AN INVESTIGATION OF PRE-SERVICE MIDDLE SCHOOL MATHEMATICS TEACHERS' ACHIEVEMENT LEVELS IN MATHEMATICAL PROOF AND THE REASONS OF THEIR WRONG INTERPRETATIONS

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Prof. Dr. Meliha ALTUNIŞIK
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science

Prof. Dr. Jale ÇAKIROĞLU
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science

Assoc. Prof. Dr. Mine ISIKSAL-BOSTAN
Supervisor

Examining Committee Members

Assoc. Prof. Dr. Erdinç ÇAKIROĞLU (METU, ELE)

Assoc. Prof. Dr. Mine IŞIKSAL-BOSTAN (METU, ELE)

Assoc. Prof. Dr. Yezdan BOZ (METU, SSME)

Assist. Prof. Dr. Çiğdem HASER (METU, ELE)

Assist. Prof. Dr. Didem AKYÜZ (METU, ELE)

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last name : Esra DEMİRAY

Signature :

ABSTRACT

AN INVESTIGATION OF PRE-SERVICE MIDDLE SCHOOL MATHEMATICS TEACHERS' ACHIEVEMENT LEVELS IN MATHEMATICAL PROOF AND THE REASONS OF THEIR WRONG INTERPRETATIONS

Demiray, Esra
M.S., Department of Elementary Science and Mathematics Education
Supervisor: Assoc. Prof. Mine IŞIKSAL-BOSTAN

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The first purpose of the study is to investigate pre-service middle school mathematics teachers' achievement levels in refutation, proof by contrapositive and proof by contradiction. The second purpose is to determine the reasons of their wrong interpretations in mentioned proof methods. The third purpose of the study is to investigate to what extent they can conduct valid proofs. As related to the third purpose, proof methods that they used in their valid proof and the reasons of conducting invalid proofs were also investigated.

The participants of the study were selected through convenience sampling. Data was collected from the pre-service middle school mathematics teachers enrolled in a state university in Ankara. To address research questions, frequencies and percentages of the data for every question was used and item based in-depth analysis was employed.

The results of the study indicated that pre-service middle school mathematics teachers' achievement levels in refutation and proof by contradiction are considerably high. However, they mostly answered to contrapositive questions wrongly. The reasons of their wrong interpretations in the mentioned proof methods were also determined.

The results of the open-ended proof questions revealed that more than half of the students can conduct valid proof for all of the statements. When students' valid proof were analyzed, it was seen that mathematical induction and direct proof were mostly used. When students' invalid proofs were analyzed, it was seen that 'using the numbers to prove the statement' and 'direct restatement of the given expression' were the common reasons of their invalid proofs.

Keywords: Mathematical proof, pre-service middle school mathematics teachers, proof by contradiction, proof by contrapositive, refutation

ÖZ

ORTAOKUL MATEMATİK ÖĞRETMEN ADAYLARININ MATEMATİKSEL İSPAT BAŞARI DÜZEYLERİNİN VE YANLIŞ ANLAMLANDIRMA NEDENLERİNİN İNCELENMESİ

Demiray, Esra
Yüksek lisans, İlköğretim Fen ve Matematik Alanları Eğitimi
Tez Danışmanı: Doç. Dr. Mine IŞIKSAL-BOSTAN

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Çalışmanın birinci amacı, ortaokul matematik öğretmen adaylarının aksine örnek verme, olmayana ergi ve çelişki ile ispat yöntemlerindeki başarı düzeylerini incelemektir. Ikinci amacı, ortaokul matematik öğretmen adaylarının yukarıda adı geçen ispat yöntemlerindeki yanlış anlamlandırmalarının nedenlerini belirlemektir. Çalışmanın üçüncü amacı, ortaokul matematik öğretmen adaylarının geçerli ispat yapabilme düzeylerini araştırmaktır. Ayrıca ortaokul matematik öğretmen adaylarının hangi ispat yöntemlerini kullandıkları ve geçersiz ispatlarının nedenleri de araştırılmıştır.

Çalışmanın katılımcıları elverişli örneklem yoluyla belirlenmiştir. Veriler Ankara'daki bir devlet üniversitesindeki ortaokul matematik öğretmen adaylarından toplanmıştır. Verilerin analizinde, her soru için frekans ve yüzdeler kullanılmıştır ve her soru derinlemesine incelenmiştir.

Çalışmanın sonuçları ortaokul matematik öğretmen adaylarının aksine örnek verme ve çelişki ile ispat yöntemlerindeki başarı düzeyleri yüksek olduğunu göstermiştir. Fakat, ortaokul matematik öğretmen adayları karşıt tersi ile ispat sorularına daha çok yanlış cevap vermişlerdir. Öğretmen adaylarının adı geçen ispat yöntemlerindeki yanlış anlamlandırmalarının nedenleri de belirlenmiştir.

Açık uçlu ispat sorularının sonuçları, verilen ifadelerin hepsinde öğrencilerin yarıdan fazlasının geçerli ispat yapabildiğini göstermiştir. Öğrencilerin geçerli ispatları analiz edildiğinde, verilen iki ifadede matematiksel tümevarım yöntemi diğer ifadede ise doğrudan ispat yöntemi en çok kullanılan ispat yöntemleri olduğu görülmüştür. Öğrencilerin geçersiz ispatları incelendiğinde, 'ifadeyi ispatlamak için sayıları kullanmak' ve 'verilen ifadenin doğrudan tekrar ifade edilmesi' en çok görülen geçersiz ispat yapma nedenleri olarak bulunmuştur.

Anahtar Kelimeler: Matematiksel ispat, ortaokul matematik öğretmen adayı, çelişki ile ispat, olmayana ergi, aksine örnek verme

To My Family

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f	Frequency
MoNE	Ministry of National Education
NCTM	National Council of Teachers of Mathematics
MPQ	Mathematical Proof Questionnaire

CHAPTER I

INTRODUCTION

Proof is a concept in the scaffold of mathematics (Heinze & Reiss, 2003) and many researchers consider proof and logical reasoning as one of the central components of mathematics (Almeida, 2000; Baylis, 1983; CadwalladerOlsker, 2011; Dickerson, 2008; Hanna, 2000; Jones, 1997; Knapp, 2005; Knuth, 1999; Ko, 2010; Mariotti, 2006; Martin & Harel, 1989; Schoenfeld, 1994). Similarly, according to Ross (1998), the essence of mathematics lies in proofs. VanSpronsen (2008) accepts proofs as the cornerstone of mathematics. As Davis and Hersh (1981) stated, "in the opinion of some, the name of the mathematics game is proof; no proof, no mathematics" (p. 147). Moreover, Heinze and Reiss (2003) stated the status of proof in mathematics as; "Mathematics is a proving discipline, which represents the main difference between mathematics and any other scientific discipline" (p.1).

Since proof is one of the core components of mathematics, it should be in the mathematics curriculum at all levels (Hanna & de Villiers, 2008; Schoenfeld, 1994). Considering reasoning and proof as one of the process standards of the National Council of Teachers of Mathematics (NCTM), students' competency in proof is an important issue for all grades. According to NCTM (2000), instructional programs from pre-kindergarten through grade 12 should enable all students to

> Recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and evaluate mathematical arguments and proofs; and select and use various types of reasoning and methods of proof (p.56).

Thus, proof is accepted as one of the fundamental parts of mathematics education by many researchers. The reflections of this situation could also be seen in the mathematics curriculum in Turkey. The general objectives of mathematics education in Turkey were revised by the Ministry of National Education (MoNE, 2005) with a focus on the necessity and the importance of proof and reasoning. Some of the objectives related to proof are that students will be able to make inferences with mathematical induction and deduction, to use mathematical terminology and language correctly in explaining and sharing mathematical ideas in a logical manner, and to express their mathematical ideas and reasoning in mathematical problem solving processes.

When mathematics curricula in elementary schools and high schools in Turkey were investigated, it was seen that reasoning is one of the fundamentals at both elementary school level and high school level, while proof is mainly studied at high school level. In the Mathematics Curriculums of the 6^{th}-8^{th} grades (MoNE, 2009), it is stated that it is necessary to show students the importance of reasoning and discussion, and to help students gain reasoning abilities. In addition, it is stated that preparing learning environments in which students can develop their reasoning abilities is important. To this end, that is, to develop students' reasoning abilities, some objectives were determined. Some of these objectives are that students are able to use reasoning in the learning process, in daily life, and in other courses including mathematics courses; students are able to make inferences and generalizations while learning mathematics; students are able to discuss whether statements are true or not; students are able to develop self-confidence in reasoning and positive attitudes towards reasoning. In the Mathematics Curriculum of the 9^{th}-12^{th} grades (MoNE, 2011), it is stated that logic is one of the learning areas in mathematics. In grade 9, propositions and proof methods are subjects within the scope of the learning topic of logic. It is also observed that proof is an important part of teaching in other learning areas such as algebra, trigonometry and linear algebra.

In the recent revisions of mathematics curriculum, the importance of reasoning and proof were also stated. In the Mathematics Curriculum of 5^{th}- 8^{th} grades (MoNE, 2013), reasoning was defined as the process of forming new knowledge by using mathematical materials such as symbols, definitions and relationships, and techniques such as induction, deduction and comparison. Students who gain and develop reasoning skills are able to justify the assertions, to conduct logical generalizations, to use mathematical patterns and relations in the analysis of a mathematical situation, suggesting predictions by using appropriate strategies.

Moreover, in the Mathematics Curriculum of 9^{th}- 12^{th} grades (MoNE, 2013), reasoning and proof were described as one of the mathematical process skills. As related to helping students to develop their reasoning skills, the following objectives were stated; to make logical generalization and assertions in mathematics and daily life, to verify their assertions and ideas, to use mathematical models, rules and relationships in their explanations, to use mathematical relations in the analysis of a mathematical case, to explain mathematical relations, suggesting meaningful and appropriate predictions, to apply general cases for specific cases, to explain their mathematical assertions by using models, propositions and relations, to use induction and deduction effectively, and to chose the most appropriate proof method in proving a mathematical proposition.

Literature review also showed that there is neither a stable and common definition of proof nor a consensus in what can be accepted as proof. For example; Saeed (1996) defined mathematical proof as “a logical sequence of statements leading from a hypothesis to a conclusion using axioms, definitions, previously proved statements and rule of inference” (p.12). Rota (1997) simply explained proof of a mathematical theorem as a sequence of steps which brings the desired conclusion. The common point in the mentioned definitions is to reach a desired conclusion by following logical steps. Proof as defined by Mingus and Grassl (1999) is “a collection of true statements linked together in a logical manner that serves as a convincing argument for the truth of a mathematical statement” (p.441). The difference of the definition provided by Mingus and Grassl (1999) from the aforementioned two definitions is that the definition involves the convincing function of proof about the truth of a statement. Similarly, Selden and Selden (2003) described proof as a concept that provides consensus on the validity of information in mathematics. Evidently, every definition of proof does not have the same focus and purpose in their formation. In other words, some definitions focus on the structure of proof, some of them focus mainly on the functions of proof such as verifying that a statement is true and some of them are broader as a means of covering some functions of proof like showing that a statement is true or false as well as its structure.

In the same vein, researchers varied in their statements of the necessary components and characteristics of a mathematical proof. Tall (1989) stated that mathematical proof must be based on two ideas, which are clearly formulated definitions, statements and agreed procedures to deduce the truth of one statement from another. On the other hand, according to Alibert and Thomas (2002), formulation of conjectures and development of proof are two fundamental aspects of mathematical proof. Stylianides and Stylianides (2009) stated the characteristics of a clear and convincing proof as using appropriate language and definitions, emphasizing key points, including clear, complete and meaningful sentences and being applicable to different areas.

Although definitions and components of proof stated by researchers somewhat differ from each other, there is an agreement on the case that proof is a comprehensive and complex theme. According to Knapp (2005), learning to prove is not a simple task. While learning to prove, students need to have content knowledge, knowledge specific to proving, problem solving skills and ability to read proof. Also, students should be using notations, symbols and examples appropriately. Similarly, Epp (2003) stated that to determine and show whether a mathematical statement is true or false is a complex cognitive activity. Competence in proof requires methodological knowledge consisting of three aspects, namely proof scheme, proof structure and logical chain. While providing or reading proof, all three aspects should be considered (Heinze & Reiss, 2003). According to Selden and Selden (2003), reading proof is a more complex task than expected; hence, in some studies, students were asked to read and write proof in order to evaluate whether they learned the subject or not.

Since proof is considered as one of the main parts in the structure of mathematics, students' deficiency in mathematical proof is a major problem (VanSpronsen, 2008). Mathematical proof is a difficult and challenging concept for many students (Almeida 2000; Gibson, 1998; Güler, Kar, Öçal & Çiltaş, 2011; Harel & Sowder, 1998; Knapp, 2005; Moore, 1994; Selden & Selden, 2003; Weber, 2001). After the 1980s, studies related to proof and justifications in mathematics education have increased. In these studies, misconceptions about proof and students' difficulties in proof are themes that were mostly investigated (Dickerson, 2008).

Since there are various points related to proof that students may have difficulty in, mathematics teachers have a critical role in detecting and overcoming these difficulties. Teachers' ability to teach mathematics by focusing on reasoning and proving directly depends on their content knowledge in mathematics (Stylianides & Stylianides, 2006). Since their knowledge of proof could affect students' proving skills, they should be competent in conducting proofs. Therefore, pre-service mathematics teachers should be trained in such a way that they have the necessary content knowledge in the subjects they will teach (Baştürk, 2009, 2011). In addition to content knowledge, mathematics teachers' attitudes and ideas about proof could affect students' perceptions of proof. As pre-service mathematics teachers are teachers of the future, pre-service mathematics teachers' achievements and their difficulties while proving are important subjects for investigation.

Considering the importance of proof in mathematics education, the inadequacy of pre-service mathematics teachers in conducting proof and the lack of the studies regarding proof methods; pre-service middle school mathematics teachers' achievement levels in some proof methods, the reasons of their wrong interpretations in proof methods, their ability to conduct valid proofs, proof methods that pre-service middle school mathematics teachers used in valid proofs and the reasons of conducting invalid proofs were determined as main research aims of this study.

1.1. Purpose of the Study

The purposes of the study are threefold. The first purpose of the study is to investigate pre-service middle school mathematics teachers' achievement levels in refutation, proof by contrapositive and proof by contradiction methods in terms of year level. The second purpose of the study is to determine the reasons of their wrong interpretations in mentioned proof methods. The third purpose of the study is to examine to what extent pre-service middle school mathematics teachers can conduct valid proofs. As related to the last purpose, proof methods that pre-service middle school mathematics teachers used in valid proofs and the reasons of their invalid proofs were also investigated.

Since pre-service middle school mathematics teachers' answers were evaluated in terms of year level, data were collected from freshmen, sophomores, juniors and seniors in elementary mathematics education program of the selected university.

Considering these aims, the research questions of this study were stated as follows.

1.2. Research Questions

1. What are pre-service middle school mathematics teachers' achievement levels in refutation, proof by contrapositive and proof by contradiction methods in terms of year level?

2. What are the reasons of pre-service middle school mathematics teachers' wrong interpretations in refutation, proof by contrapositive and proof by contradiction methods?

3. To what extent can pre-service middle school mathematics teachers conduct valid proofs?

    3.1. Which proof methods do pre-service middle school mathematics teachers use in valid proofs?

    3.2. What are the reasons of pre-service middle school mathematics teachers' invalid proofs?

1.3. Significance of the Study

Literature review showed that students in every level even at the university level have difficulties in logical reasoning and proof (Güler, Kar, Öçal & Çiltaş, 2011; Moore, 1994; Sarı, Altun & Aşkar, 2007; Selden & Selden, 2003; Weber, 2001). At this point, to develop students' perceptions of proof, taking up proof starting from elementary grades may help to overcome students' misconceptions related to proof and reasoning. Moreover, middle school mathematics teachers' knowledge in methods of proof and ability in conducting proof could affect their students' knowledge, perception, attitudes and ideas regarding proof. In this regard, middle school mathematics teachers have a critical role.

Middle school mathematics teachers should have necessary knowledge about proof in order to help middle school students to develop their reasoning skills, to gain ability in justifying their assertions, making inferences and logical generalizations while learning mathematics, and to have positive attitudes towards proof. Since pre-service middle school mathematics teachers are mathematics teachers of the future, they should also understand the importance of reasoning for middle school students and focus on developing their reasoning abilities. Therefore, to investigate pre-service middle school mathematics teachers' achievement levels, their wrong interpretations and their ability to prove are worth to investigate.

Since teacher education is an important issue in almost every society (Aslan, 2003), content of the courses in teacher education programs should be determined carefully. In this manner, this study could also contribute to the development of the content of the mathematics courses in elementary mathematics teacher education programs. In more detail, the content of mathematics courses might be modified according to the determined reasons of students' mistakes and the points they might be having difficulty in proving.

As mentioned, data were analyzed in terms of year level to reveal the differences among year levels. To this end, data were collected from freshmen, sophomores, juniors and seniors in the Elementary Mathematics Education program of the selected university. Also, it provided some insight in the indirect effects of the courses in mathematics teacher education programs on student' understanding of proof on the basis of year level. Since determining the reasons of pre-service middle school mathematics teachers' wrong interpretations related to some proof methods and the reasons of their invalid proofs were among purposes of the study, the results may help pre-service middle school mathematics teachers and teacher educators to realize pre-service teachers' common mistakes while conducting proofs and suggest solutions to correct them.

1.4. Assumptions and Limitations

The study is based on the following assumptions and has the limitations mentioned below.

It was assumed that pre-service middle school mathematics teachers who participated in the study responded to the items accurately. Another assumption was that the Mathematical Proof Questionnaire, which was administered to the participants, was appropriate to the purpose of the study.

As for the limitations of the study, the results of the study are limited to the data collected from 115 pre-service middle school mathematics teachers in a state university. Since mathematics courses taken by students in every year level are not exactly the same in terms of the instructor of the course and the textbooks used, this may be considered as a limitation for the study. Another limitation is that the instrument was administered to the students in different time periods, such as before or after the courses and in the morning or in the afternoon. This situation might have affected students' answers to the questions. Moreover, questions in Section A and Section B of Mathematical Proof Questionnaire may affect students' answers to questions in Section C. In other words, Section A and Section B involve some valid proofs and students were asked to prove given statements in the questions of Section C. Students may use given proof methods in Section A and B to prove given statements in Section C. Finally, using convenience sampling method may be regarded as an obstacle in terms of generalization. In other words, the generalizability of the results to a larger population is limited.

1.5. Definitions of Important Terms

The constitutive and operational definitions of the important terms used in the study are presented in this section.

Mathematical proof

Stylianides (2007) defined proof as a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:

1. It uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification;

2. It employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community;

3. It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community. (p. 291).

In this study, mathematical proof was defined as "a formal and logical line of reasoning that begins with a set of axioms and moves through logical steps to a conclusion" (Griffiths, 2000, p. 2).

Proof by Contradiction

Riley (2003) defined the contradiction method as “to prove $p \Rightarrow q$, this method assumes that p and negation of q is true and then deduces any contradiction” (p.18).

In this study, Question 1 and Question 3 in Section B were prepared based on proof by contrapostive.

Proof by Contrapositive

Riley (2003) defined the contrapositive method as “to prove p⇒q, this method consists of giving direct proof of the contrapositive of the statement” (p.19).

In this study, Question 2, Question 3 in Section A and Question 2 in Section B were prepared based on proof by contradiction.

Refutation

Riley (2003) defined refutation as “the process of proving a statement is false or wrong by argument or evidence” (p.19).

In this study, Question 1, Question 4 in Section A and Question 4 in Section B were prepared based on refutation.

Achievement level in proof methods

In this study, pre-service middle school mathematics teachers’ achievement levels in proof by contrapositive, proof by contradiction and refutation were measured by means of the Mathematical Proof Questionnaire, which was developed by the researcher of the study.

Valid proof

Weber (2008) stated three perspectives regarding determination of valid proof. According to the first perspective, valid proof involves a formal structure which is constructed by well-defined and explicitly stated mathematical and logical rules. The second perspective is that valid proof is convincing to a mathematician. The last perspective states that determining whether a proof is valid is a matter of social negotiation and agreement.

In this study, pre-service middle school mathematics teachers’ arguments were accepted as valid proof if they chose and applied proof methods such as direct proof, proof by contrapositive, proof by contradiction and mathematical induction for the statement correctly and deduced the desired conclusion.

Invalid proof

In this study, pre-service middle school mathematics teachers' arguments were accepted as invalid proof if they did not reach a desired conclusion by following logical steps.

Pre-service middle school mathematics teachers

Pre-service middle school mathematics teachers are students enrolled in the Elementary Mathematics Education program in a state university in Central Anatolia Region.

Year level

Year level is used to cite students' year in the Elementary Mathematics Education program. The first year students are named as freshmen, the second year students are named as sophomores, the third year students are named as juniors and lastly the fourth year students are named as seniors.

CHAPTER II

LITERATURE REVIEW

The first purpose of the study is to investigate pre-service middle school mathematics teachers' achievement levels in refutation method, proof by contrapositive and proof by contradiction in terms of year level. The second purpose of the study is to examine the reasons of pre-service middle school mathematics teachers' wrong interpretations in mentioned proof methods. The third purpose of the study is to investigate to what extent pre-service middle school mathematics teachers can conduct valid proofs. Moreover, proof methods used by pre-service middle school mathematics teachers in their valid proofs and the reasons of their invalid proofs were investigated.

This chapter includes a review of the literature related to the study. Considering the research questions of the study, the literature reviewed has been categorized into five sections, namely importance of proof in mathematics education, definitions and functions of proof, proof methods in the literature, difficulties encountered by students while proving and research studies on proof.

2.1. Importance of Proof in Mathematics Education

Proof is seen as the cornerstone of mathematics and it distinguishes mathematics from other disciplines (Heinze & Reiss, 2003; Krantz, 2007; VanSpronsen, 2008). According to Baylis (1983), proof is the essence of mathematics. Hence, it is an essential part of mathematics education (Schoenfeld, 1994). To develop students' sense and conception of proof, argumentation and the reasoning ability are among the objectives of mathematics education (Almeida, 2001; Altıparmak & Öziş, 2005; Fitzgerald, 1996; Jones, 1997; Mariotti, 2006; Martin & Harel, 1989). Moreover, teachers are expected to encourage students to provide explanations to their ideas and discuss the given statements (Altıparmak & Öziş, 2005). The National Council of Teachers of Mathematics (NCTM, 2000) stated the importance of encouraging students to participate in proof tasks as; “The ability to reason systematically and carefully develops when students are encouraged to make conjectures, are given time to search for evidence to prove or disprove them, and are expected to explain and justify their ideas” (p.112).

In addition, reasoning and proof are one of the five process standards recommended by NCTM (2000). In this manner, considering the important and essential place of proof in mathematics education, proof should take place at all levels of school mathematics curriculum as Schoenfeld (1994) stated;

> Proof is not a thing separable from mathematics as it appears to be in our curricula; it is an essential component of doing, communicating, and recording mathematics. And I believe it can be embedded in our curricula, at all levels (p.76).

Not only proof, but also reasoning and argumentation are important subjects in mathematics education research (Heinze & Reiss, 2003). Since reasoning is directly related to proof, the reasoning ability is also critically important at each level.

While it is agreed that proof is a fundamental part of mathematics education, proof is also one of the most misunderstood notions of mathematics curriculum (Schoenfeld, 1994). According to Tall (1989), mathematical proof is not adequately studied in school mathematics courses. However, according to NCTM (2000), “Reasoning and proof are not special activities reserved for special times or special topics in the curriculum but should be a natural, ongoing part of classroom discussions, no matter what topic is being studied” (p. 342).

Proof and reasoning have some benefits for students. To illustrate, proving may be effective in developing students’ reasoning ability, in enabling them to see mathematics from a broader and a different point of view, in understanding the differences between true and false results and analyze them properly (Grabiner, 2009). Similarly, Smith and Henderson (1959) stated that proof is one of the pivotal ideas in mathematics and emphasized the benefits of proving which are having opportunity to test the implications of ideas, to establish the relationship of the ideas and to find new knowledge.

Proof should be a part of school mathematics starting from early grades (Hanna & de Villiers, 2008; Schoenfeld, 1994). To introduce proof to students in early grades may help them to force their cognitive development from concrete to formal reasoning. Therefore, in the case that students are familiar with the nature of proof in early grades, they can improve their thoughts about the process of proof and be more willing to read and write proofs (Mingus & Grassl, 1999). Regarding the importance of proof and reasoning in early grades, Stylianides (2007) prepared a theoretical framework for the notion of proof in school mathematics in early grades by observing a third-grade class. This framework consists of two principles, which are the intellectual-honesty principle and the continuum principle. The intellectual-honesty principle was defined as “the notion of proof in school mathematics that should be conceptualized so that it is, at once, honest to mathematics as a discipline and honoring of students as mathematical learners” (p.3). The continuum principle was stated as “there should be continuity in how the notion of proof is conceptualized in different grade levels so that students’ experiences with proof in school have coherence” (p.3). Moreover, regarding these principles, Stylianides (2007) stated that the social mechanisms of the mathematical community have an effect on deciding whether an argument can be counted as proof. In other words, to accept an argument as proof is mainly related to what degree mathematicians are convinced.

Although the importance of proof in mathematics education is becoming more concrete, there are not so many studies on proof in elementary mathematics education (Stylianides, 2007; Stylianides & Stylianides, 2009). Similarly, Arslan (2007) pointed out that studies related to proof were mainly conducted on high school and university level. However, there has been an increase in the number of studies conducted on elementary school level since the late 1990s. Considering this situation, Arslan (2007) investigated the opinions of students in the 6^{th}, 7^{th} and 8^{th} grades regarding reasoning and proof. The results revealed that the 6^{th}, 7^{th} and 8^{th} grade students’ level of reasoning is low and their choices of proof method depend on their grade level.

As mentioned, proof and reasoning are among the limited and missing points in elementary school mathematics. To change and improve this situation, the views of elementary mathematics teachers are highly important. Similarly, according to Harel and Sowder (1998), elementary mathematics teachers' knowledge and attitudes towards proof have a critical effect on mathematics education. If students believe that giving examples can be assumed as proof in elementary school, their understanding of proof may not be built on a secure basis. According to Martin and Harel (1989), in the case that mathematics teachers have misconceptions about proof, such as empirical evidence, students may have the same misconception about proof. This may lead to a series of mistakes in students' further proof experiences.

Teachers' content knowledge is one of the factors that shape teaching, so mathematics teachers should understand the proof concepts (Jones, 1999; Stylianides & Stylianides, 2009; Stylianides & Ball, 2008). Therefore, mathematical proof is a concern not only for students in elementary and secondary schools, but also for pre-service and in-service mathematics teachers. As mentioned above, teaching of proof is one of the major problems for mathematics teachers since students have biases regarding proof and they do not think that proving is necessary. To make proof a more meaningful activity for students, the function of mathematical proof may be shown and utilized in mathematics courses (de Villiers, 1990). According to Fitzgerald (1996), mathematics teachers are also responsible for improving students' reasoning abilities and skills. Therefore, mathematics teachers have an important role in the construction of students' proof conceptions.

2.2. Definitions and Functions of Proof

In this section, definitions of proof are examined and functions of proof are explained.

2.2.1. Definitions of Proof

There are different definitions for proof in the literature since they focused on different elements of proof (Healy & Hoyles, 2000; Raman, 2003). To state differently, proof may have different meanings depending on the institutional contexts (Recio & Godino, 2001). Recio and Godino (2001) provided definitions of proof, which differed depending on contexts like daily life, experimental sciences, professional mathematics, logic and foundations of mathematics. For instance, Recio and Godino (2001) defined mathematical proof in the context of professional mathematics as; “The argumentative process that mathematicians develop to justify the truth of mathematical propositions, which is essentially a logical process” (p.94). On the other hand, Recio and Godino (2011) defined proof in the context of logic and foundations of mathematics in the following way; “In a mathematical theory, proof is a sequence of propositions, each of which is an axiom or a proposition that has been derived from axioms by inference rules” (p.95). However, they explained proof in the context of daily life as stating that “This type of informal argumentation does not necessarily produce truth, since it is based on local value considerations, which lack the objective features of proof” (p.92).

The first two proof definitions of Recio and Godino (2001) have a common point which is studying with some propositions in a logical way, but these definitions are constructed by considering different purposes of proof. The former definition is formed by emphasizing its structure in justifying the truth of mathematical propositions since its context is professional mathematics. The latter definition is formed by emphasizing logic by using terms like sequence of propositions and inference rules. On the other hand, definitions of proof in the context of professional mathematics and in the context of daily life have a contradicting point in terms of showing the truth. Since the last definition of proof was constructed in the context of daily life, producing the truth is not guaranteed.

Literature review showed that some researchers focus on the structure of proof, while some of them focus on the functions of proof in their proof definitions. The definition of proof provided by Bell (1976) was “a directed tree of statements, connected by implications, whose end point is the conclusion and whose starting points are either in the data or are generally agreed facts or principles" (p. 26). This definition describes the general structure of a proof and does not mention any of its functions. In contrast, Goetting (1995) provided a definition of proof mainly focusing on its functions. Goetting (1995) defined proof as a convincing and conclusive argument, as judged by qualified judges. Hanna and de Villiers (2008) mention both structure and functions of proof in their definition of proof:

> Mathematical proof consists of explicit chains of inference following agreed rules of deduction, and is often characterized by the use of formal notation, syntax and rules of manipulation. Yet clearly, for mathematicians proof is much more than a sequence of correct steps; it is also and, perhaps most importantly, a sequence of ideas and insights with the goal of mathematical understanding-specifically, understanding why a claim is true (p.330).

Considering this definition, it can be seen that Hanna and de Villiers (2008) explained proof as a concept that is more than making logical inferences by pursuing some rules. They also mentioned the importance of proof in understanding why a statement is true.

In some definitions, the association between proof and argument was pointed out. In this manner, to state the meaning of argument, the definition of Overton (1990) may be considered. Overton (1990) explained argument as a sequence of sentences or propositions in which premises constitute evidence for the truth of the conclusions. Hanna and de Villiers (2008) stated that some researchers accept mathematical proof as different from argumentation, whereas some others think argumentation and proof as parts of a continuum rather than as a dichotomy. In the case of considering them as a continuum, the definitions of proof involve the statement ‘argument’. For instance, Conner’s definition (2007) was “a logically correct deductive argument built up from given conditions, definitions, and theorems within an axiom system” (p.2). Heinze and Reiss (2003) explained mathematical proof as a reasoning pattern which involves deductive arguments. Hanna, de Bruyn, Sidoli and Lomas (2004) defined proof as follows: “Mathematical proof, by definition, can take a set of explicit givens (such as axioms, accepted principles or previously proven results), and use them, applying the principles of logic, to create a valid deductive argument” (p.82).

