RICCI AND COTTON FLOWS IN THREE DIMENSIONS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

KEZBAN TAŞSETEN ATA

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
PHYSICS

AUGUST 2013

Approval of the thesis:

RICCI AND COTTON FLOWS IN THREE DIMENSIONS

submitted by **KEZBAN TAŞSETEN ATA** in partial fulfillment of the requirements for the degree of **Master of Science in Physics Department, Middle East Technical University** by,

Prof. Dr. Canan Özgen
Dean, Graduate School of **Natural and Applied Sciences** ________________

Prof. Dr. Mehmet T. Zeyrek
Head of Department, **Physics** ________________

Prof. Dr. Bayram Tekin
Supervisor, **Physics Department, METU** ________________

Examining Committee Members:

Prof. Dr. Atalay Karasu
Physics Department, METU ________________

Prof. Dr. Bayram Tekin
Physics Department, METU ________________

Assoc. Prof. Dr. Seçkin Kürkçüoğlu
Physics Department, METU ________________

Assoc. Prof. Dr. Kostyantyn Zheltukhin
Mathematics Department, METU ________________

Assist. Prof. Dr. Çetin Ürtiş
Mathematics Department, TOBB-ETÜ ________________

Date: ________________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: KEZBAN TAŞSETEN ATA

Signature :

ABSTRACT

RICCI AND COTTON FLOWS IN THREE DIMENSIONS

Ata, Kezban Taşseten
M.S., Department of Physics
Supervisor : Prof. Dr. Bayram Tekin

August 2013, 92 pages

In this thesis, we give a detailed review of the Ricci and Cotton flows in 3 dimensional geometries. We especially study the flows of Thurston's 9 geometries which are used to classify 3 dimensional manifolds.

Keywords: Ricci flow, Cotton flow, Cotton solitons

ÖZ

ÜÇ BOYUTTA RICCI VE COTTON AKILARI

Ata, Kezban Taşseten
Yüksek Lisans, Fizik Bölümü
Tez Yöneticisi : Prof. Dr. Bayram Tekin

Ağustos 2013, 92 sayfa

Bu tezde 3 boyutlu geometriler için Ricci ve Cotton akılarını ayrıntılı bir şekilde çalıştık. Özellikle Thurston'un 3 boyutlu çokkatlıları sınıflandırmak için kullandığı 9 geometrinin akılarını inceledik.

Anahtar Kelimeler: Ricci akısı, Cotton akısı, Cotton solitonları

To my family

ACKNOWLEDGEMENTS

I am thankful to my supervisor Prof. Dr. Bayram Tekin for his patience and support during the writing process of this thesis. I am also thankful to Çağatay Menekay, Mehmet Şensoy and Deniz Devecioglu for their technical support.
I would like to thank to my family for their endless and unconditional love.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

This thesis is intended to study the three dimensional homogeneous manifolds under the Ricci and Cotton flows. The flows are geometric tools which are used to solve topological classification problems. The general equation of the flows is $\partial_t g_{ij} = \varepsilon_{ij}$, where ε_{ij} is a contraction of the curvature tensor. The problem underlying the introduction of these flows is if there is a locally homogeneous metric g_0 what will be $g(t)$ where t is an evolution parameter. Historically the Ricci flow introduced by Richard Hamilton [12] is aimed to prove Thurston's Geometrization Conjecture which is a more general restatement of the Poincaré Conjecture.

Thurston's Geometrization Conjecture states that any closed three dimensional manifold can be canonically decomposed into submanifolds with unique and homogeneous geometries [23]. In three dimensions if a manifold is compact and has no boundary then it is closed. The decomposition of the closed manifolds is done by the connected sum operation #. The connected sum operation in three dimensions is to cut a three-ball from each manifold and then to glue them from the two-sphere boundaries. In the case of orientable manifolds the decomposition is into a finite number prime factors [16] and is unique [17]. A three-manifold is called non-trivial if it is not isomorphic to three-sphere. A non-trivial three-manifold is called prime if there is no decomposition of it like $M_1 \# M_2$ where M_1 and M_2 are non-trivial, in other words at least one of M_1 or M_2 must be isomorphic to three-sphere to decompose a non-trivial three-manifold like that. The three-manifolds can be further decomposed by cutting them along two-tori as the result of the Torus decomposition theorem. This decomposition requires more elaborate explanation but for the moment it is enough to say that it is unique and involves finite number of submanifolds at least for compact, orientable and prime three-manifolds. These unique submanifolds have one of the so called eight model geometries.

Therefore the study of three-manifolds is reduced to the study of these geometries. Each of these geometries can be thought as having locally homogeneous Riemannian metrics on them, and when the submanifold having one of the eight model geometries is simply connected then the metric is globally homogeneous [4].

The Ricci flow is a partial differential equation used to evolve the metric g of a Riemannian manifold :

$$\partial_t g(t) = -2Ric(g(t)) \quad , g(0) = g_0 \quad .$$

$Ric(g(t))$ is the Ricci curvature tensor of the metric. The idea is to evolve the metric of a Riemannian manifold under this equation and if Thurston's Geometrization Conjecture is true then each manifold under the flow must evolve to a connected sum of the eight model geometries. However as it will be clear in Chapter 4 under the Ricci flow, for some geometries, singularities arise. One of the singularities is the shrinking to a point of a manifold with positive Ricci curvature in a finite t. This singularity is removed by a normalization term keeping the volume of the manifold constant. Let us see how the singularity arises in the case of the sphere. For an n-sphere the metric is $g = r^2 h$ where r is the radius of the n-sphere and h is the metric of the unit n-sphere and the associated Ricci curvature tensor is $Ric(g) = \frac{n-1}{r^2} g = (n-1)h$. Under the (unnormalized) Ricci flow this equation gives : $\partial_t g = -2\frac{n-1}{r^2} g \to \partial_t(r^2 h) = -2(n-1)h \to \partial_t(r^2) = -2(n-1) \to r^2(t) = r_0^2 - 2(n-1)t$, where r_0 is the initial radius of the n-sphere. Therefore in a finite $t = \frac{r_0^2}{2(n-1)}$ the n-sphere shrinks to a point. To remove this singularity a normalization term $\frac{2}{n}\mathfrak{R}$ is added to the equation [12], where $\mathfrak{R}$ is the curvature scalar. The new evolution equation is called the normalized Ricci flow equation. This process, to remove singularities by a normalization term is effective in simple cases like the one just described, but in more complicated situations is not. Thus Hamilton introduced a more general process called Ricci flow with surgery [13]. Using this process called Ricci flow with surgery Perelman proved Thurston's Geometrization Conjecture in three subsequent papers [19], [20] and [21].

The Cotton flow is another evolution equation of a metric on a Riemannian manifold introduced in [15]. This flow has two advantages over the Ricci flow which will be clear later: Firstly, it is already volume preserving and secondly, it has more fixed points among the eight model geometries. Despite these advantages the Cotton flow has a very important disadvantage, its short time existence is not proven yet and as of now it is an outstanding problem to be worked out. The short time existence and uniqueness of the Ricci flow are proven by first Hamilton [12] and later by Dennis DeTurck [8] .

In this thesis, without claiming original results, we give a somewhat detailed account of the basics of the Ricci and Cotton flows. Certain important results in the mathematical literature are often quoted without proof to keep the discussion short and to restrict the scope of the thesis to the one of a physicist. The outline of the thesis is as follows: Chapter 2 is devoted to the fundamental tools in manifolds and differential geometry. Chapter 3 is a brief introduction to the Lie groups. In Chapter 4 and 5 we study the Ricci and Cotton flows of the eight model geometries. In chapter 6 we give a brief description of the Ricci and Cotton solitons.

The list below shows the definition of some of the symbols used in the text.

$\emptyset$	the empty set.
$\in$	$p \in M$, p is an element of M.
$\cup$	$A \cup B$, the union of sets A and B.
$\cap$	$A \cap B$, the intersection of sets A and B.
$:\rightarrow$	$f : A \rightarrow B$, f is a map from the set A to the set B.
$\mathbb{R}$	the set of real numbers.
$\mathbb{C}$	the set of complex numbers.
$\mathbb{R}^n$	the set of n-tuples of real numbers.
$\mathbb{C}^n$	the set of n-tuples of complex numbers.
$\circ$	$f \circ g$, the composition of maps f and g.

CHAPTER 2

FUNDAMENTALS

This chapter is a quick review of the concepts needed in the study of geometric flows starting with the concept of a topological space. The definitions and results of this chapter are based on [9], [11], [22], [24], [25] and [3].

2.1 Manifolds

A *topological space* (χ, τ) consists of a set of points χ and a topology τ which is a choice of subsets of χ such that :

$Ti.)$ the empty set $\emptyset$ and χ belong to τ that is $\emptyset \in \tau$ and $\chi \in \tau$,

$Tii.)$ the union of any number of elements τ is again an element of τ,

$Tiii.)$ the intersection of any finite number of elements of τ is again an element of τ.

The family τ of subsets of χ is said to form a topology on χ and the elements of τ are called the *open sets* $\{\mathscr{O}\}$ of χ.

A *neighbourhood* $\mathscr{N}$ of a point $p \in \chi$ is a subset of χ which contains at least one subset of τ which contains the point p.

A topological space (χ, τ) is said to be *Hausdorff* if for each pair of distinct points of χ, such as $p, q \in \chi$ where $p \neq q$, one can find neighbourhoods $\mathscr{N}_p, \mathscr{N}_q \in \tau$ such that $p \in \mathscr{N}_p$, $q \in \mathscr{N}_q$, and $\mathscr{N}_p \cap \mathscr{N}_q = \emptyset$.

A topological space χ is said to be *compact* if every infinite sequence of points $p_1, p_2, ..., (p_i \in \chi)$ contains a subsequence of points that converges to a point $q \in \chi$.

A topological space χ is said to be *connected* if it cannot be written as $\chi = \chi_1 \cup \chi_2$ where χ_1 and χ_2 are both open and $\chi_1 \cap \chi_2 = \emptyset$.

A *loop* in a topological space χ is a continuous map $\phi : [0,1] \to \chi$ such that $\phi(0) = \phi(1)$. If any loop in χ can be shrunk to a point, then χ is said to be *simply connected*.

Let $\phi : \chi_1 \to \chi_2$ be a mapping of the topological space (χ_1, τ_1) into the topological space (χ_2, τ_2). If $p_i \in \chi_1$, then $\phi(p_i) = q_i \in \chi_2$ is called the *image* of p_i under the mapping ϕ and the set of all points $p_1, p_2, ..., \in \chi_1$ that map onto a particular point $q \in \chi_2$ is called the inverse image of q. The mapping $\phi : \chi_1 \to \chi_2$ is said to be *continuous* if the inverse image of any open set in χ_2 is an open set in χ_1.

The mapping $\phi : \chi_1 \to \chi_2$ is called a *homeomorphism* if it is continuous, one-to-one, onto and its inverse mapping $\phi^{-1} : \chi_2 \to \chi_1$ is continuous. Then χ_1 and χ_2 are said to be *homeomorphic* to each other. Homeomorphic spaces have identical topological properties.

A *differentiable manifold* M of dimension n is a Hausdorff space (χ, τ) with a collection of mappings $\phi \in \Phi$ such that $\phi : \chi \to \mathbb{R}^n$ satisfying the following properties :

Mi.) each point $p \in \chi$ lies in at least one of the open sets $\mathscr{O}$ of τ that is $\{\mathscr{O}\}$ covers M,

Mii.) each ϕ is a one-to-one mapping of an open set $\mathscr{O}$ in χ into an open set $\mathscr{U}$ in $\mathbb{R}^n$, so that M is said to be "locally looks like" $\mathbb{R}^n$, that is to say M is patched from small pieces looking like $\mathbb{R}^n$,

Miii.) if $\mathscr{O}_p \cap \mathscr{O}_q \neq \emptyset$ that is $\mathscr{O}_p$ and $\mathscr{O}_q$, are two overlapping open sets in τ, then the mappings $\phi_p \circ \phi_q^{-1}$ and $\phi_q \circ \phi_p^{-1}$ (both mappings of $(\mathscr{O}_p \cap \mathscr{O}_q)$ into $\mathbb{R}^n$) are both continuous and differentiable .

The n-tuples $(x^1, ..., x^n)$ of $\phi(p)$ in $\mathbb{R}^n$ are called the *coordinates* of a point $p \in M$. The pair $(\mathscr{O}, \phi)$ is called the *coordinate chart* of M.

Let $(\mathscr{O}_1, \phi_1)$ and $(\mathscr{O}_2, \phi_2)$ be two coordinate charts on M, where $\mathscr{O}_1 \cap \mathscr{O}_2 \neq \emptyset$, two overlapping sets. Then a point $p \in (\mathscr{O}_1 \cap \mathscr{O}_2)$ has two images in $\mathbb{R}^n$, that is $\phi_1(p) = x^i$ and $\phi_2(p) = y^j$, so p can be expressed in terms of these two images $p = \phi_1^{-1}(x) = \phi_2^{-1}(y)$.

The map $\phi_1 \circ \phi_2^{-1} : \phi_2(\mathscr{O}_1 \cap \mathscr{O}_2) \to \phi_1(\mathscr{O}_1 \cap \mathscr{O}_2)$ is called a *transition map* and is a homeomorphism between the open sets of $\mathbb{R}^n$.

If the partial derivatives of order k or less of all $\{y^j\}$ with respect to $\{x^i\}$ exist and are continuous, then $(\mathscr{O}_1, \phi_1)$ and $(\mathscr{O}_2, \phi_2)$ are said to be C^k- related.

If all the transition maps are C^k-related (k-times differentiable on their domains of definition), then M is said to be C^k. If all the transition maps are C^∞, then M is said to be *smooth*.

Let M and M' be two manifolds of the same dimension n. If a map $\phi : M \to M'$ and its inverse $\phi^{-1} : M' \to M$ are C^∞, then the map ϕ is called a *diffeomorphism*, and M and M' are said to be *diffeomorphic*.

Let M and M' be two manifolds of dimension n and n'. The *product space* $M \times M'$ is an $n+n'$ dimensional manifold which consists of all pairs (p, p') where $p \in M$ and $p' \in M'$.

A *Riemannian Manifold* (M, g) is a smooth manifold M equipped with a Riemannian metric g, that is a smooth Euclidean inner product g_p on all of the tangent spaces T_pM at a point p on M.

The manifolds of interest for this study are assumed to be smooth, Hausdorff, orientable and paracompact. The first two properties are already described, for the last two see [24] for example. A manifold is orientable if all the transition maps are orientation preserving, but since the manifolds of interest are simply connected they are orientable, so there is no additional demand on them and paracompactness is a requirement to keep the manifolds of finite volume and more importantly to define integration over the manifolds.

2.2 Tangent Vectors and Tangent Spaces

A *function* f on M is a map from M into $\mathbb{R}$. Let F denote all C^∞ functions on M, then a *tangent vector* t at a point p on M is a map $t : F \to \mathbb{R}$ which satisfies :

ti.) $t(af+bg) = at(f)+bt(g)$ where $f, g \in F$ and $a, b \in \mathbb{R}$,

tii.) $t(gh) = ft(g)+gt(f)$.

A tangent vector defines a generalized directional derivative along a (parametrized) *curve* through the point p in M which is a map $\gamma(\lambda)$ from an open set of $\mathbb{R}$ into M . Hence the tangent vector given by a curve is $t = \frac{d}{d\lambda}$ where λ is the parameter of the curve.

The set of all tangent vectors at a point p on an n-dimensional manifold M is called the tangent space T_pM. It satisfies the following axioms :

i.) $a.(t+t') = a.(t'+t) = (a.t)+(a.t')$,

ii.) $(a+b).t = (a.t)+(b.t)$,

iii.) $(ab).t = a.(b.t)$,

iv.) $1.t = t$.

Here $a,b \in \mathbb{R}$, $t,t' \in T_pM$, $+$ denotes both the addition of vectors and of real numbers and . denotes the multiplication by real numbers. Hence T_pM is a linear vector space over the field of real numbers (see Chapter 3). The manifold M and the union of all tangent spaces at all points form the *tangent bundle* TM of M, a $2n$-dimensional manifold. An element of TM is a pair (p,t) where $p \in M$ and $t \in T_pM$.

The dual space to T_pM is the *cotangent space* T_p^*M at the point p on M and is a vector space obeying the axioms of a vector space as well. T_p^*M is the space of all linear maps from T_pM to the real numbers. The elements of T_p^*M are called *dual vectors* (or one-forms, or cotangent vectors) ω and are duals to tangent vectors in the sense that $\omega(t) \equiv t(\omega)$. The action of a vector on a cotangent vector, or vice versa, is called the *contraction* of t with ω and this operation gives a real number as the result. In other words $t : T_p^*M \to \mathbb{R}$ and $\omega : T_pM \to \mathbb{R}$. The simplest example of a dual vector is the differential df of a function f where $f \in F$. Therefore $df(t) = t(df) = \frac{d}{d\lambda} f$.

2.3 Coordinate Basis/Coordinate Transformations

Any n linearly independent vectors can form a basis for T_pM and any n linearly independent one-forms can form a basis for T_p^*M. Let us assume that there is a set of n basis vectors $\{e_i\}$ which spans T_pM then, this basis induces a dual basis of n one-forms $\{e^i\}$ which spans T_p^*M, so that $e^i(e_j) = \delta^i{}_j$. If there is a local coordinate system $\{x^i\}$ around a point p on M then, it is natural to choose $e_i = \frac{\partial}{\partial x^i} (\equiv \partial_i)$ as the basis for vector fields and $e^i = dx^i$ as the basis for one-form fields where $dx^j(\frac{\partial}{\partial x^i}) = \delta^j{}_i$ as it should be. This basis is called a *coordinate basis* (or natural basis). A vector t in this basis can be written as $t = t^i \frac{\partial}{\partial x^i}$ where $t^i = \frac{dx^i}{d\lambda}$ are the components of t in this basis. A one-form ω in this basis can be written as $\omega = \omega_j dx^j$ where ω_j are the components of ω in this basis. Hence the contraction of ω with t can be expressed as :

$$\omega(t) = \omega_j dx^j (t^i \frac{\partial}{\partial x^i}) = \omega_j t^i dx^j (\frac{\partial}{\partial x^i}) = \omega_j t^i \delta^j{}_i = \omega_j t^j = \omega_i t^i \quad . \tag{2.1}$$

Let $\{x^i\}$ and $\{y^j\}$ be two coordinate systems on the overlap of any pair of coordinate charts. Then as a result of the transformation of the basis $\frac{\partial}{\partial x^i}$ and dx^i,

$$\frac{\partial}{\partial x'^i} = \frac{\partial y^j}{\partial x'^i}\frac{\partial}{\partial y^j} \quad , \quad dx'^i = \frac{\partial x'^i}{\partial y^j} dy^j \quad , \tag{2.2}$$

the components of the tangent vector $t^i(x)$ and cotangent vector $\omega_i(x)$ transform respectively as,

$$t'^i = \frac{\partial x'^i}{\partial y^j} t^j \quad , \omega'_i = \frac{\partial y^j}{\partial x'^i} \omega_j \quad . \tag{2.3}$$

Therefore, a tangent vector transforms like the basis for cotangent vectors, in a *contravariant* way and a cotangent vector transforms like the basis for vectors, in a *covariant* way.

2.4 Riemannian Normal Coordinates

Euclid's fifth postulate also known as the parallel postulate states that "Given a line and a point outside the line, there is exactly one parallel to the line through the point." In a Euclidean space there is no need to suspect this postulate. Thus a vector at a point in a Euclidean space is easily transported to another point along a line without deviating it. Then two vectors parallel at a point will stay parallel at another point when they are transported.

In non-Euclidean spaces on the other hand, that is in curved spaces, the parallel postulate is no longer true. Therefore it is not possible to transport a vector defined at a point to another point without deviating it and so it is not possible to compare vectors at different points (see [22] or [24] for example). The only way to overcome this difficulty is to define a concept called *parallel transport* along a curve and for a vector t it is defined by the equation (which will be clear later) :

$$\frac{d}{d\lambda} t^i + \Gamma^i_{jk} \frac{dx^j}{d\lambda} t^k = 0 \quad . \tag{2.4}$$

In this equation λ is the parameter of the curve, $\{x^j\}$ are the local coordinates and Γ's are the Christoffel symbols (see Section 2.7). It is clear from this equation that parallel transport depends on the curve.

A *geodesic* is a curve that parallel transports its tangent vector. The geodesic equation is thus :

$$\frac{d^2x^i}{d\lambda^2}+\Gamma^i_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda}=0 \quad . \tag{2.5}$$

The *Riemannian normal coordinates* is a coordinate system at a point p that makes $\Gamma^i_{jk}=0$. This coordinate system is obtained by exponential mapping of T_pM into M. It is always possible to find such a coordinate system at a point p to make $\Gamma^i_{jk}(p)=0$ at that point. However Γ^i_{jk} will not be necessarily zero around p, hence their derivatives $\partial_l\Gamma^i_{jk}$ will not be equal to zero.

2.5 Tensors, Relative Tensors

A *tensor* T of type (or rank) (k,l) at a point p on M is a multi-linear map from k cotangent vectors and l vectors into $\mathbb{R}$ that is $T:\underbrace{T_p^*M\times...\times T_p^*M}_{k-times}\times\underbrace{T_pM\times...\times T_pM}_{l-times}\to\mathbb{R}$. The tensor product $\otimes$ satisfies the following properties :

i) associative $(T\otimes S)\otimes Q=T\otimes(S\otimes Q)$,

ii) distributive $T\otimes(S\oplus Q)=(T\otimes S)\oplus(T\otimes Q)$,

iii) not commutative $T\otimes S\neq S\otimes T$.

Here T,S,Q are tensors of type (k,l) at a point p on M and $\oplus$ is the tensor sum. Tensors of type (k,l) on a n dimensional M form a vector space of dimension n^{k+l} at p like in the case of vectors or cotangent vectors. The tensor product is defined for tensors of different type, but the tensor sum is not. The tensor product of two tensors of type (k,l) and (k',l') produces a new tensor of the type $(k+k',l+l')$ and the tensor sum of two tensors of the type (k,l) produces a new tensor of the type (k,l). A vector is a tensor of type $(1,0)$ and a cotangent vector is a tensor of type $(0,1)$. Therefore the product of a vector with a cotangent vector is well defined, but their sum is not. In the bases $\{e_i\}$ and $\{e^j\}$ a tensor T can be expanded as :

$$T=T^{i_1...i_k}{}_{j_1...j_l}e_{i_1}\otimes...\otimes e_{i_k}\otimes e^{j_1}\otimes...\otimes e^{j_l} \quad , \tag{2.6}$$

where $T^{i_1...i_k}{}_{j_1...j_l}$ are the components of the tensor T in the bases $\{e_i\}$ and $\{e^j\}$ and they are found by the action of the tensor on the bases $T^{i_1...i_k}{}_{j_1...j_l}=T(e^{i_1},...,e^{i_k};e_{j_1},...,e_{j_l})$. In different bases the components of the tensor T will be different, but the tensor T will not, so to speak, T is an invariant. For example a vector does not change when one changes the coordinate system, but the components of the vector change. The transformation rule of the components of T between two different bases is just a generalization of the covariant and contravariant types of Section 2.3 . (In the rest of the chapter $\otimes$ is suppressed).

From a tensor of type (k,l) , a tensor of type $(k-1,l-1)$ can be produced by *contraction* on pairs of indices, where one is a subscript and the other is a superscript. The following two contractions of the tensor T show that contractions of different pairs of indices produce different tensors.

$$T^{ijk}{}_{jl} = S^{ik}{}_{l} \quad , \quad T^{ijk}{}_{jl} \neq T^{ijk}{}_{lj} \quad . \tag{2.7}$$

A *symmetric tensor* is invariant under the interchange of its arguments (for example for a second rank symmetric tensor $T(X,Y) = T(Y,X)$ or $T_{ij} = T_{ji}$ where X are Y are vector fields), yet an *antisymmetric tensor* changes sign (for example for a second rank antisymmetric tensor $T(X,Y) = -T(Y,X)$ or $T_{ij} = -T_{ji}$). A second rank symmetric (antisymmetric) tensor can be written as :

$$T_{ij} = T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji}) \quad , \quad T_{ij} = T_{[ij]} = \frac{1}{2}(T_{ij} - T_{ji}) \quad . \tag{2.8}$$

The () is the shorthand notation for symmetrization and [] is the shorthand notation for anti-symmetrization.

The *metric tensor* g is a multi-linear, type $(0,2)$ symmetric tensor so that $g(X,Y) = g(Y,X)$ where X and Y are vectors. In a basis $\{e^i\}$, $g = g_{ij}e^ie^j$ where $g_{ij} = g(e_i,e_j) = g_{ji}$. It is always possible to find a basis at some point p on M to put the metric tensor into a canonical form diagonal (-1,...,-1,0,...,0,1,...,1). The trace of the canonical form is called the signature of the metric and is an invariant. If the metric is continuous and non-degenerate (diagonal has no zero entries) then its signature will be the same at every point. If g is non-degenerate then the inverse metric g^{-1} can be constructed so that $g^{jk}g_{ij} = \delta^k{}_i$ and $g^{ij}g_{ij} = n$ where n is the dimension of the manifold.

The metric serves as a map to define the distance between two vectors of X and Y of T_pM by an operation called the *inner product* as :

$$\langle X,Y \rangle = X.Y = g(X,Y) = g(X^ie_i, Y^je_j) = X^iY^jg(e_i,e_j) = X^iY^jg_{ij} = X^iY_i \quad . \tag{2.9}$$

The metric g and its inverse g^{-1} are maps between vectors and one-forms so that $g_{ji}t^j = t_i$ and $g^{ij}t_i = t^j$. Therefore these mappings are one-to-one and invertible. As in the case of mapping between vectors and one-forms g_{ij} maps (k,l) tensor fields into $(k-1,l+1)$ tensor fields and g^{ij} maps (k,l) tensor fields into $(k+1,l-1)$ tensor fields. These operations are called index lowering and index raising, respectively.

$$g_{ij}T^{klj}{}_{mn} = T^{kl}{}_{imn} \quad , \quad g^{ij}T^{kl}{}_{imn} = T^{klj}{}_{mn} \quad . \tag{2.10}$$

An object R is called a *relative tensor* of density w if it transforms like :

$$R^{i_1...i_p}{}_{j_1...j_r} = \left|\frac{\partial y}{\partial x}\right|^w \left(\frac{\partial x^{i_1}}{\partial y^{k_1}}...\frac{\partial x^{i_p}}{\partial y^{k_p}}\right)\left(\frac{\partial y^{l_1}}{\partial x^{j_1}}...\frac{\partial y^{l_r}}{\partial x^{j_r}}\right)R^{k_1...k_p}{}_{l_1...l_r} \quad , \tag{2.11}$$

where $\left|\frac{\partial y}{\partial x}\right|$ is the Jacobian of the transformation from y coordinates to x coordinates.

