on this conic is a spherical geometry. The geometry provided by Cayley is nothing but Riemann's geometry. If the conic was real, the result would be Lobachevski's geometry. However, Salih Zeki thinks that, since Lobachevski's name is never mentioned in his work, Cayley does not know the non-Euclidean geometries.

It was Felix Klein who put forward the connection between the projective theory of distance and the non-Euclidean geometries. Klein classified types of geometry into four groups in terms of the possible properties of the absolute curve that was the basis for projection. If projection was based on a real conic, the result would be Lobachevski's two dimensional geometry. Klein called such a geometry "hyperbolic geometry" [hendese-i za'idiye]. If the conic was imaginary, projection would generate a geometry similar to Riemann's spherical geometry or Helmholtz's geometry. The name provided for this second type of geometry was "elliptic geometry" [hendese-i nakısiye]. A third type of geometry was provided if the conic was reduced to a pair of imaginary points, which was "parabolic geometry" [hendese-i mükafiye]. Lastly, if the imaginary points were circular points at infinity, projection would correspond to Euclid's geometry.

All these geometries were derived from figures on a Euclidean plane; the only modification was on the definition of distance between two points on this plane. Therefore, Klein was not dealing with space but with the definition of distance. Salih Zeki adds that the definition of distance between two points was conventional [i'tibari] and also the new geometries were conventional and qualitative [keyfi] (S.Zeki, Darülfünun Konferansları, 1331, p. 48).

What Salih Zeki regards to be interesting about the connection between the various geometries is that when the non-Euclidean geometries were accepted to be conventional and qualitative in this way, Euclidean geometry was included among them too. Cayley's way of projection was always applicable on Euclidean plane. In this sense, the distance between two points in either of the non-Euclidean geometries could be transformed to Euclidean space by means of Cayley's interpretation. The distance between two points in non-Euclidean spaces was equal to a "hyperbolic metric" or a "spherical metric" in Euclidean space.

Salih Zeki shortly introduced Beltrami's contributions to geometry in his first and second lectures in Darülfünun. In the first lecture, Beltrami was mentioned to be the geometer who exhausted the possibility of any contradictions in the non-Euclidean geometries. In the second lecture Salih Zeki pointed out that Beltrami had interpreted Lobachevski's geometry to be a case of Euclid's geometry. The fourth lecture includes a short reference to Beltrami's work too, which in a way explains the previous two references. What exhausted the possibility of any future contradictions among the theorems and consequences of non-Euclidean geometries is Beltrami's interpretation on their connection with Euclidean geometry. Beltrami established that the plane in Euclidean geometry corresponds to surfaces of constant curvature – either positive or negative – in other spaces. In other words, the planes in these geometries are Euclidean planes when distance is considered with respect to hyperbolic or elliptic metrics. By means of Beltrami's interpretation, it could be concluded that one and only one theorem in Euclid's geometry corresponds to each theorem in non-Euclidean geometries.

The next concern in Salih Zeki's lecture is to provide Klein's interpretation on practicing geometry by projective means. Klein put forward that transforming quantitative geometry to projective geometry was with respect to a conic. However, Lobachevski's geometry was the only geometry that could be derived if this conic was real. The derivation of spherical geometry or Euclid's geometry required either an imaginary conic or circular points at infinity respectively. In this sense, imaginary numbers were included in Euclid's geometry. The notion of imaginary numbers carries Salih Zeki's lecture into a philosophical debate.

Salih Zeki proposes that imaginary numbers are indispensable in analytic geometry and they are also important in physics. Without the imaginary numbers, a big deal of progress in science would not be possible such as Hamilton's theory of quaternions or the Maxwell's electromagnetic theory of light.

In spite of their importance in various branches of science, some philosophers were against their application. Salih Zeki quotes some of Russell's (1872 – 1970) ideas as an example for the philosophers that were not willing to accept the significance of the imaginary numbers.

Russell claims that imaginary numbers do have a significant role in geometry, especially because Cayley transformed quantitative geometry into projective geometry with respect to circular imaginary points. However, it is not possible to discuss the philosophical significance of imaginary numbers, since there is no philosophical theory that corresponds to the application of imaginary numbers.

Salih Zeki states that Russell was making such a claim although he had accepted the possibility of non-Euclidean geometries and non-Euclidean space. Afterwards he continues presenting Russell's ideas:

In a three dimensional Euclidean space a point can be determined in terms of Descartes' three coordinate axes. These coordinates may vary between $-\infty$ and $+\infty$, but each of them corresponds to one and only one real point in space. Space is a collection of variable three coordinates and it cannot correspond to an aggregate of imaginary quantities. Imaginary numbers may be advantageous, yet this advantage is dispensable. Even if they are applied in the positive sense in geometry, imaginary numbers are merely tools. They are included in geometry by means of real numbers and the consequences of their applications are interpreted by means of real numbers too. Moreover, since their interpretations are made in algebraic terms, they cannot include the notion of space. Another point is that a circle is a closed curve and it cannot emerge from points at infinity. To prolong every bounded and closed figure to infinity just because circular points at infinity correspond to the equations of all circles is absurd.

Salih Zeki, as a defense against Russell's ideas, states that something quantitative does not need geometry. In terms of geometry every function would have a derivation [müştakk], since a function is represented by a curve and it is possible to draw a tangent to every curve (S.Zeki, Darülfünun Konferansları, 1331, p. 55). Therefore, relying merely on geometry may bring out some false conclusions.

Once again, Salih Zeki emphasizes that geometrical concepts such as imaginary numbers or circular points at infinity are conventions for the convenience of speech. And being conventional is not limited to some particular notions; on the contrary Euclidean and non-Euclidean geometries are all conventional. However, that the notions of geometry were conventional was also objected just like the imaginary numbers. Salih Zeki asserts that this second objection was answered by Poincaré, but for the sake of following a chronological order he first evaluates the work of Sophus Lie.

The title provided for Sophus Lie's work is "Theory of Continuous Groups" [Zümre'-i Mütemadiyeler Nazariyyesi] (S.Zeki, Darülfünun Konferansları, 1331, p. 56). Salih Zeki explains what a continuous group is and then considers the application of Sophus Lie's theory to geometry. A series of transformations [silsile-i tahvilat] can be constructed upon a finite number of free variables such as x_1, x_2, $\ldots x_n$ by replacing these variables with other free variables (S.Zeki, Darülfünun Konferansları, 1331, p. 56). A group is determined by a relation between two arbitrary transformations in this series: If the transition from one transformation to the other corresponds to only one transformation in the series of transformations, then this series is a group. And if infinitesimal transformations are possible in this series than it is a continuous group.

When he explains the application of Sophus Lie's theory to geometry Salih Zeki proposes that two successive motions of a figure can always be obtained by one motion in geometry. A motion is constituted by a series of infinitesimal motions and it is a change in the coordinates of the constituting points of the figure with respect to the coordinate axes. Each transformation in the motion of this figure is a continuous group of transformation. In this sense, the equivalency of two figures – which constitutes the basis of quantitative geometry – is reduced to congruence of these two figures. In other words, the equivalency of them is determined by the application of one of them to the other, or by the theory of continuous groups which define the whole motion.

Sophus Lie regarded space as a manifold just like Helmholtz, assigned three dimensions to it and determined a point in it by three coordinates. In this manner, the motion of a point was represented by the following transformation group:

$$x^{\iota} = \varphi(x, y, z)$$

$$y^{\iota} = \psi(x, y, z)$$

$$z^{\iota} = \chi(x, y, z)$$

Helmholtz's four postulates were also accepted by Sophus Lie. He added that space is three dimensional and proposed that the possible geometries are the Euclidean and the non-Euclidean geometries. One more postulate was accepted by Sophus Lie in addition to Helmholtz's postulates. He assigned six independent motions to bodies, the number of which decreased to three if the body was considered with respect to a fixed point.

After presenting Sophus Lie's postulates, Salih Zeki asserts that he cannot examine the arguments of this mathematician in his lecture and moves on to the conclusions of Sophus Lie without explaining his arguments in detail. In two dimensional geometries, if the postulate of free mobility was true for all space, the only existing groups would be corresponding to Euclidean and non-Euclidean geometries, in agreement with Helmholtz's three postulates. Yet, if this postulate was only accepted locally then there is another group which corresponds to Helmholtz's three postulates. In this group the path of a rotating point is not closed; it is a logarithmic entity. Helmholtz's fourth postulate must be accepted, in order to exclude this possibility.

Furthermore, Sophus Lie's conclusions indicate that if in three dimensional geometries free mobility was accepted locally, then the following two situations should be distinguished: Either motion would be free in this part of space or it would be restricted. If motion was free, then while one point of a body is fixed, the rest can move freely. In this sense, Helmholtz's fourth postulate is not necessary. The first three postulates would correspond to Euclid's geometry. On the other hand, when motion is restricted, points on a line are only allowed to move on this line, while one point of a body is fixed. In this case, the number of possible groups increases and Helmholtz's fourth postulate is necessary.

According to Salih Zeki, the theory of Sophus Lie can be regarded as the end of the problem concerning the principles of geometry [mebadi'-i hendese]. Salih Zeki considers geometry to be nothing but the application of the theory of groups. However, he adds that this theory was objected too; since according to some philosopher it was distorting the notion of homogeneity [mücaneset] that they assigned to all space (S.Zeki, Darülfünun Konferansları, 1331, p. 59).

Salih Zeki ends his fourth lecture in Darülfünun by promising that he will clarify the problems concerning geometry and space in his last lecture on the history of the non-Euclidean geometries.

3.5 Lecture 5 [22 January 1915]

The fifth lecture in Darülfünun in which Salih Zeki brings his account on the history of geometry to an end, consists of paraphrasing Poincaré's expressions on space and the principles of geometry. In the previous lectures Salih Zeki introduced the change in the considerations upon space and the objections against the new tools of geometry such as imaginary numbers. Another concern which was emphasized was the status of the postulates of geometry. This lecture is held to clarify the obscurities about the basic notions of geometrical practice and in this sense consists of a presentation on the philosophy of science.

According to Salih Zeki, Poincaré distinguished the problems of space and geometry. In this lecture, by following this philosopher's ideas Salih Zeki will present the necessary grounds for the construction of a geometry and the status of the principles of this practice.