These definitions simply imply that proof is a logical, deductive argument (VanSpronsen, 2008). The same idea was also mentioned by Hanna (1989), who asserted that “a proof is an argument needed to validate a statement” (p.20). Unlike the previous definitions, Stylianides (2007) also cited that an argument is seen as proof when accepted by the classroom community. Proof is defined by Stylianides (2007) in the following way:

> Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, which has the following characteristics;
> 1. It uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification;
>
> 2. It employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community;
>
> 3. It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community (p. 291).

Although researchers define proof in various forms, they consider one essential principle: “to specify clearly the assumptions made and to provide an appropriate argument supported by valid reasoning so as to draw necessary conclusions” (Hanna & de Villiers, 2008, p.2). As similar to the definition of proof, roles and functions of proof are themes that researchers studied which are mentioned in the following section.

2.2.2. Functions of Proof

The researchers have examined the roles and functions of proof. Yopp (2011) implied that there is not a clear and strict distinction between the terms role, purpose and function in the case of mathematical proof and these terms may be used interchangeably in the studies. Volmink (1990) accepted conviction as the most important function of proof. Bell (1976) described three roles of proof as verification or justification, illumination and systemization. Verification is related to the truth of a proposition, illumination is related to explaining why a proposition is true and systemization is the organization of results in deductive way. According to de Villiers (1990), there are five important roles of proof, which are verification, explanation, systemization, discovery and communication. Verification is related to the truth of a statement; explanation means providing insight into why a statement is true; systemization refers to the organization of results into a deductive system of axioms, concepts and theorems; discovery means the invention of new results; and communication means the transmission of mathematical knowledge. Later, de Villiers (1999) added another function of proof to these five functions. It was named as intellectual challenge which is the self-realization/fulfillment derived from constructing a proof.

Hanna (2000) added some other roles of proof, which are constructing an empirical theory, exploring the meaning of a definition or the consequences of an assumption, incorporating a well-known fact into a new framework and viewing it from a fresh perspective. Similarly, Schoenfeld (1994) sought an answer to the question, “Why do we need proof in school mathematics?” To this end, Schoenfeld (1994) declared roles of proof as communicating ideas with others, thinking, exploring and understanding mathematical arguments.

Proof does not have simple roles in mathematics; it is a fundamental part of mathematics and it includes different forms (Jones, 1997). According to Altıparmak and Öziş (2005), proving can be conducted in two ways, which can be considered as functions of proof. The first way is to show that a statement is true. The second way is to explain why a statement is true. These are essential for mathematical proof. Another classification of the roles of proof was made by Almeida (2001). Almeida (2001) classified the purposes of proof into four, which are verifying a result, communicating and persuading others of the foregoing, discovering a result, and systematizing results into a deductive system. Ko (2010, p.1112) summarized these roles with the table below:

Table 2. 1. Role of Proof (Ko, 2010, p.1112)

Role of proof	Description
Verification	Verifying the truth of a statement
Explanation	Explaining why a statement is true
Communication	Communicating the importance of mathematical knowledge offered by the proof produced among members in the community
Discovery	Discovering new mathematical knowledge
Systemization	Systemizing of various results into deductive system of definitions, axioms and theorems
Intellectual challenge	Deriving the self-realization and fulfillment from constructing a proof

Yopp (2011) investigated the roles and purposes of proof and proving for university mathematicians and statisticians and determined specific roles of proof from the analysis of data. These roles were verifying that a statement is true, showing why a statement is true, increasing mathematical understanding of the statement being proven, discovering or creating mathematical knowledge, teaching critical thinking, organizing statements in an axiomatic system, and building students' awareness of mathematics as a discipline. While the roles of proof were examined in some studies in detail by considering many applications, there are also studies which examine the role of proof under more general titles. According to Hersh (1993), there are two roles of proof, namely convincing and explaining. Hersh (1993) stated that the primary role of proof is convincing in a mathematical research. However, at the high school or undergraduate level, its primary role is explaining.

As seen from the given studies, to verify that a statement is true and to show why a statement is true are two common functions of proof. Moreover, in most of the studies, these functions are stated as the main functions of proof. Systemizing the results in a deductive way comes after the mentioned two functions in terms of being common.

In this part, definitions and functions of proof were reviewed. In the following section, some proof methods in the literature will be explained since pre-service middle school mathematics teachers' achievement levels in some proof methods and which proof methods they used were aimed to investigate in this study.

2.3. Proof Methods

Mathematical proof is a concept that people have instinctively. The development of this ability depends on developing appropriate strategies. Strategies may refer to the programs related to proof and argumentation in schools. If strategies are not chosen in an appropriate way, the abilities of proving and argumentation may start to disappear, and people may prefer to memorize rather than establish cause-effect relationships (Altıparmak & Öziş, 2005). According to the secondary mathematics curriculum, students study some common proof methods until university level. In the Mathematics Curriculum of the 9^{th}-12^{th} grades (MoNE, 2011), proof methods were categorized as deduction and mathematical induction. Then, deduction was classified as direct proof and indirect proof. Moreover, subcategories of indirect proof were stated as proof by contradiction, proof by contrapositive and refutation. Altun (2007) classified proof methods in secondary school curriculum as deduction and mathematical induction and then cited subcategories of deduction as proof by contrapositive, proof by contradiction, proof with cases and refutation.

The books used as resources in Discrete Mathematics courses generally include a section on proof methods. According to the textbook of Çelik (2010), proof methods are classified in terms of showing a proposition as true or false. To show a proposition is true, direct proof and indirect proof are used. Since theorem is in the form of $p \Rightarrow q$ or some combinations of them, proving a theorem is simply related to proposition $p \Rightarrow q$. Some proof methods cited generally in the books are explained below.

Direct Proof

In direct proof, it is assumed that the statement p is true and a series of steps are conducted as each one follows the other logically and finally these steps lead to the statement q. Atayan and Hickman (2009) defined direct proof as follows: "A direct proof of a statement $A \Rightarrow B$ involves the construction of a string of statements such that $A \Rightarrow R_1$, $R_1 \Rightarrow R_2$, ..., $Rn \Rightarrow B$" (p. 8).

Proof by Contradiction and Proof by Contrapositive

Indirect proof involves proof by contrapositive and proof by contradiction methods. In proof by contrapositive, to prove the proposition $p \Rightarrow q$ is equivalent to prove the proposition $-q \Rightarrow -p$. In proof by contradiction, the logical operation $-(p \Rightarrow q) \Leftrightarrow p \wedge -q$ is used. By proving the proposition $p \wedge -q$ is false, it can be concluded that the proposition $-(p \Rightarrow q)$ is false. Therefore, the proposition $-(-(p \Rightarrow q))$ is true and by double negation equivalence the proposition $(p \Rightarrow q)$ is true. In other words, to show that $p \Rightarrow q$ is true, one assumes that $p \wedge -q$ is true and then conducts a logical contradiction, which implies that $p \wedge -q$ is false (Bloch, 2000).

Proof by Cases

Roberts (2009) defined proof by cases which is a proof method as "one proves $(p \vee q) \Rightarrow r$ by proving $p \Rightarrow r$ by any technique and proving $q \Rightarrow r$ by any technique" (p.89). For example, in this proof method, to prove a theorem for the set of integers, the first case can be accepted as the verification the theorem for the set of even integers and the second case can be accepted as the verification of the theorem for the set of odd integers.

Refutation

In order to prove a proposition is false, refutation, which means giving a counter example, is stated as a proof method. Riley (2003) defined refutation as "the process of proving a statement is false or wrong by argument or evidence" (p.19).

Mathematical Induction

Another method mentioned in the textbooks is mathematical induction. Stylianides, Stylianides and Philippou (2007) explained that to prove proposition of the form "$\forall n \in D, P(n)$" where P(n) is an open sentence and $D = \{n \mid n \in N, n \geq n_0\}$. The base step asserts P(n) for the initial value $n = n_0$. The inductive step proves that $P(k) \Rightarrow P(k+1)$ for any arbitrary k in the set D. Then, it is concluded that P(n) holds for all *n*s in set D. Formal representation of mathematical induction is

$[P(n_0) \wedge \forall k \in D, P(k) \Rightarrow P(k+1)] \Rightarrow \forall n \in D, P(n)$.

A common model for induction that can be used in teaching induction is the domino model. To prove a set of statements, that $S_1, S_2, S_3 \ldots . S_n \ldots .$ are true, assume that S_1 can be proved so that S_1 domino falls. Then, the case that S_k can be proved forces the next statement S_{k+1} to be true. In other words, S_1 falls and causes S_2 to fall, and then S_2 causes S_3 to fall and so on. Therefore, all statements fall in an order, which means all of them are proved. Hammack (2009) showed the domino model as follows (p. 153):

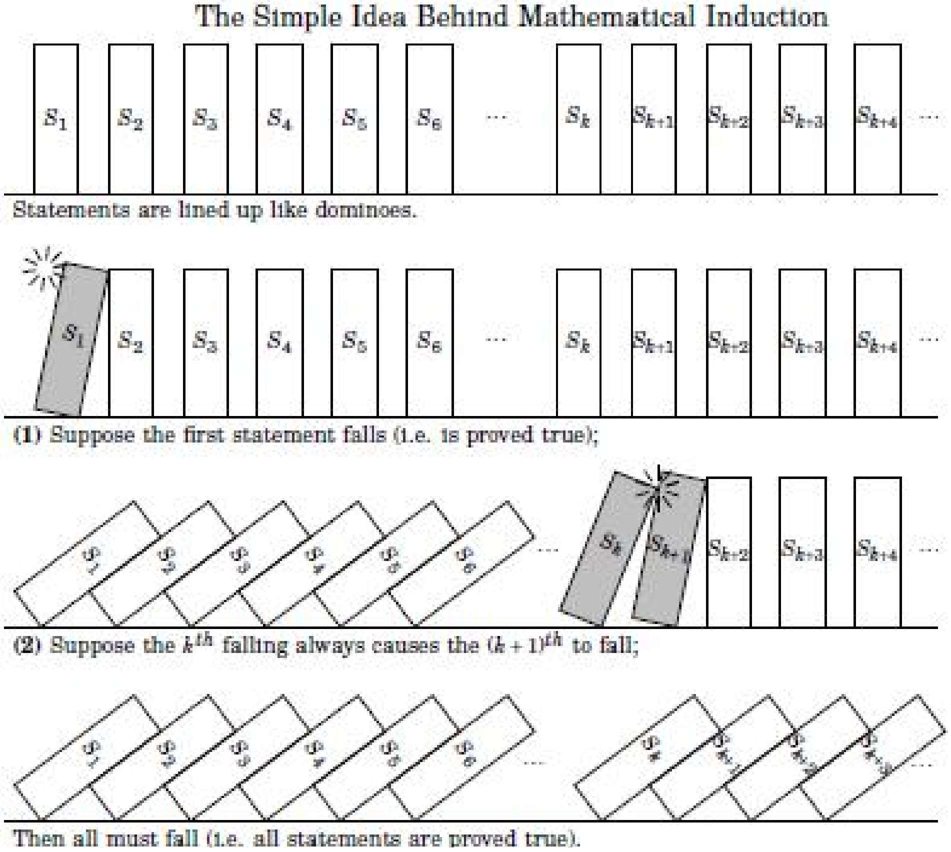

Figure 2. 1. The domino model for mathematical induction by Hammack (2009)

Proof methods which are mostly given in the literature were stated in this section. Besides, it was cited that students at all levels generally have difficulty in constructing proofs which will be reviewed in the following section.

2.4. Students' Difficulties in Proof

Even though proof is a fundamental component of mathematics, students at all levels generally have difficulty in constructing proof (Almeida 2000; Gibson, 1998; Harel & Sowder, 1998; Knapp, 2005; Moore, 1994; Sarı, Altun & Aşkar, 2007; Selden & Selden, 2003; Weber, 2001). Since students in every level have difficulties in proof, the point where students make mistakes, types of proof difficulties students experience and what can be done to address students' problems are important research areas.

According to Gibson (1998), the points where students have difficulty in proof are generally understanding the rules and the nature of the proof, conceptual understanding, proof techniques and cognitive load. Also, it is found in the study of Gibson (1998) that students who completed the first semester introduction to analysis course at a state university first try to prove by using only verbal and symbolic representations. However, when they start to experience difficulty and get stuck, they use the strategy of drawing a diagram. In general, they attempt to draw a diagram with the purpose of understanding information, judging the truthfulness of statements, discovering ideas and writing out their ideas. Moreover, interviews with students showed that the method of drawing a diagram appeals to students' thinking, make statements clear and help them to find a way to start constructing proof.

Weber (2001) concluded that the reasons underlying the difficulties students' experience can be classified into two categories. The first category is that students do not have accurate ideas about the concept of mathematical proof. For instance, if students do not know what a valid proof means, they will fail in constructing valid proof. The second one is that when students have problems in understanding theorems, it is unlikely that they can apply it in a correct way. Similarly, many students make reasoning mistakes, which generally stem from misconceptions in proving (Selden & Selden, 2003). Almeida (2001) found that students have difficulty in explaining and justifying their results for questions that ask for proof. Also, one of the major reasons of the difficulty experienced in proof is that teachers generally impose certain methods of proof and write rules instead of refining students' conception of proof and encouraging them to provide justification (Harel & Sowder, 1998).

Heinze and Reiss (2003) conducted a study with students in the 8^{th} grade in Germany, considering achievements in a geometry test in order to investigate deficiencies of students regarding proof. According to Heinze and Reiss (2003), methodological knowledge which includes proof scheme, proof structure and logical chain, is a part of competence in providing proof. Students were given one empirical argument, one circular argument and two correct proofs as one formal proof and one narrative proof. Then, they were asked to evaluate the argument and determine whether proof is universally valid or valid in some cases and whether proof had a mistake or not. The results revealed that almost all of the students evaluated formal and narrative proof correctly. Moreover, three of the students evaluated the empirical argument correctly. Also, several students evaluated empirical arguments as proof because of the deficiencies in the proof structure. It was observed that students mostly had difficulty in proof structure. For some of the students, the term ‘universal validity’ was another problem. However, it was found that students did not experience difficulties related to logical chain.

Another study was conducted by Moore (1994) with both undergraduate mathematics and mathematics education students and also graduate mathematics students. Moore (1994) determined the difficulties that students may have in learning how to conduct proof as perceptions of the nature of the proof, logic and methods of proof, problem solving skills, mathematical language and concept understanding by examining the studies in the literature. Moreover, Moore (1994) investigated the difficulties students experienced in proof by observing them in a mathematics course, which is transition to proof course, and asking professors’ and students’ perspectives on difficulties in learning proof. At the end of the observations of the course, interviews and tutorial sessions, students’ difficulties in constructing proof were determined. The first difficulty detected was that students did not know about the definitions or were unable to state them appropriately. Secondly, students had little intuitive understanding of the concepts. The third difficulty was that their images of concepts were not sufficient to construct proofs. Another area of difficulty was that students could not or did not want to form their own examples. The fifth difficulty was that students had deficiencies in using mathematical language and notations and the last area of difficulty was that students did not know how to start conducting proofs.

Another factor leading students experience difficulties in proof was the misconceptions that they held. Selden and Selden (2003) investigated misconceptions by considering some cases, by giving examples and also classified as beginning with the conclusion, thinking that names confer existence, thinking that apparent differences are real, using the converse of the theorem, believing that real number laws are universal, conservation of relationships and element set interchanges. This classification was formed through students' errors in a junior level course in abstract algebra. 'Beginning with the conclusion' involves finding a known fact by beginning with the conclusion even though the proof has an irreversible structure. The misconception which was categorized as 'names conferring existence' may be seen when existence of things such as statements and solutions are not noticed or symbols are used without thinking of their meaning in the argument. Another misconception is thinking that 'apparent differences are real'. This misconception occurs when it is difficult to recognize that two different expressions may represent the same thing. One of the most common misconceptions is 'using the converse of the theorem'. This misconception may be seen if a statement and its converse are used with equal meanings. In other words, the proposition $p \Rightarrow q$ and the converse of the proposition $p' \Rightarrow q'$ give the same implication for students. The misconception of thinking that 'real numbers are universal' occurs when it is accepted that the rules used with real numbers are universal. In other words, students who have this misconception may use the rules used with real numbers in different and inappropriate contexts. 'Conservation of relationships' means accepting the idea that doing the same thing to both sides of a relationship does not interfere with the relationship. To accept this idea in any subject is an improper generalization. The misconception named as 'element set interchanges' is about understanding statements with elements more easily compared to a different form the same statement in the sets.

In the same study, Selden and Selden (2003) also investigated reasoning errors based on misconceptions which affect the success of students in university with respect to providing proof. Reasoning errors were determined as overextended symbols, weakening the theorem, notational inflexibility, misuse of theorems, circularity, the locally unintelligible proof, substituting with abandon, ignoring and extending quantifiers, holes and using information out of the text. According to their study, overextended symbols may be observed when one symbol is used for different things. This reasoning error implies the lack in understanding mathematical structures. Another reasoning error is weakening the theorem, which occurs in the case when the way of conducting proof is stronger than the hypotheses or the structure of the proof is weaker than the conclusion. Notational inflexibility may be seen if students have problems in adapting notations to different contexts. Misuse of theorems may be seen in students who misunderstand any part of a hypothesis or conclusion in applying a theorem. Circularity, as a reasoning error, means using the conclusion to show another version of it. In other words, circularity refers to "reasoning from a statement to itself" (p.12). The locally unintelligible proof means what is written with the aim of constructing proof cannot be accepted as proof. Even though of its existence may be seen regarded proof, the assertions are not so clear. Substituting with abandon occurs when transforming one statement with another by mistake, and the reason of this error may be the confusion of the situations. Ignoring and extending quantifiers is related to having problems with the meaning of quantifiers, such as accepting a quantifier as universally quantified even though it is not. Holes, as a reasoning error, mean making connections between the statements directly without giving any argument. Using information that is not in the text may be observed when information taken from an argument is used in another one wrongly.

2.5. Research Studies on Proof

In this part of the literature review, some studies on proof were reviewed. Firstly, the studies conducted in Turkey are presented, and then some studies conducted in other countries are discussed.

2.5.1. Research Studies Conducted in Turkey

Literature review showed that there are studies related to proof which focused on different themes, such as students' views of proof, their levels of proof and their ability to prove. Specially, the view of students in different levels regarding proof is one of the mostly investigated areas in Turkey. For example, Moralı, Uğurel, Türnüklü and Yeşildere (2006) investigated the views of pre-service mathematics teachers regarding proof with 182 freshmen and 155 seniors including both pre-service elementary and secondary mathematics teachers. These researchers developed the instrument from the study of Almeida (2003). It was a five-point Likert-type scale including 20 items. The items were coded as 5 for strongly agree, 4 for agree, 3 for undecided, 2 for disagree and 1 for strongly disagree for positive statements and negative statement codes were reversed in the analysis. They conducted factor analysis to find the construct validity of the instrument. The first factor addresses students' proof competencies (items 14, 18,19 and 20), the second factor addresses students' perceptions of the importance of proof (items 6,7,8 and 17), the third factor addresses students' perceptions of proof in understanding theorems (items 11, 12, 13 and 16), the fourth factor addresses students' self-perceptions of proof (items 9 and 10), the fifth factor addresses students' general perceptions of proof (items 1, 2 and 4), the sixth factor addresses students' ideas about examples and theorems (items 3 and 5) and the seventh factor addresses students' perceptions of the relations between problem solving and mathematical proof (item 15). This study revealed that most of the pre-service mathematics teachers did not have specific ideas about proof and some of them had not formed ideas about proof exactly. Moreover, it was found that the number of the students who had marked undecided was high. Therefore, their conceptualization regarding conducting proof was lower than expected.

In the review of the literature, it is noticed that the instrument which was formed by Almeida (2001) and then adapted into Turkish by Moralı et al. (2006) was often used in the studies. Anapa and Şamkar (2010) used this instrument to investigate perceptions regarding mathematical proof of students who were attending Mathematics and Computer Sciences programs in Arts and Science Faculty and the Elementary Mathematics Education program in the Education Faculty. They applied the instrument to 444 students. The results of the study revealed that more than half of them considered themselves as being successful in mathematics at an intermediate level. Moreover, it was found that students accepted proof in mathematics teaching as important. However, they believed that proving a theorem which was already proven by a famous mathematician was unnecessary. Another point which is worth discussing is that some students agreed with the third item which was ‘It does not always help me to understand why a result is true by showing with an example’. However, in the study of Moralı et al. (2006), students were undecided about this item and thought positively about numerical proof. The reason underlying this situation was explained with the number of freshman pre-service teachers in the sample since students in higher levels were not using numerical proving.

Kayagil (2012) used the instrument in the study of Moralı et al. (2006) in order to determine the prospective elementary mathematics teachers’ views on mathematics regarding proof and investigate differences in terms of gender, high school type, grade and participation in a scientific activity. Kayagil (2012) administered the instrument to 357 students in a state university in Ankara. The results showed that pre-service elementary mathematics teachers had neither positive nor negative views about doing proof. There was no statistically significant difference in students’ views about proof in terms of gender and their participation in a scientific activity. Moreover, there was no statistically significant difference between the groups and within groups in terms of grade level and type of high school.

Another study in which the same instrument was used was conducted by Gökkurt and Soylu (2012). They investigated the views of freshmen in Elementary Mathematics Education and Science Education programs about proof. The instrument was administered to 244 freshmen, 150 students from science education and 94 students from mathematics education in Atatürk University. The results of the study showed that there was no significant difference between the views of pre-service science teachers and pre-service elementary mathematics teachers regarding proof. Similar to the study of Moralı et al. (2006), students’ views about proof was insufficient and they were undecided about their success in conducting proofs. Moreover, it could be concluded that students did not know the importance of proof in mathematics and mathematics teaching and most of them believed that proof was unnecessary.

There are also studies on views of students about proof in which different instruments were used. For example, Üzel and Özdemir (2009) conducted a study related to pre-service elementary teachers' views regarding proof. They developed an instrument by adding 20 more items to the instrument of Moralı et al. (2006) and administered it to 95 freshmen and 70 juniors in the Elementary Mathematics Education program in Balıkesir University. In this study, the results were analyzed in terms of students' school year and gender. It was found that gender and school year had an effect on two factors of the instrument, namely attitude towards proving and general aspects of proof. According to the results of the study, there was a meaningful difference between pre-service elementary mathematics teachers' attitudes toward proof and proving in favor of freshman pre-service teachers. Moreover, there was a meaningful difference between pre-service elementary mathematics teachers' attitudes toward proof and proving in favor of female students.

Baştürk (2010) investigated first-year students' conceptions of proof and proving by using a questionnaire and by conducting interviews. The questionnaire, which was based on a five-point Likert-type scale, was developed after related literature was reviewed and discussions were held with mathematicians and mathematics educators. Then, the questionnaire was applied to 37 first-year students in the Secondary Mathematics Education program in a state university located in Istanbul. As a result of missing data, the sample was reduced to 33 students. After the questionnaire, semi-structured interviews were held with 10 students. The results showed that the majority of the students did not consider giving examples as proof. Although students thought that proof was important in learning and teaching mathematics, half of them stated that they did not like proofs. Moreover, it was found that most of the students had difficulty in deciding about proof methods and how to continue in providing proof.

İskenderoğlu and Baki (2011) investigated the proof-related opinions of pre-service elementary mathematics teachers. The sample of the study was comprised of 187 pre-service elementary mathematics teachers, namely 73 freshmen, 35 sophomores, 34 juniors and 45 seniors, in Karadeniz Technical University. To collect data, the questionnaire developed by Lee (1999) was translated into Turkish. The questionnaire is based on four factors, namely belief and attitude, confidence, mental process and self-assessment. According to the results of the study, proof-related views of the pre-service elementary mathematics teachers were positive. Moreover, the average scores of students obtained from the questionnaire showed that participants often used their mental processes while constructing proof; they sometimes relied on themselves regarding proof; they frequently assessed themselves, and their proof-related attitudes and beliefs were positive.

In another study, Türker, Alkaş, Aylar, Gürel and İspir (2010) investigated the views of pre-service elementary mathematics teachers on proving. To collect data, the instrument in the study of Almeida (2001) was revised and applied to 104 pre-service elementary mathematics teachers in senior class, and then interviews were conducted with groups in different levels. According to the results, all pre-service teachers thought that mathematics could not be taught if they did not include proof. They believed that mathematical proof must be a part of the elementary mathematics education taking into consideration the levels of the students. Pre-service elementary mathematics teachers generally had positive attitudes towards proof and believed that proof had a lot of benefits in mathematics education. However, they generally tried to memorize proof rather than understand it.

It is observed that some studies also investigated different aspects of proof in addition to views or opinions of students regarding proof. For example, Köğce, Aydın and Yıldız (2010) investigated high school students' views about proof in terms of the definition of proof, the necessity of proof and their levels of proof. The instrument which contains two open-ended questions related to proof and six questions asking for proof was administered to 125 10th grade students. Students were randomly selected from two high schools in Trabzon. The results of the study regarding the definition of proof indicated that students had many different views about proof. Their answers were coded, such as 'showing the correctness of a result' and 'demonstrating the way mathematical operations are carried out in detail'. Students' answers to the necessity of proof were classified such as 'facilitating comprehension' and 'providing permanent learning'. With respect to the frequencies of the answers, the codes ‘facilitating comprehension’, ‘enabling the realization of right and wrongs’ and ‘providing permanency’ had the highest frequencies. To determine the students’ levels of proof, the classification in Miyazaki’s study (2000) was used. Miyazaki (2000) classified levels of proof as Proof A, B, C and D. Also, Proof A is accepted as the most advanced level while Proof C was described as the least advantageous level. After analysis, it was found that 46.7% of the students were classified as Proof A, 51.2% of the students were classified as Proof C. Since Proof C was described as the least advantageous level of proof, students could not use proof methods at expected levels.

Similar to the study of Köğce, Aydın and Yıldız (2010), Özer and Arıkan (2002) conducted a study about students’ levels of proving in high school mathematics courses by using the proof levels in the studies of Miyazaki (2000) and Balacheff (1988). Özer and Arıkan (2002) administered 6 open-ended questions to 110 students from the 10^{th} grade and interviewed 3 students from the 9^{th} grade in three different high schools in Istanbul. According to the answers given to the 6 open-ended questions, students’ scores were calculated and then their proof levels were examined for both Miyazaki and Balacheff proof levels. The results of the study indicted that almost all of the students could not construct proof by using the deduction and induction methods. Some students thought that proof was provided if they could show the statement as true by giving numerical values. Similarly, interviews showed that students could not provide proof by using materials. If students were not asked to use materials, they tried to prove by giving numerical values and using the induction method.

There are also studies related to the difficulties students experienced in giving proof. For example, Güler, Kar, Öçal, Çiltaş (2011) conducted a study to determine the difficulties pre-service mathematics teachers encountered in doing mathematical proofs. As an instrument, a mathematical proof test, comprised of five open-ended questions, was prepared from the literature. Questions were based on using inequality, utilization of examples and visualization. The first three of the questions were only about ‘using inequality’ and ‘utilization of examples’, the remaining of the questions were about ‘visualization’ as well as ‘utilization of inequality’ and ‘using examples'. The sample of the study was 80 seniors from the Elementary Mathematics Education program of a state university in the east of Turkey. The results showed that students had difficulty in inequality, utilization of examples and visualization. Moreover, it was found out that their operational abilities were low, which may have affected their ability in giving proof.

Some studies were conducted on proof methods. For example, Türker, Alkaş, Aylar, Güler and İşpir (2010) showed that pre-service elementary mathematics teachers in senior class could not use proof methods correctly since their knowledge of proof methods was inadequate. When they were asked to write different types of proof, they could write mostly four types of proof method. The most common answers were induction and contradiction method. Moreover, the general mistake of the students was found out to be that they classified deduction, contradiction and direct proof under the same title. Even though they knew how to apply a method, they could not be sure about the name of the proof method.

Another study on proof methods was conducted by Altıparmak and Öziş (2005). They explained the development of reasoning and proof in preschool, elementary and high school levels and mentioned the proof methods only in high school level. Altıparmak and Öziş (2005) stated that the concept of proof starts in preschool as a bridge in passing through logical thinking. Students in the 1^{st}- 5^{th} grades are in the concrete thinking process and students in the 6^{th}- 8^{th} grades are in the abstract thinking process. Students in secondary school have the ability to think in an abstract way. Therefore, in high school, students are expected to use some proof methods. Proof methods which are used in this level are stated as direct proof, contrapositive, contradiction and mathematical induction as well as geometric proofs.

2.5.2. Research Studies Conducted in Other Countries

Literature review revealed that studies conducted in other countries related to proof focused on various themes. In some of these studies, students were asked to prove the given statements with pursuing different purposes such as investigating students' ability to prove, their preparedness about teaching of proof and their levels in proving. Other themes related to proof in the studies mentioned in this section are students' understanding of validity of the arguments, their understanding about proof, empirical arguments and proof methods. Moreover, some classifications regarding students' proof given in the studies were stated in this section.

In the literature review, it is seen that students' ability in conducting proof was one of the mostly investigated themes. For example; Recio and Godino (2001) conducted a study about students' capability to build deductive proofs in their university studies. There are two problems in the questionnaire related to arithmetic and geometry. According to the frequency of answer types in the first sample, 47.5% of the students for the arithmetic question and 42.4% of the students for the geometry question gave a substantially correct mathematical proof. Another result of the study is that the mathematical content of the questions affected students' proof capacity, but not students' proof schemes. Similarly, according to the study of Riley (2003), pre-service mathematics teachers have weak understanding about truth of a conditional statement and nearly half of the participants could not write a direct mathematical proof.