The Levi-Civita symbol in 3-dimensions ε^{ijk} is a tensor density of weight +1 (ε_{ijk} is of weight -1). It is defined as $\varepsilon^{ijk} = +1$ for even permutations, -1 for odd permutations, 0 if any index is repeated.

The determinant of the metric tensor $g = \det(g_{ij})$ is a tensor density of weight +2.

2.6 Differential Forms, Exterior Derivative, Interior Product and Hodge Dual

In the previous section the antisymmetrization of a second rank tensor is defined. More generally for a tensor of type $(0,p)$ the antisymmetrization is as below :

$T_A(X_1,...,X_p) = \frac{1}{p!}$(even permutations of $T(X_1,...,X_p)$ - odd permutations of $T(X_1,...,X_p)$) ,
$T_{[i_1,...,i_p]} = \frac{1}{p!}$(even permutations of $T_{i_1,...,i_p}$- odd permutations of $T_{i_1,...,i_p}$) .

A completely antisymmetric tensor of type $(0,p)$ for $p \geq 3$ is the one changes sign under the interchange of any of its arguments/indices. Hence for a completely antisymmetric tensor T, $T_A(X_1,...,X_p) = T(X_1,...,X_p)$ and $T_{[i_1,...,i_p]} = T_{i_1,...,i_p}$.

A tensor of type $(0,p)$ is called a *p-form* if it is completely antisymmetric in its all indices. On an n-dimensional manifold (M^n) , the space of all p-forms are denoted by $\bigwedge^p(M)$ and they exist if and only if $p \leq n$. Any p-form ω in a basis $\{e^i\}$ can be written as :

$$\omega_p = \frac{1}{p!}\omega_{i_1...i_p}e^{i_1} \wedge ... \wedge e^{i_p} \quad . \tag{2.12}$$

The function $\omega_{i_1\ldots i_p}$ is antisymmetric in all indices so that $\omega_{i_1\ldots i_r\ldots i_s\ldots i_p} = -\omega_{i_1\ldots i_s\ldots i_r\ldots i_p}$. The set of all p-forms at a point on M^n form a vector space and the dimension of this space is given by :

$$dim \bigwedge^{p}(M^n) = \frac{n!}{p!(n-p)!} \quad . \tag{2.13}$$

The *wedge product* $\wedge$ is an antisymmetrized vector multiplication which produces a $(p+q)$-form from a p-form and a q-form, $\wedge : \bigwedge^p \times \bigwedge^q \to \bigwedge^{p+q}$. For given two one-forms α and β:

$$(\alpha \wedge \beta)_{ij} = \alpha_i \beta_j - \alpha_j \beta_i = 2\alpha_{[i}\beta_{j]} \quad . \tag{2.14}$$

Let α_p and β_q be a p-form and a q-form, respectively and γ be any-form, here are some of the rules for them :

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) \quad , \tag{2.15}$$

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha \quad , \tag{2.16}$$

$$(\alpha + \beta) \wedge \gamma = \alpha \wedge \gamma + \beta \wedge \gamma \quad . \tag{2.17}$$

The *exterior derivative* d is a linear map from p-forms to $(p+1)$-forms $d : \bigwedge^p \to \bigwedge^{p+1}$ with the following properties :

i.) $d(\alpha + \beta) = d\alpha + d\beta$,

ii.) $df(X) = X(f) = Xf$ where X is a vector field and f is a 0-form (a function),

iii.) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ where α is a p-form and β is a q-form,

iv.) $d^2\alpha = d(d\alpha) = 0$ Poincaré Lemma.

For a p-form ω the exterior derivative is $d\omega = \frac{1}{p!} \partial_j \omega_{i_1\ldots i_p} e^j \wedge e^{i_1} \wedge \ldots \wedge e^{i_p}$.

The *interior product* $\imath_X$ is a linear map, an anti-derivative, from p-forms to $(p-1)$-forms $\imath_X : \bigwedge^p \to \bigwedge^{p-1}$ where X is a vector with the following properties :

i.) $\imath_X(\alpha + \beta) = \imath_X \alpha + \imath_X \beta$,

ii.) $\imath_X f = 0$ where f is a 0-form,

iii.) $\imath_X dx^i = \imath_{X^j \partial_j} dx^i = X^j \delta^i_j = X^i$, *iv.)* $\iota_X(\alpha \wedge \beta) = (\iota_X \alpha) \wedge \beta + (-1)^p \alpha \wedge (\iota_X \beta)$ where α is a p-form and β is a q-form,

v.) $\iota_X^2 = 0$.

On a manifold M^n, the dimension of $\bigwedge^p = \bigwedge^{n-p}$ as a result of the equation (2.12). Therefore there is a duality (isomorphism) between these two spaces. The *Hodge dual* $*$ is a map between the two spaces $* : \bigwedge^p \to \bigwedge^{n-p}$. Its action on the basis vectors :

$$*(dx^{i_1} \wedge \ldots \wedge dx^{i_p}) = \frac{\sqrt{|g|}}{(n-p)!} \varepsilon^{i_1 \ldots i_p}{}_{j_{p+1} \ldots j_n} dx^{j_{p+1}} \wedge \ldots \wedge dx^{j_n} \quad , \tag{2.18}$$

where ε is the previously defined tensor density with weight +1.

2.7 Curvature

In Section 2.4 the notion of parallel transport of a vector along a curve is described and an equation for this procedure is written down without giving any mathematical background. The parallel transport of a vector X along a curve γ with the directional derivative $Y = \frac{d}{d\lambda}$ (its tangent vector) is a derivative giving zero rate of change in X. This derivative is called the *covariant derivative* of X along Y and the parallel transport is defined by the equation $\nabla_Y X = 0$. ∇ is called an *affine connection* and for a Riemannian manifold with several properties defined below. The choice of the affine connection is unique and is called the Levi-Civita connection. The Levi-Civita connection turns out to be a differentiation tool of the tensor fields as mentioned below.

A covariant derivative ∇ (also shown with a semicolon ;) on a manifold M is a map from (k,l) tensor fields to $(k,l+1)$ tensor fields with the following properties:

i.) ∇ is a linear map so that if S and T are both tensors of type (k,l) and a, b are constants then $\nabla(aS + bT) = a\, \nabla(S) + b\, \nabla(T)$,

ii.) ∇ satisfies Leibniz Rule $\nabla(S \otimes T) = (\nabla S) \otimes T + S \otimes (\nabla T)$,

iii.) if f is a function on M and $t \in T_p M$,then $t(f) = t^i \nabla_i f$,

iv.) ∇ commutes with traces $\nabla_X(trace T) = trace(\nabla_X T)$,

and with two additional conditions which will hold for the ongoing discussion,

v.) Torsion free : $\nabla_i \nabla_j f = \nabla_j \nabla_i f$,

vi.) Metric compatibility: $\nabla g = 0$ this condition implies that the inner product between vectors is preserved, is an invariant.

For a given coordinate system $\{x^i\}$ and its basis $\{e_i\}$ the *Christoffel symbols* Γ^i_{jk} are uniquely defined by :

$$\nabla_{e_i} e_j = \Gamma^k_{ji} e_k \quad . \tag{2.19}$$

The Christoffel symbols are not components of a tensor, so they do not obey the transformation rules of the tensors. They are symmetric under the exchange of lower indices, that is, $\Gamma^k_{ij} = \Gamma^k_{ji}$ as the result of the torsion free condition. The torsion free condition is a property of the Levi-Civita connection which hereby defines the Christoffel symbols uniquely. $\nabla_{e_i} \equiv \nabla_i$ in the rest of the chapter. Let $\{e^i\}$ to denote the dual basis, then :

$$\nabla_i e^j = -\Gamma^j_{ki} e^k \quad . \tag{2.20}$$

For a contravariant vector field $V = V^i e_i$, the covariant derivative is then described as :

$$\nabla_i V = \nabla_i V^j e_j = (\partial_i V^j + V^k \Gamma^j_{ik}) e_j \quad . \tag{2.21}$$

Although ∇ is a mapping from (k,l) tensor fields to $(k,l+1)$ tensor fields, ∇_i along the vector field e_i (or along any vector field X) is obviously a mapping from (k,l) tensor fields to (k,l) tensor fields. Hence in abstract notation this equation can be written as :

$$\nabla(V) = (\partial_i V^j + V^k \Gamma^j_{ik}) e_j \otimes e^i \quad . \tag{2.22}$$

Similar results hold for higher rank tensors. For example for a mixed tensor of type $(1,1)$, the covariant derivative can be computed to be :

$$\nabla_i T^j{}_k = \partial_i T^j{}_k + \Gamma^j_{il} T^l{}_k - \Gamma^l_{ik} T^j{}_l \quad , \tag{2.23}$$

or equivalently $\nabla T = (\partial_i T^j{}_k + \Gamma^j_{il} T^l{}_k - \Gamma^l_{ik} T^j{}_l) e_j \otimes e^k \otimes e^i$.

For a given metric the first kind *Christoffel symbols* $[ij,k]$ in terms of the components of the metric are :

$$[ij,k] = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) \quad . \tag{2.24}$$

The Christoffel symbols Γ^i_{jk} that are introduced before, are called the second kind and they can be written in terms of the components of the metric as :

$$\Gamma^i_{jk} = g^{il}[jk,l] = \frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}) \quad . \tag{2.25}$$

It is obvious from these equations that $[jk,i]$ and Γ^i_{jk} are symmetric in j and k.

The *Riemann curvature tensor* R is a tensor of type $(1,3)$ and is defined by its action on the vector fields X, Y and Z as :

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z \quad . \tag{2.26}$$

The Riemann tensor is a measure of how much a vector deviates from its original position when it is parallel transported around a closed loop due to the non-flatness of the space. Its components in a basis $\{e_i\}$ are defined by the equation :

$$R(e_i,e_j)e_k = R^l{}_{kij}e_l \quad . \tag{2.27}$$

The action of the commutator $[\nabla_i,\nabla_j]$ on a contravariant vector field A^k is a transformation given by the equation $[\nabla_i,\nabla_j]A^k = R^k{}_{lij}A^l$. This transformation is due to the parallel transport of the vector field, [3]. Since $[\nabla_i,\nabla_j](B^kA_k) = 0$, a covariant vector field B_k will transform like $[\nabla_i,\nabla_j]B_k = -R^l{}_{kij}B_l$.

By using the previous results and the definition of the covariant derivative the components of the Riemann tensor can be expressed in terms of the Christoffel symbols as :

$$R^i{}_{jkl} = \partial_k\Gamma^i_{jl} - \partial_l\Gamma^i_{jk} + \Gamma^i_{km}\Gamma^m_{jl} - \Gamma^i_{lm}\Gamma^m_{jk} \quad . \tag{2.28}$$

The lowering down of its first index with the help of the metric components gives another expression for the components of the Riemann tensor in terms of the first kind Christoffel symbols which will be helpful in proving some identities :

$$R_{hjkl} = g_{ih}R^i{}_{jkl} = \partial_k[jl,h] - \partial_l[jk,h] + g^{mn}[jk,n][hl,m] - g^{mn}[jl,n][hk,m] \quad . \tag{2.29}$$

It is obvious from this last relation that the Riemann tensor is antisymmetric in the last two indices. The other symmetries of the Riemann tensor are :

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij} \quad . \tag{2.30}$$

The symmetries of the Riemann tensor can be used to derive two relations known as the first and the second Bianchi identities which are :

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0 \quad \rightarrow R_{i[jkl]} = 0 \quad , \tag{2.31}$$

$$\nabla_m R_{klij} + \nabla_k R_{lmij} + \nabla_l R_{mkij} = 0 \quad \rightarrow \nabla_{[m}R_{kl]ij} = 0 \quad . \tag{2.32}$$

As the result of the symmetries it has, the Riemann tensor has $\frac{1}{12}n^2(n^2-1)$ independent components in n dimensions.

Before computing the Ricci tensor , let us first derive a very useful relation:
$\partial_k(g^{ij}g_{ij}) = 0 \to g^{ij}(\partial_k g_{ij}) + (\partial_k g^{ij})g_{ij} = 0,$
$\partial_k(g^{ij}g_{lj}) = 0 \to \partial_k g^{im} = -g^{ij}g^{lm}\partial_k g_{lj} = -g^{ij}g^{lm}([lk,j]+[jk,l]) = -(g^{lm}\Gamma^i_{lk} + g^{ij}\Gamma^m_{jk}),$
$\partial_k \ln\sqrt{g} = \dfrac{1}{2g}\partial_k g = \dfrac{1}{2g}gg^{ij}(\partial_k g_{ij}) = -g(\partial_k g^{ij})g_{ij} = g(g^{lm}\Gamma^i_{lk} + g^{ij}\Gamma^m_{jk})g_{ij} = g(\Gamma^l_{lk} + \Gamma^l_{lk}).$

$$\partial_i \ln\sqrt{g} = \Gamma^j_{ji} \quad . \tag{2.33}$$

The *Ricci tensor*, *Ric*, is a symmetric tensor of type (0,2) and is the trace of the Riemann tensor. Its components are formed by the contraction of the first and the third indices of the Riemann tensor :

$$R_{ij} = R^k{}_{ikj} \quad . \tag{2.34}$$

The Ricci tensor is symmetric in its indices for a Riemannian manifold because $R_{ij} = R^k{}_{ikj} = R_{kj}{}^k{}_i = R_{ji}$. Hence it has $\frac{1}{2}n(n+1)$ independent components is n dimensions. Its components in terms of the Christoffel symbols can be computed using equation (2.28) to be :

$$R_{ij} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{kl}\Gamma^l_{ij} - \Gamma^k_{jl}\Gamma^l_{ik} \quad . \tag{2.35}$$

The scalar curvature $\mathfrak{R}$ is the trace of the Ricci tensor and is formed by the contraction of the indices of the components of the Ricci tensor :

$$\mathfrak{R} = g^{ij}R_{ij} \quad . \tag{2.36}$$

The second Bianchi identity $\nabla_{[m}R_{kl]ij} = 0$ can be used to obtain another identity and to define the *Einstein tensor*. Let us first contract twice the second Bianchi identity remembering that there is metric compatibility :

$$g^{ki}g^{lj}(\nabla_m R_{klij} + \nabla_k R_{lmij} + \nabla_l R_{mkij}) = 0 \to \nabla_m \mathfrak{R} - \nabla^i R_{mi} - \nabla^j R_{mj} = 0 \to \nabla^i R_{mi} = \frac{1}{2}\nabla_m \mathfrak{R} \, .$$

Hence $\nabla_i R^i{}_j = \frac{1}{2}\nabla_j \mathfrak{R} = \frac{1}{2}\partial_j \mathfrak{R}$. Since $g_{ij}\nabla^j = \nabla_i$, then $\nabla^j(R_{ij} - \frac{1}{2}g_{ij}\mathfrak{R}) = 0$ where $R_{ij} - \frac{1}{2}g_{ij}\mathfrak{R} = G_{ij}$ is the Einstein tensor, so $\nabla^j G_{ij} = 0$. The Einstein tensor is zero $G_{ij} = 0$ for a source free gravitational field.

2.8 Weyl Tensor, Cotton Tensor and Conformal Invariance

The Riemann tensor in dimensions $n \geq 4$ can be decomposed as:

$$R_{ijkl} = C_{ijkl} + \frac{1}{n-2}(g_{ik}R_{jl} + g_{jl}R_{ik} - g_{il}R_{jk} - g_{jk}R_{il}) - \frac{\mathfrak{R}}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}). \tag{2.37}$$

It is possible to express the equation (2.37) in terms of the Schouten tensor S which will be later useful $R_{ijkl} = C_{ijkl} + g_{ik}S_{jl} + g_{jl}S_{ik} - g_{il}S_{jk} - g_{jk}S_{il}$ where,

$$S_{ij} = \frac{1}{(n-2)}\left(R_{ij} - \frac{\mathfrak{R}}{2(n-1)}g_{ij}\right) \quad , \tag{2.38}$$

and C_{ijkl} are the components of a tensor known as the *Weyl tensor*. The Weyl tensor inherits the symmetries of the Riemann tensor :

$$C_{ijkl} = -C_{jikl} = -C_{ijlk} = C_{klij} \quad , \tag{2.39}$$

$$C_{ijkl} + C_{iklj} + C_{iljk} = 0 \quad . \tag{2.40}$$

The Weyl tensor is invariant under a conformal transformation, hence it is known as the *conformal tensor*.

The necessary and sufficient condition for two spaces V and $\tilde{V}$ to be conformal spaces is that their metric components g_{ij} and $\widetilde{g_{ij}}$ are related as :

$$\widetilde{g_{ij}} = e^{2\phi}g_{ij} \quad , \tag{2.41}$$

where ϕ is any function of the coordinates (of course V and $\tilde{V}$ should have the same coordinate system). In such a conformal transformation the angle θ between two vectors A^i and B^j is given by:

$$cos\theta = \frac{\widetilde{g_{ij}}A^iB^j}{\sqrt{(\widetilde{g_{kl}}A^kA^l)(\widetilde{g_{mn}}B^mB^n)}} = \frac{e^{2\phi}g_{ij}A^iB^j}{\sqrt{(e^{2\phi}g_{kl}A^kA^l)(e^{2\phi}g_{mn}B^mB^n)}} = \frac{g_{ij}A^iB^j}{\sqrt{(g_{kl}A^kA^l)(g_{mn}B^mB^n)}} \ .$$

This equation shows that under a conformal transformation all angles are preserved though lengths may not.

In order to show the conformal invariance of the Weyl tensor , first the relations of the inverse metric, the Riemann tensor and the Christoffel symbols are derived below.

The components of the inverse metric :

$i.)\ \widetilde{g^{ij}} = e^{-2\phi} g^{ij}$,

computed by using the property $\widetilde{g^{ij}}\widetilde{g_{kj}} = g^{ij} g_{kj} = \delta^i{}_k$.

The first kind Christoffel symbols :

$$ii.)\ \widetilde{[ij,h]} = \frac{1}{2}(\partial_j \widetilde{g_{ih}} + \partial_i \widetilde{g_{jh}} - \partial_h \widetilde{g_{ij}}) = e^{2\phi}([ij,h] + g_{ih}\phi_{,j} + g_{jh}\phi_{,i} - g_{ij}\phi_{,h}),$$

computed with the help of equation (2.24) and the relation $\partial_l \widetilde{g_{ij}} = e^{2\phi}(\partial_l g_{ij} + 2g_{ij}\partial_l \phi)$.

The second kind Christoffel symbols :

$$iii.)\ \widetilde{\Gamma^k_{ij}} = \widetilde{g^{kh}}\widetilde{[ij,h]} = (e^{-2\phi} g^{kh})\left\{e^{2\phi}([ij,h] + g_{ih}\phi_{,j} + g_{jh}\phi_{,i} - g_{ij}\phi_{,h})\right\},$$
$$= \Gamma^k_{ij} + (\delta^k{}_i \phi_{,j} + \delta^k{}_j \phi_{,i} - g^{kh} g_{ij}\phi_{,h}),$$

computed by using $i.)$ and $ii.)$.

$$iv.)\ \partial_l \widetilde{[ij.h]} = \partial_l \left\{e^{2\phi}([ij,h] + g_{ih}\phi_{,j} + g_{jh}\phi_{,i} - g_{ij}\phi_{,h})\right\},$$
$$= 2e^{2\phi}\phi_{,l}([ij,h] + g_{ih}\phi_{,j} + g_{jh}\phi_{,i} - g_{ij}\phi_{,h}) +$$
$$+ e^{2\phi}(\partial_l [ij,h] + g_{ih,l}\phi_{,j} + g_{ih}\phi_{,jl} + g_{jh,l}\phi_{,i} + g_{jh}\phi_{,il} - g_{ij,l}\phi_{,h} - g_{ij}\phi_{,hl}),$$

where $(\phi_{,il} \equiv \partial_l \partial_i \phi)$.

The components of the Riemann tensor :

$$v.)\ \widetilde{R_{ijkl}} = \partial_k \widetilde{[jl,i]} - \partial_l \widetilde{[jk,i]} + \widetilde{g^{mn}}(\widetilde{[jk,m]}\widetilde{[il,n]} - \widetilde{[jl,m]}\widetilde{[ik.n]}),$$
$$= e^{2\phi}(R_{ijkl} + g_{jk}\phi_{il} + g_{il}\phi_{jk} - g_{ik}\phi_{jl} - g_{jl}\phi_{ik} + [g_{jk}g_{il} - g_{ik}g_{jl}]\Delta\phi),$$

where $\phi_{il} = \phi_{,il} - \phi_{,i}\phi_{,l} - \phi_{,m}\Gamma^m_{il}$ and $\Delta\phi = g^{mn}\phi_{,m}\phi_{,n}$. This relation is computed with the help of equation (2.29) and $ii.)$ and $iv.)$.

The components of the Ricci tensor :

$$vi.)\ \widetilde{R_{jl}} = \widetilde{g^{ik}}\widetilde{R_{ijkl}} = R_{jl} + (2-n)\phi_{jl} + (1-n)g_{jl}\Delta\phi - g_{jl}g^{ik}\phi_{ik},$$
$$= R_{jl} + (2-n)\phi_{jl} + (2-n)g_{jl}\Delta\phi - g_{jl}\Theta,$$

where $\Theta = g^{ik}(\phi_{,ik} - \phi_{,m}\Gamma^m_{ik})$. This relation is computed by using $i.)$ and $v.)$.

The curvature scalar :

$$vii.)\ \widetilde{\mathfrak{R}} = \widetilde{g^{jl}}\widetilde{R_{jl}} = e^{-2\phi}(\mathfrak{R} + [2-2n]g^{jl}\phi_{jl} + n[1-n]\Delta\phi),$$
$$= e^{-2\phi}(\mathfrak{R} - [n-1][n-2]\Delta\phi - 2[n-1]\Theta),$$

computed by using $i.)$ and $vi.)$.

The components of the Schouten tensor S :
$vii.)\ \widetilde{S_{ij}} = \frac{1}{(n-2)}(\widetilde{R_{ij}} - \frac{\widetilde{\mathfrak{R}}}{2(n-1)}\widetilde{g_{ij}}) = S_{ij} - \phi_{ij} - \frac{1}{2}g_{ij}\Delta\phi,$
computed by using the equation (2.38) and $vi.)$ and $vii.)$.

Finally the components of the Weyl tensor :
$\widetilde{R_{ijkl}} = \widetilde{C_{ijkl}} + \widetilde{g_{ik}}\widetilde{S_{jl}} + \widetilde{g_{jl}}\widetilde{S_{ik}} - \widetilde{g_{il}}\widetilde{S_{jk}} - \widetilde{g_{jk}}\widetilde{S_{il}}$,
$e^{2\phi}R_{ijkl} = \widetilde{C_{ijkl}} + e^{2\phi}(g_{ik}S_{jl} + g_{jl}S_{ik} - g_{il}S_{jk} - g_{jk}S_{il})$, hence $\widetilde{C_{ijkl}} = e^{2\phi}C_{ijkl}$. Since $\widetilde{g^{ih}}\widetilde{C_{ijkl}} = \widetilde{C^h{}_{jkl}} = e^{-2\phi}g^{ih}e^{2\phi}C_{ijkl} = C^h{}_{jkl}$, the Weyl tensor of type $(1,3)$ is invariant under the conformal transformations.

$$\widetilde{C^h{}_{jkl}} = C^h{}_{jkl} \quad . \tag{2.42}$$

A space is said to be flat if its (Gaussian) curvature K is zero at every point of the space. The curvature K is related to the Riemann tensor as :

$$K = \frac{R_{ijkl}A^iA^kB^jB^l}{(g_{ik}g_{jl} - g_{il}g_{jk})A^iA^kB^jB^l} \quad , \tag{2.43}$$

where A^i and B^j are vector components. Since the numerator and the denominator of this equation are both invariants, the curvature K is an invariant as well. If K is zero then R_{ijkl} is zero too, which is the condition for a space to be flat. If $R^h{}_{ijk}$ is zero, then clearly $C^h{}_{ijk}$ is zero as well. Before moving forward it should be emphasized that the condition for a source free gravitational field $G_{ij} = 0$ implying that both R_{ij} and $\mathfrak{R}$ are equal to zero is not a sufficient condition for a space to be flat since the equation (2.36) shows that the Riemann tensor can not be uniquely determined by the Ricci tensor and the scalar curvature for $n \geq 4$, but it has a trace free part, namely the Weyl tensor.

A space with constant curvature is the one given by the relation :

$$R_{ijkl} = c(g_{ik}g_{jl} - g_{il}g_{jk}) \quad , \tag{2.44}$$

where c is a constant. It is the necessary and sufficient condition. If this equation is multiplied by the inverse metric it gives :
$g^{ik}R_{ijkl} = R_{jl} = cg^{ik}(g_{ik}g_{jl} - g_{il}g_{jk} = c(n-1)g_{jl},$
$g^{jl}R_{jl} = \mathfrak{R} = g^{jl}c(n-1)g_{jl} = cn(n-1)$ contracted condition for constant curvature.

A flat space with $K = 0$ is then a space with constant curvature with $c = 0$ and every space with constant curvature is conformal to a flat space. Let two spaces V and $\widetilde{V}$ be conformal, where the former is a constant curvature space and the latter is a flat space. Then clearly $\widetilde{C^h{}_{ijk}} = 0$, and $C^h{}_{ijk} = \widetilde{C^h{}_{ijk}} = 0$. Therefore for $n \geq 4$ a space is conformally flat if and only if $C^h{}_{ijk} = 0$.