Salih Zeki starts his lecture by providing a general introduction on the problems he aims to discuss: All geometries, either Euclidean or non-Euclidean are based on three-dimensional space. However, measurements and figures differ from one geometry to the other which results in different geometries. Space is shapeless unless it is measured and it is the type of measurement that will bring out either Euclidean or non-Euclidean geometry. If different geometries were compared, the question about their correctness cannot be asked since one geometry cannot be truer than the other; it can only be more convenient. Moreover, measurements on space or geometries cannot reveal anything about real space. Space cannot be treated as absolute and "absolute space" is a meaningless notion,

With this introduction Salih Zeki turns to follow Poincaré's views on the relativity of space. Poincaré sets up an analogy to explain that space is relative [izafi] (S.Zeki, Darülfünun Konferansları, 1331, p. 62). In this analogy, a person at a certain point in Paris, at Pantheon Square claims to visit the same point the following day. And if she was asked whether she will be at the same point of space, her answer would be affirmative. However, this would be a mistaken answer, since until the next day Earth will move and Pantheon will be moving with it. Not only Earth will move with respect to the Sun, but also the Sun will be moving with respect to the Milky Way. As a result, the total replacement of Pantheon in one day cannot be calculated. One can at most claim that she will see the dome and the front of Pantheon again. Also, without considering Pantheon all this reasoning would be meaningless; even space would not exist. We think of space in terms of external objects and it is the only way that we can think of it.

Salih Zeki states that this analogy explains what "space is relative" means, and he provides another type of relativity. In this second example, all the distances in universe grow a thousand times in one night. The world would be similar to its initial situation but a meter length would become a kilometer. It would be quite a big change that even our bodies and beds would not escape from it. However, we would feel nothing related to this growth. Even the most precise measuring tools cannot detect anything about this change, since they would also grow with the rest of the objects.

If this is the case then we have no right to say that we know the distance between any two points. Instead of claiming that a person will be at the same distance to Pantheon in another day, Poincaré asserts that the distance between this person and Pantheon will be the same times bigger than a meter.

This change does not have to be in Euclidean terms of similarity. Salih Zeki proposes that even if the universe changes in terms of more complex laws, nothing would be apparent to us as long as all bodies were subject to the same change. Furthermore, change in shape is not important if the relations between objects are kept. The permanency of the relations between objects can be explained by considering mirrors that deform appearances and reflect objects in different shapes than they have in the external world. In this case, it would be possible to recognize the change in the forms since the external world would be standing alongside the images in the mirror. Even if the external objects were not visible, we would still have our body for comparison. However, if our body was subject to a change too, there remains nothing to be observed.

If two universes b and c were considered, and one of them was the image of the other, then an object m^{ι} in universe c would correspond to each object m in universe b. The following formulation would be true if the coordinates of m^{ι} were x^{ι}, y^{ι}, z^{ι} and the coordinates of m were x, y, z:

$$x^{\iota} = k(x, y, z)$$

$$y^{\iota} = l(x, y, z)$$

$$z^{\iota} = j(x, y, z)$$

There must be a constant relation between m and m^{ι}, and it is determined by the functions k, l, j. These functions can be chosen arbitrarily provided that they are chosen only for once.

Such two universes are not distinguishable from each other. That is to say, whatever universe b is to its dwellers, the same is true about c to its dwellers. If we suppose that we were the inhabitants of universe b, since universe c is an image of our universe , the geometry in c would also be an image of our geometry. But if a window was opened from universe b to c, we would think that the geometry in c is only a rough copy of our geometry in which curves are twisted and circles are crooked. However, the inhabitants of c would think in the same way for our geometry. Yet, it is not possible to know whose claim is true.

Salih Zeki proposes that the relativity of space is such a deep and broad notion as can be observed from the previous examples. Space is shapeless and it is the objects in space that provide it with a shape. We have no intuition [tahaddüs] of space or the distance between two points in space. In this sense, the medium that makes us think that an object has changed or kept its distance to another object is our tool for measurement [alet-i mesaha] (S.Zeki, Darülfünun Konferansları, 1331, p. 65). The only thing we can know is the relation of lengths or magnitudes to our tool for measurement; and this tool for measurement is our body. We assign positions to external objects with respect to our bodies. The possible representations for us on the external bodies are their magnitude relations [nisbet-i miktariyye] and extensional relations [nisbet-i hayyiziyye] (S.Zeki, Darülfünun Konferansları, 1331, p. 66). Those are nothing but the relations of objects to our body. That is to say, our body serves as a coordinate system for external objects.

In this system three types of coordinates can be determined for an object. These are provided by the visual sensations [ihsasat-ı basariyye], tactile sensations [ihsasat-ı lemsiyye] and sensations of motion [ihsasat-ı harekiyye] (S.Zeki, Darülfünun Konferansları, 1331, p. 66).

Based on these three types of coordinates, Salih Zeki concludes that space can be represented in three ways which are visual space, tactile space and motor space. And he provides an interesting footnote in which he claims that he has told Poincaré about this conclusion and the philosopher has affirmed it [Bu suret-i temsili müteveffa Poincaré'ye söylediğim zaman kendisinin pek ziyade hoşuna gitmişti.] (S.Zeki, Darülfünun Konferansları, 1331, p. 66).

None of these three spaces can actually represent real space or be the subject matter of Euclid's geometry. They can be named as representative space [mekan-ı temsili] and this notion be distinguished from geometrical space. Geometrical space is continuous, infinite and it has three dimensions [mütemadi, namütenahi, üç bu'dlu]. Also, it is homogeneous and isotropic [mütecanis, mütesavi'l cihed]. Salih Zeki provides short descriptions for the last two notions in this list, that is, for homogeneity and isotropy. Homogeneity of space means that all its points are identical, and its being isotropic means that all straight lines through a point in it are identical.

After he provides the properties of geometrical space, Salih Zeki considers representative space in order to put forward how the two are different. He first evaluates the visual space: The image of an external object on the retina is two dimensional and this is one of the differences of visual space from geometrical space. If visual space is examined in terms of its two dimensional images, what is in hand is pure visual space [mekan-ı basari-ı sırfi] (S.Zeki, Darülfünun Konferansları, 1331, p. 67). Another difference is that the retina is a bounded area, that is, it is not infinite. Thirdly, visual space is not homogeneous since the sensitiveness of the retina differs from point to point.

For the three dimensional images of external objects two more senses are added to the two dimensional image. These are the accommodation [itbak] and the convergence [tekarüb] of the two eyes and they are muscular sensations. When the two dimensional images become three dimensional, visual space is not pure anymore but it is the complete visual space [mekan-ı basari-i tamm] (S.Zeki, Darülfünun Konferansları, 1331, p. 67). The addition of the muscular senses does not change the fact that visual space is different than geometrical space. The complete visual space is not isotropic since it consists of two visual and two muscular senses.

Salih Zeki also evaluates these expressions on visual space in mathematical terms. He regards the visual space to be determined by four variables, two of which are purely visual and the other two muscular. The former two are independent but the latter two are dependent on each other. It is confirmed by experience that if two senses of accommodation are not distinguishable then neither are the following two senses of convergence. When we affirm that the complete visual space is three dimensional, it means that when three of these four bases are known, all four can be determined. In other words, visual space is a function of three independent variables. If the two muscular senses were not dependent on each other, we would have to assign four dimensions to the visual space.

Tactile space is more complex than the visual space and it differs from geometrical space even more than the visual space does. Still, there are other senses than sight and touching which are more helpful than these two in the generation of the notion of space. These are muscular sense [ihsasat-ı adaliyye] and they provide us with the motor space (S.Zeki, Darülfünun Konferansları, 1331, p. 68). There is a variety of muscular sensations in accordance with the number of muscles we have. Salih Zeki proposes that the idea of a representative space that is generated merely by the muscular sensations could be objected by the claim that: Muscular sensations can only give the notion of space with the addition of a sense of direction that we possess. As an answer to such an objection, Salih Zeki states that the sensations of movements that are in the same directions are connected by some ideas. The sense of direction is merely the combination of these ideas. It is acquired, that is, like the other combinations of ideas it is the result of habits that are collections of experiences. If our senses were educated in a different environment, we would have different habits than we already have. As a result, our muscular sensations would be associated by different laws.

Salih Zeki concludes his account on representative space by proposing that it is totally different than geometrical space; it is not homogeneous or isotropic, not even three dimensional. His next concern is the geometrical space.

External objects cannot be represented in geometrical space, just like a painter cannot display an object with its three dimensions on a flat surface. We cannot locate an object or point in geometrical space. Our only representations are on the movements required for the generation of this object or point. However, this does not mean that movements are actually projected in space and therefore space must exist in the first place. The representations of the mentioned movements are actually the representations of the muscular sensations that correspond to these movements and therefore they are not geometrical. These representations by no means necessitate the existence of geometrical space. In this sense, geometrical space is not imposed on our minds and none of our sensations can give us this notion; but we do think of geometrical space.

The notion of geometrical space is generated by examining the laws of the succession of sensations. This would be by arranging our impressions on the changes in the external world. External objects may change either their state or their position. The two types of change in the external world provide us with two types of impressions.

If an object has only changed its position, we can regenerate the impressions we had of it. We can move in order to correct the change in our impressions, that is, we can make the relation between our body and the object become the same as it was before the object's movement. This act would be totally conscious since it is voluntary and muscular. Still, that does not mean that we represent this motion in geometrical space. Change in position has a distinguishing property from other changes. It can be corrected by our movements. In this respect, there are two paths from one aggregate of impressions to the other. Either it is involuntary and without any muscular sensations, that consists of the cases in which our body is fixed and an object moves. Or it is voluntary and accompanied by muscular sensations, when an object is fixed and we move our body. Either of these changes in impressions corresponds to changes in positions. Therefore, the help of muscular sensations is needed for the generation of the notion of space. Also, this notion cannot be given by merely one muscular sensation but a variety of them is required.

Moreover, if a person can by no means move, he cannot acquire the notion of the changes in position since he cannot arrange any such changes of the external objects. If a creature cannot move voluntarily, it cannot even generate the notion of space. For a motionless being there would be neither a geometrical space, nor geometry.

These objects whose motions can be corrected by the movements of our body, keep their forms while they are moving. Moving without any change in form is a property of solid bodies. On the other hand, if the change of position is accompanied with a change in form, we can no longer correct this change by moving our body. Change in position together with a change in form is a complex notion. We would not even know it if we did not have the notion of change in position without any change in form. It is the solid bodies that provide us with the notion of moving without a change in form. It can be concluded that if there were no solid bodies in nature, there would be neither form nor geometry.

In some cases it can be asserted that two separate objects were subject to the same change in position. This stems from the fact that we correct both changes with the same movements of our body.