Literature review also showed that there are studies in which students were asked to prove some statements by pursuing other purposes than investigating their abilities to conduct proof. For example; Brown, Stillman, Schwarz and Kaiser (2008) investigated preparedness of pre-service mathematics teachers about teaching of proof at lower secondary school. Sample was from a university in Victoria, Australia. There were 11 secondary mathematics students, 9 of them completed data collection part about argumentation and proof and 6 of these 9 students were also interviewed. The instruments of the study were a questionnaire which contains questions with written explanation and problem-centered interviews. The purpose of the questionnaire was to collect data about students' mathematical knowledge, pedagogical content knowledge, pedagogy and mathematical beliefs. Then, in the interviews, students were asked about their reasons for being teacher, their beliefs about mathematics, their knowledge for teaching in secondary school and the contents of university courses they had. The mathematical topics of the instruments were teaching modeling, argumentation and proof in the lower secondary schools, grades 8-10. Brown et al. (2008) prepared a paper which focusing on argumentation and proof with investigating two students from interviews. According to the results of the study, pre-service mathematics teachers don't have high affinity with proving in teaching mathematics at the lower secondary schools and necessary mathematics knowledge. Also, it was observed that pre-service mathematics teachers had at least average competencies in dealing with misconceptions about the nature of the proof.

Miyazaki (2000) conducted a study in which students were asked to prove some statements to investigate their levels of proof. Miyazaki (2000) used Balacheff's idea (1988) in organizing levels of proof and formed the given table. Proof A was labeled as the most advanced level by Miyazaki (2000), since it has the most advanced categories in both the contents axis and the representation axis. Proof C was determined as the lowest level. Proof B and Proof D are labeled as intermediate levels of proof, since they have one category in common with Proof A and Proof C.

Table 2. 2. Levels of Proof (Miyazaki, 2000)

Contents / Representation	Inductive reasoning	Deductive reasoning
Functional language of demonstration	*Proof D*	*Proof A*
Other language, drawings, and/or manipulable objects	*Proof C*	*Proof B*

Another theme investigated in the studies is students' knowledge about how an argument can be accepted as valid or invalid proof. In other words, their ideas about what a mathematical proof constitutes were examined in the studies. For example, Martin and Harel (1989) asked how pre-service teachers see the role of inductive and deductive arguments in mathematical proof. The definitions of Anderson (1985) were given, for inductively valid argument 'an argument whose conclusion is not necessarily true but only highly probable' and for deductively valid argument 'where the conclusion must be true if the premises are true'. Then, the research questions were formed in the light of this idea. The instrument was administered to 101 pre-service elementary teachers registered in a sophomore-level mathematics course in Northern Illinois University. Students were asked to decide about inductive and deductive verification types for a familiar and an unfamiliar statement. A four-point scale was used as 1, 2 for low level of acceptance and 3, 4 for high level of acceptance. Students were asked to assess whether verifications for each statement can be accepted as mathematical proof. According to the results of the study, more than half of the students were rated as 3 or 4 which means high acceptance for inductive arguments. Similarly, many students rated were rated as 3 or 4 which means high acceptance for deductive arguments. A lot of students accepted both inductive and deductive arguments as mathematical proof. However, they were better in evaluating deductive arguments than inductive arguments. Moreover, one of the reasons for students' acceptance of inductive arguments as mathematical proof may be that students assume convincing arguments as mathematical proof. Moreover, students who behave in that way may not get the nature of the proof exactly (Goetting, 1995).

There were also studies in the literature which investigated students' understanding about proof and empirical arguments. For example, Stylianides and Stylianides (2009) conducted a study about prospective elementary teachers' understanding of differences between proof and empirical arguments and investigated prospective elementary teachers' abilities in construction and evaluation of proof. Prospective elementary teachers were asked to construct their own arguments, not to evaluate given arguments. The participants of the study were 39 prospective elementary teachers who will be teachers starting from kindergarten to grade 6. Students were known that they have rich experiences with proof. The study was conducted in a mathematics course taken by prospective elementary teachers. The course covers a wide range of mathematics subjects such as arithmetic, algebra, geometry, measurement etc. Also, the approach in the course was to promote prospective elementary teachers' understanding of proof. Since how prospective teachers constructed proof and how they evaluated these constructions as proof are main points of the study, students were asked to list criteria for good proofs. The list prepared by students was summarized as given below.

1. The proof is correct.

2. The proof addresses the specific question or problem that was posed

3. The proof is clear, convincing, and logical

According to Almeida (2001), students' views of proof are generally empirical. Most of the students in his study accepted justification as verifying by empirical evidence. Similarly, some students in various levels think that numerical values and examples are more convincing than mathematical proofs (Jahnke, 2007). The study of Stylianides and Stylianides (2009) revealed that students developed their understanding of proof through a mathematics course. Since in the first evaluation, the number of students who conducted proof correctly and gave an empirical argument was nearly the same. However, in the second evaluation, the number of students who gave empirical arguments decreased. Even though there is a decrease, there were students who continued to give empirical arguments for proof task in the class at the end of the course.

In the literature, it was observed that the studies about proof methods are limited. For example, Stylianides and Al-Murani (2009) investigated students' conceptions about the relationship between proof and refutation. In other words, they examined whether students have misconception about existence of a proof and a counterexample for the same statement. It was a part of the design experiment and participants were selected from two 10th grade classes in the same state school in England. There were 165 students in the 10th grade, they are divided into seven sets considering an exam at the end of the 9th grade and the highest two sets were selected for the study. Therefore, the data were collected from survey responses of 57 students and interviews with 28 students. Results of the study revealed that there are evidences related to mentioned misconception for 16 students from the interviewed 28 students. 10 of these students showed strong evidence, 6 of the students showed weak evidence for the misconception. Since proof types may be related to understanding and learning proofs (Hanna & de Villiers, 2008) and students may know about proof methods in general, but they may have some misconceptions about the relationships between them, proof methods may be assumed as an important theme.

Literature review showed that students' proof were evaluated based on some classifications. However, these categories were formed by considering different properties of proof such as proof schemes and justifications. For example, proof classification of Balacheff (1988) includes pragmatic proofs "which are having recourse to actual action or showings" and conceptual proofs "which don't involve action and rest on formulations of the properties in question and relations between them" (p.217). Balacheff (1988) explained four main types of proofs from various types of pragmatic and conceptual proofs which are naïve empiricism, crucial experiment, generic example and thought experiment. In naïve empiricism, students arrive at a conclusion about the truth of a result by trying several particular cases (Balacheff, 1988; Varghese, 2011). The crucial experiment includes checking a statement and generalization after examining a case which isn't very particular. Balacheff (1988) stated the main idea as "if it works here, it will always work" (p.219). In other words, if assertion holds in the determined case, it will be accepted as valid. According to Varghese (2011), the main difference between naïve empiricism and crucial experiment is the status of the specific example which means the crucial experiments have carefully selected extreme cases. Another level of Balacheff (1988) is the generic example in which validation of the assertion is based on the operations or transformations of an object as representing its class. The important point is that example is chosen as a representative of the class. In the last level, the thought experiment, students move from practical to intellectual justification which means from pragmatic to conceptual justification (Varghese, 2011). The thought experiment doesn't have particular situations; it involves internalizing and detaching from a particular example. Students can make logical deductions by considering properties of the situation.

Bell (1976) evaluated students' reasoning and proofs in two classes; empirical justification and deductive justification. In the study, deductive justifications were ranged from relevant to nearly complete arguments and empirical justification were ranged from testing one or two cases to testing many different cases. Empirical justifications involve "the use of examples as element of conviction" and deductive justifications involve "the use of deduction to connect data with conclusions" (Marrades and Gutierrez, 2000, p.89-90). In the study of Recio and Godino (2001), the answers of university students for two proof questions were categorized as five types. While analyzing answers for proof schemas, type 1 answers which are very deficient in terms of proof were not included. Type 2 answers in which students check the propositions with examples without serious mistakes were named as explanatory argumentative schemes. Type 3 answers in which students check the propositions and assert the validity were named as empirical inductive proof schemas. Type 4 answers in which students explain the validity of the propositions with using other theorems, propositions were named as informal deductive proof schemas. Type 5 answers in which students give substantially correct proof with appropriate symbols were named as formal deductive proof schemas.

2.6. Summary of the Literature Review

In the present chapter, the literature review related to the theme of the study was presented. First of all, the importance of proof in mathematics education was discussed. Then, to analyze the meaning of the proof, definitions of proof and functions of proof given in the studies were stated. Since some of the purposes of the study were directly related to some proof methods such as refutation proof by contrapositive and proof by contradiction, proof methods in the literature was reviewed. Lastly, the points which students may have difficulties related to proof and some research studies conducted related to proof in Turkey and in other countries were presented.

As stated, proof is an important component in both mathematics and mathematics education (Almeida, 2001; Altıparmak & Öziş, 2005; Baylis, 1983;

Heinze & Reiss, 2000; Jones, 1997; Mariotti, 2006; Martin & Harel, 1989; Schoenfeld, 1994; VanSpronsen, 2008). Besides, proof and reasoning should be a fundamental part of the mathematics courses in all levels of the schools (Hanna & de Villiers, 2008; NCTM, 2000; Schoenfeld, 1994). Since proof has such an importance in mathematics education, mathematics teachers have a critical role in terms of teaching proof and reasoning.

As can be seen above literature review, different definitions and functions of proof were stated by the researchers. According to Hanna and de Villiers (2008), the common and the essential principle in defining proof is "to specify clearly the assumptions made and to provide an appropriate argument supported by valid reasoning so as to draw necessary conclusions" (p.2).

Literature review also showed that there are not many studies about proof methods especially in Turkey. Moreover, students in every level generally have difficulty in constructing proof (Almeida 2000; Harel & Sowder, 1998; Knapp, 2005; Moore, 1994; Selden & Selden, 2003; Weber, 2001). In this regard, many studies have been conducted on proof difficulties by considering different levels of students and different purposes (Mariotti, 2006). Therefore, in this study, pre-service middle school mathematics teachers' achievement levels in refutation, proof by contrapositive and proof by contradiction and the reasons of their wrong interpretations in these methods were aimed to investigate. Also, to what extent pre-service middle school mathematics teachers can conduct valid proofs, proof methods they used and the reasons of their invalid proofs were investigated.

CHAPTER III

METHOD

This chapter introduces the methodology of the study, which includes information about research design, sample, instrumentation, data collection procedure, data analysis, and threats to internal validity and external validity.

3.1. Research Design

The aim of the study was threefold. The first purpose of the study was to examine pre-service middle school mathematics teachers' achievement levels in proof by contrapositive, proof by contradiction and refutation methods in terms of year level. The second purpose of the study was to determine the reasons underlying pre-service middle school mathematics teachers' wrong interpretations in the aforementioned proof methods. The third purpose of the study was to investigate to what extent pre-service middle school mathematics teachers could conduct valid proof. Regarding the last purpose, proof methods that pre-service middle school mathematcis teachers used in valid proofs and the reasons of conducting invalid proofs were investigated.

In this study, the survey research design was utilized since data was collected from a sample in order to describe some aspects or characteristics of the population by asking questions and the answers of the sample constitute data of the study (Fraenkel & Wallen, 2005). More precisely, a cross-sectional survey design was used due to the fact that data was collected at just one point in time from a predetermined sample (Fraenkel & Wallen, 2005). The data were analyzed by means of both descriptive statistics, such as frequencies and percentages, and item-based in-depth analysis since a detailed analysis of each item was needed to address some of the research questions.

3.2. Sample

The target population of the study is all pre-service middle school mathematics teachers in the state universities in Turkey. The accessible population of the study is pre-service middle school mathematics teachers enrolled in state universities in the Central Anatolia Region. Convenience sampling method was used in order to determine the sample of the study due to the fact that the participants of the study includes pre-service middle school mathematics teachers in all year levels of undergraduate study, which making it difficult to collect data from pre-service middle school mathematics teachers enrolled in all the universities in the Central Anatolian Region. In convenience sampling, a group of people are chosen for a particular study since they are available for it (Fraenkel & Wallen, 2005). Accordingly, the sample of the study consisted of freshmen, sophomores, juniors and seniors enrolled in the Elementary Mathematics Education program of a state university in Ankara.

The participants of the study were asked some questions in order to gather data on their demographic characteristics, such as their grade, gender, and general point average (GPA). The demographic characteristics of the participants are presented in the tables below. Students' characteristics in terms of year level and gender are given in Table 3.1.

Table 3. 1. Characteristics of the Participants by Year Level and Gender

		Gender		
		Female	Male	Total
Year level	Freshmen	16 (13,9%)	3 (2,6%)	19 (16,5%)
	Sophomores	22 (19,1%)	3 (2,6%)	25 (21,7%)
	Juniors	33 (28,7%)	6 (5,2%)	39 (33,9%)
	Seniors	20 (17,4%)	12 (10,4%)	32 (27,8%)
Total		91 (79,1%)	24 (20,9%)	115 (100,0%)

Information about students' year level and general point average (GPA) are presented in Table 3.2.

Table 3. 2. Characteristics of the Participants by Year Level and GPA

	General Point Average					
	0-1.00	1.00-2.00	2.00-3.00	3.00-3.50	3.50-4.00	Total
Freshmen	1(,9%)	4 (3,5%)	11(9,6%)	3 (2,6%)	-	19(16,5%)
Sophomores	-	2 (1,7%)	16(13,9%)	7 (6,1%)	-	25 (21,7%)
Juniors	1(,9%)	6 (5,2%)	19(16,5%)	11 (9,6%)	2 (1,7%)	39 (33,9%)
Seniors	-	-	17(14,8%)	12(10,4%)	3 (2,6%)	32 (27,8%)
Total	2(1,7%)	12(10,4%)	63(54,8%)	33(28,7%)	5 (4,3%)	115(100,0%)

As can be observed in Tables 3.1 and 3.2, 19 of the 115 participants were freshmen (16.5%), 25 of them were sophomores (21.7%), 39 of them were juniors (33.9%) and 32 of them were seniors (27.8%). As for the gender of the participants, 91 (79.1%) were females and 24 (20.9%) were males.

As can be observed in Table 3.2, the GPAs of 2 students (1.7%) were lower than 1.00, the GPAs of 12 students (10.4%) were between 1.00 and 2.00, the GPAs of 63 students (54.8%) were between 2.00 and 3.00, the GPAs of 33 students (28.7%) were between 3.00 and 3.50, and the GPAs of 5 students (4.3%) were between 3.50 and 4.00. Therefore, the majority of the students had a grade point average falling between 2.00 and 3.00.

With respect to the mathematics and mathematics education courses in the Elementary Mathematics Education program, as presented in Table 3.4 below, the participants of the study attend mathematics courses mainly in the first and the second years, while they attend mathematics education courses mostly in the third and the fourth years of the program.

Table 3. 3. Mathematics and Mathematics Education Courses

First Year	
First Term	**Second Term**
Fundamentals of Mathematics	Discrete Mathematics
Analytic Geometry	Basic Algebraic Structures
Calculus with Analytic Geometry	Calculus for Functions of Several Variables
Second Year	
Third Term	**Fourth Term**
Introduction to Differential Equations	Elementary Geometry
Instructional Principles and Methods	Measurement and Assessment
Third Year	
Fifth Term	**Sixth Term**
Basic Linear Algebra	Community Service
Methods of Teaching Mathematics I	Instructional Technology and Material Development
	Methods of Teaching Mathematics II
Fourth Year	
Seventh Term	**Eighth Term**
Research Methods	Practice Teaching in Elementary Education
School Experience	
Nature of Mathematical Knowledge for Teaching	

3.3. Instrumentation

The Mathematical Proof Questionnaire (MPQ) which consists of three sections with 11 questions was utilized as the data collection instrument. This section describes the features of MPQ and its development process.

3.3.1. Mathematical Proof Questionnaire

MPQ was designed to address the research questions of the study. MPQ includes a total of 11 questions under three sections. Section A consists of four multiple choice items, two of which are related to the proof by contrapositive and the other two to the refutation. Section B contains four discussion items. While two of the four discussion questions are related to the proof by contradiction, one of them is related to refutation and the other to the proof by contrapositive. The purposes of the questions in Section A and Section B are to investigate pre-service middle school mathematics teachers' achievement levels in refutation, proof by contrapositive and proof by contradiction methods and to determine the reasons underlying their wrong interpretations regarding these proof methods. Section C includes three open-ended proof items in which students were asked to prove the given statements. These statements can be proved by using different proof methods such as direct proof, mathematical induction and proof by cases. The purpose of Section C is to determine to what extent pre-service middle school mathematics teachers can conduct valid proof. Therefore, the proofs provided by the students were classified as valid and invalid. Subsequently, the proofs that were found to be valid were analyzed and proof methods that students used were classified. Invalid proofs of students were also examined and the reasons underlying their indirect proofs were determined. Table 3.5 presents the distribution of the questions with respect to each section of the questionnaire and the proof methods.

Table 3. 4. Questions in the Mathematical Proof Questionnaire

Proof methods	**SectionA**	**SectionB**	**SectionC**
Refutation	Q1, Q4	Q4	
Proof by contrapositive	Q2, Q3	Q2	
Proof by contradiction		Q1, Q3	
Students' proof methods, such as mathematical induction, direct proof, proof by cases			Q1, Q2, Q3

Some items of MPQ were adapted from the questions in textbooks and studies in related literature (Çelik, 2010; Galbraith, 1982; Knuth, 1999; Morris & Morris, 2009; Saeed, 1996; Velleman, 2006) and some of the items were developed by the researcher of the current study. Moreover, to analyze the students' answers in detail, a sub-question was added to each question in Section A and B. In the sub-questions, students were asked to explain the reasons for their answers.

Since some of the items of MPQ were adapted and translated into Turkish, these items were edited by an English lecturer. Subsequently, all the items in MPQ were checked by an expert in the Turkish language. The items were revised based on the feedback of these experts. After this process, the expert opinions of mathematics educators regarding MPQ were obtained for the purpose of content validation. This version of the instrument was evaluated by four mathematics educators in the Elementary Mathematics Education program of two different universities in terms of the usage of mathematical terms, and appropriateness of the items to the purposes of the study. The items were revised taking into consideration the views of the experts and then a pilot study was conducted. The last version of MPQ was obtained by making the necessary changes noticed in the pilot study.

3.3.1.1. Questions in Section A

As mentioned, Section A consists of four multiple choice questions which were prepared by reviewing the related literature (Galbraith, 1982; Knuth, 1999). Question 4 was adapted from the study of Galbraith (1982), and the other questions were prepared by the researcher by considering the structure of multiple choice questions in the studies of Galbraith (1982) and Knuth (1999). Two of the questions were related to proof by contrapositive and the other two questions were related to the refutation method. These questions are explained below.

Question 4 of Section A, which was related to the refutation method, was adapted from the study of Galbraith (1982). The purpose of the question is to investigate whether students know the meaning of counterexample and the characteristics of the refutation method. Students were asked to find the correct choice by considering the given statement. The correct choice of the question is (a). Since the number 33 corresponds to the statement S and shows that the statement S is false, it can be accepted as a counterexample.

Statement A**:** An integer is divisible by 6 if the sum of its digits is divisible by 6.

Which of the followings is correct by considering the given statement?

a) The number 33 proves that statement A is false

b) The number 30 proves that statement A is false

c) The numbers a=30 and b=33 prove that statement A is false

d) Statement A is false but the numbers a=30 and b=33 are not adequate to prove it.

e) The statement is true

- Why? State your reasons.

Figure 3. 1. Question 4 in Section A

Question 1 of Section A was prepared by the researcher by considering Question 4 which was adapted from the study of Galbraith (1982). It is related to the refutation method and students were asked to find the correct choice by considering the given statement. The correct choice is (b), which gives the appropriate counterexample and states the meaning of the refutation method. This question differs from Question 4 in terms of the content of the choices given.

Statement A: a and b are natural numbers. a.b is an even number if and only if a and b are even.

Which of the followings is correct by considering the given statement?

a) Statement A is true

b) The numbers a=5 and b=6 prove that statement A is false

c) The numbers a=5 and b=6 prove that statement A is true

d) Statement A is false but the numbers a=5 and b=6 are not adequate to prove it.

e) None of the above

- Why? State your reasons.

Figure 3. 2. Question 1 in Section A

Question 2 and Question 3 of Section A were prepared by the researcher by considering the format of the multiple choice questions in the studies of Galbraith (1982) and Knuth (1999). These questions were related to proof by contrapositive. Students were asked to find the correct assumption to start to prove. The correct choice for Question 2 is (d) and the correct choice for Question 3 is (c). These choices involve the proposition $q' \Rightarrow p'$ as an assumption to prove the proposition $p \Rightarrow q$ which is known as proof by contrapositive.

Assume that m and n are positive integers. If mn=100, then m≤10 or n≤10.

To prove the statement, which assumption can you begin with?

a) Assume that m and n are positive integers. If m>10 or n>10, then mn=100.

b) Assume that m and n are positive integers. If m≤10 or n≤10, then mn=100.

c) Assume that m and n are positive integers. If m≤10 and n≤10, then mn≠100.

d) Assume that m and n are positive integers. If m>10 and n>10, then mn≠100.

e) None of the above

- Why? State your reasons.

Figure 3. 3. Question 2 in Section A

Assume that a, b and c are real numbers and a>b. If ac≤bc, then c≤0.

To prove the statement, which assumption can you begin with?

a) Assume that a, b and c are real numbers and a>b. If c>0, then ac≤bc.

b) Assume that a, b and c are real numbers and a>b. If c≤0, then ac≤bc.

c) Assume that a, b and c are real numbers and a>b. If c>0, then ac>bc.

d) Assume that a, b and c are real numbers and a>b. If c≥0, then ac≥bc.

e) None of the above

- Why? State your reasons.

Figure 3. 4. Question 3 in Section A

3.3.1.2. Questions in Section B

Section B covers four discussion questions which were adapted from the study of Saeed (1996). These items involve the description of a mathematical situation, generalization or conclusion in the format of a dialog where there is a disagreement between the suggested ideas. Participants were asked to choose the person they agreed with and explain the reasons of their choices. As mentioned, Question 1 and Question 3 are related to proof by contradiction, Question 2 is related to proof by contrapositive and Question 4 is related to refutation. The questions in Section B are explained below.

Question 1 in Section B is related to proof by contradiction and includes the argument of a statement. The answers of students were accepted as correct if they stated that Ali was right. The aim of the question is to examine whether students can notice the contradiction method used in the proof. Results of the data also displayed students' ideas about the necessity of proof.

Ali has shown that the following statement is true for all real numbers x and y.
Statement: If $x \neq 0$ and $y \neq 0$, then $x.y \neq 0$

Ali's argument: Assume that $x \neq 0$ and $y \neq 0$ but $x.y=0$.
Since $x \neq 0$ then x^{-1} exists.
Then $x^{-1}.(x.y)=(x^{-1}.x).y=1.y=y$
Also, since $x.y = 0$, $x^{-1}.(x.y)=x^{-1}.0=0$
Therefore $y=0$. But $y \neq 0$.
Thus $x.y=0$ must be false.
Therefore, $x.y \neq 0$

Ayşe: I feel your argument is completely unnecessary. Look, everybody knows that if $x \neq 0$ and $y \neq 0$, then $x.y \neq 0$. There is no need to show it.

Ali: I agree that the above statement is familiar to everybody, but I disagree that my argument is unnecessary, Ayşe.

Questions;
- Considering the discussion above, who do you agree with?

Ali________ Ayşe_________

- Why? Explain your reasons.

Figure 3. 5. Question 1 in Section B

Question 2 in Section B is related to proof by contrapositive and Ahmet was the person presenting a valid argument in the discussion. Pınar could not realize that statement A and statement B were contrapositive, so she thought that another proof is needed for statement A. The purpose of the question is to investigate whether students could notice contrapositive statements and understand proof by contrapositive.

Statement A: If n^2 is an odd integer, then n is an odd integer.
Statement B: If n is an even integer, then n^2 is even integer.

Ahmet: I think statement A is true, Pınar.

Pınar: Let me see, if $n^2=9$, then $n=\pm 3$ is odd; if $n^2=25$, then $n\pm5$ is odd. So, statement A seems to be true Ahmet.

Ahmet: I also think that statement B is true, Pınar.

Pınar: Why?

Ahmet: Since n is even, then n=2k where k is some integer.
Therefore, $n^2 = 4k^2 = 2\ (2k^2)$ is also even.

Pınar: But Ahmet, this only show the statement B is true, but does not show that statement A is true.

Ahmet: This argument also shows that statement A is correct.

Questions;
- Considering the discussion above, who do you agree with?
Ahmet________ Pınar_________

- Why? Explain your reasons.

Figure 3. 6. Question 2 in Section B

Question 3 in Section B is concerned with proof by contradiction. Deniz applied this method correctly for the statement but Ege could not understand her proof. Therefore, the answers of students who agreed with Deniz accepted as correct. The aim of the question is to determine whether students understand the logic of proof by contradiction.

Ege: How can you show that if x is a rational number and y is an irrational number, then y-x is an irrational number?

Deniz: Suppose that y-x is not irrational (rational).
Then y-x=c/d, for some integers c and d≠0.
Since x is rational then, x=a/b for some integers a and b≠0.
Thus, (y-x)+x= c/d+a/b= (cb+ad)/db
Since cb+ad and db are both integers then (y-x)+x is rational.
But (y-x)+x= y.
Thus y is rational, which is false.
Therefore y-x is irrational as desired.

Ege: But you started out by supposing that y-x is not irrational; it does not make sense to me to suppose that y-x is not irrational in order to show just the opposite.

Deniz: I have to start out by assuming that y-x is rational because this is a correct method of proof.

Questions;
- Considering the discussion above, with who do you agree?
Ege________ Deniz_________

- Why? Explain your reasons.

Figure 3. 7. Question 3 in Section B

Question 4 in Section B is related to the refutation method. İlknur provided valid claims in the discussion since she declared that there may be a number which does not satisfy the formula. The aim of the question is to investigate whether students notice whether a counterexample is sufficient to disprove a statement or formula.

Cem and İlknur are discussing prime numbers.

Cem: I have been trying to find a formula which will always give me a prime number and I have finally succeed, İlknur.
n^2-n+41
When n=1, n^2-n+41= 41
When n=2, n^2-n+41=43
When n=3, n^2-n+41=47
When n=4, n^2-n+41=53
I tried all the numbers from 0 to 40. It just keeps giving me prime numbers. Hence, my formula is correct.

İlknur: I don't agree with you, Cem. I think, we can find at least one number greater than 40 does not satisfy your formula.

Questions;
- Considering the discussion above, who do you agree with?
Cem________ İlknur__________

- Why? Explain your reasons.

Figure 3. 8. Question 4 in Section B

3.3.1.3. Questions in Section C

Section C is comprised of three open-ended proof questions which were chosen after examining the questions in related literature (Morris & Morris, 2009; Velleman, 2006). Among the open-ended proof questions, there are statements which students were asked to prove. Statements which could be proved using different proof methods were chosen intentionally so that proof methods that students used in valid proofs could be investigated.

Table 3. 5. Questions in Section C

Question 1	Show that $1 + 2 + 3 + 4 + 5 + \ldots + n = n \cdot (n+1) / 2$.
Question2	"Assume that a and b are real numbers. If $0<a<b$, then $a^2<b^2$." Prove the given statement above.
Question3	"For all natural number n, $3 \mid n^3 - n$." Prove the given statement above.

3.3.2. Pilot Study

The pilot study of MPQ was conducted in a state university in the Western Black Sea Region at the beginning of the 2011-2012 spring semester. The purposes of the pilot study were to determine the duration of the implementation of the questionnaire, to reveal the points which may cause problems in the actual administration, and to check the validity and reliability of MPQ.

The participants of the pilot study were comprised of 21 freshman, 25 sophomore and 28 junior pre-service middle school mathematics teachers. The participants of the pilot study did not include senior students since there were no senior students in the department. MPQ which involves 14 items was administered to freshmen in the Computer course, to sophomores in the Analysis course and to juniors in the Analytic Geometry course with the permission of the instructors at the beginning of the mentioned lessons. It took between 50-80 minutes for the students to answer the questions. However, it was noticed that freshman pre-service teachers needed more time than that needed by those in the other grade levels. As it was noticed during the pilot study that most of the students could not answer the question in one hour, the number of items in the instrument was reduced. Another reason for doing so derived from the fact that as all items are related to proof, students generally became bored while answering the questions.

Considering these points, two multiple choice items and one open-ended proof question were removed from the instrument and some items were revised. One of the multiple choice questions excluded from the instrument was based on the refutation method and had similar options with Question 4 in Section B. The other multiple choice question was related to proof by contrapositive. In this question, a proof was presented in numbered steps and students were asked to find the wrong step of the given proof. Since nearly all of the students answered correctly, this question was removed from the instrument. The open-ended proof question was excluded since it was too easy for pre-service middle school mathematics teachers and there was a similar proof in one of the discussion questions.

3.3.3. Validity and Reliability Issues

Validity refers to the appropriateness, correctness, meaningfulness and usefulness of the conclusions that the researcher drew from the data collected (Fraenkel & Wallen, 2005). In other words, validity is associated with the purpose of the instrument and of the conclusions drawn from the data collected through the instrument. The instrument was submitted to experts in mathematics education for content validation. Before the pilot study, four mathematics educators in the Elementary Mathematics Education program of two different universities had evaluated the items of the instrument in terms of the appropriateness of the items in relation to the purposes of the study, the usage of mathematical terms, and the clarity of the statements.

Reliability refers to the consistency of the scores obtained from an instrument (Fraenkel & Wallen, 2005). To check the reliability of the instrument employed in the current study, the scoring observer agreement method was used. The data were assessed by a researcher and a second rater who was a graduate student in mathematics education. The inter-rater reliability was calculated and a 97% correlation was found to exist between the two ratings.

3.4. Data Collection Procedure

Before the administration of MPQ, the official permissions were taken from the Middle East Technical University Human Subjects Ethics Committee. Then, the schedule was prepared for the administration of the instrument by examining the weekly course schedule of the Elementary Mathematics Education program in the selected university. Then, the researcher asked for the permissions of the course instructors and informed them about the study.

Subsequently, MPQ was administered to 115 pre-service middle school mathematics teachers in a state university in Ankara at the end of the 2011-2012 spring semester. More specifically, the instrument was administered to 19 freshmen in the Computer Applications in Education course, 25 sophomores in the Measurement and Assessment course, 39 juniors in the Methods of Teaching Mathematics course, and 32 seniors in the Practice Teaching in Elementary Education course. Approximately one hour was given for the students to answer the questions. Also, it was administered to freshmen, sophomores, juniors and seniors once in a time. At the beginning of the administration, the participants were informed about the study. The participants were ensured that the study would expose them to no physical or psychological harm and their responses would be kept confidential.