In 3 dimensions, the Weyl tensor vanishes identically because the Riemann tensor and the Ricci tensor both have the same number of independent components, which is 6, hence the Riemann tensor can be uniquely represented by the Ricci tensor as :

$$R_{ijkl} = (g_{ik}R_{jl} + g_{jl}R_{ik} - g_{il}R_{jk} - g_{jk}R_{il}) - \frac{1}{2}\mathfrak{R}(g_{ik}g_{jl} - g_{il}g_{jk}) \quad . \tag{2.45}$$

Let us write the definition of the Weyl tensor again and use the result of second Bianchi identity :

$$C^h{}_{jkl} = R^h{}_{jkl} - \frac{1}{n-2}(\delta^h_k R_{jl} + g_{jl}R^h{}_k - \delta^h_l R_{jk} - g_{jk}R^h{}_l) + \frac{\mathfrak{R}}{(n-1)(n-2)}(\delta^h_k g_{jl} - \delta^h_l g_{jk}), \tag{2.46}$$

$\nabla_l R^h{}_{ijk} + \nabla_j R^h{}_{ikl} + \nabla_k R^h{}_{ilj} = 0.$

Therefore,

$$\nabla_l C^h{}_{ijk} + \nabla_j C^h{}_{ikl} + \nabla_k C^h{}_{ilj} = \frac{1}{n-2}(\delta^h_j \pi_{ikl} + \delta^h_k \pi_{ilj} + \delta^h_l \pi_{ijk} + g_{ik}\pi^h{}_{jl} + g_{ij}\pi^h{}_{lk} + g_{il}\pi^h{}_{kj}), \tag{2.47}$$

where,

$$\pi_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(g_{ij}\nabla_k \mathfrak{R} - g_{ik}\nabla_j \mathfrak{R}), \tag{2.48}$$

or $\pi_{ijk} = (n-2)(\nabla_k S_{ij} - \nabla_j S_{ik})$. These are the components of a tensor know as the *Cotton tensor* because it was first introduced by the French mathematician Émile Clément Cotton, [6]. Now let us raise the first index of π_{ijk} :

$$\pi^h{}_{jk} = g^{ih}\pi_{ijk} = \nabla_k(R^h{}_j - \frac{1}{2(n-1)}\delta^h{}_j \mathfrak{R}) - \nabla_j(R^h{}_k - \frac{1}{2(n-1)}\delta^h{}_k \mathfrak{R}) .$$

From the above expression it is seen that π_{ijk} and $\pi^h{}_{jk}$ are antisymmetric in j and k. Let us contract h and j in the expression $\pi^h{}_{jk}$ to see that it is traceless :

$$\begin{aligned} \pi^j{}_{jk} &= \nabla_k R^j{}_j - \nabla_j R^j{}_k - \frac{1}{2(n-1)}(\delta^j_j \nabla_k \mathfrak{R} - \delta^j_k \nabla_j \mathfrak{R}), \\ &= \nabla_k \mathfrak{R} - \frac{1}{2}\nabla_k \mathfrak{R} - \frac{1}{2(n-1)}(n\nabla_k \mathfrak{R} - \nabla_k \mathfrak{R}) = 0. \end{aligned}$$

$$g^{ij}\pi_{ijk} = 0 \quad . \tag{2.49}$$

Since π_{ijk} is antisymmetric in j and k, the contractions of π_{ijk} are zero, so the Cotton tensor vanishes for dimension $n < 3$.

Now, in the equation (2.46) let us contract h and k :

$$C^k{}_{jkl} = C_{jl} = R^k{}_{jkl} - \frac{1}{n-2}(\delta^k{}_k R_{jl} + g_{jl} R^k{}_k - \delta^k{}_l R_{jk} - g_{jk} R^k{}_l) + \frac{\mathfrak{R}}{(n-1)(n-2)}(\delta^k{}_k g_{jl} - \delta^k{}_l g_{jk}),$$

$$= R_{jl} - \frac{1}{n-2}(nR_{jl} + g_{jl}\mathfrak{R} - R_{jl} - R_{jl}) + \frac{\mathfrak{R}}{(n-1)(n-2)}(n-1)g_{jl} = 0.$$

$$C_{ij} = 0 \quad . \tag{2.50}$$

The contractions of the Weyl tensor are zero as well showing that it vanishes for dimension $n < 4$. Using these two last results one can obtain the following relations :

$$\nabla_l C^j{}_{ijk} + \nabla_j C^j{}_{ikl} + \nabla_k C^j{}_{ilj} = \frac{1}{n-2}(\delta^j_j \pi_{ikl} + \delta^j_k \pi_{ilj} + \delta^j_l \pi_{ijk} + g_{ik}\pi^j{}_{jl} + g_{ij}\pi^j{}_{lk} + g_{il}\pi^j{}_{kj}),$$

$$\nabla_j C^j{}_{ikl} = \frac{1}{n-2}(n\pi_{ikl} + \pi_{ilk} + \pi_{ilk} + \pi_{ilk}) = \frac{(n-3)}{(n-2)}\pi_{ikl} = (n-3)(\nabla_l S_{ik} - \nabla_k S_{il}).$$

For a source free space, R_{ij} and $\mathfrak{R}$ are zero and since the Riemann tensor is uniquely defined with the Ricci tensor there is no need to define another tensor. However the Riemann tensor is not conformally invariant hence one needs to define a conformally invariant tensor in order to specify conformally flat spaces. From equation (2.48) it is seen that π_{ijk} is zero for a flat space and it can be shown that the Cotton tensor is invariant under conformal transformations, [6]. Therefore in three dimensions the necessary and sufficient condition for a space to be conformally flat is that its Cotton tensor vanishes. In three dimensions the Cotton tensor has 5 independent components [10] and these are $\pi_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} - \frac{1}{4}(\nabla_k g_{ij}\mathfrak{R} - \nabla_j g_{ik}\mathfrak{R})$. The properties of the Cotton tensor can be summarized as :

i.) it is antisymmetric in its second and third indices $\pi_{ijk} = -\pi_{ikj}$,

ii.) it is traceless $g^{ij}\pi_{ijk} = 0$,

iii.) $\pi_{ijk} + \pi_{jki} + \pi_{kij} = 0 \rightarrow \pi_{[ijk]} = 0$.

The first and second properties are already derived in the text and the third property can easily be derived using the expression for the components of the Cotton tensor. Let us write again the expression for $\pi^h{}_{jk}$, $g^{ih}\pi_{ijk} = \pi^h{}_{jk} = \nabla_k R^h{}_j - \nabla_j R^h{}_k - \frac{1}{4}(\nabla_k \delta^h{}_j \mathfrak{R} - \nabla_j \delta^h{}_k \mathfrak{R})$. An equivalent form of this expression denoted by $C l^{lh}$ was first introduced by York, so the Cotton tensor is also known as the Cotton-York tensor :

$$C\prime^{lh} = -\frac{1}{2}\varepsilon^{ljk} g^{ih}\pi_{ijk} = \varepsilon^{ljk}\nabla_j (R^h{}_k - \frac{1}{4}\delta^h{}_k\mathfrak{R}) \quad .$$

This new representation $C\prime^{lh}$ of the Cotton tensor is no longer conformally invariant because of the fact that $\widetilde{\pi_{ijk}} = \pi_{ijk}$, but $\widetilde{g^{ih}} = e^{-2\phi} g^{ih}$ and so $\widetilde{C\prime^{lh}} = e^{-2\phi} C\prime^{lh}$. Since $\widetilde{g} = e^{6\phi} g$ the $\frac{5}{3}$-weight form $C^{lh} = g^{1/3} C\prime^{lh}$ of the Cotton tensor is conformally invariant. Therefore in three dimensions a space is conformally flat if and only if $C^{lh} = 0$ where,

$$C^{lh} = g^{1/3}\varepsilon^{ljk}\nabla_j (R^h{}_k - \frac{1}{4}\delta^h{}_k\mathfrak{R}) \quad . \tag{2.51}$$

The properties of the Cotton-York tensor C^{lh} are :

i.) it is symmetric in its indices $C^{lh} = C^{hl}$,

ii.) it is divergence-free $\nabla_h C^{lh} = 0$,

iii.) it is traceless $g_{lh} C^{lh} = 0$.

The derivation of i.) : If a symmetric tensor S and an antisymmetric tensor A are multiplied the result is simply zero $SA = 0$. Let us multiply the Cotton tensor with the totally antisymmetric tensor density ε_{lhm}, so to prove that the Cotton tensor is symmetric in l and h :

$$\begin{aligned}
\varepsilon_{lhm} C^{lh} &= \varepsilon_{lhm} g^{1/3}\varepsilon^{ljk}\nabla_j (R^h{}_k - \frac{1}{4}\delta^h{}_k\mathfrak{R}) \,, \\
&= g^{1/3}(\delta^j{}_h\delta^k{}_m - \delta^j{}_m\delta^k{}_h)\nabla_j (R^h{}_k - \frac{1}{4}\delta^h{}_k\mathfrak{R}) \,, \\
&= g^{1/3}\left\{\nabla_j (R^j{}_m - \frac{1}{4}\delta^j{}_m\mathfrak{R}) - \nabla_m (R^k{}_k - \frac{1}{4}\delta^k{}_k\mathfrak{R})\right\} , \\
&= g^{1/3}\left\{\frac{1}{4}\nabla_m\mathfrak{R} - \frac{1}{4}\nabla_m\mathfrak{R}\right\} = 0.
\end{aligned}$$

In the first line the identity $\varepsilon_{lhm}\varepsilon^{ljk} = \delta^j{}_h\delta^k{}_m - \delta^j{}_m\delta^k{}_h$ is used. In the third line the contacted Bianchi identity $\nabla_j (R^j{}_m - \frac{1}{2}\delta^j{}_m\mathfrak{R}) = 0$ and the contractions $R^k{}_k = \mathfrak{R}$ and $\delta^k{}_k = n$ where n is the dimension of the space are used.

The derivation of ii.) :

$$\begin{aligned}
\nabla_h C^{lh} &= g^{1/3}\varepsilon^{ljk}\nabla_h\nabla_j (R^h{}_k - \frac{1}{4}\delta^h{}_k\mathfrak{R}), \\
&= g^{1/3}\varepsilon^{ljk}(\ \nabla_j\nabla_h R^h{}_k + R_{hj}{}^h{}_m R^m{}_k + R_{hjk}{}^m R^h{}_m) = \varepsilon^{ljk} R^h{}_m R^m{}_{kjh}, \\
&= \varepsilon^{ljk} R^h{}_m \left\{(\delta^m{}_j R_{kh} + g_{kh} R^m{}_j - \delta^m{}_h R_{kj} - g_{kj} R^m{}_h) - \frac{1}{2}\mathfrak{R}(\delta^m{}_j g_{kh} - \delta^m{}_h g_{kj})\right\} = 0.
\end{aligned}$$

In the first line the expression for $[\nabla_h, \nabla_j](R^h{}_k - \frac{1}{4}\delta^h{}_k\mathfrak{R})$ and the property $[\nabla_h, \nabla_j]\mathfrak{R} = 0$ are used. In the second line the contacted Bianchi identity is used.

The derivation of iii.) :

$$
\begin{aligned}
g_{lh}C^{lh} &= g_{lh}g^{1/3}\varepsilon^{ljk}\nabla_j(R^h{}_k - \frac{1}{4}\delta^h{}_k\mathfrak{R})\ , \\
&= -\frac{1}{2}g_{lh}g^{1/3}\varepsilon^{ljk}g^{ih}\pi_{ijk} = -\frac{1}{2}g^{1/3}\delta_l{}^i\varepsilon^{ljk}\pi_{ijk}, \\
&= -\frac{1}{2}g^{1/3}\varepsilon^{ijk}\pi_{ijk} = -\frac{1}{2}g^{1/3}\pi_{[ijk]} = 0.
\end{aligned}
$$

Since the Cotton-York tensor is symmetric in its indices it is equal to its symmetric part $C^{ij} = C^{(ij)} = \frac{1}{2}(C^{ij} + C^{ji})$. This symmetrization gives another representation of the tensor :

$$
\begin{aligned}
C^{ij} &= \frac{1}{2}g^{1/3}\left\{\varepsilon^{imn}\nabla_m(R^j{}_n - \frac{1}{4}\delta^j{}_n\mathfrak{R}) + \varepsilon^{jmn}\nabla_m(R^i{}_n - \frac{1}{4}\delta^i{}_n\mathfrak{R})\right\}, \\
&= \frac{1}{2}g^{1/3}(\varepsilon^{imn}\nabla_m R^j{}_n + \varepsilon^{jmn}\nabla_m R^i{}_n)\ ,
\end{aligned}
$$

because $\varepsilon^{imn}\nabla_m\delta^j{}_n\mathfrak{R} + \varepsilon^{jmn}\nabla_n\delta^j{}_m\mathfrak{R} = 0$. Hence,

$$
C^{ij} = \frac{1}{2}g^{1/3}(\varepsilon^{imn}\nabla_m R^j{}_n + \varepsilon^{jmn}\nabla_m R^i{}_n) \quad . \tag{2.52}
$$

2.9 Coframes

Let us introduce a new orthonormal basis vectors $\{e_a\}$ which are not derived from any coordinate system, so that the inner product is simply $g(e_a, e_b) = \delta_{ab}$ (only three dimensional case is treated). Then the old basis vectors $\{e_i\}$ and the new orthonormal basis vectors will be related as $e_a = h^i{}_a e_i$, where h is an invertible $n \times n$ matrix , n is the dimension of the space. It follows that $e_i = h_i{}^a e_a$ where $h_i{}^a$ are the components of the inverse matrix h^{-1}. If the orthonormal basis one-forms are denoted by $\{e^a\}$, so that $e^a(e_b) = \delta^a{}_b$, then they will be related to the old basis one-forms $\{e^i\}$ via the following equations $e^a = h_i{}^a e^i$ and $e^i = h^i{}_a e^a$. The metrics then will be related as $g_{ij}e^i e^j = \delta_{ab}e^a e^b$ where $g(e_i, e_j) = g_{ij} = h_i{}^a h_j{}^b\delta_{ab} = g(e_a, e_b)$.

The covariant derivative in this orthonormal basis has the same structure but the Christoffel symbols Γ^i_{jk} are replaced by the spin connection $\omega_j{}^a{}_b$, so that for a tensor $X^a{}_b$ the covariant derivative is like $\nabla_i X^a{}_b = \partial_i X^a{}_b + \omega_i{}^c{}_b X^c{}_b - \omega_i{}^a{}_c X^a{}_c$. The relation between the Christoffel symbols and the spin connections can be found to be $\Gamma^i_{jk} = h^i{}_a\partial_j h_k{}^a + h^i{}_a h_k{}^b\omega_j{}^a{}_b$ or equivalently $\omega_j{}^a{}_b = h_i{}^a h^k{}_b\Gamma^i_{jk} - h^k{}_b\partial_j h_k{}^a$.

The tensors with mixed indices can be considered as vector valued forms. For example $X_i{}^a$ is a $(1,1)$ tensor, but also a vector valued one-form. Therefore spin connection can be seen as a connection one-form as $\omega_j{}^a{}_b e^j = \omega^a{}_b$. Any tensor with antisymmetric indices like the Riemann tensor then can be considered as a tensor valued p-form similarly. The Riemann and the torsion two-forms can be expressed as :

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \qquad , \tag{2.53}$$

$$T^a = de^a + \omega^a{}_b \wedge e^b \qquad . \tag{2.54}$$

These are the Cartan structure equations.

CHAPTER 3

LIE GROUPS AND LIE ALGEBRA

This Chapter is covering the basics of group theory and of Lie groups in the intention to build the bridge between the Lie groups and the geometries on which the two flows are studied. The definitions and results of this chapter are based on [11], [23] and [18].

3.1 Groups and Algebras

A *group* is a set G with a defined group multiplication operation denoted by $\odot$ which satisfies the following axioms :

$Gi.)$ if $g, h \in G$ then, $g \odot h \in G$,

$Gii.)$ for every $g, h, k \in G$, $g \odot (h \odot k) = (g \odot h) \odot k$,

$Giii.)$ there exists an element $e \in G$ such that, $g \odot e = e \odot g = g$ for every $g \in G$,

$Giv.)$ for each $g \in G$ there exist $h \in G$ such that, $g \odot h = h \odot g = e$.

The operation $\odot$ is obviously a map of $G \times G$ into G. The first axiom is called the closure property of the group, while the second axiom is called the associativity property. The element e is called the identity element of the group and h in the fourth axiom is called the inverse of g and can be denoted by g^{-1}. The identity element and inverses are unique.

If additionally for every $g, h \in G$, $g \odot h = h \odot g$, then G is called abelian (or commutative), otherwise it is called non-abelian.

Examples of groups:

The set of real numbers $\mathbb{R}$ (or complex numbers $\mathbb{C}$) is an abelian group under the regular addition operation. The identity element is zero and the inverse of each $g \in \mathbb{R}$ is $-g$. The non-zero real numbers $\mathbb{R}-\{0\}$ (or non-zero complex numbers $\mathbb{C}-\{0\}$) is an abelian group under the regular multiplication.

The n-dimensional Euclidean space $\mathbb{R}^n$ is an abelian group under the vector addition operation. The identity element is the zero vector $\mathbf{0}$ and the inverse of each $g \in \mathbb{R}^n$ is $-g$.

The general linear group over the real numbers $\mathsf{GL}(n;\mathbb{R})$ which consists of $n \times n$ invertible matrices with real entries is a group under the matrix multiplication operation. This group is not abelian since the matrix multiplication is not necessarily commutative. The identity element is the unit matrix $\mathbf{I}$ and the inverse of each $g \in \mathsf{GL}(n;\mathbb{R})$ is the inverse matrix of g.

The general linear group over the complex numbers $\mathsf{GL}(n;\mathbb{C})$ which consists of $n \times n$ invertible matrices with complex entries is a group under the matrix multiplication operation and is not abelian with the same argument as in $\mathsf{GL}(n;\mathbb{R})$. The identity element is the unit matrix $\mathbf{I}$ and the inverse of each $g \in \mathsf{GL}(n;\mathbb{C})$ is the inverse matrix of g.

A subset S of G is a *subgroup* of G if it is itself a group with the same group multiplication operation as G. The identity element of G is also the identity element of S since the identity element is unique. The trivial subgroups of G are G itself and the group with the identity element e as the only group element.

Examples of subgroups:

The special linear group over the real numbers $\mathsf{SL}(n;\mathbb{R})$ which consists of $n \times n$ invertible real matrices with determinant 1 is a subgroup of $\mathsf{GL}(n;\mathbb{R})$.

The unitary group $\mathsf{U}(n)$, $n \geq 1$,which consists of $n \times n$ invertible complex matrices u such that $u\dagger = u^{-1}$ (where $u\dagger = (u^T)^*$ is the complex conjugate of the transpose of u and u^{-1} is the inverse of u) is a subgroup of $\mathsf{GL}(n;\mathbb{C})$.

The special unitary group $\mathsf{SU}(n)$, $n \geq 2$, which consists of $n \times n$ invertible complex matrices u with determinant 1 such that $u\dagger = u^{-1}$ is a subgroup of $\mathsf{U}(n)$ which is a subgroup of $\mathsf{GL}(n;\mathbb{C})$. Therefore $\mathsf{SU}(n)$ is a subgroup of $\mathsf{GL}(n;\mathbb{C})$.

The Heisenberg group H which consists of $3\times$ 3 real matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with $a,b,c \in \mathbb{R}$ is a subgroup of $\mathsf{GL}(3;\mathbb{R})$.

Let G and G' be two groups. A mapping ϕ of G into G', $\phi : G \to G'$, is a rule associating each element g of G to some element g' of G' such that $g' = \phi(g)$. If $\phi(g_1) \odot' \phi(g_2) = \phi(g_1 \odot g_2)$ for all $g_1, g_2 \in G$ (where $\odot$ and $\odot'$ are the defined group multiplication operations of G and G', respectively), then ϕ is called a *homomorphism*. If additionally ϕ is one-to-one (faithful) and onto so that an inverse is well defined and exists, then it is called an *isomorphism*.

Let G be a group . The elements $g_1, g_2 \in G$ are said to be equivalent or conjugate ($g_1 \sim g_2$) if $g g_1 g^{-1} = g_2$ for some $g \in G$. It can be shown that this relation is reflexive (g is equivalent to itself) , symmetric (if $g_1 \sim g_2$ then $g_2 \sim g_1$) and transitive (if $g_1 \sim g_2$ and $g_2 \sim g_3$ then $g_1 \sim g_3$). Since it is an equivalence relation one can define an equivalence class. Every element of G is in one and only one equivalence class which can be denoted by $[g]$ such that $[g]$ is the set of elements equivalent to g, $[g] := \{g_G \in G \mid g_G \sim g\}$. Two equivalent classes are either equal or disjoint (have no elements in common).

A subgroup S of a group G is an *invariant subgroup (or normal subgroup)* S_N of G "if for each $s \in S_N$ and for each $g \in G$, $gsg^{-1} \in S_N$.

A *quotient group (or factor group)* G/S_N , where G is a group and S_N an invariant subgroup of G that is connected, is a discrete group formed by using the equivalence classes. The elements of the same equivalence class are considered to be the same element in the quotient group. Hence the study of continuous groups is equivalent to the study of quotient groups and invariant subgroups separately. It can be thought that G and G/S_N are homomorphic by a map associating each member of G to its equivalence class in G/S_N.

A *field* F is a mathematical structure firstly being an abelian group under the operation $+$ called addition (with the identity element f_e for the addition operation) and secondly having another operation . called the scalar multiplication satisfying the following axioms :

$Fi.$) if $f_i, f_j \in F$ then, $f_i.f_j \in F$,

$Fii.$) for every $f_i, f_j, f_k \in F$, $f_i.(f_j.f_k) = (f_i.f_j).f_k$,

$Fiii.$) the identity element for the scalar multiplication is 1 so that, $f_i.1 = 1.f_i = f_i$ for every $f_i. \in F$,

$Fiv.$) there exists an inverse $f_i{}^{-1} \in F$ for each $f_i \in F$,but $f_i \neq f_e$ so that, $f_i.(f_i{}^{-1}) = (f_i{}^{-1}).f_i = f_i$,

$Fv.$) for every $f_i, f_j, f_k \in F$, $f_i.(f_j + f_k) = f_i.f_j + f_i.f_k$ and $(f_i + f_j).f_k = f_i.f_j + f_i.f_k$.

Examples of fields:

The real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ are two commonly used fields, both are abelian groups under the usual addition for real numbers and complex numbers. The identity element of both group is zero 0 under addition. Under the usual multiplication for real numbers and complex numbers both satisfy the properties $Fi.) - Fv.)$ and the identity element zero under addition has no inverse under multiplication.

A *linear vector space* V is a mathematical structure consisting of two sets ; the elements $\{v_i\} \in V$ called vectors and the elements $\{f_j\} \in F$ with two different operations ; the vector addition $(+)$ and the scalar multiplication $(.)$ which satisfy the following axioms :

$Vi.)$ if $v_i, v_j \in V$ then, $v_i + v_j \in V$,

$Vii.)$ for every $v_i, v_j, v_k \in V$, $v_i + (v_j + v_k) = (v_i + v_j) + v_k$,

$Viii.)$ there exists an identity element $v_e \in V$ such that, $v_i + v_e = v_e + v_i = v_i$ for every $v_i \in V$,

$Viv.)$ for all $v_i \in V$ the inverse of $v_i = (-v_i)$ such that, $v_i + (-v_i) = v_i + (-v_i) = v_0$,

$Vv.)$ for all $v_i, v_j \in V$, $v_i + v_j = v_j + v_i$.

Thus a linear vector space V is an abelian group under the vector addition. Additionally V must satisfy :

$i.)$ if $v_i \in V$ and $f_j \in F$ then, $f_j.v_i \in V$,

$ii.)$ for every $v_i \in V$ and $f_j, f_k \in F$, $f_j.(f_k.v_i) = (f_j.f_k).v_i$,

$iii.)$ $1 \in F$ is the identity element for the scalar multiplication so, $1.v_i = v_i.1 = v_i$ for every $v_i \in V$,

$iv.)$ for every $v_i, v_j \in V$ and $f_k, f_l \in F$, $(f_k + f_l).v_i = f_k.v_i + f_l.v_i$ and $f_k.(v_i + v_j)$.

Examples of linear vector spaces:

Every field F is a one dimensional vector space.

The n-tuples of real numbers $\mathbb{R}^n$ is a n dimensional vector space over the field of real numbers.

The set of real $n \times n$ real matrices forms a real $n \times n$ dimensional vector space under the regular matrix addition and the scalar multiplication by the real numbers. Hence it is a vector space over the field of real numbers.

A *linear algebra* is a mathematical structure consisting of a vector space V and a field F ; and

additionally an operation $\otimes$ called vector multiplication satisfying the following properties :
$Ai.)$ if $v_i, v_j \in V$ then, $v_i \otimes v_j \in V$,
$Aii.)$ if $v_i, v_j, v_k \in V$ then, $(v_i + v_j) \otimes v_k = v_i \otimes v_k + v_j \otimes v_k$ and $v_i \otimes (v_j + v_k) = v_i \otimes v_j + v_i \otimes v_k$.

The vector space of real $n \times n$ real matrices of the previous example forms an algebra when the regular matrix multiplication is defined to be the vector multiplication operation $\otimes$. The identity of this operation is the unit matrix **I**.

Homomorphism and isomorphism can be defined between all kinds of algebraic structures similarly. If the mapping is into a set of matrices it is called a *representation* Γ. Therefore all the matrix groups discussed so far are the representations of these groups.

3.2 Lie Algebra

In the previous section additional structures are added to a set to form a group, a field, a vector space and an algebra finally. By the same procedure a *Lie algebra* can be constructed. To form a Lie algebra from an algebra first an antisymmetric multiplication $X \otimes Y \equiv [X,Y] = -[Y,X]$ which satisfies a property called *derivation* $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z + Y \otimes (X \otimes Z)$ must be defined. The derivation property will then reduce to an identity called Jacobi identity :
$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$.