The two types of changes in the impressions that were provided previously correspond to two types of phenomena. Involuntary changes that are not accompanied by muscular sensations constitute the external changes [tebdilat-ı hariciye]. And the voluntary changes that are accompanied by muscular sensations are the changes of our body, that is, internal changes [tebdilat-ı dahiliye] (S.Zeki, Darülfünun Konferansları, 1331, p. 73).

Among the external changes, those which can be corrected by internal changes are the changes of positions, and their laws constitute the subject matter of geometry. The first of these laws is homogeneity. If by an external change b_1 we pass from an aggregate of impressions b to another aggregate of impressions c, and then b_1 was corrected by an internal change c_1, we end up with our initial aggregate of impressions b. If another external change b_2, alters our impression from b to c, experience confirms that b_2 can be corrected by an internal change c_2. In this sense, c_2 corresponds to the same muscular sensations as c_1 and this is by means of the homogeneity and isotropy of space. Yet, homogeneity and isotropy are not sufficient for the possibility of geometry. The rest of the necessary laws can be generalized as "changes in positions must constitute a group [zümre]" (S.Zeki, Darülfünun Konferansları, 1331, p. 73).

If geometrical space was assigned to each of the representations separately, we would not be able to represent any image without this notion. Thus, it would be impossible to change our geometry. On the contrary, nothing can stop us from thinking representations that are similar to ours but subject to a different aggregate of laws than the laws we are accustomed to.

Creatures that are educated in an environment with a different collection of laws than the ones we are used to, would bring out a different geometry than ours. This can be explained by considering a universe surrounded by a sphere. In this universe temperature is at its maximum at the center of the sphere and decreases as the periphery is approached. At the periphery the temperature is absolute zero. If the radius of the sphere is r and the distance of a point to the center is b, then at a point in this sphere temperature would be proportional to $(r^2 - b^2)$. Moreover, all the objects in this universe have the same co-efficient of dilatation and lengths are proportional to temperature. And also, when a point moves from a point to another, it fits into the equilibrium of temperature of its surrounding.

Consequently, in such a universe as a body moves through the periphery it must shrink. This universe would look infinite to its dwellers, since the farther they get from the center the colder and smaller they will become, and they will never be able to reach the surrounding periphery.

If geometry is the laws of movements of solid bodies, then in the mentioned universe it would be: The examination of the laws of motion of solid bodies that change their form in accordance with temperature.

Salih Zeki presents a further example on the effect of the environment to geometry. This time he considers a universe in which light travels through highly refractive mediums. At every point the refractive index is proportional with $r^2 - b^2$. In such a universe, light rays cannot be straight, but they would be circular. If an object in this universe changed its position, this change would not be ordinary. It would more likely be similar to the changes in the previous example: subject to laws of temperature where motion was along with expansion or contraction. The ordinary change in position where there is no change in form can be called "Euclidean change in position". And the more complex change is "non-Euclidean change in position". If there was a sentient being somewhere near the moving object, its impressions about the object would be altered. However, this creature can move and regain its initial impressions about the object. In such a case, both the object and the intelligent creature would have accomplished a non-Euclidean change in position. This kind of change would be possible for this creature, since its limbs are also subject to the same laws as the other bodies.

When an object changes its form, it various parts do not possess the interrelations they had initially. Yet, even if the distances between these various parts would be different, the parts that were in contact would still be so after the change in form. This intelligent being will also recognize that the external changes are of two kinds and the changes in positions can be corrected by conscious movements.

The geometry that is constructed by the inhabitants of such a universe would be different than ours, since the laws of motion that does not involve a change in form would not even be known to them. Their geometry would be constructed upon the laws of a non-Euclidean change in position, namely a non-Euclidean geometry.

As a result, experience plays an important role in constructing geometries. However, geometry is not an empirical science. If it was even partially empirical, it would be approximate and temporary.

Salih Zeki continues to follow Poincaré's views on space and geometry by presenting the philosophers conclusions on this subject: Geometry is the examination of movements of solid bodies, but not the solid bodies in nature. The solid bodies in nature are not rigid, whereas the solid bodies that are considered in geometry are rigid and ideal. The concept of an ideal solid body is a result of reasoning, but its comprehension is possible with the help of experience. Geometry examines a particular group. The general idea of a particular group is, at least potentially, in our mind. It is not imposed on us as a form of sensitiveness, but as a form of understanding. Among all the possible groups we should choose the one that can corresponds to natural phenomena and this choice is guided by experience. Experience does not enforce us to choose one of these groups and does not point out the truest one, but suggests the simplest and most convenient among the possible groups.

Through the end of the fifth lecture in Darülfünun Salih Zeki provides an overall of his ideas on geometry. He asserts that constructing geometries is accomplished by accepting axioms and postulates that are preferred because of their convenience. Some of these postulates are put forth clearly, but a big number of them is assumed implicitly in geometrical proofs. The number of these assumptions should be reduced to a minimum which was achieved by Sophus Lie.

Salih Zeki adds that he does not agree with Kant's expression that these postulates are synthetic a priori judgements in our minds. If they were so, it would be impossible to understand their negations or to construct geometries on these negations. The principles of geometry are neither synthetic judgements nor empirical truths. They are conventions that are guided by experience. The only condition for choosing conventions is that they should not include any contradictions.

The last concern in Salih Zeki's lecture is to provide a theorem that was demonstrated by Sophus Lie, in order to prove that the number of different geometries does not increase with the number of postulates. Sophus Lie assumes that: 1. there is an n dimensional space, 2. it is possible for a rigid body to move in this space, and 3. the number of the conditions that are needed to locate a figure in this space is k. And he demonstrates that the number of geometries that can be constructed with these premises is limited. Furthermore, Poincaré claims that an upper limit can be assigned to k, if n was given; and since n is three a few number of geometries can be constructed based on Sophus Lie's premises.

With his fifth lecture Salih Zeki puts an end to his summary on the history of geometry. He asserts that this summary covered the major theories and geometers in history. Having introduced the general information on the history of geometry, Salih Zeki can examine the theories in detail in his following lectures. The first theory he aims to consider is the analytic theory of parallelism in Lobachevski's geometry.

CHAPTER 4

AN EVALUATİON OF *DARÜLFÜNUN KONFERANSLARI*

4.1 Salih Zeki as a Mathematician

Although the first five lectures of *Darülfünun Konferansları* would not precisely reveal the mathematical practice that was held by Salih Zeki, it is possible to figure out some of his mathematical skills. My focus in this section is Salih Zeki's ideas on the original works of geometers that appear throughout his lectures. Yet, I do not claim that such works were accessible to him during these lectures, since there is still the possibility that he follows secondary sources on the ideas of geometers. Salih Zeki does not provide descriptions of all of the geometrical works that he lectures about. However, when he does provide such descriptions he is quite careful about the ideas of the geometers. One point that may be criticized about Salih Zeki's evaluation of the original works is that he skips the general notions that are set forth by the geometers. *Darülfünun Konferansları* presents the particular illustrations of some geometers but does not introduce the general ideas which constitute the basis for examples.

Firstly, I would like to point out how Salih Zeki carefully describes the works of several geometers:

In his first lecture Salih Zeki introduces two works of Lobachevski: *Geometrical Investigations on the Theory of Parallels* and Pangeometry. Salih Zeki characterizes the first of these two works to state the geometrical theorems which can be proved without using the Euclidean postulate. Salih Zeki's own words are the following:

> Bu makalede Öklidis'in mevzu'esini isti'mal etmeksizin doğrudan doğruya diğer müte'arifat ve mevzu'at ile ispat olunabilen da'avi-i hendesiyeyi ira'e ettiği gibi…[9] (S.Zeki, Darülfünun Konferansları, 1331, p. 7)

Salih Zeki's description of *Geometrical Investigations on the Theory of Parallels* is a fair one and is also confirmed by Lobachevski himself:

> I have published a complete theory of parallels under the title *Geometrical Investigations on the Theory of Parallels*… In this work I have stated first all the theorems which can be demonstrated without the aid of the theory of parallels. (Lobachevsky, 1929, p. 362).

Salih Zeki continues his explanations on Lobachevski's study on the theory of parallels and proposes that the geometer has also provided a definition of parallelism. Lobachevski's definition of parallel lines in *Darülfünun Konferansları* is as the following:

> Bir müstevi üzerinde ka'in bir nokta-i ma'lumeden resm olunan hutut-ı müstakime 'aynı müstevi üzerinde vakı' bir hatt-ı müstakim-i ma'luma nazaran iki sınıfa tefrik olunurlar. Şöyle ki: Bir sınıf hutut-ı müstakime, hatt-ı müstakim-i ma'lumu kat' ederler. Halbuki diğer sınıf hutut-ı müstakime kat' edemezler. Bu iki sınıfın ga'ye-i müşterekesi olan hatt-ı müstakim, nokta-i ma'lumeden hatt-ı müstakim-i ma'luma muvazi resm olunan hatt-ı müstakimden 'ibarettir.[10] (S.Zeki, Darülfünun Konferansları, 1331, p. 8)

The definition provided by Salih Zeki corresponds to a theorem in Lobachevski's Theory of Parallels:

> All straight lines which in a plane go out from a point can, with reference to a given straight line in the same plane, be divided into two classes – into *cutting* and *not-cutting*.
>
> The *boundary lines* of the one and the other class of those lines will be called *parallel to the given line*. (Lobachevski, 1955, p. 13)

Another example that confirms Salih Zeki's accuracy of describing original works is his expressions on Riemann's hypotheses. Salih Zeki begins his account on Riemann's work with a quotation from the geometer's work, and adds some extra

[9] Öklidis mevzu'esi: Euclidean postulate/ isti'mal: use, apply/ müte'arifat: axioms/ mevzu'at: postulates/ da'avi-i hendesiye: geometrical theorems/ ira'e: show

[10] Müstevi: plane/ ka'in: that stands (e.g. on a plane)/ nokta-i ma'lume: a given point/ hutut-ı müstakime: straight lines/ hatt-ı müstakim-i ma'lum: a given straight line/ kat': cut/ gaye-i müştereke: boundary, limiting/ muvazi: parallel

explanations which are in agreement with Riemann's ideas. Salih Zeki asserts that the following is approximately what Riemann declares in his article:

> Hendese, mu'tayat ya'ni ma'lumat-ı evveliye olarak yalnız mekan mefhumunu değil bunun dahilinde hatt, satıh gibi teşkilat mefhum-ı esasilerini de vaz' ve kabul eder.[11] Ta'bir-i ahirle bir mekan ile bunun derununda eşkal-i hendesiyenin tersim-i mümkin olduğunu farz eyler.[12] Bu mefhumları yalnız ta'rifat ile i'ta ve takdirat ve ta'yinat-ı kemiyyesini de müte'arifat ve mevzu'at suretinde edhal eder.[13] Ancak bu ma'lumat-ı evveliye ya'ni mu'tayat miyanındaki münasebet-i mütekabile gayr-i mer'i bir şekilde bulunur ve adeta bir sır gibi mestur olduğu görülür.[14] Hatta bunların yekdiğerine merbut olup olmadıkları ve merbut iseler ne dereceye kadar merbut bulundukları ve nazari olarak merbut olabilip olamayacakları görülemez bir haldedir.[15] (S.Zeki, Darülfünun Konferansları, 1331, p. 13)

When Salih Zeki's expressions are compared to Riemann's own ideas, which are:

> It is a known fact that geometry assumes, as things given, both the notion of space and the first principles of constructions in space. She gives definitions of them which are merely nominal, while the true determinations appear in the form of axioms. The relation of these assumptions remains consequently in darkness; we neither perceive whether and how far their connection is necessary, nor a priori, whether it is possible. (Riemann, 1873)

It can be observed that Riemann's assertion that geometry assumes both the notion of space and the first principles of constructions as given, is further explained by Salih Zeki when he affirms that space and the possibility of constructing geometrical figures in it are assumed in geometry. Since Riemann constructs the basis of a space that can be measured and therefore allows geometrical constructions, it may be regarded as an appropriate explanation to mention the possibility of constructions.