3.5. Data Analysis

To investigate the research questions, an item-based analysis was conducted using the SPSS PASW program. Since the format of the questions in the instrument are different such as multiple choice, discussion and open-ended questions, rubrics were developed for each different question type. The rubrics were developed according to the answers of the students in the pilot study, analysis of the studies from which some of the questions were adapted and also by considering different answers for each question. To illustrate, the rubrics established for the multiple choice questions and discussion questions are presented below.

Table 3. 6. Rubric for Multiple Choice Questions

Codes	**Answer types**
0	No answer
1	Wrong answer, no explanation
2	Wrong answer, wrong interpretation
3	Correct answer, no explanation or unclear explanation
4	Correct answer, reason is not related to a proof method
5	Correct answer, reason is related to *refutation/ proof by contrapositive/ proof by contradiction*

Table 3. 7. Rubric for Discussion Questions

Codes	**Answer types**
0	No answer
1	Agreed with no one or both of them
2	Agreed with *name of the wrong person*, no reason was stated
3	Agreed with *name of the wrong person*, reason was stated
4	Agreed with *name of the right person*, no reason was stated
5	Agreed with *name of the right person*, reason which is not related to a proof method was stated
6	Agreed with *name of the right person*, reason which is related to *refutation/ proof by contrapositive/proof by contradiction* was stated

As previously stated, the purpose of Section A of MPQ, which consisted of 4 multiple choice questions and Section B, which included 4 discussion questions, was to determine students' achievement levels in refutation, proof by contrapositive, proof by contradiction and to reveal the reasons underlying their wrong interpretations. First, students' answers were evaluated according to the rubrics presented above. Then, the frequencies and percentages of students' answers for each item were evaluated by year levels. Subsequently, pre-service middle school mathematics teachers' wrong answers including a reason, which were coded as 2 in the multiple choice questions and coded as 3 in discussion questions, were analyzed in order to reveal the reasons behind their wrong interpretations. As previously mentioned, there is a sub-question for each question, which requires students to explain their answers. To determine the reasons underlying students' wrong interpretations, the explanations students gave to their wrong answers were taken into consideration.

Section C of MPQ consists of three open-ended proof questions. The study aimed to address the third research question by means of these questions. Firstly, the answers of students given to open-ended proof questions were examined within three categories which are no answer, invalid proof and valid proof to determine their achievement levels in conducting valid proof. Pre-service middle school mathematics teachers' arguments were accepted as valid if they chose and applied proof methods for the statement correctly and deduced the desired conclusion. After this, the valid proofs were classified into proof methods in order to determine proof methods which students used. Then, the answers of students which were coded as invalid proof were analyzed and the reasons of their invalid proof were determined.

3.6. Internal Validity and External Validity

Internal validity means that the observed differences on the dependent variable are affected by the independent variable (Fraenkel & Wallen, 2005). In other words, the reason of the difference is not an unintended variable. For every research design, different internal validity threats can be cited. Fraenkel and Wallen (2005) stated that main threats to internal validity in survey research are mortality, location and instrumentation.

Mortality: The mortality threat occurs if some subjects drop out of the study no matter what the reason is in the data collection process (Fraenkel & Wallen, 2005).

In other words, absence of participants may affect the results of the study since their data may cause a difference in the results. In this study, mortality was not a threat since cross-sectional survey, in which data were collected at one point of time, was employed.

Location: Location threat may be present if the location where the data are collected has an effect on the results of the study (Fraenkel & Wallen, 2005). In the present study, location was not a threat since data were collected from students in one university and in similarly equipped classrooms.

Instrumentation: Instrumentation threat is associated with how instruments are used. Instrument decay, data collector characteristics and data collector bias are threats related to instrumentation (Fraenkel & Wallen, 2005). Instrument decay occurs when the instrument is changed or scored differently (Fraenkel & Wallen, 2005). To eliminate this threat, the answers of the students in the current study were evaluated by two scorers according to the rubrics. Data collector characteristics can be seen as a threat to internal validity when the data are collected by different people, and data collector bias occurs when the data collector affects the results of the data to have expected outcomes more likely (Fraenkel & Wallen, 2005). To handle these threats in the current study, the application of the instrument was standardized. Data collectors were informed about the data collection procedures, there was no interaction with students during the administration, and data collection took at most one hour in all year levels.

External validity, on the other hand, is defined as "the extent that the results of a study can be generalized from a sample to a population" (Fraenkel & Wallen, 2005, p.108). It involves population generalizability and ecological generalizability. To mention population generalizability in a study, the sample should represent the intended population. In the present study, the target population of the study is all pre-service middle school mathematics teachers in the state universities in Turkey. Moreover, the accessible population of the study is pre-service middle school mathematics teachers enrolled in all state universities in the Central Anatolia Region. Since convenience sampling was used in the study, the sample of the study was determined as freshmen, sophomores, juniors and seniors enrolled in the Elementary Mathematics Education program of a state university in Ankara. However, since students in the Elementary Mathematics Teachers Education program in the selected university have relatively higher scores in the national university entrance exam, the sample may not be accepted as representative of the target population.

Frankel and Wallen (2005) defined ecological generalizability as “the degree to which the results of a study can be extended to other settings and conditions” (p.106). Since the courses in the Elementary Education programs, except for the elective courses, are virtually the same in all the state universities in Turkey and students’ scores in the national university entrance exam are considerably high, the results of the study were considered to be generalizable to the students in similar conditions.

CHAPTER IV

RESULTS

The first purpose of the study was to investigate pre-service middle school mathematics teachers' achievement levels in refutation, proof by contrapositive and proof by contradiction by year level. The second purpose of the study was to determine the reasons of their wrong interpretations in the mentioned proof methods. The third purpose of the study was to investigate to what extent pre-service middle school mathematics teachers can conduct valid proof. After determining the proof students' provide as valid and invalid, their answers were analyzed to examine proof methods that they used in their valid proofs and the reasons underlying their invalid proofs.

In this chapter, the results of the data are presented based on the different types of proof questions in the instrument, namely refutation questions, proof by contrapositive questions and proof by contradiction questions and open-ended proof questions. More specifically, the results addressing the first and second research questions of the study are presented under the following headings: *analyses of refutation, proof by contrapositive* and *proof by contradiction questions*. The results addressing the last research question of the study are presented under the title *analysis of open-ended proof questions.*

4.1. Analysis of the Refutation Questions

The Mathematical Proof Questionnaire includes three questions related to refutation. Refutation is a proof method which is used to show that a statement is false by giving a counterexample. Two of the refutation questions, Question1 (Q1) and Question 4 (Q4) in Section A, are multiple choice items. The third question, Question 4 (Q4) given in Section B, is a discussion question. In this section, the results of the analysis of the data collected through the three refutation questions are presented. Pre-service middle school mathematics teachers' achievement levels as regards the refutation method were analyzed utilizing the rubrics prepared by the researcher. Then, the reasons underlying pre-service middle school mathematics teachers' wrong interpretations regarding the refutation method were investigated. To this end, their wrong answers and wrong explanations were coded and categorized under related themes.

4.1.1. Refutation Question 1 (Section A- Q1)

The first refutation question is given below.

***Statement A*:** a and b are natural numbers. a.b is an even number if and only if a and b are even.

Which of the followings is correct by considering the given statement?

a) Statement A is true

b) The numbers a=5 and b=6 prove that statement A is false

c) The numbers a=5 and b=6 prove that statement A is true

d) Statement A is false but the numbers a=5 and b=6 are not adequate to prove it.

e) None of the above

- Why? State your reasons.

Figure 4. 1. Refutation Question 1

As seen from the question, students were asked to find the correct choice for the given statement and explain their reasons. Since refutation refers to showing a statement is false by giving counterexample, the correct choice of the question is (b) which reads as follows: "The numbers a=5 and b=6 proves that statement A is false". As seen from the question, a=5 and b=6 can be given as a counterexample to show that statement A is false. Pre-service middle school mathematics teachers' answers were assessed according to the rubric presented below.

Table 4. 1. Rubric for Refutation Question 1

Codes	Answer types
0	No answer
1	Wrong answer, no explanation
2	Wrong answer, wrong interpretation
3	Correct answer, no explanation or unclear explanation
4	Correct answer, reason is not related to a proof method (valid reasoning)
5	Correct answer, reason is related to refutation (valid reasoning)

According to the rubric, students' answers were coded as 0 if they did not answer the question. Their answers were coded as 1 if they did not write any explanation for their wrong answers; their answers were coded as 2 if they marked the wrong choice and interpreted the question wrongly. Students' correct answers were evaluated as a means of including no explanation or unclear explanation and giving valid reasoning. In more detail, answers coded as 3 are those that have either no explanation or an unclear explanation. Students' answers were coded as 4 and 5 if valid reasoning were given in the explanation. In the answers coded as 4, the reason is not related to a proof method. On other hand, the reason provided in the answers coded as 5 is related to the refutation method. To summarize, students' answers were coded as 1 and 2 if their answers were wrong and their answers were coded as 3, 4 and 5 if their answers were correct.

The analysis of 115 pre-service middle school mathematics teachers' answers in terms of year level is presented in Table 4.2.

Table 4. 2. Frequency of the Answers for Refutation Question 1

		Year level				
		Freshmen	Sophomores	Juniors	Seniors	Total
Codes	1	-	-	-	3 (9,4%)	3 (2,6%)
	2	3 (15,8%)	-	7 (17,9%)	3 (9,4%)	13 (11,3%)
	3	3 (15,8%)	9 (36,0%)	1 (2,6%)	11 (34,4%)	24 (20,9%)
	4	8 (42,1%)	2 (8,0%)	7 (17,9%)	7 (21,9%)	24 (20,9%)
	5	5 (26,3%)	14 (56,0%)	24 (61,5%)	8 (25,0%)	51 (44,3%)
Total		19(100,0%)	25 (100,0%)	39(100,0%)	32(100,0%)	115(100,0%)

Table 4.2 presents the assessment of refutation question 1. As seen from Table 4.2, 3 seniors (2.6%) among 115 students answered the refutation question wrongly without giving any explanation. 13 students (11.3%) among 115 students answered the question wrongly by giving wrong explanations. While 3 of them were freshmen, 7 of them were juniors and 3 of them were seniors. It can be inferred from Table 4.2 that all sophomore pre-service teachers answered the given item correctly.

When correct answers of the students were analyzed, it was seen that 24 students (20.9%) among 115 students had answered the refutation question correctly but had not written an explanation or stated an unclear explanation. 3 of them were freshmen, 9 of them were sophomores, 1 of them was junior and 11 of them were seniors. According to Table 4.2, 24 students (20.9%) among 115 students had answered the question correctly and suggested valid reasons which were not related to a proof method. In terms of year level, it was seen that 8 freshmen, 2 sophomores, 7 juniors and 7 seniors had answered the question correctly without mentioning a proof method. To illustrate, the answer of Participant 78 is presented below:

Participant 78 (junior):

“The statement is false for the numbers a=5 and b=6, but only values 5 and 6 are not enough to prove. We should use general terms.”

[a=5, b=6 için önerme yanlıştır ancak sadece 5 ve 6 değerlerini kullanarak bu önermeyi ispatlayamayız. Genel terimler kullanmamız gerekir.]

Similarly, it was observed that participant 56 had answered the item correctly but had not related it to a proof method.

Participant 56 (junior):

"Statement A is false because both of the numbers a and b do not have to be even. It is enough if one of them is even"

[A önermesi yanlıştır. Çünkü a, b sayılarının ikisinin de çift olmasına gerek yoktur. Birisi çift olsa yeter.]

The remaining 51 students (44.3%) among 115 students were found to have answered the refutation question correctly and supported their answers with explanations which were directly related to the refutation method. As can be seen in Table 4.2, 5 of them were freshmen, 14 of them sophomores, 24 of them were juniors and 8 of them were seniors. For example, Participant 45 answered this question correctly and her answer was related to the refutation method.

Participant 45 (junior):

"To show that a statement is false, giving a counterexample is enough. However, if we want to show the truth of this statement, we have to show it for all the numbers. Since we can't try it for all the numbers, we have to use proof methods."

[Bir önermenin yanlış olduğunu göstermek için counter example vermemiz yeterlidir. Ama bunun doğruluğunu göstermek istiyorsak, bütün sayılar için göstermek zorundayız. Hepsi için bunu deneyemeyeceğimiz için ispat yöntemlerini kullanmak zorundayız.]

Like the previous example, Participant 59 explained her correct answer by referring to the refutation method.

Participant 59 (junior):

“It was proved with counter example, which is one of the proof methods. When a counter example is found, it is proved that the statement is not true.”

[İspat yöntemlerinden olan counter example yöntemiyle kanıtlanmıştır. Olmayan bir örnek bulunduğunda, önermenin doğru olmadığını ispatlamış oluruz.]

As seen from Table 4.2, 75 students (65.2%) among 115 students answered the question by stating a correct reasoning. In terms of year level, it was seen that all sophomore pre-service teachers had answered to the refutation question correctly. However, the percentage of seniors’ correct answers was the lowest compared to other grades. Since 61.5% of juniors’ answers were coded as 5, which means that the explanation is related to the refutation method, it can be said that junior students are more successful in explaining by means of the refutation method than those in the other year levels.

Another purpose of the study was to determine the reasons underlying pre-service middle school mathematics teachers’ wrong interpretations regarding the refutation method. As stated, 16 students (13.9%) among 115 students had answered the refutation question wrongly and 13 of them (11.3%) had provided wrong reasoning. Analysis of those students’ answers in refutation question 1 revealed that the reasons behind the wrong answers could be categorized under two headings, the details of which are presented in Table 4.3.

Table 4. 3. Reasons of Wrong Interpretations for Refutation Question 1

	Reasons	Freshmen	Sophomores	Juniors	Seniors	Total
R1	Giving examples is enough to prove that a statement is true.	-	-	2	-	2
R2	Counterexample is not enough to prove that a statement is false	3	-	5	3	11

Table 4.3 shows that 2 junior pre-service teachers answered the refutation question 1 wrongly because they thought that giving examples is enough to prove that a given statement is true. For example;

Participant 48 (sophomore):

"$(A \Leftrightarrow B) \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$ this situation needs to be verified. A=2 and B=6 verify the given statement from two sides."

[$(A \Leftrightarrow B) \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$ bu durumunun sağlanması gerekiyor. A=2 ve B=6 bu önermeyi her 2 yönden doğrular.]

Another reason behind pre-service middle school mathematics teachers' wrong interpretations is that they don't understand the meaning of counterexample. In other words, they think that giving a counterexample is not enough to prove that a statement is false. As can be observed in Table 4.3, 3 freshmen, 5 juniors and 3 seniors are included in this category. For example, the explanation provided by Participant 7 is as follows:

Participant 7 (freshman):

"a=5 and b=6 are only examples. They are not enough to prove"

[a=5 ve b=6 sadece bir örnektir. İspatlamaya yetmez.]

4.1.2. Refutation Question 2 (Section A- Q4)

The second refutation question is given below.

Statement A: An integer is divisible by 6 if the sum of its digits is divisible by 6.

Which of the followings is correct by considering the given statement?

a) The number 33 proves that statement A is false

b) The number 30 proves that statement A is false

c) The numbers a=30 and b=33 prove that statement A is false

d) Statement A is false but the numbers a=30 and b=33 are not adequate to prove it.

e) The statement is true

- Why? State your reasons.

Figure 4. 2. Refutation Question 2

Similar to refutation question 1, students were asked to find the correct choice and explain their reasons for the given statement in refutation question 2. Since refutation means showing a statement is false by giving a counterexample, the correct choice of the question is alternative (a). Pre-service teachers' answers were assessed utilizing the same rubric used in the analysis of refutation question 1.

The answers of 115 pre-service middle school mathematics teachers were analyzed and the results are presented in Table 4.4.

Table 4. 4. Frequency of the Answers for Refutation Question 2

		Year level				
		Freshmen	Sophomores	Juniors	Seniors	Total
Codes	1	2 (10,5%)	2 (8,0%)	-	2 (6,3%)	6 (5,2%)
	2	3 (15,8%)	-	8 (20,5%)	6 (18,8%)	17 (14,8%)
	3	2 (10,5%)	8 (32,0%)	1 (2,6%)	12 (37,5%)	23 (20,0%)
	4	6 (31,6%)	2 (8,0%)	5 (12,8%)	2 (6,3%)	15 (13,0%)
	5	6 (31,6%)	13 (52,0%)	25 (64,1%)	10(31,3%)	54 (47,0%)
Total		19(100,0%)	25 (100,0%)	39(100,0%)	32(100,0%)	115(100,0%)

Table 4.4 presents the descriptive results of the analysis of refutation question 2. 6 students (5.2%) among 115 students answered the refutation question wrongly and did not state any explanation. The answers of 2 freshmen, 2 sophomores and 2 seniors were coded to fall in this category. Moreover, 17 students (14.8%) among 115 students answered the refutation question wrongly by providing incorrect reasoning. 3 of them were freshmen, 8 of them were juniors and 6 of them were seniors.

As can be observed in Table 4.4, 23 students (20%) among 115 students marked the correct choice for this question. However, they did not provide any explanation for their answers nor did they state their reasoning in a clear and meaningful way. 2 freshmen, 8 sophomores, 1 junior and 12 seniors were included in this category. Since the answers of 15 students (13%) among 115 students were coded as 4 and the answers of 54 students (47%) among 115 students were coded as 5, in total 69 students (60%) among 115 students had answered the refutation correctly and supported their answers with valid explanations. More specifically, while the explanations of 15 students (6 freshmen, 2 sophomores, 5 juniors and 2 seniors) were not related to any proof method, the explanations of 54 students (6 freshmen, 13 sophomores, 25 juniors and 10 seniors) were directly related to the refutation method. To illustrate, below is the answer of Participant 9, which is not related to a proof method.

> Participant 9 (freshman):
>
> “When finding whether a number is divisible by 6, the divisibility of that number by 2 and 3 must be checked.”
>
> [Bir sayının 6’ya bölünebildiğine bakılırken, 2 ve 3’e bölünebilirliğine bakmamız gerekir.]

Unlike the answer of Participant 9, Participant 94 and Participant 45 explained their answers by referring to the refutation method.

Participant 94 (senior):

“One example which refutes the given statement is enough to show that the statement is false.”

[Verilen önermeyi yanlışlayacak tek bir örnek önermenin yanlış olması için yeterlidir.]

Similarly, the answer of Participant 45 is given below.

Participant 45 (junior):

“Statement S: p⇒q, p:‘the sum of its digits is divisible by 6’ and q: ‘an integer is divisible by 6’. In the case that statement S is true, we have if p is true than q is true. However, for number 3, p is correct and q is false. Therefore, number 33 shows that statement S is false.”

[Önerme S: p⇒q’dur. p:‘bir tam sayıdaki rakamların toplamı 6’ya bölünebiliyor’ ve q:‘tamsayı 6’ya bölünebiliyor.’ Bu durumda p doğruysa q da mutlaka doğru olmalı önerme S’nin doğru olabilmesi için. Fakat, 33 sayısını düşününce, p önermesi doğru fakat q önermesi doğru değil. Bu durumda, 33 sayısı öerme S’nin yanlış olduğunu gösterir.]

In conclusion, it can be said that 69 students (60%) among 115 participants answered this question by giving valid explanations. More specifically, the percentage of the freshman pre-service teachers’ correct answers was the lowest compared to that of pre-service teachers in the other grades. On the other hand, sophomore pre-service students had the highest percentage of correct answers. When the year levels were analyzed according to the answers in relation to the refutation method, it was seen that junior students were the most successful group and senior students were the least successful group in explaining by making use of the refutation method.

As previously stated, the other purpose of this study was to determine the reasons underlying pre-service middle school mathematics teachers' wrong interpretations in relation to the refutation method. 23 students (17%) among 115 students had answered the refutation question wrongly and 17 of them (14.8%) had explained their wrong reasons. The reasons could be categorized under three headings which are presented in Table 4.5.

Table 4. 5. Reasons of Wrong Interpretations for Refutation Question 2

	Reasons	Freshmen	Sophomores	Juniors	Seniors	Total
R1	Accepting a false statement as true	-	-	1	-	1
R2	Counterexample is not enough to prove that a statement is false	2	-	6	1	9
R3	Inappropriate counterexample	1	-	1	5	7

As seen in Table 4.5, 1 junior answered the refutation question wrongly because she accepted the given statement as true which was actually false. Another reason behind pre-service middle school mathematics teachers' wrong interpretations is related to the meaning of counterexample. They accepted that giving a counterexample is not enough to prove that a statement is false. This reason was also stated in refutation question 1. Table 4.5 shows that 2 freshmen, 6 juniors and 1 senior had answered the question holding this belief. To illustrate, the answer of Participant 78 is presented below:

> Participant 78 (junior):
>
> "Statement S is false and to prove this we have to generalize for all numbers. In other words, it can be proved that statement S is false by mathematical induction."
>
> [S önermesi yanlıştır ve bunu ispatlayabilmek için tüm sayılara genelleyebilmemiz gerekir. Yani matematiksel induction ile S önermesinin yanlış olduğu ispatlanabilir.]

The last reason underlying the wrong interpretations regarding the refutation method is that pre-service teachers chose an inappropriate counterexample. In other words, some of them accepted the number 30 as a counterexample even though the number 30 did not verify the condition in the given statement. 1 freshman, 1 junior and 5 seniors made wrong interpretations based on this reason. For example, according to Participant 115, the number 30 was a counterexample for the given statement. However, she could not notice that the sum of the digits of number 30 was not divisible by 6, which meant that the number 30 could not be accepted as a counterexample.

4.1.3. Refutation Question 3 (Section B- Q4)

The third refutation question is presented below:

Case D
Cem and İlknur are discussing prime numbers

Cem: I have been trying to find a formula which will always give me a prime number and I have finally succeed, İlknur.
n^2-n+41
When n=1, n^2-n+41= 41
When n=2, n^2-n+41=43
When n=3, n^2-n+41=47
When n=4, n^2-n+41=53
I tried all the numbers from 0 to 40. It just keeps giving me prime numbers. Hence, my formula is correct.

İlknur: I don't agree with you, Cem. I think, we can find at least one number greater than 40 does not satisfy your formula.

Questions;
- Considering the discussion above, who do you agree with?
Cem________ İlknur_________
- Why? Explain your reasons.

Figure 4. 3. Refutation Question 3

As can be seen from the question, a case was given to the students and they were asked to choose the person whom they agreed with and explain their reasons for the agreement. The discussion was related to prime numbers. Since Cem asserted that he had found a formula to find prime numbers by trying only the numbers between 0 and 40, İlknur was right. There may be a counterexample which shows that the formula is not valid for all the numbers. If a counterexample is found, it refutes the statement. Therefore, students' answers were accepted as wrong if they agreed with Cem and accepted as true if they agreed with İlknur. Moreover, some students agreed with Cem because of the idea of İlknur. Since İlknur did not find a counterexample, they decided that Cem is right. However, assertion of Cem in the discussion cannot be accepted as correct. Their answers were assessed according to the rubric presented in Table 4.6.

Table 4. 6. Rubric for Refutation Question 3

Codes	**Answer types**
0	No answer
1	Agreed with no one or both of them
2	Agreed with Cem, no reason was stated
3	Agreed with Cem, reason was stated
4	Agreed with İlknur, no reason was stated
5	Agreed with İlknur, reason which is not related to a proof method was stated
6	Agreed with İlknur, reason which is related to refutation was stated

The answers of 115 pre-service middle school mathematics teachers were analyzed and the results are presented in Table 4.7.

Table 4. 7. Frequency of the Answers for Refutation Question 3

		Year level				Total
		Freshmen	Sophomores	Juniors	Seniors	
Codes	0	1 (5,3%)	1 (4,0%)	-	4 (12,5%)	6 (5,2%)
	1	-	-	2 (5,1%)	2 (6,3%)	4 (3,5%)
	3	-	3 (12,0%)	1 (2,6%)	-	4 (3,5%)
	4	3 (15,8%)	5 (20,0%)	-	3 (9,4%)	11 (9,6%)
	5	11 (57,9%)	9 (36,0%)	26 (66,7%)	16 (50,0%)	62 (53,9%)
	6	4 (21,1%)	7 (28,0%)	10 (25,6%)	7 (21,9%)	28 (24,3%)
Total		19 (100,0%)	25 (100,0%)	39 (100,0%)	32 (100,0%)	115 (100,0%)

Table 4.7 indicates the assessment of the answers given to refutation question 3. As seen, 6 students (5.2%) among 115 students did not answer the refutation question. 1 of them was a freshman, 1 of them was a sophomore and 4 of them were seniors. Moreover, the answers of 4 students (3.5%) among 115 students, 2 of whom were juniors and 2 of whom were seniors, were classified as agree with neither Cem nor İlknur. Since these students did not explain their answers, why they agreed with neither Cem nor İlknur was unknown.

All the students who agreed with Cem explained their reasons. Since the answers of 3 sophomores and 1 junior were placed in this code of the rubric, in total 4 students (3.5%) among 115 students agreed with Cem by giving explanations.

According to Table 4.7, 11 students (9.6%) among 115 students agreed with İlknur but they did not write reasons for their agreement. Among these 11 students were 3 freshmen, 5 sophomores and 3 seniors. The remaining 90 students (78.2%) among 115 students agreed with İlknur and declared their valid reasons. As mentioned in the rubric, some of these valid reasons were not related to any proof method, which were coded as 5, and some of these valid reasons were related to the refutation method, which were coded as 6. More specifically, 62 students (53.9%) among 115 students, namely 11 freshmen, 9 sophomores, 26 juniors and 16 seniors agreed with İlknur and their explanations were not found to be related to any proof method. To illustrate, the answer of Participant 98 is given below:

Participant 98 (senior):

“If it cannot be proved that the formula is true for all numbers, we cannot talk about the validity of the formula”

[Bütün sayılar için formülün doğruluğu ispatlanamıyorsa formülün geçerliliğinden bahsedemeyiz.]

The number of students who agreed with İlknur and explained it by relating it to the refutation method was 28, which corresponds to 24.3% of the sample. 4 of them were freshmen, 7 of them were sophomores, 10 of them were juniors and 7 of them were seniors. Some examples are stated below. Participant 15 noticed that one counterexample could refute the formula in the discussion.

Participant 15 (freshman):

“Cem found a formula for a particular interval and then said that it is valid for all prime numbers. However, this can’t be a proof method. By giving one counterexample, it can be shown that his proof is false.”

[Cem belli bir aralık için formül bulup, bütün asal sayılar için geçerlidir demiş, ama bu bir proof yöntemi olamaz. Sadece bir counter example ile bütün ispatının yanlış olduğu gösterilir.]

Similarly, Participant 21 suggested a counterexample so that he could show that the formula was not valid for all the numbers.

Participant 21 (freshman):

“Counter example can be given.

If $n=41$, then $41^2-41+41= 41^2$

41^2 is not a prime number.”

[Counter example verilebilir.

n=41 ise 41^2-41+41= 41^2

41^2 asal sayı değildir.]

Another example for those answers which included an explanation related to the refutation method is given below. Participant 41 also found a counterexample for the formula.

Participant 41 (sophomore):

“n^2-n+41 gave a prime number for the numbers between 0 and 40. However, for n=41, the statement 41^2-41+41= 41^2 is not a prime number. Therefore, I agree with İlknur. In fact, in this proof, it is impossible to find the result by trying all numbers. The formula that Cem found should be proved.”

[n^2-n+41 her zaman asal sayıyı 0 ile 40 aralığında vermiş. Fakat, n=41 için n^2-n+41 ifadesi bir asal sayı olmuyor. Bu yüzden bu ispatta sayıları deneyerek bulmamız imkansız olduğu için Cem’in bulduğu formülün ispatının yapılması gerekir.]

To conclude, the percentage of sophomore pre-service teachers who agreed with Cem was the highest compared to that of the participants in the other year levels, and the percentage of freshman pre-service teachers who agreed with İlknur was the highest compared to that of participants in the other year levels. In total, only 4 students (3.5%) among 115 students agreed with Cem. On the other hand, 101 students (87.8%) among 115 students agreed with İlknur.

The reasons behind pre-service middle school mathematics teachers’ wrong interpretations regarding the refutation method were also investigated. After the analysis of four students’ answers, it was observed that the reasons could be categorized under two headings, as presented in Table 4.8 below:

Table 4. 8. Reasons of Wrong Interpretations for Refutation Question 3

	Reasons	Freshmen	Sophomores	Juniors	Seniors	Total
R1	No counterexample is given	-	3	-	-	3
R2	Misunderstanding of mathematical induction	-	-	1	-	1

As presented in Table 4.8, none of the freshman and senior pre-service teachers agreed with Cem. On the other hand, 3 sophomores and 1 junior agreed with Cem. The first reason is that students thought İlknur had to find the counterexample to refute the formula of Cem. Since İlknur did not find the counterexample and only stated that there might be a counterexample, they agreed with Cem. 3 sophomore pre-service teachers agreed with Cem for this reason. For example;

Participant 31 (sophomore):

"If İlknur found that number, she was right. However, she did not find it."

[İlknur o sayıyı bulsaydı, haklıydı. Ama, bulmadı.]

Another reason of behind the wrong interpretations is related to misunderstanding of mathematical induction. 1 junior pre-service student who is Participant 75 accepted what Cem wrote as mathematical induction. However, Cem's argument is not related to mathematical induction.

Participant 75 (junior):

"For n=1, n=2, n=3, the formula is correct. If it is true for n=k, then it is true for n=k+1. If n=k, then k^2-k+41. If n=k+1, then $(k+1)^2-(k+1)+41=k^2+k+41$. He proves by using mathematical induction. Therefore, there is no need to try a greater number."

[n=1, n=2, n=3 için doğru. n=k için doğruysa n=k+1 için de doğrudur. n=k ise k^2-k+41, n=k+1ise $(k+1)^2$-(k+1)+41=k^2+k+41. Cem mathematical induction kullanarak ispat etmiştir. O yüzden, daha büyük bir değeri denemeye gerek yok.]

When the reasons of students' wrong interpretations in the three refutation questions were analyzed, it was seen that there were six reasons in total. The first and the second questions included one common reason, which was counterexample not being sufficient to prove that a statement was false. On the other hand, the reasons in the third question were not found to exist in the other two refutation questions.