Therefore a Lie algebra $\mathfrak{g}$ of a Lie group G is a vector space with a bilinear map $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying (in addition to $Ai.)$ and $Aii.)$) the following properties:
$i.)$ $[X,Y] = -[Y,X]$ for all $X,Y \in \mathfrak{g}$ anti-symmetry,
$ii.)$ $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ for all $X,Y,Z \in V$ Jacobi identity.

The map $[.,.]$ is called the *Lie bracket* (or Lie product or commutator). This definition is true for all Lie algebras, conversely any algebra with a map satisfying these properties is a Lie algebra. The matrix groups of the previous section are Lie groups and the Lie bracket is defined to be $[X,Y] = XY - YX$ where XY implies the matrix multiplication of the matrices X and Y.

For each finite-dimensional Lie algebra, there is at least one Lie group with that Lie algebra. Conversely different Lie groups may have the same Lie algebra, so the mapping is one-to-one only in one direction. Fortunately as a result of Lie's (third) theorem only one of the groups associated with a Lie algebra is simply connected and this group is called the universal covering group.

For example, $\mathsf{Spin}(n)$ is the simply connected covering group of $\mathsf{SO}(n)$. In three dimensions $\mathsf{Spin}(3) = \mathsf{SU}(2)$. This means that the algebras of $\mathsf{SO}(3)$ and $\mathsf{SU}(2)$ are same up to isomorphism, but as a group they are not.

A continuous group has two structures on it which make it to be considered as a group algebraically and as a manifold topologically :
I. it is a topological structure consisting of,
  i.) an n dimensional manifold M, and
  ii.) a continuous mapping $\phi : M \times M \to M$ associating each pair of points $p, q \in M$ to another point $r \in M$ so that this mapping is compatible with group operations,
II. it is group of continuous transformations $f : M \times G_n \to G_n$ acting on a geometric space G_n with the properties closureness, associativity, uniqueness of the identity and uniqueness of the inverses.

A Lie group is the invariant subgroup of a continuous group with an analytic mapping ϕ on its domain of definition. The smooth structure as a manifold leads to a continuous group of transformations as a group. Therefore a Lie group is a continuous group with non-countable elements and is also a smooth manifold G together with a smooth map $G \times G \to G$ that makes G into a group . In the rest of this Chapter no more emphasize will given into the manifold structure of the Lie groups since it is beyond the scope of this thesis, see [11] for an extended study of this subject.

Let us now return back to the algebras. The tangent space T_eG of a Lie group at the identity has the structure of a Lie algebra $\mathfrak{g}$, that is $\mathfrak{g} := T_e(G)$. Let $M = G$ be the underlying manifold of the Lie group G that is acted on, and define the left translation of a point $p \in M$ by $g \in G$ as $\varphi_g p = gp$. The left translation is a diffeomorphism. Consider a vector $X \in T_eG$, then this vector field is left-invariant if $\varphi_{g*}X = X$ for all $g \in G$ where φ_{g*} is the push forward map (see Appendix). Since the Lie bracket of two left-invariant vector fields is still a left-invariant vector field then, $T_e(G)$ can be identified as the space of all left-invariant vector fields on a

Lie group, so as a Lie algebra.

For example, the elements u of the unitary group $\mathsf{U}(n)$ can be written as $u = exp(ih)$ where h is an hermitian matrix.

The bases of a Lie algebra (in fact of any algebra) are called *generators* which are smallest number of elements of the algebra generating the entire group. The dimension of the generators gives the dimension of the group (the method of finding the generators is not discussed here).

For example, the algebra of the group $\mathsf{SU}(2)$ has three generators (hence is of dimension three) which are $i\sigma_i$ where σ_i are the Pauli matrices :

$$r_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad r_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad , \quad r_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad .$$

The generators $\{X_i\}$ $i = 1, 2, ..., n$ of an algebra have the commutation relations,

$$[X_i, X_j] = C_{ij}{}^{k} X_k,$$

where $C_{ij}{}^{k}$ are called the *structure constants* which determine the structure of the algebra, so nearly of the group. By Lie's second theorem these structure constants are constant and by Lie's third theorem they are antisymmetric in the subscripts $C_{ij}{}^{k} = -C_{ji}{}^{k}$.

For example, the structure constants of the generators of the algebra of the group $\mathsf{SU}(2)$ are $\{1,1,1\}$ so that $[r_1, r_2] = C_{12}{}^{3} r_3$, $C_{12}{}^{3} = 1$ (or $\{-1,-1,-1\}$ when multiplied in the reverse order) .

3.3 The Eight Model Geometries in Three Dimensions

According to Thurston's geometrization conjecture in three dimensions there are eight model geometries : H^3, E^3, S^3, $H^2 \times E$, $S^2 \times E$, Nil geometry, the geometry of $SL(2,\mathbb{R})$ and Sol geometry, [23].

A model geometry (M, G) is a smooth, simply connected manifold M together with a Lie group G acting on M with isotropy subgroups which are for a point $p \in M$ the subgroups $G_p \in G$ that leave p invariant ($G_p = \{g \in G | gp = p\}$) so that M is isotropic under the action of this subgroups. This definition is not far from the definition of the left-invariant vector fields of the previous section. The condition of isotropy subgroups implies that M admits a G-invariant Riemannian metric, so that G acts on M transitively and continuously.

The eight model geometries are all homogeneous spaces, but they have different isotropy groups. The first group, H^3,E^3 and S^3, has three-dimensional isotropy groups. In other words they are isotropic in three dimension, loosely speaking they look the same in every direction. The second group, $H^2 \times E$, $S^2 \times E$, *Nil* geometry and the geometry of $SL(2,\mathbb{R})$, has one-dimensional isotropy groups. The third group, *Sol* geometry, has zero-dimensional isotropy groups.

The five of the eight model geometries can be realized as left-invariant metrics on unimodular (left-invariant metric preserves the volume of the group) Bianchi groups which are the classifications of the Lie algebras. The Bianchi groups representing the geometries are all simply connected and they are described below :

E^3 : There are two algebras for this geometry, algebras of the groups R^3 and $\mathsf{Isom}(\mathsf{R}^2)$ with the structure constants $\{0,0,0\}$ and $\{-1,-1,0\}$, respectively. $\mathsf{Isom}(\mathsf{R}^2)$ is the symmetry group of the Euclidean plane.
S^3 : The algebra for this geometry is the one of the group $\mathsf{SU}(2)$ with structure constants $\{1,1,1\}$.
Nil geometry : The algebra for this geometry is the one of Heisenberg group with structure constants $\{-1,0,0\}$. The Heisenberg group is the simply connected covering group of the nilpotent group in 3 dimensions.
$SL(2,\mathbb{R})$: The algebra for this geometry is the one of the group $\mathsf{SL}(2,\mathbb{R})$ with structure constants $\{-1,-1,1\}$.
Sol geometry : The algebra for this geometry is the one of the group $\mathsf{E}(1,1)$ with structure constants $\{-1,0,1\}$. $\mathsf{E}(1,1)$ is the symmetry group of the plane with flat Lorentz metric.

Two of the remaining three geometries, H^3 and $H^2 \times E$, can be represented as a left-invariant metric on the Bianchi groups. However, the study of the flows of these geometries is easy computing their curvature tensors via their metric. The only geometry that can not be represented as a left-invariant metric on the Bianchi groups is $S^2 \times E$.

3.4 Curvature on Left Invariant Metrics on Lie Groups in 3-dimensions

This section follows the computations of [18]. Let g be a left-invariant metric on the unimodular, simply connected Lie group G, then there exists a left-invariant orthogonal frame $F = \{F_i\}$, called the Milnor frame, such that :

$$[F_i, F_j] = C_{ij}{}^{k} F_k \quad , \tag{3.1}$$

where $C_{ij}^{\ \ k}$ are the previously defined structure constants. Let ∇ be a uniquely defined Riemannian connection (or covariant derivative) with the identities:

$$\nabla_X Y - \nabla_Y X = [X,Y] \quad , \tag{3.2}$$

$$\langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle = 0 \quad , \tag{3.3}$$

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2}(\langle [X,Y],Z \rangle - \langle [Y,Z],X \rangle + \langle [Z,X],Y \rangle) \quad , \tag{3.4}$$

where $\langle \nabla_X Y, Z \rangle$ defines the inner product. In an orthonormal coframe $\{E_i\}$, where $[E_i, E_j] = C\prime_{ij}^{\ \ k} E_k$,

$$\left\langle \nabla_{E_i} E_j, E_k \right\rangle = \left\langle [E_i, E_j], E_k \right\rangle = \frac{1}{2}(C\prime_{ij}{}^{k} - C\prime_{jk}{}^{i} + C\prime_{ki}{}^{j}) \quad . \tag{3.5}$$

Hence $\nabla_{E_i} E_j = \sum_k \frac{1}{2}(C\prime_{ij}^{\ \ k} - C\prime_{jk}^{\ \ i} + C\prime_{ki}^{\ \ j}) E_k$. It is clear that $\nabla_{E_i} E_j = -\nabla_{E_j} E_i$ and $\nabla_{E_i} E_i = 0$ which follows also from the properties of the structure constants.

$$[F_1, F_2] = \nu F_3, \quad [F_2, F_3] = \lambda F_1, \quad [F_3, F_1] = \mu F_2, \tag{3.6}$$

where $\mu, \nu, \lambda \in \{-1, 0, 1\}$ are the structure constants. In this Milnor frame, then the metric g can be written as :

$$g = A(t)\omega^1 \otimes \omega^1 + B(t)\omega^2 \otimes \omega^2 + C(t)\omega^3 \otimes \omega^3, \tag{3.7}$$

where $\{\omega^1, \omega^2, \omega^3\}$ are the dual basis one-forms to the orthogonal basis $\{F_1, F_2, F_3\}$, that is $\omega^i(F_j) = \delta^i{}_j$. Let then the dual basis one-forms to the orthonormal basis $\{E_1, E_2, E_3\}$ be $\{e^1, e^2, e^3\}$. Clearly in this coframe , the metric will be :

$$g = e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3, \tag{3.8}$$

where the relation between these basis is :

$$e^1 = \sqrt{A}\omega^1, \qquad e^2 = \sqrt{B}\omega^2, \qquad e^3 = \sqrt{C}\omega^3, \tag{3.9}$$

and since $e^i(E_j) = \delta^i{}_j$,

$$E_1 = \frac{F_1}{\sqrt{A}}, \qquad E_2 = \frac{F_2}{\sqrt{B}}, \qquad E_3 = \frac{F_3}{\sqrt{C}}. \tag{3.10}$$

Let us now compute the equations needed to find the Ricci tensor and the curvature scalar.

$$[E_1,E_2] = \nu\sqrt{\frac{C}{AB}}E_3, \qquad [E_2,E_3] = \lambda\sqrt{\frac{A}{BC}}E_1, \qquad [E_3,E_1] = \mu\sqrt{\frac{B}{AC}}E_2,$$

$$\nabla_{E_1}E_2 = \frac{1}{2}(C\prime_{12}{}^3 - C\prime_{23}{}^1 + C\prime_{31}{}^2)E_3 = \frac{1}{2}\left(\nu\sqrt{\frac{C}{AB}} - \lambda\sqrt{\frac{A}{BC}} + \mu\sqrt{\frac{B}{AC}}\right)E_3 \,.$$

Therefore one obtains :

$$\begin{aligned}
\nabla_{F_1}F_2 &= \frac{1}{2}\left(\frac{-\lambda A + \mu B + \nu C}{C}\right)F_3 \quad , \nabla_{F_2}F_1 = \frac{1}{2}\left(\frac{-\lambda A + \mu B - \nu C}{C}\right)F_3 \quad ,\\
\nabla_{F_1}F_3 &= \frac{1}{2}\left(\frac{\lambda A - \mu B - \nu C}{B}\right)F_2 \quad ,\nabla_{F_3}F_1 = \frac{1}{2}\left(\frac{\lambda A + \mu B - \nu C}{B}\right)F_2 \quad ,\\
\nabla_{F_2}F_3 &= \frac{1}{2}\left(\frac{\lambda A - \mu B + \nu C}{A}\right)F_1 \quad ,\nabla_{F_3}F_2 = \frac{1}{2}\left(\frac{-\lambda A - \mu B + \nu C}{A}\right)F_1 \quad .
\end{aligned}$$

Using these equations first the sectional curvatures and then the components of the Ricci tensor can be computed. The usefulness of this Milnor frame is that the Ricci tensor is diagonal which will be later very handy when studying the flows. As a result, the non-zero components of the Ricci tensor in the orthonormal frame are :

$$R_{11} = \frac{1}{2ABC}(\lambda^2A^2 - \mu^2B^2 - \nu^2C^2 + 2\mu\nu BC) \quad , \tag{3.11}$$
$$R_{22} = \frac{1}{2ABC}(-\lambda^2A^2 + \mu^2B^2 - \nu^2C^2 + 2\lambda\nu AC) \quad , \tag{3.12}$$
$$R_{33} = \frac{1}{2ABC}(-\lambda^2A^2 - \mu^2B^2 + \nu^2C^2 + 2\lambda\mu AB) \quad . \tag{3.13}$$

Therefore the non-zero components of the Ricci tensor in the orthogonal frame are :

$$R_{11} = \frac{A}{2ABC}(\lambda^2A^2 - \mu^2B^2 - \nu^2C^2 + 2\mu\nu BC) \quad , \tag{3.14}$$
$$R_{22} = \frac{B}{2ABC}(-\lambda^2A^2 + \mu^2B^2 - \nu^2C^2 + 2\lambda\nu AC) \quad , \tag{3.15}$$
$$R_{33} = \frac{C}{2ABC}(-\lambda^2A^2 - \mu^2B^2 + \nu^2C^2 + 2\lambda\mu AB) \quad . \tag{3.16}$$

The curvature scalar is then :

$$\mathfrak{R} = \frac{1}{2ABC}(-\lambda^2 A^2 - \mu^2 B^2 - \nu^2 C^2 + 2\lambda\mu AB + 2\lambda\nu AC + 2\mu\nu BC) \qquad . \tag{3.17}$$

CHAPTER 4

RICCI FLOW

4.1 Ricci Flow

Richard Hamilton as mentioned in Chapter 1 suggested to use the below evolution equation in order to prove Thurston's geometrization conjecture in three dimensions [12].

$$\partial_t g = -2Ric + \frac{2}{n}\mathfrak{R}g, \tag{4.1}$$

where n is the dimension of the manifold, hence $\partial_t g = -2Ric + \frac{2}{3}\mathfrak{R}g$ in three-dimension. This equation gives a deformation of the metric with the parameter t which is different from the local coordinates of the manifold under the flow. As mentioned before $\frac{2}{3}\mathfrak{R}g$ is the normalization term to ensure that the three-sphere S^3 (in n dimension S^n) remains invariant: That is its volume is conserved under the flow. Let us drive how the volume ($\sqrt{g}$) is conserved under the normalized Ricci flow :

Multiplying the equation (4.1) by g^{ij} and using $g_{ij}g^{ij} = 3$,
$(\partial_t g_{ij})g^{ij} = -2R_{ij}g^{ij} + \frac{2}{3}\mathfrak{R}g_{ij}g^{ij} = -2\mathfrak{R} + 2\mathfrak{R} = 0.$
Since $\partial_t\sqrt{g} = \frac{1}{2}\sqrt{g}(\partial_t g_{ij})g^{ij} = 0$, volume is conserved under the flow.

The evolution of some tensors under the Ricci flow follows as :

i. Since $\partial_t(g^{ij}g_{kj}) = (\partial_t g^{ij})g_{kj} + g^{ij}(\partial_t g_{kj}) = \partial_t(g^{ij}g_{kj}) = \partial_t \delta^i{}_k = 0$ we have,
$(\partial_t g^{ij})g_{kj} = -g^{ij}(\partial_t g_{kj})$ using equation (4.1) we obtain,

$(\partial_t g^{ij}) g_{kj} g^{lk} = (\partial_t g^{ij}) \delta_j{}^{l} = \partial_t g^{il} = -g^{ij}(\partial_t g_{kj}) = -g^{ij}(-2R_{kj} + \frac{2}{3}\mathfrak{R} g_{kj}) g^{lk} = -(-2R^{li} + \frac{2}{3}\mathfrak{R} g^{li})$.

Hence:

$\partial_t g^{ij} = -(-2R^{ij} + \frac{2}{3}\mathfrak{R} g^{ij})$.

For just convenience and simplicity let us call (as it is commonly used in the relevant literature) $-2R_{ij} + \frac{2}{3}\mathfrak{R} g_{ij} = h_{ij}$ then $\partial_t g_{ij} = h_{ij}$ and $\partial_t g^{ij} = -h^{ij}$.

ii. The evolution of the the Christoffel symbols given in (2.25) is,

$\partial_t \Gamma^i_{jk} = \frac{1}{2} \partial_t g^{il} (\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}) + \frac{1}{2} g^{il} (\partial_t \partial_j g_{kl} + \partial_t \partial_k g_{jl} - \partial_t \partial_l g_{jk})$.

Around a point p in normal coordinates $\partial_i g_{jk}(p) = 0$ and $\partial_i(p) = \nabla_i(p)$. As a result the left hand side vanishes of the above equation and $\partial_t \partial_i g_{jk} = \partial_i \partial_t g_{jk} = \partial_i h_{jk} = \nabla_i h_{jk}$ since ∂_i and ∂_t commute that is $[\partial_i, \partial_t] = 0$. Hence the Christoffel symbols satisfy the following evolution equation :

$\partial_t \Gamma^i_{jk} = \frac{1}{2} g^{il} (\nabla_j h_{kl} + \nabla_k h_{jl} - \nabla_l h_{jk})$.

The Christoffel symbols can be used to compute the evolution of the Riemann tensor under the flow as in *iii*. .

iii. The evolution of the Riemann tensor given in (2.28) is,

$\partial_t R^i{}_{jkl} = \partial_t \partial_k \Gamma^i_{jl} - \partial_t \partial_l \Gamma^i_{jk} + \partial_t (\Gamma^i_{km} \Gamma^m_{jl}) - \partial_t (\Gamma^i_{lm} \Gamma^m_{jk})$,

$\partial_t R^i{}_{jkl} = \partial_k \partial_t \Gamma^i_{jl} - \partial_l \partial_t \Gamma^i_{jk} + (\partial_t \Gamma^i_{km}) \Gamma^m_{jl} + \Gamma^i_{km} \partial_t \Gamma^m_{jl} - (\partial_t \Gamma^i_{lm}) \Gamma^m_{jk} - \Gamma^i_{lm} \partial_t \Gamma^m_{jk}$.

In normal coordinates around a point p, $\Gamma^i_{jk}(p) = 0$, hence the last four expressions on the right hand side vanish and using the previous result,

$\partial_t R^i{}_{jkl} = \nabla_k \frac{1}{2} g^{im} (\nabla_j h_{lm} + \nabla_l h_{jm} - \nabla_m h_{jl}) - \nabla_l \frac{1}{2} g^{im} (\nabla_j h_{km} + \nabla_k h_{jm} - \nabla_m h_{jk})$,

$\partial_t R^i{}_{jkl} = \frac{1}{2} g^{im} \nabla_k (\nabla_j h_{lm} + \nabla_l h_{jm} - \nabla_m h_{jl}) - \frac{1}{2} g^{im} \nabla_l (\nabla_j h_{km} + \nabla_k h_{jm} - \nabla_m h_{jk})$,

$\partial_t R^i{}_{jkl} = \frac{1}{2} g^{im} (\nabla_k \nabla_j h_{lm} + \nabla_k \nabla_l h_{jm} - \nabla_k \nabla_m h_{jl} - \nabla_l \nabla_j h_{km} - \nabla_l \nabla_k h_{jm} + \nabla_l \nabla_m h_{jk})$.

iv. The evolution of the Ricci tensor given in (2.34) is,

$\partial_t R_{ij} = \partial_t R^k{}_{ikj} = \frac{1}{2} g^{km} (\nabla_k \nabla_i h_{jm} + \nabla_k \nabla_j h_{im} - \nabla_k \nabla_m h_{ij} - \nabla_j \nabla_i h_{km} - \nabla_j \nabla_k h_{im} + \nabla_j \nabla_m h_{ik})$.

Using the fact that the metric is covariantly conserved the last two expressions cancel each other, $g^{km}(-\nabla_j \nabla_k h_{im} + \nabla_j \nabla_m h_{ik}) = -\nabla_j \nabla_k h_i{}^k + \nabla_j \nabla_m h_i{}^m = 0$,we obtain,

$\partial_t R_{ij} = \frac{1}{2} g^{km} (\nabla_k \nabla_i h_{jm} + \nabla_k \nabla_j h_{im} - \nabla_k \nabla_m h_{ij} - \nabla_j \nabla_i h_{km})$.
In the volume normalized Ricci flow the last expression vanishes $g^{km} h_{km} = 0$.

v. The evolution of the curvature scalar given in (2.36) is,
$\partial_t \mathfrak{R} = \partial_t (g^{ij} R_{ij}) = \partial_t (g^{ij}) R_{ij} + g^{ij} \partial_t (R_{ij})$,
$\partial_t \mathfrak{R} = -h^{ij} R_{ij} + g^{ij} \frac{1}{2} g^{km} (\nabla_k \nabla_i h_{jm} + \nabla_k \nabla_j h_{im} - \nabla_k \nabla_m h_{ij} - \nabla_j \nabla_i h_{km})$,
$\partial_t \mathfrak{R} = -h^{ij} R_{ij} + \frac{1}{2} (\nabla_k \nabla_i h^{ik} + \nabla_k \nabla_j h^{jk} - g^{ij} \nabla^m \nabla_m h_{ij} - g^{km} \nabla^i \nabla_i h^{km})$,
$\partial_t \mathfrak{R} = -h^{ij} R_{ij} + \nabla_i \nabla_j h^{ij} - \nabla^m \nabla_m g^{ij} h_{ij}$,
$\partial_t \mathfrak{R} = -h^{ij} R_{ij} + \nabla_i \nabla_j h^{ij} - \nabla^2 g^{ij} h_{ij}$.
Where ∇^2 is the Laplace operator and in the volume normalized Ricci flow the argument of the Laplace operator vanishes because of the fact that $g^{ij} h_{ij} = 0$.

4.2 Ricci Flow on Homogeneous 3-Manifolds

In this section the results of the Ricci flow on homogeneous 3-manifolds will be derived. In order to compute the flow equations one first needs to calculate the six different components of the Ricci tensor and the scalar curvature and then to evaluate them according to the flow equation. Normally this would require to solve six different (non)homogeneous flow equations ($\partial_t g_{11}, \partial_t g_{12}, \partial_t g_{13}, \partial_t g_{22}, \partial_t g_{23}, \partial_t g_{33}$) and to interpret the results would become very difficult. However, by using the Milnor frame introduced in Chapter 3 , there are only three flow equations to solve and they are all homogeneous. Therefore, in this section it should be understood that there exists a frame for each geometry such that the metric is simply :

$$g = A(t) \omega^1 \otimes \omega^1 + B(t) \omega^2 \otimes \omega^2 + C(t) \omega^3 \otimes \omega^3 ,$$

where t is the evolution parameter of the flow and A, B, C are all functions of t and are differentiable. The components of the Ricci tensor and the curvature scalar are calculated by using the equations derived in Chapter 3 with only one difference : Since the Ricci flow is volume preserving ABC is always a constant, and without loss of generality it is set to 1, $ABC = 1$. Let us now consider the nine geometries separately.

The rest of this chapter follows [14].

I.The geometry of R^3 with the structure constants $\lambda, \mu, \nu = \{0,0,0\}$.

The metric of these geometries are flat and do not change under the Ricci flow. The components of the Ricci and tensor and the curvature scalar are all zero, $R_{11} = R_{22} = R_{33} = \mathfrak{R} = 0$, hence the initial metric g_0 stands for all $t \geq 0$.

$\partial_t g = 0 \rightarrow g(t) = g_0$.

Thus the geometry of R^3 is a fixed point of the flow.

II.The geometry of $\mathsf{SU}(2)$ with the structure constants $\lambda, \mu, \nu = \{-1,-1,-1\}$ or $\{1,1,1\}$.