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[11] Mu'tayat: that are given/ ma'lumat: that are known/ hatt: line/ satıh: surface/ teşkilat mefhum-ı esasileri: basic notions of construction, principles of construction/ vaz': put, take, consider

[12] Derun: the inside of something/ eşkal-i hendesiye: geometrical figures/ tersim-i mümkin: that can be drawn, that can be projected

[13] Ta'rifat: definitions/ i'ta: give/ takdirat ve ta'yinat-ı kemiyye: measurement and determination of quantities

[14] Miyan: between/ münasebet-i mütekabile: mutual relations/ gayr-i mer'i: invisible/ mestur: hidden, dark

[15] Merbut: connected/ nazari: theoretical

In his second lecture when Salih Zeki introduces Riemann's expression for the line element $d_l = \sqrt{\sum (d_x)^2}$, he states that the reason why Riemann preferred this expression is its simplicity (S.Zeki, Darülfünun Konferansları, 1331, p. 21). Such an explanation can be taken to be correct since in his article Riemann concludes that "[s]pace is … included in this simplest case" when he provides the mentioned formula (Riemann, 1873).

The comparison of Lobachevski's and Riemann's ideas with Salih Zeki's account on the works of these two geometers points out the mathematical sufficiency of his lectures. On the contrary, when he examines the notion of flat manifoldness in Riemann's work and Helmholtz's illustrations on the intelligent creatures that inhabit two dimensional universes, Salih Zeki's explanations are not as accurate as the previous examples.

> …Riemann'ın ta'rifine gore n bu'dlu bir zu-enva'ın her noktada ve kaffe-i istikamatta miktar-ı inhinası sıfır olur ise bu zu-enva'a bir "zu-enva'-ı müstevi" namı verilir… n bu'dlu bir zu-enva'-ı müstevi! Yoksa 'alela'de bir müstevi değil![16] (S.Zeki, Darülfünun Konferansları, 1331, p. 22)

The reason for the problem in this explanation may not actually be Salih Zeki's own ideas, since "müstevi" means both "flat" and "plane" in Ottoman Turkish. However, a flat manifold is a general notion for whom the Euclidean space stands as an example, and in this sense "zu-enva'-ı müstevi" is not a clear notion.

When Helmholtz constructs an example on how intelligent beings would come up with a geometry that would correspond to their experiences of the space surrounding them, he starts with a general notion of a two dimensional space. According to Helmholtz, intelligent beings in a two dimensional space would recognize the shortest path between two points but this would not necessarily be the straight line on a plane. "If moreover beings of this kind lived in an infinite plane, they would lay down precisely our planimetric geometry. They would maintain that only *one* straight line is possible between two points…" (Helmholtz, 1977, p. 6)

Salih Zeki's third lecture, which follows mainly Helmholtz's illustrations, starts with considering a two dimensional space. He directly terms this two dimensional space

[16] Ta'rif: definition/ n bu'dlu: n dimensional/ zu-enva': manifold/ kaffe-i istikamat: all directions/ inhina: curvature

as a “planar surface” [sath-ı müstevi] without providing a general idea on two dimensional spaces. It is possible to think of the problem to be merely caused by the ambiguous usage of the word “müstevi” again. However, when Salih Zeki asks the question of what a geometry constructed by two dimensional creatures would be like, his answer is that it would be our two dimensional plane geometry.

> Bu müstevi mühendislerin teşkil edecekleri hendese nasıl bir hendese olacaktır? Şüphesiz bizim iki bu’dlu dediğimiz hendese-i müsteviyenin ‘aynı![17] (S.Zeki, Darülfünun Konferansları, 1331, p. 32)

I have illustrated how Salih Zeki’s mathematical expressions are quite careful and accurate from some points, but also vague or inadequate in other respects. As a last remark in this section I would like to present the Ottoman term for the notion “manifold” which is a quite qualified translation. The Ottoman term probably gets its clue from the French word “variété”, since Salih Zeki provides the French term is brackets (S.Zeki, Darülfünun Konferansları, 1331, p. 14). The phrase that corresponds to “manifold” is “zu-kesirü’l-enva’” which also appears as “zu-enva’”. In this phrase “zu” means “that which has something”, “kesir” means “all” or “entire”, and “enva’” means “variety” or “kinds”. Consequently, “zu-kesirü’l-enva’” more or less means “that which allows many variations”. Although I think that this term was generated by Salih Zeki, I do not have any sufficient evidence to claim so. Still, it is a comprehensive translation of the word “manifold”.

4.2 Salih Zeki’s Views on Geometry and Its History

Salih Zeki’s account on the history of geometry and the discovery of the non-Euclidean geometries has some distinctive aspects that constitute a certain approach concerning geometry’s progress. Especially the first two lectures in Darülfünun are organized in accordance with some definite ideas on the history of geometry which I will present through some examples of Salih Zeki’s declarations. The third lecture does not provide any clear conclusions about Salih Zeki’s philosophical perspective on geometry, except that he appreciates the new

[17] Mühendis: geometer/ hendese: geometry/ hendese-i müsteviye: plane geometry

geometries and accepts the possibility of working with the notion of a fourth dimension. However, in the last two lectures the focus is mainly some philosophical expressions on geometry and the status of geometrical axioms and postulates.

The first lecture in Darülfünun starts by stating the long lasting application of Euclidean geometry in dealing with the external world and introduces the problem concerning the parallel lines, as would be expected in an account on the discovery of non-Euclidean geometries. Salih Zeki does not explain in detail what was problematic about the parallel postulate. The provided reasons about this problem are firstly that it is not related to the rest of Euclid's postulates and secondly that two lines which do not intersect one another are not necessarily parallel since they may be asymptotic.

> Fi'l-hakika evvela bu kaziyyenin, hendesede Öklidis'in sırahaten zikr eylediği mevzu'at …ile bir münasebeti görülemiyor idi [18] (S.Zeki, Darülfünun Konferansları, 1331, p. 4)
>
> Saniyen muvazi iki hatt-ı müstakim demek, her iki cihetten temdid edildikleri halde asla yekdiğerini kat' edemeyen hatt-ı müstakimler demek ise, böyle temdid edildikleri halde birbirine takarrüb eden, fakat asla yekdiğerini kat' edemeyen hatt-ı müstakimler tasavvur etmek de mümkin idi. [19] (S.Zeki, Darülfünun Konferansları, 1331, p. 5)

As an example to the studies in favor of the parallel postulate, Salih Zeki points out Legendre's work. As I have asserted in the second chapter of my thesis, Legendre was among the geometers who worked with the negations of the parallel postulate; a way of reasoning which was set by Saccheri. Although working with the three hypotheses - the hypotheses of the right angle, the obtuse angle and the acute angle - is never mentioned in Salih Zeki's lecture, he correctly presents the results of Legendre's study:

> …Legendre muvazat mevzu'esinin doğrudan doğruya bir netice'-i lazımesi olan bir da'vayı ispata kalkıştı: Ta'bir-i ahirle bir müselles-i müstakimi'l ıdla'ın üç zaviyesi mecmu'unun iki ka'imeye müsavi olduğunu – mevzu'e-i muvazatı isti'mal etmeksizin – ispata çalıştı. Fakat bu teşebbüsüyle yalnız bir neticeye destres oldu ki o da bir müselles-i müstakimi'l ıdla'ın üç zaviyesi

[18] Kaziyye: proposition/ mevzu'at: postulates

[19] Muvazi: parallel/ hatt-ı müstakim: staright line/ cihet: direction/ temdid: prolong/ kat': cut/ takarrüb: converge

mecmu'unun iki ka'imeden büyük olamayacağıydı. [20] Eğer küçük olamayacağını da ispata muvaffak olsa idi, muvazat mevzu'esini ispat etmiş olacaktı. Ne fa'ide ki buna muvaffak olamadı. Buna bedel eğer yalnız bir müselles-i müstakimi'l ıdla'da üç zaviye mecmu'unun iki ka'imeye müsavi olduğunu ispat edecek olur ise bütün müselleslerde üç zaviye mecmu'unun iki ka'imeye müsavi olması lazım geleceğini ispat eyledi. (S.Zeki, Darülfünun Konferansları, 1331, pp. 5-6)

Salih Zeki regards Legendre's failure to be that last step in the Euclidean tradition and to be the reason why finally geometers could come up with a different geometry (S.Zeki, Darülfünun Konferansları, 1331, p. 6). In his lecture Salih Zeki does not deliver any ideas concerning the role of the application of hyperbolic functions or the conviction that the problem may be about the notions of Euclid's geometry in general. On the other hand, Lobachevski explains why he is not working with Euclid's notions which points out that his leading idea was not merely that the parallel postulate was not capable of a proof:

> …most of the definitions given ordinarily in the elements of geometry… not only do not indicate the generation of the magnitudes which they define, but they do not even show that these magnitudes can exist.
>
> Instead of commencing geometry with the plane and the straight line as we do ordinarily, I have preferred to commence it with the sphere and the circle , whose definitions are not subject to the reproach of being incomplete, since they contain the generation of the magnitudes which they define. (Lobachevsky, 1929, p. 361)

It can be observed from Salih Zeki's account on the problem of parallels that he does not provide an exhaustive explanation that includes the relation between this problem and the whole Euclidean tradition. Even though, he puts forward in his fourth lecture that metrical geometry was based on the equivalency of geometrical figures which can be determined by moving one of the figures onto the other, he does not evaluate the role of the parallel postulate in such processes.