4.2. Analysis of the Contrapositive Questions

The Mathematical Proof Questionnaire includes three questions related to proof by contrapositive. In the contrapositive method, proving the proposition $p \Rightarrow q$ is equivalent to proving the proposition $-q \Rightarrow -p$. Two of the contrapositive questions, Question2 (Q2) and Question3 (Q3) in Section A, are multiple choice questions. The third question, Question 2 (Q2) in Section B, is a discussion question.

In this section, the results of the analysis of students' answers to the contrapositive questions are presented. Pre-service middle school mathematics teachers' achievement levels in proof by contrapositive were analyzed utilizing the rubrics prepared by the researcher. Moreover, the reasons underlying pre-service middle school mathematics teachers' wrong interpretations in the contrapositive method were investigated. To this end, their wrong answers were coded and the wrong explanations were categorized under related themes.

4.2.1. Contrapositive Question 1 (Section A- Q2)

The first contrapositive question is presented below:

Assume that m and n are positive integers. If mn=100, then m≤10 or n≤10.

To prove the statement, which assumption can you begin with?

a) Assume that m and n are positive integers. If m>10 or n>10, then mn=100.

b) Assume that m and n are positive integers. If m≤10 or n≤10, then mn=100.

c) Assume that m and n are positive integers. If m≤10 and n≤10, then mn≠100.

d) Assume that m and n are positive integers. If m>10 and n>10, then mn≠100.

e) None of the above

- Why? State your reasons.

Figure 4. 4. Contrapositive Question 1

As given in Figure 4.4, students were asked to choose an assumption to start proof for the given statement. They were expected to find the choice related to contrapositive method since the other choices were not appropriate for any kind of proof method. Also, they were asked to express their reasons. Since the contrapositive of the statement is logically equivalent to the statement, it could be proved instead of the given statement. Therefore, the correct choice of the question was (d) which reads as follows: "Assume that m and n are positive integers. If m>10 and n>10, then mn≠100". Pre-service middle school mathematics teachers' answers were analyzed according to the rubric below:

Table 4. 9. Rubric for Contrapositive Quesiton 1

Codes	Answer types
0	No answer
1	Wrong answer, no explanation
2	Wrong answer, wrong interpretation
3	Correct answer, no explanation or unclear explanation
4	Correct answer, reason is not related to a proof method (valid reasoning)
5	Correct answer, reason is related to contrapositive (valid reasoning)

The rubric formed for the contrapositive questions is similar to the rubric prepared for the refutation questions. These rubrics differ in code 5 of the rubric. In this rubric, the correct answers with an explanation related to the contrapositive method were coded as 5. To summarize the general structure of the rubric, it can be said that students' answers were coded as 1 and 2 if their answers were wrong and their answers were coded as 3, 4 and 5 if their answers were correct.

The frequencies of 115 pre-service middle school mathematics teachers' answers in terms of year level are presented in Table 4.10 below:

Table 4. 10. Frequency of the Answers for Contrapositive Question 1

		Year level				Total
		Freshmen	Sophomores	Juniors	Seniors	
Codes	0	-	1 (4,0%)	3 (7,7%)	-	4 (3,5%)
	1	1 (5,3%)	2 (8,0%)	-	9 (28,1%)	12 (10,4%)
	2	4 (21,1%)	8 (32,0%)	16 (41,0%)	10 (31,3%)	38 (33,0%)
	3	11 (57,9%)	12 (48,0%)	12 (30,8%)	8 (25,0%)	43 (37,4%)
	4	3 (15,8%)	1 (4,0%)	3 (7,7%)	-	7 (6,1%)
	5	-	1 (4,0%)	5 (12,8%)	5 (15,6%)	11 (9,6%)
Total		19(100,0%)	25 (100,0%)	39(100,0%)	32(100,0%)	115(100,0%)

Table 4.10 presents the results of the analysis of the answers given to contrapositive question 1. According to Table 4.10, 1 sophomore and 3 juniors did not answer the contrapositive question. 12 students (10.4%) among 115 students answered the question wrongly, but they did not state any explanation for their wrong answers. 1 freshman, 2 sophomores and 9 seniors were coded in this category. 38 students (33%) among 115 students answered the contrapositive question wrongly with a wrong reasoning. 4 freshmen, 8 sophomores, 16 juniors and 10 seniors were coded in this category.

When the correct answers of the students were investigated, it was seen that 43 students (34.7%) among 115 students marked the correct choice of the questions without giving any explanation or a clear explanation. 11 of them were freshmen, 12 of them were sophomores, 12 of them were juniors and 8 of them were seniors. The answers of 7 students (6.1%) among 115 students were correct and their reasoning were not related to any proof methods. 3 freshmen, 1 sophomore and 3 juniors explained their answers without referring to any proof method. As an example, the answer of Participant 4 can be given.

Participant 4 (freshman):

“If n and m are both greater than zero, then mn is greater than 100”

[n ve m’in ikisi de sıfırdan büyük olursa mn 100’den büyük olur.]

The remaining 11 students (9.6%) among 115 students answered the question correctly by providing an explanation based on valid reasoning in relation to the contrapositive method. The answers of 1 sophomore, 5 juniors and 5 seniors were coded in this category. As an example, the answers of Participant 37 and Participant 52 are presented below.

Participant 37 (sophomore):

“Proof by contrapositive”

[Olmayana ergi yöntemiyle ispat]

Similarly, Participant 52 answered the question correctly by explaining through proof by contrapositive.

Participant 52 (junior):

“ $p \Rightarrow q \equiv p' \vee q \equiv q \vee p' \equiv q' \Rightarrow p'$

If $m>10$ and $n>10$, then $mn \neq 100$

p: $mn=100$ p’:$mn \neq 100$

q: $m \leq 10 \vee n \leq 10$ q’: $m>10 \wedge n>10$”

To conclude, as seen in Table 4.10, only 18 students (15.7%) among 115 students answered the question correctly with suggesting valid reasoning. Moreover, the percentage of freshman pre-service teachers’ correct answers was the highest compared to the percentage of those in other year levels. On the other hand, the percentage of junior pre-service students’ correct answers was the lowest compared to the percentage of those in other year levels. Senior students had the highest percentage (15.6%) in mentioning proof by contrapositive in their explanations.

In this study, another purpose was to investigate the reasons behind pre-service middle school mathematics teachers’ wrong interpretations regarding the contrapositive method. As stated, 50 students (43.4%) among 115 students had answered the question wrongly and 38 of them explained their reasons. According to the results obtained from the analysis of 38 students’ answers, the reasons could be categorized under four headings.

Table 4. 11. Reasons of Wrong Interpretations for Contrapositive Question 1

	Reasons	Freshmen	Sophomores	Juniors	Seniors	Total
R1	Lack of information related to contradiction method and contrapositive method	5	6	10	5	26
R2	Trying to prove $q \Rightarrow p$ or $q \Rightarrow p'$ in order to prove the statement $p \Rightarrow q$	-	1	2	2	5
R3	Accepting a true statement as false	-	-	3	2	5
R4	Trying to prove by using direct proof instead of using given choices	-	1	1	1	3

As stated in the Table 4.11, 26 students (5 freshmen, 6 sophomores, 10 juniors and 5 seniors) answered the contrapositive question wrongly since they had deficiencies related to proof methods involving proof by contrapositive and proof by contradiction. In other words, they thought that one of the choices in the question was related to contradiction even though this choice was not appropriate for the contradiction method. For example, Participant 78 chose one of the wrong choices and explained it as an assumption for contradiction.

Participant 78 (junior):

"By starting with this choice (Assume that m and n are positive integers. If $m \leq 10$ and $n \leq 10$, then $mn \neq 100$), we can prove the statement with contradiction method"

[Bu seçenekle başlandığında (m ve n positif tam sayılar olmak üzere $m \leq 10$ ve$n \leq 10$ ise $mn \neq 100$ olur), contradiciton yöntemiyle ifadeyi ispatlayabiliriz.]

The second reason was that students accepted the propositions $q \Rightarrow p$ and $q \Rightarrow p'$ as a proof method in order to prove the statement in the form $p \Rightarrow q$ and selected the choice appropriate to this idea. In fact, the propositions $q \Rightarrow p$ and $q \Rightarrow p'$ cannot be used to prove the statement $p \Rightarrow q$ since they are not logically equivalent. 5 students (1 sophomore, 2 juniors and 2 seniors) answered the question wrongly because of this misunderstanding. To illustrate, the answer of Participant 94 is presented below. Participant 94 firstly determined the propositions and then implied that proving $p \Rightarrow q$ was equivalent to proving $q \Rightarrow p$, which is not true.

Participant 94 (senior):

"Assume that $p: m \leq 10$ $q: n \leq 10$ $r: mn=100$

$(p \vee q) \Rightarrow r$ can be used to prove $r \Rightarrow (p \vee q)$"

Similarly, Participant 75 accepted the proposition $q \Rightarrow p'$ as appropriate to prove the statement $p \Rightarrow q$ and selected the choice appropriate to that.

Participant 75 (junior):

"$m, n \in Z^+$, if $m \leq 10$ or $n \leq 10$ then $mn \neq 100$.

$0 \leq m \leq 10$

$0 \leq n \leq 10$

$0 \leq mn \leq 100 \Rightarrow mn \neq 100$"

The third reason is that students accepted the given statement as false even though it was a true statement. Therefore, these students tried to find counterexamples to refute the statement. As presented in Table 4.11, 5 students (3 juniors and 2 seniors) answered the question wrongly with this idea. For instance, Participant 114 could not see that the given statement was true, so he cited that counterexamples could be given.

Participant 114 (senior):

"This is not a true statement. So,

$m=12$ and $n=12$

$mn=12.12=144\neq100$

Therefore, if $mn=100$ then $m\leq10$ and $n\leq10$"

[Bu doğru bir önerme değil. Bu nedenle,

$m=12$ ve $n=12$

$mn=12.12=144\neq100$

Buradan, $mn=100$ ise $m\leq10$ ve $n\leq10$ olur]

The last reason is that students tried to provide proof by using direct proof instead of using the given choices. 3 students (1 sophomore, 1 junior and 1 senior) answered the question wrongly as a result of this idea. As an example, the answer of Participant 106 is given below:

Participant 106 (senior):

"Firstly, we assume that $mn=100$; we try to find $m\leq10$ or $n\leq10$. The sentences above are statement sentences. We cannot start like this"

[İlk olarak, $mn=100$ olduğunu farzederiz; $m\leq10$ veya $n\leq10$ olduğunu bulmaya çalışırız.Yukarıdaki verilen cümleler statement (durum) cümleleridir. Bu şekilde başlayamayız.]

4.2.2. Contrapositive Question 2 (Section A- Q3)

The second contrapositive question is as follows:

Assume that a, b and c are real numbers and a>b. If ac≤bc, then c≤0.

To prove the statement, which assumption can you begin with?

a) Assume that a, b and c are real numbers and a>b. If c>0, then ac≤bc.

b) Assume that a, b and c are real numbers and a>b. If c≤0, then ac≤bc.

c) Assume that a, b and c are real numbers and a>b. If c>0, then ac>bc.

d) Assume that a, b and c are real numbers and a>b. If c≥0, then ac≥bc.

e) None of the above

- Why? State your reasons.

Figure 4. 5. Contrapositive Question 2

In this question, students were asked to choose an assumption to start proof for the given statement. Since one of the choices was appropriate to the contrapositive method, the correct answer was (c), which reads as follows: "Assume that a, b and c are real numbers and a>b. If c>0, then ac>bc". In this choice, the proposition q'⇒p' was aimed to prove instead of the proposition p⇒q, which is known as the contrapositive method. Their answers were analyzed according to the rubric used in the contrapositive question 1.

The results obtained from the analyses of the answers of 115 pre-service middle school mathematics teachers are presented in Table 4.12.

Table 4. 12. Frequency of the Answers for Contrapositive Question 2

		Year level				
		Freshmen	Sophomores	Juniors	Seniors	Total
Codes	0	1 (5,3%)	1 (4,0%)	1 (2,6%)	1 (3,1%)	4 (3,5%)
	1	2 (10,5%)	2 (8,0%)	1 (2,6%)	7 (21,9%)	12 (10,4%)
	2	9 (47,4%)	6 (24,0%)	16 (41,0%)	12 (37,5%)	43 (37,4%)
	3	5 (26,3%)	10 (40,0%)	11 (28,2%)	7 (21,9%)	33 (28,7%)
	4	2 (10,5%)	1 (4,0%)	4 (10,3%)	2 (6,3%)	9 (7,8%)
	5	-	5 (20,0%)	6 (15,4%)	3 (9,4%)	14 (12,2%)
Total		19(100,0%)	25 (100,0%)	39(100,0%)	32(100,0%)	115(100,0%)

Table 4.12 displays the descriptive results obtained from the assessment of contrapositive question 2. 1 student from each year level, totaling to 4 students (3.5%) among 115 students, did not answer to the question. As can be seen in Table 4.12, 12 students (10.4%) among 115 students answered the question wrongly without specifying any reason. 2 freshmen, 2 sophomores, 1 junior and 7 seniors fall in this category. The number of students who answered the question wrongly and also wrote an explanation is 43 (37.4%), which constitutes more than one third of the participants. 9 of them were freshmen, 6 of them were sophomores, 16 of them were juniors and 12 of them were seniors.

The results revealed that 33 students (28.7%) among 115 students marked the correct choice in the question but they did not substantiate their ideas. 5 freshmen, 10 sophomores, 11 juniors and 7 seniors answered the question in this way. Moreover, 9 students (7.8%) among 115 students answered the question correctly without referring to any proof method. 2 freshmen, 1 sophomore, 4 juniors and 2 seniors were evaluated in this category. For example, Participant 19 explained her correct answer without relating it to the contrapositive method.

Participant 19 (freshman):

“a>b and c>0 gives the result ac<bc”

[a>b ve c>0, ac<bc sonucunu verir.]

The last code refers to the answers that are correct and have an explanation related to the contrapositive method. 14 students (12.2%) among 115 students answered in this way. In terms of year level, 5 of them were sophomores, 6 of them were juniors and 3 of them were seniors. To illustrate, the explanations of Participant 97 and Participant 37 are related to the contrapositive method.

Participant 97 (senior):

“p: $ac \leq bc$ q: $c \leq 0$

$p \Rightarrow q \equiv p' \vee q \equiv q \vee p' \equiv q' \Rightarrow p'$ ”

Participant 37 also explained by mentioning the contrapositive method.

Participant 37 (sophomore):

“Proof by contrapositive,

Assume $c>0$, then $ac>bc$”

As seen in Table 4.12, it can be said that 23 students (20%) among 115 students answered this question basing the response on correct and valid reasoning. The percentage of sophomores’ correct answers was the highest compared to that of participants in the other year levels. The percentages of freshman and seniors pre-service teachers’ correct answers are close to each other and constitute the lowest values compared to those of participants in the other year levels. Sophomore pre-service teachers have the highest percentage (20%) in explaining a response using proof by contrapositive.

Another purpose of the study was to determine the reasons underlying pre-service middle school mathematics teachers’ wrong interpretations regarding the contrapositive method. As previously stated, 55 students (47.8%) among 115 students answered the contrapositive question wrongly and 43 of them stated their wrong reasons. Analysis of 43 students' answers showed that the reasons could be categorized under three headings. The reasons are presented in Table 4.13.

Table 4. 13. Reasons of Wrong Interpretations for Contrapositive Question 2

	Reasons	Freshmen	Sophomores	Juniors	Seniors	Total
R1	Lack of information related to contradiction method and contrapositive method	5	3	5	6	19
R2	Trying to prove q⇒p or q'⇒p in order to prove the statement p⇒q	4	-	9	3	16
R3	Trying to prove by using direct proof instead of using given choices	-	3	2	3	8

All of the reasons are also stated in contrapositive question 1. As presented in Table 4.13, 19 students (5 freshmen, 3 sophomores, 5 juniors and 6 seniors) answered the contrapositive question wrongly since they had deficiencies related to the contradiction method and the contrapositive method. In other words, they thought that one of the choices in the question was related to contradiction or contrapostive even though this choice was not appropriate for the mentioned proof methods. For example, Participant 7 chose one of the wrong choices and explained it as an assumption for contradiction.

> Participant 17 (freshman):
>
> "To prove by contradiction, we have to prove the converse situation. It is enough for this situation too"
>
> [Contradiction ile ispatı yapmamız için ters durumu ispatlamalıyız. Bu durum için de yeterli olur.]

Similarly, Participant 115 could not find the proposition q'⇒p' in the choices of the question.

Participant 115 (senior):

"We can convert $A \Rightarrow B$ to $B' \Rightarrow A'$. All of the choices include the expression 'if…then'. However, there is no the statement $B' \Rightarrow A'$ in any of them."

[$A \Rightarrow B$'yi $B' \Rightarrow A'$ diye çevirebiliriz. Şıkların hepsi 'ise' ifadesini içeriyor fakat hiçbirinde $B' \Rightarrow A'$ ifadesi yok]

The second reason is that students intended to prove the propositions $q \Rightarrow p$ or $q' \Rightarrow p$ in order to prove the given statement in the form of $p \Rightarrow q$ and chose the choice appropriate to this idea. 16 students (4 freshmen, 9 juniors and 3 seniors) answered the question wrongly because of this misunderstanding. For example, the answer of Participant 4 is presented below. Participant 4 thought that the converse of the proposition $p \Rightarrow q$ was the proposition $q \Rightarrow p$ and also they could be assumed to be equivalent in providing proof. Therefore, Participant 4 chose the option appropriate to the assumption $q \Rightarrow p$.

Participant 4 (freshman):

"It should be the choice b by considering the idea that if its converse is true, it is also true.

b) Assume that a, b and c are real numbers and $a>b$. If $c \leq 0$, then $ac \leq bc$. "

[Tersi doğruysa, kendisi de doğrudur mantığıyla b seçeneği olmalı.

b) a, b ve c reel sayılar ve $a>b$ olsun. Eğer $c \leq 0$ ise $ac \leq bc$ olur.]

As another example for the second reason, the answer of Participant 101 can be given. Participant 101 accepted that the statement $p \Rightarrow q$ could be proven by starting with q. Therefore, she selected the option which stated the proposition $q' \Rightarrow p$ as an assumption to prove the statement $p \Rightarrow q$.

Participant 101 (senior):

“By starting from the converse, the proof can be constructed.”

[Tersinden başlayarak ispata gidilebilir.]

The last reason is that students tried to provide proof by using the direct proof instead of using the given choices. Although students were asked to select the assumption they could begin with to prove the statement and none of the options were related to direct proof, some students attempted to provide proof by using direct proof. 8 students’ explanations (3 sophomores, 2 juniors and 3 seniors) were related to direct proof. For instance, Participant 55 explained by referring to direct proof but marked a choice which was not appropriate to direct proof as an assumption.

Participant 55 (junior):

“It can be proved by ‘direct proof’ method”

[‘Direct proof’ yöntemiyle ispatlanabilir.]

4.2.3. Contrapositive Question 3 (Section B- Q2)

The third contrapositive question is given below.

Case B
Statement A: If n^2 is an odd integer, then n is an odd integer.
Statement B: If n is an even integer, then n^2 is even integer.

Ahmet: I think statement A is true, Pınar.

Pınar: Let me see, if n^2=9, then n=± 3 is odd; if n^2=25, then n±5 is odd. So, statement A seems to be true Ahmet.

Ahmet: I also think that statement B is true, Pınar.

Pınar: Why?

Ahmet: Since n is even, then n=2k where k is some integer.
Therefore, $n^2 = 4k^2 = 2\ (2k^2)$ is also even.

Pınar: But Ahmet, this only show the statement B is true, but does not show that statement A is true.

Ahmet: This argument also shows that statement A is correct.

Questions;
- Considering the discussion above, who do you agree with?
Ahmet________ Pınar_________
- Why? Explain your reasons.

Figure 4. 6. Contrapositive Question 3

As can be seen, a question including a case was presented to the students and they were asked to choose the person whom they agreed with and explained their reasons for the agreement. The discussion was about the proofs of the two statements. Ahmet claims that the proof of statement B also proves statement A. However, Pınar does not think the same way. Since statement B is the contrapositive of statement A, when statement B is proved, it also serves as a proof for statement A. More specifically, statement A can be described as the proposition $p \Rightarrow q$ where p is "n^2 is an odd integer" and q is "n is an odd integer". Similarly, statement B can be described as the proposition $q' \Rightarrow p'$ where p' is "n^2 is an even integer" and q' is "n is an even integer". Therefore, students' answers were accepted as wrong if they agreed with Pınar and accepted as true if they agreed with Ahmet. Their answers were assessed according to the rubric given below.

Table 4. 14. Rubric for Contrapositive Question 3

Codes	**Answer types**
0	No answer
1	Agreed with no one or both of them
2	Agreed with Pınar, no reason was stated
3	Agree with Pınar, reason was stated
4	Agreed with Ahmet, no reason was stated
5	Agreed with Ahmet, reason which is not related to a proof method was stated
6	Agreed with Ahmet, reason which is related to contrapositive was stated

The results of the analyses of the answers of 115 pre-service middle school mathematics teachers are presented in Table 4.15.

Table 4. 15. Frequency of the Answers for Contrapositive Question 3 in terms of Year Level

		Year level				
		Freshmen	Sophomores	Juniors	Seniors	Total
Codes	0	1 (5,3%)	-	2 (5,1%)	1 (3,1%)	4 (3,5%)
	1	-	1 (4,0%)	1 (2,6%)	1 (3,1%)	3 (2,6%)
	2	3 (15,8%)	8 (32,0%)	-	5 (15,6%)	16 (13,9%)
	3	10 (52,6%)	15 (60,0%)	21 (53,8%)	13 (40,6%)	59 (51,3%)
	4	-	1 (4,0%)	-	4 (12,5%)	5 (4,3%)
	5	5 (26,3%)	-	13 (33,3%)	6 (18,8%)	24 (20,9%)
	6	-	-	2 (5,1%)	2 (6,3%)	4 (3,5%)
Total		19(100,0%)	25 (100,0%)	39(100,0%)	32(100,0%)	115(100,0%)

Table 4.15 presents the results obtained from the assessment of contrapositive question 3. All sophomore pre-service teachers answered the question while 2 juniors, 1 freshman and 1 senior did not answer the question. The answers of 3 students (2.6%) among 115 students (1 sophomore, 1 junior and 1 senior) fell within the classification of 'agree with neither Pınar nor Ahmet'. According to Table 4.15,

16 students (13.9%) among 115 students stated that they agreed with Pınar but they did not cite their reasoning. 3 of them were freshmen, 8 of them were sophomores and 5 of them were seniors. On the other hand, 59 students (51.3%) among 115 students, which constitutes nearly half of the students, agreed with Pınar and they wrote their reasons. 10 freshmen, 15 sophomores, 21 juniors and 13 seniors were coded in this category.

When the answers of the students who agreed with Ahmet were analyzed, it was seen that 1 sophomore and 4 seniors, totaling to 5 students (4.3%) among 115 students, agreed with Ahmet without stating any explanation. 21 students (20.9%) among 115 students agreed with Ahmet and supported their agreement with valid reasons but were not related to a proof method. The answers of 5 freshmen, 13 juniors and 6 seniors fell within this category. For example, Participant 15 agreed with Ahmet and her explanation was not related to contrapositive method.

> Participant 15 (freshman):
>
> "Pınar tried to prove only by giving an example. This is not a correct method of proof. Ahmet proved statement B. However, I think, we can prove statement A by giving n=2k+1."
>
> [Pınar sadece bir örnek göstererek doğruluğunu kanıtlamayı denedi. Bu ispat için doğru bir yol değildir. Ahmet B'yi kanıtlamış. Fakat A'yı bence n=2k+1 vererek kanıtlayabiliriz.]

On the other hand, none of the freshman and sophomore pre-service teachers mentioned the contrapositive method in the explanations of their agreement with Ahmet. As seen, only 2 juniors and 2 seniors (3.5%) among 115 students agreed with Ahmet and wrote an explanation related to the contrapositive method. For instance, Participant 52 agreed with Ahmet and her explanation was related to the contrapositive method.

Participant 52 (junior):

"p: n is even q: n^2 is even

p⇒q was proved.

p⇒q ≡ p'∨q ≡ q∨p' ≡ q'⇒p'

Thus, if n^2 is odd then n is odd. Therefore, Ahmet is right."

[p: n is even q: n^2 is even

p⇒q ispatlandı.

p⇒q ≡ p'∨q ≡ q∨p' ≡ q'⇒p'

Yani, n^2 tek ise n de tektir. Bu yüzden, Ahmet haklı.]

Likewise, Participant 111 explained the agreement with Ahmet by showing the equivalence of contapositive statements given in the discussion.

Participant 111 (senior):

"n is even ⇒ n^2 is even

(n^2 is even)' ⇒ (n is even)'

n^2 is odd ⇒ n is odd

Two proof is equivalent"

[n çift ⇒ n^2 çift

(n^2 çift)' ⇒ (n çift)'

n^2 tek ⇒ n tek

İki ispat biribirine eşit.]

In summary, 75 students (65.2%) among 115 students agreed with Pınar, which is accepted as wrong answer. 33 students (28.7%) among 115 students agreed with Ahmet, which is accepted as correct answer. However, only 4 of them (3.5%) explained their agreement with Ahmet by referring to the contrapositive method. The percentage of junior students' correct answers was the highest compared to the percentage of those in the other year levels, whereas the percentage of sophomore students' correct answers was the lowest compared that of participants in the other year levels.

As mentioned, 59 students (51.3%) among 75 students who agreed with Pınar suggested explanations. After the analysis of 59 students' answers, one reason for their wrong interpretations in proof by contrapositive was found as given in Table 4.16.

Table 4. 16. Reason of Wrong Interpretations for Contrapositive Question 3

	Reason	Freshmen	Sophomores	Juniors	Seniors	Total
R1	No relation between the statements A and B (Statements should be proved separately)	10	15	21	13	59

The common reason is that there is no relation between statements A and B. Therefore, they claimed that statements A and B should be proved separately. Table 4.16 shows that 59 students answered the question wrongly because of this idea. For instance, Participant 5 thought that the given proof was valid for only even numbers and another proof for odd numbers was needed.

Participant 5 (freshman):

"Because this proof is only for even numbers.

For odd numbers, we should try (2k+1)"

[Çünkü o ispat sadece çift sayılar içindir.

Tek sayılar için (2k+1)'i denemeliyiz.]

Another example is that Participant 20 accepted statement A and B as different.

> Participant 20 (sophomore):
>
> “The statements A and B are different from each other. Therefore, they should be proved separately.”
>
> [A ve B önermeleri birbirlerinden farklıdır. Bu yüzden ayrı ayrı ispatlanmalıdır.]

Similar to Participant 20, Participant 30 cited that statements A and B were different and also emphasized that proofs of the statements could not be exactly the same.

> Participant 30 (sophomore):
>
> “Because the statements are different, one of them starts with an even number, one of them starts with an odd number. The proof of statement A can’t be thought together with the proof of statement B.”
>
> [Çünkü ayrı ifadeler, biri çift sayıdan biri tek sayıdan yola çıkıyor. A önermesinin ispatı B ifadesinin ispatıyla beraber düşünülemez.]

In summary, there were five reasons for students’ wrong interpretations in proof by contradiction in total. Moreover, nearly the same reasons were found in the first and the second contrapositive questions. The only difference is that there was one more reason about understanding whether a statement was true or not in the first contrapostive question. In the third contrapositive question, one reason which is “no relation between the statements A and B” was determined. This reason, in the third question specifically, showed that students had difficulty noticing and understanding the equivalence of the contrapositive statements.

4.3. Analysis of Contradiction Questions

The Mathematical Proof Questionnaire includes two questions related to contradiction. Contradiction is a proof method in which to prove the proposition $p \Rightarrow q$, p and negation of q is assumed as true and then a contradiction is formed. Both of the contradiction questions -Question 1 (Q1) and Question 3 (Q3) in Section B- are discussion questions.

In this section, the results of the contradiction questions are presented. Pre-service middle school mathematics teachers' achievement levels in the contradiction method were analyzed based on the rubrics prepared by the researcher. Then, the reasons of underlying pre-service middle school mathematics teachers' wrong interpretations in the contradiction method were investigated.

4.3.1. Contradiction Question 1 (Section B- Q1)

The first contradiction question is given below.

Case A
Ali has shown that the following statement is true for all real numbers x and y.
Statement: If $x \neq 0$ and $y \neq 0$, then $x.y \neq 0$
Ali's argument is as follows;

Assume that $x \neq 0$ and $y \neq 0$ but $x.y=0$.
Since $x \neq 0$ then x^{-1} exists.
Then $x^{-1}.(x.y)=(x^{-1}.x).y=1.y=y$
Also, since $x.y = 0$, $x^{-1}.(x.y)=x^{-1}.0=0$
Therefore $y=0$. But $y \neq 0$.
Thus $x.y=0$ must be false.
Therefore, $x.y \neq 0$

Ayşe: I feel your argument is completely unnecessary. Look, everybody knows that if $x \neq 0$ and $y \neq 0$, then $x.y \neq 0$. There is no need to show it.
Ali: I agree that the above statement is familiar to everybody, but I disagree that my argument is unnecessary, Ayşe.

Questions;
- Considering the discussion above, who do you agree with?

Ali________ Ayşe_________
- Why? Explain your reasons.

Figure 4. 7. Contradiction Question 1

In this question, students were asked to select the person whom they agreed with and explained their reasons. The discussion is about the proof of a statement. Ali proved the statement by using the contradiction method, but Ayşe claimed that there was no need to provide proof. Since the proof of the statement constructed by Ali was correct and proof was not unnecessary for even familiar or easy statements, students' answers were accepted as correct if they agreed with Ali and accepted as wrong if they agreed with Ayşe. Their answers were assessed according to a rubric given in Table 4.17.

Table 4. 17. Rubric for Contradiction Question 1

Codes	**Answer types**
0	No answer
1	Agreed with no one or both of them
2	Agreed with Ayşe, no reason was stated
3	Agree with Ayşe, reason was stated
4	Agreed with Ali, no reason was stated
5	Agreed with Ali, reason which is not related to a proof method was stated
6	Agreed with Ali, reason which is related to contradiction was stated

The results of 115 pre-service middle school mathematics teachers' answers are presented in Table 4.18.