The non-zero components of the Ricci tensor are,

$$R_{11} = \frac{A}{2}(A^2 - B^2 - C^2 + 2BC) ,$$

$$R_{22} = \frac{B}{2}(-A^2 + B^2 - C^2 + 2AC) ,$$

$$R_{33} = \frac{C}{2}(-A^2 - B^2 + C^2 + 2AB) ,$$

and the curvature scalar is,

$$\mathfrak{R} = \frac{1}{2}(-A^2 - B^2 - C^2 + 2AB + 2AC + 2BC) .$$

The flow equations yield :

a) $\partial_t g_{11} = -2R_{11} + \frac{2}{3}\mathfrak{R} g_{11}$,

$$\frac{d}{dt}A = \frac{2A}{3}(-2A^2 + B^2 + C^2 + AB + AC - 2BC) = \frac{2A}{3}[-A(2A - B - C) + (B - C)^2] ,$$

b) $\partial_t g_{22} = -2R_{22} + \frac{2}{3}\mathfrak{R} g_{22}$,

$$\frac{d}{dt}B = \frac{2B}{3}(A^2 - 2B^2 + C^2 + AB - 2AC + BC) = \frac{2B}{3}[-B(2B - A - C) + (A - C)^2] ,$$

c) $\partial_t g_{33} = -2R_{33} + \frac{2}{3}\mathfrak{R} g_{33}$,

$$\frac{d}{dt}C = \frac{2C}{3}(A^2 + B^2 - 2C^2 - 2AB + AC + BC) = \frac{2C}{3}[-C(2C - A - B) + (A - B)^2] .$$

These three flow equations are all symmetric in A,B and C that is for example if A and B are interchanged (or A and C, or B and C) the equations would not change at all. In addition, when $A = B = C \rightarrow \frac{d}{dt}A = \frac{d}{dt}B = \frac{d}{dt}C = 0$, which means that the metric does not change any more under the flow at the point where the three functions of t are equal to each other. This metric, the one with $A = B = C$ hence $A = B = C = 1$ since ABC is set to 1, is the metric of the round sphere. Therefore once the metric is evolved to the metric of the round sphere it stays there forever. The round sphere is then called the fixed point of the Ricci flow. When $A \neq B \neq C$, clearly flow equations will not be equal to zero. It is not possible to solve the equations analytically, so it is necessary to make some estimates. Therefore let us investigate the difference between the flow equations :

$$\frac{d}{dt}(A-B) = \frac{2}{3}[-2(A^3-B^3)+C(A^2-B^2)+C^2(A-B)] \ ,$$
$$\frac{d}{dt}(A-C) = \frac{2}{3}[-2(A^3-C^3)+B(A^2-C^2)+B^2(A-C)] \ ,$$
$$\frac{d}{dt}(B-C) = \frac{2}{3}[-2(B^3-C^3)+A(B^2-C^2)+A^2(B-C)] \ .$$

Using the symmetry of the flow equations it can be assumed that initially $A_O \geq B_O \geq C_O$ where A_O stands for the initial value of A and similarly for B and C. If at $t = \tau$, $A_\tau = B_\tau$, then:

$$\frac{d}{dt}(A-B)|_{t=\tau} = \frac{2}{3}[-2(A_\tau^3-B_\tau^3)+C_\tau(A_\tau^2-B_\tau^2)+C_\tau^2(A_\tau-B_\tau)] = 0 \ ,$$

which means that the flow of the difference between the functions A and B stops at $t = \tau$ if $A_\tau = B_\tau$. Similarly, if $B_\tau = C_\tau$, then the flow of the difference between the functions B and C stops at $t = \tau$. Therefore if initially, $A_O \geq B_O \geq C_O$ then at all $t \geq 0$, $A \geq B \geq C$ until the point they become exactly equal to each other (another conclusion is that A is evolving more rapidly than B and C and similarly B is evolving more rapidly than C unless they are initially equal).

| We note that the second derivatives yield no different result to these conclusions :

$$\frac{d^2}{dt^2}(A-B)|_{t=\tau} = \frac{2}{3}[-6(A_\tau^2\frac{dA}{dt}|_{t=\tau}-B_\tau^2\frac{dB}{dt}|_{t=\tau})+2C_\tau(A_\tau\frac{dA}{dt}|_{t=\tau}-B_\tau\frac{dB}{dt}|_{t=\tau})+\frac{dC}{dt}|_{t=\tau}(A_\tau^2-B_\tau^2)+C_\tau^2\frac{d}{dt}(A-B)|_{t=\tau}+2C_\tau\frac{dC}{dt}|_{t=\tau}(A_\tau-B_\tau)] = 0 \ ,$$

which proves the claim that at all $t \geq 0$, $A \geq B \geq C$ until the point they become exactly equal to each other. ∎

Since $A \geq B \geq C$ one gets :

$$\frac{d}{dt}C = \frac{2C}{3}[-C\underbrace{(2C-A-B)}_{\text{negative or zero}}+(A-B)^2] \geq 0 \rightarrow C \geq C_0 \ .$$

Hence, $t \geq 0$, $A \geq B \geq C \geq C_0$, using this condition the flow of $(A-C)$ becomes :

$$\frac{d}{dt}(A-C) = \frac{2}{3}(A-C)[-2(A^2+C^2+AC)+B(A+C)+B^2] \leq \frac{2}{3}(A-C)(-3C_0^2) \ .$$

By direct integration one obtains :

$$(A-C) \leq (A_0-C_0)exp(-2C_0^2 t) \ .$$

Therefore $(A-C)$ vanishes exponentially (since $A \geq B \geq C$,$(A-B)$ and $(B-C)$ vanish at least exponentially), so the geometry converges to the round sphere exponentially where $A = B = C = 1$. The curvature scalar hence converges to the constant $\frac{3}{2}$ because :

$$\mathfrak{R}|_{A=B=C} = \frac{3}{2} \ .$$

III.The geometry of $\mathsf{SL}(2,\mathbb{R})$ with the structure constants $\lambda, \mu, \nu = \{-1,-1,1\}$.

The non-zero components of the Ricci tensor are,

$$R_{11} = \frac{A}{2}(A^2-B^2-C^2-2BC) \ ,$$
$$R_{22} = \frac{B}{2}(-A^2+B^2-C^2-2AC) \ ,$$
$$R_{33} = \frac{C}{2}(-A^2-B^2+C^2+2AB) \ ,$$

and the curvature scalar is,

$$\mathfrak{R} = \frac{1}{2}(-A^2 - B^2 - C^2 + 2AB - 2AC - 2BC) \ .$$

The flow equations yield :

$$\frac{d}{dt}A = \frac{2A}{3}(-2A^2 + B^2 + C^2 + AB - AC + 2BC) = \frac{2A}{3}[-A(2A - B + C) + (B + C)^2] \ ,$$
$$\frac{d}{dt}B = \frac{2B}{3}(A^2 - 2B^2 + C^2 + AB + 2AC - BC) = \frac{2B}{3}[-B(-A + 2B + C) + (A + C)^2] \ ,$$
$$\frac{d}{dt}C = \frac{2C}{3}(A^2 + B^2 - 2C^2 - 2AB - AC - BC) = \frac{2C}{3}[-C(A + B + 2C) + (A - B)^2] \ .$$

These equations are symmetric in A and B. It is not possible to solve the equations analytically, so let us assume that initially $A_0 \geq B_0$. The difference between their flow shows that if at any $t = \tau$, $A_\tau = B_\tau$ then :

$$\frac{d}{dt}(A - B) = \frac{2}{3}[-2(A^3 - B^3) - C(A^2 - B^2) + C^2(A - B)] \ ,$$
$$\frac{d}{dt}(A - B)|_{t=\tau} = \frac{2}{3}[-2(A_\tau^3 - B_\tau^3) - C_\tau(A_\tau^2 - B_\tau^2) + C_\tau^2(A_\tau - B_\tau)] = 0 \ .$$

Therefore if initially $A_0 \geq B_0$ then $A \geq B$ at all $t \geq 0$ (second derivative also shows that the system is stable). This condition implies that :

$$\begin{aligned}
\frac{d}{dt}B &= \frac{2B}{3}(A^2 - 2B^2 + C^2 + AB + 2AC - BC), \\
&= \frac{2}{3}(A^2B - 2B^3 + BC^2 + AB^2 + 2ABC - B^2C), \\
&\geq \frac{2}{3}(\underbrace{A^2B - 2B^3 + AB^2}_{\geq 0} + \underbrace{ABC}_{=1} + \underbrace{ABC - B^2C}_{\geq 0}) \\
&\geq \frac{2}{3} \ .
\end{aligned}$$

By direct integration one obtains :

$$B \geq \frac{2}{3}t + B_0 \ , \qquad B \text{ is an increasing function.}$$

This condition -lower bound- on B gives the following conditions on A and C :

$$
\begin{aligned}
&A \geq B \geq \frac{2}{3}t + B_0 \rightarrow AB \geq \left(\frac{2}{3}t + B_0\right)^2 , \\
&C = \frac{1}{AB} \rightarrow C \leq \left(\frac{2}{3}t + B_0\right)^{-2} .
\end{aligned}
$$

Thus, C vanishes (or shrinks) while A and B grow linearly in t. Therefore there must be a $t = \tau$ where $A_\tau = C_\tau$ and thereafter $A \geq C$. This can be used to find an upper bound for A after $t \geq \tau$:

$$
\begin{aligned}
\frac{d}{dt}A &= \frac{2B}{3}(-2A^2 + B^2 + C^2 + AB - AC + 2BC) , \\
&= \frac{2}{3}[AC\underbrace{(C-A)}_{\leq 0}\underbrace{-2A^3 + A^2B + AB^2}_{\leq 0} + \underbrace{2ABC}_{=2}] , \\
&\leq \frac{4}{3} .
\end{aligned}
$$

By direct integration one obtains :

$$
\begin{aligned}
&A \leq \left(\frac{4}{3}(t-\tau) + A_\tau\right) \rightarrow B \leq A \leq \left(\frac{4}{3}(t-\tau) + A_\tau\right) , \\
&C = \frac{1}{AB} \rightarrow C \geq \left(\frac{4}{3}(t-\tau) + A_\tau\right)^{-2} .
\end{aligned}
$$

By using this condition $A_\tau = C_\tau \geq B_\tau$, the flow of the difference $(A-B)$ after $t \geq \tau$ becomes:

$$
\begin{aligned}
\frac{d}{dt}(A-B) &= \frac{2}{3}[-2(A^3 - B^3) - C(A^2 - B^2) + C^2(A-B)] , \\
&= \frac{2}{3}(A-B)[-2\underbrace{(A^2 + B^2 + AB)}_{\geq 3B_\tau^2}\underbrace{-C(A+B)}_{\leq 0} + \underbrace{C^2}_{\leq B_\tau^2}] , \\
&\leq \frac{2}{3}(A-B)(-5B_\tau^2) .
\end{aligned}
$$

By direct integration one gets :

$$
(A-B) \leq exp\left(\frac{-10B_\tau^2(t-\tau)}{3}\right)(A_0 - B_0) .
$$

Thus, A and B are approaching to each other exponentially as they grow and C vanishes. This behaviour is called a pancake degeneracy in the relevant literature. The scalar curvature in the absolute value can be written as below :

$$
\begin{aligned}
|\mathfrak{R}| &= \frac{1}{2}(A^2+B^2+C^2-2AB+2AC+2BC)\ , \\
&= \frac{1}{2}C^2+\frac{1}{2}(A-B)^2+AC+BC\ .
\end{aligned}
$$

Each term on the right hand side is bounded from above.

$$
\begin{aligned}
& C \leq \left(\frac{2}{3}t+B_0\right)^{-2} \rightarrow C^{-2} \leq \left(\frac{2}{3}t+B_0\right)^{-4}\ , \\
& (A-B) \leq exp\left(\frac{-10B_\tau^2(t-\tau)}{3}\right)(A_0-B_0) \rightarrow (A-B) \leq exp\left(\frac{-10B_\tau^2(t-\tau)}{3}\right)^2 (A_0-B_0)^2\ , \\
& B \leq A \leq \left(\frac{4}{3}(t-\tau)+A_\tau\right) \rightarrow BC \leq AC \leq \left(\frac{4}{3}(t-\tau)+A_\tau\right)\left(\frac{2}{3}t+B_0\right)^{-2}\ .
\end{aligned}
$$

The slower decay rate is due to the AC and BC terms, so the curvature scalar decays at least by the factor t^{-1} . As a result $0 \leq |\mathfrak{R}| \leq t^{-1}$ the curvature scalar dies off.

IV.The geometry of $\mathsf{Isom}(\mathsf{R}^2)$ with the structure constants $\lambda,\mu,\nu = \{-1,-1,0\}$.

The non-zero components of the Ricci tensor are,

$$
\begin{aligned}
R_{11} &= \frac{A}{2}(A^2-B^2) = \frac{A}{2}(A-B)(A+B)\ , \\
R_{22} &= \frac{B}{2}(-A^2+B^2) = -\frac{B}{2}(A-B)(A+B)\ , \\
R_{33} &= \frac{C}{2}(-A^2-B^2+2AB) = -\frac{C}{2}(A-B)^2\ ,
\end{aligned}
$$

and the curvature scalar is,

$$
\mathfrak{R} = \frac{1}{2}(-A^2-B^2+2AB) = -\frac{1}{2}(A-B)^2\ .
$$

The flow equations yield :

$$\frac{d}{dt}A = -\frac{2A}{3}(2A^2 - B^2 - AB) = -\frac{2A}{3}(2A+B)(A-B) ,$$
$$\frac{d}{dt}B = -\frac{2B}{3}(-A^2 + 2B^2 - AB) = -\frac{2B}{3}(2B+A)(B-A) ,$$
$$\frac{d}{dt}C = \frac{2C}{3}(A-B)^2 .$$

These equations are symmetric in A and B. It is not possible to solve the equations analytically, so let us assume that initially $A_0 \geq B_0$. The difference between their flow shows that if at any $t = \tau$, $A_\tau = B_\tau$:

$$\frac{d}{dt}(A-B) = -\frac{4}{3}(A-B)(A^2 + AB + B^2) ,$$
$$\frac{d}{dt}(A-B)|_{t=\tau} = -\frac{4}{3}(A_\tau - B_\tau)(A_\tau^2 + A_\tau B_\tau + B_\tau^2) = 0 .$$

Therefore if initially $A_0 \geq B_0$ then $A \geq B$ at all $t \geq 0$. This condition implies that :

$$\frac{d}{dt}A = -\frac{2A}{3}(2A+B)\underbrace{(A-B)}_{\geq 0} \leq 0 \qquad A \text{ is a non-increasing function ,}$$
$$\frac{d}{dt}B = \frac{2B}{3}(2B+A)(A-B) \geq 0 \qquad B \text{ is a non-decreasing function ,}$$

$$\frac{d}{dt}C = \frac{2C}{3}(A-B)^2 \geq 0 \qquad C \text{ is a non-decreasing function .}$$

Hence $B_0 \leq B \leq A \leq A_0$ and since $ABC = 1$ these conditions can be used to find an upper bound for C :

$$C = \frac{A_0 B_0 C_0}{AB} \leq \frac{A_0 B_0 C_0}{B_0 B_0} \rightarrow C_0 \leq C \leq \frac{A_0 C_0}{B_0} .$$

By using these conditions -upper/lower bounds- the flow of the difference $(A-B)$ becomes :

$$\frac{d}{dt}(A-B) = -\frac{4}{3}(A-B)\underbrace{(A^2 + AB + B^2)}_{\geq 3B_0^2} ,$$

$$\leq (A-B)(-4B_0^2) \ .$$

By direct integration one obtains :

$$(A-B) \leq (A_0 - B_0) exp(-4B_0^2 t) \ .$$

Similarly, a lower bound can be found for $(A-B)$:

$$\frac{d}{dt}(A-B) = -\frac{4}{3}(A-B)\underbrace{(A^2+AB+B^2)}_{\leq 3A_0^2} \ ,$$
$$\geq (A-B)(-4A_0^2) \ .$$

$$(A-B) \geq (A_0 - B_0) exp(-4A_0^2 t) \ .$$

Thus, A and B approach to each other exponentially. Since all the flow equations include the term $(A-B)$, at the point where $A=B$, $\frac{d}{dt}A = \frac{d}{dt}B = \frac{d}{dt}C = 0$ which means that the geometry exponentially approaches to a fixed point, to the flat one because $\mathfrak{R} \to 0$ as A and B approach to each other and this can be shown by using the bounds as :

$$|\mathfrak{R}| = \frac{1}{2}(A-B)^2 \ ,$$
$$\frac{1}{2}(A_0 - B_0)^2 exp(-8B_0^2 t) \geq |\mathfrak{R}| \geq \frac{1}{2}(A_0 - B_0)^2 exp(-8A_0^2 t) \text{ or,}$$
$$|\mathfrak{R}_o| exp(-8B_0^2 t) \geq |\mathfrak{R}| \geq |\mathfrak{R}_o| exp(-8A_0^2 t) \ .$$

The curvature scalar exponentially decays to zero.

V. The geometry of $\mathsf{E}(1,1)$ with the structure constants $\lambda, \mu, \nu = \{-1, 0, 1\}$.

The non-zero components of the Ricci tensor are,

$R_{11} = \frac{A}{2}(A^2 - C^2)$,

$R_{22} = \frac{B}{2}(-A^2 - C^2 - 2AC) = -\frac{B}{2}(A+C)^2$,

$R_{33} = \frac{C}{2}(-A^2 + C^2)$,

and the curvature scalar is,

$\mathfrak{R} = -\frac{1}{2}(A+C)^2$.

The flow equations yield :

$\frac{d}{dt}A = -\frac{2A}{3}(2A^2 + AC - C^2)$,

$\frac{d}{dt}B = \frac{2B}{3}(A+C)^2$,

$\frac{d}{dt}C = -\frac{2C}{3}(2C^2 + AC - A^2)$.

These equations are symmetric in A and C. It is not possible to solve the equations analytically, so let us assume that initially $A_0 \geq C_0$. The difference between their flow shows that if at any $t = \tau$, $A_\tau = C_\tau$ then :

$$\frac{d}{dt}(A - C) = -\frac{4}{3}(A+C)^2(A-C) ,$$
$$\frac{d}{dt}(A - C)|_{t=\tau} = -\frac{4}{3}(A_\tau + C_\tau)^2(A_\tau - C_\tau) = 0 .$$

Therefore if initially $A_0 \geq C_0$ then $A \geq C$ at all $t \geq 0$. This condition implies that :

$$\frac{d}{dt}A = -\frac{2}{3}\underbrace{[2A^3 + AC(A-C)]}_{\geq 2A^3} \leq -\frac{4}{3}A^3 \qquad (A \text{ is a non-increasing function}).$$

By direct integration one gets :

$$A \leq A_0(1 + \frac{8}{3}A_0^2 t)^{-1/2} .$$

This condition -upper bound on A- is applicable to C as well since $A \geq C$. Therefore, A and C vanish (or shrink) by the factor $t^{-1/2}$. By the same reasoning :

$$\frac{d}{dt}C = -\frac{2}{3}\underbrace{[2C^3 + AC(C-A)]}_{\leq 2C^3} \geq -\frac{4}{3}C^3 \ .$$

By direct integration one obtains :

$$C \geq C_0(1+\frac{8}{3}C_0^2 t)^{-1/2} \ .$$

This condition -lower bound on C- is applicable to A as well since $A \geq C$. Consequently :

$$C_0(1+\frac{8}{3}C_0^2 t)^{-1/2} \leq C \leq A \leq A_0(1+\frac{8}{3}A_0^2 t)^{-1/2} \ .$$

Let us now investigate the difference between their flow using these limits :

$$\begin{aligned}\frac{d}{dt}(A-C) &= -\frac{4}{3}\underbrace{(A+C)^2}_{\geq 4C^2}(A-C) \\ &\leq -\frac{16}{3}C_0^2(1+\frac{8}{3}C_0^2 t)^{-1}(A-C) \ .\end{aligned}$$

By direct integration one gets :

$$(A-C) \leq (A_0 - C_0)(1+\frac{8}{3}C_0^2 t)^{-2} \ .$$

Thus, while A and C shrink by the factor $t^{-1/2}$ they approach to each other by the factor t^{-2}. Since $B = \dfrac{1}{AC}$ it will expand by the factor t. The behaviour of B can be found more explicitly using the upper and lower bounds for A and C as :

$$\frac{A_0B_0C_0}{A_0^2}(1+\frac{8}{3}A_0^2t) \leq B = \frac{A_0B_0C_0}{AC} \leq \frac{A_0B_0C_0}{C_0^2}(1+\frac{8}{3}C_0^2t) \quad \text{or,}$$

$$\frac{B_0C_0}{A_0}(1+\frac{8}{3}A_0^2t) \leq B \leq \frac{A_0B_0}{C_0}(1+\frac{8}{3}C_0^2t) \ .$$

Hence B is growing linearly in t while A and C shrink. This behaviour is called a cigar degeneracy in the relevant literature. In order to see what happens to the scalar curvature, first the flow of $(A+C)$ must be investigated :

$$\begin{aligned}\frac{d}{dt}(A+C) &= -\frac{4}{3}(A^3+C^3) = -\frac{4}{3}(A+C)\underbrace{[(A-C)^2+AC]}_{\geq AC}, \\ &\leq -\frac{4}{3}C_0^2(1+\frac{8}{3}C_0^2t)^{-1}(A+C) \ .\end{aligned}$$

By direct integration one obtains :

$$(A+C) \leq (A_0+C_0)(1+\frac{8}{3}C_0^2t)^{-1/2} \qquad \text{gives an upper bound for } (A+C)$$

Similarly a lower bound can be found :

$$\begin{aligned}\frac{d}{dt}(A+C) &= -\frac{4}{3}(A+C)\underbrace{(A^2+C^2-AC)}_{\leq A^2+C^2}, \\ &\geq -\frac{2}{3}A_0^2(1+\frac{8}{3}A_0^2t)^{-1}(A+C) \ .\end{aligned}$$

By direct integration one gets :

$$(A+C) \geq (A_0+C_0)(1+\frac{8}{3}A_0^2t)^{-1/4} \ .$$

These upper and lower bounds can now be used to find the behaviour of the scalar curvature :

$$|\mathfrak{R}| = \frac{1}{2}(A+C)^2 \ ,$$

$$\frac{(A_0+C_0)}{2}(1+\frac{8}{3}C_0^2 t)^{-1} \geq |\mathfrak{R}| \geq \frac{(A_0+C_0)}{2}(1+\frac{8}{3}A_0^2 t)^{-1/2} \text{ or },$$

$$|\mathfrak{R}_o|(1+\frac{8}{3}C_0^2 t)^{-1} \geq |\mathfrak{R}| \geq |\mathfrak{R}_o|(1+\frac{8}{3}A_0^2 t)^{-1/2} \ .$$

$|\mathfrak{R}_o|$ is the initial value of the scalar curvature in the absolute value. The scalar curvature dies off as t evolves.

VI. The geometry of Heisenberg with the structure constants $\lambda, \mu, \nu = \{-1, 0, 0\}$.

The non-zero components of the Ricci tensor are,

$R_{11} = \frac{1}{2}A^3$,

$R_{22} = -\frac{1}{2}A^2 B$,

$R_{33} = -\frac{1}{2}A^2 C$,

and the curvature scalar is,

$\mathfrak{R} = \frac{-1}{2}A^2$.

The flow equations yield :

$\frac{d}{dt}A = -\frac{4}{3}A^3$,

$\frac{d}{dt}B = \frac{2}{3}A^2 B$,

$\frac{d}{dt}C = \frac{2}{3}A^2 C$.

In this case it is possible to solve equations analytically,so by direct integration starting from A we have :

$$A = A_0(1+\frac{8}{3}A_0^2 t)^{-1/2} \ ,$$

$$B = B_0(1+\frac{8}{3}A_0^2 t)^{1/4} ,$$
$$C = C_0(1+\frac{8}{3}A_0^2 t)^{1/4} .$$

Thus B and C grow by the factor $t^{1/4}$ while A shrink by the factor $t^{-1/2}$. This is again a pancake degeneracy as in the case of $SL(2,R)$, but with an important difference. This time B and C do not converge as they grow, but they diverge by the factor $t^{1/4}$ unless they start equal to each other because :

$$(B-C) = (B_0-C_0)(1+\frac{8}{3}A_0^2 t)^{1/4} .$$

The scalar curvature can be easily computed as :

$$\mathfrak{R} = \frac{-1}{2}A^2 = \frac{-1}{2}A_0^2(1+\frac{8}{3}A_0^2 t)^{-1} ,$$
or, $\mathfrak{R} = \mathfrak{R}_o(1-\frac{16}{3}\mathfrak{R}_o t)^{-1}$ where $\mathfrak{R}_o = \frac{-1}{2}A_0^2$.

The scalar curvature vanishes by the factor t^{-1} as the geometry approaches to the pancake degeneracy.

Non-Bianchi Classes

VII.The geometry of H^3 with the metric $g = \kappa g_{H^3}$.

The metrics of the geometries in this class are constant multiples of the hyperbolic metric g_{H^3}, so all have constant negative curvature. Since g is a constant $\partial_t g = 0$. Therefore the initial metric g_0 stands for all $t \geq 0$ showing that it is a fixed point of the flow.

VIII.The geometry of $S^2 \times \mathbb{R}$ with the metric $g = Kg_R + \kappa g_{S^2}$.

The metrics of the geometries in this class are of the above form where g_R is the metric of $\mathbb{R}$ and g_{S^2} is the metric of the two-sphere with constants K and κ respectively. In a basis $\{d\psi, d\theta, sin\theta d\phi\}$ the components of the Ricci tensor and the curvature scalar can be computed directly by using the equations (2.25), (2.28), (2.34) and (2.36) :

$g = Kd\psi^2 + \kappa(d\theta^2 + sin^2\theta d^2\phi)$,
$g_{\phi\phi,\theta} = 2\kappa sin\theta cos\theta d\theta$ the only non-zero derivative of the components of the metric,
$\Gamma^{\theta}_{\phi\phi} = -sin\theta cos\theta \quad \Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = cos\theta / sin\theta$ the non-zero Christoffel symbols,
$R_{\psi\psi} = 0 \quad R_{\theta\theta} = 1 \quad R_{\phi\phi} = sin^2\theta$ the components of the Ricci tensor,
$\mathfrak{R} = \frac{2}{\kappa}$ the curvature scalar.

$$\partial_t g = -2Ric + \frac{2}{3}\mathfrak{R}g \ .$$

The flow equations yield :
a) $\partial_t g_{\psi\psi} = -2R_{\psi\psi} + \frac{2}{3}\mathfrak{R}g_{\psi\psi}$,
$\frac{d}{dt}K = \frac{2}{3}K\frac{2}{\kappa} = \frac{4K}{3\kappa}$,
b) $\partial_t g_{\theta\theta} = -2R_{\theta\theta} + \frac{2}{3}\mathfrak{R}g_{\theta\theta}$,
$\frac{d}{dt}\kappa = -2 + \frac{2}{3}\frac{2}{\kappa}\kappa = -\frac{2}{3}$.

It is possible to solve these equations analytically,so by direct integration one obtains :

$$\kappa = \kappa_0 - \frac{2}{3}t \ ,$$
$$K = K_0\kappa_0^2(\kappa_0 - \frac{2}{3}t)^{-2} \ .$$

From these solutions it is seen that while the two-sphere shrinks with the parameter t, when $t = \frac{3}{2}\kappa_0$ the sphere becomes a point, $\mathbb{R}$ grows with t^{-2}. In a finite $t = \frac{3}{2}\kappa_0$ hence the radius of S^2 becomes infinite because :

$$\mathfrak{R} = \frac{2}{\kappa} = 2\left(\kappa_0 - \frac{2}{3}t\right)^{-1} .$$

This is called a curvature singularity. However it is possible to overcome this unexpected result by simply realizing that the normalized Ricci flow equation is constructed so that to preserve the volume of the n-sphere. In this case the sphere is not a three-sphere but a two-sphere , thus the correct flow equation must be $\partial_t g = -2Ric + \frac{2}{2}\mathfrak{R}g = -2Ric + \mathfrak{R}g$. Under this re-normalization the new flow equations read :

$$\text{a)}\quad \partial_t g_{\psi\psi} = -2R_{\psi\psi} + \mathfrak{R}g_{\psi\psi} ,$$
$$\frac{d}{dt}K = K\frac{2}{\kappa} ,$$
$$\text{b)}\quad \partial_t g_{\theta\theta} = -2R_{\theta\theta} + \frac{2}{3}\mathfrak{R}g_{\theta\theta} ,$$
$$\frac{d}{dt}\kappa = -2 + \frac{2}{\kappa}\kappa = 0 .$$

By direct integration one gets :

$$\kappa = \kappa_0 ,$$
$$K = K_0 exp\left(\frac{2}{\kappa_0}t\right) .$$

Therefore as the volume of the two-sphere is preserved under the flow the metric of the $\mathbb{R}$ does exist for all $t \geq 0$. The scalar curvature stays constant because :

$$\mathfrak{R} = \frac{2}{\kappa} = \frac{2}{\kappa_0} .$$

IX.The geometry of $H^2 \times \mathbb{R}$ with the metric $g = Kg_R + \kappa g_{H^2}$.