> …iki şeklin müsavatı mes'elesi, ki bütün hendese-i mikyasiyenin esasını teşkil eder, bu iki şekilden birinin diğeri üzerine vaz' ve tatbikine müsa'it bir hareketle hall olunabilir.[21] (S.Zeki, Darülfünun Konferansları, 1331, p. 57)

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[20] Muvazat mevzu'esi: parallel postulate/ netice-i lazıme: necessary result/ da'va: theorem/ müselles-i mütakimi'l ıdla': rectilinear triangle/ zaviye: angle/ mecmu': sum/ ka'ime: right angle/ isti'mal: use, apply/ destres: obtain

[21] Müsavat: equivalency/ hendese-i mikyasiye: quantitative geometry/ vaz': put/ tatbik: application, comparison/ istihrac: deduce/ redd: negation/ tenakuz: contradiction

Instead he prefers to put forward that the parallel postulate was problematic and the rejection of this postulate generated the non-Euclidean geometries. In Salih Zeki's lectures the parallel postulate is presented in terms of its logical bearing to Euclid's geometry. He does not examine the problem of parallels elaborately and points out the logical independency of the parallel postulate from the rest of Euclid's axioms and postulates several times. According to Salih Zeki the first geometers to work in a non-Euclidean fashion thought that if the parallel postulate could be deduced from the rest of the axioms and postulates logically, then in a geometry which was constructed upon the negation of the parallel postulate contradictive results would be inevitable:

> Eğer mevzu'e-i Öklidis'in diğer mevzu'e ve müte'arifelerden mantık tarikiyle istihracı mümkin ise bu mevzu'enin reddi ve diğerlerinin kabulu halinde teşkil edilecek olan bir hendesede tenakuza tesadüf etmemek mümkin değildir.[22] (S.Zeki, Darülfünun Konferansları, 1331, p. 6)

The above explanation is claimed to be made by Gauss, Lobachevski and Bolyai independently. Furthermore, Salih Zeki thinks that all three geometers rejected the parallel postulate and constructed geometries on the rest of Euclid's postulates; and the results of these new geometries did not include any contradictions:

> …bu üç zat ayrı ayrı mevzu'e-i Öklidis'i inkar ile diğer müte'arifat ve mevzu'at üzerine birer hendese te'sis eylemişler ve bu hendeselerde hiçbir tenakuza tesadüf etmemişlerdir.[23] (S.Zeki, Darülfünun Konferansları, 1331, p. 7)

Salih Zeki's emphasis on discovering geometries merely by rejecting the parallel postulate stems from his idea that the aim of the geometers was to prove that this postulate was logically independent from the rest of Euclid's geometry. And its logical independence was the reason why the parallel postulate was rejected in the new geometries (S.Zeki, Darülfünun Konferansları, 1331, p. 11).

With respect to Salih Zeki's account, after the appearance of the non-Euclidean geometries the course of geometry continues with the question whether the new geometries would bring out any contradictive results. It was true that there were no contradictions among their conclusions, yet this fact would not exhaust the possibility of future contradictions. When Riemann's geometry was considered the

[22] Mevzu'e-i Öklidis: Euclidean postulate/ mevzu'e: postulate/ müte'arife: axiom/ tarik: way, path

[23] İnkar: deny

solution was easy, since Riemann's geometry was nothing but Euclid's spherical geometry. In this sense, any claim about the inconsistency of Riemann's geometry would necessitate affirming that Euclid's spherical geometry included contradictions too. However, there was no apparent relation between Lobachevski's and Euclid's geometries which would save the former geometer's work from inconsistency. This problem about Lobachevski's geometry had to wait until Beltrami provided a map between the two geometries. At the end, the consistency of Lobachevski's geometry was also reduced to relative grounds and its inconsistency could not be affirmed unless the same was accepted for Euclid's geometry (S.Zeki, Darülfünun Konferansları, 1331, pp. 27-28).

When Salih Zeki's considerations in his first two lectures are summarized we have the following story: The problem about the parallel postulate is related to its logical independence from the rest of Euclid's principles, and this is the reason why it cannot be proved by means of Euclid's geometry. However, the fact that the parallel postulate is not capable of any proof was the inspiration for some geometers who came up with the new geometries. The Euclidean tradition ended with the rejection of the parallel postulate owing to its logical independence. The geometers who practiced with the non-Euclidean geometries aimed to prove the logical independency of the mentioned postulate in the first place. This requirement was accomplished when they were able to construct geometries that did not result in any contradictions. Still, the world of geometry was worried about the possibility of the inconsistency of non-Euclidean geometries but this problem was solved by means of Beltrami's studies.

If Salih Zeki's story is the case for the discovery of the new geometries, then the change from Euclidean to non-Euclidean geometries is merely a logical step, namely the rejection of a single postulate. The main concerns of Salih Zeki's account on the progress of geometry are logical aspects – the independence of axioms and the consistency of theories. In this sense, the significance of Beltrami's work in history is that it provided the various geometrical systems a relative consistency.

Obviously, Salih Zeki's account, with respect to his first and second lectures constitutes an example for how the axiomatization of geometry affected writing its history. The logical independence of the parallel postulate and the consistency of geometrical systems are notions that came along with the attempts to axiomatize geometry. These notions were not present when Lobachevski, Bolyai or Beltrami were studying. Actually, Janos Bolyai was not even interested in negating the parallel postulate. Bolyai's aim was to put forward the Euclidean theorems which he termed to be absolute, since they did not need the parallel postulate in their demonstrations.

Although Salih Zeki puts forward an account on the discovery of non-Euclidean geometries which regards geometry's progress to be axiomatic, he correctly points out a difference between the studies of Lobachevski-Bolyai and Riemann-Helmholtz. Salih Zeki's description of the studies of Lobachevski and Bolyai is an example how the axiomatic approach distorts actual history. However, he is correct in affirming that Riemann and Helmholtz were not concerned with a specific non-Euclidean geometry but the notion of "space" in general.

> Riemann'ın… mütala'aname[si]nin münderecatı ne "muvazat nazariyesi" ne de doğrudan doğruya "hendese-i gayr-i Öklidisiye" mes'elesidir. Bu muhtıranın mevzu'u mevzu'at-ı hendesiyenin münasebat-ı mütekabilesi veya açıkçası "mekan" mes'elesidir.[24] (S.Zeki, Darülfünun Konferansları, 1331, p. 13)

In this sense, contrary to the fact that he ascribes an inappropriate meaning to the non-Euclidean studies preceding Riemann and Helmholtz, Salih Zeki successfully points out the change in reasoning in geometrical studies that came along with the studies of these two geometers. Moreover, he provides a description of Helmholtz work, which is confirmed by Helmholtz's own explanations.

> Saha-ı rü'yet dahilinde bir cismin mevzi'ini ta'yin için icra-i taharriyat ettiği sırada mekan tasavvurunun menşe'i hakkında da ba'zı tedkikatta bulunmuştur.[25] Maksadı da'avi-i hendesiye miyanında hangilerinin hakayık-ı tecrübiyeyi ifade eylediklerini, ve hangilerinin hakayık-ı i'tibariyeden ya'ni

[24] Mütala'aname: reading, remark (Salih Zeki uses this word synonymously with "muhtıra")/ münderecat: content/ muvazat nazariyyesi: theory of parallels/ hendese-i gayr-i Öklidisiyye: non-Eucilidean geometry/ münasebat-ı mütekabile: mutual relations

[25] Saha-ı rü'yet: field of vision/ mevzi': position/ ta'yin: determination/ taharriyat: investigation, research/ menşe'i: origin/ tedkikat: examination

ta'rifat ile bu ta'rifatın netaicinden 'ibaret olduklarını bilmek idi.[26] (S.Zeki, Darülfünun Konferansları, 1331, p. 24)

In his article On the Facts Underlying Geometry (1868), Helmholtz provides a similar explanation and says that his "…investigations on spatial intuitions in the field of vision" led to his interest in the origin of general intuitions of space. And as a result he wanted to answer the questions on "…how much of the propositions of geometry has an objectively valid sense?" and "how much is on the contrary only definition or the consequence of definitions, or depends on the form of description?" (Helmholtz, 1977, p. 39)

Apparently, Salih Zeki is careful about the content of Riemann's and Helmholtz's works. When he provides an overall description of Riemann's geometry Salih Zeki states the following:

> Bu hendesede yalnız Öklidis'in mevzu'e-i muvazatı değil, diğer bir mevzu'esi daha redd edilmiş bulunuyor idi: İkişer noktası müşterek iki hatt-ı müstakimin bu iki nokta arasında mutlaka yekdiğerine muntabık olacağı inkar olunuyor idi.[27] (S.Zeki, Darülfünun Konferansları, 1331, pp. 23-24)

The above quotation affirms that Riemann's geometry not only excluded Euclid's parallel postulate, but also the straight lines in this geometry could have two common points without coinciding in between them. This is an appropriate statement about Riemann's geometry since it did not include the notion of parallel lines. Also, the straight lines in Riemann's geometry were great circles on a sphere and distinct great circles intersect at two antipodal points of the sphere.

While he examines Riemann's and Helmholtz's works, Salih Zeki provides an interesting expression on the notions of "real space" and "geometrical space" which in a way contradicts his ideas in the following lectures.

> [Riemann ve Helmholtz] bir zu-enva'da muhtelif münasebet-i kemiyyenin mümkin olabildiğini göstermek ve mekan-ı hakiki ya'ni mekan-ı hendesiyi de

[26] Hakayık: truths, facts/ tecrübi: experiential/ i'tibari: conventional/ ta'rifat: definitions/ netaic: results

[27] Muntabık: coinciding

> bu zu-enva'ın en basit bir sureti olmak üzere ira'e eylemek istiyorlar idi.[28] (S.Zeki, Darülfünun Konferansları, 1331, p. 12)

In this explanation Salih Zeki identifies real space with geometrical space, while in his fifth lecture one of his concerns is to put forward the difference between these two notions in terms of Poincaré's views on geometry. On the other hand, Salih Zeki's aim may be to point out the empirical aspect that determines the appropriate geometry for the external world in Riemann's and Helmholtz's approaches. In his fourth and fifth lectures Salih Zeki terms the principles of geometry to be a matter of convenience in relation to the external world and he emphasizes that these principles are nothing but conventions. As a result, his appreciation of the empirical aspect on geometry may be the reason why he considers the real space and geometrical space from the same point of view.