Table 4. 18. Frequency of the Answers for Contradiction Question 1

		Year level				
		Freshmen	Sophomores	Juniors	Seniors	Total
Codes	0	1 (5,3%)	1 (4,0%)	-	1 (3,1%)	3 (2,6%)
	1	-	1 (4,0%)	-	-	1 (,9%)
	2	1 (5,3%)	2 (8,0%)	-	2 (6,3%)	5 (4,3%)
	3	1 (5,3%)	2 (8,0%)	1 (2,6%)	-	4 (3,5%)
	4	3 (15,8%)	3 (12,0%)	-	6 (18,8%)	12 (10,4%)
	5	12 (63,2%)	15 (60,0%)	35 (89,7%)	23 (71,9%)	85 (73,9%)
	6	1 (5,3%)	1 (4,0%)	3 (7,7%)	-	5 (4,3%)
Total		19(100,0%)	25(100,0%)	39(100,0%)	32(100,0%)	115 (100,0%)

Table 4.18 presents the results obtained from the assessment of contradiction question 1. As can be seen in Table 4.18, all junior pre-service teachers answered the question while 1 freshman, 1 sophomore and 1 senior did not answer the question. Moreover, 1 student (4%) among 25 sophomore pre-service teachers stated that she agreed with neither Ayşe nor Ali.

As presented in Table 4.18, 5 students (4.3%) among 115 students agreed with Ayşe and no reasons for their answers were written. The answers of 1 freshman, 2 sophomores and 2 seniors were coded to fall in this category. Moreover, 4 students (3.5%) agreed with Ayşe and explained their reasoning. 1 freshman, 2 sophomores and 1 junior were placed in this category.

Regarding the answers in which there was an agreement with Ali, it was seen that 12 students (10.4%) among 115 students agreed with Ali without supporting their answers. 3 of them were freshmen, 3 of them were sophomores and 6 of them were seniors. Although 85 students (73.9%) among 115 students agreed with Ali, they explained their reasoning without using a proof method. 12 freshmen, 15 sophomores, 35 juniors and 23 seniors fell within this category of the rubric. For example, Participant 15 agreed with Ali and her explanation was not related to a proof method. She explained her response by mentioning the importance of proof.

> Participant 15 (freshman):
>
> “A mathematical proof was constructed. Even though everyone knows that the statement is true, it can be proved only by giving proof.”
>
> [Matematiksel bir ispat yapılmıştır. Herkes bilse de önermenin doğru olduğunu, doğru olduğu sadece ispatı verilerek kanıtlanabilir.]

Participant 98 also agreed with Ali and explained without mentioning a proof method. Her explanation focused on the necessity of proof.

Participant 98 (senior):

"Because there is a lot of known truth and the proof for all of them is necessary."

[Çünkü bilinen bir çok doğru vardır ve bunların hepsi için ispat gereklidir.]

As given in Table 4.18, only 5 students (4.3%) among 115 students agreed with Ali and explained their reasoning by relating to the contradiction method. While there were no seniors who noticed the contradiction method in the discussion of the question, 1 freshman, 1 sophomore and 3 juniors answered by mentioning contradiction. As example, the answer of Participant 41 is given below:

Participant 41 (sophomore):

"Ali started to prove by accepting that the statement xy=0 is true which is the converse of the given statement xy≠0. And, he found that xy=0 is false at the end of the proof. (by contradiction method)
Therefore, it is proved that the given statement is true."

[Ali, en başta verilen xy≠0 önermesinin tersi olan, xy=0 önermesini doğru kabul ederek ispata başlamış ve ispatın sonunda xy=0'ın yanlış olduğunu bulmuştur. (by contadiction method).
Bu durumda, verilen ilk önermenin doğru olduğunu ispat etmiştir.]

Similarly, Participant 50 noticed the contradiction method stated by Ali in the discussion.

Participant 50 (junior):

"Ayşe accepted the statement 'if x≠0 and y≠0, then x.y ≠ 0' by generalization. Ali used his reasoning as a mathematician and showed that the theorem is true by forming a contradiction."

[Ayşe ‘x≠0 ve y≠0 ise x.y ≠ 0’ olduğunu genelleyerek kabul etmiştir.
Ali bir matematikçi olarak mantığını kullanmıştır ve contradiction yaratarak teoremin doğru olduğunu göstermiştir.]

In total, 9 students (7,8%) among 115 students agreed with Ayşe, which is accepted as a wrong answer. 102 students (88.6%) among 115 students agreed with Ali, which is accepted as a correct answer. More specifically, only 5 of them (4.3%) explained by relating to the contradiction method. The percentage of sophomores' giving the correct answers was found to be the lowest compared to that of participants in the other year levels and the percentage of juniors' correct answers was found to be the highest compared to that of participants in the other year levels.

The reasons of pre-service middle school mathematics teachers' wrong interpretations regarding contradiction method were also investigated. As previously mentioned, 4 students (3.5%) among 9 students who agreed with Ayşe explained their reasons. After the analysis of 4 students' answers, it was found that reasons could be categorized under two headings as presented in Table 4.19.

Table 4. 19. Reasons of Wrong Interpretations for Contradiction Question 1

	Reasons	Freshmen	Sophomores	Juniors	Seniors	Total
R1	Accepting proof as unnecessary	1	2	-	-	3
R2	Developmental aspects	-	-	1	-	1

The first reason is that students accepted the proof as unnecessary for this statement. 1 freshman and 2 sophomores agreed with Ayşe because of this idea. As an example, the answer of Participant 35 is stated below:

Participant 35 (sophomore):

“I think, there is no need to prove these basic subjects.”

[Bu kadar temel konuları ispatlamaya gerek yok bence.]

Likewise, Participant 9 thought that proof is not needed for the given statement.

Participant 9 (freshman):

"To say the same things in different languages is unnecessary, I think."

[Aynı şeyleri farklı dillerde söylemek gereksiz bence.]

Although these reasons are not basically different, the second reason also includes the developmental aspects. Only 1 senior student, Participant 82, explained his response with this idea.

Participant 82 (junior):

"In fact, the ages of Ali and Ayşe are important in this situation. If Ayşe is a high school student or elementary school student, the way Ayşe reacted is normal in this situation. Trying to prove some accepted truths may be unnecessary for a person at that age."

[Aslında bu durumda Ayşe ve Ali'nin yaşları önemlidir. Eğer Ayşe bir lise öğrencisi ya da ilkokul öğrencisi ise bu durmda Ayşe'nin böyle tepki vermesi normal. Bazı kabul edilen gerçekleri ispatlamaya çalışmak o yaştaki bir insana göre gereksiz olabilir.]

4.3.2. Contradiction Question 2 (Section B- Q3)

The second contradiction question is given below.

Case D

Ege: How can you show that if x is a rational number and y is an irrational number, then y-x is an irrational number?

Deniz: Suppose that y-x is not irrational (rational).
Then y-x=c/d, for some integers c and d≠0.
Since x is rational then, x=a/b for some integers a and b≠0.
Thus, (y-x)+x= c/d+a/b= (cb+ad)/db
Since cb+ad and db are both integers then (y-x)+x is rational.
But (y-x)+x= y.
Thus y is rational, which is false.
Therefore y-x is irrational as desired.

Ege: But you started out by supposing that y-x is not irrational; it does not make sense to me to suppose that y-x is not irrational in order to show just the opposite.

Deniz: I have to start out by assuming that y-x is rational because this is a correct method of proof.

Questions;
- Considering the discussion above, with who do you agree?
Ege________ Deniz_________
- Why? Explain your reasons.

Figure 4. 8. Contradiction Question 2

In this question, students were asked to select the person who they agreed with and explain their reasons for their agreement. The discussion is about the proof of a statement. Deniz proved the statement by using the contradiction method, but Ege claimed that the assumption of Deniz was not meaningful. Since the proof of the statement constructed by Deniz was correct, students' answers were accepted as correct if they agreed with Deniz and accepted as wrong if they agreed with Ege. Their answers were assessed utilizing the rubric presented in Table 4.20.

Table 4. 20. Rubric for Contradiction Question 2

Codes	Answer types
0	No answer
1	Agreed with no one or both of them
2	Agreed with Ege, no reason was stated
3	Agree with Ege, reason was stated
4	Agreed with Deniz, no reason was stated
5	Agreed with Deniz, reason which is not related to a proof method was stated
6	Agreed with Deniz, reason which is related to contradiction was stated

The results of the analyses of the answers of 115 pre-service middle school mathematics teachers are presented in Table 4.21.

Table 4. 21. Frequency of the Answers for Contradiction Question 2

		Year level				Total
		Freshmen	Sophomores	Juniors	Seniors	
Codes	0	1 (5,3%)	5 (20,0%)	1 (2,6%)	3 (9,4%)	10 (8,7%)
	1	-	-	-	1 (3,1%)	1 (,9%)
	2	-	3 (12,0%)	-	1 (3,1%)	4 (3,5%)
	3	1 (5,3%)	-	5 (12,8%)	1 (3,1%)	7 (6,1%)
	4	-	5 (20,0%)	2 (5,1%)	6 (18,8%)	13 (11,3%)
	5	4 (21,1%)	3 (12,0%)	7 (17,9%)	4 (12,5%)	18 (15,7%)
	6	13 (68,4%)	9 (36,0%)	24 (61,5%)	16 (50,0%)	62 (53,9%)
Total		19(100,0%)	25(100,0%)	39(100,0%)	32(100,0%)	115(100,0%)

Table 4.21 presents the assessment of contradiction question 2. According to Table 4.21, 1 freshman, 5 sophomores, 1 junior and 3 seniors did not answer the contradiction question. In addition, 1 student among 32 seniors stated that she agreed with none of them. As mentioned, students' answers were accepted as wrong if they agreed with Ege and accepted as correct if they agreed with Deniz. 4 students (3.5%) among 115 students only stated that they agreed with Ege and there was no explanation. 3 of them were sophomores and 1 of them was a senior. On the other hand, 7 students (6.1%) among 115 students gave an explanation for the agreement with Ege. 1 freshman, 5 juniors and 1 senior fell within this category.

When students' correct answers were examined, it was seen that 13 students (11.3%) among 115 students agreed with Deniz without explaining their reasoning. 5 of these students were sophomores, 2 of them were juniors and 6 of them were seniors. However, there were 80 students who provided answers based on valid reasoning. 18 students (15.7%) among 80 students agreed with Deniz by writing explanations which were not related to any proof method. The answers of 4 freshmen, 3 sophomores, 7 juniors and 4 seniors were not related to any proof method. The remaining 62 students (53.9%) agreed with Deniz and their explanations were related to the contradiction method. 13 freshmen, 9 sophomores, 24 juniors and 16 seniors were placed in this category. For example, Participant 1, Participant 36 and Participant 56 agreed with Deniz and explained by mentioning the contradiction method.

Participant 1 (freshman):

"This proof method is contradiction. We start to prove by assuming the converse of the given. Since this is a proof method, what Deniz does is correct because the end of the proof contradicts with the thing which she accepted as the converse at the beginning. Finally, the situation wanted is reached."

[Bu proof yöntemi contradictiondır. Ispatlamaya verilenlerin tersini kabul ederek başlarız. Bu bir ispat yöntemi olduğu için Deniz'in yaptıkları doğrudur. Çünkü başında tersini kabul ettiği şey ile çelişiyor. Sonunda istenilen duruma ulaşılıyor.]

Similarly, Participant 36 noticed the contradiction method given in the discussion.

Participant 36 (sophomore):

"Deniz used 'prove by contradiction' method. She started correctly."

[Deniz 'prove by contradiction' yöntemini kullanmıştır. Doğru başlamıştır.]

As another example, the answer of the Participant 56 can be given:

Participant 56 (junior):

“By using the contradiction method, $(p \Rightarrow q)' \equiv (p' \vee q)' \equiv p \wedge q'$

I mean, we can start by assuming that y-x is rational in the question”

[Contradiction methodunu kullanarak, $(p \Rightarrow q)' \equiv (p' \vee q)' \equiv p \wedge q'$

Yani, y-x’in rasyonel olduğunu kabul ederek başlayabiliriz soruda]

In summary, 11 students (9.6%) among 115 students agreed with Ege, which is accepted as a wrong answer. 93 students (80.9%) among 115 students agreed with Deniz, which is accepted as a correct answer. In more detail, 13 of them (11.3%) did not state their reasons, 18 of them (15.7%) explained without relating to a proof method and finally 62 of them (53.9%) explained their reasons by relating to the contradiction method. The percentage of junior pre-service teachers’ wrong answers was the highest compared to that of participants in the other year levels and the percentage of freshman pre-service teachers’ correct answers was the highest compared that of participants in the other year levels.

As mentioned, 7 students (6.1%) among 11 students who agreed with Ege explained their reasons. When the wrong reasons regarding the contradiction method were analyzed, the reasons behind those wrong answers were categorized under two headings as given in Table 4.22.

Table 4. 22. Reasons of Wrong Interpretations for Contradiction Question 2

	Reasons	Freshmen	Sophomores	Juniors	Seniors	Total
R1	Lack of information related to contradiction	1	-	1	-	2
R2	Misunderstanding of the assumption in contradiction method	-	-	4	1	5

The first reason is about students' lack of information related to the contradiction method. 2 students (1 freshman and 1 junior) answered the question wrongly since they did not understand the function of the contradiction in the given proof. For example, Participant 14 agreed with Ege.

> Participant 14 (freshman):
>
> "She found the thing that she assumed at the beginning as true, but she said false and was in contradiction with herself."
>
> [Başta kabul ettiği şeyi doğru bulmuş ama yanlış demiş, kendiyle çelişmiş.]

The second reason is that students misunderstood the assumption at the beginning of the proof in the contradiction method. 5 students (4 juniors and 1 senior) answered the question wrongly because of this misunderstanding. For example,

> Participant 65 (junior):
>
> "A true proof method does not start by assuming the converse. Deniz is not right."
>
> [Doğru bir ispat yöntemi aksini kabul ederek başlamaz. Deniz haksızdır.]

Similarly, Participant 62 answered the question wrongly due to the problem related to the assumption in the beginning of the proof

> Participant 62 (junior):
>
> "We had to assume that x is not a rational number too."
>
> [x'in de rasyonel olmadığını kabul etmemiz gerekirdi.]

4.4. Analysis of Open-ended Proof Questions

The Mathematical Proof Questionnaire includes three open-ended questions which are Question 1 (Q1), Question 2 (Q2) and Question 3 (Q3) given in Section C. In this section, the results of open-ended proof questions are presented. In these questions, students were asked to prove the given statements. Since one of the purposes was to investigate to what extent pre-service middle school mathematics teachers could provide valid proofs for the given statements, the answers of the students were initially analyzed as whether they conducted valid or invalid proofs. Then, the valid proofs were categorized according to the proof methods that students used. Lastly, the reasons underlying students' wrong interpretations in conducting proof were investigated through the analysis of their invalid proofs.

4.4.1. Open-ended Proof Question 1 (Section C- Q1)

The first proof question is given below.

Q1) Show that 1 + 2 + 3 + 4 + 5 +….. + n = n . (n+1) / 2.

Figure 4. 9. Open-ended Proof Question 1

As seen in Figure 4.9, students were asked to prove a given statement. Firstly, their proofs were evaluated as invalid and valid. The results of 115 pre-service middle school mathematics teachers' answers are presented in Table 4.23.

Table 4. 23. Frequency of the Answers for Open-ended Proof Question 1

	Year level				Total
	Freshmen	Sophomores	Juniors	Seniors	
No answer	4 (21,1%)	1 (4,0%)	-	9 (28,1%)	14 (12,2%)
Invalid proof	2 (10,5%)	5 (20,0%)	7 (17,9%)	10 (31,3%)	24 (20,9%)
Valid proof	13 (68,4%)	19 (76,0%)	32 (82,1%)	13 (40,6%)	77 (67,0%)
Total	19(100,0%)	25(100,0%)	39(100,0%)	32(100,0%)	115(100,0%)

Table 4.23 presents the results obtained from the assessment of open-ended proof question 1. As seen in Table 4.23, 14 students (12.2%) among 115 students did not write any answer. While proofs of 24 students (20.9%) among 115 students were invalid, those of 77 students (67%) among 115 students were classified as valid. When proofs were analyzed in terms of year level, it was seen that seniors had the highest percentage (31.3%) in writing invalid proofs. On the other hand, juniors had the highest percentage (82.1%) in conducting valid proofs compared to that of participants in the other year levels.

Valid proofs of 77 students were examined and classified under three categories, namely mathematical induction, proof by using Gauss method and visual proof.

Table 4. 24. Frequency of the Proof Methods in Valid Proofs for Open-ended Proof Question1

Proof methods	Year level				Total
	Freshmen	Sophomores	Juniors	Seniors	
Mathematical induction	7 (36,8%)	9 (36,0%)	25 (64,1%)	9 (28,1%)	50(43,5%)
Proof by using Gauss method	6 (31,6%)	10 (40,0%)	6 (15,4%)	4 (12,5%)	26(22,6%)
Visual proof	-	-	1 (2,6%)	-	1 (,9%)
Total	13 (68,4%)	19 (76,0%)	32 (82,1%)	13(40,6%)	77(67,0%)

As stated in the Table 4.24, 50 students (43.5%) used mathematical induction, 26 students (22,6%) used Gauss method and only 1 junior (2.6%) used visual proof to prove the given statement. It can also be said that mathematical induction was mostly used by juniors (64.1%) and proof through Gauss method was mostly used by sophomores (40%). As an example of mathematical induction, the answer of Participant 67 can be given.

Participant 67 (junior):

"For n=1, $\frac{1.2}{2} = 1$

Assume that 1+2+3…..n = $\frac{n.(n+1)}{2}$ is true for n.

And it must be true for n+1.

1+2+3….n+(n+1) = $\frac{(n+1).(n+2)}{2}$

$\frac{n.(n+1)}{2}$ + (n+1) = $\frac{n^2+3n+2}{2}$

$\frac{n.(n+1)+2n+2}{2} = \frac{n^2+3n+2}{2}$

The proof is complete."

As an example of proof by using Gauss method, the answer of Participant 34 can be stated.

Participant 34 (sophomore):

"1+2+3….n

1+2+3+4+……+(n-2)+(n-1)+n

= (n+1) + (n+1) + ….. + (n+1)

Since there are n terms, it can be said that there are n/2 terms when two terms were added.

= $\frac{n}{2}.(n+1)$"

[1+2+3….n

1+2+3+4+……+(n-2)+(n-1)+n

= (n+1) + (n+1) + ….. + (n+1)

n terim olduğundan, ikişerli topladığımızda n/2 tane terim vardır.

$= \frac{n}{2}.(n+1)$]

Only 1 student, Participant 73, gave visual proof which is given below.

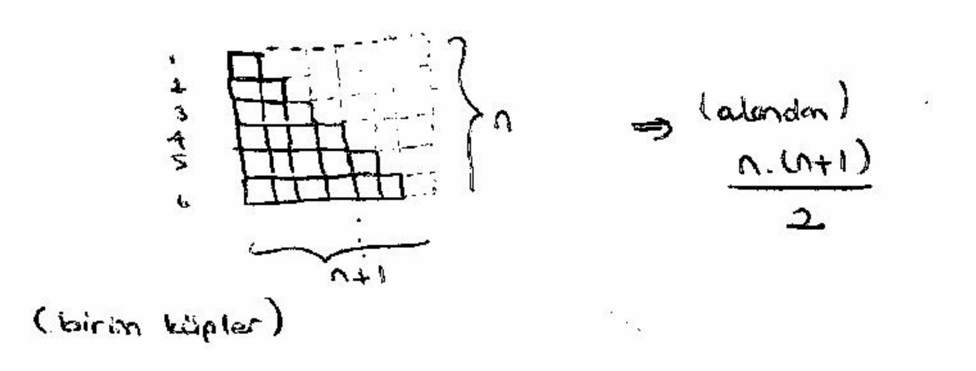

Figure 4. 10. Answer of Participant 73

Lastly, the reasons of pre-service middle school mathematics teachers' indirect proofs were investigated. As previously stated, 24 students among 115 students (20.9%) conducted invalid proof. After the analysis of 24 students' answers, the reasons were categorized under three headings.

Table 4. 25. Reasons in Open-ended Proof Question 1

	Reasons	Freshmen	Sophomores	Juniors	Seniors	Total
R1	Using numbers to prove the statement	1	1	0	2	4
R2	Direct restatement of the given expression	1	1	2	4	8
R3	Incomplete proof	0	3	5	4	12

One of the reasons is that students tried to prove by giving numbers for the statement and accepted this application as a proof. For example,

Participant 4 (freshman):

“For example, let’s add the numbers from 1 to 5.

1+2+3+4+5=15

$\frac{n.(n+1)}{2} = \frac{5.6}{2} = 15$

Since they are equal, it can be said that the formula is true.”

Another reason is that what students wrote cannot be accepted as a part of proof. They simply rewrite the given in the statement. 1 freshman, 1 sophomore, 2 juniors and 3 seniors tried to provide proof in this way. For example;

Participant 30 (sophomore):

“1+2+3….+(n+1)= $\frac{(n-1).n}{2}$

1+2+3….+n= $\frac{n.(n+1)}{2}$ ”

The most common reason of students’ invalid proofs is that they gave incomplete proofs. None of the freshman pre-service teachers gave an incomplete proof while 3 sophomores, 5 juniors and 4 seniors supplied incomplete proofs. For example, Participant 18 did not show the last step of the mathematical induction.

Participant18 (sophomore):

“n=1 $1=\frac{(1+1).1}{2}$

n=2 $1+2=\frac{(2+1).2}{2}$

For all integers n, formula is true $\Rightarrow$ For all integers (n+1), formula is true.”

4.4.2. Open-ended Proof Question 2 (Section C- Q2)

The second proof question is given below.

Q2) “Assume that a and b are real numbers. If $0<a<b$, then $a^2<b^2$.”

Prove the given statement above.

Figure 4. 11. Open-ended Proof Question 2

Students were asked to prove the given statement. Their answers were assessed in a similar way to how the answers to open-ended proof question 1 were assessed. The results of the analyses of 115 pre-service middle school mathematics teachers’ answers as invalid and valid are presented in Table 4.26.

Table 4. 26. Frequency of the Proof Methods in Valid Proofs for Open-ended Proof Question2

	Year level				Total
	Freshmen	Sophomores	Juniors	Seniors	
No answer	3 (15,8%)	2 (8,0%)	4 (10,3%)	11 (34,4%)	20 (17,4%)
Invalid proof	2 (10,5%)	6 (24,0%)	14 (35,9%)	8 (25,0%)	30 (26,1%)
Valid proof	14 (73,7%)	17 (68,0%)	21(53,8%)	13 (40,6%)	65 (56,5%)
Total	19(100,0%)	25 (100,0%)	39(100,0%)	32(100,0%)	115(100,0%)

Table 4.26 presents the results obtained from the assessment of open-ended proof question 2. According to Table 4.26, 20 students (17.4%) among 115 students did not try to prove the given statement. Moreover, proofs given by 30 students

(26.1%) among 115 students were invalid and proofs by 65 students (56.5%) among 115 students were valid. In terms of year level, it was seen that juniors had the highest percentage (35.9%) in conducting invalid proofs. However, freshmen had the highest percentage (73.7%) in providing valid proofs compared to that of participants in the other year levels.

As previously mentioned, valid proofs of 65 students were examined in terms of the proof methods that they used. Proof methods were classified under two items which are direct proof and proof by contradiction.

Table 4. 27. Frequency of the Proof Methods in Valid Proofs for Open-ended Proof Question2

Proof methods	Year level				Total
	Freshmen	Sophomores	Juniors	Seniors	
Direct proof	14(73,7%)	14 (56,0%)	16 (41,0%)	12 (37,5%)	56(48,7%)
Proof by contradiction	-	3 (12,0%)	5 (12,8%)	1 (3,1%)	9 (7,8%)
Total	14(73,7%)	17 (68,0%)	21(53,8%)	13(40,6%)	65(56,5%)

Table 4.27 shows that 56 students (48.7%) used direct proof and 9 students (2.6%) used proof by contradiction for the statement. Moreover, direct proof was the most frequently used method among freshman and sophomore students. For example, Participant 5 used direct proof as given below.

Participant 5 (freshman):

“$0<a<b$ multiply each term by a

$0<a^2<ba$ then also multiply each term by b

$0<ab<b^2$ then $0<a^2<ba<b^2$

So, $a^2<b^2$”

As another example for direct proof, the proof of Participant 29 is stated below.

Participant 29 (sophomore):

"Let n be reel number and a=n and b=n+1

Then, $a^2=n^2$ and $b^2=(n+1)^2= n^2+2n+1$

Then $n^2<n^2 + 2n + 1$

So, $a^2<b^2$"

As an example to contradiction method, the answer of Participant 51 can be given.

Participant 51 (sophomore):

"Assume that $0<a<b$, $0<a$ and $0<b$ but $a^2>b^2$.

If $a^2>b^2$ then $a^2- b^2 >0$

$a^2- b^2 = (a-b) (a+b)$

Since $a<b$ then $a-b<0$. However $a+b>0$.

Since $a^2 > b^2 = a^2- b^2 >0$

$$= \underbrace{(a-b)}_{\text{neg.}} \underbrace{(a+b)}_{\text{pos.}} > 0 \text{ we have } a^2- b^2 \not> 0$$

Proof by contradiction shows that if $0<a<b$ then $a^2<b^2$"

As stated, the proofs of 30 students (26.1%) among 115 students were accepted as invalid proofs. After the analysis of these students' answers, the reasons behind students' invalid proofs were categorized under three headings, which are given in Table 4.28.

Table 4. 28. Reasons in Open-ended Proof Question 2

	Reasons	Freshmen	Sophomores	Juniors	Seniors	Total
R1	Using numbers to prove the statement	0	3	1	0	4
R2	Direct restatement of the given expression	1	3	8	5	17
R3	Trying to prove $p \Rightarrow q$, by starting with the proposition q or p'	1	0	5	3	9

The first and the second reasons are also stated in open-ended proof question 1. In the first reason, it was stated that students' proofs were accepted as invalid since they tried to verify the statement by trying some numbers. 3 sophomore and 1 junior pre-service teachers provided invalid proof because of this idea. For example;

Participant 28 (sophomore):

"Assume that a=1 and b=3

$a^2=1$ $b^2=9$

$b^2>a^2$"

The second reason is that students simply rewrote the givens in the statement. In this question, 1 freshman, 3 sophomores, 8 juniors and 5 seniors answered the question in that way. For instance;

Participant 26 (sophomore):

"a>0, b>0 and b>a

$a^2>0$ $b^2>0$"

The third reason of students' invalid proofs is that they tried to prove the given statement in the form of $p \Rightarrow q$, by starting with proposition q or p'. Starting to prove with q or p' is not meaningful since this is not appropriate to any proof method. The proofs of 1 freshman, 5 juniors and 3 seniors are placed in this category. To give an example, the answer of Participant 58 can be given. She tried to prove by starting the proposition q of the given statement in the form of $p \Rightarrow q$.

Participant 58 (junior):

“$b^2 > a^2$

b.b>a.a divide both sides to b (b>0 a>0)

$b > \frac{a}{b}.a$ If a>b, then $\frac{a}{b} > 1$ and $\frac{a}{b}.a > b$.

So $b > \frac{a}{b}.a$

b.b > a.a

$b^2 > a^2$”

[$b^2 > a^2$

b.b>a.a her iki tarafı b’ye bölelim (b>0 a>0)

$b > \frac{a}{b}.a$ Eğer a>b olsaydı $\frac{a}{b} > 1$ ve $\frac{a}{b}.a > b$ olurdu.

Bu yüzden $b > \frac{a}{b}.a$

b.b > a.a

$b^2 > a^2$]

As another example, the answer of Participant 67 can be given. He attempted to prove $-p \Rightarrow -q$, but could not follow a logical way.

Participant 67 (junior):

“$0 < a < b \Rightarrow a^2 < b^2$

Assume that a>b and a<0. Then $a^2 > b^2$.

It is not true. Therefore, $0 < a < b \Rightarrow a^2 < b^2$”

4.4.3. Open-ended Proof Question 3 (Section C- Q3)

The third proof question is given below.

Q3) "For all natural number n, $3 \mid n^3 - n$."

Prove the given statement above.

Figure 4. 12. Open-ended Proof Question 3

In this proof question, students were asked to prove a given statement. The analyses of 115 pre-service middle school mathematics teachers' answers were evaluated as either invalid or valid, which is presented in Table 4.29.

Table 4. 29. Frequency of the Proof Methods in Valid Proofs for Open-ended Proof Question3

	Year level				Total
	Freshmen	Sophomores	Juniors	Seniors	
No answer	2 (10,5%)	5 (20,0%)	2 (5,1%)	11 (34,4%)	20 (17,4%)
Invalid proof	4 (21,1%)	4 (16,0%)	10 (25,6%)	9 (28,1%)	27 (23,5%)
Valid proof	13 (68,4%)	16 (64,0%)	27(69,2%)	12 (37,5%)	68 (59,1%)
Total	19(100,0%)	25 (100,0%)	39(100,0%)	32(100,0%)	115(100,0%)

Table 4.29 presents the results obtained from the assessment of open-ended proof question 3. Table 4.29 shows that 20 students (17.4%) among 115 students did not write any answer for the question. The remaining 95 students' answers were evaluated as either invalid or valid. The answers of 27 students (23.5%) were coded as invalid proofs and the answers of 68 students (59.1%), which means more than half of the students, were coded as valid proofs. In the analysis of the answers in terms of year level, it was observed that seniors had the highest percentage (28.1%) in conducting invalid proofs. On the other hand, juniors had the highest percentage (69.2%) in providing valid proofs. Moreover, in the analyses of valid proofs, it was seen that the percentage of freshmen, sophomores and juniors was close to each other and the percentage of seniors was nearly half of other year levels.

Then, valid proofs of 68 students were investigated in terms of the proof methods that they chose. Three methods were used by students in proving this statement, namely mathematical induction, direct proof and proof by cases.

Table 4. 30. Frequency of the Proof Methods in Valid Proofs for Open-ended Proof Question3

Proof methods	Year level				Total
	Freshmen	Sophomores	Juniors	Seniors	
Mathematical induction	10 (56,6%)	8 (32,0%)	15(38,5%)	4 (12,5%)	37(32,2%)
Direct proof	2 (10,5%)	8 (32,0%)	11(28,2%)	7 (21,9%)	28(24,3%)
Proof by cases	1 (5,3%)	-	1 (2,6%)	1 (3,1%)	3 (2,6%)
Total	13 (72,4%)	16 (64,0%)	27(69,3%)	12(37,5%)	68(59,1%)

As presented in Table 4.30, 37 students (32.2%) among 115 students used mathematical induction, 28 students (24.3%) among 115 students used direct proof and only 3 students (2.6%) among 115 students used proof by cases. Moreover, mathematical induction was mostly used by freshmen, direct proof was mostly used by sophomores. To give an example for mathematical induction, the answer of Participant 45 is presented below.