The metrics of the geometries in this class are of the above form where g_R is the metric of

$\mathbb{R}$ and g_{H^2} is the metric of the hyperbolic plane with constants K and κ respectively. In a basis $\{d\psi, d\theta, sinh\theta d\phi\}$ the components of the Ricci tensor and the curvature scalar can be computed directly :

$g = Kd\psi^2 + \kappa(d\theta^2 + sinh^2\theta d^2\phi)$,

$g_{\phi\phi,\theta} = 2\kappa sinh\theta cosh\theta$ the only non-zero derivative of the components of the metric,

$\Gamma^{\theta}_{\phi\phi} = -sinh\theta cosh\theta \quad \Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = \frac{cosh\theta}{sinh\theta}$ the non-zero Christoffel symbols,

$R_{\psi\psi} = 0 \quad R_{\theta\theta} = -1 \quad R_{\phi\phi} = -sinh^2\theta$ the components of the Ricci tensor,

$\mathfrak{R} = -\frac{2}{\kappa}$ the curvature scalar.

$$\partial_t g = -2Ric + \frac{2}{3}\mathfrak{R}g \ .$$

The flow equations read :

a) $\partial_t g_{\psi\psi} = -2R_{\psi\psi} + \frac{2}{3}\mathfrak{R}g_{\psi\psi}$,

$\frac{d}{dt}K = \frac{2}{3}(-\frac{2}{\kappa})K = -\frac{4K}{3\kappa}$,

b) $\partial_t g_{\theta\theta} = -2R_{\theta\theta} + \frac{2}{3}\mathfrak{R}g_{\theta\theta}$,

$\frac{d}{dt}\kappa = 2 + \frac{2}{3}(-\frac{2}{\kappa})\kappa = \frac{2}{3}$.

It is possible to solve these equations analytically, so by direct integration one obtains :

$$\kappa = \kappa_0 + \frac{2}{3}t \ ,$$
$$K = K_0\kappa_0^2(\kappa_0 + \frac{2}{3}t)^{-2} \ .$$

From these solutions it is seen that while the hyperbolic plane grows linearly with t, $\mathbb{R}$ shrinks with t^{-2} giving a pancake degeneracy to the geometry. The scalar curvature vanishes with t^{-1} because :

$$\mathfrak{R} = -\frac{2}{\kappa} = -2(\kappa_0 + \frac{2}{3}t)^{-1} \ .$$

It is also possible in this case to prevent the hyperbolic plane to expand by re-normalization. Let us again write the flow equation for $n = 2$, the new flow equations read :

$$\partial_t g = -2Ric + \Re g \ ,$$

$$\frac{d}{dt}K = -\frac{2K}{\kappa} \ ,$$
$$\frac{d}{dt}\kappa = 0 \ .$$

The solutions of these equations are:

$$\kappa = \kappa_0 \ ,$$
$$K = K_0 exp\left(-\frac{2}{\kappa_0}t\right) \ .$$

Therefore hyperbolic plane remains fixed for all $t \geq 0$ while $\mathbb{R}$ shrinks exponentially. The scalar curvature scalar stays constant because :

$$\Re = -\frac{2}{\kappa} = -\frac{2}{\kappa_0} \ .$$

CHAPTER 5

COTTON FLOW

The Cotton tensor, as mentioned in Chapter 2 is the conformal tensor which takes the place of the Weyl tensor in three dimensions. Thus a three dimensional space is conformally flat if and only if the Cotton tensor vanishes. The Cotton tensor is represented as before as :

$$C^{ij} = g^{1/3}\varepsilon^{mni}\nabla_m(R^j{}_n - \frac{1}{4}\delta^j{}_n\Re) \quad . \tag{5.1}$$

This chapter follows [15].

5.1 Cotton Flow

The Cotton flow first introduced in [15] is an evolution equation of the metric as in the case of the Ricci flow. This evolution equation can be stated as in the case of the Ricci flow as :

$$\partial_t g_{ij} = \kappa C_{ij} \qquad , \tag{5.2}$$

where κ is a positive constant. The choice of κ is arbitrary except being positive, so it can be set to 1 as long as the evolution parameter t is properly scaled. The Cotton flow is already volume preserving since the Cotton tensor is traceless. One can compute the flow equations by using equation (5.1). However, it is easier to compute flow equations by using differential forms. The flow equation can be written in terms of forms as :

$$\partial_t e^a = *C^a \quad , \tag{5.3}$$

where C^a is the Cotton two-form and $*$ denotes the Hodge dual.

The evolution of some tensors under the Cotton flow follows as :

i. $\partial_t g_{ij} = C_{ij}$,
$\partial_t (g^{ij} g_{kj}) = \partial_t (\delta^i{}_k) = 0$,
$\partial_t (g^{ij} g_{kj}) = (\partial_t g^{ij}) g_{kj} + g^{ij} (\partial_t g_{kj}) = 0$,
$(\partial_t g^{ij}) g_{kj} = -g^{ij} C_{kj}$,
$(\partial_t g^{ij}) g_{kj} g^{kl} = -C_k{}^i g^{kl}$,
$(\partial_t g^{ij}) \delta_j{}^l = -C^{li}$,
$\partial_t g^{il} = -C^{il}$.

ii. The evolution of the the Christoffel symbols given in (2.25) is,
$\partial_t \Gamma^i_{jk} = \frac{1}{2} \partial_t g^{il} (\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}) + \frac{1}{2} g^{il} (\partial_t \partial_j g_{kl} + \partial_t \partial_k g_{jl} - \partial_t \partial_l g_{jk})$.
Around a point p in normal coordinates $\partial_l g_{jk}(p) = 0$ and $\partial_i(p) = \nabla_i(p)$. As a result the left hand side of the above equation vanishes and $\partial_t \partial_i g_{jk} = \partial_i \partial_t g_{jk} = \partial_i C_{jk} = \nabla_i C_{jk}$ since ∂_i and ∂_t commute. Therefore, under the Cotton flow the evolution of the the Christoffel symbols is,
$\partial_t \Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j \partial_t g_{kl} + \partial_k \partial_t g_{jl} - \partial_l \partial_t g_{jk})$,
or, $\partial_t \Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j C_{kl} + \partial_k C_{jl} - \partial_l C_{jk})$,
or, $\partial_t \Gamma^i_{jk} = \frac{1}{2} g^{il} (\nabla_j C_{kl} + \nabla_k C_{jl} - \nabla_l C_{jk})$.
Notice that as a result of the above equation $\partial_t \Gamma^i_{ik} = 0$, although this could be seen by noticing that $\partial_t \Gamma^i_{ik} = \partial_t \partial_k ln\sqrt{g} = \partial_k \partial_t ln\sqrt{g} = 0$ since g is independent of t.

iii. The evolution of the Ricci tensor given in (2.34) is,
$R_{ij} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik}$ the Ricci tensor in the normal coordinates ,
$\partial_t R_{ij} = \partial_t (\partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik})$,
$\partial_t R_{ij} = \partial_k \partial_t \Gamma^k_{ij} - \partial_j \partial_t \Gamma^k_{ik}$,
using the previous results for Γ's one can write,
$\partial_t R_{ij} = \frac{1}{2} g^{km} \nabla_k (\nabla_i C_{jm} + \nabla_j C_{im} - \nabla_m C_{ij})$,
$\quad = \frac{1}{2} \nabla_k (\nabla_i C_j{}^k + \nabla_j C_i{}^k - \nabla^k C_{ij})$,
or, $\partial_t R_{ij} = \frac{1}{2} \left((\nabla_k \nabla_i - \nabla_i \nabla_k) C_j{}^k + (\nabla_k \nabla_j - \nabla_j \nabla_k) C_i{}^k - \nabla^2 C_{ij} \right)$ since $\nabla_i \nabla_k C_j{}^k$ and $\nabla_j \nabla_k C_i{}^k$ are zero as the result of the divergence-free property of the Cotton tensor, hence
$\partial_t R_{ij} = \frac{1}{2} ([\nabla_k, \nabla_i] C_j{}^k + [\nabla_k, \nabla_j] C_i{}^k - \nabla^2 C_{ij})$,

or, $\partial_t R_{ij} = \frac{1}{2}(R^k{}_{lki}C_j{}^l - R^l{}_{jki}C_l{}^k + R^k{}_{lkj}C_i{}^l - R^l{}_{ikj}C_l{}^k - \nabla^2 C_{ij})$.

In this last line if the expression of the Riemann tensor in three dimension in terms of the Ricci tensor (see section 2.7) is used and if the order of the indices is changed, one finds the following result ,

$\partial_t R_{ij} = 3R_{l(i}C_{j)}{}^l - R^{lm}C_{lm}g_{ij} - \frac{1}{2}\mathfrak{R}C_{ij} - \frac{1}{2}\nabla^2 C_{ij}$.

iv. The evolution of the curvature scalar given in (2.36) is,

$$\partial_t\mathfrak{R} = \partial(g^{ij}R_{ij}) ,$$
$$= (\partial_t g^{ij})R_{ij} + g^{ij}(\partial_t R_{ij}) ,$$
$$= -C^{ij}R_{ij} + g^{ij}(\frac{3}{2}R_{li}C_j{}^l + \frac{3}{2}R_{lj}C_i{}^l - R^{lm}C_{lm}g_{ij} - \frac{1}{2}\mathfrak{R}C_{ij} - \frac{1}{2}\nabla^2 C_{ij}) ,$$

since C_{ij} is traceless the last two expressions on the right hand side give zero,

$$\partial_t\mathfrak{R} = -C^{ij}R_{ij} + \frac{3}{2}R_{li}C^{il} + \frac{3}{2}R_{lj}C^{jl} - 3R^{lm}C_{lm} ,$$

the indices in the last line are dummy indices $R_{li}C^{il} = R_{lj}C^{jl} = R^{lm}C_{lm}$, hence the result is,

$$\partial_t\mathfrak{R} = -C^{ij}R_{ij} .$$

v. The evolution of the Cotton tensor given in (2.52) is,

$$\partial_t C^{ij} = \frac{g^{1/3}}{2}[\varepsilon^{mni}\partial_t(\partial_m R^j{}_n + \Gamma^j_{mk}R^k{}_n - \Gamma^k_{mn}R^j{}_k) + \varepsilon^{mnj}\partial_t(\partial_m R^i{}_n + \Gamma^i_{mk}R^k{}_n - \Gamma^k_{mn}R^i{}_k)] ,$$
$$\partial_t C^{ij} = \frac{g^{1/3}}{2}\{\varepsilon^{mni}[\partial_m\partial_t R^j{}_n + \Gamma^j_{mk}\partial_t R^k{}_n - \Gamma^k_{mn}\partial_t R^j{}_k + (\partial_t\Gamma^j_{mk})R^k{}_n - (\partial_t\Gamma^k_{mn})R^j{}_k] +$$
$$+ \varepsilon^{mnj}[\partial_m\partial_t R^i{}_n + \Gamma^i_{mk}\partial_t R^k{}_n - \Gamma^k_{mn}\partial_t R^i{}_k + (\partial_t\Gamma^i_{mk})R^k{}_n - (\partial_t\Gamma^k_{mn})R^i{}_k]\} ,$$
$$\partial_t C^{ij} = \frac{g^{1/3}}{2}[\varepsilon^{mni}\nabla_m\partial_t R^j{}_n + \varepsilon^{mnj}\nabla_m\partial_t R^i{}_n + \varepsilon^{mni}(\partial_t\Gamma^j_{mk})R^k{}_n + \varepsilon^{mnj}(\partial_t\Gamma^i_{mk})R^k{}_n] ,$$
$$\partial_t C^{ij} = g^{1/3}(\varepsilon^{mn(i}\nabla_{|m|}\partial_t R^{j)}{}_n + \varepsilon^{mn(i}\partial_t\Gamma^{j)}_{mk}R^k{}_n) .$$

vi. $\partial_t(R_{ij}R^{ij}) = (\partial_t R_{ij})R^{ij} + R_{ij}(\partial_t R^{ij})$,

$$\partial_t(R_{ij}R^{ij}) = (\partial_t R_{ij})R^{ij} + R_{ij}[\partial_t(g^{im}g^{jn}R_{mn})] ,$$
$$\partial_t(R_{ij}R^{ij}) = 4R_{ik}C^k{}_jR^{ij} - 3\mathfrak{R}R_{ij}C^{ij} - R^{ij}\nabla^2 C_{ij} .$$

5.2 Flow Equations

First one needs to compute connection 1-forms $\omega^a{}_b$. For this purpose torsion-free condition $de^a + \omega^a{}_b \wedge e^b = 0$ and metric compatibility $\nabla_c g_{ab} = 0$ will be used.

$de^1 = d(\sqrt{A}\omega^1) = \sqrt{A}d\omega^1 = \sqrt{A}\lambda\,\omega^2 \wedge \omega^3 = \sqrt{\frac{A}{BC}}\lambda e^2 \wedge e^3$.

Similarly,

$de^2 = d(\sqrt{B}\omega^2) = \sqrt{B}d\omega^2 = \sqrt{B}\mu\,\omega^3 \wedge \omega^1 = \sqrt{\frac{B}{AC}}\mu e^3 \wedge e^1$,

$de^3 = d(\sqrt{C}\omega^3) = \sqrt{C}d\omega^3 = \sqrt{C}\nu\,\omega^1 \wedge \omega^2 = \sqrt{\frac{C}{AB}}\nu e^1 \wedge e^2$.

Now in the Milnor frame $g_{ab} = \delta_{ab}$, hence the metric compatibility becomes $\nabla_c \delta_{ab} = 0$.

$\nabla\delta_{ab} = \nabla(e_a, e_b) = (\nabla e_a).e_b + e_a.(\nabla e_b) = \omega^c{}_a e_c.e_b + e_a.\omega^d{}_b e_d = \omega^c{}_a \delta_{cb} + \delta_{ad}\omega^d{}_b = \omega^b{}_a + \omega^a{}_b$.

Therefore, $\omega^b{}_a = -\omega^a{}_b$. As a result of this equation, $\omega^1{}_1 = \omega^2{}_2 = \omega^3{}_3 = 0$.

Let us write the torsion-free condition and expand it for each parameter :

$1 : de^1 + \omega^1{}_1 \wedge e^1 + \omega^1{}_2 \wedge e^2 + \omega^1{}_3 \wedge e^3 = 0$, but $\omega^1{}_1 = 0$,

$2 : de^2 + \omega^2{}_1 \wedge e^1 + \omega^2{}_2 \wedge e^2 + \omega^2{}_3 \wedge e^3 = 0$, but $\omega^2{}_2 = 0$,

$3 : de^3 + \omega^3{}_1 \wedge e^1 + \omega^3{}_2 \wedge e^2 + \omega^3{}_3 \wedge e^3 = 0$, but $\omega^3{}_3 = 0$.

Let us now expand the connection 1-forms :

$\omega^1{}_2 = -\omega^2{}_1 = a_1 e^1 + a_2 e^2 + a_3 e^3$,

$\omega^1{}_3 = -\omega^3{}_1 = b_1 e^1 + b_2 e^2 + b_3 e^3$,

$\omega^2{}_3 = -\omega^3{}_2 = c_1 e^1 + c_2 e^2 + c_3 e^3$,

where $\{a_i\}$, $\{b_i\}$ and $\{c_i\}$ are just constants. As a result the connection 1-forms are :

$\omega^1{}_2 = \frac{1}{2\sqrt{ABC}}[\lambda A + \mu B - \nu C]e^3$,

$\omega^1{}_3 = \frac{1}{2\sqrt{ABC}}[-\lambda A + \mu B - \nu C]e^2$,

$\omega^2{}_3 = \frac{1}{2\sqrt{ABC}}[-\lambda A + \mu B + \nu C]e^1$.

Curvature 2-forms can be computed by the relation $R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$. For example ,

$R^1{}_2 = d\omega^1{}_2 + \omega^1{}_1 \wedge \omega^1{}_2 + \omega^1{}_2 \wedge \omega^2{}_2 + \omega^1{}_3 \wedge \omega^3{}_2 = d\omega^1{}_2 + \omega^1{}_3 \wedge \omega^3{}_2$, where,

$$\begin{aligned} d\omega^1{}_2 &= \frac{1}{2\sqrt{ABC}}[\lambda A + \mu B - \nu C]de^3 , \\ &= \frac{1}{2\sqrt{ABC}}[\lambda A + \mu B - \nu C]\sqrt{\frac{C}{AB}}\nu e^1 \wedge e^2 , \end{aligned}$$

$$= \frac{\nu}{2AB}[\lambda A + \mu B - \nu C] e^1 \wedge e^2 \ .$$

$$\omega^1{}_3 \wedge \omega^3{}_2 = \Big(\frac{1}{2\sqrt{ABC}}[-\lambda A + \mu B - \nu C] e^2\Big) \wedge \Big(-\frac{1}{2\sqrt{ABC}}[-\lambda A + \mu B + \nu C] e^1\Big) \ ,$$
$$= \frac{1}{4ABC}[\lambda^2 A^2 + \mu^2 B^2 - \nu^2 C^2 - 2\lambda\mu AB] e^1 \wedge e^2 \ .$$

As a result :

$$R^1{}_2 = \frac{1}{4ABC}[\lambda^2 A^2 + \mu^2 B^2 - 3\nu^2 C^2 - 2\lambda\mu AB + 2\lambda\nu AC + 2\mu\nu BC] e^1 \wedge e^2 \ .$$

Similarly,

$$R^1{}_3 = \frac{1}{4ABC}[\lambda^2 A^2 - 3\mu^2 B^2 + \nu^2 C^2 + 2\lambda\mu AB - 2\lambda\nu AC + 2\mu\nu BC] e^1 \wedge e^3 \ ,$$
$$R^2{}_3 = \frac{1}{4ABC}[-3\lambda^2 A^2 + \mu^2 B^2 + \nu^2 C^2 + 2\lambda\mu AB + 2\lambda\nu AC - 2\mu\nu BC] e^2 \wedge e^3 \ .$$

Ricci 1-forms can be computed by the relation $Ric_a = \iota_b R^b{}_a$. For example :

$$Ric_1 = \iota_2 R^2{}_1 + \iota_3 R^3{}_1 \ ,$$
$$= \frac{1}{4ABC}[\lambda^2 A^2 + \mu^2 B^2 - 3\nu^2 C^2 - 2\lambda\mu AB + 2\lambda\nu AC + 2\mu\nu BC] \iota_2(e^2 \wedge e^1) +$$
$$+ \frac{1}{4ABC}[\lambda^2 A^2 - 3\mu^2 B^2 + \nu^2 C^2 + 2\lambda\mu AB - 2\lambda\nu AC + 2\mu\nu BC] \iota_3(e^3 \wedge e^1) \ ,$$

where, $\iota_2(e^2 \wedge e^1) = (\iota_2 e^2) \wedge e^1 + (-1)^1 e^2 \wedge (\iota_2 e^1) = e^1 \ ,$

$\iota_3(e^3 \wedge e^1) = (\iota_3 e^3) \wedge e^1 + (-1)^1 e^3 \wedge (\iota_3 e^1) = e^1 \ .$

Hence :

$$Ric_1 = \frac{1}{2ABC}[\lambda^2 A^2 - \mu^2 B^2 - \nu^2 C^2 + 2\mu\nu BC] e^1 \ .$$

Similarly,

$$Ric_2 = \frac{1}{2ABC}[-\lambda^2 A^2 + \mu^2 B^2 - \nu^2 C^2 + 2\lambda\nu AC] e^2 \ ,$$
$$Ric_3 = \frac{1}{2ABC}[-\lambda^2 A^2 - \mu^2 B^2 + \nu^2 C^2 + 2\lambda\mu AB] e^3 \ .$$

Finally the curvature scalar can be computed from the equation $\mathfrak{R} = \iota_a (Ric)^a$ (or simply from the previous chapter),

$$\mathfrak{R} = \frac{1}{2ABC}[-\lambda^2 A^2 - \mu^2 B^2 - \nu^2 C^2 + 2\lambda\mu AB + 2\lambda\nu AC + 2\mu\nu BC] \ .$$

The Cotton 2-form by the inspection of equation (5.1) is in the following form :
$C^a = d[(Ric)^a - \frac{1}{4}\Re e^a] + \omega^a{}_b \wedge [(Ric)^b - \frac{1}{4}\Re e^b]$.
For example :
$C^1 = d[(Ric)^1 - \frac{1}{4}\Re e^1] + \omega^1{}_2 \wedge [(Ric)^2 - \frac{1}{4}\Re e^2] + \omega^1{}_3 \wedge [(Ric)^3 - \frac{1}{4}\Re e^3]$.
Hence :
$C^1 = \frac{1}{2(ABC)^{3/2}}[-\lambda^2 A^2(-2\lambda A + \mu B + \nu C) - (\mu B + \nu C)(\mu B - \nu C)^2] e^2 \wedge e^3$,
$C^2 = \frac{1}{2(ABC)^{3/2}}[-\mu^2 B^2(\lambda A - 2\mu B + \nu C) - (\lambda A + \nu C)(\lambda A - \nu C)^2] e^3 \wedge e^1$,
$C^3 = \frac{1}{2(ABC)^{3/2}}[-\nu^2 C^2(\lambda A + \mu B - 2\nu C) - (\lambda A + \mu B)(\lambda A - \mu B)^2] e^1 \wedge e^2$.

The last thing to compute to find flow equations is to find $*C^a$.
$*C^1 = \frac{1}{2(ABC)^{3/2}}[-\lambda^2 A^2(-2\lambda A + \mu B + \nu C) - (\mu B + \nu C)(\mu B - \nu C)^2] * (e^2 \wedge e^3)$,

$*(e^2 \wedge e^3) = \frac{\sqrt{|g|}}{(3-2)!}\varepsilon^{23}{}_1 e^1 = e^1$.

Therefore :
$*C^1 = \frac{1}{2(ABC)^{3/2}}[-\lambda^2 A^2(-2\lambda A + \mu B + \nu C) - (\mu B + \nu C)(\mu B - \nu C)^2] e^1$.
Similarly,
$*C^2 = \frac{1}{2(ABC)^{3/2}}[-\mu^2 B^2(\lambda A - 2\mu B + \nu C) - (\lambda A + \nu C)(\lambda A - \nu C)^2] e^2$,
$*C^3 = \frac{1}{2(ABC)^{3/2}}[-\nu^2 C^2(\lambda A + \mu B - 2\nu C) - (\lambda A + \mu B)(\lambda A - \mu B)^2] e^3$.

Finally, the flow equations are :
$\partial_t e^1 = *C^1, \qquad e^1 = \sqrt{A}\omega^1, \qquad \partial_t e^1 = \partial_t \sqrt{A}\omega^1 = \frac{1}{2\sqrt{A}}\frac{dA}{dt}\omega^1$.
Hence :
$\frac{dA}{dt} = \frac{A}{(ABC)^{3/2}}[-\lambda^2 A^2(-2\lambda A + \mu B + \nu C) - (\mu B + \nu C)(\mu B - \nu C)^2]$.