After the first non-Euclidean geometries and the approaches of Riemann and Helmholtz, *Darülfünun Konferansları* concentrates on projective geometry in terms of Cayley's and Klein's works. Salih Zeki asserts that the studies in terms of projective geometry can be described to have a different motive than constructing non-Euclidean geometries or investigating space in general. This can be observed when he characterizes Cayley's study to have a new definition of "distance" as a starting point.

> Bu risalede mü'ellif bütün bu'da ait tasavvuratı sırf tersimi olan mebadi üzerine bina etmek istiyor idi.[29] (S.Zeki, Darülfünun Konferansları, 1331, p. 47)

Also, Salih Zeki points out the general idea of Cayley's study by stating that Cayley reformulated the definition of distance with respect to circular points or a straight line at infinity.

> …Cayley, evvel emirde namütenahide vakı' nikat-ı da'ireviye ve namütenahide vakı' hatt-ı müstakim vasıtasıyla bu'd mefhumuna sırf irtisami

[28] Zu-enva': manifold/ münasebet-i kemiyye: quantitative relations/ mekan-ı hakiki: real space/ mekan-ı hendesi: geometrical space/ ira'e: show

[29] Mü'ellif: writer/ bu'd: distance (also means "dimension")/ sırf: pure/ tersimi: projective/ mebadi: principles

> bir suret verileceğini ispat eyledi.[30] (S.Zeki, Darülfünun Konferansları, 1331, p. 47)

Salih Zeki states that Cayley's studies were improved by Klein's contributions, since it was the latter geometer who put forward a relation between projective geometry and non-Euclidean geometries. In doing so Salih Zeki is correct since it is a historical fact that Cayley did not provide such a relation but Klein did. However, Salih Zeki's consideration of Klein's study brings back the idea of consistency of non-Euclidean geometries into *Darülfünun Konferansları*. Firstly, Salih Zeki explains Klein's interpretation of non-Euclidean geometries by means of projective geometry.

> …Klein evvela irtisama esas olan münhani-i mutlak, hakiki bir mahrutiye olduğuna göre Lobachevski'nin iki bu'dlu hendesesi istihsal edileceğini ispat eylemiş ve buna "hendese-i za'idiye" namını vermiştir.[31] Saniyen bu mahrutiye mevhum olduğuna göre gerek Riemann'ın hendese'-i küreviyesi, gerek Helmholtz'un bir hendesesine müşabih "hendese-i nakısiye" namını verdiği diğer bir hendese istihsal olunacağını ira'e eylemiştir.[32] Salisen mahrutiye bir çift nokta'-i mevhumeye müncerr olduğu halde "hendese'-i mükafiye" tesmiye eylediği bir hendese vücuda geleceğini izah etmiştir.[33] Rabi'an bu çift nokta'-i mevhume namütenahide vakı' nikat-ı da'ireviyeden 'ibaret olduğu halde de Öklidis'in hendese'-i 'adiyesi istihsal edileceğini meydana koymuştur.[34] (S.Zeki, Darülfünun Konferansları, 1331, p. 48)

In the above quotation Salih Zeki explains that if projection was with respect to a real conic, the result would be Lobachevski's geometry for which Klein provided the term "hyperbolic geometry". If the conic was imaginary, projection would bring out Riemann's spherical geometry or "elliptic geometry". When the reference of projection was two imaginary points, one would obtain a "parabolic geometry". And lastly, if the two imaginary points were circular, the projected geometry would be Euclid's geometry.

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[30] Namütenahi: infinity (also means "infinite")/ nikat-ı da'ireviyye: circular points/ sırf irtisami: pure projective

[31] İrtisam: projection/ münhani-i mutlak: absolute curve/ mahrutiye: conic/ hendese-i za'idiye: hyperbolic geometry

[32] Mevhum: imaginary/ hendese-i küreviye: spherical geometry/ müşabih: similar/ hendese-i nakısiye: elliptic geometry

[33] Bir çift nokta-i mevhume: a pair of imaginary points/ müncerr: reduce/ hendese-i mükafiye: parabolic geometry/ tesmiye: give a name

[34] Hendese-i 'adiye: elementary geometry

When he evaluates Klein’s interpretation, Salih Zeki proposes that one of the advantageous aspects of such a relation between various types of geometries is that it exhausts the possibility of any contradictions among the results of non-Euclidean geometries:

> …bu tefsirin bir fa’idesi daha vardı ki o da hendese’-i gayr-i Öklidisiyelerin da’avi ve netaici miyanında tenakuz imkanını külliyen ref’ eylemesidir.[35] Fi’l-hakika bu hendeselerden birinin bir da’vasına, şu tefsir mucibince, mutlaka Öklidis hendesesinin bir, ve yalnız bir, da’vası tevafuk eder.[36] (S.Zeki, Darülfünun Konferansları, 1331, p. 49)

It can be observed that Salih Zeki assigns the same meaning to Klein’s work as he did to Beltrami’s interpretation. Klein’s work finds its place in the history of geometry by means of the relative consistency it provides for the non-Euclidean geometries.

Another argument in *Darülfünun Konferansları* asserts that since the projection of Euclid’s geometry was in terms of an imaginary conic, imaginary numbers had entered into the field of classical geometry. Salih Zeki thinks that any mathematical tool should be allowed into the realm of geometry if it leads to progress. In this sense, he criticizes Russell’s views on the philosophical irrelevance of imaginary numbers and provides Poincaré’s ideas as an answer to him.

Russell, in his *Essay on the Foundations of Geometry* accepts the significance of imaginary numbers for Cayley’s studies in terms of projective geometry. At the same time, he proposes that he cannot assign a philosophical meaning to them (Russell, 1897, p. 43). Russell puts forward that the geometrical interpretation of imaginary numbers is with respect to the rules of Algebra, and this is the reason why they do not lead to any contradictions. On the other hand, he claims that “…only a knowledge of space, not a knowledge of Algebra, can assure us that any given set of quantities will have a spatial correlate, and in the absence of such a correlate, operations with these quantities have no geometrical import” (Russell, 1897, p. 46). Consequently, the application of imaginary numbers in Cayley’s geometry is only a technical move from Russell’s point of view.

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[35] Da’avi: theorems/ netaic: results/ miyan: among (also means “between”)/ tenakuz: contradictions/ ref’: remove

[36] Mucibince: in accordance with/ tevafuk: correspond (also means “agreement” and “congruence”)

In *Darülfünun Konferansları*, Salih Zeki presents Russell's views on imaginaries and claims that one cannot reject the philosophical significance of imaginary numbers while appreciating the non-Euclidean geometries. Salih Zeki puts forward that imaginary numbers are "conventions" that were preferred for the convenience of speech.

> Hülasa kemiyyat-ı mevhume, nikat-ı da'ireviye… gibi feylesofların ma'nadan hali buldukları ta'birat ihtisar-ı ifade için icad edilmiş i'tibarattan 'ibarettirler.[37] (S.Zeki, Darülfünun Konferansları, 1331, p. 55)

By supporting the application of imaginary numbers and putting forward that they are conventions like the rest of geometrical notions, Salih Zeki's lectures evolve through philosophical considerations on geometry and its principles. The fifth lecture in Darülfünun consists of following Poincaré's expressions on these subjects. In his last lecture Salih Zeki reveals his philosophical approach through geometry. As I have presented in the previous chapter he explains the origin of geometrical space and proposes that it is a result of the correlation of visual, tactile and motor sensations. He points out that motion is necessary for the origin of geometrical space in terms of which geometrical constructions are possible, in other words, the principles of these constructions are the rules of motion in geometrical space. The principles of geometry are neither empirical nor a priori, since the former destroys geometry's certainty and the latter leaves no room for the idea of different types of geometries. Geometrical practice is in terms of a constructed space and the rules of this practice are nothing but conventions.

In conclusion, throughout his lectures Salih Zeki presents an axiomatic interpretation of the history of geometry. He characterizes the change from Euclid's geometry to the first non-Euclidean geometries to be merely a logical step, where the negation of Euclid's parallel postulate brings out new systems of geometry. In *Darülfünun Konferansları* Beltrami's model of non-Euclidean geometry, that is, his map between a region of the Euclidean plane and a non-Euclidean surface is regarded to show that if the non-Euclidean geometries were not consistent, neither could Euclid's geometry be. Similarly, Klein's consideration that Euclid's geometry and non-Euclidean geometries can all be generated in terms of projection and that each

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[37] Kemiyyat-ı mevhume: imaginary numbers/ nikat-ı da'ireviye: circular points/ feylesof: philosopher/ ma'nadan hali: lacking any meaning/ ta'birat: terms/ ihtisar: speaking or writing briefly/ i'tibarat: conventions

Euclidean theorem corresponds to a non-Euclidean theorem, is evaluated in terms of the same logical concerns. According to Salih Zeki, Klein's work ended the doubts about the consistency of the new geometries. Apparently, Salih Zeki is providing a history of axioms and his answer to the question on the nature of these axioms is that they are conventions.

4.3 Conclusive Remarks

In this section I will provide the works of geometers that were mentioned or examined in the first five lectures of *Darülfünun Konferansları*. Furthermore, I will point out the secondary sources that were probably followed by Salih Zeki during these lectures. I have to emphasize that he "probably" used the secondary sources that I will provide, since *Darülfünun Konferansları* does not include any citations of them. However, these secondary sources are highly confirmed by the expressions in Salih Zeki's lectures and the organization of his account.

Firstly, I would like to put forward the original works of geometers that were introduced in Salih Zeki's lectures. He appears to have a special interest in Lobachevski's works, considering that three out of the fourteen lectures in *Darülfünun Konferansları* concentrate on the *Theory of Parallels*. Salih Zeki's sixth lecture is entitled as "Lobachevski Hendesesi / Muvazat Nazariye-i Tahlilesi"[38] (S.Zeki, Darülfünun Konferansları, 1331, p. 79). The titles of the ninth and tenth lectures are "Lobachevski Hendesesi'nin Tefsiri / Kısm-ı Musattahat"[39] (S.Zeki, Darülfünun Konferansları, 1331, p. 130) and "Lobachevski Hendesesi'nin Tefsiri / Maba'd"[40] (S.Zeki, Darülfünun Konferansları, 1331, p. 146) respectively.