Participant 45 (junior):

“$3 \mid n^3 - n$

For n=1, $1^3-3=0$ and the number 3 divides 0.

Assume that it is true for n=n. $3 \mid n^3-n$

For n=n+1, $(n+1)^3-(n+1)= n^3+3n^2+3n+1-1$

$$= n^3-n+3(n^2+n)$$

So, $n^3-n+3(n^2+n)$ is divided by 3, $3(n^2+n)$ is a multiple of 3.

Thus, we showed that the statement is true for n=n+1

By induction, for all natural number n, $3 \mid n^3 – n$.”

As an example for direct proof, the answer of Participant 35 is stated as follows.

> Participant 35 (sophomore):
>
> "n^3- n = (n^2-1).n=n.(n-1).(n+1)
>
> n-1, n, n+1 are three consecutive numbers. One of the consecutive numbers is divided by 3. Since one of the factors is divided by 3, n^3-n is divided by 3, too."

For proof by cases, the answer of Participant 9 is given below.

> Participant 9 (freshman):
>
> "n.(n^2-1) = n^3-n
>
> Assume that n=3k. Since 3k(9k^2-1) is a multiple of 3, the statement 3 | n^3-n is verified.
>
> Assume that n=3k-1. Then, (3k-1) - ((3k-1)^2-1)=(3k-1).(9k^2-6k+1-1)
>
> = (3k-1). (3k^2-2k).3
>
> The statement 3 | n^3-n is verified.
>
> Assume that n=3k-2. Then, (3k-2) – ((3k-2)^2- 1)=(3k-2).(9k^2-6k+4-1)
>
> = (3k-2).(3k^2-2k+!).3
>
> The statement 3 | n^3-n is verified.
>
> 3 | n^3-n is true."

Another purpose of the study was to investigate the reason of pre-service middle school mathematics teachers' invalid proofs. Therefore, invalid proofs of 27 students were examined and the reasons were categorized under the related headings.

Table 4. 31. Reasons in Open-ended Proof Question 3

	Reasons	Freshmen	Sophomores	Juniors	Seniors	Total
R1	Using numbers to prove the statement	0	1	0	3	4
R2	Direct restatement of the given expression	0	0	3	3	6
R3	Incomplete proof	4	1	6	1	12
R4	Stating that they could not remember the proof	0	2	1	2	5

The first and the second reasons are common in all of the open-ended proof questions. The third reason of this question is also stated in the reasons of the first open-ended proof question. The first reason is that students tried to give numbers to prove the statement. The answers of 1 sophomore and 3 seniors were placed in this category. For example; Participant 108 tried numbers for the statement.

Participant 108 (senior):

“n^3-n=n(n^2-1)=n.(n-1).(n+1)

For n=1, 1.0.2=0

For n=2, 2.1.3=6

For n=3, 3.2.4=24

……

$3 \mid n^3-n$”

The second reason is that the givens of the statement were rewritten by the students. 3 juniors and 3 seniors answered the question in this way. For example;

Participant 71 (junior):

“$3 \mid n^3 - n$ $\quad \frac{n^3 - n}{3}$

$n.(n^2-1) = n.(n-1).(n+1)$”

The third reason is the most common one for this proof question in which students left the proof incomplete. 4 freshmen, 1 sophomore, 5 juniors and 1 senior could not complete the proof. For example, the answer of Participant 100 is given below.

Participant 100 (senior):

“Assume that $3 \mid n^3$-n for n natural number. For k as an integer, we have

$\frac{n^3 - n}{3} = k.$

$n^3-n=3k$

$n(n^2-1)=3k \quad n.(n-1).(n+1)=3k$

For n=0, k=0

For n=1, k=0

If $k = \frac{n.(n-1).(n+1)}{3}$ is true for n=n

For n=n+1, we have $k = \frac{(n+1).n.(n+2)}{3}$ „

As the fourth reason, students wrote that they did not remember the proof of the statement. 2 sophomores, 1 junior and 2 seniors answered as such. For example;

Participant 102 (senior):

"$3.a = n^3 - n$

$3.a = n(n^2-1)$

I don't remember how to prove the statement $3 \mid n^3-n$."

[$3.a = n^3 - n$

$3.a = n(n^2-1)$

$3 \mid n^3-n$ ifadesinin nasıl ispatlanacağını hatırlamıyorum.]

4.5. Summary of the Results

The first purpose of the study was to investigate pre-service middle school mathematics teachers' achievement levels in refutation, proof by contrapositive and proof by contradiction by year level. In refutation questions, freshmen and sophomores generally have the highest percentage of correct answers while seniors were the least successful group. In contrapositive questions, freshmen, sophomores and juniors were the most successful group in one of the questions respectively. In contadiction question, juniors and freshmen have the highest percentage in correct answers. Similar to refutation questions, seniors could not be the most successful group in contrapositive and contradiction questions. Moreover, compared to proof by contradiction and refutation methods, pre-service middle school mathematics teachers mostly answered to contrapositive questions wrongly.

The second purpose of the study was to determine the reasons of their wrong interpretations in the mentioned proof methods. According to the results, some common reasons were found in each proof method. For example, 'counterexample is not enough to prove that a statement is false' was a common reason in two of the refutation questions.

The third purpose of the study was to investigate to what extent pre-service middle school mathematics teachers can conduct valid proof. More than half of the students could conduct valid proof for all of the statements. The percentages of valid proofs were given; 67.0% for the first statement, 59.1% for the second statement and 56.5% for the third statement. Moreover, in all of the open-ended proof questions, seniors have the lowest percentage in conducting valid proofs. On the other hand, juniors are the most successful group in the open-ended proof questions 1 and 3, and freshmen are the most successful group in the open-ended proof question 2. Moreover, proof methods that pre-service middle school mathematics teachers used in their valid proofs and the reasons underlying their invalid proofs were investigated. According to the results, mathematical induction was mostly used in two of the statements and direct proof was mostly used in the other one. Besides, proof by contradiction, proof by cases, and visual proof were used by students. There were five reasons of invalid proofs in total. 'Using the numbers to prove the statement', 'direct restatement of the given expression' and 'incomplete proof" were the common reasons for all of three statements.

CHAPTER V

DISCUSSION, IMPLICATIONS and RECOMMENDATIONS

This study involves three purposes related to mathematical proof. The first purpose of the study was to investigate pre-service middle school mathematics teachers' achievement levels in proof by contrapositive, proof by contradiction and refutation methods in terms of year level. The second purpose of the study was to investigate the reasons of pre-service middle school mathematics teachers' wrong interpretations in the above-mentioned proof methods. The third purpose of the study was to determine to what extent pre-service middle school mathematics teacher could conduct valid proofs. Then, as related to the third purpose, proof methods used by pre-service middle school mathematics teachers in valid proofs and the reasons of their invalid proofs were investigated.

In this chapter, the findings of the study are discussed under two sections to address the research questions. In the first section, pre-service middle school mathematics teachers' achievement levels in refutation, proof by contrapositive and proof by contradiction by year level, and also the reasons behind their wrong interpretations as regards the mentioned proof methods are discussed with reference to the previously conducted studies in the related literature. In the second section, to what extent pre-service middle school mathematics teachers could conduct valid proofs, proof methods they used in proving the given statements, and the reasons of conducting invalid proofs are discussed with references to previously conducted studies reported in the literature. Moreover, this chapter addresses implications and recommendations for the further studies.

5.1. Discussion of the Results

The results of the pre-service middle school mathematics teachers' answers are discussed according to the content and structure of the proof questions in MPQ, so the first heading was determined as *discussion of refutation, proof by contrapositive and proof by contradiction questions* and the second heading was determined as *discussion of open-ended proof questions*. The first and second research questions are discussed under the first heading and the last research question is discussed under the second heading.

5.1.1. Discussion of Refutation, Proof by Contrapositive and Proof by Contradiction Questions

As previously mentioned, the first and second research questions are discussed in this section. Firstly, pre-service middle school mathematics teachers' achievement levels in refutation, proof by contrapositive, and proof by contradiction and then the reasons behind their wrong interpretations in the mentioned proof methods are discussed.

The results of pre-service middle school mathematics teachers' answers to the three refutation questions showed that the majority of them answered the questions correctly. In other words, their achievement levels in the refutation questions were found to be considerably high. More specifically, 99 students among 115 students answered the first refutation question correctly, 92 students among 115 students answered the second refutation question correctly, and 101 students among 115 students answered the third refutation question correctly. This finding is consistent with some studies which assert that students generally understand the meaning of refutation and recognize the counterexamples (Riley, 2003; Lewis, 1986; Saeed, 1996; Williams, 1979). The achievement of students in refutation questions may derive from the fact that refutation is easier than other proof methods for most of the students. More precisely, students don't have to suggest a logical argument to refute the statement, they just have to find an example which does not verify the statement.

Students' understandings of proof are generally empirical (Almeida, 2001; Jahnke, 2007). In other words, some students think that giving examples which verify the true statement can be accepted as proof. Considering students' tendency to giving empirical argument, they are expected to be successful in refutation method since they suggest an example to show that the statement is false.

Moreover, the majority of pre-service middle school mathematics teachers in the present study were aware of the fact that one counterexample was sufficient to prove that a statement is false. Similarly, in the study of Lewis (1986), it was found that 96% of the college students knew that a counterexample was enough for disproving. Thus, in this study, it was not sprising that students' success was high in refutation questions.

As stated, one multiple choice question related to refutation was taken from the study of Galbraith (1982) and the other one was prepared by the researcher considering the structure of the multiple choice questions in the study of Galbraith (1982). The aim of the questions in the study of Galbraith (1982) was to investigate whether the first year undergraduate students and graduate students in a teacher training course could use counterexamples effectively. Unlike the findings of this study, the results of Galbraith's study (1982) showed that less than half of the first-year undergraduate students and more than half of the graduate students answered the questions correctly. In other words, there was an increase in the achievement in refutation questions as year levels increased in Galbraith's study (1982). However, in the present study, a decrease in students' achievement levels was observed as year levels increased. The reason of this difference might be the content of the courses students were attending at the time when the instrument was administered. In this study, seniors are the least successful group. Since they had taken a course in which refutation was explained three years ago, they might forget what is needed when showing that the given statement is false or the meaning of the refutation.

According to the results of pre-service middle school mathematics teachers' answers to three contrapositive questions, it was found that nearly half of them answered the first and the second contrapositive questions correctly and nearly one third of them answered the third question correctly. The number of students who answered the question correctly was 61 out of 115 for the first contrapositive question. For the second contrapositive question, 56 students among 115 students answered correctly. In the third refutation question, 33 students among 115 students answered correctly. Students' achievement levels in contradiction questions are found considerably low. The reason behind this finding may be the case that students might be confusing the logical equivalence in this method or have problems in applications. Additionally, students may not notice the contrapositive statements in the third refutation question which involves a discussion about the proofs of two contrapositive statements.

The results of pre-service middle school mathematics teachers' answers to both of the contradiction questions showed that the majority of them answered the questions correctly. In other words, their achievement levels regarding contradiction questions were found to be considerably high. More specifically, 102 students among 115 students answered the first contradiction question correctly and 93 students among 115 students answered the second contradiction question correctly.

It was also seen that only 5 students among 102 students who answered correctly explained by relating to contradiction method in the first contradiction question. However, in the second contradiction question, more than half of the students who answered correctly explained by relating to contradiction method. As seen the number of students who answered correctly and explained by relating to contradiction in the first contradiction question is considerably less than the second contradiction question. This may be resulted from the case that students could not see the mentioned proof method in the first question easily and focused on the content of the given discussion. In the first conradiction question, Ali proved the statement by using the contradiction method, but Ayşe argued that there is no need to prove because she thought that it was an obvious statement. Therefore, most of the students stated that they agreed with Ali since proof is not unnecessary for familiar or easy statements instead of explaining with the contradiction method.

In summary, compared to proof by contradiction and refutation methods, pre-service middle school mathematics teachers mostly answered to contrapositive questions wrongly.

When students' achievement levels in refutation, contrapositive and contradiction questions were analyzed in terms of year level, it was seen that the results were not similar for each item. In the first and the second refutation questions, sophomores were the most successful group; in the third refutation question, freshmen were the most successful group. In the first contrapositive question, freshmen had the highest achievement level; in the second contrapositive question, sophomores had the highest achievement level and in the third contrapositive question, juniors had the highest achievement level. In the first contradiction question, juniors were the most successful group in the first contradiction question; in the second contradiction question, freshmen were the most successful group. It was seen that freshmen, sophomores and juniors were evaluated as the most successful group in some of these questions. Especially freshmen and sophomores were more successful. The main reason behind this might be related to the Discrete Mathematics course, which is taken by students in the second semester of the first year. That is, freshmen and sophomores had recently taken this course in which there was a chapter related to proof and proof methods. On the contrary, senior pre-service teachers were not the most successful group in any of these questions. This result could have stemmed from the fact that senior students had not taken any mathematics course during the last three semesters of their education. Even though they had courses related to mathematics education, they were not taking a pure mathematics course in the semester the data were collected. Therefore, they might have forgotten some of the characteristics of proof and proof methods. Moreover, senior pre-service middle school mathematics teachers might have started to think as a teacher since they had taken the courses School Experience and Practice Teaching in Elementary Education recently. Since finding the correct answer of the question is main concern in the school courses because of the exams in the educational system, both students and teachers might be moved away reasoning and proof which require deep mathematical thinking. After these courses, senior pre-service middle school students might have ignored to think about the questions mathematically and focused on the answer of the question.

To address the second research question, the reasons behind students' wrong answers were investigated. The findings related to this research question were discussed as follows. According to the analysis of the data collected through refutation questions, six reasons for students' wrong interpretations were found. The first reason is that some students think that giving examples is enough to prove a given statement as true. This finding was supported by many research studies mentioned in the literature review (Almeida, 2001; Jahnke, 2007; Stylianides and Stylianides, 2009). This reason for wrong interpretations could be regarded critical because it shows that some students could not see the difference between a valid proof and an empirical argument. The second reason is that giving counterexample is not adequate to prove a given statement as false for some students. This reason was also observed in some other studies (Goetting, 1995; Lewis, 1986). Also, in the studies of Goetting (1995) and Lewis (1986), it was observed that there is need for more than one counterexample to accept a statement as false for some students. In this study, two of the refutation questions were multiple choice questions and these questions involve a choice pointing this idea. Unfortunately, many students selected the choice stating counterexample is not enough to prove that the statement is false even though there is a choice stating the opposite situation. The third reason behind students' wrong answers to refutation questions is about accepting a false statement as true. Only 1 junior pre-service teacher answered wrongly because of this idea. This result is supported by Williams (1979) and it was stated that minority of eleven school high school students accepted a false statement as true since they could not find any counterexample in order to show it is false. In the current study, this situation might be resulted from the fact that students could not understand the given statement. In other words, students might confuse the logical connectives 'and' and 'or' so that they selected the choice stating that the given statement is true. The fourth reason is finding inappropriate counterexamples. This reason is found through refutation question 2. This situation may have resulted from the fact that students did not care about the characteristics of the given statement and used their previous knowledge in finding counterexamples. The next two reasons have not exactly stemmed from common misunderstandings. In other words, these reasons may be regarded as specific to the given case. The fifth reason for giving a wrong answer is that no counterexample was given to refute the statement. 3 sophomore pre-service middle school mathematics teachers answered with this notion. These students wrote that a counterexample should be found to refute the statement and assuming the presence of a counterexample was not sufficient to refute. The last reason is that a student thought what is given in the discussion was related to mathematical induction. Only 1 senior answered in this way; this student may not have read the discussion carefully enough to notice there is no mathematical induction in the case.

Five reasons have been detected from three contrapositive questions. The first reason is lack of information related to proof by contradiction and proof by contrapositive. This finding is consistent with the results of the study of Atwood (2001), who stated that students have difficulty in using the words converse, contrapositive, contradiction and counterexample, and that they may use them interchangeably, which is not correct. In this study, it might be related to the case that students generally memorize proof methods instead of understanding the structure of them. Therefore, they might use proof by contrapositive and proof by contradiction inaccurately and interchangeably. The second reason is that students tried to prove the proposition $q \Rightarrow p$ or $q \Rightarrow p'$ as in order to prove a statement in the form of $p \Rightarrow q$. This finding of the present study might be due to the fact that students do not understand meaning of logical equivalences formed in indirect proofs. In other words, they might think that the propositions $q \Rightarrow p$ or $q \Rightarrow p'$ are logically equivalent to the proposition $p \Rightarrow q$ even though they are not. The third reason causing students to answer the questions wrongly is accepting a true statement as false and trying to prove with this idea. As Williams (1979) stated, some students have trouble in deciding whether the given statement is true or false. In this study, students might have this reason for misinterpretation in the multiple choice questions. Since two of the four multiple choice questions are related to refutation method which involve false statements, students might think that the remaining two multiple choice questions also have false statements. The fourth reason is that students attempted to prove with direct proof instead of using the given choices in the questions. This situation may result form the fact that majority of the proofs in the textbooks are given as direct proofs (Atwood, 2001). Therefore, students may have tendency to use direct proofs since they are more familiar with this method. The fifth reason is that students could not see the relation between proofs of the statements A⇒B and B'⇒A'. To state differently, they think that statement A⇒B and statement B'⇒A' should be proved separately. In fact, since they are contrapositive statement, proof of the statement A⇒B may be accepted as a proof for the statement B'⇒A'. This was consistent with the findings of some studies (Williams, 1979; Saeed, 1996) which stated that students could not see the logical equivalence of contrapositive statements. According to Williams (1979), nearly half of the high school students did not see the contrapositive statements and suggested that different proofs should be conducted for each statement. Similarly, about 12% of the undergraduate mathematics students could see that a statement and its contrapositive are logically equivalent (Saeed, 1996).

Pre-service middle school mathematics teachers' wrong answers in contradiction questions were also investigated and the reasons for their wrong interpretations were classified. According to the analysis, there were four items which lead students to answer wrongly. The first reason is that students accepted the given proof for the statement in the question as unnecessary. This finding was in agreement with the study of Saeed (1996) from which both of the contradiction questions were adapted. In the study of Saeed (1996), nearly one third of the undergraduate mathematics students argued that the statement was intuitively obvious and there was no need for prove. Moreover, in the study of Williams (1979), half of the high school students could not see the need to prove a statement which may be accepted as obvious. This situation of the participant in the present study might be related to students' idea that proof is needed for difficult statements since it is a complex activity. As the second reason, a student's thinking of the given case from a developmental perspective might caused her to answer the contradiction question wrongly. One junior student answered wrongly because of this idea. This finding could be stemmed from two reasons. Firstly, the participants of the study are pre-service middle school mathematics teachers so that she might perceive the given case from the view of a teacher. Secondly, juniors took courses Methods of Teaching Mathematics I and II previous and the current semester of the application of the instrument. In this manner, students generally study mathematical subjects in the mathematics education courses for students in the middle school level. Therefore, she might consider the relation between proof and the level of middle school students while answering the question. The third reason is lack of information related to proof by contradiction. Pre-service middle school mathematics teachers' insufficient information related to proof by contradiction causes some mistakes in practice. This finding is consistent with previous studies (Atwood, 2001; Saeed, 1996). In this study, explanations of pre-service middle school mathematics teachers who classified in this category indicated that they have negative ideas about the word 'contradiction'. In other words, proof which ended with a contradiction might be seemed to them as invalid and also might give students idea that there is something wrong about proof. The last reason is that students misunderstood the assumptions in the proof by contradiction. This result was consistent with some studies (Antonini & Mariotti, 2008; Atwood, 2001; Hoyles & Küchemann, 2002; Saeed, 1996). Atwood (2001) stated three potential points which students may have difficulty in. These points are the starting assumptions, the derivation of contradiction and interpretation of contradiction in the proof. In the study of Atwood (2001), the last one was the most common difficulty. However, in this study, most of the students had difficulty in the structure of the assumptions. Moreover, in this study, some pre-service middle school mathematics teachers might have difficulty in assuming the statement which was wanted to be proved is false and forming contradictions. Similar to the participants in the study of Antonini and Mariotti (2008), students in the present study might think that it does not make sense to start proof with the opposite of the statement. Therefore, the assumption in the contradiction method might lead them to think the given proof in the question is invalid.

5.1.2. Discussion of Open-ended Proof Questions

The findings related to the last research question are discussed in this chapter. To address research question, students were asked to prove three statements given to them. Firstly, their arguments were assessed as valid or invalid in order to investigate to what extent they could conduct valid proof. Then, their valid proofs were classified based on the proof methods that they used. Lastly, the reasons of conducting invalid proofs were determined through the analysis of invalid proofs.

According to the results, more than half of the students provided valid proof for all of the statements. The percentages of valid proofs for each statement were 67.0%, 59.1% and 56.5% respectively. Moreover, for each statement, nearly one fourth of the participants wrote invalid proofs. In terms of year levels, it was seen that seniors were the least successful group in conducting valid proof for each statement. The reason of this finding might be that they forget some of the characteristics of proof and proof methods.

When students' valid proofs were analyzed, it was seen that mathematical induction was mostly used in two of the statements and in the other one, direct proof was mostly used. Other than these methods, proof by contradiction, proof by cases, and visual proof were used by students. This finding could be because of three reasons. Firstly, as Atwood (2001) stated, studies on proof methods in common textbooks in various mathematics courses showed that the majority of proofs were given as direct proof. Following direct proof, mathematical induction and proof by contradiction were generally given in the textbooks. Therefore, participants of the present study might want to use the direct proof and induction which are the methods they are more familiar with. Secondly, according to Antonini and Mariotti (2008), students have much more difficulties in indirect proofs than direct proofs. Therefore, pre-service middle school mathematics teachers might prefer to use direct proof in conducting proofs since they have difficulty in understanding the structure of indirect proof. Lastly, according to Saeed (1996), some students accept indirect arguments such as proof by contrapositive and proof by contradiction as nonconvincing. In this manner, participants of this study might think that direct proof and induction are much easier to construct and understand and choose to prove by using these methods instead of indirect proof methods.

However, it was also seen that the number of students who gave visual proof was 1, the number of students who used proof by contradiction was 9 and the number of students who used proof by cases was 3. Visual proof was given by a junior student, which may have resulted from the fact that students took teaching mathematics method course during the two semesters in the third year. Therefore, the student might have learned the visual proof of the given statement recently in these courses and might have discussed showing the given statement to middle school students as a visual proof.

In the study, students' invalid proofs were also analyzed and the reasons were classified under five headings. The most common reason was direct restating some parts of the given expression. This finding was consistent with the study of Moore (1994) which stated that students may not know how to start conducting proof. Students in this study might be writing the givens in the statement in order to find a way to prove but then fail to provide a valid proof. The second common reason of pre-service teachers' invalid proof was that they left the proof incomplete. This situation was generally seen in the last step of mathematical induction. To state differently, they accept that the statement is true for n=k, but they cannot show that it is also true for n=k+1 by using the information in the previous step. Therefore, they could not end up with a valid proof for the statements. Moreover, this reason might be related to the fact that pre-service middle school mathematics teachers generally memorize the proofs instead of learning how proofs can be conducted.

In the following section, some implications related to the results of the study and some recommendations for further research are presented.

5.2. Implications and Recommendations for Further Research Studies

The purposes of the study can be listed under two parts. In the first part, pre-service middle school mathematics teachers' achievement levels in proof by contrapositive, proof by contradiction and refutations methods in terms of year level and the reasons of their wrong interpretations in mentioned proof methods were investigated. In the second part, to what extent pre-service middle school mathematics teachers could conduct valid proof, proof methods that pre-service teachers used and the reasons of their invalid proofs were investigated through open-ended proof questions. According to the findings of the study, some implications are stated for pre-service and in-service mathematics teachers, teacher educators, textbook writers and curriculum developers related to proof.

The results of the study can help teacher educators to gain insight into pre-service middle school mathematics teachers' achievement levels in determining proof methods, their wrong interpretations and the reasons behind these mistakes. The results of the study could be attributed to the courses in the Elementary Mathematics Education program. In more detail, instructors may focus on the proofs and proof methods in the mathematics courses, such as Discrete Mathematics, Calculus and Analysis. If there is a gap in students' understanding regarding the structure of mathematical proof, this might affect students' achievement in the courses of the next semesters negatively. To avoid this situation, some additional sources such as textbooks and articles may be offered to the students in order to develop their understanding of proof, to inform them about different proof methods and to make students aware of the characteristics of proof methods and the differences among them by teacher educators.

As stated previously, pre-service middle school mathematics teachers were asked to prove the given statements. The results revealed that the percentages of the students who suggested valid proofs for the given three statements are 67.0%, 59.1% and 56.5%. To state differently, at least one fourth of the students failed to provide valid proof for each statement. This percentage cannot be considered to be low since the lack of knowledge in proof will affect their teaching when they are in-service teachers. To avoid this, students' understanding of proof should be developed through both mathematics and mathematics education courses in the program by integrating proof into the courses effectively.

In addition, teacher educators may arrange some elective courses related to proof for the pre-service middle school mathematics teachers so that they can have the opportunity to study proof. In the elective courses related to proof, meaning and nature of proof, proof in mathematics education, proof methods, and difficulties related to proof might be included. Moreover, to help pre-service middle school teachers understand the status of proof and reasoning in mathematics education in Turkey, studies conducted in Turkey might be examined by students in these elective courses. In this manner, the findings of this study may help pre-service middle school mathematics teachers to see the characteristics of a valid proof, the meaning of the mentioned proof methods in the study, in which conditions their proofs might be accepted as invalid and their wrong interpretations regarding proof.

The results of the study showed that seniors were not the most successful group in each question of the instrument. To avoid this situation, elective courses related to proof might be offered for seniors to enhance their reasoning ability. Moreover, it can be inferred that there was a decrease in students' achievement levels as year levels increased in most of the questions. Similarly, the analysis regarding students' achievement levels in proof by contrapositive, proof by contradiction and refutation methods showed that they were the least successful in contrapositive questions. According to these findings, it can be inferred that students generally have difficulty in different areas of proof concept. According to Baştürk (2010), to have an idea about undergraduate students' conceptions of proof and their difficulties in proof, their education before higher education should be examined. Baştürk (2010) also stated the fact that university entrance exam has a definite effect on the secondary school mathematics courses in Turkey, causing students to have inadequate knowledge regarding proof until university. Therefore, most of them first meet proof at the university level. This situation may lead students to experience some difficulties in transition from the mathematics at the secondary level to mathematics at the university level. Therefore, curriculum developers should modify the objectives of mathematics programs that are related to proof and reasoning by considering the case that proof is an important concept at all levels (NCTM, 2000; Schoenfeld, 1994).

The findings revealed that proof is a complicated and difficult concept for students even at the university level. Therefore, to develop students' proof and reasoning ability at early grades is an important issue. At this point, in-service mathematics teachers should be aware of the importance of proof and reasoning in mathematics teaching and prepare appropriate environments in the classrooms which help students to develop their reasoning skills and mathematical thinking. In this regard, seminars and training programs may be prepared for teachers in order to inform them about helping students to gain reasoning abilities, to understand the meaning and necessity of proof. In addition to teachers, textbook writers may benefit from the results of this study. Since majority of the proofs in the textbooks are given as direct proofs (Atwood, 2001), different proof methods such as proof by contradiction, proof by contrapositive and proof by cases may be included in the textbooks.

Lastly, some recommendations are offered for further studies on the basis of the finding of the present study.

There are some limitations for generalizability since convenience sampling was used in the study. The participants of the study were freshmen, sophomores, juniors and seniors in Elementary Mathematics Education program of a state university. Therefore, further research might be conducted with the pre-service middle school mathematics teachers enrolled in randomly selected universities in Turkey in order to generalize the findings of the study to the population.

Additionally, further studies might be conducted with in-service middle school mathematics teachers in order to investigate their perceptions of proof so that pre-service and in-service middle school mathematics teachers might be compared. Since proof is an important subject in secondary school mathematics, the same instrument might also be administered to pre-service and in-service secondary mathematics teachers. Moreover, for the further studies, pre-service middle school mathematics teachers' ideas, beliefs and attitudes towards proof might be investigated in order to see the relationship among their ideas, beliefs, attitudes and their achivement levels in proof.

A longitudinal study might be conducted to investigate direct effects of the courses on students' proof achievement levels, their wrong interpretations and their ability to conduct valid proofs.

In this study, pre-service middle school mathematics teachers' achievement levels in proof by contrapositive, proof by contradiction and refutation methods and the reasons for their wrong interpretations were investigated through multiple choice and discussion questions. Therefore, other proof methods might be added to the instrument. Moreover, to analyze the answers of the students in-depth, the follow-up interviews might be conducted with the participants in the further studies.

References

Alibert, D., & M. Thomas, (1991). Research on mathematical proof. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 215-230). Hingham, MA: Kluwer Academic Publishers.

Almeida, D. (2000). A survey of mathematics undergraduates' interaction with proof: some implications for mathematics education. *International Journal of Mathematical Education in Science and Technology, 31*(6), 869-890.

Almeida, D. (2001). Pupils' proof potential. *International Journal of Mathematical Education in Science and Technology, 32*(1), 53-60.

Almeida, D. (2003). Engendering Proof Attitudes: Can the Genesis of Mathematical Knowledge Teach Us Anything? *International Journal of Mathematical Education in Science and Education, 34*(4), 479-488.

Altıparmak, K., & Öziş, T. (2005). Matematiksel İspat ve Matematiksel Muhakemenin Gelişimi Üzerine Bir İnceleme, *Ege Eğitim Dergisi, 6*(1), 25-37.

Altun, M. (2007). *Eğitim Fakülteleri ve Matematik Öğretmenleri İçin Ortaöğretimde Matematik Öğretimi*, Bursa.

Anapa, P., & Şamkar, H. (2010). Investigation of undergraduate students' perceptions of mathematical proof. *Procedia Social and Behavioral Sciences, 2,* 2700-2706.

Anderson, J. R. (1985). *Cognitive Psychology and Its Implications* (2nd ed.). New York: W. H. Freeman.