The Cotton flow preserves volume density as mentioned in the first section, therefore it can be assumed that ABC=1. With this last property, the first flow equation is :
$\frac{dA}{dt} = A[-\lambda^2 A^2(-2\lambda A + \mu B + \nu C) - (\mu B + \nu C)(\mu B - \nu C)^2]$.
Similarly,

$$\frac{dB}{dt} = B[-\mu^2 B^2(\lambda A - 2\mu B + \nu C) - (\lambda A + \nu C)(\lambda A - \nu C)^2] ,$$
$$\frac{dC}{dt} = C[-\nu^2 C^2(\lambda A + \mu B - 2\nu C) - (\lambda A + \mu B)(\lambda A - \mu B)^2] .$$

5.3 Cotton Entropy

In Chapter 1 it is shown that the Cotton tensor is covariantly conserved. Therefore it is natural to seek a geometrical invariance and it was shown that this invariance is of Chern-Simons form [7]. Therefore Chern-Simons action can be defined as the entropy functional F of the flow:

$$F = -\frac{1}{4}\int (\omega^a{}_b \wedge d\omega^b{}_a + \frac{2}{3}\omega^a{}_b \wedge \omega^b{}_c \wedge \omega^c{}_a) e^1 \wedge e^2 \wedge e^3. \tag{5.4}$$

This equation when all the symmetries of $\omega^a{}_b$ and of the wedge product are used, gives the following result :

$$F = \frac{1}{4}\int (\lambda^3 A^3 + \mu^3 B^3 + \nu^3 C^3 - \lambda^2 \mu A^2 B - \lambda^2 \nu A^2 C - \lambda \mu^2 A B^2 - \mu^2 \nu B^2 C - \lambda \nu^2 A C^2 - \mu \nu^2 B C^2 + $$
$$+ 4\lambda\mu\nu ABC) e^1 \wedge e^2 \wedge e^3 ,$$
$$= \frac{1}{4}\int (2\lambda\mu\nu ABC + [\lambda A + \mu B - \nu C][-\lambda A + \mu B - \nu C][-\lambda A + \mu B + \nu C]) e^1 \wedge e^2 \wedge e^3 .$$

As it is seen this functional is negative for some geometries although it is non-decreasing. Hence this functional is not an appropriate entropy in the usual sense. For some geometries a simpler functional can be found to understand the behaviour of the flow. Firstly it should be emphasized that since the volume ABC is conserved by the Cotton flow, its derivative with respect to t must be zero, that is $\frac{dABC}{dt} = 0$. But also $\frac{dABC}{dt} = \frac{dA}{dt}BC + A\frac{dB}{dt}C + AB\frac{dC}{dt} = 0$, dividing this result by the invariant ABC gives $\frac{1}{A}\frac{dA}{dt} + \frac{1}{B}\frac{dB}{dt} + \frac{1}{C}\frac{dC}{dt} = 0$.

$$\frac{d}{dt}\left(\frac{1}{A^2}\right) = \left(-\frac{2}{A^3}\right)\frac{dA}{dt} \rightarrow \left(-\frac{1}{2}A^2\right)\frac{d}{dt}\left(\frac{1}{A^2}\right) = \frac{1}{A}\frac{dA}{dt} ,$$
$$\frac{1}{A}\frac{dA}{dt} + \frac{1}{B}\frac{dB}{dt} + \frac{1}{C}\frac{dC}{dt} = \left(-\frac{1}{2}A^2\right)\frac{d}{dt}\left(\frac{1}{A^2}\right) + \left(-\frac{1}{2}B^2\right)\frac{d}{dt}\left(\frac{1}{B^2}\right) + \left(-\frac{1}{2}C^2\right)\frac{d}{dt}\left(\frac{1}{C^2}\right) = 0 .$$

$$\frac{d}{dt}\left(\frac{1}{A^2}\right) = \left(-\frac{2}{A^3}\right)\frac{dA}{dt} = 2\lambda^2(-2\lambda A+\mu B+\nu C)+\frac{2}{A^2}(\mu B+\nu C)(\mu B-\nu C)^2\ ,$$
$$\frac{d}{dt}\left(\frac{1}{B^2}\right) = \left(-\frac{2}{B^3}\right)\frac{dB}{dt} = 2\mu^2(\lambda A-2\mu B+\nu C)+\frac{2}{B^2}(\lambda A+\nu C)(\lambda A-\nu C)^2\ ,$$
$$\frac{d}{dt}\left(\frac{1}{C^2}\right) = \left(-\frac{2}{C^3}\right)\frac{dC}{dt} = 2\nu^2(\lambda A+\mu B-2\nu C)+\frac{2}{C^2}(\lambda A+\mu B)(\lambda A-\mu B)^2\ .$$

Therefore :

$$\frac{d}{dt}\left(\frac{\mu^2\nu^2}{A^2}\right) = \mu^2\nu^2\frac{d}{dt}\left(\frac{1}{A^2}\right) = 2\lambda^2\mu^2\nu^2(-2\lambda A+\mu B+\nu C)+\frac{2\mu^2\nu^2}{A^2}(\mu B+\nu C)(\mu B-\nu C)^2$$

.

Similarly ,

$$\frac{d}{dt}\left(\frac{\lambda^2\nu^2}{B^2}\right) = \lambda^2\nu^2\frac{d}{dt}\left(\frac{1}{B^2}\right) = 2\lambda^2\mu^2\nu^2(\lambda A-2\mu B+\nu C)+\frac{2\lambda^2\nu^2}{B^2}(\lambda A+\nu C)(\lambda A-\nu C)^2$$

,

$$\frac{d}{dt}\left(\frac{\lambda^2\mu^2}{C^2}\right) = \lambda^2\mu^2\frac{d}{dt}\left(\frac{1}{C^2}\right) = 2\lambda^2\mu^2\nu^2(\lambda A+\mu B-2\nu C)+\frac{2\lambda^2\mu^2}{C^2}(\lambda A+\mu B)(\lambda A-\mu B)^2$$

.

When this results are added up gives the result :

$$\frac{d}{dt}\left(\frac{\mu^2\nu^2}{A^2}+\frac{\lambda^2\nu^2}{B^2}+\frac{\lambda^2\mu^2}{C^2}\right) = \frac{2\mu^2\nu^2}{A^2}(\mu B+\nu C)(\mu B-\nu C)^2+\frac{2\lambda^2\nu^2}{B^2}(\lambda A+\nu C)(\lambda A-\nu C)^2+$$
$$+\frac{2\lambda^2\mu^2}{C^2}(\lambda A+\mu B)(\lambda A-\mu B)^2\ .$$

The inspection of the right-hand side of the above equation shows that for the case $\mathsf{SU}(2)$ and $\mathsf{Isom}(\mathsf{R}^2)$ with $\{-1,-1,-1\}$ and $\{-1,-1,0\}$ respectively, this equation is non-increasing (≤ 0). Hence the equation $\frac{d}{dt}\left(-\frac{\mu^2\nu^2}{A^2}-\frac{\lambda^2\nu^2}{B^2}-\frac{\lambda^2\mu^2}{C^2}\right)\geq 0$ for $\mathsf{SU}(2)$ and $\mathsf{Isom}(\mathsf{R}^2)$ and can be used as the new entropy functional $f(t)$ for these geometries.

$$f(t) = \left(-\frac{\mu^2\nu^2}{A^2}-\frac{\lambda^2\nu^2}{B^2}-\frac{\lambda^2\mu^2}{C^2}\right). \tag{5.5}$$

5.4 Cotton Flow on Homogeneous 3-Manifolds

Let us now consider the nine geometries separately under the Cotton flow.

I.The geometry of R^3 with the structure constants $\lambda, \mu, \nu = \{0,0,0\}$.

The metric of these geometries are flat ($\mathfrak{R} = 0$) and do not change under the Cotton flow because the Cotton tensor is conformally invariant. The initial metric g_0 stands for all $t \geq 0$. Hence R^3 is a fixed point of the Cotton flow.

II.The geometry of $\mathsf{SU}(2)$ with the structure constants $\lambda, \mu, \nu = \{-1,-1,-1\}$ or $\{1,1,1\}$.

The curvature scalar is,

$$\mathfrak{R} = \frac{1}{2}[-A^2 - B^2 - C^2 + 2AB + 2AC + 2BC] .$$

The entropy functional reads :

$$f(t) = \left(-\frac{1}{A^2} - \frac{1}{B^2} - \frac{1}{C^2}\right) ,$$

$$\frac{df}{dt} = \frac{2}{A^2}(B+C)(B-C)^2 + \frac{2}{B^2}(A+C)(A-C)^2 + \frac{2}{C^2}(A+B)(A-B)^2 .$$

The flow equations yield :

$$\frac{dA}{dt} = A[-A^2(2A-B-C) + (B+C)(B-C)^2] ,$$

$$\frac{dB}{dt} = B[-B^2(-A+2B-C) + (A+C)(A-C)^2] ,$$

$$\frac{dC}{dt} = C[-C^2(-A-B+2C) + (A+B)(A-B)^2] .$$

These three flow equations are all symmetric in A,B and C and when $A = B = C \to \frac{d}{dt}A = \frac{d}{dt}B = \frac{d}{dt}C = 0$, like in the case of the Ricci flow the round sphere is hence the fixed point of the Cotton flow as well. When $A \neq B \neq C$, flow equations will not be equal to zero. It is not possible to solve the flow equations analytically, so one needs to make some estimates. Let us investigate the difference between flow equations :

$$\frac{d}{dt}(A-B) = (A-B)[-2(A+B)(A^2+B^2) + C(A+B)^2 + C^3] ,$$

$$\frac{d}{dt}(A-C) = (A-C)[-2(A+C)(A^2+C^2) + B(A+C)^2 + B^3] ,$$

$$\frac{d}{dt}(B-C) = (B-C)[-2(B+C)(B^2+C^2)+A(B+C)^2+A^3] \; .$$

Using the symmetry let us assume that initially $A_O \geq B_O \geq C_O$. If at $t = \tau$, $A_\tau = B_\tau$, then :

$$\frac{d}{dt}(A-B)|_{t=\tau} = (A_\tau - B_\tau)[-2(A_\tau + B_\tau)(A_\tau^2 + B_\tau^2) + C_\tau(A_\tau + B_\tau)^2 + C_\tau^3] = 0 \; .$$

This means that the flow of the difference between the functions A and B stops at $t = \tau$ if $A_\tau = B_\tau$ (second derivative test gives the same result). Similarly, if $B_\tau = C_\tau$, then the flow of the difference between the functions B and C stops at $t = \tau$. Therefore if initially, $A_O \geq B_O \geq C_O$ then at all $t \geq 0$, $A \geq B \geq C$. Now let us investigate the flow equation of C using these conditions :

$$\frac{dC}{dt} = C[C^2 \underbrace{(A+B-2C)}_{\geq 0} + (A+B)(A-B)^2] \geq 0 \; .$$

This shows that C is a non-decreasing function of t, that is $C \geq C_0$. Hence, $A \geq B \geq C \geq C_0$, and using this condition the flow of $(A-C)$ becomes :

$$\begin{aligned}
\frac{d}{dt}(A-C) &= (A-C)[-2(A+C)(A^2+C^2)+B(A+C)^2+B^3] \; , \\
&\leq (A-C)[-2(C_0+C_0)(C_0^2+C_0^2)+C_0(C_0+C_0)^2+C_0^3] \; , \\
&\leq -3C_0^3(A-C) \; .
\end{aligned}$$

By direct integration one obtains :
$(A-C) \leq exp(-3C_0^3 t)(A_0 - C_0)$

Therefore $(A-C)$ vanishes exponentially (B is squeezed between A and C), so the geometry converges to the round sphere exponentially. The curvature scalar hence converges to the constant $\frac{3}{2}$ ($\mathfrak{R} \to \frac{3}{2}$) . The entropy functional $f(t) \geq -3$ and $\frac{df}{dt} \geq 0$.

III.The geometry of $\mathsf{SL}(2,\mathbb{R})$ with the structure constants $\lambda, \mu, \nu = \{-1,-1,1\}$.

The curvature scalar is,
$\mathfrak{R} = \frac{1}{2}[-A^2 - B^2 - C^2 + 2AB - 2AC + 2BC]$.

The flow equations yield :

$$\frac{dA}{dt} = A[-A^2(2A - B + C) + (B - C)(B + C)^2] ,$$
$$\frac{dB}{dt} = B[-B^2(-A + 2B + C) + (A - C)(A + C)^2] ,$$
$$\frac{dC}{dt} = C[C^2(A + B + 2C) + (A + B)(A - B)^2] .$$

These equations are symmetric in A and B. However in this case it is not possible to talk about a fixed point of the flow because the flows of the functions do not stop even if $A = B = C$ at some $t \geq 0$. It is not possible to solve the equations analytically, so it is necessary to make some estimates. Firstly, since C is strictly positive it is an increasing function of t , so $C \geq C_0$ and it can be shown that in a finite $t, C \to \infty$:

$$\begin{aligned}\frac{dC}{dt} &= C[C^2(A + B + 2C) + (A + B)(A - B)^2] ,\\ &\geq 2C^4 \quad \text{equality occurs only if } A = B = 0 ,\\ &\geq 2C_0^4 .\end{aligned}$$

By direct integration one obtains :

$C \geq 2C_0^4 t + C_0$, this proves the claim that in a finite $t, C \to \infty$.

Secondly, it can be shown that if initially $A_0 \geq B_0$ then for all $t \geq 0, A \geq B$:

$$\begin{aligned}\frac{d}{dt}(A - B) &= -(A - B)[2(A + B)(A^2 + B^2) + C(A + B)^2 + C^3] ,\\ &= -(A - B)[2(A + B)(A^2 + B^2) + C(A^2 + B^2) + C^3 + 2ABC] ,\\ &= -(A - B)[2(A + B)(A^2 + B^2) + C(A^2 + B^2) + C^3 + 2] \qquad \text{since } ABC = 1 .\end{aligned}$$

Since at the point $A = B$ the flow of $(A-B)$ will stop as it is seen from the above equation, that is $\frac{d}{dt}(A-B)|_{A=B}= 0$, this proves the claim.. Using two conditions on the functions $A \geq B$ and $C \geq C_0$, the difference between A and B becomes :

$$\frac{d}{dt}(A-B) = -(A-B)\underbrace{[2(A+B)(A^2+B^2)+C(A^2+B^2)+C^3+2]}_{\geq (C_0^3+2)},$$
$$\leq -(A-B)(C_0^3+2)\ .$$

By direct integration one obtains :
$(A-B) \leq (A_0-B_0)exp[-(C_0^3+2)t]$.

Hence A and B are approaching to each other exponentially as C grows. Now let us see what happens to A during the flow :

$$\frac{dA}{dt} = A[(B^3-A^3)+C(B^2-A^2)+A^2(B-A)-BC^2-C^3],$$
$$\leq -ABC^2-AC^3,$$
$$\leq -AC^3\ .$$

This shows that A is strictly decreasing. Since $C \to \infty$ in a finite t , there must be a $t = \tau$ after which $C \geq A+B$, so that :

$$\frac{dC}{dt}|_{t\geq\tau}= \left\{C[C^2\underbrace{(A+B+2C)}_{\geq 3C_\tau}+(A+B)(A-B)^2]\right\}|_{t\geq\tau}\ ,$$
$$\geq 3C_\tau^4\ .$$

Therefore :
$\frac{dA}{dt}|_{t\geq\tau}\leq -AC_\tau^3$.

By direct integration one obtains :

$A_{t\geq\tau} \leq exp[-C_{\tau}^{3}(t-\tau)]A_0$.

This shows that as $C\to\infty, A\to 0$, so does B since $A\geq B$. Hence this geometry gives cigar degeneracy under the Cotton flow. The curvature scalar $\mathfrak{R}$ is diverging as shown below :

$$\mathfrak{R} = -\frac{1}{2}\underbrace{(A-B+C)^2}_{\geq C^2} ,$$
$$\leq -\frac{1}{2}C^2 .$$

Since $C\to\infty$ in a finite t the scalar curvature is diverging, showing that a curvature singularity arises.

IV.The geometry of $\mathsf{Isom}(\mathsf{R}^2)$ with the structure constants $\lambda,\mu,\nu = \{-1,-1,0\}$.

The curvature scalar is,

$$\mathfrak{R} = \frac{1}{2}[-A^2-B^2+2AB] ,$$

The flow equations yield :

$$\frac{dA}{dt} = A[-2A^3+A^2B+B^3] ,$$
$$\frac{dB}{dt} = B[A^3+AB^2-2B^3] ,$$
$$\frac{dC}{dt} = C[(A+B)(A-B)^2] .$$

There is a symmetry between A and B , so let us look at their difference in order to make some estimates :

$$\frac{d}{dt}(A-B) = -2(A-B)(A+B)(A^2+B^2) .$$

If at some $t, A=B$, then $\frac{d}{dt}(A-B)=0$, so if initially $A_0\geq B_0$, then for all $t, A\geq B$. Using this condition now it is possible to make other estimates about the flow equations :

$$\frac{dA}{dt} = A\underbrace{[-A^3 + A^2B - A^3 + B^3]}_{\leq 0} \leq 0 \ , \quad A \text{ is a non-increasing function .}$$

$$\frac{dB}{dt} = B\underbrace{[A^3 - B^3 + AB^2 - B^3]}_{\geq 0} \geq 0 \ , \quad B \text{ is a non-decreasing function .}$$

Therefore $A_0 \geq A \geq B \geq B_0$ and by using this condition it is possible to find an upper bound and a lower bound for the flow of $(A - B)$ as below :

$$\frac{d}{dt}(A-B) = -2(A-B)\underbrace{(A+B)(A^2+B^2)}_{\geq 4B_0^3} \ ,$$
$$\leq -8B_0^3(A-B) \ .$$

By direct integration one obtains :
$(A-B) \leq exp(-8B_0^3 t)(A_0 - B_0)$.

Similarly the lower bound for $(A - B)$ can be found :

$$\frac{d}{dt}(A-B) = -2(A-B)\underbrace{(A+B)(A^2+B^2)}_{\leq 4A_0^3} \ ,$$
$$\geq -8A_0^3(A-B) \ .$$

By direct integration one obtains :
$(A-B) \geq exp(-8A_0^3 t)(A_0 - B_0)$.

These are the upper and lower bounds for $(A-B)$ and they show that in the limit $t \to \infty, A = B$. Similarly the upper and lower bounds for the flow of $(A+B)$ can be found in order to see the evolution of the geometry.

$$
\begin{aligned}
\frac{d}{dt}(A+B) &= -2(A+B)\underbrace{(A^3+B^3)}_{\geq 2B_0^3} , \\
&\leq -4B_0^3(A+B) . \\
(A+B) &\leq exp(-4B_0^3 t)(A_0+B_0) .
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}(A+B) &= -2(A+B)\underbrace{(A^3+B^3)}_{\leq 2A_0^3} , \\
&\geq -4A_0^3(A+B) . \\
(A+B) &\geq exp(-4A_0^3 t)(A_0+B_0) .
\end{aligned}
$$

Now using these bounds it is clear that A approaches to a constant value in the limit $t \to \infty$:

$$\frac{dA}{dt} = -A[(A-B)\underbrace{(2A^2+AB+B^2)}_{\geq 0}] ,$$

where $exp(-8B_0^3 t)(A_0-B_0) \geq (A-B) \geq exp(-8A_0^3 t)(A_0-B_0)$, so the upper and lower bounds of A can be obtained by direct integration as :

$$\frac{(A_0-B_0)}{8A_0^3} exp(-8A_0^3 t) \geq \ln A \geq \frac{(A_0-B_0)}{8B_0^3} exp(-8B_0^3 t) .$$

This last expression proves the claim. Since $(A-B)$ vanishes exponentially, B approaches to a constant value as well. Since $ABC = 1$ there is no suspect that C will also approach to a constant value, but let us see it anyway by using the bounds of $(A-B)$ and $(A+B)$ as below :

$$\frac{dC}{dt} = C[(A+B)(A-B)^2] ,$$

where $exp(-16B_0^3 t)(A_0-B_0)^2 \geq (A-B)^2 \geq exp(-16A_0^3 t)(A_0-B_0)^2$ and ,

$exp(-4B_0^3 t)(A_0+B_0) \geq (A+B) \geq exp(-4A_0^3 t)(A_0+B_0)$ which will give the result,

$$\frac{(A_0+B_0)(A_0-B_0)^2}{(-20B_0^3)} exp(-20B_0^3 t) \geq \ln C \geq \frac{(A_0+B_0)(A_0-B_0)^2}{(-20A_0^3)} exp(-20A_0^3 t).$$

Therefore A, B and C are approaching to constant values in the limit $t \to \infty$, so that the geometry becomes flat. The curvature scalar can be squeezed between two values by using $(A-B)$

to prove that it approaches zero in the limit $t \to \infty$,

$$|\mathfrak{R}| = \frac{1}{2}(A-B)^2 \,,$$
$$|\mathfrak{R}_o| exp(-16B_0^3 t) \geq |\mathfrak{R}| \geq |\mathfrak{R}_o| exp(-16A_0^3 t) \text{ , where } |\mathfrak{R}_o| = \frac{1}{2}(A_0 - B_0)^2 \,.$$

Hence the curvature scalar exponentially vanishes. Like in the case of the other geometries the evolution of this geometry is computed analysing the relations between the functions. However, it is possible in this case to solve the equations analytically as it is done in [15] realizing that there is a second conserved quantity besides ABC. Now let us find it :

$$C(A-B)\frac{d}{dt}(A-B) + 2(A^2+B^2)\frac{dC}{dt} = 0, \quad \text{from the flow equations.}$$

In the above equation when the replacement $\left\{ (A^2+B^2) = (A-B)^2 + 2AB = (A-B)^2 + \frac{2}{C} \right\}$ is made, the equation becomes :

$$C(A-B)\frac{d}{dt}(A-B) + 2\left[(A-B)^2 + \frac{2}{C}\right]\frac{dC}{dt} = 0 \,.$$

The integrating factor for this equation is $2C^3$ and integration of this equation gives :

$$C^4(A-B)^2 + \frac{8}{3}C^3 = k \,, \quad \text{where } k \text{ is a constant.}$$

The above function is then a conserved quantity of this geometry under the Cotton flow and it can be used for the direct integration of A,B and C as below :

$$C^4(A-B)^2 + \frac{8}{3}C^3 = k \to (A-B)^2 = \frac{k}{C^4} - \frac{8}{3C},$$
$$(A+B)^2 = (A-B)^2 + 4AB = (A-B)^2 + \frac{4}{C} = \frac{k}{C^4} + \frac{4}{3C} \,.$$

Hence, $\frac{dC}{dt} = C[(A+B)(A-B)^2] = \left(\frac{k}{C^3} - \frac{8}{3}\right)\sqrt{\frac{k}{C^4} + \frac{4}{3C}}$,

$$A = \frac{1}{2}[(A+B) + (A-B)] = \frac{1}{2}\left(\sqrt{\frac{k}{C^4} + \frac{4}{3C}} + \sqrt{\frac{k}{C^4} - \frac{8}{3C}}\right),$$

$$B = \frac{1}{2}[(A+B)-(A-B)] = \frac{1}{2}\left(\sqrt{\frac{k}{C^4}+\frac{4}{3C}} - \sqrt{\frac{k}{C^4}-\frac{8}{3C}}\right) .$$

Once C is integrated A and B can be computed easily, see [15] for a solution.

V.The geometry of $\mathsf{E}(1,1)$ with the structure constants $\lambda, \mu, \nu = \{-1, 0, 1\}$.

The curvature scalar is,

$$\mathfrak{R} = \frac{1}{2}[-A^2 - C^2 - 2AC] .$$

The flow equations yield :

$$\frac{dA}{dt} = A[-2A^3 - A^2C - C^3] ,$$
$$\frac{dB}{dt} = B[(A-C)(A+C)^2] ,$$
$$\frac{dC}{dt} = C[A^3 + AC^2 + 2C^3] .$$

In this case there is an anti-symmetry between A and C, so their behaviour are opposite to each other. A is a strictly decreasing function and C is a strictly increasing function, so that $A < A_0$ and $C > C_0$. Using these two conditions, let us look at the flows :

$$\frac{dA}{dt} = A[\underbrace{-2A^3 - A^2C}_{<0} - \underbrace{C^3}_{>C_0^3}] ,$$
$$< -AC_0^3 ,$$

$A < A_0 exp(-C_0^3 t)$, this upper bound shows that in the limit $t \to \infty$, A vanishes.

$$\frac{dC}{dt} = C[\underbrace{A^3 + AC^2}_{>0} + \underbrace{2C^3}_{>2C_0^3}] ,$$
$$> C2C_0^3 ,$$

$C > C_0 exp(2C_0^3 t)$, this shows that in the limit $t \to \infty$, C diverges.

The behaviour of B is not easy to estimate. Firstly one needs to make estimates about $(C-A)$ and $(A+C)$.

$$\frac{d}{dt}(C-A) = 2[\underbrace{(A+C)}_{>C_0}\underbrace{(A^3+C^3)}_{>C_0^3}] ,$$
$$> 2C_0^4.$$

$(C-A) > 2C_0^4 t + (C_0 - A_0)$, so A and C are diverging linearly in t.

$$\frac{d}{dt}(A+C) = 2(A+C)\underbrace{(C^2+A^2)}_{>C_0^2}\underbrace{(C-A)}_{>2C_0^4 t} ,$$
$$> 4C_0^6 t.$$

$(A+C) > (A_0+C_0)exp(4C_0^6 t)$, so $(A+C)$ is increasing exponentially.

Finally it is clear that B is vanishing (shrinking) too since its flow equation consists of $(A+C)$ and $(C-A)$ terms :

$$\frac{dB}{dt} = -B[(C-A)(A+C)^2] ,$$
$$= -B[(C^2-A^2)(A+C)] , \quad \text{by suppressing } (C^2-A^2) > 0 \text{ term just for simplicity ,}$$
$$< -B(A_0+C_0)exp(4C_0^6 t).$$

$lnB < -\frac{(A_0+C_0)}{4C_0^6} exp(4C_0^6 t)$, this shows that B is vanishing.

Since two dimensions vanish while the other grows this geometry gives a cigar degeneracy under the Cotton flow. The curvature scalar diverges as well,

$\mathfrak{R} = -\frac{1}{2}(A+C)^2$, since $(A+C)$ diverges in the limit $t \to \infty$.

It is yet possible to solve the equations analytically, [15], because there is another conserved quantity in this case too.

$B(A+C)\frac{d}{dt}(A+C) + 2(A+C)^2\frac{dB}{dt} = 0$, from the flow equations. The integrating constant is $2B^3$, so $B^4(A+C)^2 = k$ where k is a constant. The conserved quantity can be used again for direct integration of the equations as follows :

$$B^4(A^2+C^2)=k \to (A^2+C^2)=\frac{k}{B^4} \to (A+C)^2=\frac{k}{B^4}+\frac{2}{B}, \qquad (A-C)^2=\frac{k}{B^4}-\frac{2}{B}.$$

Hence, $\dfrac{dB}{dt}=B[(A-C)(A+C)^2]=\left(\dfrac{k}{B^3}+2\right)\sqrt{\dfrac{k}{B^4}-\dfrac{2}{B}}$,

$$A=\frac{1}{2}[(A+C)+(A-C)]=\left(\sqrt{\frac{k}{B^4}+\frac{2}{B}}+\sqrt{\frac{k}{B^4}-\frac{2}{B}}\right),$$
$$C=\frac{1}{2}[(A+C)-(A-C)]=\left(\sqrt{\frac{k}{B^4}+\frac{2}{B}}-\sqrt{\frac{k}{B^4}-\frac{2}{B}}\right).$$

Once B is integrated A and C can be computed easily, see [15] for a solution.

VI.The geometry of Heisenberg with the structure constants $\lambda,\mu,\nu=\{-1,0,0\}$.

The curvature scalar is,
$\mathfrak{R}=\dfrac{-1}{2}A^2$.