Salih Zeki does not provide the titles of all the works of Lobachevski in the languages they were published in, instead he gives the Ottoman translations of

[38] Lobachevski's Geometry/ The Analytic Theory of Parallels

[39] Interpretation of Lobachevski's Geometry/ on Surfaces

[40] Interpretation of Lobachevski's Geometry/ Later on

them. Three works appear in their original form, one of which is “Géométrie Imaginaire” (S.Zeki, Darülfünun Konferansları, 1331, p. 7). Secondly, he submits the name “Pangéométrie” together with its Ottoman translation. Also, “*Geometrische Untersuchungen zur Therie der Parallellinien*”[41] is given in a letter written by Gauss which is quoted by Salih Zeki (S.Zeki, Darülfünun Konferansları, 1331, p. 9).

Lobachevski’s works that are mentioned by Salih Zeki are first the articles he published in Russian and he terms them as “Hendese’nin Mebadi-i Cedidesi ve Muvazat Nazariye-i Mükemmelesi”[42]. The next work is “Hendese-i Mevhume[43] – Géométrie Imaginaire” which is described to be a long and analytic article published in1837. Salih Zeki states that Lobachevski’s 1840 work was on the theory of parallels and that it was published in German, but the title of this work is not included in the account on Lobachevski’s works. It later on appears in a letter written by Gauss, as I have mentioned in the previous paragraph. Finally, Salih Zeki proposes that Lobachevski altered “Hendese-i Mevhume” into “Hendese-i Cami’e - Pangeometry” in 1855 (S.Zeki, Darülfünun Konferansları, 1331, pp. 7-8). It can be observed that the titles and dates of Lobachevski’s works are correctly presented in *Darülfünun Konferansları*.

Salih Zeki states that Janos Bolyai’s work was published as an appendix to his father’s *Tentamen*[44] (S.Zeki, Darülfünun Konferansları, 1331, p. 8). However, he does not mention the title of Bolyai’s work which is “The Science of Absolute Space” (Bonola, 1955). Still, Salih Zeki’s statement is correct as a historical fact.

Riemann’s study is also among the works that are not provided in their original titles. Salih Zeki calls Riemann’s habilitation dissertation ““…hendesenin esasını teşkil

[41] *Geometrical Researches on the Theory of Parallels* (1840)

[42] This Ottoman phrase corresponds to “New foundations of Geometry with a Complete Theory of Parallels” (1835-37).

[43] Imaginary Geometry

[44] The complete title of Wolfgang Bolyai’s work is *Tentamen Juventutem Studiosam in Elementa Mathesos* (Essay on the Elements of Mathematics for Studios Youths).

eden faraziye"ye dair bir mütala'aname…"[45] (S.Zeki, Darülfünun Konferansları, 1331, p. 12). According to Salih Zeki, Riemann read this article in Göttingen in 1854, which is correct, but he does not indicate that this was the lecture that Riemann delivered in order to complete his habilitation.[46]

Salih Zeki asserts that Helmholtz published his first work related to non-Euclidean geometries after Riemann's death. He does not point out a certain date and only states an Ottoman title which is "Hendesenin Mü'esses Bulunduğu Mebadiye da'ir Muhtıra".[47] The second work of Helmholtz in *Darülfünun Konferansları* is "Hendeseye Esas Olan Hadisata da'ir Makale"[48] which was published in 1868 (S.Zeki, Darülfünun Konferansları, 1331, p. 12).

The last original work that is mentioned in Salih Zeki's account is Cayley's "A Sixth Memoir upon Quantics". Salih Zeki provides this English title together with an Ottoman translation of it, that is, "Kemmiyyata da'ir Altı Muhtıra" (S.Zeki, Darülfünun Konferansları, 1331, p. 47).

In his first five lectures Salih Zeki does not examine the content of Lobachevski's work directly, but through a secondary source. Bolyai's work is only introduced and not studied, since Salih Zeki thinks that it would only be a repetition of what he explains about Lobachevski's geometry. It is not very clear whether Riemann's original work is examined or the explanations are taken from a secondary source. Salih Zeki's views regarding Riemann's geometry are confirmed by the geometer's own article, however this does not exhaust the possibility that Salih Zeki may have studied Riemann's work through other sources. The same is true for Helmholtz. What can be asserted about Helmholtz's works in *Darülfünun Konferansları* is that Salih Zeki studies "On the Facts Underlying Geometry" in comparison with

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[45] The title given by Salih Zeki is the Ottoman phrase for "On the Hypothesis that Constitutes the Bases of Geometry".

[46] The original title of Riemann's habilitation dissertation is "Über die Hypothesen welche der Geometrie zu Grunde Liegen" and it is translated into English as "On the Hypotheses which Lie at the Bases of Geometry".

[47] This title would correspond to "on the actual foundations of geometry". The phrase that is mentioned by Salih Zeki actually stands for Helmholtz's "Über die Tatsachlichen Grundlagen der Geometries"(1866).

[48] On the Facts Underlying Geometry (1868).

Riemann's hypotheses. Secondly, Helmholtz's 1870 lecture in Heidelberg, "On the Origin and Significance of Geometrical Axioms" constitutes the content of Salih Zeki's third lecture. And this is not among the works of Helmholtz that were introduced in Salih Zeki's account on the publications of the geometer.

It can clearly be observed that Salih Zeki is following two major works of Poincaré in his fifth lecture and actually he presents these works nearly word by word. The fifth lecture starts with presenting Poincaré's expressions in his *Science and Method* (1908). He mainly follows the section entitled as "The Relativity of Space". Afterwards, the fifth lecture presents Poincaré's "Space and Geometry" in his *Science and Hypothesis* (1901).

I have pointed out that Salih Zeki criticizes Russell's views on the philosophical bearing of imaginary numbers and also he thinks that geometrical axioms are conventions which Russell would not agree with. However, Salih Zeki's ideas in his first four lectures, the logical concerns he presents and especially the organization of the content of his lectures suggest that the source he uses is Russell's *An Essay on the Foundations of Geometry* (1897).

I would like to propose my claim by comparing Russell's and Salih Zeki's expressions. The following ideas and quotations from Russell's *essay* are paraphrased in *Darülfünun Konferansları* and sometimes they are exactly translated into Ottoman Turkish.

Russell in a chapter entitled as "A Short History of Metageometry", provides a story which starts by pointing out Legendre's studies to be the last step before the rejection of the parallel postulate (Russell, 1897, p. 7). Similarly, Salih Zeki states that Legendre's study should be accounted for since this geometer's failure encouraged the following geometers to reject the mentioned postulate (S.Zeki, Darülfünun Konferansları, 1331, p. 6).

Salih Zeki provides some explanations that he claims to be declared by Gauss, Lobachevski and Bolyai, which are exact translations of the following statements:

> If the axiom of parallels is logically deducible from the others, we shall, by denying it and maintaining the rest, be led to contradictions. (Russell, 1897, p. 8) (S.Zeki, Darülfünun Konferansları, 1331, p. 6)

> These three mathematicians [Gauss, Bolyai and Lobachevski], accordingly, attacked the problem indirectly: they denied the axiom of parallels, and yet obtained a logically consistent Geometry. (Russell, 1897, p. 8) (S.Zeki, Darülfünun Konferansları, 1331, p. 7)

Although Salih Zeki claims that he regards geometry's progress to be of three periods in accordance with Klein's classification (S.Zeki, Darülfünun Konferansları, 1331, p. 12), what he provides in *Darülfünun Konferansları* differs than Klein's classification and agrees with Russell's consideration of the subject. Klein determines the first period of non-Euclidean geometry to be in terms of elementary geometry and to include the studies of Lobachevski and Bolyai (Klein, 1894, p. 85). The second period in Klein's classification is described "from the point of view of projective geometry" (Klein, 1894, p. 85). And the studies of Riemann and Helmholtz constitute the last period in this classification. On the other hand, Salih Zeki gives these three periods in a different sequence than Klein, where Riemann and Helmholtz constitute the second era and Cayley's studies find their place in the last period. And it is not Klein but Russell who regards Gauss's studies to be a part of the first period. Also, the philosophical significance of the era which started with Riemann's studies is emphasized in Russell's account and not in Klein's (Russell, 1897, p. 8) (S.Zeki, Darülfünun Konferansları, 1331, p. 13).

Both Russell and Salih Zeki consider Lobachevski's and Bolyai's studies to be indistinguishable except the difference in their postulates:

> Only the initial postulates, which are more explicit than Lobatschewsky's demand a brief attention. (Russell, 1897, p. 12)
>
> Şu kadar ki Bolyai'nin kabul eylediği mevzu'at Lobachevski'nin mevzu'atına nisbetle daha vazihtirler.[49] (S.Zeki, Darülfünun Konferansları, 1331, p. 9)

When Salih Zeki concludes his ideas about the first period in non-Euclidean geometries he provides a translation of the following explanation:

> Buraya kadar verdiğim izahattan da tezahür ediyor ki gerek Lobachevski, gerek Bolyai'nin nazarları bir noktaya ma'tuf idi: Bu nokta da mevzu'e-i Öklidis'in mantıken diğer mevzu'e ve müte'arifelere gayr-i tabi' bulunduğunu ispat etmekten 'ibaret idi.[50] (S.Zeki, Darülfünun Konferansları, 1331, p. 11)

[49] Vazih: clear

[50] Tezahür: appear, manifest/ ma'tuf: directed/ gayr-i tabi': independent

And similarly, if not exactly, Russell proposes the following claim:

> It is important to remember that, throughout the period we have just reviewed, the purpose of the hyperbolic Geometry is indirect: not the truth of the latter, but the logical independence of the axiom of parallels from the rest, is the guiding motive of the work. (Russell, 1897, pp. 12-13)

Salih Zeki's concern about the logical consistency of the new geometries can also be found Russell's *essay*:

> Of course, it remained possible that, by further development, latent contradictions might have been revealed in these systems. This possibility, however, was removed by the more direct and constructive work of the second period… (Russell, 1897, p. 13) (S.Zeki, Darülfünun Konferansları, 1331, p. 11)

Moreover, Salih Zeki's explanations on the notion of curvature correspond exactly to the statements of Russell:

> Tul fikri esasen hatt-ı müstakimden iktibas edildiği halde münhanileri gayet asgar kısımlara taksim etmek sayesinde bunlara da tatbik edilebilmiştir[51]… Bunun gibi inhina fikri de da'ireden iktibas edilmiş ve yine asgar-ı namütenahi kısımlara taksim etmek sayesinde diğer münhaniyata tatbik olunabilmiştir.[52] (S.Zeki, Darülfünun Konferansları, 1331, p. 15)
>
> Just as the notion of *length* was originally derived from the straight line, and extended to other curves by dividing them into infinitesimal straight lines, so the notion of *curvature* was derived from the circle, and extended to other curves by dividing them into infinitesimal circular ares. (Russell, 1897, p. 17)

As a last example, I would like to show that Salih Zeki's description of Cayley's work is again an expression which appears in Russell's *essay*.