Antonini, S. & Mariotti, A.M. (2008). Indirect proof: What is specific to this way of proving? *ZDM-The International Journal of Mathematics Education, 40*, 401-412.

Arslan, Ç. (2007). *İlköğretim Öğrencilerinde Muhakeme Etme ve İspatlama Düşüncesinin Gelişimi*, Unpublished doctoral dissertation, Uludağ Üniversity, Bursa.

Aslan, K. (2003). Eğitim fakültelerinin yeniden yapılandırılmalarına ilişkin bir değerlendirme. *Balıkesir Üniversitesi Sosyal Bilimler Enstitüsü Dergisi*, *6*(9), 23-37.

Atayan, S., & Hickman, B. (2009). *Key Concepts: Fundamentals of Logic and Techniques for Mathematical Proofs.* Retrieved from Lecture Notes Online Web site: http://home.uchicago.edu/~hickmanbr/uploads/PROOFS_Notes.pdf.

Atwood, P. R. (2001). *Learning to Construct Proofs in a First Course on Mathematical Proof. Doctoral Dissertation.* Western Michigan University.

Balacheff, N. (1988). Aspects of Proof in Pupils' Practice of School Mathematics. In D. Pimm (ed.), *Mathematics, Teachers and Children* (pp. 216-230). London: Hodder and Stoughton.

Baştürk, S. (2009). Ortaöğretim matematik öğretmen adaylarına göre fen edebiyat fakültelerindeki alan eğitimi. *İnönü Üniversitesi Eğitim Fakültesi Dergisi*, *10*(3), 137-160.

Baştürk, S. (2010). First-year secondary school mathematics students' conceptions mathematical proofs and proving. *Educational Studies*, *36*(3), 283-298.

Baştürk, S. (2011). Matematik öğretmen adaylarının eğitim fakültesindeki eğitim-öğretim sürecini değerlendirmeleri. *Uluslararası İnsan Bilimleri Dergisi, 1*(8), 58-94.

Baylis, J. (1983). Proof - The essence of mathematics, part 1', *International Journal of Mathematics Education and Science Technology, 14,* 409-414.

Bell, A. W. (1976). A study of pupils' proof-explanations in mathematical situations. *Educational Studies in Mathematics, 7*, 23-40.

Bloch, Ethan D. (2000). *Proofs and Fundamentals. A First Course in Abstract Algebra*. Birkhauser, Boston, Basel, Berlin.

Brown, J., Stillman, G., Schwarz, B. & Kaiser, G. (2008). The case of mathematical proof in lower secondary school: Knowledge and competencies of pre-service teachers. In M. Goos, R. Brown and K. Makar (Eds.), *Navigating currents and charting directions, Proceedings of the 31st annual conference of the Mathematics Education Research Group of Australasia, Brisbane* (Vol. 1, pp. 85–91) Adelaide: MERGA.

CadwalladerOlsker, T. (2011) What Do We Mean by Mathematical Proof? *Journal of Humanistic Mathematics, 1*(1), 33-60.

Conner, A. (2007). *Student teachers' conceptions of proof and facilitation of argumentation in secondary mathematics classrooms.* Doctoral dissertation, The Pennsylvania State University.

Çelik, B. (2010). *Soyut Matematik I.* Dora Basım Yayın.

Davis, P. J., & Hersh, R. (1981). *The mathematical experience*. Boston: Birkhauser.

de Villers, M. (1990). The Role and Function of Proof with Sketchpad. *Pythagoras*, vol.24, p.17 -24.

de Villiers, M. (1999). *Rethinking Proof with Geometer's Sketchpad*, Key Curriculum Press.

Dickerson, D. S. (2008). *High school mathematics teachers' understandings of the purposes of mathematical proof.* Unpublished doctoral dissertation, Syracuse University, Syracuse.

Epp, S. S. (2003). The role of logic in teaching proof. *American Mathematical Monthly, 110*(10), 886-899

Fitzgerald, J.F. (1996). Proof in Mathematics Education. *Journal of Education, 178*(1), 35-45.

Fraenkel, J. R., & Wallen, N. E. (2005). *How to design and evaluate research in education* (6th ed.). Boston: McGraw Hill.

Galbraith, P. (1982). The mathematical vitality of secondary mathematics graduates and prospective teachers: A comparative study. *Educational Studies in Mathematics*, 13, 89-112.

Gibson, D.(1998). *Students' use of diagrams to develop proofs in an introductory analysis course. Students' proof schemes*. In E. Dubinsky, A. Schoenfeld, & J.Kaput (Eds.), Research in Collegiate Mathematics Education, III, 284–307. AMS.

Goetting, M. (1995). *The college students' understanding of mathematical proof.* Doctoral dissertation, University of Maryland, College Park.

Gökkurt, B., & Soylu, Y. (2012). Üniversite Öğrencilerinin Matematiksel İspat Yapmaya Yönelik Görüşleri, *Journal of Research in Education and Teaching 1*(4), 56-64.

Grabiner, J. (2009). Why proof? Some lessons from the history of mathematics. *ICMI Study 19: Proof and Proving in Mathematics Education* p.147-169.

Griffiths, P. A. (2000). Mathematics at the turn of the millennium. *American Mathematical Monthly, 107*(1), 1-14.

Güler, G., Kar, T., Öçal, M., & Çiltaş, A. (2011). Prospective Mathematics Teachers' Difficulties in Proof. *Procedia Social and Behavioral Sciences,* 15, 336-340.

Hanna, G. (1989). More than Formal Proof. *For the Learning of Mathematics, 9*(1), 20-23.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics, 44,* 5-23.

Hanna, G., de Bruyn, Y., Sidoli, N., & Lomas, D. (2004). Teaching proof in the context of physics. *International Reviews on Mathematical Education, ZDM, 36*(3), 82-90.

Hanna, G., de Villiers, M. (2008). ICMI study 19: Proof and proving in mathematics education. *ZDM-The International Journal of Mathematics Education, 40*(2), 1-8. Advance online publication. Retrieved March 20, 2012, doi: 10.1007/s11858-008-0080-5.

Hammack, R. (2009). *Book of Proof,* Virginia Commonwealth University.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *CBMS Issues in Mathematics Education, 7,* 234-283.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education, 31*(4), 396-428.

Heinze, A., & Reiss, K. (2003). Reasoning and proof: Methodological knowledge as a component of proof competence. In M.A. Mariotti (Ed.), *Proceedings of the Third Conference of the European Society for Research in Mathematics Education,* Bellaria, Italy.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics, 24*, 389-399.

Hoyles, C. and D. Kuchemann (2002). Students' understandings of logical implication. *Educational Studies in Mathematics, 51,* 193-223.

İskenderoğlu, T. A., Baki, A. (2011). Quantitative Analysis of Pre-service Elementary Mathematics Teachers' Opinions about Doing Mathematical Proof. *Educational Sciences: Theory and Practice, 11*(4), 2285-2290.

Jahnke, H. N. (2007). Proofs and hypotheses. *ZDM-The International Journal of Mathematics Education, 39*(1-2), 79-86.

Jones, K. (1997). Student Teachers' Conceptions of Mathematical Proof. *Mathematics Education Review,* 9, 21-32.

Kayagil, S. (2012). İlköğretim Matematik Öğretmen Adaylarinin İspat Yapmaya Yönelik Görüşleri Ve Bu Görüşlerin Bazi Değişkenlere Göre İncelenmesi. *International Journal Of New Trends In Arts, Sports & Science Education (IJTASE),* 1, 134-141.

Knapp, J. (2005). *Learning to prove in order to prove to learn.* Cited from http://mathpost.asu.edu/~sjgm/issues/2005_spring/SJGM_knapp.pdf.

Knuth, E. (1999). *The nature of secondary school mathematics teachers' conceptions of proof.* Unpublished doctoral dissertation, University of Colorado, Colorado.

Ko, Y. Y. (2010). Mathematics teachers' conceptions of proof: Implications for educational research. *International Journal of Science and Mathematics Education, 8*(6), 1109-1129.

Köğce, D., Aydın, M., & Yıldız, C. (2010). The views of high school students about proof and their levels of proof (The case of Trabzon). *Procedia Social and Behavioral Sciences,* 2, 2544-2549.

Krantz, S. G. (2007). *The History and Concept of Mathematical Proof*. Retrieved from www.math.wustl.edu/~sk/eolss.pdf.

Lee, W. I. (1999). *The relationship between students' proof writing ability and Van Hiele Levels of geometric thought in a college geometric course.* Unpublished Doctoral Dissertation. University of Northern Colorado, Greeley, Colorado, USA.

Lewis, S.M. (1986). *University Mathematics Students' Perception of Proof and Its Relationship to Achievement.* Dissertation Abstract, Oregon State University.

Mariotti, M.A. (2006) Proof and proving in mathematics education. A. Gutiérrez & P. Boero (eds) *Handbook of Research on the Psychology of Mathematics Education*, Sense Publishers, Rotterdam, The Netherlands. ISBN: 9077874194, pp. 173-204.

Marrades, R., & Gutierrez, A. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment, *Educational Studies in Mathematics 44* (1/2), 87-125.

Martin, W. G., & Harel G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education*, *20*(1), 41-51.

Mingus, T. Y. & Grassl, R. M. (1999). Preservice teacher beliefs about proofs. *School Science and Mathematics*, *99*(8), 438-444.

Ministry of National Education [MoNE]. (2005), *İlköğretim Ders Programları: Matematik Programı 6-7-8 (Elementary Curricula Programs: Mathematics Curriculum Program for Middle Grades)*. Ankara: MEB.

Ministry of National Education. [MoNE]. (2009), *İlköğretim Matematik Dersi Öğretim programı*. Retrieved from website: http://ttkb.meb.gov.tr/www/ogretim-programlari/icerik/72.

Ministry of National Education. [MoNE]. (2011), *Ortaöğretim Matematik Dersi Öğretim programı*. Retrieved from website: http://ttkb.meb.gov.tr/www/ogretim-programlari/icerik/72.

Ministry of National Education. [MoNE]. (2013), *Ortaokul Matematik Dersi 5-8 Sınıflar Öğretim Programı*. Retrieved from website: http://ttkb.meb.gov.tr/www/ogretim-programlari/icerik/72.

Ministry of National Education. [MoNE]. (2013), *Ortaöğretim Matematik Dersi 9-12 Sınıflar Öğretim Programı*. Retrieved from website: http://ttkb.meb.gov.tr/www/ogretim-programlari/icerik/72.

Miyazaki, M. (2000). Levels of proof in lower secondary school mathematics. *Educational Studies in Mathematics*, 4, 47-68.

Moore, R. C. (1994). Making the Transition to Formal Proof. *Educational Studies in Mathematics*, 27, 249-266.

Moralı, S., Uğurel, I., Türnüklü, E., & Yeşildere, S. (2006). The views of mathematics teachers on proving. *Kastamonu Education Journal*, *14*(1), 147-160.

Morris, D.W., Morris, J. (2009). Proofs and Concepts: The Fundamentals of Abstract Mathematics. University of Lethbridge.

National Council of Teacher of Mathematics (2000). Principles and standard for school mathematics. Reston, VA: National Council of Teacher of Mathematics.

Overton, W. F. (Ed.). (1990). *Reasoning, necessity, and logic*: *Developmental perspectives*. Hillsdale, N.J.: L. Erlbaum Associates.

Özer, Ö., & Arıkan, A. (2002, September). *Students' levels of doing proof in high school mathematics classes*. Paper presented at the meeting of 5th National Science and Mathematics Education Congress (pp. 1083-1089). Middle East Technical University, Ankara.

Raman, M. (2003). What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics*, 52, 319-325.

Recio, A. M. & Godino, J. D. (2001). Instutional and Personal Meanings of Mathematical Proof, *Educational Studies in Mathematics*, *48*(1), 83-89.

Riley, K. J. (2003). *An investigation of prospective secondary mathematics teachers' conceptions of proof and refutations.* Doctoral dissertation, Montana State University.

Roberts, C. (2009). *Introduction to Mathematical Proofs: A Transition.* Boca Raton, Florida: Taylor & Francis Group.

Ross K. A. (1998) Doing and Proving: The place of Algorithms and Proof in School Mathematics *American Mathematical Monthly*, 252-255.

Rota, G. (1997). The Phenomenology of Mathematical Proof. *Synthese. 111* (2): 183-196.

Saeed, R. M. (1996). *An exploratory study of college students' understanding of mathematical proof and the relationship of this understanding to their attitudes toward mathematics*, Unpublished doctoral dissertation, Ohio University.

Sarı, M., Altun, A., & Aşkar, P. (2007). Üniversite öğrencilerinin analiz dersi kapsamında matematiksel kanıtlama süreçleri: Örnek olay çalışması. *Ankara Üniversitesi Eğitim Bilimleri Fakültesi Dergisi*, *40*(2), 295-319.

Schoenfeld, A. H. (1994). What do we know about mathematics curricula? *Journal of Mathematical Behavior*, *13*(1), 55-80.

Selden, A., & J. Selden (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education*, *34*(1), 4-36.

Smith E. P., & Henderson K. (1959). Proof. The growth of mathematical ideas. *24th Year Book of the NCTM*.

Stylianides, A. J. (2007). The notion of proof in the context of elementary school mathematics. *Educational Studies in Mathematics, 65*(1), 1-20.

Stylianides, A. J., & Al-Murani, T. (2009). Can a proof and a counterexample coexist? Students' conceptions about the relationship between proof and refutation. *Research in Mathematics Education, 12*(1), 21-36.

Stylianides, A. J., & Ball, D. L. (2008). Understanding and describing mathematical knowledge for teaching: Knowledge about proof for engaging students in the activity of proving. *Journal of Mathematics Teacher Education, 11*, 307-332.

Stylianides, A. J**.**, & Stylianides, G. J. (2006). Content knowledge for mathematics teaching: The case of reasoning and proving. In J. Novotná, H. Moraová, M. Krátká, & N. Stehliková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 201-208). Prague, Czech Republic.

Stylianides, A., & Stylianides, G. (2009). Proof constructions and evaluations. *Educational Studies in Mathematics*, *72*, 237-253.

Stylianides, G. J., Stylianides, A. J. & Philippou, G. N. (2007). Preservice teachers' knowledge of proof by mathematical induction. *Journal of Mathematics Teacher Education*, *10*, 145-166.

Tall, D. (1989). The nature of mathematical proof. *Mathematics Teaching*, 127, 28-32.

Türker, B., Alkaş, Ç., Aylar, E., Gürel, R., & Akkuş İspir, O. (2010). The Views of Elementary Mathematics Education Preservice Teachers on Proving. *International Journal of Human and Social Sciences 5*(7), 423-427.

Üzel, D., & Özdemir, E. (2009). Elementary mathematics teacher candidates' attitudes towards proof and proving. *e-Journal of New World Sciences Academy Education Sciences, 4*(4), 1226–1236.

Yopp, D. (2011). How some research mathematicians and statisticians use proof in undergraduate mathematics. *Journal of Mathematical Behavior, 30,* 115–130.

VanSpronsen, H. D. (2008). Proof Processes of Novice Mathematics Proof Writers. Unpublished Doktoral Dissertation, University of Montana. R. F. (1992). *Probing understanding*. The Falmer Press, London.

Varghese, T. (2011). Possible Student Justification of Proofs. *School Science and Mathematics, 111*(8), 409-415.

Velleman, D.J. (2006). *How to Prove It.* Cambridge Universtiy Press.

Volmink, J.D. (1990). The Nature and Role of Proof in Mathematics Education. *Pythagoras, 23*, 7-10.

Weber, K. (2001). Student difficulty in constructing proofs. The need for strategic knowledge. *Educational Studies in Mathematics, 48*(1), 101-119.

Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *Journal for Research in Mathematics Education, 39*(4), 431-459.

Williams, E. (1979). *An Investigation of Senior High School Students' Understanding of the Nature of Mathematical Proof.* Doctoral Dissertation, University of Alberta, Edmonton.

APPENDICES

APPENDIX A- MATHEMATICAL PROOF QUESTIONNAIRE

Değerli Matematik Öğretmeni Adayı,

Bu çalışma ortaokul matematik öğretmen adaylarında matematiksel ispat kavramını araştırmak amacıyla hazırlanmıştır. Sorulara vereceğiniz yanıtlar, bilimsel bir araştırmada kullanılacak ve gizli tutulacaktır. Lütfen soruları dikkatlice okuyarak eksiksiz yanıtlayınız.

Teşekkür ederim.

Esra DEMİRAY
Orta Doğu Teknik Üniversitesi

KİŞİSEL BİLGİLER

1. Üniversite: ……………………………………………….

2. Sınıf: ☐ 1. Sınıf ☐ 2. Sınıf ☐ 3. Sınıf ☐ 4. Sınıf

3. Yaş: ☐ 18 ve 18 altı ☐ 19-21 ☐ 22-25 ☐ 26 ve 26 üstü

4. Cinsiyet: ☐ Kız ☐ Erkek

5. Not Ortalaması: ……………..

BÖLÜM A

1) *Önerme A*: a ve b iki doğal sayı olsun. a.b'nin bir çift sayı olması ancak ve ancak a ve b sayılarının çift olması durumunda mümkündür.

Bu önermeye göre aşağıdakilerden hangisi doğrudur?

a) A önermesi doğrudur.

b) a=5 ve b=6 sayıları A önermesinin yanlış olduğunu ispatlar.

c) a=2 ve b=6 sayıları A önermesinin doğru olduğunu ispatlar.

d) A önermesi yanlıştır ama a=5 ve b=6 sayıları bunu ispatlamaya yetmez.

e) Hiçbiri

- Neden? Gerekçelerinizi belirtiniz.

2) m ve n pozitif tamsayılar olmak üzere mn=100 ise, m≤10 veya n≤10 olur.

İfadeyi ispatlamak için aşağıdaki varsayımlardan hangisi ile başlanabilir?

a) m ve n pozitif tamsayılar olmak üzere m>10 veya n>10 ise mn=100 olur.

b) m ve n pozitif tamsayılar olmak üzere m≤10 veya n≤10 ise mn=100 olur.

c) m ve n pozitif tamsayılar olmak üzere m≤10 ve n≤10 ise mn≠100 olur.

d) m ve n pozitif tamsayılar olmak üzere m>10 ve n>10 ise mn≠100 olur.

e) Hiçbiri

- Neden? Gerekçelerinizi belirtiniz.

3) a,b ve c reel sayılar ve a>b olmak üzere ac≤bc ise c≤0 olur.

İfadeyi ispatlamak için aşağıdaki varsayımlardan hangisi ile başlanabilir?

a) a,b ve c reel sayılar ve a>b olmak üzere, c>0 ise ac≤bc olur.

b) a,b ve c reel sayılar ve a>b olmak üzere, c≤0 ise ac≤bc olur.

c) a,b ve c reel sayılar ve a>b olmak üzere, c>0 ise ac>bc olur.

d) a,b ve c reel sayılar ve a>b olmak üzere, c≥0 ise ac≥bc olur

e) Hiçbiri

- Neden? Gerekçelerinizi belirtiniz.

4) *Önerme A:* Bir tam sayıdaki rakamların toplamı 6'ya bölünebiliyorsa, bu tam sayı 6'ya bölünebilir.

Aşağıdakilerden hangisi doğrudur?

a) 33 sayısı A önermesinin yanlış olduğunu gösterir.

b) 30 sayısı A önermesinin yanlış olduğunu gösterir.

c) 30 ve 33 sayıları A önermesinin yanlış olduğunu gösterir.

d) A önermesi yanlıştır fakat ne 30 ne de 33 sayısı bunu ispatlamaya yeterlidir.

e) A önermesi doğrudur.

- Neden? Gerekçelerinizi belirtiniz.

BÖLÜM B

1) Durum A

Ali aşağıdaki önermenin her x ve y reel sayısı için doğru olduğunu göstermektedir.

Önerme: $x \neq 0$ ve $y \neq 0$ ise $x.y \neq 0$ olur

Ali'ın ispatı:

$x \neq 0$, $y \neq 0$ ve $x.y = 0$ olduğunu kabul edelim.

$x \neq 0$ olduğu için x^{-1} vardır.

Buradan, $x^{-1}.(x.y)=(x^{-1}.x).y=1.y=y$

Ayrıca, $x.y = 0$ olduğu için, $x^{-1}.(x.y)=x^{-1}.0=0$

Buradan, $y=0$ olur. Ama, $y \neq 0$ kabul etmiştik.

Böylece, $x.y=0$ yanlış olmalı.

Sonuç olarak, $x.y \neq 0$ elde ederiz.

Ayşe:

İspatın tamamen gereksiz bence. Bak, herkes $x \neq 0$ ve $y \neq 0$ ise $x.y \neq 0$ olduğunu bilir, bunu göstermeye gerek yok.

Ali:

Önermenin herkese tanıdık olduğu konusunda sana katılıyorum. Fakat, ispatımın gereksiz olduğuna katılmıyorum, Ayşe.

Sorular;

- Yukarıdaki tartışmada kiminle aynı fikirdesiniz?

Ali____X____ Ayşe_________

- Neden? Gerekçelerinizi belirtiniz.

2)Durum B

Önerme A: n^2 tek tam sayı ise n de bir tek tamsayıdır.

Önerme B: n çift tam sayı ise, n^2 de çift tam sayıdır.

Ahmet: Bence önerme A doğru, Pınar.

Pınar: Bakalım, $n^2=9$ ise $n=\pm 3$ tektir; $n^2=25$ ise $n\pm5$ tektir. Böylece, önerme A doğru görünüyor Ahmet.

Ahmet: Ben önerme B'nin de doğru olduğunu düşünüyorum.

Pınar: Neden?

Ahmet: n çift olduğu için n=2k, k= tam sayı olur.

Böylece, $n^2 = 4k^2 = 2\ (2k^2)$ çifttir.

Burada, önerme B'nin doğru olduğunu ispatlayarak, önerme A'nın da doğru olduğunu ispatlamış olduk.

Pınar: Fakat Ahmet, bu ispatın sadece önerme B'nin doğru olduğunu gösterir, önerme A'nın doğru olduğunu göstermez.

Ahmet: Bu ispat, önerme A'nın da doğru olduğunu gösterir.

Sorular;

- Yukarıdaki tartışmada kiminle aynı fikirdesiniz?

Ahmet___X____ Pınar_________

- Neden? Gerekçelerinizi belirtiniz.

3) Durum C

Ege: x rasyonel sayı ve y irrasyonel sayı ise, y-x sayısının irrasyonel olduğunu nasıl ispatlarsın?

Deniz: y-x sayısının irrasyonel olmadığını (rasyonel) kabul edelim.

Buradan, c ve d≠0 tamsayıları için y-x=c/d olur.

x rasyonel ise, a ve b≠0 tamsayıları için x=a/b olur.

Böylece, (y-x)+x= c/d+a/b= (cb+ad)/db

cb+ad ve db tamsayı olduğu için (y-x)+x rasyonel sayıdır.

Fakat, (y-x)+x= y olur.

y rasyonel sayıdır ki bu ifade yanlıştır.

Böylece beklendiği gibi y-x irrasyonel sayıdır.

Ege: Fakat y-x sayısının irrasyonel olmadığını kabul ederek başlamıştın; bana bunun tersini göstermek için y-x sayısının irrasyonel olmadığını kabul etmek mantıklı gelmiyor.

Deniz: Bu doğru bir ispat yöntemi olduğu için y-x sayısının rasyonel olduğunu kabul ederek başlamak zorundayım.

Sorular;

- Yukarıdaki tartışmada kiminle aynı fikirdesiniz?

Ege________ Deniz_____X_____

- Neden? Gerekçelerinizi belirtiniz.

4) Durum D

Cem ve İlknur asal sayıları tartışıyorlar.

Cem: Her zaman asal sayı veren bir formül bulmaya çalışıyordum ve sonunda başardım, İlknur.

n^2-n+41

n=1 olduğunda, n^2-n+41= 41

n=2 olduğunda, n^2-n+41=43

n=3 olduğunda, n^2-n+41=47

n=4 olduğunda, n^2-n+41=53

0'dan 40'a kadar bütün sayıları denedim, formül her zaman asal sayı verdi. Bu nedenle, formülüm doğru.

İlknur: Seninle aynı fikirde değilim. Bence, 40'tan büyük ve formülü sağlamayan en az bir sayı bulabiliriz.

Sorular;

- Yukarıdaki tartışmada kiminle aynı fikirdesiniz?

Cem________ İlknur___X____

- Neden? Gerekçelerinizi belirtiniz.

BÖLÜM C

1) $1 + 2 + 3 + 4 + 5 + \ldots + n = n \cdot (n+1) / 2$ olduğunu ispatlayınız.

2) “a ve b reel sayılar olmak üzere, $0<a<b$ ise $a^2<b^2$ olur.” ifadesini ispatlayınız.

3) “Her n doğal sayısı için $3 \mid n^3 - n$ olur.” ifadesini ispatlayınız.

APPENDIX B

RUBRICS OF MATHEMATICAL PROOF QUESTIONNAIRE

SECTION A

Questions 1-4

	Answer types
0	No answer
1	Wrong answer, no explanation
2	Wrong answer, wrong interpretation
3	Correct answer, no explanation or unclear explanation
4	Correct answer, reason is not related to a proof method (valid reasoning)
5	Correct answer, reason is related to refutation (valid reasoning)

Questions 2- 3

	Answer types
0	No answer
1	Wrong answer, no explanation
2	Wrong answer, wrong interpretation
3	Correct answer, no explanation or unclear explanation
4	Correct answer, reason is not related to a proof method (valid reasoning)
5	Correct answer, reason is related to proof by contrapositive (valid reasoning)

SECTION B

Question 1

	Answer types
0	No answer
1	Agreed with no one or both of them
2	Agreed with Ayşe, no reason was stated
3	Agree with Ayşe, reason was stated
4	Agreed with Ali, no reason was stated
5	Agreed with Ali, reason which is not related to a proof method was stated
6	Agreed with Ali, reason which is related to proof by contradiction was stated

Question 2

	Answer types
0	No answer
1	Agreed with no one or both of them
2	Agreed with Pınar, no reason was stated
3	Agree with Pınar, reason was stated
4	Agreed with Ahmet, no reason was stated
5	Agreed with Ahmet, reason which is not related to a proof method was stated
6	Agreed with Ahmet, reason which is related to proof by contrapositive was stated

Question 3

	Answer types
0	No answer
1	Agreed with no one or both of them
2	Agreed with Ege, no reason was stated
3	Agree with Ege, reason was stated
4	Agreed with Deniz, no reason was stated
5	Agreed with Deniz, reason which is not related to a proof method was stated
6	Agreed with Deniz, reason which is related to proof by contradiction was stated

Question 4

	Answer types
0	No answer
1	Agreed with no one or both of them
2	Agreed with Cem, no reason was stated
3	Agreed with Cem, reason was stated
4	Agreed with İlknur, no reason was stated
5	Agreed with İlknur, reason which is not related to a proof method was stated
6	Agreed with İlknur, reason which is related to refutation was stated

SECTION C

Question 1

0- No answer

1- Invalid proof

For example: “Let’s add the numbers from 1 to 5.

1+2+3+4+5=15

$\frac{n.(n+1)}{2} = \frac{5.6}{2} = 15$

Since they are equal, it can be said that the formula is true.”

2- Valid proof

For example: “For n=1, $\frac{1.2}{2} = 1$

Assume that 1+2+3…..n = $\frac{n.(n+1)}{2}$ is true for n.

And it must be true for n+1.

1+2+3….n+(n+1) = $\frac{(n+1).(n+2)}{2}$

$\frac{n.(n+1)}{2}$ + (n+1) = $\frac{n^2+3n+2}{2}$

$\frac{n.(n+1)+2n+2}{2} = \frac{n^2+3n+2}{2}$

The proof is complete.”

Question 2

0- No answer

1- Invalid proof

For example: "Assume that a=1 and b=3

$a^2=1$ $b^2=9$

$b^2>a^2$"

2- Valid proof

For example: "$0<a<b$ multiply each term by a

$0<a^2<ba$ then also multiply each term by b

$0<ab<b^2$ then $0<a^2<ba<b^2$

So, $a^2<b^2$"

Question 3

0- No answer

1- Invalid proof

For example: “n^3-n=n(n^2-1)=n.(n-1).(n+1)

For n=1, 1.0.2=0

For n=2, 2.1.3=6

For n=3, 3.2.4=24

……

3 | n^3-n.”

2- Valid proof

For example: “3 | n^3- n

For n=1, 1^3-3=0 and the number 3 divides 0.

Assume that it is true for n=n. 3 | n^3-n

For n=n+1, $(n+1)^3$-(n+1)= n^3+3n^2+3n+1-1

= n^3-n+3(n^2+n)

So, n^3-n+3(n^2+n) is divided by 3, 3(n^2+n) is a multiple of 3.

Thus, we showed that the statement is true for n=n+1

By induction, for all natural number n, 3 | n^3 – n.”

APPENDIX C

TEZ FOTOKOPİ İZİN FORMU

ENSTİTÜ

Fen Bilimleri Enstitüsü ☐

Sosyal Bilimler Enstitüsü ☒

Uygulamalı Matematik Enstitüsü ☐

Enformatik Enstitüsü ☐

Deniz Bilimleri Enstitüsü ☐

YAZARIN

Soyadı: Demiray
Adı: Esra
Bölümü: İlköğretim Fen ve Matematik Eğitimi

TEZİN ADI (İngilizce) :
An Investigation of Pre-Service Middle School Mathematics Teachers' Achievement Levels in Mathematical Proof and The Reasons of Their Wrong Interpretations

TEZİN TÜRÜ : Yüksek Lisans ☒ Doktora ☐

1. Tezimin tamamı dünya çapında erişime açılsın ve kaynak gösterilmek şartıyla tezimin bir kısmı veya tamamının fotokopisi alınsın. ☐

2. Tezimin tamamı yalnızca Orta Doğu Teknik Üniversitesi kullanıcılarının erişimine açılsın. (Bu seçenekle tezinizin fotokopisi ya da elektronik kopyası Kütüphane aracılığı ile ODTÜ dışına dağıtılmayacaktır.) ☐

3. Tezim bir (1) yıl süreyle erişime kapalı olsun. (Bu seçenekle tezinizin fotokopisi ya da elektronik kopyası Kütüphane aracılığı ile ODTÜ dışına dağıtılmayacaktır.) ☒

Yazarın imzası .... Tarih ....