The flow equations yield :
$$\frac{dA}{dt}=-2A^4 ,$$
$$\frac{dB}{dt}=A^3B ,$$
$$\frac{dC}{dt}=A^3C .$$

These equations can be directly integrated starting from the first one :

$$A(t)=\left(6t+\frac{1}{A_0^3}\right)^{-1/3}, \qquad B(t)=\left(6t+\frac{1}{A_0^3}\right)^{1/6}B_0 , \qquad C(t)=\left(6t+\frac{1}{A_0^3}\right)^{1/6}C_0 .$$

As t evolves, B and C diverge (grow) whereas A vanishes. Hence this geometry gives pancake degeneracy under the Cotton flow. The curvature scalar vanishes as well since :
$$|\mathfrak{R}|=\frac{1}{2}A^2 ,$$
$$=\frac{1}{2}\left(6t+\frac{1}{A_0^3}\right)^{-2/3} .$$

VII. The geometries of $H^3, S^2 \times \mathbb{R}, S^2 \times \mathbb{R}$ The metrics of these geometries are flat, so as in the case of R^3 the initial metric g_0 stands for all $t \geq 0$ for all three geometries where their metric are the ones in Chapter 4. These geometries are fixed points of the Cotton flow.

CHAPTER 6

RICCI AND COTTON SOLITONS

In Chapters 4 and 5 the flows of three dimensional homogeneous geometries are studied. In some cases (such as $\mathbb{R}^3$ for both Ricci and Cotton flow) the geometry is called as a fixed point of the flow because the initial metric (up to a scale factor) is preserved under the flow. Solitons can be thought as the fixed points of the flows and also as the generalized fixed points of the flows, because solitons are considered to be the solutions of the flow which evolve along the symmetries of the flow. The solutions of the Ricci flow are called solitons because it can be considered as a heat equation for metrics. Thus it is expected that under the Ricci flow the curvature diffuses over the manifold as t evolves just like the heat diffuses as time evolves. The first section of this Chapter is based on [4] and [5] and the second section is based on [2].

6.1 Ricci Solitons

The first kind of symmetries of the Ricci flow are scalings defined by $\widetilde{g_{ij}} = \kappa g_{ij}$ where κ is the scaling constant. It is not difficult to show that this is actually a symmetry of the flow.
$\widetilde{g_{ij}} = \kappa g_{ij} \rightarrow \widetilde{g^{ij}} = \kappa^{-1} g^{ij}$ since $\widetilde{g^{ij}}\widetilde{g_{mj}} = \delta^i{}_m = g^{ij} g_{mj}$,
$\widetilde{\Gamma^{ij}_m} = \Gamma^{ij}_m$, as a result of the equation (2.25),
$\widetilde{R_{ij}} = R_{ij}$, as a result of the equation (2.35),
$\widetilde{\mathfrak{R}} = \kappa^{-1}\mathfrak{R}$, as a result of the equation (2.36).
Therefore the fixed points of the Ricci flow are Ricci solitons and they can be classified into two categories as the fixed points of the unnormalized and of the normalized Ricci flow . The first class of metrics which satisfies the equation $\partial_t g = -2Ric$ are the Ricci flat metrics so that the Ricci tensor vanishes (source free gravitational field). The second class of metrics which

satisfies the equation $\partial_t g = -2Ric + \frac{2}{3}\mathfrak{R}g$ are the Einstein metrics so that the Ricci tensor is $Ric = kg$ where k is a constant. The Ricci flat metrics are Einstein metrics with $k = 0$. Thus Einstein metrics are Ricci solitons.

The second kind of symmetries of the Ricci flow are diffeomorphisms so that $Ric[g(t)] = \phi_t^*[Ric(g_0)]$ (see A.A). Therefore the generalized fixed points of the unnormalized Ricci flow which are only allowed to change by a diffeomorphism and a rescaling are Ricci solitons. A solution $g(t)$ of the Ricci flow on M^n is a Ricci soliton if there exists a positive function $\sigma(t)$ that is a t dependent scale constant and a one-parameter diffeomorphisms ϕ_t of M^n such that :

$$g(t) = \sigma(t)\phi_t^*(g_0) \quad , \tag{6.1}$$

for all $t \in (\alpha, \omega)$. Let us now differentiate the equation (5.1) with respect to the parameter t, $\partial_t(g(t)) = \frac{d}{dt}[\sigma(t)\phi_t^*(g_0)] = \frac{d}{dt}(\sigma(t))\phi_t^*(g_0) + \sigma(t)\frac{d}{dt}[\phi_t^*(g_0)]$, where $\partial_t(g(t)) = -2Ric[g(t)]$ by the definition of the unnormalized Ricci flow and define $\frac{d}{dt}(\sigma(t)) = \dot{\sigma}(t)$ and using the definition of the Lie derivative $\frac{d}{dt}[\phi_t^*(g_0)] = \phi_t^*(\mathfrak{L}_X g_0)$ where $X = X(\phi_t(p)) = \frac{d}{dt}(\phi_t(p))$ a t dependent vector field one obtains,

$$-2Ric[g(t)] = \dot{\sigma}(t)\phi_t^* g_0 + \sigma(t)\phi_t^*(\mathfrak{L}_X g_0) \quad . \tag{6.2}$$

If at some $t = \tau$, $\dot{\sigma}(\tau)$ is $> 0, = 0$ or $0 <$ then $g(t)$ is called expanding, steady or shrinking, respectively. The Ricci flow is invariant under the diffeomorphisms $-2Ric[g(t)] = \phi_t^*[-2Ric(g_0)]$, hence all the pull backs can be dropped to get the following equation :

$$-2Ric(g_0) = \dot{\sigma}(t)g_0 + \sigma(t)\mathfrak{L}_X g_0 \quad . \tag{6.3}$$

In the equation (6.3) g_0 is called the Ricci soliton. Defining the vector field $Y = \sigma(t)X$ and $\sigma(t) = 1 + 2\lambda t$ for some constant λ the equation (6.3) becomes :

$$-2Ric(g_0) = 2\lambda g_0 + \mathfrak{L}_X g_0 \quad , \tag{6.4}$$

in which case by rescaling λ can be set to be equal to $\{-1, 0, 1\}$ corresponding to the shrinking, steady and expanding solitons, respectively. If X vanishes then the Ricci soliton is an Einstein metric. Let us use the expansion of the Lie derivative of the metric to write the equation (6.4) in the following form $\quad -2R_{ij} = 2\lambda g_{ij} + \nabla_i X_j + \nabla_j X_i$. The pair (g, X) is a Ricci soliton structure. If the vector field X is the gradient of some function f , then this equation becomes :

$$R_{ij} + \lambda g_{ij} + \nabla_i \nabla_j f = 0 \quad . \tag{6.5}$$

The pair (g,X) satisfying equation (6.5) is called a gradient Ricci soliton. Perelman in [19] defined a functional F,

$$F(f,g) = \int (\mathfrak{R} + |\nabla f|^2) e^{-f} d\mu \tag{6.6}$$

where $|\nabla f|^2 = df(\nabla f)$, $d\mu$ is the volume form and f is a scalar function on the manifold. The Ricci flow is the gradient flow (steepest descent) of this functional. The idea motivating a functional for a flow is that, it should be constant for the solitons of the flow. The evolution of the functional F is given by,

$$\partial_t F(f,g) = \int \left[-v_{ij}(R_{ij} + \nabla_i \nabla_j f) + \left(\frac{g^{ij} v_{ij}}{2} - k \right) (2\Delta f - |\nabla f|^2 + \mathfrak{R}) \right] e^{-f} d\mu \quad , \tag{6.7}$$

where $\partial_t g_{ij} = v_{ij}$ and $\partial_t f = k$. If the new volume form $e^{-f} d\mu$ is conserved then $\partial_t (e^{-f} d\mu) = \left(\frac{g^{ij} v_{ij}}{2} - k \right) e^{-f} d\mu = 0$, so the right hand side of (6.7) must be zero, so that it reduces to,

$$\partial_t F(f,g) = \int -v_{ij}(R_{ij} + \nabla_i \nabla_j f) e^{-f} d\mu \quad . \tag{6.8}$$

The gradient flow is hence given by $v_{ij} = \partial_t g_{ij} = -R_{ij} - \nabla_i \nabla_j f$ which is a steady gradient soliton. The new volume form is conserved if $\partial_t f = \frac{g^{ij} v_{ij}}{2} = \ln \sqrt{g}$.

6.2 Cotton Solitons

A Cotton soliton can be analogously defined. Any metric which is invariant under the flow, a fixed point, is a Cotton soliton. The generalized fixed points of the Cotton flow, as in the case of the Ricci flow, can be constructed to obtain the following equation :

$$C(g_0) = 2\lambda g_0 + \mathfrak{L}_X g_0 \quad . \tag{6.9}$$

The Cotton soliton is the pair (g,X) and it is said to be shrinking, steady or expanding if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively. One solution to this equation is given for locally conformally flat manifolds where X is a Killing vector field and $\lambda = 0$ defining a steady soliton, [2]. However this solution is trivial since conformally flat manifolds are fixed points of the Cotton flow, so they are Cotton solitons.

The functional for the Cotton flow is the Chern-Simons action as mentioned in Chapter 5, [7]:

$$F = -\frac{1}{2}\int \varepsilon^{ijk}\Gamma^{l}_{im}(\partial_j\Gamma^{m}_{kl} + \frac{2}{3}\Gamma^{m}_{jn}\Gamma^{n}_{kl})d^3x \quad . \tag{6.10}$$

The evolution of this functional is, [15] :

$$\partial_t F = \int v_{ij}C^{ij}d\mu \quad . \tag{6.11}$$

Hence the steepest descent of this functional is given by $v_{ij} = \partial_t g_{ij} = C_{ij}$. For the locally conformally flat manifolds, for which $C_{ij} = 0$, the functional is constant.

CHAPTER 7

CONCLUSION

In this thesis, we firstly briefly reviewed the Ricci calculus of Riemannian manifolds in Chapter 2. In Chapters 1 and 3 we tried to make the connections between the model geometries and the flows. The flows are just evolutions equations of the metric that can be used to prove geometrization conjectures. As we mentioned in Chapter 3 the flow equations become simpler when we use the Milnor frame on manifolds rather than any other frame.

Secondly, in Chapters 4 and 5 we studied the Ricci and Cotton flows on the model geometries. The results of these chapters coincide with the previous works which we followed [14, 15], with a few different computational details. We saw that some geometries remained unchanged under the flows and we called them as the fixed points of the flows. However in some geometries singularities showed up. In the case of the Ricci flow shrinking of the whole manifold is removed by adding a normalization term to the flow equation. In the Cotton flow such a singularity has never appeared because it is volume preserving in all cases. In other type of singularities we did not offer any solution. There is no doubt that one expects these model geometries do not change at all or to evolve to a connected sum of them under the flows if they are to be model, but this is not the case although Perelman was able to prove Thurston's geometrization conjecture using the Ricci flow.

Finally, in Chapter 6 we briefly mentioned of the Ricci and Cotton solitons. Extended study about the Ricci solitons can be found in [4]. The Cotton solitons on the other hand are not extensively studied and we only gave one trivial example, another example for Riemannian metrics and examples for Lorentzian metrics can be found in [2].

REFERENCES

[1] Belgun, F. A. (2006). Null-geodesics in complex conformal manifolds and the Lebrun correspondance *J. Reine Angew. Math.* , *2001*(536), 43-63.

[2] Calvino-Louzao, E., Garcia-Rio, E., & Vazquez-Lorenzo, R. (2012). A note on compact Cotton solitons *Class. Quantum Grav.*, *29*(20), 205014.

[3] Carroll S. M. (2004). *Spacetime and Geometry: An Introduction to General Relativity*. San Francisco: Addison Wesley.

[4] Chow, B., & Knopf, D. (2004). *The Ricci Flow: An Introduction*. Mathematical Surveys and Monographs, *110*. Providence: American Mathematical Society.

[5] Chow, B., Chu, S., Glickenstein, D., Guenther, C., Isenberg, J., Ivey, T., Knopf, D., Lu, P., Luo, F., & Ni, L. (2007). *The Ricci Flow: Techniques and Applications Part I:Geometric Aspects*, Mathematical Surveys and Monographs, *135*. Providence: American Mathematical Society.

[6] Cotton, É (1899). Sur les variétés à trois dimensions, *Ann. Fac. d. Sc. Toulouse(2)*, (1), 385-438.

[7] Deser, S., Jackiw, R., & Templeton, S. (1982). Topologically Massive Gauge Theories *Annals of Physics*, *140*(2), 372-411.

[8] DeTurck, D. M. (1983). Deforming metrics in the direction of their Ricci tensors *J. Differ. Geom.*, *18*(1), 157-162.

[9] Eisenhart, L. P. (1960). *Riemannian Geometry*(4th ed.). Princeton: Princeton University Press.

[10] Garcia, A. A., Hehl, F. W., Heinicke, C., & Marcias, A. (2004). The Cotton tensor in Riemannian spacetimes *Class. Quantum Grav.*, *21*(4), 1099-1118.

[11] Gilmore, R., (2005). *Lie Groups, Lie Algebras, and Some of Their Applications*. New York: Dover Publications, Inc..

[12] Hamilton, R. S. (1982). Three-Manifolds with Positive Ricci Curvature *J. Differ. Geom.*, *17*(2), 255-306.

[13] Hamilton, R. S. (1997). Four-Manifolds with Positive Isotropic Curvature *Comm. Anal. Geom.*, *5*(1), 1-92.

[14] Isenberg, J., & Jackson, M. (1992). Ricci flow of locally homogeneous geometries on closed manifolds *J. Differ. Geom.*, *35*(3), 723-741.

[15] Kişisel, A. U. Ö. , Sarıoğlu, Ö., & Tekin, B. (2008). Cotton flow *Class. Quantum Grav.*, *25*(16), 165019.

[16] Kneser, H. (1929). Geschlossen Flachen im dreidimensionalen Mannigfaltigkeiten *Jahresbericht der Deutschen Mathematiker-Vereinigung* , *38*, 248-259.

[17] Milnor, J. (1962). A Unique Decomposition Theorem for 3-Manifolds *Amer. J. Math.*, *84*(1), 1-7.

[18] Milnor, J. (1976). Curvatures of left invariant metrics on Lie groups *Adv. Math.*, *21*(3), 293-329.

[19] Perelman, G. (2002, November 11). The entropy formula for the Ricci flow and its geometric applications. Retrieved 2013, March 14, [arXiv:math/0211159 [math.DG]].

[20] Perelman, G. (2003, March 10) Ricci flow with surgery on three-manifolds. Retrieved 2013, March 14, [arXiv:math/0303109 [math.DG]].

[21] Perelman, G. (2003, July 17) Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. Retrieved 2013, March 14, [arXiv:math/0307245 [math.DG]].

[22] Schutz, B. F. , (1980). *Geometrical Methods of Mathematical Physics*. Cambridge: Cambridge University Press.

[23] Thurston, W. P. (1997). *Three-Dimensional Geometry and Topology Vol. 1*. Princeton Mathematical Series, *35*. Princeton: Princeton University Press.

[24] Wald, R. M. (1984). *General Relativity*. Chicago: The University of Chicago Press.

[25] York, J. W. (1971). Gravitational Degrees of Freedom and the Initial-Value Problem *Phys. Rev. Lett.*, *26*(26), 1656-1658.

APPENDIX A

MAPS BETWEEN MANIFOLDS AND LIE DERIVATIVE

This chapter is based on [3].

Let M and N be two manifolds with dimensions m and n, respectively and assume that there is a map $\phi : M \to N$ between these manifolds. A function $f : N \to \mathbb{R}$ on N can be pulled back via the composition map $\phi^* f = f \circ \phi : M \to \mathbb{R}$ to be a function on M. A vector $X \in T_p M$ can be pushed forward by a map $\phi_* : T_p M \to T_{\phi(p)} N$ so that the action of $\phi_* X \in T_{\phi(p)} N$ on a function f on N is equivalent to the action of X on the pull back of f by ϕ :

$$\underbrace{(\phi_* X)\,(f)}_{\text{both are on } N} = \underbrace{(X)\,(f \circ \phi)}_{\text{both are on } M}.$$

A one-form ω on N can be pulled back by the map ϕ^* so that the action of $\phi^* \omega$ on X is equivalent to the action of $\phi_* X$ on ω :

$$\underbrace{(\omega)\,(\phi_* X)}_{\text{both are on } N} = \underbrace{(\phi^* \omega)\,(X)}_{\text{both are on } M}.$$

If there are coordinate bases for vectors $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial y^j}$ on M and N, respectively then the maps (ϕ_*) and (ϕ^*) can be considered as matrix operators with entries $(\phi_*)^j{}_i = \frac{\partial y^j}{\partial x^i}$ and $(\phi^*)_i{}^j = \frac{\partial y^j}{\partial x^i}$ so that $(\phi_* X)^j = (\phi_*)^j{}_i X^i$ and $(\phi^* \omega)_i = (\phi^*)_i{}^j \omega_j$.

Since a vector is a $(1,0)$ tensor, a $(k,0)$ tensor T on M can be pushed forward so that the action of $(\phi_* T)$ on the one-forms ω on N is equivalent to the action of T on the pulled back one-forms $(\phi^*\omega)$ on M :

$$(\phi_*T)(\omega^{(1)},\omega^{(2)},...,\omega^{(k)}) = T(\phi^*\omega^{(1)},\phi^*\omega^{(2)},...,\phi^*\omega^{(k)}) \ .$$

Similarly since a one-form is a $(0,1)$ tensor, a $(0,l)$ tensor T can be pulled back so that the action of (ϕ^*T) on the vectors X on M is equivalent to the action of T on the pushed forward vectors (ϕ_*X) on N :

$$(\phi^*T)(X^{(1)},X^{(2)},...,X^{(l)}) = T(\phi_*X^{(1)},\phi_*X^{(2)},...,\phi_*X^{(l)}) \ .$$

If ϕ is a diffeomorphism then mixed tensors can be pulled back and pushed forward by using the maps ϕ and ϕ^{-1}. In the case $M = N$, a one-parameter family of diffeomorphisms ϕ_t where $t \in \mathbb{R}$ defines a map $\phi_t : \mathbb{R}\times M \to M$ satisfying $\phi_s \circ \phi_t = \phi_{s+t}$ so that ϕ_0 is the identity map (i.e. $\phi_0 \circ \phi_t = \phi_{0+t} = \phi_t$). This family of mappings induces integral curves for each point p of M, so that M is filled up with integral curves. On an integral curve $\phi_t(p)$ a tensor field T will have different values $T(p)$ at p and $T(\phi_t(p))$ at $\phi_t(p)$. The difference between the pulled back value of $T(\phi_t(p))$ to p , $\phi_t^*[T(\phi_t(p))]$ and $T(p)$ defines the change of the tensor T along this integral curve. For a (k,l) tensor this change can be written as :

$$\Delta_t T^{i_1...i_k}{}_{j_1...j_l}(p) = \phi_t^*[T^{i_1...i_k}{}_{j_1...j_l}(\phi_t(p))] - T^{i_1...i_k}{}_{j_1...j_l}(p) \ .$$

At each point of these curves a tangent vectors X to these curves at $t = 0$ can be defined. These tangent vectors can be thought as the generators of the integral curves so that every smooth vector field generates a unique family of integral curves. Therefore the change of a tensor along an integral curve is equivalent to its change along the associated vector field. The Lie derivative $\mathfrak{L}$ of a tensor T along a vector field X is defined as $\mathfrak{L}_X = \lim|_{t\to 0}\left(\frac{\Delta_t T^{i_1...i_k}{}_{j_1...j_l}}{t}\right)$. The Lie derivative $\mathfrak{L}$ is then a map from (k,l) tensor fields to (k,l) tensor fields which is linear in its arguments ($\mathfrak{L}_X(T+S) = \mathfrak{L}_X T + \mathfrak{L}_X S$) and obeys Leibniz rule ($\mathfrak{L}_X(TS) = (\mathfrak{L}_X T)S + T(\mathfrak{L}_X S)$).

The Lie derivative of a function f is simply the directional derivative of the function $\mathfrak{L}_X f = X(f)X^i\partial_i f$.

The Lie derivative of a vector field Y^j is $\mathfrak{L}_X Y^j = [X,Y]^j = X^i\partial_i Y^j - Y^i\partial_i X^j$ where $[X,Y]$ is previously defined as Lie bracket, so $\mathfrak{L}_X Y = -\mathfrak{L}_Y X$.

The Lie derivative of a one-form field ω_j is $\mathfrak{L}_X \omega_j = X^i \partial_i \omega_j + (\partial_j X^i)\omega_i$.

The Lie derivative of mixed tensors can be generalized using previous results, but for mixed tensors the partial derivatives can be replaced by the covariant derivatives because all the Christoffel symbols would vanish. Now let us expand the Lie derivative of the metric tensor g_{ij} :

$$\begin{aligned}\mathfrak{L}_X g_{ij} &= X^k \nabla_k g_{ij} + (\nabla_i X^l) g_{lj} + (\nabla_j X^l) g_{il}, \\ &= \nabla_i X_j + \nabla_j X_i \ , \\ &= 2\nabla_{(i} X_{j)} \ .\end{aligned}$$

The Lie derivative of a tensor along the vector field X is change in the tensor via the diffeomorphism induced by X. Thus a diffeomorphism is a symmetry of a tensor T if the tensor stays unchanged after being pulled back so that $\phi^* T = T$ and $\mathfrak{L}_X T = 0$.

The symmetries of the metric tensor are called isometries and the vector fields generating these isometries are called Killing vectors, so for a Killing vector X, $\mathfrak{L}_X g_{ij} = 0 \rightarrow \nabla_{(i} X_{j)} = 0$, the latter equation is called Killing's equation.

APPENDIX B

COORDINATE-INVARIANT FORM OF THE COTTON TENSOR

This Chapter follows [15]. In [1] a Cotton 3-form is defined by the equation,

$$C(X,Y)(Z) = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \quad , \tag{B.1}$$

$$(\nabla_X S)(Y,Z) = \partial_X (S(Y,Z)) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z) \quad , \tag{B.2}$$

where X, Y, Z are the vector fields on M and S is the $(0,2)$ Schouten tensor in three dimensions given by the equation,

$$S = Ric - \frac{1}{4}\Re g \quad . \tag{B.3}$$

In the Milnor frame $\{F_1, F_2, F_3\}$ the Schouten tensor is clearly diagonal since the Ricci tensor is diagonal. By using the equations (3.14)-(3.17), its components can be computed as follows :

$$\begin{aligned}
S(F_1,F_1) &= Ric(F_1,F_1) - \frac{1}{4}\Re g(F_1,F_1) \quad , \\
&= R_{11} - \frac{1}{4}\Re g_{11} \quad , \\
&= \frac{A}{2ABC}(\lambda^2A^2 - \mu^2B^2 - \nu^2C^2 + 2\mu\nu BC) - \frac{A}{8ABC}(-\lambda^2A^2 - \mu^2B^2 - \nu^2C^2 + 2\lambda\mu AB + \\
&\quad + 2\lambda\nu AC + 2\mu\nu BC) \quad , \\
&= \frac{A}{8ABC}(5\lambda^2A^2 - 3\mu^2B^2 - 3\nu^2C^2 - 2\lambda\mu AB - 2\lambda\nu AC + 6\mu\nu BC) \quad .
\end{aligned}$$

$$\begin{aligned}
S(F_2,F_2) &= R_{22} - \frac{1}{4}\Re g_{22} \quad , \\
&= \frac{B}{2ABC}(-\lambda^2A^2 + \mu^2B^2 - \nu^2C^2 + 2\lambda\nu AC) - \frac{B}{8ABC}(-\lambda^2A^2 - \mu^2B^2 - \nu^2C^2 + 2\lambda\mu AB + \\
&\quad + 2\lambda\nu AC + 2\mu\nu BC) \quad , \\
&= \frac{B}{8ABC}(-3\lambda^2A^2 + 5\mu^2B^2 - 3\nu^2C^2 - 2\lambda\mu AB + 6\lambda\nu AC - 2\mu\nu BC) \quad .
\end{aligned}$$

$$
\begin{aligned}
S(F_3,F_3) &= R_{33} - \frac{1}{4}\mathfrak{R}g_{33} \quad , \\
&= \frac{C}{2ABC}(-\lambda^2A^2 - \mu^2B^2 + \nu^2C^2 + 2\lambda\mu AB) - \frac{C}{8ABC}(-\lambda^2A^2 - \mu^2B^2 - \nu^2C^2 + 2\lambda\mu AB + \\
&+ 2\lambda\nu AC + 2\mu\nu BC) \quad , \\
&= \frac{C}{8ABC}(-3\lambda^2A^2 - 3\mu^2B^2 + 5\nu^2C^2 + 6\lambda\mu AB - 2\lambda\nu AC - 2\mu\nu BC) \quad .
\end{aligned}
$$

In the equation (B.2) , $(\nabla_X S)(Y,Y) \equiv \partial_{F_i}(S(F_j,F_j)) = 0$. Therefore the only non-vanishing components of the Cotton 3-form are $C(F_i,F_j)(F_k)$ for $i \neq j \neq k$. Hence, using the antisymmetry of the Cotton in X and Y, the non-vanishing component $C_{123} = C(F_1,F_2)(F_3)$ is,
$C(F_1,F_2)(F_3) = -C(F_2,F_1)(F_3) = -S(\nabla_{F_1}F_2,F_3) - S(F_2,\nabla_{F_1}F_3) + S(\nabla_{F_2}F_1,F_3) + S(F_1,\nabla_{F_2}F_3)$.
The other components are similarly expanded. Let us compute C_{123} using the equations of Chapter 3 for each $\nabla_{F_i}F_j$,

$$
\begin{aligned}
C_{123} = &-s\left(\frac{1}{2}\left(\frac{-\lambda A + \mu B + \nu C}{C}\right)F_3, F_3\right) - s\left(F_2, \frac{1}{2}\left(\frac{\lambda A - \mu B - \nu C}{B}\right)F_2\right) + \\
&+ s\left(\frac{1}{2}\left(\frac{-\lambda A + \mu B - \nu C}{C}\right)F_3, F_3\right) + s\left(F_1, \frac{1}{2}\left(\frac{\lambda A - \mu B + \nu C}{A}\right)F_1\right) \quad ,
\end{aligned}
$$

$$
C_{123} = \frac{1}{2}\left(\frac{\lambda A - \mu B + \nu C}{A}\right)s(F_1,F_1) - \frac{1}{2}\left(\frac{\lambda A - \mu B - \nu C}{B}\right)s(F_2,F_2) - \nu s(F_3,F_3).
$$