> Bu risalede mü'ellif bütün bu'da 'a'it tasavvuratı sırf tersimi olan mebadi üzerine bina etmek istiyor idi.[53] Mesafe, zaviye ve ila-ahirihi mefhum-ı kemmi veya mikyasiyeleri suver-i irtisamiyyeye irca' ediyor[54], ve kendi zamanına

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[51] Tul: length/ hatt-ı müstakim: straight line/ iktibas: borrow/ münhani: curve/ asgar: small

[52] İnhina: curvature/ asgar-ı namütenahi: infinitely small, infinitesimal/ münhaniyat: curves

[53] Bu'd: distance/ sırf tersimi: purely projective/ mebadi: principles

[54] Mesafe: length/ zaviye: angle/ ila-ahirihi: etc., et cetera/ mefhum-ı kemmi: quantitative notions/ mikyasi: quantitative, measurable/ suver-i irtisamiye: projective aspects, projective manners

> kadar istihsalı müyesser olamayan bir sadelik, 'adeta bir "vahdet-i usul" vücuda getiriyor idi.[55] (S.Zeki, Darülfünun Konferansları, 1331, p. 47)
>
> It begins by reducing all so-called metrical notions - distance, angle, etc.- to projective forms, and obtains, from this reduction, a methodological unity and simplicity before impossible. (Russell, 1897, p. 28)

In conclusion, Russell's ideas in his *An Essay on the Foundations of Geometry* spread into the first four lectures in *Darülfünun Konferansları.* Russell's book stands as the source in terms of which Salih Zeki defines the problem of parallels or considers the logical consistency of the non-Euclidean geometries. In other words, Russell's *essay* is the source which shapes the story provided by Salih Zeki into an axiomatic history of geometry. Salih Zeki does not agree with Russell's views on imaginary numbers and takes Poincaré's side when the issue is the nature of geometry and its axioms. However, Russell's *essay* constitutes one of the main sources of Salih Zeki's lectures, together with Poincaré's *Science and Method* and *Science and Hypothesis*.

[55] İstihsal: obtain/ müyesser: can be accomplished/ vahdet-i usul: unity of method

CHAPTER 5

CONCLUSION

This thesis was an attempt to evaluate Salih Zeki's lectures on non-Euclidean geometries. Secondly, I aimed to make *Darülfünun Konferansları* accessible. In this sense, I provided an English summary of the mentioned lectures and tried not to leave out anything that would count as the subject matter of these lectures. The mathematical notions that took place in through the lectures were given together with Salih Zeki's Ottoman terms for them in brackets. Furthermore, I provided quite a lot of quotations from Salih Zeki's lectures with the English meaning of the obsolete words that may not be found in Turkish dictionaries in our day, so that the reader could have the chance to figure out how geometry was narrated in Ottoman Turkish.

I devoted a chapter for a summary of the history of non-Euclidean geometries in the western world, in order to provide the required knowledge for following Salih Zeki's lectures. This summary also stands as an illustration of an appropriate way of writing geometry's history. In the same chapter, the evaluation of the "standard account" on the history of geometry showed that it is a result or regarding geometry to be axiomatic. I objected the standard account since its expressions do not constitute sufficient or even correct explanations on geometry's progress. When this progress is regarded to be axiomatic, the history of geometry is already determined. Such an outlook would classify historical facts under logical notions imposed on geometry's progress. If the history of geometry is merely a study of axioms, then logic should be able to answer all the questions that may be directed to this history. And indeed it answers some of them. One of the most obvious questions that are asked to geometry's history is: Why was the parallel postulate problematic? Another decisive question would be: How did the first non-Euclidean geometries appear? An axiomatic approach would answer the former by stating that the parallel postulate is logically independent from the rest of Euclid's geometry which makes its demonstration impossible within this system. And the answer of the second question is the rejection of the parallel postulate. These questions can be answered in terms of an axiomatic account since it decides what kind of answer they should have from the beginning.

Lobachevski's and Bolyai's intentions were different at their starting points; yet they ended up with the same type of geometry. If the historian leaves aside considering these works in terms of solely logical grounds, she would come up with fruitful explanations. A mathematician may not distinguish these works, since they would provide her with the same geometrical practice. However, for a historian this may be the very significant point about two different geometrical works. Lobachevski set forth a new definition of parallelism, in which the basic notions of Euclid's parallelism were violated. He clearly rejected the Euclidean parallelism. On the other hand, Bolyai was only after the propositions that did not include the application of the parallel postulate, since he regarded them to be absolutely true. The different approaches of these two geometers should be considered as two different confirmations of the possibility of practicing geometry in a presupposed hyperbolic space. Also, since it is a historical fact that Lobachevski had an axiom of parallelism from the beginning but Bolyai did not, this should be convincing the historian that the practice or space in hand either agrees with Euclid's parallel postulate or it may not. In other words, geometrical practice does not flow from a set of axioms, but has some presuppositions. In this sense, an axiomatic structure can only be assigned to a geometrical system and its progress subsequent to accepting it. And the progress of geometry is more complex than a clear logical path. That it can be presented in terms of axioms, does not necessarily mean that geometry is an axiomatic discipline, and neither its history is.

The course of geometry should be regarded as a continuum, and not a very regular one. The history of geometry includes some gaps that cannot be explained by an axiomatic view and also some facts that would look contradictive from a logical point of view. The negations of the parallel postulate were already examined in the 18th century, and the possibility of triangles with an angle sum less than two right angles could not be rejected. However, a geometrical system that would agree with the hypothesis of the acute angle did not pop up immediately. It required a belief in working with non-Euclidean terms some mathematical improvements. If the focus of an account on the history of geometry was the logical aspects of the practice, the existence of a spherical geometry together with the rejection of the hypothesis of the acute angle does not make sense (Gray, 2003). An axiomatic account cannot explain why the geometers were bothered with the hypothesis of the obtuse angle, while they already had a spherical geometry. Or why Euclid's spherical geometry persisted even though the hypothesis of the obtuse angle was disproved.

The account I provided in the second chapter of my thesis, presented the historical facts in a way that does not distort actual history. And the comparison of the two different perspectives reveals the criteria that should be granted in writing geometry's history. I also pointed out that the axiomatization of geometry at the beginning of the 20th century was the reason why the history of geometry was provided in terms of standard-axiomatic accounts. Chapter two and the next chapter in which I summarized Salih Zeki's lectures together constitute the required grounds for the evaluation of *Darülfünun Konferansları*; and this is the subject matter of the fourth chapter.

When considered as a mathematician Salih Zeki's explanations on the content of the original works are pretty qualified. And it would be unfair to underestimate his efforts to present non-Euclidean geometries, but his account is another example of writing geometry's history with the presupposition that it is an axiomatic discipline. Salih Zeki can deal with the mathematical stuff concerning non-Euclidean geometries. However, he does not point out the change in the mathematical methods that served geometry's progress. One cannot learn from Salih Zeki's account that the early non-Euclidean studies, which did not mean to be so, were in terms of classical geometry, or that the hyperbolic functions blended into the geometrical practice in the 19th century. On the other hand, he is aware of the fact that Lobachevski and Bolyai directly put forward geometrical systems. He can explain that Riemann and Helmholtz investigated space in general as a preliminary to their geometries, and that Cayley's work departed from by the definition of distance.

The problem about the inquiry that was carried out in *Darülfünun Konferansları* is that whenever Salih Zeki describes a progressive step in history of geometry, his account cannot escape from the effect of axiomatic ideas. His ideas regarding the origin of the new geometries and the geometrical developments that confirmed the possibility of practicing in terms of them are all logical explanations. In fact, according to Salih Zeki, Beltrami's model or Klein's interpretations in terms of projective geometry were merely affirming the consistency of the new systems.

A more comprehensive account which is not stuck with axiomatic concerns would be able to point out the relations that were set up by Beltrami and Klein, to be confirming the possibility of applying the new geometries to the external world. Euclid's geometry was a tool for classical physics, in other words, classical physics interpreted the external world by means of Euclid's geometry. Salih Zeki's account is not capable of concluding that Beltrami's and Klein's studies supplied the new geometries with the opportunity of entering into the field of physics. He points out that each theorem of Euclid's geometry corresponds to a theorem of the non-Euclidean geometries. However, his aim is to explain the consistency of the new geometries relatively to Euclid's system. Salih Zeki concludes that if the new geometries included any contradiction, Euclid's geometry had to include contradictions too because of the one-to-one correspondence between the theorems of the new and the old geometries. If Salih Zeki's views were the case for geometry, then the appreciation of the new geometries stems from realizing that they are logically consistent. On the contrary, the logical consistency of a new theory cannot sweep away the faith in the customarily adapted system. In the same way, it cannot be the fact that non-Euclidean theories were accepted because of their logical consistency. It is more appropriate to put forward that they found a place in the scientific practice since it was demonstrated that whatever can be done in terms of Euclid's geometry, could also be practiced in terms of non-Euclidean geometry.

In conclusion, *Darülfünun Konferansları* is not a sufficient account on the history of geometry. Salih Zeki's ideas on geometry's progress are leaded by some prejudices on the geometrical discipline. It is not clear whether he constructs his account on axiomatic ideas on purpose or else the structure of his account only happens to be so by means of the secondary sources he makes use of. Either way, Salih Zeki's lectures turn out to be under the effect of the axiomatization of geometry.

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APPENDIX

TEZ FOTOKOPİSİ İZİN FORMU

ENSTİTÜ

Fen Bilimleri Enstitüsü ☐

Sosyal Bilimler Enstitüsü ☐

Uygulamalı Matematik Enstitüsü ☐

Enformatik Enstitüsü ☐

Deniz Bilimleri Enstitüsü ☐

YAZARIN

Soyadı : Kadıoğlu

Adı : Dilek

Bölümü : Felsefe

TEZİN ADI : SALİH ZEKİ'S *DARÜLFÜNUN KONFERANSLARI* AND HIS TREATMENT OF THE DISCOVERY OF NON-EUCLIDEAN GEOMETRIES

TEZİN TURÜ : Yüksek Lisans ☐ Doktora ☐

1. Tezimin tamamından kaynak gösterilmek şartıyla fotokopi alınabilir. ☐
2. Tezimin içindekiler sayfası, özet, indeks sayfalarından ve/veya bir bölümünden kaynak gösterilmek şartıyla fotokopi alınabilir. ☐
3. Tezimden bir bir (1) yıl süreyle fotokopi alınamaz. ☐

TEZİN KÜTÜPHANEYE TESLİM TARİHİ: